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COUNTABLY INFINITE-DIMENSIONAL SPACES OF FOURIER QUASICRYSTALS AND CRYSTALLINE MEASURES

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Abstract. Meyer asked what the dimension of the space of crystalline measures with support and spectrum in a given locally finite set $\Lambda \subset \mathbb{R}$ can be. We construct a set Λ on which the Fourier quasicrystals with support and spectrum in Λ form a space of countably infinite Hamel dimension: every such Fourier quasicrystal is a finite combination of a fixed sequence, and the layers of fixed polynomial growth are finite-dimensional, with unbounded dimensions. This determines the Fourier quasicrystals on Λ, not all of its crystalline measures. The blocks of the construction are Poisson formulas on zero sets of Lee–Yang exponential polynomials, with masses that grow polynomially at near-crossings of their sheets; every finite union of blocks is determined by a uniqueness theorem for measures on cones and an algebraic splitting on $\mathbb{P}^1 \times \mathbb{P}^1$, and the infinite union by compactness. We also construct a symmetric locally finite set Λ for which the space $\mathcal{M}_\Lambda$ of all crystalline measures has dimension $\aleph_0$, with finite-dimensional layers of fixed Hermite–Sobolev order. Its blocks use clusters of atoms that imitate derivatives of Dirac masses, placed at random scales, and no nonzero element is a Fourier quasicrystal. The existence of sets with each of these two properties (for the second, without the symmetry) has been formalised and machine-checked in Lean 4, as recorded in Palomar [Pal26c, Pal26b].

1. Introduction

1.1. **Fourier quasicrystals and Meyer's question.** The Poisson summation formula $\sum_{n\in\mathbb{Z}} \varphi(n) = \sum_{n\in\mathbb{Z}} \widehat{\varphi}(n)$ says that the Dirac comb $\delta_{\mathbb{Z}} = \sum_{n\in\mathbb{Z}} \delta_n$ is its own Fourier transform. Following Meyer [Mey16, Mey17a], a *crystalline measure* is a locally finite complex Radon measure $\mu = \sum_\lambda c_\lambda \delta_\lambda$ on $\mathbb{R}$ with locally finite support, which is a tempered distribution and whose distributional Fourier transform $\widehat{\mu}$ is again a measure with locally finite support. Each crystalline measure is a generalized Poisson summation formula,

$$\mu(\widehat{\varphi}) = \sum_s \widehat{\mu}(\{s\})\, \varphi(s) \qquad (\varphi \in C_c^\infty(\mathbb{R})); \tag{1}$$

the right-hand side is a finite sum, and the left-hand side is the distributional pairing, equal to $\lim_{R\to\infty} \sum_\lambda \chi(\lambda/R) c_\lambda \widehat{\varphi}(\lambda)$ for any $\chi \in C_c^\infty$ equal to 1 near 0 (Lemma 3.1). Such formulas go back to Kahane–Mandelbrojt [KM58] and Guinand [Gui59], and were revived by the theory of mathematical quasicrystals [Lag00, BG13, Dys09] and by Meyer's work [Mey16, Mey17a, Mey17b, Mey21, Mey22]. Only μ itself is required to be tempered. If $|\mu|$ and $|\widehat{\mu}|$ are also tempered, that is, if the total variations of μ and $\widehat{\mu}$ grow at most polynomially, μ is a *Fourier quasicrystal* [LO17, KS20, OU20b]; this is the class in which much of the theory takes place [LO17, KS20, OU20b, ACV24, Fav24a], and every crystalline measure whose support and spectrum are separated at a polynomial rate belongs to it [Fav24a, Theorem 1].

For a locally finite set $\Lambda \subset \mathbb{R}$, let $\mathcal{M}_\Lambda$ be the space of crystalline measures with support and spectrum ($= \operatorname{supp} \widehat{\mu}$) in Λ, and $\mathcal{M}_\Lambda^{\mathrm{str}} \subseteq \mathcal{M}_\Lambda$ the subspace of Fourier quasicrystals. Meyer asked: "what is the dimension of $\mathcal{M}_\Lambda$?" [Mey17a, Section 6]. Kulikov, Nazarov and Sodin [KNS25, Section 2.3] consider the two-set version

$$\mathcal{N}(A,B) = \{\mu :\ \mu \text{ crystalline},\ \operatorname{supp}\mu \subseteq B,\ \operatorname{supp}\widehat{\mu} \subseteq A\}, \tag{2}$$

so that $\mathcal{M}_\Lambda = \mathcal{N}(\Lambda,\Lambda)$; as they do, we put the spectrum first. The known answers are small or very large. The integers carry one crystalline measure up to scalars, $\mathcal{M}_{\mathbb{Z}} = \mathbb{C}\delta_{\mathbb{Z}}$; so does

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$\{0\} \cup \{\pm\sqrt{n} : n \geq 1\}$, by the interpolation theorem of Radchenko and Viazovska [RV19] (see [LR21a, Section 1]); for the shifted square roots $\{\pm\sqrt{n+\theta} : n \geq 0\}$, $0 < \theta < 1$, about which Meyer also asks [Mey17a, Section 6], the dimension is 2, 1 or 0, according as $\theta \leq \frac{1}{8}$, $\frac{1}{8} < \theta \leq \frac{3}{8}$ or $\theta > \frac{3}{8}$ [Pal26a]; uniformly discrete supports and spectra allow only generalized Dirac combs [Cór89, LO13, LO15]; and Kulikov, Nazarov and Sodin proved that $\mathcal{N}(\Lambda, M)$ is finite-dimensional for their subcritical pairs and infinite-dimensional for their supercritical pairs [KNS25, Section 2.3, Remark 7.5]. For the supercritical pairs that are separated at a polynomial rate, these infinite-dimensional spaces consist of Fourier quasicrystals by Favorov's theorem (Section 2).

1.2. **Three possible sizes.** "Infinite-dimensional" hides two different possibilities. For Fourier quasicrystals the natural grading is by growth. For an integer $N \geq 0$ let the *strong layer* $X_N(\Lambda)$ consist of the $\mu \in \mathcal{M}_\Lambda$ with

$$\sum_{x\in\Lambda} \frac{|\mu(\{x\})|}{(1+|x|)^N} + \sum_{\xi\in\Lambda} \frac{|\widehat{\mu}(\{\xi\})|}{(1+|\xi|)^N} < \infty,$$

so that $\mathcal{M}_\Lambda^{\mathrm{str}} = \bigcup_N X_N(\Lambda)$. The strong layers are Banach spaces, and a classical argument of Mackey [Mac45] shows (Proposition 3.8) that the Hamel dimension of $\mathcal{M}_\Lambda^{\mathrm{str}}$ is

(i) finite, if the dimensions of the $X_N(\Lambda)$ are bounded;
(ii) $\aleph_0$, if they are all finite but unbounded;
(iii) $\mathfrak{c} = 2^{\aleph_0}$, if some $X_N(\Lambda)$ is infinite-dimensional.

(Countable dimension does not mean countably many measures: the coefficients are complex, so even a one-dimensional space has uncountably many elements. The dimension counts linearly independent directions.) In case (iii) one layer contains infinitely many independent measures, and their convergent infinite combinations produce continuum many. In case (ii) every Fourier quasicrystal on Λ is a finite combination of a fixed sequence, each new member of which grows faster than the previous ones. Constructing infinitely many Fourier quasicrystals on one set cannot distinguish (ii) from (iii): one has to determine *all* of them. The question is whether case (ii) occurs at all.

For the space $\mathcal{M}_\Lambda$ of all crystalline measures the same trichotomy holds with the strong layers replaced by the layers of fixed Hermite order. Let

$$H_s = \Big\{ \sum_{n\geq 0} u_n h_n : \sum_{n\geq 0} (n+1)^{2s} |u_n|^2 < \infty \Big\} \qquad (s \in \mathbb{R})$$

be the Hermite–Sobolev scale built on the Hermite functions h_n, so that $\mathcal{S}(\mathbb{R}) = \bigcap_p H_p$ and $\mathcal{S}'(\mathbb{R}) = \bigcup_p H_{-p}$ [Sim71]. We say that a tempered distribution has *Hermite order at most* p if it lies in H_{-p}; large Hermite order means large growth or low regularity. The layers

$$V_p(\Lambda) = \mathcal{M}_\Lambda \cap H_{-p}, \qquad r_p(\Lambda) = \dim V_p(\Lambda),$$

are closed subspaces of the Hilbert spaces H_{-p}, they exhaust $\mathcal{M}_\Lambda$, and $\dim \mathcal{M}_\Lambda$ is finite, $\aleph_0$ or $\mathfrak{c}$ according as the r_p are bounded, finite and unbounded, or not all finite (Proposition 3.4). In case (ii) no series in the fixed sequence with infinitely many nonzero coefficients converges in $\mathcal{S}'(\mathbb{R})$ (Corollary 3.6).

1.3. **The main result.**

Theorem A. *There exist a locally finite set $\Lambda \subset \mathbb{R}$, Fourier quasicrystals $\nu_1, \nu_2, \ldots$ with support and spectrum in Λ, and integers $2 \leq s_1 < s_2 < \cdots$ such that:*

(i) *every Fourier quasicrystal with support and spectrum in Λ is a finite linear combination of the ν_k, and the ν_k are linearly independent; in particular $\dim_{\mathbb{C}} \mathcal{M}_\Lambda^{\mathrm{str}} = \aleph_0$;*
(ii) *for every integer $N \geq 0$,*

$$2\,\#\{i :\ s_i + 3 \leq N\} \leq \dim X_N(\Lambda) \leq 2\,\#\{i :\ s_i \leq N\};$$

(iii) *for every $i \geq 1$, the measures ν_{2i-1} and ν_{2i} belong to $X_{s_i+3}(\Lambda)$ but not to $X_{s_i-1}(\Lambda)$.*

So case (ii) occurs for Fourier quasicrystals: $\mathcal{M}^{\mathrm{str}}_\Lambda$ is a complete space of summation formulas with polynomially growing masses that is finite-dimensional at every fixed growth exponent and infinite-dimensional overall. The existence assertion of part (i) has been formalised in Lean 4 (Section 1.6). Theorem A determines $\mathcal{M}^{\mathrm{str}}_\Lambda$, but not the space $\mathcal{M}_\Lambda$ of all crystalline measures on Λ, which may be larger; we do not know whether $\mathcal{M}_\Lambda$ is also countably infinite-dimensional (Question 9.1).

1.4. **How measures imitate derivatives.** If derivatives of Dirac masses were allowed, case (ii) would be easy. The tempered distributions T with $\operatorname{supp} T \subseteq \mathbb{Z}$ and $\operatorname{supp} \widehat{T} \subseteq \mathbb{Z}$ are the finite combinations of the $x^j \delta^{(k)}_{\mathbb{Z}}$, $j,k \geq 0$ (Example 3.10, a special case of a theorem of Lev and Reti [LR21b, Theorem 1.1]); this space has countable dimension, and the k-th derivatives enter at higher and higher order. A measure cannot contain derivatives, and indeed $\mathcal{M}_\mathbb{Z} = \mathbb{C}\delta_\mathbb{Z}$.

The starting point of this paper is that clusters of atoms can imitate derivatives. For $a \in \mathbb{R}$,

$$\frac{\delta_{a+\varepsilon} - \delta_{a-\varepsilon}}{2\varepsilon} \longrightarrow -\delta'_a \qquad (\varepsilon \to 0) \quad \text{in } \mathcal{S}'(\mathbb{R}), \tag{3}$$

since pairing with φ gives the difference quotient $(\varphi(a+\varepsilon)-\varphi(a-\varepsilon))/(2\varepsilon) \to \varphi'(a)$. In the same way, $m+1$ atoms within distance ε of a, with masses of size ε^{-m} given by divided differences, imitate $\delta^{(m)}_a$. Two features of (3) matter below. The more atoms a cluster has, the higher the derivative it can imitate, hence the higher the order. And the masses are large while the distribution stays bounded: if the clusters tighten at a polynomial rate, the masses grow polynomially and the measure can still be a Fourier quasicrystal; if they tighten faster, the measure stays tempered but its total variation does not. Meyer already observed, in a variant of Guinand's construction, that "a derivative of the Dirac mass at $\pm 1/2$ peeps out from the Fourier transform" [Mey16]; Olevskii and Ulanovskii note that clustering zeros of exponential polynomials produce derivatives and unbounded coefficients [OU20a].

Theorem A is proved in Section 4, with proofs of the technical lemmas in Appendix A. Its building blocks are Poisson formulas on the zero sets of exponential polynomials, in the spirit of Kurasov and Sarnak [KS20] and Olevskii and Ulanovskii [OU20a]. The atoms of a block lie on s sheets, the real zeros of $\prod_{j\leq s} p_{a_j}(e(x/\lambda), e(\beta x/\lambda - \frac{1}{4}))$, where $p_a(z,w) = 1 - aw - z(w-a)$ is a Lee–Yang polynomial, $e(t) = e^{2\pi i t}$, the a_j are distinct numbers in $(0,1)$, and β is a Liouville number. The sheets never meet, but along a sequence of points they come within a distance of order $|x|^{-1}$ of one another, and there the masses are of size $|x|^{s-1}$. These clusters imitate derivatives at a polynomial rate, so the block measures are Fourier quasicrystals whose growth exponent is determined by s up to a bounded error (Figure 1); the arithmetic of β controls the near-crossings from both sides. The blocks are placed at distinct prime scales and overlaid with growing inner radii, a device of Kolountzakis [Kol16, Section 2], so that the union stays locally finite. Infinitely many independent Fourier quasicrystals give only a lower bound, and a union of sets may well support new ones; the heart of the proof is to show that the union carries *nothing else*. Every finite union of blocks is determined completely, by a uniqueness theorem for measures on cones at a scale whose square is a quadratic irrational and an algebraic splitting of rational functions on $\mathbb{P}^1 \times \mathbb{P}^1$, and the infinite union is then handled by a compactness argument.

1.5. **All crystalline measures.** None of the measures of Theorem A needs masses that grow faster than a polynomial. If arbitrary crystalline measures are allowed, the clusters can tighten at any rate, and the countable case also occurs for the whole space $\mathcal{M}_\Lambda$, on a set where every crystalline measure is determined.

Theorem B. *There exist a symmetric locally finite set $\Lambda \subset \mathbb{R}$ and nonzero crystalline measures $\mu_1, \mu_2, \ldots$ with support and spectrum in Λ such that:*

(i) *every $\mu \in \mathcal{M}_\Lambda$ is a finite linear combination of the μ_k, and the μ_k are linearly independent; in particular $\dim_\mathbb{C} \mathcal{M}_\Lambda = \aleph_0$;*

(ii) *every layer $V_p(\Lambda)$ is finite-dimensional, and the dimensions grow linearly: for every integer $p \geq 1$,*

$$2\lfloor (p-4)/18 \rfloor_+ \leq r_p(\Lambda) \leq 12p - 2, \tag{4}$$

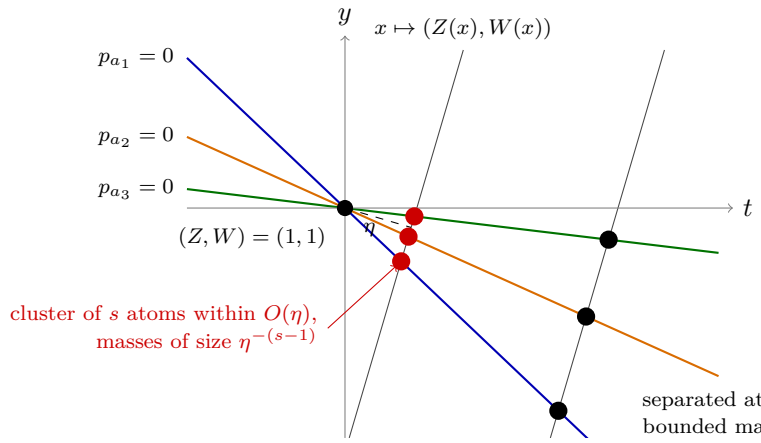


FIGURE 1. The mechanism of Theorem A near the point $(Z, W) = (1, 1)$, in the coordinates $t = x/\lambda$ and $y = \beta x/\lambda - \frac{1}{4}$ modulo 1 (schematic, $s = 3$). All sheets $\{p_{a_j} = 0\}$ pass through this point, with different slopes. The real line is mapped to the line $x \mapsto (Z(x), W(x))$ of slope β, which never meets the point but returns arbitrarily close to it. When it passes at distance η, it crosses the s sheets at s atoms within distance $O(\eta)$ of one another, and the masses, whose denominator $D_\beta P$ is of size η^{s-1}, are of size $\eta^{-(s-1)}$. At these returns η is of order $1/|x|$, from above and from below, by the arithmetic of β (Lemmas 4.1 and 4.3).

where $\lfloor x \rfloor_+ = \max\{0, \lfloor x \rfloor\}$*;*

(iii) *the Hermite orders of the* μ_k *grow linearly: for every* $i \geq 1$*, the measures* μ_{2i-1} *and* μ_{2i} *belong to* $H_{-(18i+4)}$ *but not to* H_{-i}*.*

No nonzero element of this space is a Fourier quasicrystal; in fact neither μ nor $\widehat{\mu}$ is strongly tempered for any nonzero $\mu \in \mathcal{M}_\Lambda$ (Theorem 8.1). Single crystalline measures with this property were constructed by Mazáč, Richard and Strungaru [MRS26, Theorem 4.11, Remark 5.2]; here it holds for every element of an infinite-dimensional space, as a consequence of the mechanism above: a cluster that imitates a derivative of high Hermite order needs masses that are large compared with the distances within the cluster. Theorems A and B thus show that the countable case occurs both among Fourier quasicrystals and among crystalline measures, on different sets, and in Theorem B the whole space $\mathcal{M}_\Lambda$ is determined. The existence assertion of part (i), without the symmetry of Λ, has been formalised in Lean 4 (Section 1.6). Adding the point 0 to Λ, as in Meyer's normalization, changes $\mathcal{M}_\Lambda$ by at most two dimensions (Proposition 3.9).

The set Λ of Theorem B is a union of rescaled copies of one-dimensional "blocks" (Figure 3), and μ_{2i-1}, μ_{2i} are the two copies of the measure of the i-th block. In a block, the atoms $n \pm \gamma_j$ have phases $\gamma_j \to \frac{1}{2}$ at a doubly exponential rate, so that near every half-integer the atoms of two adjacent cells form a cluster, and cells farther out carry more atoms (Figure 2). We show that each block carries exactly one crystalline measure up to scalars, that the speed of clustering controls its Hermite order, and that the block can avoid an arbitrarily large central interval (Section 5, with the proof of the main analytic input in Appendix B). The blocks are again overlaid with growing inner radii, and the exhaustion (Section 6) rests on a short implication. If $T \in \mathcal{M}_\Lambda$ has Hermite order at most p, then its component on each block has Hermite order at most a fixed multiple of p (we obtain $6p$); since the measure of the i-th block has Hermite order larger than P_i, this component vanishes as soon as $P_i \geq 6p$. As $P_i \to \infty$, only finitely many blocks contribute at each order, while every block whose measure has Hermite order at most p does (this holds when $18P_i + 4 \leq p$). The difficulty lies in the phrase "the component of T on a block": one has to show that restricting the spectrum of T to one block gives a tempered distribution whose inverse Fourier transform lives on the reciprocal block, with no additional distribution at the half-integer seams where the atoms accumulate. Random scalings of the blocks, periodic annihilators and the large gaps between the blocks are used for this. One feature

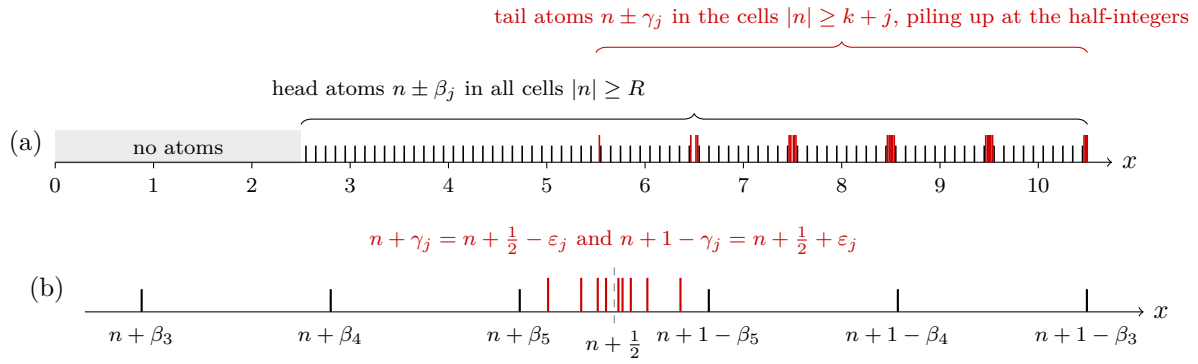


FIGURE 2. One block $C(P, R)$ of Section 5, schematically, with $R = 3$, $k = 2R - 1 = 5$, and only $x \geq 0$ shown (the set is symmetric). (a) There are no atoms in the central interval. The head phases $\beta_j = (2j - 1)/(4k)$ occur in all cells $|n| \geq R$; the tail phases $\gamma_j = \frac{1}{2} - \varepsilon_j$ occur in the cells $|n| \geq k + j$, so that farther cells carry more of them. (b) Near a half-integer $n + \frac{1}{2}$, the tail atoms of the two adjacent cells form a cluster that tightens from both sides. The distances ε_j are greatly exaggerated: in the construction they decrease doubly exponentially. Such clusters play the role of the derivatives of a Dirac mass at $n + \frac{1}{2}$ in the heuristic (3).

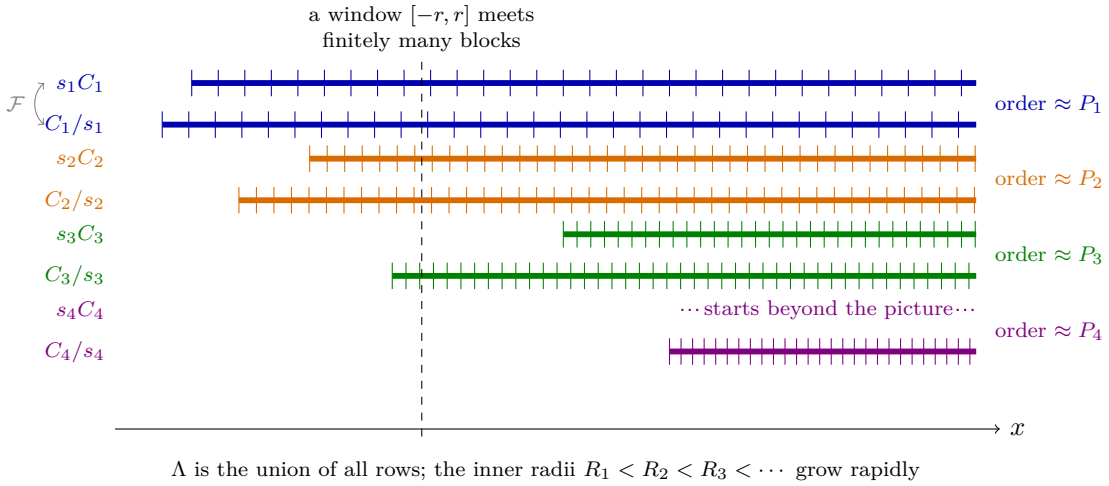


FIGURE 3. The set Λ of Section 6, schematically ($x \geq 0$ shown). The block $C_i = C(P_i, R_i)$ has inner radius about R_i and carries one crystalline measure, of Hermite order between P_i and $18P_i + 4$. It is used twice, dilated by s_i and by $1/s_i$, and the Fourier transform exchanges the two copies: $\mathcal{F}e_{(i,+)} = \zeta_i e_{(i,-)}$. Because the inner radii grow rapidly, every bounded window meets only finitely many blocks, so Λ is locally finite. The main work (Section 6) is to show that Λ carries no crystalline measures other than finite combinations of the block measures.

of the construction is forced: in any countably infinite-dimensional $\mathcal{N}(A, B)$, the elements of Hermite order at most p are determined by their masses in a bounded window, whose size tends to infinity with p (Lemma 3.7), so new elements must appear farther and farther out. What we do not supply, for either theorem, is a general geometric characterization of the sets with countably infinite-dimensional spaces (Question 9.4).

The Hermite orders of the blocks can be prescribed, which gives any rate of growth of the layer dimensions, and the construction has a natural consequence for pairs of sets.

Theorem C. *For every nondecreasing unbounded $F : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ with $F(0) = 0$ there is a symmetric locally finite Λ with $\dim_{\mathbb{C}} \mathcal{M}_\Lambda = \aleph_0$ and $F(\lfloor (p-4)/18 \rfloor_+) - 1 \le r_p(\Lambda) \le F(6p)$ for all $p \ge 1$.*

The same construction yields $\dim \mathcal{M}_\Lambda = \mathfrak{c}$ when the block orders stay bounded (Corollary 6.7). Together with $\mathcal{M}_{N^{-1}\mathbb{Z}}$, of dimension N^2 (Example 3.10), and the subsets considered below, the dimensions 0, N^2, every even integer, $\aleph_0$ and $\mathfrak{c}$ all occur for spaces $\mathcal{M}_\Lambda$.

Corollary D. *There exist disjoint nonempty locally finite sets $A, B \subset \mathbb{R}$ with $\dim_{\mathbb{C}} \mathcal{N}(A, B) = \aleph_0$ and $\mathcal{M}_A = \mathcal{M}_B = \{0\}$. Moreover, for the set Λ of Theorem B and all $A', B' \subseteq \Lambda$, the space $\mathcal{N}(A', B')$ is spanned by the μ_k it contains, so every finite dimension and $\aleph_0$ occur as $\dim \mathcal{N}(A', B')$.*

Taking products with the lattice $\mathbb{Z}^{d-1}$ gives the analogous sets in $\mathbb{R}^d$ for every $d \ge 2$, for Theorems A and B alike (Proposition 7.5).

1.6. **Formal verification.** The existence statements of Theorems A and B have been formalised in Lean 4 with Mathlib and checked by Palomar, a public registry of machine-checked Lean results; each record fixes a statement of record, a repository at a pinned commit, the toolchain and the admitted axioms (`propext`, `Classical.choice` and `Quot.sound`).

- *Fourier quasicrystals* [Mar26, Pal26c]: there is a locally finite Λ with $\dim_{\mathbb{C}} \mathcal{M}_\Lambda^{\mathrm{str}} = \aleph_0$, with $\mathcal{M}_\Lambda^{\mathrm{str}}$ defined through the strong layers above and Mathlib's Fourier transform. This is the existence assertion of Theorem A(i). The formal witness is built from blocks of the kind used in Section 4, but with a different choice of the primes, sheet numbers and parameters, and with radii selected by classical choice; it is therefore not the set of Theorem 4.12.
- *All crystalline measures* [MS26, Pal26b]: there is a locally finite Λ with $\dim_{\mathbb{C}} \mathcal{M}_\Lambda = \aleph_0$, every element of $\mathcal{M}_\Lambda$ being a unique finite combination of a fixed sequence. This is the existence and finite-synthesis assertion of Theorem B(i), without the symmetry of Λ.

The explicit bounds in parts (ii) and (iii) of both theorems, the identification of the bases, Theorems C and 8.1, Corollary D, and Section 7 are not part of either statement of record. Appendix E describes the two statements of record and indicates where the main steps of the proofs are formalised.

1.7. **Organization.** The paper has two parts, which share the framework of Section 3: the layers, and the trichotomies for Hermite layers and for strong layers.

Part I, Fourier quasicrystals (Section 4 and Appendix A). The logical chain is

$$\text{Lems. 4.1–4.6} \longrightarrow \text{Prop. 4.9} \longrightarrow \text{Cor. 4.10} \longrightarrow \text{Thm. 4.12} \longrightarrow \text{Thm. A},$$

where Proposition 4.9 also uses the uniqueness theorem Proposition 4.7 and the splitting Lemma 4.8; a more detailed guide is given at the beginning of Section 4.

Part II, all crystalline measures (Sections 5–8). Section 5 constructs the blocks; its main analytic input, Theorem 5.1, is proved in Appendix B. Section 6 proves Theorems B and C, with technical lemmas in Appendices C and D. The logical chain is

$$\text{Thm. 5.1} \longrightarrow \text{Prop. 5.3} \longrightarrow \text{Thm. 6.1} \longrightarrow \text{Cor. 6.7} \longrightarrow \text{Thms. B and C}.$$

Section 7 derives Corollary D, a reformulation in terms of Schwartz space and the analogues in higher dimensions, and Section 8 shows that no nonzero measure of Part II is a Fourier quasicrystal.

Section 2 relates our results to the literature, Section 9 collects open questions, and Appendix E describes the two formal developments.

2. Relation to the literature

We discuss the literature under three headings: what was known about the dimension of spaces of crystalline measures, why the restriction to measures is essential, and what is new in the constructions. Work on strong temperedness and on the classification of summation formulas is discussed in Section 8, together with Theorem 8.1.

2.1. **What was known about dimensions.** Poisson-type summation formulas not coming from lattices go back to Kahane–Mandelbrojt [KM58] and Guinand [Gui59], and interest in them was renewed by the theory of mathematical quasicrystals; see Lagarias [Lag00], Baake and Grimm [BG13], and Dyson's proposal to classify one-dimensional quasicrystals [Dys09]. Meyer [Mey16, Mey17a] introduced crystalline measures in the form used here, gave a new proof of Guinand's formula, and constructed crystalline measures that are not generalized Dirac combs; see also [Mey21, Mey22, Mey23, Mey23b] and, for formulas involving $r_k(n)$, Lev and Reti [LR21a]. Further constructions use stable and Lee–Yang polynomials: Kurasov and Sarnak [KS20] found positive crystalline measures and Fourier quasicrystals that are not generalized Dirac combs; Olevskii and Ulanovskii characterized Fourier quasicrystals with unit masses [OU20b] and showed that pairs of exponential polynomials generate Poisson-type formulas [OU20a]; and Alon, Cohen and Vinzant [ACV24] completed the description of Fourier quasicrystals with positive integer masses (see also [AV24, AKKV25]). Modular forms [AKM25, BSS23] and inner functions [SZ26] give further examples, and Tselishchev [Tse26] found a new crystalline-type measure with unit masses, with an application to tilings of the line by translates of a function. These works construct individual measures or explicit families, often under restrictions on the masses, such as positivity or integrality, which do not define vector spaces. Meyer's question [Mey17a, Section 6] concerns instead the whole space $\mathcal{M}_\Lambda$ attached to a prescribed set. (Crystalline measures should not be confused with measures with pure point diffraction; see Richard and Strungaru [RiS17].)

Three kinds of answers were known. The first is *rigidity*. If the support and the spectrum are uniformly discrete, the structure is rigid: Córdoba [Cór89] proved that the support is a finite union of translated lattices if the masses take finitely many values, and Lev and Olevskii [LO13, LO15] proved that in dimension one every such measure is a generalized Dirac comb, that is, a finite combination of translated and modulated lattice combs; this answered a problem of Lagarias [Lag00, Problem 4.1]. In dimension one, uniform discreteness of the spectrum can be relaxed to bounded density [LO17, Theorem 2.2]. Lev and Reti [LR21b] extended the structure theorem to tempered distributions, and Favorov [Fav16, Fav21, Fav23] proved further rigidity and uniqueness theorems.

The second is *interpolation*. By duality, $\mathcal{N}(A, B)$ is the space of linear relations between the restrictions $\varphi|_A$ and $\widehat{\varphi}|_B$ of Schwartz functions (Proposition 3.2), so Fourier interpolation theorems compute dimensions. Radchenko and Viazovska [RV19] proved that an even Schwartz function is determined by its values and those of its Fourier transform at $\pm\sqrt{n}$, and that the data map is an isomorphism onto a closed subspace of codimension one, cut out by Poisson's formula [RV19, Theorem 2]. This was extended to powers of integers and perturbed nodes by Ramos and Sousa [RS22, RS23], whose Fredholm-index argument yields a dichotomy between interpolation and a one-dimensional defect [RS23, Theorem 5.3]; to zeros of zeta and L-functions by Bondarenko, Radchenko and Seip [BRS23]; and to spheres in $\mathbb{R}^d$ by Stoller [Sto21] and Ramos and Stoller [RSt22]. For Meyer's shifted square roots $\{\pm\sqrt{n+\theta} : n \geq 0\}$, $0 < \theta < 1$, which lie in the critical case of [KNS25] and are not covered by the perturbation results of [RS23], the space has been determined in a formalised classification [Pal26a]: it is spanned by an even measure with Fourier eigenvalue 1 when $\theta \leq \frac{1}{8}$ and an odd measure with eigenvalue $-\mathrm{i}$ when $\theta \leq \frac{3}{8}$, both built from generalized eta products. Density thresholds were obtained by Kulikov [Kul21] and Adve [Adv23], critical and asymmetric pairs were treated by Lysen [Lys25], the interpolation basis was studied in [BBRSS25], and Bondarenko and Seip [BS26] showed, with nonzero Fourier-invariant functions vanishing on $\{\sqrt{n}/(\log(e+n))^\beta\}$, $0 \leq \beta \leq \frac{1}{2}$, that uniqueness depends on the function class. See Cohn [Coh24] and Viazovska [Via18, Via23] for surveys, including the connection with sharp sphere packings, and Meyer [Mey22, Section 8] for the relation with the problem of recovering f from $f|_\Lambda$ and $\widehat{f}|_F$.

The third is *infinite dimension without a determination of the space*. Kulikov, Nazarov and Sodin [KNS25] proved that supercritical discrete pairs are uniqueness pairs and subcritical pairs are not, and deduced that $\mathcal{N}(\Lambda, M)$ is finite-dimensional for subcritical and infinite-dimensional for supercritical pairs [KNS25, Section 2.3, Remark 7.5]. Their model pairs, such as $\Lambda = M = \{\mathrm{sign}(j)\sqrt{a|j|} : j \in \mathbb{Z}\}$ with $0 < a < 1$, are separated at a polynomial

rate, and every supercritical pair contains a supercritical subpair with this property [KNS25, Claim 1]; for such pairs every element of $\mathcal{N}(\Lambda, M)$ is a Fourier quasicrystal by Favorov's theorem [Fav24a, Theorem 1] (Section 8). Thus infinite-dimensional spaces of Fourier quasicrystals were known, again without a determination of their dimension. The closest precedent for an infinite-dimensional space of relations built from finite-dimensional pieces is due to Stoller [Sto21, Proposition 7.1]: for $d \geq 5$, the relations between the restrictions of a Schwartz function on $\mathbb{R}^d$ and of its Fourier transform to the spheres $\sqrt{n}\, S^{d-1}$ form an infinite-dimensional space containing all finite sums of suitable modular forms of weights $d/2 + m$, $m \geq 0$, each weight contributing finitely many dimensions. There the grading is by modular weight and the whole relation space is not determined. By Proposition 3.4, deciding between dimensions $\aleph_0$ and $\mathfrak{c}$ requires control of every layer, and we are not aware of an earlier example in which the whole space was determined and found to be countably infinite-dimensional, either for crystalline measures or for Fourier quasicrystals.

2.2. **Why the restriction to measures is essential.** For tempered distributions the countable case occurs already for the lattice: the distributions with support and spectrum in $\mathbb{Z}$ form a space of countable dimension (Example 3.10), a special case of the structure theorem of Lev and Reti [LR21b, Theorem 1.1], by which tempered distributions with uniformly discrete support and spectrum are obtained from Poisson's formula by finitely many shifts, modulations, differentiations, multiplications by polynomials and linear combinations. Derivatives raise the Hermite order, and countable dimension comes from adding more and more of them. For measures nothing of this kind is available, and $\mathcal{M}_{\mathbb{Z}} = \mathbb{C}\delta_{\mathbb{Z}}$. Derivatives do appear at the edge of the theory of measures. Guinand's formula with nodes $\pm\sqrt{n}$ and weights $r_3(n)$ contains a derivative of the Dirac mass at the origin [Gui59, Mey16]; Meyer observed that a natural variant of his construction is not crystalline, because "a derivative of the Dirac mass at $\pm 1/2$ peeps out from the Fourier transform" [Mey16]; and Olevskii and Ulanovskii [OU20a] noted that multiple zeros of exponential polynomials, together with the failure of uniform discreteness, lead to derivatives in the formula and to unbounded coefficients. Our construction turns this phenomenon into a mechanism: clusters of atoms imitate derivatives, but remain measures (Section 1.4).

2.3. **What is new in the constructions.** The blocks of our main construction (Section 4) belong to a well-studied family. Kurasov and Sarnak [KS20] obtained Fourier quasicrystals with positive integer masses from Lee–Yang polynomials restricted to lines in a torus, and Alon, Cohen and Vinzant [ACV24] completed the description of Fourier quasicrystals with positive integer masses in these terms. Olevskii and Ulanovskii [OU20a] proved that every pair (ψ, φ) of exponential polynomials, where the zeros of φ are simple and form a uniformly discrete set, gives a Poisson-type formula with masses $\psi(\lambda)/\varphi'(\lambda)$; they remark that without these assumptions the formula may contain derivatives and the masses may be unbounded, and set this case aside. Our block measures have masses of exactly this form, with zero sets that are not uniformly discrete and masses that grow polynomially; related constructions with Blaschke factors are Meyer's curved model sets [Mey23c] and the measures of Szehr and Zarouf [SZ26]. Every element of the spaces of Theorem A satisfies the growth hypotheses of Gonçalves and Vedana [GV25], so each individual summation formula is covered by their classification; the classification concerns single formulas, not the space attached to a prescribed set. What is new in Section 4 is the control of the growth exponent of every element of a block, from above and from below, through the near-crossings of its sheets, and the complete determination of the Fourier quasicrystals on unions of blocks, which uses a uniqueness theorem for measures on cones (Proposition 4.7) and an algebraic splitting lemma on $\mathbb{P}^1 \times \mathbb{P}^1$ (Lemma 4.8).

Overlaying crystalline measures whose supports and spectra avoid larger and larger intervals is a device of Kolountzakis [Kol16, Section 2], who used it to give a simpler construction of crystalline measures, found by Lev and Olevskii [LO16], whose support meets every arithmetic progression in finitely many points; the device is also used in [Fav24b, MRS26]. We use it in both constructions; the growing gaps keep the union locally finite and produce infinitely many independent measures. Without an exhaustion the overlay gives only a lower bound for the

dimension, which cannot distinguish $\aleph_0$ from $\mathfrak{c}$. For all crystalline measures, what is new is, first, the blocks: whole spaces of crystalline measures that are one-dimensional and of prescribed Hermite order, produced by clustering atoms (Section 5); and second, the exhaustion: a proof that the union of the blocks carries no crystalline measures other than finite combinations of the block measures (Section 6). The sets of Theorems B and C are not uniformly discrete, and the coalescence of their atoms is essential.

3. The functional-analytic framework

3.1. **Conventions.** We write $\widehat{f} = \mathcal{F}f$, $\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi \mathrm{i}x\xi}\,dx$, on $\mathcal{S} = \mathcal{S}(\mathbb{R})$ and extend $\mathcal{F}$ to $\mathcal{S}' = \mathcal{S}'(\mathbb{R})$ by $(\mathcal{F}T)(\varphi) = T(\mathcal{F}\varphi)$, using the complex-bilinear pairing. Thus $\mathcal{F}^2T$ is the reflection $\check{T}(\varphi) = T(\varphi(-\cdot))$. For $h \in \mathbb{R}$ let $\tau_h f(x) = f(x+h)$, extended to $\mathcal{S}'$ by transposition, so that $\mathcal{F}(e^{2\pi \mathrm{i}\omega x}T) = \tau_{-\omega}\mathcal{F}T$. For $t > 0$ let

$$(D_tT)(\varphi) = \sqrt{t}\,T(\varphi(t\,\cdot)),$$

so that $D_tf(x) = t^{-1/2}f(x/t)$ on functions, $D_t\delta_x = \sqrt{t}\,\delta_{tx}$, and $\mathcal{F}D_t = D_{1/t}\mathcal{F}$.

Let $(h_n)_{n\ge 0}$ be the Hermite functions, normalized in $L^2(\mathbb{R})$, so that $\mathcal{F}h_n = (-\mathrm{i})^n h_n$, and let H_s be the Hermite–Sobolev spaces defined in the introduction, with norm $\|u\|_s^2 = \sum_n (n+1)^{2s}|u_n|^2$. Then $\mathcal{S} = \bigcap_{p\in\mathbb{N}} H_p$ with its Fréchet topology, $\mathcal{S}' = \bigcup_p H_{-p}$, and the pairing of H_{-p} with H_p extends the distributional pairing, $|T(\varphi)| \le \|T\|_{-p}\,\|\varphi\|_p$ [Sim71], [RS80, Section V.3 and its appendix]. For integers $p \ge 0$ the norm $\|\cdot\|_p$ is equivalent to $\big(\sum_{a+b\le 2p} \big\|x^a\partial^b\varphi\big\|_{L^2}^2\big)^{1/2}$. The Fourier transform is unitary on every H_s. Consequently translations, modulations $e^{2\pi \mathrm{i}ax}$, and dilations D_t with t in a compact subset of $(0,\infty)$ are bounded on $H_{\pm p}$ for every integer p, with constants that are locally uniform in the parameters; translation by h has norm at most $C_p(1+|h|)^{2p}$ on $H_{\pm p}$. A tempered distribution has *Hermite order at most* p if it belongs to H_{-p}. The word *order* without qualification refers to the classical order of a distribution (the number of derivatives in a local representation).

A set $C \subset \mathbb{R}$ is *locally finite* if it meets every bounded interval in a finite set; it is then closed and at most countable. A tempered distribution T is *locally atomic on* C if there are coefficients $a_c \in \mathbb{C}$ such that $T(f) = \sum_{c\in C} a_c f(c)$ for all $f \in C_c^\infty(\mathbb{R})$ (a finite sum for each f); the a_c are the *masses* of T. For locally finite A, B we let $\mathcal{N}(A,B)$ be the space of $T \in \mathcal{S}'$ that are locally atomic on B and whose Fourier transform is locally atomic on A; this is the space (2) of crystalline measures with support in B and spectrum in A. We put $\mathcal{M}_\Lambda = \mathcal{N}(\Lambda,\Lambda)$,

$$V_p(A,B) = \mathcal{N}(A,B) \cap H_{-p}, \qquad r_p(A,B) = \dim V_p(A,B),$$

where r_p is a Hilbert dimension (which coincides with the Hamel dimension when it is finite), and $V_p(\Lambda) = V_p(\Lambda,\Lambda)$, $r_p(\Lambda) = r_p(\Lambda,\Lambda)$. Unless stated otherwise, dim of $\mathcal{N}(A,B)$ denotes the complex Hamel dimension. A measure ν is *strongly tempered* if $\int (1+|x|)^{-N}\,d|\nu|(x) < \infty$ for some $N \ge 0$; equivalently, $|\nu|([-R,R])$ grows at most polynomially in R.

3.2. **The annihilator description.** For a locally finite set C let $I_C = \{\varphi \in \mathcal{S} : \varphi(c) = 0 \text{ for all } c \in C\}$. It is essential that I_C is defined by the vanishing of values: distributions supported in C may contain derivatives of Dirac masses, and these are not annihilated by I_C.

Lemma 3.1. *Let C be locally finite. A tempered distribution T is locally atomic on C if and only if $T(\varphi) = 0$ for every $\varphi \in I_C$. In that case $T(\varphi) = \lim_{R\to\infty} \sum_{c\in C} a_c\chi(c/R)\varphi(c)$ for every $\varphi \in \mathcal{S}$ and every $\chi \in C_c^\infty$ equal to 1 near 0.*

Proof. Suppose T is locally atomic on C. For $\varphi \in \mathcal{S}$ let $\chi_R(x) = \chi(x/R)$; then $T(\chi_R\varphi) = \sum_c a_c\chi_R(c)\varphi(c)$, and $\chi_R\varphi \to \varphi$ in $\mathcal{S}$, which proves the last formula; in particular $T(\varphi) = 0$ if $\varphi \in I_C$. Conversely, suppose $T(I_C) = 0$. For $c \in C$ choose $\eta_c \in C_c^\infty$ with $\eta_c(c) = 1$ whose support contains no other point of C. For $f \in C_c^\infty$, the sum $\sum_c f(c)\eta_c$ is finite and $f - \sum_c f(c)\eta_c \in I_C$. Hence $T(f) = \sum_c T(\eta_c)f(c)$, and $T(\eta_c)$ does not depend on the choice of η_c. □

Proposition 3.2. *For locally finite $A, B \subset \mathbb{R}$,*

$$\mathcal{N}(A,B) = (I_B + \mathcal{F}I_A)^\perp = \{T \in \mathcal{S}' : T(\varphi) = 0 \text{ for all } \varphi \in I_B + \mathcal{F}I_A\}.$$

In particular $\mathcal{N}(A,B) \neq \{0\}$ if and only if the closure of $I_B + \mathcal{F}I_A$ in $\mathcal{S}$ is a proper subspace.

Proof. Apply Lemma 3.1 to T and $C = B$, and to $\mathcal{F}T$ and $C = A$; the second condition reads $T(\mathcal{F}\psi) = 0$ for $\psi \in I_A$. The last assertion follows from the Hahn–Banach theorem. □

Thus an element of $\mathcal{N}(A,B)$ is the same as a continuous linear functional on $\mathcal{S}$ that depends only on $\varphi|_B$ and also only on $(\mathcal{F}^{-1}\varphi)|_A$; it encodes the summation formula (1).

3.3. **Layers and the trichotomy.** Lemma 3.3, Proposition 3.4, Lemma 3.5 and Corollary 3.6 use only that $\mathcal{N}(A,B)$ is a weak*-closed subspace of $\mathcal{S}'$, and they hold, with the same proofs, for every such subspace (for the upper bound in Proposition 3.4 one uses that $\mathcal{S}'$ has cardinality $\mathfrak{c}$). Lemma 3.7 is specific to $\mathcal{N}(A,B)$.

Lemma 3.3. *Each $V_p(A,B)$ is a closed subspace of H_{-p}. The spaces $V_p(A,B)$ increase with p and their union is $\mathcal{N}(A,B)$. Moreover $r_p(A,B)$ equals the codimension in H_p of the H_p-closure of $I_B + \mathcal{F}I_A$.*

Proof. For fixed $\varphi \in \mathcal{S}$ the map $T \mapsto T(\varphi)$ is continuous on H_{-p}, so by Proposition 3.2 $V_p(A,B)$ is an intersection of closed hyperplanes. The inclusions $H_{-p} \subset H_{-p-1}$ preserve the distribution represented, and $\mathcal{S}' = \bigcup_p H_{-p}$. Finally $V_p(A,B)$ is the annihilator in $H_{-p} = (H_p)'$ of $I_B + \mathcal{F}I_A$, hence is isomorphic to the dual of the quotient of H_p by the closure of $I_B + \mathcal{F}I_A$. □

Proposition 3.4. *Let $A, B \subset \mathbb{R}$ be locally finite. Exactly one of the following holds.*

(i) *All $r_p(A,B)$ are finite and bounded. Then $V_p(A,B) = \mathcal{N}(A,B)$ for all large p, and $\dim \mathcal{N}(A,B) = \sup_p r_p(A,B) < \infty$.*
(ii) *All $r_p(A,B)$ are finite and $\sup_p r_p(A,B) = \infty$. Then $\dim \mathcal{N}(A,B) = \aleph_0$.*
(iii) *Some $r_p(A,B)$ is infinite. Then $\dim \mathcal{N}(A,B) = \mathfrak{c}$.*

Proof. If all layers are finite-dimensional, the union of bases of the layers is a countable spanning set of $\mathcal{N}(A,B)$. If the ranks are bounded, the increasing sequence of layers stabilizes, which gives (i); otherwise $\mathcal{N}(A,B)$ is infinite-dimensional and countably spanned, which gives (ii). In case (iii) let (e_n) be an orthonormal sequence in an infinite-dimensional layer V_p. For $0 < t < 1$ the vector $v_t = \sum_n t^n e_n$ lies in V_p, and pairing a relation between $v_{t_1}, \dots, v_{t_k}$ (t_j distinct) with $e_0, \dots, e_{k-1}$ gives a Vandermonde system; hence the v_t are linearly independent. This is Mackey's argument [Mac45, Theorem I-1]; see also [Lac73]. Conversely, an element of $\mathcal{N}(A,B)$ is determined by its masses on the countable set B (Lemma 3.1), so $|\mathcal{N}(A,B)| \le |\mathbb{C}^B| \le \mathfrak{c}$. □

3.4. **The countable case.**

Lemma 3.5. *Let $T_n \in \mathcal{N}(A,B)$ and suppose that $T_n(\varphi)$ converges for every $\varphi \in \mathcal{S}$. Then there is p such that all T_n and the limit $T = \lim_n T_n$ belong to $V_p(A,B)$.*

Proof. Since $\mathcal{S}$ is a Fréchet space, the Banach–Steinhaus theorem shows that $\{T_n\}$ is equicontinuous: there are C and p with $|T_n(\varphi)| \le C\,\|\varphi\|_p$ for all n and φ. Hence all T_n, and T, lie in the ball of radius C of H_{-p}, and $T \in \mathcal{N}(A,B)$ because $\mathcal{N}(A,B)$ is an annihilator. □

Corollary 3.6. *Suppose $\dim \mathcal{N}(A,B) = \aleph_0$, and let (ν_k) be a basis of $\mathcal{N}(A,B)$ such that each H_{-p} contains only finitely many ν_k. If the partial sums of $\sum_k c_k\nu_k$ converge in $\mathcal{S}'$, then $c_k = 0$ for all but finitely many k.*

Proof. By Lemma 3.5 all partial sums lie in some V_p, hence so does $c_k\nu_k$ for every k, and $c_k = 0$ whenever $\nu_k \notin H_{-p}$. □

Lemma 3.7. *Let $V_p(A,B)$ be finite-dimensional. Then there is $R > 0$ such that every $T \in V_p(A,B)$ is determined by its masses at the points of $B \cap (-R,R)$; in particular $r_p(A,B) \le \#(B \cap (-R,R))$. If $\dim \mathcal{N}(A,B) = \aleph_0$ and R_p denotes the infimum of such R, then $R_p \to \infty$.*

Proof. For $R \in \mathbb{N}$ let K_R be the subspace of $T \in V_p(A,B)$ with no mass in $B \cap (-R,R)$. These finite-dimensional spaces decrease in R, so they are eventually constant, and their intersection is $\{0\}$ because an element of $\mathcal{N}(A,B)$ is determined by its masses. In case (ii) the ranks r_p are unbounded, so $R_p \to \infty$. □

In words: in a countably infinite-dimensional space the elements of bounded Hermite order are detected in a bounded window, and new independent elements must appear farther and farther from the origin. In the construction of Section 6 the elements of Hermite order at most p are combinations of the generators of finitely many blocks, whose inner radii tend to infinity.

3.5. **Strong layers.** For $T \in \mathcal{N}(A,B)$ we write $T(\{x\})$ for the mass of T at x. For an integer $N \geq 0$ put

$$\|T\|_{(N)} = \sum_{x \in B} \frac{|T(\{x\})|}{(1+|x|)^N} + \sum_{\xi \in A} \frac{|\mathcal{F}T(\{\xi\})|}{(1+|\xi|)^N} \in [0,\infty],$$

and let $X_N(A,B) = \{T \in \mathcal{N}(A,B) : \|T\|_{(N)} < \infty\}$ and $\mathcal{N}^{\mathrm{str}}(A,B) = \bigcup_N X_N(A,B)$. Thus $\mathcal{N}^{\mathrm{str}}(A,B)$ is the space of Fourier quasicrystals with support in B and spectrum in A. We write $X_N(\Lambda) = X_N(\Lambda,\Lambda)$ and $\mathcal{M}^{\mathrm{str}}_\Lambda = \mathcal{N}^{\mathrm{str}}(\Lambda,\Lambda)$. If $\|T\|_{(N)} < \infty$, then T is the measure $\sum_x T(\{x\})\delta_x$ and $T(\varphi) = \sum_x T(\{x\})\varphi(x)$ for $\varphi \in \mathcal{S}$, with absolute convergence, by Lemma 3.1 and dominated convergence; the same holds for $\mathcal{F}T$.

Proposition 3.8. *Let $A, B \subset \mathbb{R}$ be locally finite. Each $X_N(A,B)$ is a Banach space with the norm $\|\cdot\|_{(N)}$, and $X_N(A,B) \subseteq X_{N+1}(A,B)$. Exactly one of the following holds.*

(i) *The dimensions of the $X_N(A,B)$ are finite and bounded. Then* $\dim \mathcal{N}^{\mathrm{str}}(A,B) = \sup_N \dim X_N(A,B) < \infty$.
(ii) *The dimensions of the $X_N(A,B)$ are finite and unbounded. Then* $\dim \mathcal{N}^{\mathrm{str}}(A,B) = \aleph_0$.
(iii) *Some $X_N(A,B)$ is infinite-dimensional. Then* $\dim \mathcal{N}^{\mathrm{str}}(A,B) = \mathfrak{c}$.

Proof. Since $(1+|x|)^{-N-1} \leq (1+|x|)^{-N}$, the layers increase with N, and $\|\cdot\|_{(N)}$ is a norm because an element of $\mathcal{N}(A,B)$ is determined by its masses. Let (T_n) be a Cauchy sequence in $X_N(A,B)$. The records $(T_n(\{x\}))_{x\in B}$ and $(\mathcal{F}T_n(\{\xi\}))_{\xi\in A}$ converge in the weighted ℓ^1 norms to limits (c_x) and (d_ξ), and we put $\mu = \sum_x c_x\delta_x$ and $\nu = \sum_\xi d_\xi\delta_\xi$. For $\varphi \in \mathcal{S}$ the function $(1+|x|)^N\mathcal{F}\varphi$ is bounded, so $T_n(\mathcal{F}\varphi) = \sum_x T_n(\{x\})\mathcal{F}\varphi(x) \to \mu(\mathcal{F}\varphi)$, and likewise $\mathcal{F}T_n(\varphi) \to \nu(\varphi)$. As $T_n(\mathcal{F}\varphi) = \mathcal{F}T_n(\varphi)$, we get $\mathcal{F}\mu = \nu$. Hence $\mu \in X_N(A,B)$ and $T_n \to \mu$, so $X_N(A,B)$ is complete. It is separable, being isometric to a subspace of a weighted ℓ^1 space over the countable set $B \sqcup A$.

If all layers are finite-dimensional, cases (i) and (ii) follow as in the proof of Proposition 3.4. In case (iii), an infinite-dimensional separable Banach space has Hamel dimension $\mathfrak{c}$ [Lac73], so $\dim \mathcal{N}^{\mathrm{str}}(A,B) \geq \mathfrak{c}$, and the upper bound follows from $|\mathcal{N}(A,B)| \leq \mathfrak{c}$. □

The Hermite layers V_p and the strong layers X_N measure different things, growth in the sense of distributions and growth of the total variation. Theorem A gives a set for which Proposition 3.8 is in case (ii). By Theorem 8.1, the set of Theorem B lies in case (ii) of Proposition 3.4 and in case (i), with $\mathcal{M}^{\mathrm{str}}_\Lambda = \{0\}$, of Proposition 3.8.

3.6. **Covariance and finite modifications.**

Proposition 3.9. *Let $A, B \subset \mathbb{R}$ be locally finite.*

(a) *For $a, b \in \mathbb{R}$ and $t > 0$, the map induced by $f(x) \mapsto e^{2\pi i a x} t^{-1/2} f((x-b)/t)$ is a linear isomorphism $\mathcal{N}(A,B) \to \mathcal{N}(a + t^{-1}A,\, b + tB)$ which maps V_p onto V_p for every integer p.*
(b) *$\mathcal{F}$ is a linear isomorphism $\mathcal{N}(A,B) \to \mathcal{N}(-B,A)$ mapping V_p onto V_p. In particular $\mathcal{F}\mathcal{M}_\Lambda = \mathcal{M}_\Lambda$ if $\Lambda = -\Lambda$.*
(c) *Let $A_1, B_1 \subset \mathbb{R}$ be locally finite with $k = \#(A_1 \triangle A) + \#(B_1 \triangle B) < \infty$. Then $\mathcal{N}(A \cap A_1, B \cap B_1)$ has codimension at most k in both $\mathcal{N}(A,B)$ and $\mathcal{N}(A_1,B_1)$, and the same holds for every layer V_p. In particular $\mathcal{N}(A,B)$ and $\mathcal{N}(A_1,B_1)$ are both finite-dimensional, both of dimension $\aleph_0$, or both of dimension $\mathfrak{c}$.*

Proof. (a) The map sends an atom of T at x to $b+tx$, and by the computation $\mathcal{F}[e^{2\pi i a x}t^{-1/2}f((\cdot-b)/t)](\xi) = t^{1/2}e^{-2\pi i b(\xi-a)}\widehat{f}(t(\xi-a))$ an atom of $\mathcal{F}T$ at ξ to $a+\xi/t$; its inverse is of the same form, and it preserves H_{-p} by the boundedness of translations, modulations and dilations. (b) $\mathcal{F}$ preserves H_{-p} and $\mathcal{F}^2$ is the reflection. (c) $\mathcal{N}(A\cap A_1, B\cap B_1)$ is the kernel of the map on $\mathcal{N}(A,B)$ recording the masses of T on the finite set $B\setminus B_1$ and of $\mathcal{F}T$ on $A\setminus A_1$; the same argument applies to $\mathcal{N}(A_1,B_1)$ and to each layer. □

3.7. **Crystalline distributions on $\mathbb{Z}$.** If derivatives of Dirac masses are admitted, countable dimension occurs already for the lattice. The following is a special case of [LR21b, Theorem 1.1]; we include the short argument.

Example 3.10. Let $\mathcal{D}_{\mathbb{Z}}$ be the space of $T\in\mathcal{S}'$ with $\operatorname{supp}T\subseteq\mathbb{Z}$ and $\operatorname{supp}\widehat{T}\subseteq\mathbb{Z}$. Then $\mathcal{D}_{\mathbb{Z}}$ is the linear span of the linearly independent distributions $x^j\delta_{\mathbb{Z}}^{(k)}$, $j,k\geq 0$; hence $\dim\mathcal{D}_{\mathbb{Z}}=\aleph_0$, and every layer $\mathcal{D}_{\mathbb{Z}}\cap H_{-p}$ is finite-dimensional. On the other hand $\mathcal{M}_{\mathbb{Z}}=\mathbb{C}\delta_{\mathbb{Z}}$.

Indeed, $\mathcal{F}(x^jT)=(\mathrm{i}/2\pi)^j\partial^j\mathcal{F}T$ and $\mathcal{F}(\delta_{\mathbb{Z}}^{(k)})=(2\pi\mathrm{i}\xi)^k\delta_{\mathbb{Z}}$, so $\operatorname{span}\{x^j\delta_{\mathbb{Z}}^{(k)}\}$ is invariant under $\mathcal{F}$. Conversely, if $T\in\mathcal{D}_{\mathbb{Z}}$, then $\widehat{T}$ has finite order K, so it is annihilated by $(e^{2\pi\mathrm{i}\xi}-1)^{K+1}$, i.e. $(\tau_1-I)^{K+1}T=0$; an induction on the exponent, using that $(\tau_1-I)^{L-1}T$ is a 1-periodic distribution supported in $\mathbb{Z}$, shows that T is a polynomial combination of the $\delta_{\mathbb{Z}}^{(k)}$. Linear independence follows by comparing the coefficients of $\delta_n^{(k)}$, which are polynomials in n. Each layer is a closed subspace of H_{-p} of countable Hamel dimension and is therefore finite-dimensional (Proposition 3.4). Finally, if $T\in\mathcal{D}_{\mathbb{Z}}$ is a measure, the difference equation shows that its masses at $n\in\mathbb{Z}$ are given by a polynomial $q(n)$, so $T=q(x)\delta_{\mathbb{Z}}$, and $\mathcal{F}T=q(\mathrm{i}\partial/2\pi)\delta_{\mathbb{Z}}$ is a measure only if q is constant.

More generally, for $N\geq 1$ the space $\mathcal{M}_{N^{-1}\mathbb{Z}}$ consists of the N-periodic measures supported in $N^{-1}\mathbb{Z}$, and has dimension N^2. Indeed, if $\mu\in\mathcal{M}_{N^{-1}\mathbb{Z}}$, the continuous function $e^{2\pi\mathrm{i}N\xi}-1$ vanishes on $N^{-1}\mathbb{Z}$, so it annihilates the measure $\widehat{\mu}$, i.e. $\tau_N\mu=\mu$. Conversely, the Fourier transform of $\sum_{t\in\mathbb{Z}}\delta_{a+Nt}$ is $N^{-1}e^{-2\pi i a\xi}\sum_{k\in\mathbb{Z}}\delta_{k/N}$, so the N^2 linearly independent measures $\sum_{t\in\mathbb{Z}}\delta_{r/N+Nt}$, $0\leq r<N^2$, belong to $\mathcal{M}_{N^{-1}\mathbb{Z}}$ and span it.

4. A countably infinite-dimensional space of Fourier quasicrystals

In this section we prove Theorem A, the main result. The architecture, which the construction of Theorem B in Sections 5 and 6 shares, has three parts: blocks whose complete spaces are of small dimension and of prescribed order, an assembly with growing gaps, and an exhaustion argument showing that the union carries nothing else. The order of an element is its position in the strong layers X_N of Section 3.5, and the blocks come from Poisson formulas on the zero sets of exponential polynomials, in the spirit of Kurasov and Sarnak [KS20] and Olevskii and Ulanovskii [OU20a]. Table 1 compares the two constructions. The notation of this section is independent of that of Sections 5 and 6.

	Theorem A (this section)	Theorem B (Sections 5–6)
order of an element	growth exponent of $\lvert\mu\rvert$ and $\lvert\widehat{\mu}\rvert$	Hermite order
block	s sheets of zeros of a product of Lee–Yang polynomials	clusters of atoms at the half-integers
order of a block	between s and $s+3$	between P and $18P+4$
separation of blocks	distinct prime scales and generic parameters	random scales
exhaustion	complete description of finite unions, then compactness	the component of an element on each block

Table 1. The two constructions.

The mechanism. We write $e(t)=e^{2\pi\mathrm{i}t}$. For $0<a<1$ put

$$p_a(z,w)=1-aw-z(w-a)=(1-aw)\bigl(1-zB_a(w)\bigr),\qquad B_a(w)=\frac{w-a}{1-aw}.$$

Along a line $(z,w)=(e(t),e(\beta t-\frac14))$ in the torus, p_a has only real simple zeros. A product of s such factors with distinct parameters has s sheets of zeros. The sheets never meet, but they come close where the line passes near the point $(1,1)$, since $p_b-p_a=(b-a)(z-w)$. There the derivative of the product, which is the denominator of the residues in the Poisson formula, is of size $|z-w|^{s-1}$, so the masses become large. For a suitable slope β they grow like a power $|t|^{s-1}$, with the exponent controlled from above and from below (Lemma 4.4). The number of sheets thus plays the role that, for the blocks of Section 5 (Theorem 5.1), the clustering parameter P plays for the Hermite order; see Figure 1.

Plan. The proof uses three tools, each for one purpose. Continued fractions (Section 4.1) choose the slope β so that the sheets approach one another at a controlled polynomial rate, from above and from below; this fixes the growth exponent of a block (Lemma 4.4). Poisson formulas on Lee–Yang curves (Section 4.2) turn every crystalline measure on a block, or on a finite union of blocks, into a rational function R/Q whose numerator ranges over a finite-dimensional space; on one block, finite linear algebra then cuts the space down to a line (Lemma 4.5). For a finite union of blocks (Section 4.4) two further inputs are needed. A uniqueness theorem (Proposition 4.7), proved by harmonic analysis at a scale whose square is a quadratic irrational, separates the part of an element that lives on the cones from the part that lives on the roots. An algebraic splitting lemma (Lemma 4.8), proved on $\mathbb{P}^1\times\mathbb{P}^1$ with a group action and the vanishing of $H^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O})$, splits the rational function of the root part into the contributions of the individual blocks; this is needed because the zero sets of the denominators of different blocks intersect, so that partial fractions are not available. Finally, a compactness argument (Section 4.5) passes from finite to infinite unions. In terms of dependencies: Lemma 4.1 is used in Lemmas 4.3 and 4.6 and in Proposition 4.7; Lemmas 4.2 and 4.3 give Lemma 4.4, and Lemma 4.2 gives Lemma 4.5; Lemmas 4.5 and 4.6, Proposition 4.7 and Lemma 4.8, together with Lemmas 4.2 and 4.3, give Proposition 4.9, which with Lemma 4.4 gives Corollary 4.10; and Corollary 4.10 with Lemma 4.11 gives Theorem 4.12, hence Theorem A. The proofs of Lemmas 4.1, 4.3 and 4.6, Proposition 4.7 and Lemma 4.8 are given in Appendix A.

Throughout this section $c=2^{1/4}$, $\Gamma_{\mathbb{Q}}=(\mathbb{Q}+\beta\mathbb{Q})/c$ with β as in Lemma 4.1, and

$$C_\beta=\{k+\beta n:\ k,n\in\mathbb{Z},\ k,n\ge0\}\cup\{k+\beta n:\ k,n\in\mathbb{Z},\ k,n\le0\}.$$

Since β is irrational, the labels (k,n) of the points of C_β are unique.

4.1. **An arithmetic choice.** Let $\beta=[3;1,a_2,a_3,\dots]$ be the continued fraction with partial quotients $a_j=2q_{j-1}^j$ for $j\ge2$, where p_j/q_j are the convergents of β.

Lemma 4.1. (i) *Every q_j is odd, and $|2k\beta-u|\ge1/(4|k|)$ for all integers $k\ne0$ and all odd integers u.*
(ii) *There are integers n of arbitrarily large modulus, and integers k, with $|\beta n-k-\frac14|\le1/|n|$.*
(iii) *For every $j\ge1$, $\varepsilon_j=\beta q_j-p_j$ satisfies $0<|\varepsilon_j|<q_j^{-j-2}$.*
(iv) *β is transcendental.*

Part (i) bounds the near-crossings of the sheets from below, so that the masses are at most polynomial; part (ii) produces near-crossings, so that the masses really grow; part (iii) drives the no-return theorem (Proposition 4.7); and part (iv) is used to choose the parameters of the blocks. The proof is given in Appendix A.

4.2. **Poisson formulas on Lee–Yang curves.** We call a polynomial $Q(z,w)$ *admissible* if it is a product of factors $p_a(\rho z^n,\sigma w^n)$ with $a\in(0,1)$, $n\ge1$ and $|\rho|=|\sigma|=1$. It has bidegree (S,S), where S is the sum of the exponents n, and nonzero coefficients at 1 and at z^Sw^S. We write

$$V_S=\operatorname{span}\{z^iw^j:\ 0\le i,j\le S,\ (i,j)\ne(S,S)\}.$$

For $\alpha>0$ and $\varphi_0\in\mathbb{R}$ put $\gamma(x)=(e(\alpha x),e(\alpha\beta x+\varphi_0))$. The following lemma is a version of the Poisson formulas of [KS20, OU20a] that does not assume that the zero set is uniformly discrete.

Lemma 4.2. *Let Q be admissible of bidegree (S,S), and suppose that the real zeros of $f=Q\circ\gamma$ are simple; let A be the set of these zeros. For $R\in V_S$ the function $G_R=(R/Q)\circ\gamma$ is meromorphic in $\mathbb{C}$, with poles only at points of A, all simple. Put $\mu_R=\sum_{x\in A}\operatorname{Res}_x(G_R)\,\delta_x$.*

(a) *μ_R is a crystalline measure with support in A and spectrum in αC_β. If $\sum c_{kn} z^k w^n$ and $\sum c'_{kn} z^{-k} w^{-n}$ $(k,n \ge 0)$ are the expansions of R/Q at $(0,0)$ and at (∞,∞), then $c'_{00} = 0$,*

$$\mathcal{F}\mu_R(\{\alpha(k+\beta n)\}) = -\frac{e(n\varphi_0)\, c_{kn}}{2\pi \mathrm{i}}, \qquad \mathcal{F}\mu_R(\{-\alpha(k+\beta n)\}) = \frac{e(-n\varphi_0)\, c'_{kn}}{2\pi \mathrm{i}}$$

for $k,n \ge 0$ (with $(k,n) \ne (0,0)$ in the second formula), and $|c_{kn}| + |c'_{kn}| \le C_R(1+k+n)^S$.

(b) *Let $u : \mathbb{Z}^2 \to \mathbb{C}$ vanish outside $\mathbb{Z}^2_{\ge 0} \cup \mathbb{Z}^2_{\le 0}$ and satisfy $\sum_{i,j} Q_{ij}\, u(k-i, n-j) = 0$ for all $(k,n) \in \mathbb{Z}^2$. Then there is a unique $R \in V_S$ with $e(-n\varphi_0)\, \mathcal{F}\mu_R(\{\alpha(k+\beta n)\}) = u(k,n)$ for all $(k,n) \in \mathbb{Z}^2_{\ge 0} \cup \mathbb{Z}^2_{\le 0}$.*

(c) *$R \mapsto \mu_R$ is a linear bijection of V_S onto $\mathcal{N}(\alpha C_\beta, A)$.*

Proof. (a) For $|u|, |v| < 1$ we have $|B_a(v)| < 1$, hence $|p_a(u,v)| = |1 - av|\, |1 - uB_a(v)| \ge (1-a)(1 - \max\{|u|,|v|\})$. Consequently $|Q| \ge \kappa(1-r)^S$ on the polydisc $\{|z|, |w| \le r\}$, $r < 1$, for some $\kappa > 0$. Moreover $p_a(u,v) = -uv\, p_a(1/u, 1/v)$, so that $Q(z,w) = \zeta_Q z^S w^S Q^\sharp(1/z, 1/w)$ with $|\zeta_Q| = 1$ and $Q^\sharp$ admissible (replace ρ, σ by their conjugates). Thus Q has no zeros in $\{|z|, |w| < 1\}$ or in $\{|z|, |w| > 1\}$. For $\operatorname{Im} x > 0$ the point $\gamma(x)$ lies in the first of these sets, so G_R is holomorphic there and $G_R(x) = \sum_{k,n \ge 0} c_{kn} e(n\varphi_0) e(\alpha(k+\beta n)x)$, with absolute convergence. Writing $R(z,w) = z^S w^S R^\sharp(1/z, 1/w)$, where $R^\sharp(0,0) = R_{SS} = 0$, we get $G_R(x) = \sum_{k,n \ge 0} c'_{kn} e(-n\varphi_0) e(-\alpha(k+\beta n)x)$ for $\operatorname{Im} x < 0$, with c' the Taylor coefficients of $\zeta_Q^{-1} R^\sharp / Q^\sharp$; in particular $c'_{00} = 0$. Cauchy's estimate on the torus of radius $r = 1 - 1/(k+n+2)$ gives the coefficient bound. Since the cone has $O((1+T)^2)$ points in $[-T, T]$, the two series converge in $\mathcal{S}'(\mathbb{R})$ to tempered distributions $G_\pm$, and $G_R(\cdot \pm \mathrm{i}y) \to G_\pm$ in $\mathcal{S}'$ as $y \downarrow 0$ (pair with a Schwartz function and use dominated convergence). On the other hand, near the real axis G_R is meromorphic with simple poles at the points of A, so on every bounded interval the Sokhotski–Plemelj formula gives $G_+ - G_- = -2\pi\mathrm{i} \sum_{x \in A} \operatorname{Res}_x(G_R) \delta_x$. Hence $\mu_R = (G_- - G_+)/(2\pi\mathrm{i})$ is tempered, and the formulas for $\mathcal{F}\mu_R$ follow from $\mathcal{F}(e(\omega x)) = \delta_\omega$.

(b) Write $u = u^+ + u^-$, where u^+ is the restriction of u to $\mathbb{Z}^2_{\ge 0}$ and u^- that to $\mathbb{Z}^2_{\le 0} \setminus \{0\}$. The convolution $\widetilde{R} = Q * u^+ = -Q * u^-$ is supported in $\mathbb{Z}^2_{\ge 0}$ by the first expression and in $\{(k,n) : k, n \le S\}$ by the second, and its coefficient at (S,S) is $-Q_{SS} u^-(0,0) = 0$. Thus $\widetilde{R} \in V_S$, $\sum u^+(k,n) z^k w^n$ is the Taylor series of $\widetilde{R}/Q$ at the origin (as $Q_{00} \ne 0$), and $\sum u^-(-k,-n) z^{-k} w^{-n}$ is minus its expansion at infinity (as $Q_{SS} \ne 0$). By (a), $R = -2\pi\mathrm{i}\widetilde{R}$ has the required property. Uniqueness follows because u determines $\widetilde{R} = Q * u^+$.

(c) If $\mu_R = 0$, then $u = 0$ in (b), so $R = 0$. Conversely let $T \in \mathcal{N}(\alpha C_\beta, A)$. Since f vanishes on A and T is locally atomic there, $fT = 0$. As $f = \sum Q_{ij} e(j\varphi_0) e(\alpha(i+\beta j)x)$, the equation $\mathcal{F}(fT) = 0$ says that $u(k,n) = e(-n\varphi_0)\mathcal{F}T(\{\alpha(k+\beta n)\})$ satisfies the recurrence of (b). If R is given by (b), then $\mathcal{F}(T - \mu_R) = 0$, so $T = \mu_R$. □

4.3. **Blocks.** A *block* is given by an odd prime m, an integer $s \ge 2$ and distinct parameters $a_1, \dots, a_s \in (0,1)$. We put $\lambda = cm$ and

$$Z(x) = e(x/\lambda), \quad W(x) = e(\beta x/\lambda - \tfrac{1}{4}), \quad P = \prod_{j \le s} p_{a_j}, \quad A = \{x \in \mathbb{R} :\ P(Z(x), W(x)) = 0\},$$

and we let $\mathcal{C} = C_\beta/\lambda$ and $\mathcal{C}_0 = C_\beta/c$. Since m is an integer, $\mathcal{C}_0 \subset \mathcal{C}$; precisely, $(k+\beta n)/\lambda \in \mathcal{C}_0$ if and only if m divides both k and n. For $0 < a < 1$ let $\psi_a : \mathbb{R} \to \mathbb{R}$ be the continuous function with $\psi_a(0) = 0$ and $B_a(e(y)) = e(\psi_a(y))$; it is increasing, $\psi_a(y+1) = \psi_a(y) + 1$, and $\psi'_a(y) = (1-a^2)/|1 - ae(y)|^2$.

Lemma 4.3. *Let $D_\beta = z\partial_z + \beta w \partial_w$.*

(i) *P is admissible of bidegree (s,s). The zeros of $P(Z(x), W(x))$ in $\mathbb{C}$ are real and simple, and A is the disjoint union of the sets $A_j = \{x :\ x/\lambda + \psi_{a_j}(\beta x/\lambda - \frac{1}{4}) \in \mathbb{Z}\}$. Moreover $\#(A \cap [-r, r]) \le s(2(1+\beta)r/\lambda + 2)$.*

(ii) *$A \cap (-\delta\lambda, \delta\lambda) = \varnothing$, where $\delta = 1/(8(1+\beta))$.*

(iii) *There is $\kappa > 0$ with $|D_\beta P(Z(x), W(x))| \ge \kappa(1+|x|)^{-(s-1)}$ for all $x \in A$.*

(iv) *There are* $x \in A$ *of arbitrarily large modulus with* $|D_\beta P(Z(x), W(x))| \le C|x|^{-(s-1)}$ *and* $|Z(x) - 1| + |W(x) - 1| \le C/|x|$.
(v) *There is* $\kappa' > 0$ *with* $|x - x'| \ge \kappa'(1 + |x|)^{-1}$ *for all distinct* $x, x' \in A$.

The proof is given in Appendix A. We apply Lemma 4.2 to P with $\alpha = 1/\lambda$ and $\varphi_0 = -\frac{1}{4}$, so that $\gamma = (Z, W)$ and μ_R is defined for $R \in V_s$. The strong layers X_N were defined in Section 3.5.

Lemma 4.4. *For* $R \in V_s$ *we have* $\mu_R \in X_{s+3}(\mathcal{C}, A)$. *If* $R(1,1) \ne 0$, *then* $\sum_{x\in A} |\mu_R(\{x\})|(1 + |x|)^{-N} = \infty$ *for every* $N \le s - 1$.

Proof. Along the real line the derivative of $P(Z, W)$ is $2\pi\mathrm{i}\lambda^{-1} D_\beta P(Z, W)$, so the mass of μ_R at $x \in A$ is $\lambda R(Z, W)/(2\pi\mathrm{i}\, D_\beta P(Z, W))$, evaluated at $(Z(x), W(x))$. By Lemma 4.3(iii) these masses are $O((1 + |x|)^{s-1})$, and by Lemma 4.3(i) the physical sum converges for $N \ge s + 1$. By Lemma 4.2(a) the Fourier masses at $\pm(k + \beta n)/\lambda$ are $O((1 + k + n)^s) = O((1 + |\xi|)^s)$, and $\mathcal{C}$ has $O((1 + r)^2)$ points in $[-r, r]$, so the Fourier sum converges for $N \ge s + 3$. If $R(1,1) \ne 0$, then at the points of Lemma 4.3(iv) we have $R(Z, W) \to R(1,1)$, and the masses are at least $\kappa''|x|^{s-1}$ for some $\kappa'' > 0$; for $N \le s - 1$ the corresponding summands do not tend to zero. □

Lemma 4.5. *Let* $b = \lfloor s/2 \rfloor/2$ *and* $B = \mathcal{C} \setminus ([-b/\lambda, b/\lambda] \setminus \mathcal{C}_0)$. *There are a finite set* $E \subset A$ *and* $R_0 \in V_s$ *with* $R_0(1,1) \ne 0$ *such that*

$$\mathcal{N}(B, A \setminus E) = \mathbb{C}\mu_{R_0}.$$

Proof. By Lemma 4.2(c), for every finite $E \subset A$ the space $\mathcal{N}(B, A \setminus E)$ consists of the μ_R, $R \in V_s$, that have no mass on E and whose Fourier transforms vanish on $\mathcal{C} \setminus B$. The points of $\mathcal{C} \setminus B$ are the $\pm(k + \beta n)/\lambda$ with $(k, n) \in \mathbb{Z}^2_{\ge 0} \setminus m\mathbb{Z}^2$ and $k + \beta n \le b$; for them $k, n < s$, since $b < s/2$ and $\beta > 1$. By Lemma 4.2(a), the mass of $\mathcal{F}\mu_R$ at $(k + \beta n)/\lambda$ is a nonzero multiple of c_{kn}, and division by $Q = P$ shows that c_{kn} equals R_{kn}/P_{00} plus a combination of the $R_{k'n'}$ with $k' \le k$, $n' \le n$, $(k', n') \ne (k, n)$. Similarly the mass at $-(k + \beta n)/\lambda$ is a nonzero multiple of $R_{s-k,s-n}$ plus a combination of the $R_{s-k',s-n'}$ with $k' \le k$, $n' \le n$, $(k', n') \ne (k, n)$. The first family of conditions involves only the coefficients R_{ij} with $i + \beta j \le b$, the second only those with $i + \beta j \ge s(1 + \beta) - b > b$, and each family is triangular with nonzero diagonal. Hence these conditions are linearly independent; let $H \subset V_s$ be the subspace they define. The monomial $z^h w^h$, $h = \lfloor s/2 \rfloor$, lies in V_s and in H: the expansion of $z^h w^h/P$ at the origin involves only monomials $z^k w^n$ with $k, n \ge h$, its expansion at infinity only monomials $z^{-k} w^{-n}$ with $k, n \ge s - h \ge h$, and $h(1 + \beta) > b$. As $z^h w^h$ takes the value 1 at $(1, 1)$, the subspace $H_0 = \{R \in H : R(1,1) = 0\}$ has codimension one in H.

The functionals $R \mapsto R(Z(x), W(x))$, $x \in A$, separate the points of V_s: if they all vanish, then $\mu_R = 0$, and $R = 0$ by Lemma 4.2(c). Choose a set E of $\dim H_0$ points of A whose functionals restrict to a basis of H_0^*. Their common kernel in H is a line ℓ with $\ell \cap H_0 = \{0\}$. Since μ_R has no mass at x if and only if $R(Z(x), W(x)) = 0$, we obtain $\mathcal{N}(B, A \setminus E) = \{\mu_R : R \in \ell\} = \mathbb{C}\mu_{R_0}$ for any nonzero $R_0 \in \ell$, and $R_0(1,1) \ne 0$. □

We call $\mu = \mu_{R_0}$ the *block measure*. The set B is symmetric, so $\mathcal{F}^{-1}\mu$ has support in B and spectrum in $A \setminus E$. By Lemma 4.4, μ and $\mathcal{F}^{-1}\mu$ belong to X_{s+3}, while the physical record of μ, which is the spectral record of $\mathcal{F}^{-1}\mu$, has infinite N-weighted variation for $N \le s - 1$. The spectral gap of a block is only partial: B contains the fixed coarse cone $\mathcal{C}_0$, and only its other points avoid $[-b/\lambda, b/\lambda]$. The coarse cone is shared by all blocks; including it from the start is what allows different blocks to be combined.

4.4. **Finite unions.** We now consider blocks $i = 1, \dots, r$, with data $m_i, s_i, a_{i,j}$ and λ_i, P_i, A_i, $\mathcal{C}_i$, B_i, E_i and μ_i as above, and with pairwise distinct primes m_i. We put $L_i = (A_i \setminus E_i) \cup B_i$ and $L_{[r]} = \bigcup_{i \le r} L_i$. Three further ingredients are needed; their proofs are given in Appendix A. The first is a choice of the parameters that separates the roots of different blocks arithmetically.

We call blocks $1, \dots, r$ *independent* if their root sets $A_1, \dots, A_r$ are pairwise disjoint and disjoint from $\Gamma_{\mathbb{Q}}$, and no difference of two distinct points of $A_1 \cup \dots \cup A_r$ lies in $\Gamma_{\mathbb{Q}}$.

Lemma 4.6. *Let blocks $1,\dots,r-1$ be independent, let m_r be an odd prime and let $s_r \geq 2$. Then the parameters $a_{r,1},\dots,a_{r,s_r}$ can be chosen, successively and each outside a countable subset of $(0,1)$, so that blocks $1,\dots,r$ are independent.*

The second is a uniqueness theorem: at a scale whose square is a quadratic irrational, no nonzero crystalline measure has both its support and its spectrum in finitely many translates of the cone. It is used to separate the part of an element that lives on the cones from the part that lives on the roots. The hypothesis on the scale cannot be weakened to irrationality: since $\mathbb{Z} \subset C_\beta$, the measure $\delta_{\mathbb{Z}/\sqrt{2}}$ has support in $C_\beta/\sqrt{2}$, and its Fourier transform $\sqrt{2}\,\delta_{\sqrt{2}\mathbb{Z}}$ is supported in $\mathbb{Z}/\sqrt{2} \subset C_\beta/\sqrt{2}$. This is why the blocks are placed at the scales cm with $c = 2^{1/4}$.

Proposition 4.7. *Let $\lambda > 0$ be such that λ^2 is a quadratic irrational, and let $F_1, F_2 \subset \mathbb{Z} + \beta\mathbb{Z}$ be finite. If T is a crystalline measure with support in $(C_\beta + F_1)/\lambda$ and spectrum in $(C_\beta + F_2)/\lambda$, then $T = 0$.*

The third is an algebraic splitting of rational functions with sparse Taylor series. The zero sets of the denominators of different blocks intersect in $\mathbb{C}^2$, so the splitting does not reduce to partial fractions; it is proved on $\mathbb{P}^1 \times \mathbb{P}^1$ with a group action and the vanishing of $H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O})$.

Lemma 4.8. *Let $m_1,\dots,m_r$ be distinct odd primes, $M = m_1 \cdots m_r$ and $n_i = M/m_i$. For each i let P_i be a product of s_i factors p_a with distinct $a \in (0,1)$, let $Q_i(z,w) = P_i(z^{n_i}, -\mathrm{i}w^{n_i})$, $Q = \prod_i Q_i$ and $S = \sum_i n_i s_i$. If $R \in V_S$ and the Taylor coefficients of R/Q at the origin vanish outside $\bigcup_i n_i \mathbb{Z}^2_{\geq 0}$, then*

$$\frac{R}{Q} = \sum_{i \leq r} \frac{R_i(z^{n_i}, -\mathrm{i}w^{n_i})}{Q_i(z,w)} \qquad \text{with } R_i \in V_{s_i}.$$

Proposition 4.9. *Let blocks $1,\dots,r$ be independent, with pairwise distinct primes m_i. Then*

$$\mathcal{M}_{L_{[r]}} = \operatorname{span}\{\mu_i,\ \mathcal{F}^{-1}\mu_i :\ i \leq r\},$$

and these $2r$ measures are linearly independent.

Proof. Put $M = \prod_i m_i$, $n_i = M/m_i$, $\lambda_* = cM$, $\Gamma_* = (\mathbb{Z} + \beta\mathbb{Z})/\lambda_*$ and $\mathcal{C}_* = C_\beta/\lambda_*$. Then $\lambda_i = \lambda_*/n_i$, so $\mathcal{C}_i = n_i C_\beta/\lambda_* \subset \mathcal{C}_* \subset \Gamma_* \subset \Gamma_{\mathbb{Q}}$. In the variables $z = e(x/\lambda_*)$, $w = e(\beta x/\lambda_*)$ we have $Z_i = z^{n_i}$ and $W_i = -\mathrm{i}w^{n_i}$, so $P_i(Z_i, W_i) = Q_i(z,w)$ with Q_i as in Lemma 4.8. The polynomial $Q = \prod_i Q_i$ is admissible, and by Lemma 4.3 and the independence of the blocks, the real zeros of $f = Q(z(x), w(x))$ are simple and form the set $A_* = \bigcup_i A_i$. Thus Lemma 4.2 applies to Q with $\alpha = 1/\lambda_*$, $\varphi_0 = 0$. Note that $L_{[r]} \subset A_* \cup \mathcal{C}_*$, that $A_* \cap \Gamma_* = \varnothing$, and that $\mathcal{C}_* \subset \Gamma_*$.

Let $T \in \mathcal{M}_{L_{[r]}}$, and write $f = \sum_{\omega \in \Omega} f_\omega e(\omega x)$ with a finite $\Omega \subset \Gamma_*$.

Step 1: clearing the roots. The measure $U = fT$ is supported in $\mathcal{C}_*$, since f vanishes on A_*. Its Fourier transform $\mathcal{F}U = \sum_\omega f_\omega \tau_{-\omega}\mathcal{F}T$ is supported in $(A_* \cup \mathcal{C}_*) + \Omega$. The trigonometric polynomial $g(x) = \prod_{\omega \in \Omega} f(x - \omega)$ vanishes on $A_* + \Omega$, so $Y = g\,\mathcal{F}U$ is supported in finitely many Γ_*-translates of $\mathcal{C}_*$. As the frequencies of g lie in Γ_*, the measure $\mathcal{F}^{-1}Y$ is a finite combination of Γ_*-translates of U, so it is supported in finitely many Γ_*-translates of $\mathcal{C}_*$ as well. Since $\lambda_*^2 = M^2\sqrt{2}$ is a quadratic irrational, Proposition 4.7 gives $Y = 0$.

Step 2: the cone part. For $y \in \Gamma_*$ we have $y - \omega \in \Gamma_*$ for $\omega \in \Omega$, so $g(y) \neq 0$, and therefore $\mathcal{F}U(\{y\}) = 0$, i.e. $\sum_\omega f_\omega \mathcal{F}T(\{y - \omega\}) = 0$. The points of Γ_* that carry mass of $\mathcal{F}T$ lie in $\mathcal{C}_*$. Hence $u(k,n) = \mathcal{F}T(\{(k + \beta n)/\lambda_*\})$ vanishes outside $\mathbb{Z}^2_{\geq 0} \cup \mathbb{Z}^2_{\leq 0}$ and satisfies the recurrence of Lemma 4.2(b) for Q. By that lemma there is a measure $\rho = \mu_R$, $R \in V_S$, supported in A_*, such that $\mathcal{F}\rho$ is the restriction of $\mathcal{F}T$ to Γ_*.

Step 3: the literal decomposition. The measure $\sigma = \mathcal{F}(T - \rho) = \mathcal{F}T - \mathcal{F}\rho$ is the restriction of $\mathcal{F}T$ to $\mathbb{R} \setminus \Gamma_*$, so it is supported in A_* and $f\sigma = 0$. Its Fourier transform $\mathcal{F}\sigma = (T - \rho)^\vee$ is supported in $(-A_*) \cup \mathcal{C}_*$ and, by $\mathcal{F}(f\sigma) = 0$, satisfies $\sum_\omega f_\omega \mathcal{F}\sigma(\{y - \omega\}) = 0$ for all $y \in \mathbb{R}$. A coset $y + \Gamma_*$ with $y \notin \Gamma_*$ contains no point of $\mathcal{C}_*$ and, by the independence of the blocks, at most one point of $-A_*$. A nonzero measure carried by one point cannot satisfy this recurrence, because the term with the largest frequency in Ω is not cancelled. Hence $\mathcal{F}\sigma$ is supported in

$\Gamma_* \cap ((-A_*) \cup \mathcal{C}_*) = \mathcal{C}_*$, and $T - \rho$ is supported in $\mathcal{C}_*$. Since ρ is supported in A_*, we conclude that

$$\rho = T|_{A_*}, \qquad T - \rho = T|_{\mathcal{C}_*}, \qquad \mathcal{F}\rho = (\mathcal{F}T)|_{\mathcal{C}_*}, \qquad \sigma = (\mathcal{F}T)|_{A_*}.$$

In particular ρ and σ are measures with support in $\bigcup_i (A_i \setminus E_i)$ and spectrum in $\bigcup_i B_i$; for σ this uses that the sets B_i are symmetric.

Step 4: splitting. Let $\varrho \in \{\rho, \sigma\}$. By Lemma 4.2(c), $\varrho = \mu_R$ for some $R \in V_S$, and the Taylor coefficients of R/Q at the origin are multiples of the masses of $\mathcal{F}\varrho$ at the points $(k + \beta n)/\lambda_*$, $k, n \geq 0$. These masses vanish unless the point lies in some $\mathcal{C}_i$, that is, unless $(k, n) \in n_i \mathbb{Z}^2_{\geq 0}$ for some i. By Lemma 4.8, $R/Q = \sum_i R_i(z^{n_i}, -\mathrm{i}w^{n_i})/Q_i(z, w)$ with $R_i \in V_{s_i}$. Along the real line the i-th summand is the function $(R_i/P_i)(Z_i, W_i)$, so $\varrho = \sum_i \varrho_i$, where ϱ_i is the measure of Lemma 4.2 for P_i and R_i; it has support in A_i and spectrum in $\mathcal{C}_i$. As the A_i are disjoint, $\varrho_i = \varrho|_{A_i}$ has no mass on E_i. Let $\xi \in \mathcal{C}_i \setminus B_i$. Then $\xi \notin \mathcal{C}_0$, since $\mathcal{C}_0 \subseteq B_i$, and as $\mathcal{C}_i \cap \mathcal{C}_j = \mathcal{C}_0$ for $j \neq i$ (the primes are distinct), $\xi \notin \mathcal{C}_j$. Thus ξ lies in no B_j, so $\mathcal{F}\varrho(\{\xi\}) = 0$, and $\mathcal{F}\varrho_j(\{\xi\}) = 0$ for $j \neq i$; hence $\mathcal{F}\varrho_i(\{\xi\}) = 0$. Therefore $\varrho_i \in \mathcal{N}(B_i, A_i \setminus E_i) = \mathbb{C}\mu_i$ by Lemma 4.5.

Applying this to ρ and to σ we get $T|_{A_*} = \sum_i \theta_i \mu_i$ and $\mathcal{F}(T|_{\mathcal{C}_*}) = \sum_i \theta_i' \mu_i$, that is, $T = \sum_i (\theta_i \mu_i + \theta_i' \mathcal{F}^{-1}\mu_i)$. Conversely every μ_i and every $\mathcal{F}^{-1}\mu_i$ lies in $\mathcal{M}_{L_{[r]}}$. The coefficients θ_i and θ_i' are determined by the physical and the Fourier records of T on the disjoint sets A_i, which gives linear independence. □

Corollary 4.10. *In the situation of Proposition 4.9, for every $N \geq 0$*

$$\operatorname{span}\{\mu_i, \mathcal{F}^{-1}\mu_i : \ s_i + 3 \leq N\} \subseteq X_N(L_{[r]}) \subseteq \operatorname{span}\{\mu_i, \mathcal{F}^{-1}\mu_i : \ s_i \leq N\}.$$

Proof. The first inclusion is Lemma 4.4. Let $T = \sum_i (\theta_i \mu_i + \theta_i' \mathcal{F}^{-1}\mu_i) \in X_N(L_{[r]})$. The physical record of T on A_i is θ_i times that of μ_i, and the Fourier record of T on A_i is θ_i' times the physical record of μ_i. By Lemma 4.4, $\theta_i = \theta_i' = 0$ whenever $N \leq s_i - 1$. □

There is no cancellation in this argument: the masses that diverge are those of the private root set A_i of a block, which no other block can reach.

4.5. **The infinite union.** It remains to exclude elements of X_N that involve infinitely many blocks. Unlike the construction of Theorem B, where this is the main difficulty (Section 6), here a soft compactness argument suffices, because Proposition 4.9 describes every finite union completely. For an integer $N \geq 0$ write $w_N(x) = (1 + |x|)^{-N}$, and let $\mathcal{Q}_N$ be the set of pairs (μ, ν) of complex Radon measures on $\mathbb{R}$ with $\nu = \mathcal{F}\mu$ and $\|w_N\mu\| + \|w_N\nu\| \leq 1$, where $\|\cdot\|$ is the total variation. We give $\mathcal{Q}_N$ the topology induced by the weak* topology of $(w_N\mu, w_N\nu)$ in $C_0(\mathbb{R})^* \times C_0(\mathbb{R})^*$. For a closed set $L \subset \mathbb{R}$ and $R > 0$ let $\mathcal{A}_R(L)$ be the set of pairs in $\mathcal{Q}_N$ such that μ and ν are supported in $L \cup \{|x| \geq R\}$.

Lemma 4.11. *$\mathcal{Q}_N$ is compact and metrizable, and every $\mathcal{A}_R(L)$ is closed. If L is locally finite, then $\bigcap_R \mathcal{A}_R(L)$ consists of the pairs $(T, \mathcal{F}T)$ with $T \in X_N(L)$ and $\|T\|_{(N)} \leq 1$. Consequently, if Ψ is a continuous function on $\mathcal{Q}_N$ that vanishes on these pairs, then $\sup_{\mathcal{A}_R(L)} |\Psi| \to 0$ as $R \to \infty$.*

Proof. The unit ball of $C_0(\mathbb{R})^* \times C_0(\mathbb{R})^*$ is weak* compact and metrizable, since $C_0(\mathbb{R})$ is separable. For $\varphi \in \mathcal{S}$ we have $\varphi/w_N \in C_0(\mathbb{R})$, so $\mu \mapsto \mu(\varphi)$ is continuous; hence the condition $\nu = \mathcal{F}\mu$ is closed, and so are the support conditions. A Radon measure carried by a locally finite closed set is locally atomic on it, which identifies the intersection. If the last assertion failed, there would be $\epsilon > 0$ and pairs in $\mathcal{A}_R(L)$, $R \to \infty$, with $|\Psi| \geq \epsilon$; a convergent subsequence would have a limit in the intersection of the decreasing closed sets $\mathcal{A}_R(L)$, where $\Psi = 0$, a contradiction. □

Theorem 4.12. *There are blocks $i = 1, 2, \ldots$, with pairwise distinct primes m_i and strictly increasing sheet numbers $s_i \geq i$, such that blocks $1, \ldots, r$ are independent for every r, the set $\Lambda = \bigcup_i L_i$ is locally finite, and*

$$\operatorname{span}\{\mu_i, \mathcal{F}^{-1}\mu_i : \ s_i + 3 \leq N\} \subseteq X_N(\Lambda) \subseteq \operatorname{span}\{\mu_i, \mathcal{F}^{-1}\mu_i : \ s_i \leq N\} \qquad (N \geq 0). \tag{5}$$

Consequently $\mathcal{M}^{\mathrm{str}}_\Lambda = \bigoplus_i \operatorname{span}\{\mu_i, \mathcal{F}^{-1}\mu_i\}$, and this space has Hamel dimension $\aleph_0$.

Proof. We construct the blocks recursively, together with points $r_i \in A_i$, functions $\phi_i \in C_c^\infty(\mathbb{R})$ and radii R_i. Let $R_0 = 1$, and let $(h_l)_{l\geq 0}$ be the Hermite functions. At stage $i \geq 1$:

(a) choose an odd prime m_i different from $m_1, \dots, m_{i-1}$ with $\delta c m_i > R_{i-1}$, where δ is as in Lemma 4.3(ii);
(b) choose an integer $s_i \geq \max\{i, 2, s_{i-1}+1\}$ with $\lfloor s_i/2 \rfloor > 2cm_iR_{i-1}$;
(c) choose the parameters $a_{i,j}$ by Lemma 4.6, and then E_i and μ_i by Lemma 4.5;
(d) choose $r_i \in A_i \setminus E_i$ with $\mu_i(\{r_i\}) \neq 0$, normalize $\mu_i(\{r_i\}) = 1$, and choose ϕ_i with $\phi_i(r_i) = 1$, supported in $(r_i - \frac{1}{2}, r_i + \frac{1}{2})$ and containing no other point of $L_{[i]}$;
(e) choose $R_i \geq \max\{R_{i-1}+1, |r_i|+1\}$ such that $\sup_{\mathcal{A}_{R_i}(L_{[i]})} |\Psi_{N,l}| < 2^{-i}$ for all $N, l \leq i$, where

$$\Psi_{N,l}(\mu,\nu) = \mu(h_l) - \sum_{k\leq N} \Big(\mu(\phi_k)\, \mu_k(h_l) + \nu(\phi_k)\, (\mathcal{F}^{-1}\mu_k)(h_l) \Big).$$

By (a) and Lemma 4.3(ii), the roots of block k lie outside $[-R_{k-1}, R_{k-1}]$, and by (b) the points of B_k in $[-R_{k-1}, R_{k-1}]$ lie in $\mathcal{C}_0 \subseteq L_1$. As the R_k increase, it follows that $L_{[i']} \cap (-R_k, R_k) = L_{[k]} \cap (-R_k, R_k)$ for $k \leq i'$, and $\Lambda \cap (-R_k, R_k) = L_{[k]} \cap (-R_k, R_k)$ for every k. In particular Λ is locally finite, $\Lambda \subseteq L_{[k]} \cup \{|x| \geq R_k\}$, and, since the support of ϕ_k lies in $(-R_k, R_k)$, the function ϕ_k isolates r_k in every $L_{[i']}$, $i' \geq k$, and in Λ.

Step (e) is possible by Lemma 4.11. Indeed, $\Psi_{N,l}$ is continuous on $\mathcal{Q}_N$. Let $N \leq i$ and $T \in X_N(L_{[i]})$. By Corollary 4.10 and $s_k \geq k$, T lies in $W_N = \operatorname{span}\{\mu_k, \mathcal{F}^{-1}\mu_k : k \leq N\}$. The point r_k lies in no $A_{k'}$ with $k' \neq k$ and in no $B_{k'}$, because A_k is disjoint from the other root sets and from $\Gamma_{\mathbb{Q}} \supseteq B_{k'}$. Since moreover ϕ_k isolates r_k, the generators other than μ_k have no mass at r_k, and those other than $\mathcal{F}^{-1}\mu_k$ have Fourier transforms without mass at r_k, so the numbers $T(\phi_k)$ and $\mathcal{F}T(\phi_k)$ are the coefficients of μ_k and $\mathcal{F}^{-1}\mu_k$ in T. Hence $\Psi_{N,l}(T, \mathcal{F}T) = 0$.

Let $T \in X_N(\Lambda)$ with $\|T\|_{(N)} \leq 1$. Then $(T, \mathcal{F}T) \in \mathcal{A}_{R_i}(L_{[i]})$ for every i, so $|\Psi_{N,l}(T, \mathcal{F}T)| < 2^{-i}$ for all $i \geq \max\{N, l\}$, and therefore $\Psi_{N,l}(T, \mathcal{F}T) = 0$ for every l. As the h_l span a dense subspace of $\mathcal{S}$, the distribution T coincides with $\sum_{k\leq N}(T(\phi_k)\mu_k + \mathcal{F}T(\phi_k)\mathcal{F}^{-1}\mu_k)$, so $T \in W_N$. The argument of Corollary 4.10, applied to the records of T on the sets A_k, now gives the right-hand inclusion in (5), and the left-hand inclusion is Lemma 4.4. The μ_i and $\mathcal{F}^{-1}\mu_i$ are linearly independent by their records on the disjoint sets A_i, and every element of $\mathcal{M}^{\mathrm{str}}_\Lambda$ lies in some $X_N(\Lambda)$. Hence the μ_i and $\mathcal{F}^{-1}\mu_i$ form a basis of $\mathcal{M}^{\mathrm{str}}_\Lambda$, which has Hamel dimension $\aleph_0$; in accordance with Proposition 3.8, the layers $X_N(\Lambda)$ are finite-dimensional and their dimensions are unbounded. □

Proof of Theorem A. Take Λ and the s_i as in Theorem 4.12, $\nu_{2i-1} = \mu_i$ and $\nu_{2i} = \mathcal{F}^{-1}\mu_i$. These are Fourier quasicrystals with support and spectrum in Λ, and part (i) is the last assertion of Theorem 4.12. Part (ii) follows from (5) and the linear independence of the ν_k. For part (iii), the norms $\|\nu_{2i-1}\|_{(N)}$ and $\|\nu_{2i}\|_{(N)}$ on Λ are equal to $\|\mu_i\|_{(N)}$ on $(\mathcal{C}_i, A_i)$, which is finite for $N = s_i + 3$ and infinite for $N = s_i - 1$ by Lemma 4.4, since $\mu_i = \mu_{R_0}$ with $R_0(1,1) \neq 0$. □

Remark 4.13. (a) No random choices are made: the primes, the sheet numbers, the parameters (outside countable sets) and the radii are chosen by deterministic conditions, although the existence of suitable radii is obtained by compactness. The formal proof of the existence statement in Theorem A(i) [Mar26, Pal26c] uses a set of the same kind, with a different schedule of primes, sheet numbers and parameters; there too the radii come from a compactness statement and are selected by classical choice, so the formal witness is not given by an algorithm. (b) The set Λ is not separated at a polynomial rate: it contains the cone $\mathcal{C}_0 = C_\beta/c$, and since β is a Liouville number, distinct points $k + \beta n$ of C_β can be closer than any power of their size. This is consistent with Question 9.2. The atoms of a single block, on the other hand, are separated at a polynomial rate by Lemma 4.3(v), and the clusters of Lemma 4.3(iv) consist of atoms at mutual distances of order $|x|^{-1}$, bounded below by Lemma 4.3(v). (c) Theorem 4.12 determines the space of Fourier quasicrystals on Λ, not the space $\mathcal{M}_\Lambda$ of all crystalline measures, which might be larger; see Question 9.1.

5. One-dimensional blocks of prescribed Hermite order

We now turn to the second part of the paper, on all crystalline measures. Sections 5 and 6 prove Theorems B and C; the notation is independent of that of Section 4. In this section we construct, for all positive integers P and R, a symmetric locally finite set $C(P,R)$ which avoids $[-R+\frac12, R-\frac12]$ and carries, up to scalars, exactly one crystalline measure, whose Hermite order lies between P and $18P+4$. The point is that the Hermite order and the inner radius can be chosen independently. Figure 2 shows such a block. Heuristically, the clusters of atoms near the half-integers imitate derivatives of Dirac masses there, as in (3), and the speed at which the clusters tighten determines the Hermite order; Theorem 5.1 below makes this precise. As in Kolountzakis's construction [Kol16, Section 2], the gap around the origin is produced by a finite-dimensional argument (Lemma 5.2).

5.1. **Half-endpoint carriers.** For a sequence $\gamma = (\gamma_j)_{j\ge1}$ of distinct numbers in $(0,\frac12)$ put

$$L_\gamma = \{n \pm \gamma_j : \ j \ge 1,\ n \in \mathbb{Z},\ |n| \ge j\}.$$

This set is locally finite because of the condition $|n| \ge j$: the cell $[n, n+1)$ contains at most $2(|n|+2)$ of its points. If $\gamma_j \to \frac12$, it is not uniformly discrete: as the cells move outward, they contain increasingly tight finite clusters around the half-integers, formed by the points $n+\gamma_j$ and $n+1-\gamma_j$; there is no accumulation in any bounded interval.

For positive integers P and k we fix the parameters

$$b = 1 + \frac{1}{6P}, \quad Q = 2^{64}k, \quad d = 6P\log Q, \quad \varepsilon_j = \tfrac{1}{16}e^{-db^j}, \quad \gamma_j = \tfrac12 - \varepsilon_j \quad (j \ge 1), \tag{6}$$

and the *head phases* $\beta_j = (2j-1)/(4k)$, $1 \le j \le k$. Note that $b/(b-1) = 6P+1$, that $\varepsilon_1 = \frac{1}{16}Q^{-(6P+1)} \le 2^{-452}$, that $\varepsilon_{j+1}/\varepsilon_j = Q^{-b^j} \le 2^{-64}$, and that $\varepsilon_1 < \frac{1}{4k}$, so that all the β_j and γ_j are distinct. Let $\gamma' = (\beta_1, \dots, \beta_k, \gamma_1, \gamma_2, \dots)$; thus in $L_{\gamma'}$ the phase β_j occurs in the cells $|n| \ge j$ and the phase γ_j in the cells $|n| \ge k+j$.

The analytic input of the construction is the following theorem, which is proved in Appendix B.

Theorem 5.1. *Let $P, k \ge 1$ and let γ, γ' be as above.*

(a) *$\mathcal{M}_{L_\gamma}$ is one-dimensional; it is spanned by an odd measure μ with $\mathcal{F}\mu = -\mathrm{i}\mu$.*
(b) *$\mathcal{M}_{L_\gamma} \cap H_{-p} = \{0\}$ for $1 \le p \le P$.*
(c) *$\mathcal{M}_{L_\gamma} \subset H_{-(18P+2)}$.*
(d) (Head removal.) *There is a linear bijection $\mathcal{R} : \mathcal{M}_{L_{\gamma'}} \to \mathcal{M}_{L_\gamma}$ such that $\mathcal{R}$ maps $V_p(L_{\gamma'})$ into $V_p(L_\gamma)$ and $\mathcal{R}^{-1}$ maps $V_p(L_\gamma)$ into $V_{p+2}(L_{\gamma'})$, for every integer $p \ge 0$.*
(e) (Finite heads carry nothing.) *If w belongs to the space W_k of (7) below and neither w nor $\mathcal{F}w$ has mass at the points $n \pm \beta_j$ with $j \le k$ and $|n| < j$, then $w = 0$.*

Here $\mathcal{R}$ is built from finitely many integer translations and multiplications by a trigonometric polynomial (Section 5.2), so it preserves every H_{-p} and preserves strong temperedness on both sides; the content of (d) is that its inverse loses at most two Hermite orders.

5.2. **Outline of the proof of Theorem 5.1.** The proof is given in Appendix B; we describe its structure here. Part (e) concerns finitely many lattice combs and is proved by a triangularity argument: in suitable coordinates the conditions on w and on $\mathcal{F}w$ are complementary triangular systems, and a determinant computation shows that they admit only $w = 0$ (Lemma B.2). Parts (a)–(c) are proved in Zak coordinates. After a shift by $\frac12$, the Zak transform of $T \in \mathcal{M}_{L_\gamma}$ is a compactly supported distribution V on a small square, and the atoms of T and of $\mathcal{F}T$ become point masses on the grid $\{0, \pm\varepsilon_j\}^2$. We test V against products of the Newton-type functions $\cos(\pi t)D_i(t)$ and $\sin(\pi t)D_i(t)$, where $D_i = \prod_{l\le i}(1-\cos 2\pi t - a_l)$ and $a_l = 1-\cos 2\pi\varepsilon_l$ vanishes at the first i pairs of points. In the resulting moment arrays the support condition on T says that the array $\mathbf{A}$ of V is upper triangular, the support condition on $\mathcal{F}T$ says that the array $\mathbf{B}$ of $e^{-2\pi\mathrm{i}xy}V$ is lower triangular, and $\mathbf{B} = E\mathbf{A}$ for an explicit bounded operator E (the *gauge*). The proof has four stages.

1. *Uniqueness* (Appendix B.5). With geometric weights the two triangularity conditions leave a single free parameter: after elimination they reduce to $\mathcal{K}u = 0$ for an operator $\mathcal{K} = c_0 S + \mathcal{E}$ on $\ell^2(\mathbb{N}_0)$, where S is the backward shift and $\|\mathcal{E}\| < |c_0|/10$. Hence $\dim \mathcal{M}_{L_\gamma} \le 1$.
2. *Exclusion at low order* (Appendix B.6). If $T \in H_{-p}$, then V has distribution order at most $2p$, and when $6p < b/(b-1) = 6P+1$ this allows moments with weights $\prod_{l\le i} \varepsilon_l$, which shrink as fast as the clusters. In these weights the relevant part of $E - I$ carries a factor ε_{i+1}^2, which is small compared with the coupling $\tan(\pi\varepsilon_{i+1})$ created by the support conditions, and a contraction argument gives $T = 0$. This is (b).
3. *Construction* (Appendix B.7). Keeping only the first N phases and adding the half-integers outside $[-N+\frac12, N-\frac12]$ gives a finite system, whose space of crystalline measures is a line spanned by a combination T_N of lattice combs. Its moments decay like $\prod_{l\le m} \varepsilon_l$ uniformly in N, and an interpolation argument converts this into $\sup_N \|T_N\|_{-(2L+2)} < \infty$ whenever $b^L > 4$. A weak limit is a nonzero element of $\mathcal{M}_{L_\gamma}$; with $L = 9P$ this gives (a) and (c).
4. *Head transfer* (Appendix B.8). The map $\mathcal{R}$ in (d) is the composition of an operator $\sum_{|l|\le k} c_l \tau_l$, whose symbol has simple zeros exactly at $e^{\pm 2\pi i \beta_j}$, $j \le k$, with multiplication by the trigonometric polynomial with the same coefficients. Each factor has a right inverse built from bounded Green sequences that loses one Hermite order, and a finite correction in W_k, controlled by (e), makes the inverse exact.

The parameters (6) enter as follows: in (b) through the condition $6p < b/(b-1)$, and in (c) through $b^L > 4$ with $L = 9P$, since $b^{9P} = (1+\frac{1}{6P})^{9P} \ge (\frac76)^9 = 40353607/10077696 > 4$ by the monotonicity of $(1+1/x)^x$. The arguments are also carried out in the formal development [MS26] (Appendix E). For odd $k \le 11$ we have, in addition, checked (e) by computing the rank of the corresponding $4k^2 \times 4k^2$ matrix exactly, modulo a prime $p \equiv 1 \pmod{16k^2}$; this is not needed for the proof.

5.3. **Exchanging finitely many holes.** Let $k \ge 1$. The points $n \pm \beta_j$, $n \in \mathbb{Z}$, $j \le k$, form the lattice coset $G_k = \{x_m = (m+\frac12)/(2k) : m \in \mathbb{Z}\}$. Let

$$W_k = \operatorname{span}\Big\{ \sum_{n\in\mathbb{Z}} e^{2\pi i n b} \delta_{n+a} : \ a, b \in \{\pm\beta_j : j \le k\} \Big\}. \tag{7}$$

These modulated translated combs are crystalline with support and spectrum in G_k, since the Fourier transform of $\sum_n e^{2\pi i n b}\delta_{n+a}$ has mass $e^{-2\pi i a(\ell+b)}$ at $\ell + b$. One checks that $\dim W_k = 4k^2$, that $W_k \subset H_{-1}$, and that W_k consists of the measures $\sum_m c_m \delta_{x_m}$ with $c_{m+4k^2} = -c_m$.

Lemma 5.2. *Let $Y \subset \mathbb{R} \setminus G_k$ be locally finite, let $H^0, H^1 \subset G_k$ be finite sets with $\#H^0 = \#H^1 = 2k^2$, and put $\Lambda^j = (G_k \setminus H^j) \cup Y$. Assume that, for $j = 0, 1$, the map $\mathcal{L}_j : W_k \to \mathbb{C}^{H^j} \times \mathbb{C}^{H^j}$ recording the masses of w and of $\mathcal{F}w$ at the points of H^j is injective. Then there is a linear bijection $\mathcal{M}_{\Lambda^0} \to \mathcal{M}_{\Lambda^1}$ which maps $V_p(\Lambda^0)$ onto $V_p(\Lambda^1)$ for every $p \ge 1$. In particular $r_p(\Lambda^0) = r_p(\Lambda^1)$ for all $p \ge 1$.*

Proof. Since $\dim W_k = 4k^2$, the maps $\mathcal{L}_j$ are bijective. For $T \in \mathcal{M}_{\Lambda^0}$ let $\mathcal{E}_1 T \in \mathbb{C}^{H^1} \times \mathbb{C}^{H^1}$ record the masses of T and $\mathcal{F}T$ at H^1, and put $S_1 T = T - \mathcal{L}_1^{-1}\mathcal{E}_1 T$. Then $S_1 T$ and $\mathcal{F}S_1 T$ have no mass at H^1, and they are supported in $G_k \cup Y$; hence $S_1 T \in \mathcal{M}_{\Lambda^1}$. Define $S_0 : \mathcal{M}_{\Lambda^1} \to \mathcal{M}_{\Lambda^0}$ symmetrically. The difference $S_0 S_1 T - T$ lies in W_k and has no mass, on either side, at H^0, so it vanishes by the injectivity of $\mathcal{L}_0$; similarly $S_1 S_0 = I$. Finally, each mass is a continuous functional on H_{-p} (pair with a fixed bump function), and $W_k \subset H_{-1}$, so S_0 and S_1 preserve V_p for $p \ge 1$. (The norms of these maps may grow rapidly with k; only the qualitative preservation of each V_p is used.) □

We apply the lemma with the triangular holes $H^0 = \{n \pm \beta_j : j \le k,\ |n| < j\}$ and the central holes $H^1 = \{n \pm \beta_j : j \le k,\ |n| \le R-1\}$, where $k = 2R-1$. Injectivity of $\mathcal{L}_0$ is Theorem 5.1(e). For $\mathcal{L}_1$, note that $H^1 = \{x_m : m \in I_0\}$ with $I_0 = \{-k^2, \dots, k^2-1\}$, and that the elements of W_k are determined by $(c_m)_{-k^2 \le m < 3k^2}$. If $w = \sum c_m \delta_{x_m} \in W_k$ has no mass on H^1, then $c_m = 0$ for $m \in I_0$, and by Poisson summation the mass of $\mathcal{F}w$ at x_s is $\frac{1}{2k}\sum_{m\in I_1} e^{-2\pi i (m+1/2)(s+1/2)/(4k^2)} c_m$ with $I_1 = \{k^2, \dots, 3k^2-1\}$. Requiring these to vanish for $s \in I_0$ gives a square system which,

after removing nonzero row and column factors, has Vandermonde matrix $(z_m^{s+k^2})_{s\in I_0, m\in I_1}$ with distinct nodes $z_m = e^{-2\pi i m/(4k^2)}$, $m \in I_1$. Hence $c = 0$ and $\mathcal{L}_1$ is injective. (This matrix is extremely ill-conditioned, as its nodes fill only half of the unit circle; its invertibility is exact, not numerical.)

5.4. **The blocks.** For positive integers P, R put $k = 2R - 1$, take the parameters (6), and let

$$C(P,R) = \{n \pm \beta_j : \ 1 \le j \le k, \ |n| \ge R\} \cup \{n \pm \gamma_j : \ j \ge 1, \ |n| \ge k + j\}. \tag{8}$$

Proposition 5.3. *For all positive integers P, R the following hold.*

(i) $\mathcal{M}_{C(P,R)} = \mathbb{C}\mu_{P,R}$ *for a nonzero* $\mu_{P,R}$ *with* $\mathcal{F}\mu_{P,R} = \zeta\mu_{P,R}$, $\zeta^4 = 1$.
(ii) $C(P,R) \cap [-R + \frac{1}{2}, R - \frac{1}{2}] = \varnothing$.
(iii) $\mathcal{M}_{C(P,R)} \cap H_{-p} = \{0\}$ *for* $1 \le p \le P$, *and* $\mu_{P,R} \in H_{-(18P+4)}$.
(iv) $\#(C(P,R) \cap [-r,r]) \le 32(1+r)^2$ *for* $r \ge 0$.
(v) *The set of phases* $\Theta \subset (-\frac{1}{2}, \frac{1}{2})$ *of* $C(P,R)$ *has closure* $K = \Theta \cup \{\frac{1}{2}\}$ *in* $\mathbb{R}/\mathbb{Z}$; K *is countable, its only accumulation point is* $\frac{1}{2}$, *and* $K \cap (-\frac{1}{4k}, \frac{1}{4k}) = \varnothing$.

Proof. By Theorem 5.1(a)–(c), $\mathcal{M}_{L_\gamma}$ is a line contained in $H_{-(18P+2)}$ which meets H_{-p} only in 0 for $p \le P$. By Theorem 5.1(d), $\mathcal{M}_{L_{\gamma'}}$ is a line contained in $H_{-(18P+4)}$, again meeting H_{-p} only in 0 for $p \le P$, since $\mathcal{R}$ is injective and preserves Hermite orders. Now $L_{\gamma'} = (G_k \setminus H^0) \cup Y$ and $C(P,R) = (G_k \setminus H^1) \cup Y$ with $Y = \{n \pm \gamma_j : |n| \ge k + j\}$ and H^0, H^1 as above, and $\#H^0 = \#H^1 = 2k^2$. Lemma 5.2 transfers (i) and (iii) to $C(P,R)$; the line is invariant under $\mathcal{F}$ because $C(P,R)$ is symmetric (Proposition 3.9(b)), and $\mathcal{F}^4 = I$. Points of the first set in (8) satisfy $|x| \ge R - \beta_k > R - \frac{1}{2}$, and points of the second satisfy $|x| > k + \frac{1}{2} \ge R$, which gives (ii). For (iv), points of the first set in $[-r,r]$ exist only if $k < 2r$, and there are at most $2k(2r+2)$ of them; points of the second set satisfy $j \le r + 1$, and there are at most $4(r+1)^2$ of them. Finally (v) holds because $\gamma_j \to \frac{1}{2}, \pm\frac{1}{2}$ coincide on $\mathbb{R}/\mathbb{Z}$, and $\beta_1 = \frac{1}{4k}$ is the smallest phase. □

6. The construction and complete exhaustion

6.1. **The construction.** Fix a sequence $(P_i)_{i\ge1}$ of positive integers, the *profile*. For integers $R_i \ge 2^i$, chosen below, let

$$C_i = C(P_i, R_i), \qquad k_i = 2R_i - 1,$$

and let μ_i span $\mathcal{M}_{C_i}$, with $\mathcal{F}\mu_i = \zeta_i\mu_i$ (Proposition 5.3). Thus

$$\mathbb{C}\mu_i \cap H_{-p} = \{0\} \quad (p \le P_i), \qquad \mu_i \in H_{-(18P_i+4)}. \tag{9}$$

Let $\Theta_i \subset (-\frac{1}{2}, \frac{1}{2})$ be the set of phases of C_i, and $K_i = \Theta_i \cup \{\frac{1}{2}\} \subset \mathbb{R}/\mathbb{Z}$ its closure; we fix an enumeration $\Theta_i = \{\theta_{i,1}, \theta_{i,2}, \dots\}$.

For $s = (s_i) \in [1,2]^{\mathbb{N}}$ we use *labels* $\ell = (i, \sigma)$, $i \ge 1$, $\sigma \in \{+,-\}$; we call i the *index* of ℓ, write $\ell^* = (i, -\sigma)$, and put

$$t_{i,+} = s_i, \quad t_{i,-} = s_i^{-1}, \quad \Gamma_\ell = t_\ell C_i, \quad e_\ell = D_{t_\ell}\mu_i, \quad \Lambda = \bigcup_\ell \Gamma_\ell \tag{10}$$

(Figure 3). Then $\operatorname{supp} e_\ell \subseteq \Gamma_\ell$ and, since $\mathcal{F}D_t = D_{1/t}\mathcal{F}$,

$$\mathcal{F}e_\ell = \zeta_i e_{\ell^*}, \tag{11}$$

so $e_\ell \in \mathcal{M}_\Lambda$. We also use the periodic sets

$$\widetilde{K_\ell} = (K_i + \mathbb{Z})/t_\ell \subset \mathbb{R}, \qquad \text{of period } \varpi_\ell = 1/t_\ell.$$

The set $\widetilde{K_\ell}$ contains $\Gamma_{\ell^*} = C_i/t_\ell$, but it is much larger, and it contains the *seam* $(\mathbb{Z} + \frac{1}{2})/t_\ell$, which does not meet Λ. The union of the $\widetilde{K_\ell}$ may be dense.

Theorem 6.1. *For every profile (P_i) there are deterministic integers $R_i \ge 2^i$ and a measurable set $\Omega \subset [1,2]^{\mathbb{N}}$ of product measure at least* 13/16 *such that for every $s \in \Omega$ the sets Γ_ℓ are pairwise*

disjoint, Λ is symmetric and locally finite, and the following holds. If $p \geq 1$ and $T \in V_p(\Lambda)$, then for every label $\ell = (i, \sigma)$ there is $c_\ell \in \mathbb{C}$ with

$$\mathcal{F}^{-1}(\mathcal{F}T\,|_{\Gamma_\ell}) = c_\ell\, e_{\ell^*}, \qquad c_\ell = 0 \text{ whenever } P_i \geq 6p. \tag{12}$$

If moreover only finitely many i satisfy $P_i < 6p$, then $T = \sum_\ell c_\ell e_{\ell^*}$, a finite sum.

Here $\mathcal{F}T|_{\Gamma_\ell}$ is the restriction of the measure $\mathcal{F}T$ to the set Γ_ℓ; that it is a tempered distribution is part of the proof.

6.2. **Strategy of the proof.** Theorem 6.1 turns into countable dimension through one implication, which we state first. Fix $T \in V_p(\Lambda)$, write $S = \mathcal{F}T$, and call $U_\ell = \mathcal{F}^{-1}(S|_{\Gamma_\ell})$ the *component* of T on the label ℓ. Then

$$T \in H_{-p} \implies U_\ell \in H_{-6p} \text{ for every } \ell \text{ (Lemma 6.3)}, \qquad U_\ell = c_\ell e_{\ell^*} \text{ and } P_i \geq 6p \implies c_\ell = 0,$$

because $\mathbb{C}e_{\ell^*} \cap H_{-6p} = \{0\}$ when $6p \leq P_i$. When $P_i \to \infty$, only finitely many components survive at each fixed order, so T is a finite combination of the e_ℓ; and the generators admitted at order p, those with $18P_i + 4 \leq p$, make the layer dimensions unbounded (Corollary 6.7). The hard part is to justify the phrase "component of T": the restriction $S|_{\Gamma_\ell}$ must be a tempered distribution, and its inverse Fourier transform must live on the reciprocal block Γ_{ℓ^*}, without additional distributions at the seam. A priori T is an arbitrary tempered distribution, and nothing prevents its masses on different blocks from conspiring to form a new crystalline measure; ruling this out is the content of the proof. It has a preparatory step and three main steps.

Step 0: splitting. The points of different blocks are separated at a polynomial rate; this is where the random scales s_i are first used. Smooth selectors then cut S into its restrictions S_ℓ to the sets Γ_ℓ, at the cost of a fixed loss of Hermite order, and give $T = \sum_{i \leq M} U_\ell + U_{>M}$ with $U_\ell = \mathcal{F}^{-1}S_\ell$, where $U_{>M}$ comes from the blocks of index $> M$. Nothing is known yet about where U_ℓ lives.

Step 1: isolating a block up to the seam. Multiply T by a smooth function G_M which vanishes on the first M blocks and whose Fourier coefficients have mean one. Then $G_M T$ lives beyond the next gap, which is far away, while $\mathcal{F}(G_M T)$ is a combination of translates of S. Because the scales are random, no translate of an atom accidentally lands on another atom (nonresonance), and this leaves a convolution identity for the masses of S_ℓ. It shows that U_ℓ is supported in its periodic set $\widetilde{K_\ell}$, up to a polynomial whose coefficients are antiperiodic: a residue created by the accumulation at the seam.

Step 2: removing the residues. Antiperiodic terms change sign under translation by half a period, while everything supported in the sets $\widetilde{K_{\ell'}}$ is unaffected by suitable translations. Using Kronecker's theorem we choose two large translations that agree, modulo all periods, except at the block under consideration, where they differ by half a period. Comparing the two isolates the leading coefficient of the residue and shows that it is supported in $\widetilde{K_\ell}$. The error is controlled because the translated test functions stay inside the large gap before the next block.

Step 3: recovering the atoms. Shrinking test functions at the points of $\widetilde{K_\ell}$ show that U_ℓ has the same masses as T on Γ_{ℓ^*} and nothing at the remaining points of $\widetilde{K_\ell}$, in particular nothing at the seam. Thus U_ℓ is a crystalline measure on one rescaled block, and Proposition 5.3 identifies it.

Order of choices. Every step produces error terms that must tend to zero. All of them are paid for by the gaps: the inner radius R_{M+1} is chosen after all the parameters of stage M have been fixed, and so large that the error terms of stage M are at most 2^{-M}. The probabilistic events used (separation, nonresonance, Kronecker nets) have probability bounds that hold for every continuation of the sequence of radii; this is why a single deterministic sequence (R_i) and a set of scales of positive measure can be chosen. The proofs of the technical lemmas, the construction of the parameters, and a full table of the parameters are in Appendix D.

Table 2 lists the objects attached to a label $\ell = (i, \sigma)$, and Table 3 the three choices that define Λ.

Object	Role	What is known about its support
$\Gamma_\ell = t_\ell C_i$	dilated block; carries e_ℓ	by definition
$\widetilde{K}_\ell = (K_i + \mathbb{Z})/t_\ell$	periodic set containing Γ_{ℓ^*} and the seam $(\mathbb{Z} + \frac{1}{2})/t_\ell$	by definition
$S_\ell = \chi_{\{\ell\}}\mathcal{F}T$	restriction of $\mathcal{F}T$ to Γ_ℓ (Step 0)	contained in Γ_ℓ
$U_\ell = \mathcal{F}^{-1}S_\ell$	component of T on ℓ	unknown after Step 0; in $\widetilde{K}_\ell$ up to the residue after Step 1; in $\widetilde{K}_\ell$ after Step 2; atomic on Γ_{ℓ^*} after Step 3
W_ℓ, $v_{\ell,r}$	$U_\ell = W_\ell + \sum_r x^r v_{\ell,r}$ (27); $v_{\ell,r}$ is the residue at the seam	W_ℓ in $\widetilde{K}_\ell$; $v_{\ell,r}$ antiperiodic, shown to be supported in $\widetilde{K}_\ell$ in Step 2

TABLE 2. The objects attached to a label $\ell = (i, \sigma)$.

Choice	Role	When it is chosen
P_i	Hermite order of the i-th block, between P_i and $18P_i + 4$	given (the profile)
R_i	inner radius of the i-th block; keeps Λ locally finite, and the gap before the block pays for all errors of stage $i-1$	after all parameters of stage $i-1$
s_i	scale of the i-th block; separates blocks (Lemma 6.2) and prevents resonances (Lemma 6.4)	at random, in a set Ω of probability at least $\frac{13}{16}$

TABLE 3. The three choices defining Λ. The auxiliary parameters of stage M (cutoffs, probe radii, translations) depend only on $R_1, \dots, R_M$ (Proposition 6.6 and Table 5).

6.3. **Tools.** For $t > 0$ the dilation D_t is bounded with bounded inverse on H_{-p} for every integer p, with constants uniform for t in compact subsets of $(0, \infty)$; this follows from the norm equivalence of Section 3, $xD_t = tD_t x$, $\partial D_t = t^{-1}D_t\partial$, and duality. Hence $e_\ell \in H_{-(18P_i+4)}$ and $\mathbb{C}e_\ell \cap H_{-p} = \{0\}$ for $p \le P_i$.

By Proposition 5.3(ii) and (iv) and $R_i \ge 2^i$, only $O(\log(2+r))$ blocks meet $[-r, r]$, even after dilation by $t_\ell \in [\frac{1}{2}, 2]$, and each contributes $O((1+r)^2)$ points. Thus Λ is locally finite for every choice of radii and scales, with $\#(\Lambda \cap [-r, r]) = O((1+r)^2 \log(2+r))$, uniformly over all sequences (R_i) with $R_i \ge 2^i$.

Lemma 6.2 (Separation)**.** *Let (R_i) be any sequence with $R_i \ge 2^i$. For almost every s there is $c_s \in (0, 1]$ such that*

$$|x - y| \ge c_s(1 + |x| + |y|)^{-6} \qquad (x \in \Gamma_\ell,\ y \in \Gamma_{\ell'},\ \ell \ne \ell'). \tag{13}$$

In particular the sets Γ_ℓ are pairwise disjoint for almost every s. Within one block the atoms coalesce, and no separation is claimed there.

Lemma 6.3 (Selectors)**.** *Assume (13). For every set E of labels there is $\chi_E \in C^\infty(\mathbb{R})$ which equals 1 on a neighbourhood of $\bigcup_{\ell\in E}\Gamma_\ell$ and 0 on a neighbourhood of $\bigcup_{\ell\notin E}\Gamma_\ell$, with $|\chi_E^{(j)}(x)| \le C_{j,s}(1+|x|)^{6j}$ for all j, where $C_{j,s}$ does not depend on E. Consequently*

$$\|\chi_E U\|_{-6p} \le C_{p,s}\,\|U\|_{-p} \qquad (U \in H_{-p}), \tag{14}$$

uniformly in E.

Splitting. For $T \in V_p(\Lambda)$ write $S = \mathcal{F}T$, $S_\ell = \chi_{\{\ell\}}S$, $U_\ell = \mathcal{F}^{-1}S_\ell$, and, with E_M the set of labels of index $> M$, $S_{>M} = \chi_{E_M}S$ and $U_{>M} = \mathcal{F}^{-1}S_{>M}$. Since S is locally atomic on Λ, S_ℓ is the restriction of the measure S to Γ_ℓ, and $S_{>M}$ its restriction to the blocks of index $> M$. Since $\sum_{\ell\notin E_M}\chi_{\{\ell\}} + \chi_{E_M} = 1$ near Λ,

$$T = \sum_{\ell\notin E_M} U_\ell + U_{>M}, \qquad \|U_\ell\|_{-6p} + \|S_{>M}\|_{-6p} \le C_{p,s}\,\|T\|_{-p}\,. \tag{15}$$

The measure $S_{>M}$ is supported in $\{|x| > (R_{M+1} - \frac{1}{2})/2\}$. For $\theta \in \Theta_i$ (an isolated point of K_i), the masses $c_{\ell,\theta}(n)$ of S_ℓ at the points $t_\ell(n+\theta)$ satisfy

$$|c_{\ell,\theta}(n)| \le C_{p,s,\ell,\theta}\, \|T\|_{-p}\, (1+|n|)^{12p}, \tag{16}$$

as one sees by testing $S_\ell \in H_{-6p}$ against translates of a fixed bump narrower than the distance from θ to $K_i \setminus \{\theta\}$.

Annihilators. For each i choose a nonnegative smooth 1-periodic function g_i of mean one whose zero set is exactly $K_i + \mathbb{Z}$ (normalize a sum of flat positive bumps on the complementary intervals). Its Fourier coefficients $a_i(m)$ decay faster than any power, $a_i(0) = 1$, and $g_i(t_\ell x)$ vanishes exactly on $\widetilde{K}_\ell$. At stage M put

$$G_M(x) = \prod_{i\le M} g_i(s_i x)\, g_i(x/s_i) = \sum_{k\in\mathbb{Z}^{2M}} A_M(k) e^{2\pi \mathrm{i}\omega_k x}, \tag{17}$$
$$A_M(k) = \prod_{i\le M} a_i(k_{i,+}) a_i(k_{i,-}), \qquad \omega_k = \sum_{i\le M} (k_{i,+} s_i + k_{i,-}/s_i).$$

The coefficients $A_M(k)$ decay faster than any power of $|k|$, and the derivatives of G_M are bounded uniformly for $(s_1,\dots,s_M) \in [1,2]^M$. Every point of Γ_ℓ with ℓ of index $\le M$ lies in $\widetilde{K}_{\ell^*}$ and is a zero of a factor of G_M, so $G_M T$ is supported in the union of the blocks of index $> M$. The series $\mathcal{F}(G_M T) = \sum_k A_M(k) \tau_{-\omega_k} S$ converges absolutely in H_{-p}.

Nonresonance. The *catalogue* of stage M consists of the targets $z = t_\ell(n+\theta_{i,j})$ with $\ell = (i,\sigma)$, $i \le M$, $j \le M$ and $|n| \le M$; a target that is not a point of Λ carries mass zero. Let $d_M > 0$ be the minimum, over $i,j \le M$, of the distances of $\theta_{i,j}$ from $K_i \setminus \{\theta_{i,j}\}$ and from 0 in $\mathbb{R}/\mathbb{Z}$. For a target $z = t_\ell(n+\theta)$, a point $y \in \Lambda$ and $k \in \mathbb{Z}^{2M}$, the pair (y,k) is *expected* if $y = t_\ell(n'+\theta) \in \Gamma_\ell$ and all coordinates of k vanish except $k_{i,\sigma} = n - n'$; then $y + \omega_k = z$.

Lemma 6.4 (Nonresonance)**.** *Let $M \ge 1$ and fix $R_1,\dots,R_M$. There are $c_M \in (0,1]$ and $\beta_M = 2(M+1)(2M+6)$, depending only on $R_1,\dots,R_M$ and on the profile, such that for every deterministic continuation $(R_i)_{i>M}$ with $R_i \ge 2^i$ the event A_M that*

$$|y + \omega_k - z| \ge c_M (1+L)^{-\beta_M} \tag{18}$$

for every dyadic $L \ge 1$, every target z, every k with $\|k\|_\infty \le L$ and every $y \in \Lambda$ such that (y,k) is not expected, has probability at least $1 - 2^{-M-4}$.

An example. The simplest instance of Lemma 6.4 involves a single block i, its two reciprocal sectors $\Gamma_{(i,+)} = s_i C_i$ and $\Gamma_{(i,-)} = C_i/s_i$, and a target $z = s_i(n+\theta)$ in the first. In Step 1, $\mathcal{F}(G_M T)$ is a combination of translates of S by the frequencies ω_k. The translate of an atom $y = s_i(n'+\theta)$ of the same sector, with the same phase, by $\omega = (n-n')s_i$ lands exactly on z; these expected pairs produce the convolution identity (26) below. The translate of an atom $y = u/s_i$ of the reciprocal sector by $\omega = m s_i$ lands at distance $|u/s_i + (m-n-\theta)s_i|$ from z. After multiplication by s_i this is $|u + (m-n-\theta)s_i^2|$, whose derivative in $s_i \in [1,2]$ has modulus at least $2|m-n-\theta| \ge 2\,\mathrm{dist}(\theta,\mathbb{Z}) > 0$; so the probability that the distance is less than ε is at most $2\varepsilon/\,\mathrm{dist}(\theta,\mathbb{Z})$. At scale L only polynomially many pairs (y,k) matter, and the sublevel estimate of Appendix D gives the analogous bound when several scales are involved; this yields Lemma 6.4. We do not know an explicit choice of the scales that avoids all resonances (Remark 6.8).

Antiperiodic remainders. We will meet distributions that satisfy an antiperiodic difference equation off a periodic set.

Lemma 6.5 (Antiperiodic lift)**.** *Let $\varpi \in [\frac{1}{2}, 2]$, let $\Pi \subset \mathbb{R}$ be closed and ϖ-periodic, and let $U \in H_{-q}$ satisfy $(\tau_\varpi + I)^{\Delta+1} U = 0$ on $\mathbb{R}\setminus\Pi$. Then*

$$U = W + \sum_{r=0}^{\Delta} x^r v_r, \qquad \operatorname{supp} W \subseteq \Pi, \qquad \tau_\varpi v_r = -v_r, \tag{19}$$

where the v_r are obtained from U by a continuous linear operation depending only on ϖ and Δ, not on Π. The v_r lose only polynomially many Hermite orders: for $q = 6p$ and $\Delta \le 12p$ one

may take

$$\|v_r\|_{-m_p} \le C_p \|U\|_{-6p}, \qquad m_p = 1000(p+1)^3. \tag{20}$$

The proof, given in Appendix C, uses a cutoff and a periodization; it does not divide by a flat annihilator. Put $N_p = 2m_p + 2$.

The parameters. Fix a smooth bump ϕ supported in $(-1,1)$ and equal to 1 near 0; the *bump of radius r centred at z* is $x \mapsto \phi((x-z)/r)$. Let θ_R be a smooth function vanishing on $[-R/4, R/4]$ and equal to 1 outside $[-R/3, R/3]$, with the usual rescaled derivative bounds, and let $f^{(d)}_{y,r}(x) = \frac{(x-y)^r}{r!}\phi((x-y)/d)$.

Proposition 6.6. *Fix a profile (P_i). There are deterministic integers $R_i \ge 2^i$ and, for each $M \ge 1$, deterministic parameters: a cutoff $L_M \ge M$; radii $d^A_M, d^J_M \in (0, 1/M]$; a smooth partition of unity $(\kappa_{M,j})_{j \le J_M}$ of $[-M, M]$ by functions whose supports have diameter less than $\min\{1/M, \min_{i \le M} 1/(100k_i)\}$; numbers $1 \le \underline{h}_M \le \overline{h}_M$ with $6(\overline{h}_M + M + 2) \le R_{M+1}$; and $\eta_M \in (0, 1/(1000k_M)]$. There is also a measurable set $\Omega \subset [1,2]^{\mathbb{N}}$ with $\mathbb{P}(\Omega) \ge 13/16$ such that every $s \in \Omega$ satisfies (13), the sets $\widetilde{K}_\ell$ are pairwise disjoint, and for every $M \ge 1$ and every $p \le M$, with $R = R_{M+1}$:*

(P1) Nonresonance. *For every target z of the stage-M catalogue, every k with $\|k\|_\infty \le L_M$ and every $y \in \Lambda$ such that (y,k) is not expected, $|y + \omega_k - z| \ge 10 d^A_M$. For every bump f of radius d^A_M centred at a target and every $S \in H_{-p}$,*

$$\Big| \sum_{\|k\|_\infty > L_M} A_M(k)\, (\tau_{-\omega_k} S)(f) \Big| \le 2^{-M} \|S\|_{-p}. \tag{21}$$

(P2) Separation of the periodic sets. *For distinct labels ℓ, ℓ' of index $\le M$, the sets $\widetilde{K}_\ell \cap [-M-3, M+3]$ and $\widetilde{K}_{\ell'} \cap [-M-3, M+3]$ are at distance at least $10 d^J_M$.*

(P3) Translations. *For every vector (a_ℓ), $a_\ell \in [0, 2\varpi_\ell)$, indexed by the labels of index $\le M$, there is $h \in [\underline{h}_M, \overline{h}_M]$ with $h - a_\ell \in 2\varpi_\ell \mathbb{Z} + [-\eta_M, \eta_M]$ for all these ℓ. For every such h, every $\varphi = \kappa_{M,j}\psi$ with $\operatorname{supp}\psi \subset [-M, M]$ and $\|\psi\|_{C^{N_p}} \le 1$, every $\Delta \le 12p$, and all ϖ_ℓ-antiperiodic $v_{\ell,r} \in H_{-m_p}$ $(r \le \Delta)$ of norm at most $\mathcal{A}$,*

$$\Big| h^{-\Delta} \sum_\ell \sum_{r \le \Delta} (x^r v_{\ell,r})(\varphi(\cdot - h)) - \sum_\ell v_{\ell,\Delta}(\varphi(\cdot - a_\ell)) \Big| \le \frac{2^{-M}\mathcal{A}}{1 + J_M}. \tag{22}$$

(P4) Gaps. *For bumps f of radius d^A_M centred at targets, for $|h| \le \overline{h}_M$ and ψ as in* (P3)*, and for $|y| \le M+1$ and $0 \le r \le 12p$,*

$$\big\| \theta_R G_M \mathcal{F}^{\pm 1} f \big\|_p \le 2^{-M}, \tag{23}$$

$$\big\| \theta_R \mathcal{F}^{\pm 1}\big((\kappa_{M,j}\psi)(\cdot - h)\big) \big\|_{6p} \le \frac{2^{-M}}{1 + J_M}, \tag{24}$$

$$\big\| \theta_R \mathcal{F}^{\pm 1} f^{(d^J_M)}_{y,r} \big\|_{6p} \le 2^{-M}. \tag{25}$$

We fix (R_i), Ω and $s \in \Omega$ as in Proposition 6.6 from now on, and we abbreviate $f_{y,r} = f^{(d^J_M)}_{y,r}$.

6.4. **Step 1: phase convolution and the seam.** Let $T \in V_p(\Lambda)$, fix a label $\ell = (i, \sigma)$, a phase $\theta \in \Theta_i$ and an integer n, and let $M \ge p$ be so large that $z = t_\ell(n + \theta)$ belongs to the catalogue of stage M. Test $\mathcal{F}(G_M T)$ against the bump f of radius d^A_M centred at z. By (P1), every unexpected translated atom misses the support of f, so the part of $\sum_k A_M(k)(\tau_{-\omega_k} S)(f)$ with $\|k\|_\infty \le L_M$ equals $\sum_{|m| \le L_M} a_i(m) c_{\ell,\theta}(n-m)$, because all other factors of $A_M(k)$ equal $a_{i'}(0) = 1$; the omitted part is bounded by (21). On the other hand $\mathcal{F}(G_M T)(f) = (G_M T)(\theta_R \mathcal{F} f) = T(\theta_R G_M \mathcal{F} f)$ with $R = R_{M+1}$, because $G_M T$ is supported in the union of the blocks of index $> M$, and by (23) this is at most $2^{-M}\|T\|_{-p}$. Hence the truncated sum has modulus at most $2^{1-M}\|T\|_{-p}$. Letting

$M \to \infty$, and using (16) and the rapid decay of a_i, we obtain

$$\sum_{m\in\mathbb{Z}} a_i(m)\, c_{\ell,\theta}(n-m) = 0 \qquad (n\in\mathbb{Z},\ \theta\in\Theta_i). \tag{26}$$

The left-hand side of (26) is the mass at $t_\ell(n+\theta)$ of

$$Z_\ell := \mathcal{F}(g_i(t_\ell\,\cdot)\,U_\ell) = \sum_{m\in\mathbb{Z}} a_i(m)\,\tau_{-mt_\ell} S_\ell,$$

where the series converges absolutely in H_{-6p}, and pairing with a bump that isolates $t_\ell(n+\theta)$ from the rest of $t_\ell(K_i+\mathbb{Z})$ commutes with the summation. Each term is supported in $t_\ell(K_i+\mathbb{Z})$, so (26) shows that Z_ℓ vanishes near every isolated point of $t_\ell(K_i+\mathbb{Z})$: it is supported in the seam lattice $t_\ell(\mathbb{Z}+\frac12)$. Since $Z_\ell \in H_{-6p}$, it has order at most $12p$ near each lattice point, so $Z_\ell = \sum_{m\in\mathbb{Z}}\sum_{r\le 12p} c_{m,r}\delta^{(r)}_{t_\ell(m+1/2)}$, and testing against translated bumps shows that the $c_{m,r}$ grow at most polynomially in m. Since $\mathcal{F}^{-1}(\delta_a^{(r)}) = (-2\pi\mathrm{i}x)^r e^{2\pi\mathrm{i}ax}$, we get $g_i(t_\ell x)U_\ell = \sum_{r\le 12p} x^r w_r$ with $w_r = (-2\pi\mathrm{i})^r \sum_m c_{m,r} e^{2\pi\mathrm{i}t_\ell(m+1/2)x}$, and $\tau_{\varpi_\ell} w_r = -w_r$. Since $(\tau_{\varpi_\ell}+I)(x^r w)$ has degree $r-1$ in x when w is ϖ_ℓ-antiperiodic, $(\tau_{\varpi_\ell}+I)^{12p+1}[g_i(t_\ell x)\,U_\ell] = 0$. The multiplier $g_i(t_\ell x)$ is ϖ_ℓ-periodic, commutes with τ_{ϖ_ℓ}, and does not vanish on $\mathbb{R}\setminus\widetilde{K_\ell}$. Hence $(\tau_{\varpi_\ell}+I)^{12p+1}U_\ell = 0$ on $\mathbb{R}\setminus\widetilde{K_\ell}$, and Lemma 6.5 gives

$$U_\ell = W_\ell + \sum_{r=0}^{12p} x^r v_{\ell,r}, \qquad \operatorname{supp} W_\ell \subseteq \widetilde{K_\ell}, \qquad \tau_{\varpi_\ell} v_{\ell,r} = -v_{\ell,r}, \qquad \|v_{\ell,r}\|_{-m_p} \le \mathcal{A}, \tag{27}$$

with $\mathcal{A} = C_p C_{p,s}\,\|T\|_{-p}$ independent of ℓ. The large derivatives of the flat function g_i do not enter these bounds: the lift is applied to U_ℓ after the difference equation has been obtained.

6.5. **Step 2: elimination of the antiperiodic remainders.** We show by downward induction on $\Delta = 12p, 12p-1, \dots, 0$ that $\operatorname{supp} v_{\ell,\Delta} \subseteq \widetilde{K_\ell}$ for every label ℓ. Suppose this holds for all degrees $> \Delta$, and absorb the corresponding terms into W_ℓ, so that $U_\ell = W'_\ell + \sum_{r\le\Delta} x^r v_{\ell,r}$ with $\operatorname{supp} W'_\ell \subseteq \widetilde{K_\ell}$. Fix a label ℓ_0 and $\psi \in C_c^\infty$ with $\operatorname{supp}\psi \cap \widetilde{K_{\ell_0}} = \varnothing$, normalized by $\|\psi\|_{C^{N_p}} \le 1$. Let M be large: ℓ_0 has index at most M, $p \le M$, $\operatorname{supp}\psi \subset [-M, M]$, and η_M is smaller than the distance from $\operatorname{supp}\psi$ to $\widetilde{K_{\ell_0}}$. By (15),

$$\sum_{\ell\notin E_M}\sum_{r\le\Delta} x^r v_{\ell,r} = T - \sum_{\ell\notin E_M} W'_\ell - U_{>M}. \tag{28}$$

Let $\varphi = \kappa_{M,j}\psi$ be one of the pieces of ψ. For every label $\ell \neq \ell_0$ of index $\le M$ choose $a_\ell \in [0, 2\varpi_\ell)$ such that $\operatorname{supp}\varphi + a_\ell$ lies in a gap of $\widetilde{K_\ell}$, at distance at least $1/(20k_i)$ from it; this is possible since each $\widetilde{K_\ell}$ has a gap of length at least $1/(4k_i)$ in every period (Proposition 5.3(v)) and $\operatorname{supp}\varphi$ has diameter less than $1/(100k_i)$. Put $a_{\ell_0} = 0$ in a first phase vector and $a_{\ell_0} = \varpi_{\ell_0}$ in a second one, the other coordinates being equal, and let h, h' be the corresponding translations given by (P3).

Pair (28) with $\varphi(\cdot - h)$. On the right, $W'_\ell(\varphi(\cdot-h)) = 0$ for all $\ell \notin E_M$, because $\operatorname{supp}\varphi + h$ avoids every $\widetilde{K_\ell}$ of index $\le M$ (here $\eta_M < 1/(20k_i)$ is used); $T(\varphi(\cdot-h)) = 0$, because T is locally atomic on Λ, whose points of index $\le M$ lie in $\bigcup_\ell \widetilde{K_\ell}$ and whose other points lie outside $[-R_{M+1}/3, R_{M+1}/3] \supset \operatorname{supp}\varphi + h$; and $|U_{>M}(\varphi(\cdot-h))| = |S_{>M}(\theta_R \mathcal{F}^{-1}(\varphi(\cdot-h)))| \le C_{p,s}\,\|T\|_{-p}\, 2^{-M}(1+J_M)^{-1}$ by (15) and (24). The same holds for h'. On the left we use (22) for both translations. Subtracting the two resulting identities, all terms with $\ell \neq \ell_0$ cancel, and since $v_{\ell_0,\Delta}(\varphi(\cdot - \varpi_{\ell_0})) = -v_{\ell_0,\Delta}(\varphi)$ we obtain $|v_{\ell_0,\Delta}(\varphi)| \le C\,2^{-M}(1+J_M)^{-1}(\|T\|_{-p} + \mathcal{A})$. Summing over the at most J_M pieces gives $|v_{\ell_0,\Delta}(\psi)| \le C\,2^{-M}(\|T\|_{-p} + \mathcal{A})$. Letting $M \to \infty$ gives $v_{\ell_0,\Delta}(\psi) = 0$, which completes the induction. Consequently

$$\operatorname{supp} U_\ell \subseteq \widetilde{K_\ell} \qquad \text{for every label } \ell. \tag{29}$$

Only finite sums and the term $U_{>M}$ were used; no convergence of the series $\sum_\ell W_\ell$ or $\sum_\ell \sum_r x^r v_{\ell,r}$ is assumed.

6.6. **Step 3: recovery of the atoms.** Fix a label ℓ_0 and a point $y \in \widetilde{K}_{\ell_0}$, and let M be large: ℓ_0 has index at most M, $|y| \le M$, $p \le M$. By (P2), the probes $f_{y,r}$, of radius d_M^J, do not meet any $\widetilde{K}_\ell$ with $\ell \neq \ell_0$ of index $\le M$. Pairing (15) with $f_{y,r}$ and using (29), we get $U_{\ell_0}(f_{y,r}) = T(f_{y,r}) - U_{>M}(f_{y,r})$, and $|U_{>M}(f_{y,r})| = |S_{>M}(\theta_R \mathcal{F}^{-1} f_{y,r})| \le C_{p,s}\,\|T\|_{-p}\,2^{-M}$ by (25). As $M \to \infty$ the radius d_M^J tends to zero, and since Λ is locally finite, $T(f_{y,r})$ is eventually $T(\{y\})$ if $r = 0$ and $y \in \Lambda$, and 0 otherwise.

If y is an isolated point of $\widetilde{K}_{\ell_0}$, then near y the distribution U_{ℓ_0} is a finite sum $\sum_r u_r \delta_y^{(r)}$, and $U_{\ell_0}(f_{y,r}) = (-1)^r u_r$ once d_M^J is small. Hence U_{ℓ_0} has, at y, no derivative terms and the same mass as T; this mass is zero unless $y \in \Lambda$, in which case $y \in \Gamma_{\ell_0^*}$, because the points of Λ of index $\le M$ lie in the pairwise disjoint sets $\widetilde{K}_\ell$ and those of larger index are far away. If y is a seam point, then $y \notin \Lambda$; near y only finitely many points of $\widetilde{K}_{\ell_0}$ belong to $\Gamma_{\ell_0^*}$, since the phase γ_j of C_i occurs only in the cells $|n| \ge k_i + j$, and U_{ℓ_0} vanishes near the other isolated points. Thus, in a small neighbourhood of y, U_{ℓ_0} is supported at $\{y\}$, and the same probes show that it vanishes there. Consequently U_{ℓ_0} is locally atomic on $\Gamma_{\ell_0^*} = C_i/t_{\ell_0}$, while $\mathcal{F}U_{\ell_0} = S_{\ell_0}$ is locally atomic on $\Gamma_{\ell_0} = t_{\ell_0} C_i$.

6.7. **Conclusion.**

Proof of Theorem 6.1. By the previous subsection and Proposition 3.9(a), $D_{t_\ell} U_\ell \in \mathcal{M}_{C_i} = \mathbb{C}\mu_i$, so $U_\ell = c_\ell e_{\ell^*}$ for some $c_\ell \in \mathbb{C}$; this is (12), since $U_\ell = \mathcal{F}^{-1}(S|_{\Gamma_\ell})$. As $U_\ell \in H_{-6p}$ and $\mathbb{C}e_{\ell^*} \cap H_{-6p} = \{0\}$ when $6p \le P_i$, we get $c_\ell = 0$ whenever $P_i \ge 6p$. If only finitely many i satisfy $P_i < 6p$, choose M larger than all of them. Then $S_\ell = 0$ for all labels of index $> M$, so the locally atomic measure $S_{>M}$ has no mass at any point and vanishes, and (15) gives $T = \sum_{\ell \notin E_M} c_\ell e_{\ell^*}$. Disjointness of the Γ_ℓ follows from (13), and symmetry of Λ from that of the C_i. □

Corollary 6.7. *In the situation of Theorem 6.1, assume $P_i \to \infty$. Then*

$$\mathcal{M}_\Lambda = \bigoplus_\ell \mathbb{C}e_\ell \qquad \textit{(algebraic direct sum)}, \tag{30}$$

every element of $V_p(\Lambda)$ is a combination of the e_ℓ with $P_i < 6p$, and

$$2\,\#\{i : 18P_i + 4 \le p\} \le r_p(\Lambda) \le 2\,\#\{i : P_i < 6p\} \qquad (p \ge 1). \tag{31}$$

In particular $\dim \mathcal{M}_\Lambda = \aleph_0$. *If $P_i \not\to \infty$, then* $\dim \mathcal{M}_\Lambda = \mathfrak{c}$.

Proof. The e_ℓ belong to $\mathcal{M}_\Lambda$ by (11) and are linearly independent because their supports Γ_ℓ are pairwise disjoint. If $P_i \to \infty$, then for each p only finitely many i satisfy $P_i < 6p$, so Theorem 6.1 gives (30) and the upper bound in (31). The lower bound holds for every profile, since $e_\ell \in H_{-(18P_i+4)} \subseteq H_{-p}$ when $18P_i + 4 \le p$. Hence all r_p are finite and unbounded, and Proposition 3.4 gives $\dim \mathcal{M}_\Lambda = \aleph_0$. If $P_i \le q$ for infinitely many i, the lower bound shows that $V_{18q+4}(\Lambda)$ is infinite-dimensional, and Proposition 3.4 gives $\dim \mathcal{M}_\Lambda = \mathfrak{c}$. □

6.8. **Proof of Theorems B and C.** For Theorem B take $P_i = i$ in Corollary 6.7 and enumerate the basis as $\mu_{2i-1} = e_{(i,+)}$, $\mu_{2i} = e_{(i,-)}$. Then (i) is (30), (ii) is (31) because $\#\{i : 18i + 4 \le p\} = \lfloor (p-4)/18 \rfloor_+$ and $\#\{i : i < 6p\} = 6p - 1$, and (iii) follows from (9) and the boundedness of dilations. For Theorem C, given a nondecreasing unbounded F with $F(0) = 0$, put $P_i = \min\{q \ge 1 : F(q) \ge 2i\}$. Then $P_i \le q$ if and only if $2i \le F(q)$, so $\#\{i : P_i \le q\} = \lfloor F(q)/2 \rfloor$ and $P_i \to \infty$. By Corollary 6.7, $\dim \mathcal{M}_\Lambda = \aleph_0$ and

$$r_p(\Lambda) \ge 2\Big\lfloor \tfrac{1}{2} F\big(\lfloor (p-4)/18 \rfloor_+\big) \Big\rfloor \ge F\big(\lfloor (p-4)/18 \rfloor_+\big) - 1, \qquad r_p(\Lambda) \le 2\Big\lfloor \tfrac{1}{2} F(6p-1) \Big\rfloor \le F(6p).$$

□

Remark 6.8. The constants 18 and 6 come from the Hermite order of the blocks in Proposition 5.3 and from the loss in (14); no optimality is claimed. The scales s are chosen at random only to avoid resonances; we do not know an explicit admissible sequence.

7. Consequences: pairs, Schwartz space and higher dimensions

7.1. **Disjoint support and spectrum, and all subpairs.** Let Λ, (P_i) and (e_ℓ) be as in Corollary 6.7, with $P_i \to \infty$. Recall that $\operatorname{supp} e_\ell \subseteq \Gamma_\ell$ and $\mathcal{F}e_\ell = \zeta_i e_{\ell^*}$, where Γ_{ℓ^*} is the reciprocal sector.

Proposition 7.1. *For all subsets $A', B' \subseteq \Lambda$,*

$$\mathcal{N}(A', B') = \bigoplus_{\ell:\ \operatorname{supp} e_\ell \subseteq B',\ \operatorname{supp} \mathcal{F}e_\ell \subseteq A'} \mathbb{C}e_\ell. \tag{32}$$

In particular, for unions of sectors $B_I = \bigcup_{\ell\in I} \Gamma_\ell$ and $A_J = \bigcup_{\ell\in J} \Gamma_\ell$,

$$\mathcal{N}(A_J, B_I) = \bigoplus_{\ell\in I,\ \ell^*\in J} \mathbb{C}e_\ell,$$

so every finite dimension and $\aleph_0$ occur among the spaces $\mathcal{N}(A', B')$, $A', B' \subseteq \Lambda$, and every even dimension and $\aleph_0$ occur among the spaces $\mathcal{M}_{B_I}$ with $I = I^*$.

Proof. Since $A', B' \subseteq \Lambda$, every $T \in \mathcal{N}(A', B')$ lies in $\mathcal{M}_\Lambda$ and is therefore a finite combination $\sum c_\ell e_\ell$ by (30). The measures e_ℓ have pairwise disjoint supports, and so do the $\mathcal{F}e_\ell$. If $c_\ell \neq 0$, then every atom of e_ℓ is an atom of T, so $\operatorname{supp} e_\ell \subseteq B'$, and likewise $\operatorname{supp} \mathcal{F}e_\ell \subseteq A'$. Conversely every e_ℓ with these two properties lies in $\mathcal{N}(A', B')$. For unions of sectors, $\operatorname{supp} e_\ell \subseteq \Gamma_\ell$ is nonempty, so $\operatorname{supp} e_\ell \subseteq B_I$ if and only if $\ell \in I$, and similarly $\operatorname{supp} \mathcal{F}e_\ell \subseteq A_J$ if and only if $\ell^* \in J$. Taking $J = I^*$ with I finite of any cardinality, or infinite, gives all finite dimensions and $\aleph_0$; taking $J = I = I^*$ gives the diagonal statement. □

Theorem 7.2. *For a choice of orientations $\eta = (\eta_i) \in \{+,-\}^{\mathbb{N}}$ put*

$$B_\eta = \bigcup_{i\geq 1} \Gamma_{(i,\eta_i)}, \qquad A_\eta = \bigcup_{i\geq 1} \Gamma_{(i,-\eta_i)}, \qquad u_i = e_{(i,\eta_i)}.$$

Then A_η and B_η are disjoint, nonempty, symmetric and locally finite, $A_\eta \cup B_\eta = \Lambda$, and

$$\mathcal{N}(A_\eta, B_\eta) = \bigoplus_{i\geq 1} \mathbb{C}u_i, \qquad \dim \mathcal{N}(A_\eta, B_\eta) = \aleph_0,$$

$$\mathcal{M}_\Lambda = \mathcal{N}(A_\eta, B_\eta) \oplus \mathcal{N}(B_\eta, A_\eta), \qquad \mathcal{M}_{A_\eta} = \mathcal{M}_{B_\eta} = \{0\},$$

$$\#\{i : 18P_i + 4 \leq p\} \leq r_p(A_\eta, B_\eta) \leq \#\{i : P_i < 6p\} \qquad (p \geq 1).$$

Proof. Everything except the rank bounds follows from Proposition 7.1: $\operatorname{supp} u_i \subseteq \Gamma_{(i,\eta_i)} \subseteq B_\eta$ and $\operatorname{supp} \mathcal{F}u_i \subseteq \Gamma_{(i,-\eta_i)} \subseteq A_\eta$, whereas $e_{(i,-\eta_i)}$ is supported in A_η, which is disjoint from B_η; and an element of $\mathcal{M}_{B_\eta}$ would have to be a combination of u_i's whose Fourier transforms lie in A_η, hence zero. The lower rank bound counts the u_i with $18P_i + 4 \leq p$. For the upper bound, an element of $V_p(A_\eta, B_\eta) \subseteq V_p(\Lambda)$ is a combination of the e_ℓ with $P_i < 6p$ (Corollary 6.7), and only the u_i can occur, one for each such i. □

Proof of Corollary D. Take $P_i = i$ and any η in Theorem 7.2, and apply Proposition 7.1 with $\mu_{2i-1} = e_{(i,+)}$ and $\mu_{2i} = e_{(i,-)}$. □

The upper bound $6p-1$ is not obtained by halving the bound $12p-2$ of Theorem B: projections onto the summands in the decomposition of $\mathcal{M}_\Lambda$ need not preserve the Hermite order, and the argument uses instead that an element of Hermite order at most p involves only blocks with $P_i < 6p$. By Proposition 3.9(a), translating A_η and B_η independently, and dilating them by reciprocal factors, gives non-symmetric examples with the same layer dimensions.

7.2. **Schwartz space modulo the relations.** The countable case has a reformulation in terms of Schwartz space. Let $\mathcal{J}(A, B)$ be the closure of $I_B + \mathcal{F}I_A$ in $\mathcal{S}$. By the Hahn–Banach theorem and Proposition 3.2, the dual of the Fréchet space $\mathcal{S}/\mathcal{J}(A, B)$ is $\mathcal{N}(A, B)$. Let $\omega = \mathbb{C}^{\mathbb{N}}$ be the Fréchet space of all sequences, whose dual is the space of finitely supported sequences.

Proposition 7.3. *Let $A, B \subset \mathbb{R}$ be locally finite. Then $\dim \mathcal{N}(A,B) = \aleph_0$ if and only if $\mathcal{S}/\mathcal{J}(A,B)$ is isomorphic to ω. More precisely, if $(T_k)_{k\ge1}$ is a Hamel basis of $\mathcal{N}(A,B)$ in case* (ii) *of Proposition 3.4, then $\varphi \mapsto (T_k(\varphi))_{k\ge1}$ is a continuous linear surjection of $\mathcal{S}$ onto ω with kernel $\mathcal{J}(A,B)$. In particular this holds for the sets of Theorems B and C and Corollary D.*

Proof. If $\mathcal{S}/\mathcal{J} \cong \omega$, then $\mathcal{N}(A,B) \cong \omega'$ has dimension $\aleph_0$. Conversely let (T_k) be a Hamel basis in case (ii). The map $\Phi : \mathcal{S}/\mathcal{J} \to \omega$, $\Phi(\varphi + \mathcal{J}) = (T_k(\varphi))_k$, is continuous, and it is injective because $\mathcal{J} = \mathcal{N}(A,B)^\perp$ by the Hahn–Banach theorem. Every continuous seminorm on $\mathcal{S}/\mathcal{J}$ is dominated by a multiple of the quotient seminorm q_p of some $\|\cdot\|_p$, and the functionals in $(\mathcal{S}/\mathcal{J})'$ that are bounded with respect to q_p form the space $V_p(A,B)$, which is finite-dimensional in case (ii). By Eidelheit's theorem [Eid36], see [MV92, Satz 26.27], Φ is surjective, and by the open mapping theorem it is an isomorphism. □

Remark 7.4. More generally, a Fréchet space E whose dual has countably infinite Hamel dimension is isomorphic to ω: for each zero-neighbourhood U, the functionals bounded on U form a Banach space of countable Hamel dimension, which is therefore finite-dimensional, and Eidelheit's theorem applies as above. Thus in case (ii) the countably many independent summation formulas T_k are independent in the strongest sense: their values on Schwartz functions can be prescribed arbitrarily, and they are the only relations. This is a dual analogue of the codimension-one statement of Radchenko and Viazovska [RV19, Theorem 2]: it does not describe the image of the data map $\varphi \mapsto (\varphi|_\Lambda, \widehat{\varphi}|_\Lambda)$, but it shows that, modulo the closure of $I_\Lambda + \mathcal{F}I_\Lambda$, Schwartz space is isomorphic to ω.

7.3. **Higher dimensions.** For $d \ge 1$ we use $\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi}\,dx$ on $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$. For locally finite $A, B \subset \mathbb{R}^d$ (meeting every bounded set in a finite set) we let $\mathcal{N}(A,B)$ be the space of tempered distributions on $\mathbb{R}^d$ that are locally atomic on B and whose Fourier transforms are locally atomic on A, and we put $\mathcal{M}_\Lambda = \mathcal{N}(\Lambda,\Lambda)$; Fourier quasicrystals and $\mathcal{M}^{\mathrm{str}}_\Lambda$ are defined as in Section 3.5, with the weights $(1+|x|)^{-N}$ on $\mathbb{R}^d$. Let $\delta_{\mathbb{Z}^{d-1}} = \sum_{n\in\mathbb{Z}^{d-1}} \delta_n$, so that $\mathcal{F}\delta_{\mathbb{Z}^{d-1}} = \delta_{\mathbb{Z}^{d-1}}$ by Poisson's formula.

Proposition 7.5. *Let $A, B \subset \mathbb{R}$ be locally finite and $d \ge 2$. Then $\mu \mapsto \mu \otimes \delta_{\mathbb{Z}^{d-1}}$ is a linear isomorphism of $\mathcal{N}(A,B)$ onto $\mathcal{N}(A\times\mathbb{Z}^{d-1}, B\times\mathbb{Z}^{d-1})$, and $\mu \otimes \delta_{\mathbb{Z}^{d-1}}$ is a Fourier quasicrystal if and only if μ is one.*

Proof. Since $\mathcal{F}(\mu\otimes\delta_{\mathbb{Z}^{d-1}}) = \widehat{\mu}\otimes\delta_{\mathbb{Z}^{d-1}}$, the map is well defined; it is linear and injective. Let $T \in \mathcal{N}(A\times\mathbb{Z}^{d-1}, B\times\mathbb{Z}^{d-1})$ and $2 \le j \le d$. The function $e^{2\pi i\xi_j} - 1$ vanishes on $A\times\mathbb{Z}^{d-1}$, where $\mathcal{F}T$ is locally atomic, so $(e^{2\pi i\xi_j}-1)\mathcal{F}T = 0$; that is, T is invariant under the translation by the j-th unit vector. Hence the mass of T at (b,n) does not depend on $n \in \mathbb{Z}^{d-1}$; call it c_b, and put $\mu = \sum_{b\in B} c_b\delta_b$. Let $\psi \in C_c^\infty(\mathbb{R}^{d-1})$ be supported in the ball of radius $\frac12$ about 0, with $\psi(0) = 1$. Then $\mu(\varphi) = T(\varphi\otimes\psi)$ for $\varphi \in C_c^\infty(\mathbb{R})$, and the right-hand side is continuous on $\mathcal{S}(\mathbb{R})$, so μ is tempered; as $\mu\otimes\delta_{\mathbb{Z}^{d-1}}$ and T agree on $C_c^\infty(\mathbb{R}^d)$, they are equal. Hence $\mathcal{F}T = \widehat{\mu}\otimes\delta_{\mathbb{Z}^{d-1}}$ and $\widehat{\mu}(\varphi) = \mathcal{F}T(\varphi\otimes\psi)$, and since $\mathcal{F}T$ is locally atomic on $A\times\mathbb{Z}^{d-1}$, $\widehat{\mu}$ is locally atomic on A. Thus $\mu \in \mathcal{N}(A,B)$. Finally $|\mu\otimes\delta_{\mathbb{Z}^{d-1}}| = |\mu|\otimes\delta_{\mathbb{Z}^{d-1}}$, and $\sum_{n\in\mathbb{Z}^{d-1}}(1+|(x,n)|)^{-N}$ is comparable to $(1+|x|)^{d-1-N}$ for $N > d$, so $\mu\otimes\delta_{\mathbb{Z}^{d-1}}$ is strongly tempered if and only if μ is; the same holds for the Fourier transforms. □

Corollary 7.6. *For every $d \ge 1$ there are a symmetric locally finite set $\Lambda \subset \mathbb{R}^d$ with $\dim \mathcal{M}_\Lambda = \aleph_0$, disjoint locally finite sets $A, B \subset \mathbb{R}^d$ with $\dim \mathcal{N}(A,B) = \aleph_0$ and $\mathcal{M}_A = \mathcal{M}_B = \{0\}$, and a locally finite set $\Lambda' \subset \mathbb{R}^d$ with $\dim \mathcal{M}^{\mathrm{str}}_{\Lambda'} = \aleph_0$.*

Proof. Apply Proposition 7.5 to the sets of Theorems B and A and Corollary D; the product of a symmetric set with $\mathbb{Z}^{d-1}$ is symmetric. □

Remark 7.7. Define $H_s(\mathbb{R}^d)$ through the products h_α of Hermite functions and the weights $(|\alpha|+1)^{2s}$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Since $\max\{\alpha_1+1, |\alpha'|+1\} \le |\alpha|+1 \le (\alpha_1+1)(|\alpha'|+1)$ for $\alpha = (\alpha_1,\alpha')$, the map of Proposition 7.5 sends $\mathcal{N}(A,B)\cap H_{-p}$ into $H_{-(p+q)}$ when $\delta_{\mathbb{Z}^{d-1}} \in H_{-q}(\mathbb{R}^{d-1})$, and its inverse sends elements of H_{-p} to H_{-p}. Hence the layer dimensions in $\mathbb{R}^d$ and in $\mathbb{R}$ agree up to the fixed shift q of the order, and the layer bounds of Theorems B and C transfer with this shift. The proof of Proposition 3.4 applies verbatim in $\mathbb{R}^d$.

8. The boundary with Fourier quasicrystals

8.1. **Background.** Recall that a measure ν is *strongly tempered* if $\int (1+|x|)^{-N}\, d|\nu|(x) < \infty$ for some N; a crystalline measure μ is a Fourier quasicrystal if μ and $\widehat{\mu}$ are both strongly tempered. The difference between temperedness of μ and of $|\mu|$ is discussed in general by Baake and Strungaru [BS23]. For crystalline measures, Favorov [Fav24a, Theorem 1] (see also [Fav23]) proved that a tempered measure with polynomially separated support, $|x-x'| \geq c\min\{1, |x|^{-h}, |x'|^{-h}\}$, has tempered total variation; in particular a crystalline measure with polynomially separated support and spectrum is a Fourier quasicrystal. He constructed crystalline measures that are not Fourier quasicrystals [Fav24a, Fav24b], and Mazáč, Richard and Strungaru [MRS26] studied almost periodicity of crystalline measures in several distributional categories, constructing in particular a crystalline Fourier eigenmeasure that is not strongly tempered, hence not strongly tempered on either side [MRS26, Theorem 4.11, Remark 5.2]. These works answer, for several notions of almost periodicity, the question raised by Meyer [Mey17b, Mey21] whether every crystalline measure is an almost periodic distribution. Boyvalenkov and Favorov [BF26] studied measures of unbounded total variation that are positive or translation bounded and have pure point Fourier transform; they showed that the squared masses of $\widehat{\mu}$ form a translation bounded measure, and that $\widehat{\mu}$ itself is translation bounded if the spectrum is locally linearly independent over $\mathbb{Z}$, which gives conditions for a crystalline measure to be a Fourier quasicrystal.

Gonçalves [Gon23] classified the Fourier summation pairs (μ, a), $\int \widehat{\varphi}\, d\mu = \sum_\lambda a(\lambda)\varphi(\lambda)$, in which the left-hand side is an absolutely convergent integral, under the hypotheses $\int (1+t^2)^{-1}\, d|\mu|(t) < \infty$ and $\sum_\lambda |a(\lambda)| e^{-c|\lambda|} < \infty$, in terms of almost periodic holomorphic functions in the upper half-plane, Hermite–Biehler functions and de Branges spaces. Gonçalves and Vedana [GV25] removed the restriction on the polynomial degree, requiring $\int (1+t^2)^{-c_1}\, d|\mu|(t) + \sum_\lambda |a(\lambda)| e^{-c_2|\lambda|} < \infty$ for some $c_1, c_2 > 0$, and remark that they are unaware of a summation formula valid for all compactly supported smooth φ that violates this condition. A crystalline measure ν gives such a formula in two ways, with $(\mu, a) = (\nu, \widehat{\nu})$ or, by Fourier inversion, with the roles of ν and $\widehat{\nu}$ exchanged; if ν is not strongly tempered, the left-hand side has to be read as the distributional pairing (1). In Favorov's example [Fav24b] of a crystalline measure that is not a Fourier quasicrystal, the Fourier transform appears to be strongly tempered, so the exchanged formula may well satisfy the hypotheses of [GV25]. Alfes, Kiefer and Mazáč [AKM25] relate Fourier eigenmeasures to modular-type transformation laws.

8.2. **No element is strongly tempered.** On our sets the failure of strong temperedness is a property of every nonzero element of an infinite-dimensional space.

Theorem 8.1. *For Λ as in Theorem B or C, no nonzero $\mu \in \mathcal{M}_\Lambda$ is strongly tempered, and neither is its Fourier transform. In particular, for $A', B' \subseteq \Lambda$, no nonzero element of $\mathcal{N}(A', B')$ is a Fourier quasicrystal.*

Consequently every nonzero element of the spaces of Theorems B and C and Corollary D gives a summation formula in which neither ν nor $\widehat{\nu}$ is strongly tempered, so that it violates the hypotheses of [Gon23, GV25] in either reading. The super-polynomial clustering of our sets is necessary for this, by Favorov's theorem quoted above.

The theorem follows from an exclusion statement for a single half-endpoint carrier, in which strong temperedness is assumed on *both* sides; the notation is that of Section 5 and Appendix B.

Lemma 8.2. *Let (ε_j) and (δ_j) be strictly decreasing sequences of positive numbers tending to zero, with $64\varepsilon_1 \leq 10^{-12}$ and $64\delta_1 \leq 10^{-12}$, and put $\alpha_j = \frac{1}{2} - \varepsilon_j$ and $\alpha'_j = \frac{1}{2} - \delta_j$. If $T \in \mathcal{N}(L_{\alpha'}, L_\alpha)$ and both T and $\mathcal{F}T$ are strongly tempered, then $T = 0$.*

Idea of the proof. In the Zak coordinates of Appendix B, the two support conditions say that the moment array of T vanishes on one side of the diagonal and that of $\mathcal{F}T$ on the other, and the two arrays are related by the gauge E. Strong temperedness on both sides allows us to use moments whose weights absorb the products of the small distances ε_j and δ_j; in these weights the weighted contraction of Proposition B.9 applies, and its only solution is zero. For merely

tempered T these weights are not available, in accordance with the existence of the nonzero block measure of Theorem 5.1(a).

Lemma 8.3. *In the situation of Lemma 8.2, let V be as in Lemma B.3, with $\eta = \varepsilon$ and $\eta' = \delta$, and let $\phi_{i,e}$ and $\phi'_{j,f}$ be built from ε and δ as in (49). Let $K = [-10^{-2}, 10^{-2}]^2$. There are C and r such that for all $f \in C^\infty(\mathbb{R}^2)$, $i, j \ge 0$ and $e, f' \in \{0,1\}$,*

$$|V(f\,\phi_{i,e} \otimes \phi'_{j,f'})| \le C\,\|f\|_{C^r(K)}\; 32^{i+j}(1+i+j)^r \min\Big\{\prod_{l\le i}\varepsilon_l^2,\ \prod_{l\le j}\delta_l^2\Big\}.$$

Proof. Write $T = \sum_\lambda c_\lambda \delta_\lambda$, with $\sum_\lambda |c_\lambda|(1+|\lambda|)^{-N} < \infty$, and $\lambda = m_\lambda + \frac12 + x_\lambda$ with $m_\lambda \in \mathbb{Z}$ and $x_\lambda \in \Xi_\varepsilon$. Since T is strongly tempered, $T(\psi) = \sum_\lambda c_\lambda \psi(\lambda)$ for all $\psi \in \mathcal{S}$, with absolute convergence, and the definition of V gives

$$V(F) = \sum_\lambda c_\lambda (-1)^{m_\lambda} \chi(x_\lambda) \int \chi(y) F(x_\lambda, y)\, e^{-2\pi \mathrm{i}(m_\lambda + \frac12)y}\, dy \qquad (F \in C^\infty(\mathbb{R}^2)).$$

Since V is supported in $[-10^{-12}, 10^{-12}]^2$, we may replace F by $(\chi_0 \otimes \chi_0)F$, with χ_0 as in Section B.5, so that only the values of F on K matter. Integrating by parts $r \ge N$ times in y gives

$$|V(F)| \le C \sum_\lambda |c_\lambda|(1+|m_\lambda|)^{-r} \max_{\nu \le r} \sup_y |\partial_y^\nu F(x_\lambda, y)|,$$

an estimate in which F is evaluated only at the atoms x_λ and no derivative in x occurs. For $F = f\,\phi_{i,e} \otimes \phi'_{j,f'}$, we have $\phi_{i,e}(x_\lambda) = 0$ unless $x_\lambda = \pm\varepsilon_l$ with $l > i$, and then

$$|\phi_{i,e}(x_\lambda)| \le |D_i(x_\lambda)| \le \prod_{l' \le i} a_{l'} \le 20^i \prod_{l \le i} \varepsilon_l^2,$$

since $0 \le z(x_\lambda) \le a_{l'} \le 2\pi^2 \varepsilon_{l'}^2$ for $l' \le i$. The function $\phi'_{j,f'}$ is a trigonometric polynomial whose coefficients have ℓ^1 norm at most 2^j and whose frequencies are at most $j + \frac12$ in modulus, so its derivatives of order $\nu \le r$ are bounded by $2^j(2\pi(j+1))^r$. This gives the bound with $\prod_{l\le i}\varepsilon_l^2$. For the other bound we write $V(F) = (gV)(\bar g F)$ with $\bar g = e^{2\pi \mathrm{i} xy}$ and use the analogous representation of gV through the atoms of $\mathcal{F}T$, which follows from (48) in the same way; it involves derivatives in x only, and evaluations at the recentred atoms $y_\xi \in \Xi_\delta$ of $\mathcal{F}T$. The derivatives of $\bar g$ are bounded on the chart. □

Proof of Lemma 8.2. Use the moments (50) with $\kappa = 10^{-3}$ and the weights $\omega_i = \prod_{l\le i} 64\varepsilon_l$ and $\omega'_j = \prod_{l\le j} 64\delta_l$, which are of the form considered in Section B.6 with $\rho = 10^{-12}$. Since $\min\{P^2, Q^2\} \le PQ$, Lemma 8.3 shows that $|M(fV)_{ie,jf'}| \le C\kappa^{-2}\,\|f\|_{C^r(K)}\, 2^{-i-j}(1+i+j)^r$. Hence the arrays $M(fV)$ lie in $\mathbb{H}$, continuously in $f \in C^r(K)$, and Lemma B.6 gives $M(gV) = E\,M(V)$. By Lemma B.5, $\mathbf{A} = M(V)$ and $\mathbf{B} = M(gV)$ satisfy (51) and (52), so Proposition B.9 gives $\mathbf{A} = 0$. By Lemma B.4, $V = 0$, and by (45) all clusters of T vanish; hence $T = 0$. □

Proposition 8.4. *For all positive integers P, R, no nonzero element of $\mathcal{M}_{C(P,R)}$ is strongly tempered, and no nonzero element has strongly tempered Fourier transform.*

Proof. Let μ span $\mathcal{M}_{C(P,R)}$. Since $\mathcal{F}\mu = \zeta\mu$, strong temperedness of μ or of $\mathcal{F}\mu$ implies strong temperedness of both. The inverse of the hole exchange of Lemma 5.2 subtracts an element of W_k, a finite combination of modulated translated lattice combs, which is strongly tempered on both sides; it therefore produces an element of $\mathcal{M}_{L_{\gamma'}}$ that is strongly tempered on both sides. The map $\mathcal{R}$ of Theorem 5.1(d) is built from finitely many integer translations and multiplications by a trigonometric polynomial, so it maps this element to an element of $\mathcal{M}_{L_\gamma}$ that is strongly tempered on both sides. The sequence (ε_j) of (6) satisfies $64\varepsilon_1 \le 2^{-446} < 10^{-12}$, so this element vanishes by Lemma 8.2 (with $\delta = \varepsilon$). Since both maps are injective, $\mu = 0$. □

Proof of Theorem 8.1. Let $0 \ne T \in \mathcal{M}_\Lambda$, so that $T = \sum_\ell c_\ell e_\ell$ is a finite sum with some $c_\ell \ne 0$ (Corollary 6.7). Each e_ℓ is a dilate of a block measure and hence, by Proposition 8.4, not strongly tempered. The supports of the e_ℓ are pairwise disjoint, so $|T| = \sum_\ell |c_\ell|\,|e_\ell| \ge |c_\ell|\,|e_\ell|$, and $|T|$ is

not tempered. Since $\mathcal{F}T = \sum_\ell c_\ell \zeta_{i(\ell)} e_{\ell^*}$, where $i(\ell)$ is the index of ℓ, is of the same form, $|\mathcal{F}T|$ is not tempered either. The statement for subpairs follows from $\mathcal{N}(A', B') \subseteq \mathcal{M}_\Lambda$. □

Remark 8.5. Since every element of $\mathcal{M}_\Lambda$ is a *finite* combination of the e_ℓ, no infinite combination can produce a strongly tempered element; cf. Corollary 3.6. Theorem 8.1 concerns the sets constructed in Sections 5 and 6. It does not mean that countable dimension requires masses that grow faster than every polynomial: on the different set of Theorem A (Section 4), the Fourier quasicrystals form a space of dimension $\aleph_0$.

9. Questions

Question 9.1. Let Λ be the set of Theorem A. Is $\dim \mathcal{M}_\Lambda = \aleph_0$? More generally, is there a locally finite set Λ for which both $\mathcal{M}_\Lambda$ and $\mathcal{M}_\Lambda^{\mathrm{str}}$ have dimension $\aleph_0$?

For the sets of Theorems B and C we have $\mathcal{M}_\Lambda^{\mathrm{str}} = \{0\}$ (Theorem 8.1), and for the set of Theorem A the space $\mathcal{M}_\Lambda^{\mathrm{str}}$ is determined but $\mathcal{M}_\Lambda$ is not. If A and B are separated at a polynomial rate, every element of $\mathcal{N}(A, B)$ is a Fourier quasicrystal by Favorov's theorem [Fav24a, Theorem 1], so that $\mathcal{N}(A, B) = \mathcal{N}^{\mathrm{str}}(A, B)$. If, moreover, the growth exponents $\inf\{N : \int (1 + |x|)^{-N} d|\mu| < \infty\}$ of the elements of $\mathcal{N}(A, B)$ and of their Fourier transforms were bounded, then $\mathcal{N}(A, B)$ would be contained in a single strong layer $X_N(A, B)$, and Proposition 3.8 would exclude the countable case. Theorem A shows that these exponents can be unbounded on one set, but its set is not separated at a polynomial rate (Remark 4.13). This suggests the following question.

Question 9.2. Let $A, B \subset \mathbb{R}$ be locally finite and separated at a polynomial rate. Is $\dim \mathcal{N}(A, B)$ always finite or equal to $\mathfrak{c}$?

Under strong separation assumptions every element of $\mathcal{N}(A, B)$ is a generalized Dirac comb: when A and B are uniformly discrete [LO15], and when the support is uniformly discrete and the spectrum is relatively separated [LO17, Theorem 2.2]; see [LR21b, Theorem 6.1] for distributions. A natural test case for Question 9.2 and for Question 9.4 below is Meyer's set $\Lambda = \bigcup_{j \geq 0} \{x \in 2^{-j}\mathbb{Z} : \ |x| > 2^{j-2}\}$, which carries Kolountzakis's crystalline measure and for which (Λ, Λ) is a uniqueness pair for $\mathcal{S}$ [Mey22, Section 8]; it is separated at a polynomial rate.

Question 9.3. Can Theorem B be achieved with a set Λ given by an explicit formula, without the probabilistic choice of the scales s_i of Section 6 and the adaptive choice of the radii R_i? Can the layer dimensions be prescribed exactly: for which nondecreasing unbounded F is there a locally finite Λ with $r_p(\Lambda) = F(p)$ for all $p \geq 1$?

Theorem C realizes every growth rate up to the changes of order $p \mapsto \lfloor (p-4)/18 \rfloor_+$ and $p \mapsto 6p$, which come from the window $[P, 18P + 4]$ for the Hermite order of a block and from the loss in (14); in particular the minimal Hermite orders of the basis elements can be made to grow arbitrarily slowly. The question is whether these losses can be removed, for instance by blocks whose Hermite order is known exactly. In the construction of Theorem A no random choices are made, but the radii are again chosen adaptively, by a compactness argument.

Question 9.4. Find geometric conditions on Λ that characterize each of the three alternatives of Propositions 3.4 and 3.8. Lemma 3.7 shows that countable dimension requires new elements to appear at larger and larger distances, and Favorov's theorem [Fav24a, Theorem 1] shows that measures that are not Fourier quasicrystals, such as those of Theorem 8.1, require a super-polynomial coalescence of atoms. The set of Theorem A also contains points at super-polynomially small distances, in the cone C_β/c, although the atoms of each of its blocks are separated at a polynomial rate; we do not know whether such points are necessary for $\dim \mathcal{M}_\Lambda^{\mathrm{str}} = \aleph_0$.

Meyer [Mey23] describes the theory of crystalline measures in several variables as being in its infancy; see also [Mey23b] and, for Fourier interpolation in higher dimensions, [Sto21, RSt22]. Products with lattices give analogues of Theorems B, C and A and of Corollary D in $\mathbb{R}^d$ (Proposition 7.5), but every element of the resulting spaces is periodic in $d - 1$ directions.

Question 9.5. Is there, for some $d \geq 2$, a locally finite set $\Lambda \subset \mathbb{R}^d$ with $\dim \mathcal{M}_\Lambda = \aleph_0$, or with $\dim \mathcal{M}_\Lambda^{\mathrm{str}} = \aleph_0$, such that no nonzero element of $\mathcal{M}_\Lambda$ is invariant under a nonzero translation?

Appendix A. Proofs for Section 4

We use the notation of Section 4, and the following standard facts on the convergents p_j/q_j of β [HW08, Chapter X]: $\gcd(p_j,q_j)=1$, $q_{j+1}=a_{j+1}q_j+q_{j-1}$ and $|q_j\beta-p_j|<1/q_{j+1}$.

A.1. **The slope.**

Proof of Lemma 4.1. (i) We have $q_0=q_1=1$, and $q_j\equiv q_{j-2}\pmod 2$ for $j\ge 2$ because a_j is even; hence every q_j is odd. Let $k\neq 0$ and u be integers, u odd, and suppose that $|2k\beta-u|<1/(4|k|)$. Write $u/(2k)=p/q$ in lowest terms with $q>0$. Since $\gcd(u,2k)$ is odd, q is even, and $q\le 2|k|$. Then $|\beta-p/q|<1/(8k^2)\le 1/(2q^2)$, so p/q is a convergent of β by Legendre's theorem [HW08, Theorem 184], and q is odd, a contradiction.

(ii) Let $j\ge 1$. Since q_j is odd, there is an integer r with $|r/q_j-\frac14|=1/(4q_j)$, and since $\gcd(p_j,q_j)=1$, there is $n\in\{1,\dots,q_j\}$ with $np_j\equiv r\pmod{q_j}$. Put $k=(np_j-r)/q_j$ and $\varepsilon_j=\beta q_j-p_j$. Then

$$\beta n-k-\tfrac14=(r/q_j-\tfrac14)+n\varepsilon_j/q_j,$$

so $|\beta n-k-\frac14|\le 1/(4q_j)+1/q_{j+1}\le 3/(4n)$, because $n\le q_j$ and $q_{j+1}\ge a_{j+1}q_j\ge 2q_j$. As $j\to\infty$ the bound $1/(4q_j)+1/q_{j+1}$ tends to zero, whereas $\beta n-\frac14\notin\mathbb{Z}$ for every n; hence the integers n obtained in this way are unbounded.

(iii) Since β is irrational, $\varepsilon_j\neq 0$, and $|\varepsilon_j|<1/q_{j+1}\le 1/(a_{j+1}q_j)=1/(2q_j^{j+2})$.

(iv) By (iii), $|\beta-p_j/q_j|<q_j^{-j-3}$ for all $j\ge 1$, and $q_j\to\infty$. If β, which is irrational, were algebraic of degree d, then $|\beta-p/q|\ge c/q^d$ for all rationals p/q by Liouville's theorem [HW08, Theorem 191], which fails for $j+3>d$. □

A.2. **Geometry of the sheets.**

For $0<a<1$ the function ψ_a satisfies $\psi_a(-y)=-\psi_a(y)$, $\psi_a(y+1)=\psi_a(y)+1$ and $\psi_a(\frac12)=\frac12$, and

$$\psi_a'(y)=\frac{1-a^2}{|1-ae(y)|^2}\in\Big[\frac{1-a}{1+a},\frac{1+a}{1-a}\Big]$$

is decreasing on $[0,\frac12]$. Hence $\psi_a(y)-y$ vanishes at 0 and $\frac12$ and is first increasing and then decreasing on $[0,\frac12]$, so that $\psi_a(y)\ge y$ on $[0,\frac12]$, $\psi_a(y)\le y$ on $[-\frac12,0]$, and $|\psi_a(y)-y|\le\frac12$ for all y. We write $K_a=(1+a)/(1-a)$ and $g_j(x)=x/\lambda+\psi_{a_j}(\beta x/\lambda-\frac14)$, so that $A_j=\{x: g_j(x)\in\mathbb{Z}\}$ and $\lambda^{-1}\le g_j'\le\lambda^{-1}(1+\beta K_{a_j})$.

Proof of Lemma 4.3. (i) P is admissible of bidegree (s,s). As in the proof of Lemma 4.2(a), P has no zeros in $\{|z|,|w|<1\}$ or in $\{|z|,|w|>1\}$, and $(Z(x),W(x))$ lies in one of these sets when $x\notin\mathbb{R}$; so all zeros of $P(Z(x),W(x))$ are real. For real x we have $|W|=1$, hence $1-aW\neq 0$, and $p_a(Z,W)=(1-aW)(1-ZB_a(W))$ vanishes if and only if $ZB_a(W)=1$; for $a=a_j$ this means $e(g_j(x))=1$, that is, $x\in A_j$. Since $p_b-p_a=(b-a)(z-w)$ and $p_a(w,w)=1-w^2$, a common zero of p_a and p_b, $a\neq b$, satisfies $Z=W=\pm1$. Then $x/\lambda=u/2$ with $u\in\mathbb{Z}$, and $Z=W$ gives $(\beta-1)u/2-\frac14\in\mathbb{Z}$, so $2\beta u$ is an odd integer, which is impossible. Hence the sets A_j are disjoint. At a point of A_j, with $a=a_j$,

$$\frac{d}{dx}\,p_a(Z(x),W(x))=-(1-aW)\,\frac{d}{dx}e(g_j(x))=-2\pi\mathrm{i}\,(1-aW)\,g_j'(x)\neq 0, \tag{33}$$

and the other factors of P do not vanish there, so the zeros are simple. Finally, $g_j(r)-g_j(-r)\le 2(1+\beta)r/\lambda+1$ because $|\psi_a(y)-y|\le\frac12$, so $A_j\cap[-r,r]$ has at most $2(1+\beta)r/\lambda+2$ points.

(ii) Let $|x|<\delta\lambda$ and $t=x/\lambda$. Since $\beta\delta<\frac18$, the number $y=\beta t-\frac14$ lies in $(-\frac38,-\frac18)$, so $-\frac12<\psi_{a_j}(y)\le y<-\frac18$ and $g_j(x)=t+\psi_{a_j}(y)\in(-\frac12-\delta,-\frac18+\delta)$, an interval in $(-1,0)$. Hence $x\notin A$.

(iii) Let $x\in A_j$, $a=a_j$, $t=x/\lambda$, $Z=Z(x)$ and $W=W(x)$. Since $\frac{d}{dx}p(Z(x),W(x))=\frac{2\pi\mathrm{i}}{\lambda}D_\beta p(Z,W)$ for every polynomial p, (33) and $p_b(Z,W)=(b-a)(Z-W)$ give

$$D_\beta P(Z,W)=-(1-aW)(1+\beta\psi_a'(\beta t-\tfrac14))\prod_{b\neq a}(b-a)\,(Z-W)^{s-1}, \tag{34}$$

the product being over the parameters $b \neq a$ of the block. The first factors have modulus at least $(1-a)\prod_{b\neq a}|b-a| > 0$. To bound $\eta = |Z-W|$ from below, note that $p_a(Z,W) = 0$ and $p_a(W,W) = 1-W^2$ give $(Z-W)(W-a) = 1-W^2$, so that $|1-W^2| \le 2\eta$, and also $|Z^2-W^2| \le 2\eta$. Using $|1-e(v)| \ge 4\,\mathrm{dist}(v,\mathbb{Z})$ we obtain $\mathrm{dist}(2y,\mathbb{Z}) \le \eta/2$ and $\mathrm{dist}(2t-2y,\mathbb{Z}) \le \eta/2$, where $y = \beta t - \frac14$. Choose integers u, u' with $|2t-u| \le \eta$ and $|2y-u'| \le \eta/2$. Since $2y = 2\beta t - \frac12$,

$$|2\beta u - (2u'+1)| \le 2|2y-u'| + 2\beta|2t-u| \le (1+2\beta)\eta.$$

If $u \neq 0$, Lemma 4.1(i) gives $(1+2\beta)\eta \ge 1/(4|u|)$, and $|u| \le 2|t| + \eta \le 2(1+|t|)$; if $u = 0$, then $(1+2\beta)\eta \ge 1$. In both cases

$$|Z-W| \ge \frac{c_\beta}{1+|t|}, \qquad c_\beta = \frac{1}{8(1+2\beta)}. \tag{35}$$

Since $\lambda > 1$, we have $1+|t| \le 1+|x|$, and (iii) follows from (34) and (35).

(iv) By Lemma 4.1(ii) there are integers n of arbitrarily large modulus and integers k with $|\vartheta| \le 1/|n|$, where $\vartheta = \beta n - k - \frac14$. Fix such n and j, and put $g(t) = t + \psi_{a_j}(\beta t - \frac14)$, so that $g(n) = n + k + \psi_{a_j}(\vartheta)$ with $|\psi_{a_j}(\vartheta)| \le K_{a_j}|\vartheta|$. Since $g' \ge 1$, there is t_j with $g(t_j) = n+k$ and $|t_j - n| \le K_{a_j}/|n|$. The point $x_j = \lambda t_j$ lies in A_j, and $Z(x_j) = e(t_j - n)$ and $W(x_j) = e(\beta(t_j-n)+\vartheta)$ are within $2\pi(1+\beta K_{a_j})/|n|$ of 1. By (34), $|D_\beta P(Z(x_j), W(x_j))| \le C|n|^{-(s-1)}$, and $|x_j|$ is comparable to $\lambda|n|$. This proves (iv); note that the s points $x_1, \dots, x_s$ all lie within distance $C/|n|$ of λn.

(v) Let $x \neq x'$ be points of A. If both lie in A_j, then $g_j(x)$ and $g_j(x')$ are distinct integers, and $|x-x'| \ge \lambda/(1+\beta K_{a_j})$. Let $x \in A_j$ and $x' \in A_k$ with $k \neq j$, and let Z, W, t be as in (iii). Then $e(g_k(x)) = ZB_{a_k}(W)$, and $p_{a_k}(Z,W) = (a_k - a_j)(Z-W)$ gives

$$|1-e(g_k(x))| = \frac{|p_{a_k}(Z,W)|}{|1-a_kW|} \ge \frac{|a_k-a_j|\,c_\beta}{2(1+|t|)}$$

by (35). As $g_k(x') \in \mathbb{Z}$ and $|1-e(v)| \le 2\pi\,\mathrm{dist}(v,\mathbb{Z})$, we get $|x-x'| \ge \lambda\,\mathrm{dist}(g_k(x),\mathbb{Z})/(1+\beta K_{a_k}) \ge \kappa'(1+|x|)^{-1}$. □

A.3. **Choice of the parameters.**

Proof of Lemma 4.6. For $a \in (-1,1)$ and $y \in \mathbb{R}$ put

$$\psi_a(y) = y + \frac{1}{\pi}\arctan\frac{a\sin 2\pi y}{1-a\cos 2\pi y}.$$

This is the continuous function with $\psi_a(0) = 0$ and $B_a(e(y)) = e(\psi_a(y))$, it is real-analytic in (a,y), $\psi_0(y) = y$, and $\partial_a\psi_a(y)|_{a=0} = \pi^{-1}\sin 2\pi y$. Let $\lambda = cm_r$ and $n \in \mathbb{Z}$. Since $\partial_t(t + \psi_a(\beta t - \frac14)) = 1 + \beta\psi_a'(\beta t - \frac14) > 0$ and $\psi_a(y) - y$ is bounded, the equation $t + \psi_a(\beta t - \frac14) = n$ has a unique solution $t_n(a)$, which depends real-analytically on $a \in (-1,1)$ by the implicit function theorem. For $a \in (0,1)$, the root set of a sheet with parameter a is $\{x_n(a) : n \in \mathbb{Z}\}$, where $x_n(a) = \lambda t_n(a)$.

We have $t_n(0) = (n+\frac14)/(1+\beta)$ and $(1+\beta)t_n'(0) = -\pi^{-1}\sin 2\pi y_n$, where $y_n = \beta t_n(0) - \frac14$. If $2y_n$ were an integer k, then $2\beta(n+\frac14) = (k+\frac12)(1+\beta)$, that is, $\beta(2n-k) = k + \frac12$, which is impossible. Hence $t_n'(0) \neq 0$, and x_n is not constant. Moreover, for $n \neq n'$,

$$x_n(0) - x_{n'}(0) = \frac{cm_r(n-n')}{1+\beta} \notin \Gamma_{\mathbb{Q}} :$$

otherwise $\sqrt{2}\,m_r(n-n') = (q_1+\beta q_2)(1+\beta) = q_1 + (q_1+q_2)\beta + q_2\beta^2$ for some $q_1, q_2 \in \mathbb{Q}$, and since β is transcendental (Lemma 4.1(iv)), the numbers $1, \beta, \beta^2$ are linearly independent over $\mathbb{Q}(\sqrt{2})$, which gives $q_1 = q_2 = 0$ and then $n = n'$.

We choose $a_{r,1}, \dots, a_{r,s_r}$ in turn. When $a_{r,j}$ is chosen, let $\mathcal{Y}$ be the union of $\{0\}$, the root sets of blocks $1, \dots, r-1$ and the root sets of the sheets with parameters $a_{r,1}, \dots, a_{r,j-1}$; it is countable. We require that $a = a_{r,j}$ is different from $a_{r,1}, \dots, a_{r,j-1}$, that $x_n(a) \notin \rho + \Gamma_{\mathbb{Q}}$ for all $n \in \mathbb{Z}$ and $\rho \in \mathcal{Y}$, and that $x_n(a) - x_{n'}(a) \notin \Gamma_{\mathbb{Q}}$ for all $n \neq n'$. For fixed n, ρ and $\gamma \in \Gamma_{\mathbb{Q}}$, the function $a \mapsto x_n(a) - \rho - \gamma$ is real-analytic on $(-1,1)$ and not constant; for fixed $n \neq n'$ and γ, the function $a \mapsto x_n(a) - x_{n'}(a) - \gamma$ is real-analytic and does not vanish at $a = 0$. In both cases the zero set in $(0,1)$ is countable, and as $\Gamma_{\mathbb{Q}}$ is countable, the requirements exclude only

countably many a. If they hold at every step, then, by induction, the roots of the new sheets lie outside $\Gamma_{\mathbb{Q}}$ (take $\rho = 0$), differ from all earlier roots by elements outside $\Gamma_{\mathbb{Q}}$, and in particular are distinct from them because $0 \in \Gamma_{\mathbb{Q}}$, and distinct roots of one sheet differ by elements outside $\Gamma_{\mathbb{Q}}$. Thus blocks $1, \dots, r$ are independent. □

A.4. **No return to the cone.**

Proof of Proposition 4.7. Put $S_1 = (C_\beta + F_1)/\lambda$, $S_2 = (C_\beta + F_2)/\lambda$ and $\Theta = \lambda^2$, and let $T \in \mathcal{N}(S_2, S_1)$. Every point $z \in S_1 \cup S_2$ has the form $z = (p + \beta q)/\lambda$ with unique integers p, q. Since $C_\beta + F_i$ is contained in finitely many translates of the two quadrants, and $|k + \beta n| = |k| + \beta|n|$ when k, n have the same sign, there is K_0 with $|q| \le K_0(1 + |z|)$ for all such z. Since Θ is a quadratic irrational, Liouville's theorem [HW08, Theorem 191] gives $c_\Theta > 0$ with $|n\Theta - m| \ge c_\Theta/|n|$ for all integers m and $n \ne 0$.

Fix $t \in S_1$, write $\lambda t = p_0 + \beta q_0$, and choose an integer $k \ge 1$ with $\mathcal{F}T \in H_{-k}$. Put $L = 2k + 1$ and $M = 4k + 2$. For $j \ge 1$ let $b = q_j$ and $\varepsilon = \beta q_j - p_j$, so that $0 < |\varepsilon| < b^{-j-2}$ by Lemma 4.1(iii); thus ε tends to zero faster than every power of b as $j \to \infty$. All statements below are for sufficiently large j, and C denotes constants that do not depend on j.

Step 1: a test function. No point of S_1 lies within b^{-4} of $t - \ell\lambda b$ for $1 \le \ell \le L$. Indeed, let $z = (p' + \beta q')/\lambda \in S_1$ with $|z - (t - \ell\lambda b)| \le b^{-4}$. Then $|q'| \le Cb$, and since $\beta b = p_j + \varepsilon$,

$$\lambda b\,(z - t + \ell\lambda b) = \bigl[b(p' - p_0) + p_j(q' - q_0)\bigr] + \varepsilon(q' - q_0) + \ell\Theta b^2.$$

The bracket is an integer, so the sum of the first and the last term has modulus at least $c_\Theta/(Lb^2)$, while $|\varepsilon(q' - q_0)| \le C|\varepsilon|b < c_\Theta/(2Lb^2)$. Hence $\lambda b^{-3} \ge c_\Theta/(2Lb^2)$, which is false for large b.

Let $\phi \in C_c^\infty((-1,1))$ with $\phi(0) = 1$, $f_b(x) = \phi(b^4(x - t))$ and

$$\eta_b(x) = \sum_{\ell=0}^{L} (-1)^\ell \binom{L}{\ell} f_b(x + \ell\lambda b).$$

Since S_1 is locally finite, the interval $(t - b^{-4}, t + b^{-4})$ contains no point of S_1 other than t, and by Step 1 the other bumps contain no point of S_1. Hence $T(\eta_b) = T(\{t\})$. With $h_b = \mathcal{F}^{-1} f_b$ and $g_b(\xi) = 1 - e(-\lambda b\xi)$ we have $\mathcal{F}^{-1}\eta_b = g_b^L h_b$, and since $h_b(\xi) = b^{-4} e(t\xi)(\mathcal{F}^{-1}\phi)(\xi/b^4)$,

$$|h_b^{(r)}(\xi)| \le C_{r,N}\, b^{-4}(1 + |\xi|/b^4)^{-N} \qquad (r, N \ge 0). \tag{36}$$

Step 2: the spectral grid. Put $R = b^8|\varepsilon|^{-1/(2M)}$ and $w = K|\varepsilon|R/b$ with $K = 12K_0/\lambda$. If $\xi = (p + \beta q)/\lambda \in S_2$ and $|\xi| \le 2R$, then $|q| \le K_0(1 + 2R) \le 3K_0R$ and $\lambda b\xi = (bp + p_j q) + \varepsilon q$, so

$$\operatorname{dist}(\xi, (\lambda b)^{-1}\mathbb{Z}) \le |q||\varepsilon|/(\lambda b) \le w/4.$$

Moreover $bw = Kb^8|\varepsilon|^{1-1/(2M)} \to 0$ and $R/b^4 = b^4|\varepsilon|^{-1/(2M)} \to \infty$. Let $\chi_1, \chi_2 \in C_c^\infty((-2,2))$ be equal to 1 on $[-1,1]$, and put $\chi_R(\xi) = \chi_1(\xi/R)$ and $\chi_{\mathrm{gr}}(\xi) = \sum_{v \in \mathbb{Z}} \chi_2((\xi - v/(\lambda b))/w)$. As $4w < 1/(\lambda b)$, the windows $|\xi - v/(\lambda b)| < 2w$ are disjoint, χ_{gr} equals 1 within distance w of the grid, and $|\chi_{\mathrm{gr}}^{(r)}| \le C_r w^{-r}$. Let

$$K_b = K_{\mathrm{in}} + K_{\mathrm{out}}, \qquad K_{\mathrm{in}} = \chi_R\,\chi_{\mathrm{gr}}\, g_b^L h_b, \qquad K_{\mathrm{out}} = (1 - \chi_R)\, g_b^L h_b.$$

Then $K_b \in \mathcal{S}$, and $K_b - g_b^L h_b = \chi_R(\chi_{\mathrm{gr}} - 1) g_b^L h_b$ belongs to $C_c^\infty(\mathbb{R})$ and vanishes at every point of S_2. Since $\mathcal{F}T$ is locally atomic on S_2,

$$T(\{t\}) = T(\eta_b) = \mathcal{F}T(\mathcal{F}^{-1}\eta_b) = \mathcal{F}T(K_b). \tag{37}$$

Step 3: estimates. By Section 3 it suffices to show that $\|\xi^u \partial^r K_{\mathrm{in}}\|_{L^2}$ and $\|\xi^u \partial^r K_{\mathrm{out}}\|_{L^2}$ tend to zero for $u + r \le 2k$; then $\|K_b\|_k \to 0$, and (37) and $|\mathcal{F}T(K_b)| \le \|\mathcal{F}T\|_{-k}\,\|K_b\|_k$ give $T(\{t\}) = 0$.

The function K_{in} is supported in the union of the $O(bR)$ windows that meet $[-2R, 2R]$, a set of measure $O(bRw)$. On a window, $|g_b| \le 4\pi\lambda bw$ and, for $v \le 2k < L$, $|(g_b^L)^{(v)}| \le C(bw)^{L-v} b^v = C(bw)^L w^{-v}$. A derivative that falls on χ_R, χ_{gr} or h_b costs at most Cw^{-1}, as $w < 1 < R$, and $|h_b| \le Cb^{-4}$. Hence, using $bw = K|\varepsilon|R$,

$$\|\xi^u \partial^r K_{\mathrm{in}}\|_{L^2} \le CR^u (bw)^L w^{-r} b^{-4} (bRw)^{1/2} \le C|\varepsilon|^{L-r+\frac{1}{2}}\, b^{r-4} R^{u+L-r+1}.$$

Here $u+L-r+1\le M$, so substituting $R=b^8|\varepsilon|^{-1/(2M)}$ lowers the exponent of $|\varepsilon|$ by at most $\frac12$, and we obtain the bound $C|\varepsilon|^{L-r}b^{2k-4+8M}\le Cb^{2k-4+8M}|\varepsilon|$, which tends to zero.

On the support of K_{out} we have $|\xi|\ge R$, and there $|(g_b^L)^{(v)}|\le Cb^v$. By (36) with $N=2k+2$, $|\xi^u\partial^r K_{\mathrm{out}}(\xi)|\le Cb^{r-4}|\xi|^u(|\xi|/b^4)^{-N}$, and therefore

$$\|\xi^u\partial^r K_{\mathrm{out}}\|_{L^2}\le Cb^{r-4+4N}R^{u-N+\frac12}\le Cb^{8k-2}(R/b^4)^{-3/2}=Cb^{8k-8}|\varepsilon|^{3/(4M)},$$

which tends to zero. Thus $T(\{t\})=0$ for every $t\in S_1$, and $T=0$. □

A.5. **Sparse splitting.** Let $X=\mathbb{P}^1\times\mathbb{P}^1\supset\mathbb{C}^2$. For an effective divisor D on X we write $\mathcal{O}(D)$ for the sheaf of rational functions h with $\operatorname{div}h+D\ge0$ (together with 0). The group $G_i=\mu_{m_i}\times\mu_{m_i}$ acts on X by $(z,w)\mapsto(\zeta z,\zeta' w)$, and on rational functions by $gF=F\circ g^{-1}$; the polar divisor of gF is the image under g of that of F. The groups G_i commute. Let D_i be the closure in X of the curve $\{Q_i=0\}\subset\mathbb{C}^2$, and $D=\sum_i D_i$. Since Q_i has nonzero coefficients at 1, z^{d_i}, w^{d_i} and $z^{d_i}w^{d_i}$, where $d_i=n_is_i$, the divisor of the bihomogenization of Q_i has no component at infinity; as Q_i is square-free (Step 1 below), this divisor is D_i.

Proof of Lemma 4.8. Step 1: the components. Let $a\in(0,1)$, $|\rho|=|\sigma|=1$ and $n\ge1$. The polynomial $p_a(\rho z^n,\sigma w^n)=(1-a\sigma w^n)-\rho z^n(\sigma w^n-a)$ is irreducible: over $\mathbb{C}(w)$ it is a multiple of $z^n-h(w)$ with $h=(1-a\sigma w^n)/(\rho(\sigma w^n-a))$, and h has a simple zero at every root of $1-a\sigma w^n$, which is not a root of σw^n-a because $a^2\ne1$; so z^n-h is Eisenstein at such a root and irreducible over $\mathbb{C}(w)$, and since the coefficients $1-a\sigma w^n$ and $-\rho(\sigma w^n-a)$ are coprime, the polynomial is primitive in $\mathbb{C}[w][z]$ and irreducible by Gauss's lemma. Its coefficients at 1, z^n and w^n are 1, $a\rho$ and $-a\sigma$, so two such polynomials with the same n are associates only if they have the same (a,ρ,σ); polynomials with different n have different bidegrees. As n_j is prime to m_j, the map $\zeta\mapsto\zeta^{n_j}$ is a bijection of μ_{m_j}. Hence Q is square-free, D is reduced, and a nontrivial $g\in G_j$ maps every component of D_j to a curve that is not a component of D, nor of $g'D_i$ for any $i\ne j$ and $g'\in G_i$. Finally D_k is G_j-invariant for $k\ne j$, because m_j divides n_k.

Step 2: no base points. For every i and every point $\mathrm{P}\in X$ there is $g\in G_i$ with $\mathrm{P}\notin gD_i$. Let $\mathrm{P}=(z,w)$ with $z,w\in\mathbb{C}\setminus\{0\}$, and put $U=z^{n_i}$ and $V=-\mathrm{i}w^{n_i}$. If the points $(\zeta z,w)$, $\zeta\in\mu_{m_i}$, all lay on D_i, then for every $\varrho\in\mu_{m_i}$ there would be a parameter a of block i with $\varrho U=(1-aV)/(V-a)$. If $V=\pm1$, the right-hand side equals V for every a, which is impossible since $m_i\ge3$. Otherwise the right-hand side is injective in a, so at least three parameters a satisfy $|1-aV|^2-|U|^2|V-a|^2=0$; this real quadratic polynomial in a then vanishes identically, which gives $|U|=|V|=1$. But for $|V|=1$,

$$\operatorname{Im}\frac{1-aV}{V-a}=\frac{(a^2-1)\operatorname{Im}V}{|V-a|^2},$$

so all the points ϱU would lie in one open half-plane bounded by $\mathbb{R}$, whereas their sum is zero. At the remaining points one of the coordinates is 0 or ∞. On the lines $z=0$, $z=\infty$, $w=0$ and $w=\infty$ the factors of block i are, up to nonzero constants, $1-aV$, $a-V$, $1+aU$ and $a+U$, which vanish only for $V>1$, $0<V<1$, $U<-1$ and $-1<U<0$, respectively. At a point of one of these lines that is not a corner, the relevant coordinate V or U is nonzero, and a rotation of w or of z by an element of μ_{m_i} moves it off the corresponding half-line, since $m_i\ge3$. The four corners do not lie on D_i, since the corresponding coefficients of Q_i are nonzero.

Step 3: mixed differences. Put $F=R/Q$. On $\mathbb{C}^2$ the poles of F lie on $\{Q=0\}$ and are simple, and at infinity F is the quotient of the bihomogenizations of R and Q in bidegree (S,S); hence $F\in H^0(X,\mathcal{O}(D))$. An element $(\zeta,\zeta')\in G_j$ multiplies the coefficient of z^kw^n in the Taylor series of F at the origin by $\zeta^{-k}\zeta'^{-n}$, and a monomial with $(k,n)\in n_l\mathbb{Z}^2$ is invariant under G_j for every $j\ne l$. Hence for $i\ne j$, $g_i\in G_i$ and $g_j\in G_j$, every monomial of the Taylor series is annihilated by g_i-1 or by g_j-1, and

$$(g_i-1)(g_j-1)F=0. \tag{38}$$

Step 4: the poles of a difference. Let $g\in G_i$ and $\Delta=gF-F$. By (38), Δ is G_j-invariant for every $j\ne i$. Its poles are simple and lie on $D\cup gD=D_i\cup gD_i\cup\bigcup_{j\ne i}D_j$. If Δ had a pole along

a component Y of some D_j, $j \neq i$, it would also have a pole along $g'Y$ for every $g' \in G_j$, which is impossible for $g' \neq 1$ by Step 1. Hence

$$gF - F \in H^0(X, \mathcal{O}(D_i + gD_i)) \quad (g \neq 1), \qquad gF - F \in H^0(X, \mathcal{O}(D_i)) \quad (g = 1). \tag{39}$$

Step 5: splitting off one block. Fix i and let $D' = \sum_{j \neq i} D_j$, which is G_i-invariant. Given $\mathrm{P} \in X$, choose $g \in G_i$ with $\mathrm{P} \notin gD_i$ (Step 2). On the open set $X \setminus gD_i$ we have $F = gF - (gF - F)$, where the poles of gF lie on $gD = gD_i + D'$, so that gF is a section of $\mathcal{O}(D')$ there, and $-(gF - F)$ is a section of $\mathcal{O}(D_i)$ there by (39). Thus F is a global section of the subsheaf $\mathcal{O}(D_i) + \mathcal{O}(D')$ of the constant sheaf of rational functions. Since D_i and D' have no common component and X is smooth, $\mathcal{O}(D_i) \cap \mathcal{O}(D') = \mathcal{O}_X$ [Har77, Chapter II, Proposition 6.3A], so the sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}(D_i) \oplus \mathcal{O}(D') \to \mathcal{O}(D_i) + \mathcal{O}(D') \to 0,$$

with maps $h \mapsto (h, -h)$ and $(a, b) \mapsto a + b$, is exact. As $H^1(X, \mathcal{O}_X) = 0$ [Har77, Chapter III, Exercise 5.6(a)], the long exact sequence of cohomology gives $F = F_i + F^{(i)}$ with $F_i \in H^0(X, \mathcal{O}(D_i))$ and $F^{(i)} \in H^0(X, \mathcal{O}(D'))$.

Step 6: the remainder is constant. Do this for every i and put $G = F - \sum_i F_i$. Along a component of D_i the functions $F - F_i = F^{(i)}$ and F_j, $j \neq i$, have no poles. So G has no poles on X, and it is constant.

Step 7: invariance. Let $j \neq i$ and $g \in G_j$. Then $(g-1)F_i = (g-1)F - \sum_{k \neq i}(g-1)F_k$. The left-hand side has poles only on D_i, which is G_j-invariant. On the right-hand side, $(g-1)F$ has poles only on $D_j \cup gD_j$ by (39), $(g-1)F_k$ has poles only on D_k for $k \neq j$, and $(g-1)F_j$ only on $D_j \cup gD_j$. Hence $(g-1)F_i$ is a constant $c(g)$. Since constants are invariant, $c(gg') = c(g) + c(g')$, so c is a homomorphism from the finite group G_j to $(\mathbb{C}, +)$, and $c = 0$. Thus F_i is invariant under $H_i = \prod_{j \neq i} G_j$.

Step 8: descent. Since F_i has at most simple poles along D_i and Q_i is square-free, $J_i = F_i Q_i$ has no poles on $\mathbb{C}^2$, so it is a polynomial. For all but finitely many w_0, the function $F_i(\cdot, w_0)$ is regular at $z = \infty$ and $Q_i(\cdot, w_0)$ has degree d_i; hence J_i has degree at most d_i in z, and likewise in w. As F_i and Q_i are H_i-invariant, so is J_i, and since $n_i = \prod_{j \neq i} m_j$, every monomial $z^k w^n$ of J_i has $n_i \mid k$ and $n_i \mid n$. Thus $J_i = R_i'(z^{n_i}, -\mathrm{i}w^{n_i})$ for a polynomial $R_i'(Z, W)$ of bidegree at most (s_i, s_i). Let τ_i be the quotient of the coefficients of $Z^{s_i} W^{s_i}$ in R_i' and in P_i, and $R_i = R_i' - \tau_i P_i \in V_{s_i}$; then $F_i = R_i(z^{n_i}, -\mathrm{i}w^{n_i})/Q_i + \tau_i$, and

$$\frac{R}{Q} = \sum_{i \leq r} \frac{R_i(z^{n_i}, -\mathrm{i}w^{n_i})}{Q_i(z, w)} + G', \qquad G' = G + \sum_i \tau_i.$$

All terms are regular at (∞, ∞), and their values there are the quotients of the top coefficients of numerator and denominator. These vanish for R/Q, because $R \in V_S$, and for every summand, by the choice of R_i. Hence $G' = 0$. □

Appendix B. Proof of Theorem 5.1

Throughout this appendix P, k, b, d, ε_j, γ_j and β_j are as in (6). The explicit formula $\varepsilon_j = \frac{1}{16} e^{-db^j}$ enters only through two exponent computations, in Stages 2 and 3 below. Apart from these, we use only that

$$0 < \varepsilon_{j+1} \leq \tfrac{1}{4}\,\varepsilon_j \leq \tfrac{1}{4} \cdot 10^{-12} \qquad (j \geq 1), \tag{40}$$

which holds because $\varepsilon_1 \leq 2^{-452}$ and $\varepsilon_{j+1}/\varepsilon_j \leq 2^{-64}$. In Section B.8 we also use that $\varepsilon_1 < \frac{1}{4k}$ (Section 5): the β_j and γ_j are distinct, the γ_j lie within ε_1 of $\frac{1}{2}$, and $\beta_j \leq \beta_k = \frac{1}{2} - \frac{1}{4k}$.

Plan. Part (e) is a statement about finitely many lattice combs; it is proved in Section B.2 by a triangularity argument, which is used again twice. Parts (a)–(c) concern the infinite carrier L_γ. In Zak coordinates (Section B.3) the support condition on T and the support condition on $\mathcal{F}T$ become triangularity conditions, called *flags*, on two arrays of moments $\mathbf{A}$ and $\mathbf{B}$, and the Fourier transform becomes an explicit bounded operator E with $\mathbf{B} = E\mathbf{A}$, the *gauge* (Section B.4). The proof then has four stages.

- *Stage 1, uniqueness* (Section B.5): the flags leave at most one free parameter, so $\dim \mathcal{M}_{L_\gamma} \le 1$.
- *Stage 2, exclusion* (Section B.6): for elements of Hermite order at most P, moments with weights that shrink as fast as the clusters can be used, and in these weights the flags leave no free parameter. This proves (b).
- *Stage 3, construction* (Section B.7): the free parameter of Stage 1 is realized by a limit of finite approximations whose Hermite norms of order $18P+2$ are uniformly bounded. This proves (a) and (c).
- *Stage 4, head transfer* (Section B.8): inserting the finite head $\beta_1, \dots, \beta_k$ costs two Hermite orders. This proves (d).

The same arguments are carried out in the formal development (see the table in Appendix E); the account below is self-contained. Table 4 lists the objects that the reader has to keep track of.

Object	Meaning	Introduced in
V, gV	Zak transforms of T and $\mathcal{F}T$, cut off near the clusters at the half-integers	Lemma B.3
$\phi_{i,e}$; $M(U)$; $\mathbb{H}$	Newton test functions vanishing at the first i cluster points; the weighted moment array of U; the Hilbert space of arrays	(49), (50)
$\mathbf{A}$, $\mathbf{B}$	the moment arrays $M(V)$ and $M(gV)$ of T and $\mathcal{F}T$	Lemma B.5
flags	triangularity of $\mathbf{A}$ and $\mathbf{B}$, which encodes the support conditions	(51), (52)
E	the gauge, $\mathbf{B} = E\mathbf{A}$, which encodes the Fourier transform	Lemma B.6
$\mathcal{H}_A, \mathcal{H}_B, \mathcal{D}, \mathcal{C}, \mathcal{G}$; $P_\bullet$, Q	subspaces of $\mathbb{H}$ given by sets of index pairs, and the projections onto them	before (54)
Σ_A, Σ_B	graph maps, which encode the diagonal relations of the flags	before (54)
$\mathcal{T}$; $a = (u, c)$	the kernel operator, and its unknown: u free, c eliminated	(54)
$\mathcal{K}$	the Schur complement, acting on u	Stage 1

TABLE 4. The objects of Appendix B.

B.1. **Two facts about the Hermite scale.** We use the norm equivalence of Section 3: for integers $p \ge 0$, $\|\varphi\|_p$ is equivalent to $\sum_{a+b\le 2p} \left\| x^a \varphi^{(b)} \right\|_{L^2}$. Two consequences are used repeatedly. First, if $m \in C^\infty(\mathbb{R})$ is 1-periodic, then multiplication by m is bounded on H_p and, by duality, on H_{-p}, for every integer $p \ge 0$; since $\mathcal{F}$ is unitary on $H_{\pm p}$, the same holds for $\mathcal{F}^{-1} m \mathcal{F}$. Second, for $a \in [-1, 1]$ and bounded coefficients (c_n),

$$\left| \sum_{n\in\mathbb{Z}} c_n \varphi(n+a) \right| \le C \sup_n |c_n| \, \|\varphi\|_1 . \tag{41}$$

Indeed, by the Sobolev inequality on unit intervals, $\sum_n (1+n^2)|\varphi(n+a)|^2 \le C \int (1+x^2)(|\varphi|^2 + |\varphi'|^2)\, dx \le C \|\varphi\|_1^2$, and it remains to apply the Cauchy–Schwarz inequality.

B.2. **Finitely many phases, and part (e).** For $a, b \in \mathbb{R}$ let $w_{a,b} = \sum_{n\in\mathbb{Z}} e^{2\pi i n b} \delta_{n+a}$. By (41), $w_{a,b} \in H_{-1}$, and by Poisson summation

$$\mathcal{F} w_{a,b} = \sum_{l\in\mathbb{Z}} e^{-2\pi i a (l+b)} \delta_{l+b} = e^{-2\pi i a b} \, w_{b,-a}. \tag{42}$$

Thus $w_{a,b}$ is a crystalline measure with support $a + \mathbb{Z}$ and spectrum $b + \mathbb{Z}$. We write $S(\{x\})$ for the mass at x of a locally atomic distribution S.

Lemma B.1. *Let $A, B \subset \mathbb{R}$ be finite sets, each consisting of numbers that are pairwise incongruent modulo 1, and let $T \in \mathcal{S}'$ be locally atomic on $A+\mathbb{Z}$ with $\mathcal{F}T$ locally atomic on $B+\mathbb{Z}$. Then $T = \sum_{a\in A,\, b\in B} c_{a,b} w_{a,b}$ for unique coefficients $c_{a,b} \in \mathbb{C}$.*

Proof. The trigonometric polynomial $p(\xi) = \prod_{b\in B}(e^{2\pi i\xi} - e^{2\pi i b}) = \sum_{l=0}^{m} p_l e^{2\pi i l\xi}$, $m = \#B$, vanishes on $B+\mathbb{Z}$, so $p\,\mathcal{F}T = 0$; since $\mathcal{F}^{-1}(e^{2\pi i l\xi}\mathcal{F}T) = \tau_l T$, this says $\sum_l p_l \tau_l T = 0$. For $a \in A$ let $t_a(n) = T(\{n+a\})$. The mass of $\tau_l T$ at $n+a$ is $t_a(n+l)$, so $\sum_l p_l t_a(n+l) = 0$ for all n. This is a linear recurrence whose characteristic polynomial $\prod_{b\in B}(w - e^{2\pi i b})$ has m distinct roots, so $t_a(n) = \sum_b c_{a,b} e^{2\pi i n b}$ for unique $c_{a,b}$. The distributions T and $\sum c_{a,b} w_{a,b}$ are locally atomic on $A+\mathbb{Z}$ with the same masses, so they coincide. □

Lemma B.2 (Finite critical caps)**.** *Let $r \geq 1$, let $\alpha_1, \dots, \alpha_r \in (0, \frac{1}{2})$ be distinct and $\alpha'_1, \dots, \alpha'_r \in (0, \frac{1}{2})$ be distinct, and let $T = \sum c_{a,b} w_{a,b}$, the sum over $a \in \{\pm\alpha_j\}$ and $b \in \{\pm\alpha'_j\}$. Suppose that T has no mass at the points $n \pm \alpha_j$ with $|n| < j$, and $\mathcal{F}T$ has no mass at the points $n \pm \alpha'_j$ with $|n| < j$ $(1 \leq j \leq r)$. Then $T = 0$.*

Proof. Let $\Xi = \{\pm\alpha_j\}$ and $\Xi' = \{\pm\alpha'_j\}$, and let U be the measure $\sum_{a,b} c_{a,b}\, \delta_a \otimes \delta_b$ on the finite grid $\Xi \times \Xi'$. Put $g(x,y) = e^{-2\pi i x y}$. By the definition of $w_{a,b}$ and by (42), for all functions ψ on Ξ, ψ' on Ξ', and all $n \in \mathbb{Z}$,

$$\sum_{a\in\Xi} \psi(a)\, T(\{n+a\}) = U(\psi(x)e^{2\pi i n y}), \qquad \sum_{b\in\Xi'} \psi'(b)\, \mathcal{F}T(\{n+b\}) = (gU)(e^{-2\pi i n x}\psi'(y)). \tag{43}$$

Moments. Let $s(t) = \sin 2\pi t$, $z(t) = 1 - \cos 2\pi t$, $z_j = z(\alpha_j)$, $D_i = \prod_{l\leq i}(z - z_l)$, and $\phi_{i,0} = D_i$, $\phi_{i,1} = sD_i$ for $0 \leq i < r$; put $\kappa_i = s(\alpha_{i+1}) > 0$. For a measure u on Ξ let $m(u) \in \mathbb{C}^r \otimes \mathbb{C}^2$ have the entries $m(u)_{i,e} = \kappa_i^{-e} u(\phi_{i,e})$. This map is bijective: the z_j are distinct, so the restrictions to Ξ of the D_i, $i < r$, span the even functions on Ξ, and since s is odd and does not vanish on Ξ, the sD_i span the odd ones. Define m' on Ξ' in the same way, and put $\mathbf{A} = (m \otimes m')(U)$ and $\mathbf{B} = (m \otimes m')(gU)$, with entries $\mathbf{A}_{ie,jf}$ and $\mathbf{B}_{ie,jf}$.

Flags. We claim that

$$\mathbf{A}_{ie,jf} = 0 \ \text{ if } i \geq j+f, \qquad \mathbf{B}_{ie,jf} = 0 \ \text{ if } j \geq i+e. \tag{44}$$

The function $\phi'_{j,f}$ is a trigonometric polynomial with frequencies in $[-j-f, j+f]$. By (43), $U(\phi_{i,e}(x)e^{2\pi i n y}) = \sum_a \phi_{i,e}(a) T(\{n+a\})$. For $|n| \leq j+f$ the mass $T(\{n \pm \alpha_l\})$ vanishes unless $l \leq |n| \leq j+f$, while $\phi_{i,e}(\pm\alpha_l) = 0$ for $l \leq i$; hence all terms vanish if $i \geq j+f$. The second assertion follows in the same way from the second identity in (43).

The gauge. Let $X = m \circ x \circ m^{-1}$ be the matrix of multiplication by x in the coordinates m, and define X' likewise. The function $x\phi_{i,e}$ vanishes at $\pm\alpha_l$ for $l \leq i$, and the functions on Ξ with this property are spanned by the $\phi_{i',e'}$ with $i' \geq i$; hence $X_{(i,e),(i',e')} = 0$ for $i' < i$. Evaluating at $\pm\alpha_{i+1}$, where the $\phi_{i',e'}$ with $i' > i$ vanish, and using $\phi_{i,0}(\pm\alpha_{i+1}) = D_i(\alpha_{i+1}) \neq 0$ and $\phi_{i,1}(\pm\alpha_{i+1}) = \pm\kappa_i D_i(\alpha_{i+1})$, we find that $x\phi_{i,0} - (\alpha_{i+1}/\kappa_i)\phi_{i,1}$ and $x\phi_{i,1} - \alpha_{i+1}\kappa_i\phi_{i,0}$ lie in the span of the $\phi_{i',e'}$ with $i' > i$. Thus the diagonal block of X at level i is $\alpha_{i+1}Q$, with $Q = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. On the finite grid, multiplication by $x^n y^n$ is represented by $X^n \otimes X'^n$, so $\mathbf{B} = E\mathbf{A}$ with $E = \exp(-2\pi \mathrm{i} X \otimes X')$. Consequently the block $(E\mathbf{A})_{(i,j)} = ((E\mathbf{A})_{ie,jf})_{e,f}$ involves only the blocks $\mathbf{A}_{(i',j')}$ with $i' \geq i$ and $j' \geq j$, and E acts from (i,j) to (i,j) by

$$\exp(-2\pi \mathrm{i}\alpha_{i+1}\alpha'_{j+1} Q \otimes Q) = \cos\theta_{ij}\, I - \mathrm{i}\sin\theta_{ij}\, Q \otimes Q, \qquad \theta_{ij} = 2\pi\alpha_{i+1}\alpha'_{j+1} \in (0, \tfrac{\pi}{2}).$$

Conclusion. Suppose $\mathbf{A} \neq 0$, and choose a nonzero block $\mathbf{A}_{(i,j)}$ with $i+j$ maximal. Then $i \leq j$ by (44), and $(E\mathbf{A})_{(i,j)} = (\cos\theta_{ij} I - \mathrm{i}\sin\theta_{ij} Q \otimes Q)\mathbf{A}_{(i,j)}$, since all other contributing blocks have a larger value of $i+j$. If $i < j$, all four entries of $\mathbf{B}_{(i,j)}$ vanish by (44), and the 4×4 matrix is invertible, so $\mathbf{A}_{(i,j)} = 0$. If $i = j$, the entries of $\mathbf{A}_{(i,i)}$ with $f = 0$ vanish, and the entries of $\mathbf{B}_{(i,i)}$ with $e = 0$ vanish; in terms of the remaining entries this reads $-\mathrm{i}\sin\theta_{ii}\, \mathbf{A}_{i1,i1} = 0$ and $\cos\theta_{ii}\, \mathbf{A}_{i0,i1} = 0$, so again $\mathbf{A}_{(i,i)} = 0$. This contradiction shows that $\mathbf{A} = 0$; hence $U = 0$ and $T = 0$. □

Proof of Theorem 5.1(e). An element $w \in W_k$ is a combination of the $w_{a,b}$ with $a, b \in \{\pm\beta_j : j \leq k\}$, and the hypothesis of (e) is that of Lemma B.2 with $r = k$ and $\alpha_j = \alpha'_j = \beta_j$. □

B.3. **Zak coordinates.** For $\Phi \in \mathcal{S}(\mathbb{R}^2)$ let $\widehat{\Phi}^{(2)}(x,n) = \int \Phi(x,y)e^{-2\pi i n y}\,dy$ and

$$\mathcal{Z}^t\Phi(t) = \sum_{n\in\mathbb{Z}} (-1)^n\, \widehat{\Phi}^{(2)}(t-n-\tfrac12, n).$$

This is a continuous linear map $\mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R})$, and for $T \in \mathcal{S}'(\mathbb{R})$ we define $\mathcal{Z}T \in \mathcal{S}'(\mathbb{R}^2)$ by $\mathcal{Z}T(\Phi) = T(\mathcal{Z}^t\Phi)$. For $T \in \mathcal{S}(\mathbb{R})$, $\mathcal{Z}T(x,y) = \sum_n (-1)^n T(x+n+\frac12)e^{-2\pi i n y}$; thus $\mathcal{Z}$ is the Zak transform, recentred at the half-integers.

For a sequence $\eta = (\eta_j)_{j\ge1}$ with $0 \le \eta_j \le 10^{-12}$ and $\eta_j \to 0$ we put $\Xi_\eta = \{0\} \cup \{\pm\eta_j : j \ge 1\}$. We fix an even function $\chi \in C_c^\infty(-\frac14,\frac14)$ equal to 1 on $[-10^{-6}, 10^{-6}]$ and, for a distribution S and $c \in \mathbb{R}$, we call $S_c(\psi) = S((\chi\psi)(\cdot - c))$ the *cluster* of S at c. If S is locally atomic and its only atoms in $c + \operatorname{supp}\chi$ lie in $c + \Xi_\eta$, then $S_c = \sum_{x\in\Xi_\eta} S(\{c+x\})\delta_x$.

Lemma B.3. *Let η, η' be as above, let $T \in \mathcal{S}'$ be supported in $\frac12 + \Xi_\eta + \mathbb{Z}$ with $\mathcal{F}T$ supported in $\frac12 + \Xi_{\eta'} + \mathbb{Z}$, and put*

$$V = e^{-\pi i y}\,\chi(x)\chi(y)\,\mathcal{Z}T, \qquad g(x,y) = e^{-2\pi i x y}.$$

Then V is supported in $\Xi_\eta \times \Xi_{\eta'}$, and for all $\psi \in C^\infty(\mathbb{R})$ and $n \in \mathbb{Z}$,

$$V\big(\psi(x)\,e^{2\pi i(n+\frac12)y}\big) = (-1)^n\, T_{n+\frac12}(\psi), \tag{45}$$

$$(gV)\big(e^{-2\pi i(n+\frac12)x}\,\psi(y)\big) = i(-1)^n\,(\mathcal{F}T)_{n+\frac12}(\psi). \tag{46}$$

Moreover, $T(\varphi) = V(G_\varphi)$ for $\varphi \in \mathcal{S}$, where $G_\varphi(x,y) = \sum_n (-1)^n \varphi(x+n+\frac12)e^{2\pi i(n+\frac12)y}$. If $T \in H_{-p}$ for an integer $p \ge 0$, then

$$|V(q)| \le C_p\,\|T\|_{-p}\,\|q\|_{C^{2p}} \qquad (q \in C^\infty(\mathbb{R}^2)), \tag{47}$$

where the C^{2p} norm is taken on $\operatorname{supp}\chi \times \operatorname{supp}\chi$.

Proof. For $T \in \mathcal{S}$, Poisson summation applied to $t \mapsto T(x+\frac12+t)e^{-2\pi i t(y+\frac12)}$ gives

$$\mathcal{Z}T(x,y) = e^{2\pi i(x+\frac12)(y+\frac12)}\,\mathcal{Z}(\mathcal{F}T)(y,-x), \tag{48}$$

and since both sides depend continuously on $T \in \mathcal{S}'$, (48) holds on $\mathcal{S}'$. Also $\mathcal{Z}T(x,y+1) = \mathcal{Z}T(x,y)$.

If $\Phi \in C_c^\infty(\mathbb{R}^2)$ vanishes on $N \times \mathbb{R}$, where N is a 1-periodic open neighbourhood of $\Xi_\eta + \mathbb{Z}$, then $\mathcal{Z}^t\Phi$ vanishes on $\frac12 + N$, which is a neighbourhood of $\operatorname{supp} T$; hence $\mathcal{Z}T$ is supported in $(\Xi_\eta + \mathbb{Z}) \times \mathbb{R}$. Applying this to $\mathcal{F}T$ and using (48), we see that $\mathcal{Z}T$ is also supported in $\mathbb{R} \times (\Xi_{\eta'} + \mathbb{Z})$. This gives the support of V.

Let $\varrho \in C_c^\infty(-\frac34,\frac34)$ with $\varrho = 1$ on $[-\frac14,\frac14]$ and $\sum_k \varrho(y+k) = 1$. Then $\int \varrho(y)e^{-2\pi i m y}\,dy = \delta_{m0}$ for $m \in \mathbb{Z}$, so for $\psi \in C_c^\infty(\mathbb{R})$ we have $\mathcal{Z}^t(\psi \otimes \varrho e^{2\pi i n\cdot}) = (-1)^n\psi(\cdot - n - \frac12)$, that is, $\mathcal{Z}T(\psi \otimes \varrho e^{2\pi i n\cdot}) = (-1)^n T(\psi(\cdot - n - \frac12))$. Since $\chi = \varrho$ near $\Xi_{\eta'} + \mathbb{Z}$ and $\mathcal{Z}T$ is supported in $\mathbb{R} \times (\Xi_{\eta'} + \mathbb{Z})$, we may replace ϱ by χ; with $\chi\psi$ in place of ψ this is (45). For (46), (48) gives $gV = i\,e^{\pi i x}\chi(x)\chi(y)\,\mathcal{Z}(\mathcal{F}T)(y,-x)$. After the change of variables $(u,v) = (y,-x)$, and since χ is even, the left side of (46) becomes $i\,\mathcal{Z}(\mathcal{F}T)\big((\chi\psi)(u)\,\chi(v)e^{2\pi i n v}\big)$, and the previous argument, applied to $\mathcal{F}T$ (whose Fourier transform $\check{T}$ is supported in $\frac12 + \Xi_\eta + \mathbb{Z}$, because Ξ_η is symmetric and $-\frac12 \equiv \frac12$ modulo 1), gives (46).

For $\varphi \in \mathcal{S}$ we have $T(\varphi) = \sum_n T(\chi(\cdot - n - \frac12)\varphi)$, because $\sum_n \chi(\cdot - n - \frac12) = 1$ near $\operatorname{supp} T$, and by (45) the n-th term is $(-1)^n V(\varphi(x+n+\frac12)e^{2\pi i(n+\frac12)y})$. Summing gives $T(\varphi) = V(G_\varphi)$, since the series defining G_φ converges in $C^\infty(\mathbb{R}^2)$.

Finally, $V(q) = T(\mathcal{Z}^t\Phi)$ with $\Phi = e^{-\pi i y}\chi(x)\chi(y)q$. Since Φ is supported in $(-\frac14,\frac14)^2$, we have $\mathcal{Z}^t\Phi(t) = (-1)^n\widehat{\Phi}^{(2)}(t-n-\frac12,n)$ for $|t-n-\frac12| < \frac12$. For $a+b \le 2p$ we expand $(x+n+\frac12)^a = \sum_{l\le a}\binom{a}{l} n^l (x+\frac12)^{a-l}$ and use $n^l\widehat{\Phi}^{(2)}(x,n) = (2\pi i)^{-l}\widehat{\partial_y^l\Phi}^{(2)}(x,n)$ together with Parseval's identity in y; this gives $\big\|t^a\partial^b\mathcal{Z}^t\Phi\big\|_{L^2} \le C\sum_{l\le a}\big\|\partial_y^l\partial_x^b\Phi\big\|_{L^2} \le C\,\|q\|_{C^{2p}}$, and (47) follows from the norm equivalence. ∎

B.4. **Newton moments and the gauge.** Let η be a nonincreasing sequence with $0 \le \eta_j \le 10^{-12}$ and $\eta_j \to 0$; zeros are allowed. Put

$$z(t) = 1 - \cos 2\pi t, \qquad a_j = z(\eta_j), \qquad D_i = \prod_{l \le i} (z - a_l),$$
$$\phi_{i,0} = \cos(\pi t)\, D_i, \qquad \phi_{i,1} = \sin(\pi t)\, D_i \qquad (i \ge 0). \tag{49}$$

The function $\phi_{i,e}$ vanishes at $\pm\eta_l$ for $l \le i$. It is a trigonometric polynomial with frequencies in $\{\pm\frac12, \dots, \pm(i+\frac12)\}$, and its coefficients at the extreme frequencies $i+\frac12$ and $-(i+\frac12)$ are h_i and h_i if $e=0$, and $-\mathrm{i}h_i$ and $\mathrm{i}h_i$ if $e=1$, where $h_i = (-1)^i 2^{-i-1}$. Since D_i is a polynomial of degree exactly i in $\cos 2\pi t$, the $\phi_{i,e}$ span the trigonometric polynomials with half-integer frequencies.

Given two such sequences, η for the variable x and η' for y, positive weights (ω_i) and (ω'_j), and $\kappa > 0$, we define, for a distribution U on $\mathbb{R}^2$ with compact support in $(-10^{-3}, 10^{-3})^2$, the *moment array*

$$M(U)_{ie,jf} = \frac{U(\phi_{i,e}(x)\, \phi'_{j,f}(y))}{\omega_i\, \omega'_j\, \kappa^{e+f}} \qquad (i,j \ge 0,\ e,f \in \{0,1\}), \tag{50}$$

where ϕ' is built from η'. Write $I = \mathbb{N}_0 \times \{0,1\}$ and $\mathbb{H} = \ell^2(I) \otimes \ell^2(I) = \ell^2(I \times I)$.

Lemma B.4. *If U is a distribution with compact support in $(-\frac14, \frac14)^2$ such that $U(\phi_{i,e} \otimes \phi'_{j,f}) = 0$ for all i, j, e, f, then $U = 0$.*

Proof. The products $\phi_{i,e}(x)\phi'_{j,f}(y)$ span the trigonometric polynomials in (x,y) with half-integer frequencies, so $e^{\pi\mathrm{i}(x+y)}U$ annihilates every $e^{2\pi\mathrm{i}(mx+m'y)}$, $m, m' \in \mathbb{Z}$. Its periodization to $\mathbb{R}^2/\mathbb{Z}^2$ therefore vanishes, and since its support lies inside a single period cell, $e^{\pi\mathrm{i}(x+y)}U = 0$. □

Lemma B.5 (Flags)**.** *Let ε, δ be nonincreasing sequences with values in $[0, 10^{-12}]$ tending to 0, put $\alpha_j = \frac12 - \varepsilon_j$ and $\alpha'_j = \frac12 - \delta_j$, and let $T \in \mathcal{N}(L_{\alpha'}, L_\alpha)$, where $L_\alpha = \{n \pm \alpha_j : j \ge 1,\ |n| \ge j\}$ as in Section 5 (if $\varepsilon_j = 0$, the corresponding points are half-integers). Let V be as in Lemma B.3 with $\eta = \varepsilon$ and $\eta' = \delta$, and let $\mathbf{A} = M(V)$ and $\mathbf{B} = M(gV)$, built with ε in x and δ in y, for arbitrary weights and κ. Then*

$$\mathbf{A}_{ie,jf} = 0 \quad (i > j), \qquad \mathbf{B}_{ie,jf} = 0 \quad (j > i), \tag{51}$$

and, with $t_i = \tan(\pi\varepsilon_{i+1})$ and $t'_i = \tan(\pi\delta_{i+1})$,

$$\mathbf{A}_{i1,i0} = \mathrm{i}t_i \mathbf{A}_{i0,i1}, \quad \mathbf{A}_{i1,i1} = -\frac{\mathrm{i}t_i}{\kappa^2}\mathbf{A}_{i0,i0}, \quad \mathbf{B}_{i0,i1} = -\mathrm{i}t'_i \mathbf{B}_{i1,i0}, \quad \mathbf{B}_{i1,i1} = \frac{\mathrm{i}t'_i}{\kappa^2}\mathbf{B}_{i0,i0}. \tag{52}$$

Proof. Since $n + \alpha_l = n + \frac12 - \varepsilon_l$ and $(n+1) - \alpha_l = n + \frac12 + \varepsilon_l$, the cluster of T at $n + \frac12$ is $\sum_{x \in \Xi_n} T(\{n + \frac12 + x\})\delta_x$, where

$$\Xi_n = \{-\varepsilon_l :\ l \le |n|\} \cup \{\varepsilon_l :\ l \le |n+1|\}; \tag{53}$$

the clusters of $\mathcal{F}T$ are described in the same way, with δ in place of ε. Expanding $\phi'_{j,f}$ in the exponentials $e^{2\pi\mathrm{i}(n+\frac12)y}$, $-j-1 \le n \le j$, and using (45), we see that $V(\phi_{i,e} \otimes \phi'_{j,f})$ is a combination of the numbers $T_{n+\frac12}(\phi_{i,e})$ with $-j-1 \le n \le j$. For these n the set Ξ_n contains only points $\pm\varepsilon_l$ with $l \le j+1$, at which $\phi_{i,e}$ vanishes if $i \ge j+1$. This proves the first part of (51); the second follows in the same way from (46), by expanding $\phi_{i,e}(x)$ in the exponentials $e^{-2\pi\mathrm{i}(n+\frac12)x}$, $-i-1 \le n \le i$.

Now let $i = j$. Then $\phi_{i,e}$ vanishes at every point of the sets Ξ_n with $-i-1 \le n \le i$ except at $\varepsilon_{i+1} \in \Xi_i$ and at $-\varepsilon_{i+1} \in \Xi_{-i-1}$, and these two clusters are extracted by the extreme frequencies $\pm(i+\frac12)$ of $\phi'_{i,f}$. Put $X_+ = (-1)^i T(\{i+\frac12+\varepsilon_{i+1}\})$, $X_- = (-1)^{i+1} T(\{-i-\frac12-\varepsilon_{i+1}\})$, $c = \cos\pi\varepsilon_{i+1}$, $s = \sin\pi\varepsilon_{i+1}$ and $D = D_i(\varepsilon_{i+1})$. By (45) and the values of the extreme coefficients,

$$V(\phi_{i,0} \otimes \phi'_{i,0}) = h_i c D(X_+ + X_-), \qquad V(\phi_{i,0} \otimes \phi'_{i,1}) = -\mathrm{i}h_i c D(X_+ - X_-),$$
$$V(\phi_{i,1} \otimes \phi'_{i,0}) = h_i s D(X_+ - X_-), \qquad V(\phi_{i,1} \otimes \phi'_{i,1}) = -\mathrm{i}h_i s D(X_+ + X_-),$$

and the first two relations of (52) follow after normalization. For $\mathbf{B}$ the roles of the variables are exchanged. By (46), the frequency $i+\frac12$ of $\phi_{i,e}(x)$ extracts the cluster of $\mathcal{F}T$ at $-i-\frac12$, whose only

point not annihilated by $\phi'_{i,f}$ is $-\delta_{i+1}$, and the frequency $-(i+\frac12)$ extracts the cluster at $i+\frac12$, with the point δ_{i+1}. With $Y_+ = \mathrm{i}(-1)^i\,\mathcal{F}T(\{i+\frac12+\delta_{i+1}\})$ and $Y_- = \mathrm{i}(-1)^{i+1}\,\mathcal{F}T(\{-i-\frac12-\delta_{i+1}\})$, the same computation gives $(gV)(\phi_{i,1}\otimes\phi'_{i,0}) = \mathrm{i}h_ic'D'(Y_+-Y_-)$, $(gV)(\phi_{i,0}\otimes\phi'_{i,1}) = h_is'D'(Y_+-Y_-)$, $(gV)(\phi_{i,0}\otimes\phi'_{i,0}) = h_ic'D'(Y_++Y_-)$ and $(gV)(\phi_{i,1}\otimes\phi'_{i,1}) = \mathrm{i}h_is'D'(Y_++Y_-)$, where c', s', D' are formed with δ_{i+1}. This gives the last two relations. □

Lemma B.6 (The gauge)**.** *In the setting of* (50)*, suppose that* $\omega_{i+1} = \lambda_{i+1}\omega_i$ *with* $0 < \lambda_i \le 10^{-3}$*, and let* S *be the backward shift on* $\ell^2(\mathbb{N}_0)$*,* $(Su)_i = u_{i+1}$*. Put*

$$J = \operatorname{diag}(a_{i+1}) + \operatorname{diag}(\lambda_{i+1})\,S, \qquad \mathsf{s} = \begin{pmatrix} 0 & \kappa(2-J) \\ J/\kappa & 0 \end{pmatrix}$$

on $\ell^2(I) = \ell^2(\mathbb{N}_0)\oplus\ell^2(\mathbb{N}_0)$ *(parities* 0 *and* 1*), and define* J' *and* s' *from* η' *and* λ' *in the same way. Assume that* $\|\mathsf{s}\|, \|\mathsf{s}'\| \le \frac12$*, and put* $X = \frac{1}{2\pi}\arcsin\mathsf{s}$*,* $X' = \frac{1}{2\pi}\arcsin\mathsf{s}'$ *and* $E = \exp(-2\pi\mathrm{i}X\otimes X')$*. Let* $K = [-10^{-2}, 10^{-2}]^2$*, and let* U *be a distribution with compact support in* $(-10^{-3}, 10^{-3})^2$ *such that* $M(fU)\in\mathbb{H}$ *and* $\|M(fU)\|_{\mathbb{H}} \le C\,\|f\|_{C^N(K)}$ *for all* $f\in C^\infty(\mathbb{R}^2)$*, for some* C *and* N*. Then* $M(gU) = E\,M(U)$*.*

Proof. The identities

$$zD_i = D_{i+1} + a_{i+1}D_i, \qquad \sin(2\pi t)\cos(\pi t) = (2-z)\sin(\pi t), \qquad \sin(2\pi t)\sin(\pi t) = z\cos(\pi t)$$

show that, entry by entry,

$$M(\sin(2\pi x)\,U) = (\mathsf{s}\otimes I)M(U), \qquad \text{hence} \qquad M(p(\sin 2\pi x)\,U) = (p(\mathsf{s})\otimes I)M(U)$$

for every polynomial p; the entries on the right are finite sums. On $[-10^{-2}, 10^{-2}]$ we have $2\pi x = \arcsin(\sin 2\pi x)$, and the Taylor series of arcsin converges there in every C^N norm, since $|\sin 2\pi x| < \frac{1}{10}$; it also converges in operator norm at s. By the assumed continuity of $f\mapsto M(fU)$ in $C^N(K)$, $M(xU) = (X\otimes I)M(U)$; likewise $M(yU) = (I\otimes X')M(U)$, and the same holds with fU in place of U. The exponential series of g converges in every $C^N(K)$ norm, and that of E in operator norm, so

$$M(gU) = \sum_n \frac{(-2\pi\mathrm{i})^n}{n!}(X\otimes X')^n M(U) = E\,M(U).$$

□

The kernel equation. We use the following subspaces of $\mathbb{H}$, described by the index pairs $((i,e),(j,f))$ they contain: $\mathcal{H}_A$, the pairs with $i<j$ and the diagonal pairs ($i=j$) of parity $ef\in\{00,01\}$; $\mathcal{H}_B$, the pairs with $i>j$ and the diagonal pairs of parity $ef\in\{00,10\}$; $\mathcal{D}$ and $\mathcal{G}$, the diagonal pairs of parity 00 and 11; and $\mathcal{C} = \mathcal{H}_A\ominus\mathcal{D}$. Let $P_A, P_B, P_\mathcal{D}, P_\mathcal{C}, P_\mathcal{G}$ be the orthogonal projections, and $Q = I - P_B$, so that $Q\mathbb{H} = \mathcal{C}\oplus\mathcal{G}$. Given sequences (t_i) and (t'_i), define the *graph maps* Σ_A and Σ_B by

$$\Sigma_A:\ e_{i0,i0}\mapsto -\frac{\mathrm{i}t_i}{\kappa^2}e_{i1,i1}, \quad e_{i0,i1}\mapsto \mathrm{i}t_ie_{i1,i0}; \qquad \Sigma_B:\ e_{i0,i0}\mapsto \frac{\mathrm{i}t'_i}{\kappa^2}e_{i1,i1}, \quad e_{i1,i0}\mapsto -\mathrm{i}t'_ie_{i0,i1},$$

and zero on all other basis vectors. Then (51) and (52) say exactly that $\mathbf{A} = (I+\Sigma_A)P_A\mathbf{A}$ and $\mathbf{B} = (I+\Sigma_B)P_B\mathbf{B}$. If moreover $\mathbf{B} = E\mathbf{A}$, then, applying Q and using $QP_B = 0$ and $Q\Sigma_B = \Sigma_B$, we get $QE\mathbf{A} = \Sigma_BP_BE\mathbf{A}$, that is,

$$\mathcal{T}a = 0, \qquad \mathcal{T} = (Q-\Sigma_BP_B)\,E\,(I+\Sigma_A)\big|_{\mathcal{H}_A}, \qquad a = P_A\mathbf{A}. \tag{54}$$

We write $a = (u,c)$ with $u = P_\mathcal{D}a$, identified with the sequence $(u_i) = (\mathbf{A}_{i0,i0})\in\ell^2(\mathbb{N}_0)$, and $c = P_\mathcal{C}a$. The operator $\mathcal{T}$ maps $\mathcal{D}\oplus\mathcal{C}$ to $\mathcal{C}\oplus\mathcal{G}$, and $Q|_{\mathcal{H}_A}$ is the map $(u,c)\mapsto(c,0)$.

How the kernel equation is used. The diagonal part $u\in\mathcal{D}$ of the unknown is free, and its remaining part $c\in\mathcal{C}$ is eliminated. Since $\mathcal{T}$ is a small perturbation of $Q|_{\mathcal{H}_A}$, the equation splits into two components:

$$\begin{aligned} &\mathcal{C}\text{-component:} && \mathcal{T}_{\mathcal{C}\mathcal{D}}u + \mathcal{T}_{\mathcal{C}\mathcal{C}}c = 0, \quad \mathcal{T}_{\mathcal{C}\mathcal{C}}\approx I && \implies c \text{ is determined by } u;\\ &\mathcal{G}\text{-component:} && \mathcal{T}_{\mathcal{G}\mathcal{D}}u + \mathcal{T}_{\mathcal{G}\mathcal{C}}c = 0 && \implies \text{one equation for } u\in\ell^2(\mathbb{N}_0). \end{aligned}$$

The dimension of the space of solutions is therefore decided by the $\mathcal{G}$-component alone. In Stage 1, with the weights $\omega_i = \vartheta^i$, it becomes $\mathcal{K}u = 0$, where $\mathcal{K}$ is a perturbation of a multiple of the backward shift, whose kernel is one-dimensional. In Stage 2, with weights that shrink as fast as the clusters, its leading part is a positive diagonal operator, and it forces $u = 0$.

B.5. **Stage 1: at most one dimension.** Let ε be a nonincreasing sequence with values in $[0, 10^{-12}]$ tending to 0, let $\gamma_j^\varepsilon = \frac{1}{2} - \varepsilon_j$, and consider $L_{\gamma^\varepsilon} = \{n \pm \gamma_j^\varepsilon : \ j \geq 1, \ |n| \geq j\}$. In this subsection $\vartheta = 2^{-12}$, and we use (50) with $\eta = \eta' = \varepsilon$, $\omega_i = \omega_i' = \vartheta^i$ and $\kappa = \vartheta^{1/2}$; thus $\lambda_i = \vartheta$ in Lemma B.6, and $t_i = t_i' = \tan(\pi\varepsilon_{i+1}) \leq \vartheta^3$ and $a_i \leq \vartheta^6$. For $T \in \mathcal{M}_{L_{\gamma^\varepsilon}}$ we let V be as in Lemma B.3, $\mathbf{A} = M(V)$ and $\mathbf{B} = M(gV)$.

Proposition B.7. *In this setting $\mathbf{A}, \mathbf{B} \in \mathbb{H}$ and $\mathbf{B} = E\mathbf{A}$, and there is a vector $\bar{u} = \bar{u}(\varepsilon) \in \ell^2(\mathbb{N}_0)$ with $\bar{u}_0 = 1$, depending only on ε, such that $u = (\mathbf{A}_{i0,i0})_{i\geq 0}$ satisfies $u = u_0\,\bar{u}$ for every $T \in \mathcal{M}_{L_{\gamma^\varepsilon}}$. If $u_0 = 0$, then $T = 0$; in particular $\dim \mathcal{M}_{L_{\gamma^\varepsilon}} \leq 1$. Finally $\|\bar{u}\| \leq \frac{11}{10}$ and $\|\mathbf{A}\| \leq 2|u_0|$.*

Proof. Membership. Let $\chi_0 \in C_c^\infty(-10^{-3}, 10^{-3})$ equal 1 near $[-10^{-12}, 10^{-12}]$. For $|t| \leq 10^{-3}$ we have $z(t) \leq 2\pi^2 t^2 < \vartheta/4$, so $|z(t) - a_l| < \vartheta/2$, and the Leibniz rule gives $\|\chi_0 \phi_{i,e}\|_{C^N} \leq C_N(1+i)^N(\vartheta/2)^i$. Since V has compact support, it has finite order N, and $|M(fV)_{ie,jf}| \leq C\,\|f\|_{C^N}\,(1+i)^N(1+j)^N 2^{-i-j}\kappa^{-2}$. Hence $M(fV) \in \mathbb{H}$, continuously in $f \in C^N$, and Lemma B.6 gives $\mathbf{B} = E\mathbf{A}$ (the hypothesis $\|\mathsf{s}\| \leq \frac{1}{2}$ is verified below). By Lemma B.5 and (54), $a = P_A\mathbf{A}$ satisfies $\mathcal{T}a = 0$, and $\mathbf{A} = (I + \Sigma_A)a$.

Estimates. We have $\|J\| \leq \vartheta + \vartheta^6 < 2\vartheta$ and, since $\vartheta/\kappa = \kappa$, $\|\mathsf{s}\| \leq \kappa(2 + 2\vartheta) < 3\kappa$. As the Taylor coefficients of arcsin are nonnegative, $\|X\| \leq \frac{1}{2\pi}\arcsin(3\kappa) < \frac{\kappa}{2}$, so that $Z_0 = -2\pi\mathrm{i}X \otimes X$ satisfies $\|Z_0\| < \frac{\pi}{2}\kappa^2 < 2\vartheta$. Hence

$$\|E - I\| < 3\vartheta, \qquad \|E - I - Z_0\| < 4\vartheta^2, \qquad \|\Sigma_A\|, \|\Sigma_B\| \leq \vartheta^2,$$

and, writing $\mathcal{T} - Q = Q(E-I)(I+\Sigma_A) + Q\Sigma_A - \Sigma_B P_B E(I+\Sigma_A)$ on $\mathcal{H}_A$,

$$\|\mathcal{T} - Q|_{\mathcal{H}_A}\| \leq 3\vartheta(1+\vartheta^2) + \vartheta^2 + \vartheta^2(1+3\vartheta)(1+\vartheta^2) < 4\vartheta. \tag{55}$$

The leading term. Write $\mathcal{T}$ in block form with respect to $\mathcal{D} \oplus \mathcal{C} \to \mathcal{C} \oplus \mathcal{G}$. By (55), $\mathcal{T}_{\mathcal{C}\mathcal{C}}$ is invertible with $\left\|\mathcal{T}_{\mathcal{C}\mathcal{C}}^{-1}\right\| < 2$, and $\|\mathcal{T}_{\mathcal{C}\mathcal{D}}\|, \|\mathcal{T}_{\mathcal{G}\mathcal{C}}\| < 4\vartheta$. If $\mathcal{T}(u, c) = 0$, then $c = -\mathcal{T}_{\mathcal{C}\mathcal{C}}^{-1}\mathcal{T}_{\mathcal{C}\mathcal{D}}u$ and $\mathcal{K}u = 0$, where

$$\mathcal{K} = \mathcal{T}_{\mathcal{G}\mathcal{D}} - \mathcal{T}_{\mathcal{G}\mathcal{C}}\mathcal{T}_{\mathcal{C}\mathcal{C}}^{-1}\mathcal{T}_{\mathcal{C}\mathcal{D}} : \ \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0),$$

$\mathcal{D}$ and $\mathcal{G}$ being identified with $\ell^2(\mathbb{N}_0)$ through the diagonal index. For $u \in \mathcal{D}$ we have $(I+\Sigma_A)u = u + \Sigma_A u$ with $\Sigma_A u \in \mathcal{G}$, and $P_{\mathcal{G}}u = 0$, so

$$\mathcal{T}_{\mathcal{G}\mathcal{D}}u = P_{\mathcal{G}}Z_0 u + P_{\mathcal{G}}(E - I - Z_0)u + \Sigma_A u + P_{\mathcal{G}}(E-I)\Sigma_A u - P_{\mathcal{G}}\Sigma_B P_B E(I+\Sigma_A)u,$$

where the last four terms have total norm less than $7\vartheta^2\,\|u\|$. In $P_{\mathcal{G}}Z_0|_{\mathcal{D}}$ only the block X_{10} of X from parity 0 to parity 1 enters: $(P_{\mathcal{G}}Z_0 u)_i = -2\pi\mathrm{i}\sum_l (X_{10})_{il}^2 u_l$. Now $\arcsin \mathsf{s} = \mathsf{s}\,h(\mathsf{s}^2)$ with $h(w) = \sum_m \binom{2m}{m}\frac{w^m}{4^m(2m+1)}$, and $\mathsf{s}^2 = \Psi \oplus \Psi$ with $\Psi = J(2-J)$, so $X_{10} = Y\,h(\Psi)$ with $Y = J/(2\pi\kappa)$. From $\|\Psi\| < 5\vartheta$ we get $\|h(\Psi) - I\| < \vartheta$, and with $\|Y\| < \kappa/3$ this gives $\|X_{10} - Y\| < \kappa\vartheta/3$ and $\|X_{10}\| < \kappa/2$. For the entrywise squares $M \circ M$ of bounded matrices, $\|M \circ M - N \circ N\| \leq \|M - N\|\,\|M + N\|$, so replacing X_{10} by Y costs less than $2\vartheta^2$; and the entrywise square of $Y = (\vartheta S + \mathrm{diag}(a_{i+1}))/(2\pi\kappa)$ is $(\vartheta^2 S + \mathrm{diag}(a_{i+1}^2))/(4\pi^2\vartheta)$. Altogether

$$\mathcal{K} = c_0 S + \mathcal{E}, \qquad c_0 = -\frac{\mathrm{i}\vartheta}{2\pi}, \qquad \|\mathcal{E}\| < 3\vartheta^2 + 7\vartheta^2 + 2\,(4\vartheta)^2 = 42\vartheta^2, \tag{56}$$

so that $\|\mathcal{E}\|/|c_0| < 270\vartheta < \frac{1}{10}$. The leading term of $\mathcal{K}$ is thus a multiple of the backward shift.

The kernel. Let S^* be the forward shift, so that $SS^* = I$. Then $\mathcal{K}S^* = c_0 I + \mathcal{E}S^*$ is invertible. If $\mathcal{K}u = 0$, write $u = u_0 e_0 + S^*Su$; then $u_0\mathcal{K}e_0 + \mathcal{K}S^*(Su) = 0$, so $Su = -u_0(\mathcal{K}S^*)^{-1}\mathcal{K}e_0$ and $u = u_0\bar{u}$ with

$$\bar{u} = e_0 - S^*(\mathcal{K}S^*)^{-1}\mathcal{K}e_0. \tag{57}$$

Since $Se_0 = 0$, $\|\mathcal{K}e_0\| = \|\mathcal{E}e_0\|$, and therefore $\|\bar{u} - e_0\| \leq \|\mathcal{E}\|/(|c_0| - \|\mathcal{E}\|) < \frac{1}{10}$. If $u_0 = 0$, then $u = 0$, hence $c = 0$, $a = 0$ and $\mathbf{A} = 0$, so that $V = 0$ by Lemma B.4; by (45) all clusters

of T vanish, and $T = 0$ because T is supported in $\frac{1}{2} + \Xi_\varepsilon + \mathbb{Z}$. Finally, $\|c\| \le 8\vartheta\,\|u\|$, so $\|\mathbf{A}\| \le (1+\vartheta^2)(1+8\vartheta)\frac{11}{10}|u_0| < 2|u_0|$. □

Lemma B.8. *In the setting of Proposition B.7:*

(i) *there is an absolute constant C_* with $\|\bar u(\varepsilon) - e_0\| \le C_* \sup_j \varepsilon_j$;*

(ii) *for $h \ge 0$ let $\varepsilon^{[h]} = (\varepsilon_{h+1}, \varepsilon_{h+2}, \dots)$; then $(u_{h+i})_{i\ge0} = u_h\,\bar u(\varepsilon^{[h]})$, and the restriction of $\mathbf{A}$ to the index pairs with $\max(i,j) \ge h$ has norm at most $2|u_h|$.*

Proof. (i) For the zero sequence, $J = \vartheta S$ annihilates e_0, so X annihilates $e_{(0,0)}$, E fixes $e_{00,00}$, and $\Sigma_A = \Sigma_B = 0$; hence $\mathcal{T} e_{00,00} = Q e_{00,00} = 0$, and $\mathcal{K}(0)e_0 = 0$. In general $\|J(\varepsilon) - J(0)\| \le 2\pi^2 \sup_j \varepsilon_j^2$ and $\|\Sigma_A\|, \|\Sigma_B\| \le 4 \sup_j \varepsilon_j/\vartheta$. The operators X, E, $\mathcal{T}$, $\mathcal{T}_{\mathcal{C}\mathcal{C}}^{-1}$ and $\mathcal{K}$ are obtained from J, Σ_A and Σ_B by norm-convergent power series, products and inverses whose arguments stay in fixed balls, so $\|\mathcal{K}(\varepsilon) - \mathcal{K}(0)\| \le C \sup_j \varepsilon_j$ with an absolute constant C. By (57), $\|\bar u(\varepsilon) - e_0\| \le \|(\mathcal{K}S^*)^{-1}\|\,\|(\mathcal{K}(\varepsilon) - \mathcal{K}(0))e_0\| \le C_* \sup_j \varepsilon_j$.

(ii) The operators J and s, and hence X, E, Σ_A, Σ_B and $\mathcal{T}$, have the following property: each entry of the image involves only entries at indices that are at least as large, coordinatewise. Consequently the subarray $(\mathbf{A}_{(h+i)e,(h+j)f})_{i,j\ge0}$ satisfies (51), (52) and $\mathbf{B} = E\mathbf{A}$ for the sequence $\varepsilon^{[h]}$, with the same weights, and its diagonal of parity 00 is $(u_{h+i})_{i\ge0}$. The proof of Proposition B.7 used only these relations, so $(u_{h+i}) = u_h \bar u(\varepsilon^{[h]})$. For the second assertion, let U_h be the set of index pairs with $\max(i,j) \ge h$. By the same property, the restriction to U_h of the equation $\mathcal{T}_{\mathcal{C}\mathcal{C}} c = -\mathcal{T}_{\mathcal{C}\mathcal{D}} u$ involves only $c|_{U_h}$ and $(u_l)_{l\ge h}$, and the compression of $\mathcal{T}_{\mathcal{C}\mathcal{C}}$ to U_h is within 4ϑ of the identity. Hence $\|c|_{U_h}\| \le 8\vartheta\,\|(u_l)_{l\ge h}\| \le 9\vartheta|u_h|$, and $\|\mathbf{A}|_{U_h}\| \le (1+\vartheta^2)(\frac{11}{10} + 9\vartheta)|u_h| < 2|u_h|$. □

B.6. **Stage 2: exclusion at low Hermite order.** In this subsection $\rho = 10^{-12}$ and $\kappa = \rho^{1/4}$. Let ε and δ be nonincreasing sequences of positive numbers tending to 0, and let the weights be

$$\omega_i = \prod_{l\le i} \lambda_l, \qquad \omega'_j = \prod_{l\le j} \lambda'_l, \qquad \text{where} \quad \varepsilon_l \le \lambda_l \le \min\{\rho, 64\varepsilon_l\}, \quad \delta_l \le \lambda'_l \le \min\{\rho, 64\delta_l\}.$$

These weights shrink as fast as the clusters do.

Proposition B.9 (Weighted contraction)**.** *Let $\mathbf{A} \in \mathbb{H}$ satisfy (51) and (52), with $t_i = \tan \pi\varepsilon_{i+1}$ and $t'_i = \tan \pi\delta_{i+1}$, and suppose that $\mathbf{B} = E\mathbf{A}$ satisfies them as well, where E is the operator of Lemma B.6 for the present weights. Then $\mathbf{A} = 0$.*

Proof. Here $J = D_\lambda J_1$ with $D_\lambda = \operatorname{diag}(\lambda_{i+1})$ and $J_1 = S + \operatorname{diag}(a_{i+1}/\lambda_{i+1})$; since $a_l \le 2\pi^2\varepsilon_l^2$ and $\lambda_l \ge \varepsilon_l$, we have $\|J_1\| < 2$ and $\|J\| < 2\rho$. Hence $\|\mathsf{s}\| < 3\kappa$, the operator $\Psi = J(2-J)$ has norm less than 5ρ, and

$$X = \frac{1}{2\pi}\,\mathsf{s}\,h(\mathsf{s}^2) = \begin{pmatrix} 0 & \kappa U_1 \\ D_\lambda V_1/\kappa & 0 \end{pmatrix}, \qquad X^2 = \begin{pmatrix} \Upsilon & 0 \\ 0 & \Upsilon \end{pmatrix}, \qquad \Upsilon = D_\lambda W_1,$$

with $U_1 = (2-J)h(\Psi)/2\pi$, $V_1 = J_1 h(\Psi)/2\pi$ and $W_1 = J_1(2-J)h(\Psi)^2/4\pi^2$, all of norm less than 1; in particular $\|X\| \le \kappa$ and $\|\Upsilon\| \le \rho$. The same holds for X', with λ'. Consequently $\|E - I\| \le e^{2\pi\kappa^2} - 1 < 7\sqrt{\rho}$, and $\|\Sigma_A\|, \|\Sigma_B\| \le \tan(\pi\rho)/\kappa^2 < 4\sqrt{\rho}$. As in (55),

$$\|\mathcal{T} - Q|_{\mathcal{H}_A}\| \le d_0 := 16\sqrt{\rho}.$$

The key point is that the diagonal entries of parity 11 of $(E-I)\mathbf{A}$ carry the factor $\lambda_{i+1}\lambda'_{i+1}$:

$$P_{\mathcal{G}}(E - I) = \kappa^{-2} D_{\lambda\lambda'} Z, \qquad \|Z\| < 8, \tag{58}$$

where $D_{\lambda\lambda'} = \operatorname{diag}(\lambda_{i+1}\lambda'_{i+1})$. Indeed, in $(X \otimes X')^n$ an output of parity 11 arises, in the first factor, through the block $D_\lambda V_1 \Upsilon^q/\kappa$ (from parity 0) if $n = 2q+1$, and through $\Upsilon^q = D_\lambda W_1 \Upsilon^{q-1}$ (from parity 1) if $n = 2q \ge 2$, and similarly in the second factor. In both cases the diagonal factors D_λ and $D_{\lambda'}$ appear on the left, and summing the exponential series gives (58) with $\|Z\| \le \sum_{q\ge0} \frac{(2\pi)^{2q+1}\rho^{2q}}{(2q+1)!} + \kappa^2 \sum_{q\ge1} \frac{(2\pi)^{2q}\rho^{2q-2}}{(2q)!} < 8$.

Now let $a = (u, c)$ satisfy $\mathcal{T}a = 0$, as in (54). The $\mathcal{C}$-component of this equation gives $\|c\| \le d_0(1-d_0)^{-1}\|u\|$. For the $\mathcal{G}$-component, note that $P_{\mathcal{G}}a = 0$, $P_{\mathcal{G}}\Sigma_A = -\mathrm{i}\,\mathrm{diag}(t_i/\kappa^2)P_{\mathcal{D}}$, $P_{\mathcal{D}}\Sigma_A = 0$ and $P_{\mathcal{G}}\Sigma_B P_B = \mathrm{i}\,\mathrm{diag}(t_i'/\kappa^2)P_{\mathcal{D}}$. Writing $E = I + (E-I)$ we obtain the exact identity

$$P_{\mathcal{G}}\mathcal{T} = -\mathrm{i}D_0 P_{\mathcal{D}} + \mathcal{R}, \qquad D_0 = \mathrm{diag}\Big(\frac{t_i + t_i'}{\kappa^2}\Big),$$

$$\mathcal{R} = P_{\mathcal{G}}(E-I)(I+\Sigma_A) - \mathrm{i}\,\mathrm{diag}(t_i'/\kappa^2)P_{\mathcal{D}}(E-I)(I+\Sigma_A).$$

The diagonal operator D_0 has positive entries but an unbounded inverse. However, $t_i \ge \pi\varepsilon_{i+1} \ge \pi\lambda_{i+1}/64$, similarly for t_i', and $\lambda\lambda'/(\lambda+\lambda') \le \rho/2$, so

$$\frac{\lambda_{i+1}\lambda_{i+1}'}{t_i + t_i'} \le \frac{64\rho}{2\pi} < 11\rho,$$

and by (58) the composition $D_0^{-1}\mathcal{R}$ is bounded, with

$$\big\|D_0^{-1}\mathcal{R}\big\| \le (88\rho + 7\sqrt{\rho})(1 + 4\sqrt{\rho}) < 8\sqrt{\rho} =: e_0.$$

The equation $P_{\mathcal{G}}\mathcal{T}a = 0$ reads $\mathrm{i}D_0 u = \mathcal{R}a$. Dividing coordinatewise, $u = -\mathrm{i}D_0^{-1}\mathcal{R}a$, so that $\|u\| \le e_0(\|u\| + \|c\|) \le e_0(1-d_0)^{-1}\|u\|$. Since $e_0 < 1 - d_0$, we get $u = 0$, then $c = 0$, $a = 0$ and $\mathbf{A} = (I + \Sigma_A)a = 0$. □

Lemma B.10. *Let ε be as in (6), let $m \ge 0$ be an integer with $3m < b/(b-1)$, let $T \in \mathcal{M}_{L_\gamma}$, and let V be as in Lemma B.3 with $\eta = \eta' = \varepsilon$. Suppose that $|V(q)| \le C\|q\|_{C^m}$ for all $q \in C^\infty(\mathbb{R}^2)$. Then, for the weights $\lambda = \lambda' = \varepsilon$, $M(fV) \in \mathbb{H}$ and $\|M(fV)\|_{\mathbb{H}} \le C'\|f\|_{C^m}$ for all $f \in C^\infty(\mathbb{R}^2)$.*

Proof. We first show that V has no transversal derivatives along the isolated lines $x = \pm\varepsilon_l$. Let $x_0 = \pm\varepsilon_l$ and let $\psi \in C_c^\infty$ equal 1 near x_0, with support meeting Ξ_ε only at x_0. The distribution $\psi(x)V$ has compact support in the line $\{x_0\}\times\mathbb{R}$, so $\psi(x)V = \sum_{r\le N}\delta_{x_0}^{(r)} \otimes S_r$ for some compactly supported distributions S_r on $\mathbb{R}$ [Hör03, Section 2.3]. By (45), the value of this distribution at $\varphi(x)e^{2\pi\mathrm{i}(n+\frac12)y}$ depends on φ only through $\varphi(x_0)$, because the clusters of T are combinations of point masses. Hence $S_r(e^{2\pi\mathrm{i}(n+\frac12)\cdot}) = 0$ for all n and $r \ge 1$, and since $\operatorname{supp} S_r \subset (-\frac14, \frac14)$, this forces $S_r = 0$ for $r \ge 1$. Thus $\psi(x)V = \delta_{x_0}\otimes S_0$, and $D_i(x)\psi(x)V = 0$ if $D_i(x_0) = 0$. The same argument, applied to gV with (46), handles the lines $y = \pm\varepsilon_l$. Consequently $D_i(x)D_j(y)V$ is supported in $\Xi^{>i}\times\Xi^{>j}$, where $\Xi^{>i} = \{0\}\cup\{\pm\varepsilon_l : l > i\} \subset [-\varepsilon_{i+1}, \varepsilon_{i+1}]$.

For $i \ge 1$ let $\chi_i \in C_c^\infty$ equal 1 near $[-\varepsilon_{i+1}, \varepsilon_{i+1}]$ and be supported in $|t| < \frac12(\varepsilon_i + \varepsilon_{i+1})$, with $\big\|\chi_i^{(r)}\big\|_\infty \le C_r(\varepsilon_i - \varepsilon_{i+1})^{-r}$, and let χ_0 be as in Stage 1. Then $(fV)(\phi_{i,e}\otimes\phi_{j,f}) = V(f\cdot(\chi_i\phi_{i,e}\otimes\chi_j\phi_{j,f}))$. On the support of χ_i we have $z(t) \le a_i$, so $|z(t) - a_l| \le a_l \le 2\pi^2\varepsilon_l^2$ for $l \le i$, and differentiating one of the factors of D_i costs at most a factor $C\varepsilon_i^{-2}$. Hence

$$\|\chi_i\phi_{i,e}\|_{C^m} \le C_m(1+i)^m(\varepsilon_i - \varepsilon_{i+1})^{-m}\varepsilon_i^{-2m}\,20^i\prod_{l\le i}\varepsilon_l^2,$$

and $|M(fV)_{ie,jf}| \le C\|f\|_{C^m}\kappa^{-2}\beta_i\beta_j$ with $\beta_i = (1+i)^m(\varepsilon_i - \varepsilon_{i+1})^{-m}\varepsilon_i^{-2m}20^i\prod_{l\le i}\varepsilon_l$. Since $\varepsilon_i - \varepsilon_{i+1} \ge \varepsilon_i/2$ and $\sum_{l\le i}\log\varepsilon_l = -d(b^{i+1}-b)/(b-1) - i\log 16$,

$$\log\beta_i \le -d\Big(\frac{b}{b-1} - 3m\Big)b^i + O(1+i),$$

so that $(\beta_i) \in \ell^2$ when $3m < b/(b-1)$. □

Proof of Theorem 5.1(b). Let $1 \le p \le P$ and $T \in \mathcal{M}_{L_\gamma} \cap H_{-p}$. By (47), V satisfies the hypothesis of Lemma B.10 with $m = 2p$, and $3m = 6p < 6P + 1 = b/(b-1)$. Hence the moment arrays for the weights $\lambda = \lambda' = \varepsilon$ lie in $\mathbb{H}$, continuously in the multiplier, and Lemma B.6 gives $\mathbf{B} = E\mathbf{A}$; by Lemma B.5 the hypotheses of Proposition B.9 hold, with $\delta = \varepsilon$ (note that $\varepsilon_1 \le \rho$). Thus $\mathbf{A} = 0$, so $V = 0$ by Lemma B.4, and $T = 0$ by (45). □

B.7. **Stage 3: existence and admission.** For $N \geq 0$ let $\varepsilon^{(N)} = (\varepsilon_1, \dots, \varepsilon_N, 0, 0, \dots)$, and let $L^{(N)} = L_{\gamma^{\varepsilon^{(N)}}}$ in the notation of Section B.5:

$$L^{(N)} = \{n \pm \gamma_j : \ j \leq N, \ |n| \geq j\} \cup \{n + \tfrac{1}{2} : \ n \geq N \text{ or } n \leq -N-1\}.$$

Thus $L^{(N)}$ consists of the points of L_γ with phase index at most N, together with the half-integers outside $[-N + \frac{1}{2}, N - \frac{1}{2}]$.

Lemma B.11. *$\mathcal{M}_{L^{(N)}}$ is spanned by a crystalline measure T_N with $\mathcal{F}T_N = -\mathrm{i}T_N$ and $\check{T}_N = -T_N$, normalized so that its moment array in the weights of Section B.5 has $u_0 = \mathbf{A}_{00,00} = 1$.*

Proof. For $N = 0$ the set $L^{(0)}$ consists of the half-integers, $\mathcal{M}_{L^{(0)}} = \mathbb{C} w_{1/2,1/2}$ by Lemma B.1, and the assertions below about $w_{1/2,1/2}$ apply directly; so let $N \geq 1$. Let $\mathcal{A} = \{\pm\gamma_j : j \leq N\} \cup \{\frac{1}{2}\}$, a set of $2N+1$ numbers that are pairwise incongruent modulo 1. By Lemma B.1, every $T \in \mathcal{M}_{L^{(N)}}$ is a combination $\sum_{a,b \in \mathcal{A}} c_{a,b} w_{a,b}$, and such a combination belongs to $\mathcal{M}_{L^{(N)}}$ if and only if neither it nor its Fourier transform has mass at the $2N^2 + 2N$ points of $(\mathcal{A} + \mathbb{Z}) \setminus L^{(N)}$. These are $4N^2 + 4N < (2N+1)^2$ linear conditions, so $\mathcal{M}_{L^{(N)}} \neq \{0\}$.

Let $T = \sum c_{a,b} w_{a,b} \in \mathcal{M}_{L^{(N)}}$ with $c_{1/2,1/2} = 0$. The masses of T at the half-integers $n + \frac{1}{2}$ are $\sum_{b \neq 1/2} c_{1/2,b} e^{2\pi i n b}$, a combination of $2N$ distinct characters which vanishes for the $2N$ consecutive values $-N \leq n \leq N-1$; hence $c_{1/2,b} = 0$ for all b. By (42) the masses of $\mathcal{F}T$ at the points $n + \frac{1}{2}$ are $\sum_{a \neq 1/2} c_{a,1/2} e^{-2\pi i a(n+1/2)}$, and the same argument gives $c_{a,1/2} = 0$. Then T satisfies the hypotheses of Lemma B.2 with $r = N$ and $\alpha_j = \alpha'_j = \gamma_j$, so $T = 0$. Thus $T \mapsto c_{1/2,1/2}$ is injective on $\mathcal{M}_{L^{(N)}}$, and $\dim \mathcal{M}_{L^{(N)}} = 1$.

Since $L^{(N)}$ is symmetric, $\mathcal{M}_{L^{(N)}}$ is invariant under $\mathcal{F}$ and under the reflection $T \mapsto \check{T}$. By (42), $\mathcal{F} w_{1/2,1/2} = -\mathrm{i} w_{1/2,1/2}$, and $\check{w}_{1/2,1/2} = w_{-1/2,-1/2} = -w_{1/2,1/2}$, while neither operation maps any other $w_{a,b}$, $a, b \in \mathcal{A}$, to a multiple of $w_{1/2,1/2}$; comparing the coefficients of $w_{1/2,1/2}$ gives $\mathcal{F}T = -\mathrm{i}T$ and $\check{T} = -T$ for $T \in \mathcal{M}_{L^{(N)}}$. Finally, Proposition B.7 applies to $\varepsilon^{(N)}$, so a nonzero element of $\mathcal{M}_{L^{(N)}}$ has $u_0 \neq 0$, and we may normalize $u_0 = 1$. □

Lemma B.12. *Let V_N be the distribution of Lemma B.3 for T_N, and let the $\phi_{i,e}$ be built from $\varepsilon^{(N)}$. Then, for all N, $i, j \geq 0$ and e, f,*

$$|V_N(\phi_{i,e} \otimes \phi_{j,f})| \leq 2\, \vartheta^{i+j} C_*^m \prod_{l \leq m} \varepsilon_l, \qquad m = \max(i,j),$$

where C_* is the constant of Lemma B.8.

Proof. By Lemma B.8, applied to $\varepsilon^{(N)}$, $|u_{h+1}| = |u_h|\, |\bar{u}(\varepsilon^{(N),[h]})_1| \leq C_* \varepsilon^{(N)}_{h+1} |u_h|$, so that $|u_h| \leq C_*^h \prod_{l \leq h} \varepsilon_l$, and every entry of $\mathbf{A}$ with $\max(i,j) = m$ is bounded by $2|u_m|$. Undoing the normalization (50), with $\kappa \leq 1$, gives the claim. □

To convert these moment bounds into bounds for $V_N(f)$ with f smooth, we interpolate f on the grid that supports V_N. We use $C^{r,r}$ norms, which control the partial derivatives $\partial_x^\alpha \partial_y^{\alpha'}$ with $\alpha, \alpha' \leq r$.

Lemma B.13. *Let $t_1 > t_2 > \cdots > t_n \geq 0$ with $t_{l+1} \leq \frac{1}{2} t_l$, let $L \geq 1$, and let $G \in C^L([0, t_1])$. Then, for all s and i with $s + i \leq n$, the divided differences of G satisfy*

$$|G[t_s, \dots, t_{s+i}]| \leq \frac{4^i \, \|G\|_{C^L}}{\prod_{l=s}^{s+i-L-1} t_l},$$

an empty product being 1.

Proof. For $i \leq L$ this is the mean value bound $|G[t_s, \dots, t_{s+i}]| \leq \left\|G^{(i)}\right\|_\infty / i!$. For $i > L$ we use $G[t_s, \dots, t_{s+i}] = (G[t_s, \dots, t_{s+i-1}] - G[t_{s+1}, \dots, t_{s+i}])/(t_s - t_{s+i})$ and $t_s - t_{s+i} \geq \frac{1}{2} t_s$, and argue by induction on i. The second term contributes at most $2 \cdot 4^{i-1} \|G\|_{C^L} / \prod_{l=s}^{s+i-L-1} t_l$, and so does the first, because its bound $2 \cdot 4^{i-1} \|G\|_{C^L} / (t_s \prod_{l=s}^{s+i-L-2} t_l)$ is at most this, as $t_s \geq t_{s+i-L-1}$. □

Lemma B.14. *Let $L \ge 1$ and $c = \frac{1}{100}$. Every $f \in C^{2L+1,2L+1}([-c,c]^2)$ can be written on $[-c,c]^2$ as $f(x,y) = \sum_{e,e' \in \{0,1\}} s_e(x) s_{e'}(y) G_{ee'}(z(x), z(y))$, with $s_0 = \cos \pi \cdot$, $s_1 = \sin \pi \cdot$ and $\|G_{ee'}\|_{C^{L,L}([0,z(c)]^2)} \le C_L \|f\|_{C^{2L+1,2L+1}}$.*

Proof. In one variable, write $f = f_0 + f_1$ with f_0 even and f_1 odd. Then f_0/s_0 is even and of class C^{2L+1}, and $f_1/s_1 = (f_1(x)/x)(x/\sin \pi x)$ is even and of class C^{2L}, since $f_1(x)/x = \int_0^1 f_1'(\sigma x)\, d\sigma$. An even function $v \in C^{2L}([-c,c])$ is of the form $v(x) = W(x^2)$ with $\|W\|_{C^L([0,c^2])} \le C \|v\|_{C^{2L}}$, as one sees by induction from $W'(x^2) = v'(x)/2x = \frac{1}{2}\int_0^1 v''(\sigma x)\, d\sigma$. Finally $z(x) = 2\sin^2 \pi x$ is a function of x^2 with nonvanishing derivative at 0, so it is a smooth change of variables. In two variables we apply these operations in each variable separately; they are linear and commute with differentiation in the other variable. □

Proposition B.15. *Let $L \ge 1$ be an integer with $b^L > 4$. Then $\sup_N \|T_N\|_{-(2L+2)} < \infty$.*

Proof. Let $f \in C^\infty(\mathbb{R}^2)$, and write $f = \sum s_e(x) s_{e'}(y) G_{ee'}(z(x), z(y))$ as in Lemma B.14. The distribution V_N is a finite combination of point masses at the points of $\Xi_{\varepsilon^{(N)}} \times \Xi_{\varepsilon^{(N)}}$: indeed T_N is a combination of combs $w_{a,b}$, and Poisson summation in y shows that $\mathcal{Z} w_{a,b}$ is a sum of point masses on $(a - \frac{1}{2} + \mathbb{Z}) \times (b + \frac{1}{2} + \mathbb{Z})$. The z-coordinates are $a_1 > \cdots > a_N > a_{N+1} = 0$, and $a_{l+1} \le \frac{1}{2} a_l$ by (40). On these points $G_{ee'}$ coincides with its tensor Newton interpolant $\sum_{i,j \le N} N_{ij}^{ee'} D_i(x) D_j(y)$, where $N_{ij}^{ee'}$ is the divided difference of $G_{ee'}$ at the nodes $a_1, \dots, a_{i+1}$ in the first variable and $a_1, \dots, a_{j+1}$ in the second; and $s_e D_i = \phi_{i,e}$. Applying Lemma B.13 in each variable, and Lemma B.14,

$$|N_{ij}^{ee'}| \le \frac{C\, 16^{\max(i,j)} \|f\|_{C^{2L+1,2L+1}}}{\Pi_{(i-L)_+}\, \Pi_{(j-L)_+}}, \qquad \Pi_q = \prod_{l \le q} a_l.$$

Since $a_l < 1$, Π_q decreases with q; with $m = \max(i,j)$ and Lemma B.12 we obtain

$$|V_N(f)| \le \sum_{e,e'} \sum_{i,j} |N_{ij}^{ee'}|\, |V_N(\phi_{i,e} \otimes \phi_{j,e'})| \le C \|f\|_{C^{2L+1,2L+1}} \sum_{m \ge 0} (2m+1)(16 C_*)^m \frac{\prod_{l \le m} \varepsilon_l}{\Pi^2_{(m-L)_+}}.$$

Since $a_l \ge 8\varepsilon_l^2$, the logarithm of the m-th term is at most $\sum_{l \le m} \log \varepsilon_l - 4 \sum_{l \le m-L} \log \varepsilon_l + O(1+m) = -\frac{d\, b^{m+1}}{b-1}(1 - 4b^{-L}) + O(1+m)$, and the series converges because $b^L > 4$. Thus $|V_N(f)| \le C \|f\|_{C^{2L+1,2L+1}([-c,c]^2)}$, uniformly in N.

Now let $\varphi \in \mathcal{S}$ and $r = 2L+1$. By Lemma B.3, $T_N(\varphi) = V_N(G_\varphi)$, and for $\alpha, \alpha' \le r$,

$$|\partial_x^\alpha \partial_y^{\alpha'} G_\varphi(x,y)| \le \sum_n (2\pi(|n|+1))^{\alpha'} \sup_{|x| \le c} |\varphi^{(\alpha)}(x + n + \tfrac{1}{2})| \le C \sum_{\sigma \le 1} \left\| (1+|x|)^{\alpha'+1} \varphi^{(\alpha+\sigma)} \right\|_{L^2},$$

by the Sobolev inequality on unit intervals and the Cauchy–Schwarz inequality. Since $(\alpha' + 1) + (\alpha + 1) \le 2(r+1)$, the norm equivalence bounds the right side by $C\|\varphi\|_{r+1}$. Hence $|T_N(\varphi)| \le C \|\varphi\|_{2L+2}$ for all N. □

Proof of Theorem 5.1(a) and (c). Let $L = 9P$. Then $b^L = (1 + \frac{1}{6P})^{9P} \ge (\frac{7}{6})^9 > 4$, by the monotonicity of $(1 + 1/x)^x$. By Proposition B.15, a subsequence of (T_N) converges weakly in $H_{-(2L+2)}$ to some T; hence $T_N \to T$ and $\mathcal{F}T_N \to \mathcal{F}T$ in $\mathcal{S}'$ along the subsequence. Let I be a bounded open interval. For large N, $L^{(N)} \cap I = L_\gamma \cap I$, because the half-integers in $L^{(N)}$ have modulus at least $N + \frac{1}{2}$ and the points of L_γ in I have bounded phase index. The distributions on I that are combinations of the point masses at the finitely many points of $L_\gamma \cap I$ form a finite-dimensional, hence closed, subspace. Therefore T and $\mathcal{F}T$ are locally atomic on L_γ, and $T \in \mathcal{M}_{L_\gamma}$. By Lemma B.3, $1 = u_0 = V_N(\cos \pi x \cos \pi y) = T_N(\Phi_0)$ with $\Phi_0 = \mathcal{Z}^t(e^{-\pi i y} \chi(x)\chi(y) \cos \pi x \cos \pi y) \in \mathcal{S}$, so $T(\Phi_0) = 1$ and $T \ne 0$; moreover $\mathcal{F}T = -iT$ and $\check{T} = -T$, by Lemma B.11. By Proposition B.7, applied to ε itself, $\dim \mathcal{M}_{L_\gamma} \le 1$. Hence $\mathcal{M}_{L_\gamma} = \mathbb{C}T \subset H_{-(18P+2)}$, which proves (a) and (c). □

B.8. **Stage 4: the head.** Recall that $\gamma' = (\beta_1, \dots, \beta_k, \gamma_1, \gamma_2, \dots)$. Let

$$p_k(x) = \prod_{j \le k} (z(x) - z(\beta_j)) = \sum_{|l| \le k} c_l e^{2\pi i l x},$$

a real even trigonometric polynomial with $c_{\pm k} = (-\frac{1}{2})^k$, whose zeros are exactly the points of $\{\pm\beta_j : j \le k\} + \mathbb{Z}$, all of them simple. Put $A_k T = p_k T$ and $B_k T = \mathcal{F}^{-1}(p_k \mathcal{F} T) = \sum_l c_l \tau_l T$. Since p_k is 1-periodic, A_k and B_k commute, and since p_k is even, $\mathcal{F} A_k = B_k \mathcal{F}$ and $\mathcal{F} B_k = A_k \mathcal{F}$. Let $\mathcal{R} = A_k B_k$, so that $\mathcal{F}\mathcal{R} = \mathcal{R}\mathcal{F}$. We write $\Lambda^{\mathrm{tail}} = \{n \pm \gamma_j : \ |n| \ge k + j\}$, the part of $L_{\gamma'}$ with phases γ_j.

Lemma B.16. *$\mathcal{R}$ maps $\mathcal{M}_{L_{\gamma'}}$ injectively into $\mathcal{M}_{L_\gamma}$, and $\|\mathcal{R}T\|_{-p} \le C_p \|T\|_{-p}$ for every integer $p \ge 0$.*

Proof. The bound follows from Section B.1. Let $T \in \mathcal{M}_{L_{\gamma'}}$. The measure $A_k T$ has no mass on $\{\pm\beta_j\} + \mathbb{Z}$, so it is locally atomic on Λ^{tail}, and B_k translates it by integers l with $|l| \le k$. Since $|n - l| \ge j$ when $|n| \ge k + j$, $\mathcal{R}T = B_k A_k T$ is locally atomic on L_γ. Since $\mathcal{F}T \in \mathcal{M}_{L_{\gamma'}}$ and $\mathcal{F}\mathcal{R}T = \mathcal{R}\mathcal{F}T$, the same holds for $\mathcal{F}\mathcal{R}T$. Thus $\mathcal{R}T \in \mathcal{M}_{L_\gamma}$.

Suppose that $\mathcal{R}T = 0$, and put $H = A_k T$, so that $B_k H = 0$. For a phase $\pm\gamma_j$ let $t(n)$ be the mass of H at $n \pm \gamma_j$. Then $\sum_l c_l t(n + l) = 0$ for all n, a recurrence of order $2k$ with nonzero extreme coefficients, and $t(n) = 0$ for $|n| \le k$, because $|n| < k + j$. Hence $t = 0$ and $H = 0$, i.e. T is supported in $\{\pm\beta_j\} + \mathbb{Z}$. Then $B_k \mathcal{F}T = \mathcal{F} A_k T = 0$, and the same recurrence argument shows that $\mathcal{F}T$ has no mass at the points $n \pm \gamma_j$, so $\mathcal{F}T$ is also supported in $\{\pm\beta_j\} + \mathbb{Z}$. By Lemma B.1, $T \in W_k$, and T satisfies the hypothesis of Theorem 5.1(e), since $L_{\gamma'}$ does not contain the points $n \pm \beta_j$ with $|n| < j$. Hence $T = 0$. □

Lemma B.17. *Let $D = \sum_{|l| \le k} c_l \tau_l$, where $\sum_l c_l w^l$ has $2k$ simple zeros on the unit circle. There are bounded sequences $(g_t^+)_{t \ge k}$ and $(g_t^-)_{t \le -k}$ such that, as formal Laurent series,*

$$\Big(\sum_l c_l w^l\Big)\Big(\sum_{t \ge k} g_t^+ w^t\Big) = 1 = \Big(\sum_l c_l w^l\Big)\Big(\sum_{t \le -k} g_t^- w^t\Big).$$

Let $\chi_+ + \chi_- = 1$ be a smooth partition of unity with $\chi_+ = 0$ on $(-\infty, -\frac{1}{4}]$ and $\chi_+ = 1$ on $[\frac{1}{4}, \infty)$, and put

$$\mathcal{Q}H = \sum_{t \le -k} g_t^- \, \tau_t(\chi_+ H) + \sum_{t \ge k} g_t^+ \, \tau_t(\chi_- H) \qquad (H \in \mathcal{S}').$$

Then $\mathcal{Q}$ is continuous on $\mathcal{S}'$, $D\mathcal{Q} = I$, $\mathcal{Q}$ commutes with multiplication by 1-periodic smooth functions, and $\|\mathcal{Q}H\|_{-(p+1)} \le C_p \|H\|_{-p}$ for every integer $p \ge 0$.

Proof. Let $q(w) = w^k \sum_l c_l w^l$, a polynomial of degree $2k$ with $q(0) \ne 0$ and simple zeros on the unit circle. By partial fractions, the Taylor coefficients of $1/q$ at 0 and its Laurent coefficients at ∞ are bounded; after the shift by w^k they give g^+ and g^-. The translates $\tau_t(\chi_+ H)$, $t \le -k$, are supported in $[|t| - \frac{1}{4}, \infty)$, and the $\tau_t(\chi_- H)$, $t \ge k$, in $(-\infty, \frac{1}{4} - t]$, so both sums are locally finite, and $D\mathcal{Q}H = \chi_+ H + \chi_- H = H$ by the two formal identities. Multiplication by a 1-periodic function commutes with integer translations and with multiplication by $\chi_\pm$. The transpose of $\mathcal{Q}$ is $\mathcal{Q}^t\varphi = \chi_+ \sum_{t \le -k} g_t^- \varphi(\cdot - t) + \chi_- \sum_{t \ge k} g_t^+ \varphi(\cdot - t)$. For the first sum, the term $x^a \chi_+^{(\sigma)}(x) \varphi^{(b-\sigma)}(x + |t|)$, with $a + b \le 2p$, is supported where $x \ge -\frac{1}{4}$; there, with $y = x + |t|$, we have $1 + |x| \le C(1 + |y|)$ and $1 + |t| \le C(1 + |y|)$, so its L^2 norm is at most $C(1 + |t|)^{-2} \left\| (1 + |y|)^{a+2} \varphi^{(b-\sigma)} \right\|_{L^2} \le C(1 + |t|)^{-2} \|\varphi\|_{p+1}$ by the norm equivalence. Summing over t, and treating the second sum in the same way, gives $\|\mathcal{Q}^t\varphi\|_p \le C \|\varphi\|_{p+1}$; this proves the continuity of $\mathcal{Q}$ and, by duality, the bound. □

Proof of Theorem 5.1(d). By Lemma B.16 it remains to show that every $H \in V_p(L_\gamma)$ is $\mathcal{R}T$ for some $T \in V_{p+2}(L_{\gamma'})$. Let $\mathcal{Q}_B$ be the operator of Lemma B.17 for $D = B_k$, and $\mathcal{Q}_A = \mathcal{F}^{-1} \mathcal{Q}_B \mathcal{F}$. Since $\mathcal{F} A_k \mathcal{F}^{-1} = B_k$, we have $B_k \mathcal{Q}_B = I$ and $A_k \mathcal{Q}_A = I$; moreover $\mathcal{Q}_A$ commutes with

$B_k = \mathcal{F}^{-1} p_k \mathcal{F}$, because $\mathcal{Q}_B$ commutes with multiplication by p_k. Put $T_0 = \mathcal{Q}_A \mathcal{Q}_B H \in H_{-(p+2)}$. Then

$$A_k T_0 = \mathcal{Q}_B H, \qquad B_k T_0 = \mathcal{Q}_A B_k \mathcal{Q}_B H = \mathcal{Q}_A H, \qquad \mathcal{R} T_0 = A_k \mathcal{Q}_A H = H.$$

Since L_γ has no points in $[-\frac{1}{2}, \frac{1}{2}]$, $\chi_\pm H$ is the restriction of H to $\{\pm x > 0\}$. An atom of H at $n \pm \gamma_j$ with $n \geq j$ is moved by $\mathcal{Q}_B$ to the points $n + |t| \pm \gamma_j$, $t \leq -k$, and an atom with $n \leq -j$ to the points $n - t \pm \gamma_j$, $t \geq k$; all of these lie in Λ^{tail}. Thus $\mathcal{Q}_B H$ is locally atomic on Λ^{tail}, and so is $\mathcal{F}\mathcal{Q}_A H = \mathcal{Q}_B \mathcal{F} H$, since $\mathcal{F} H \in \mathcal{M}_{L_\gamma}$.

From $p_k T_0 = \mathcal{Q}_B H$ we read off the support of T_0. Away from the zeros of p_k, $T_0 = p_k^{-1} \mathcal{Q}_B H$ is locally atomic on Λ^{tail}. Near a zero $x_0 \in \{\pm\beta_j\} + \mathbb{Z}$, the right-hand side vanishes, because Λ^{tail} lies within ε_1 of the half-integers and $\beta_j \leq \frac{1}{2} - \frac{1}{4k}$; since x_0 is a simple zero of p_k, T_0 is a multiple of δ_{x_0} near x_0. Thus T_0 is locally atomic on $\Lambda^{\mathrm{tail}} \cup (\{\pm\beta_j\} + \mathbb{Z})$, and the same argument, applied to $p_k \mathcal{F} T_0 = \mathcal{F} B_k T_0 = \mathcal{F}\mathcal{Q}_A H$, shows that so is $\mathcal{F} T_0$. The only points of this set outside $L_{\gamma'}$ are the points $n \pm \beta_j$ with $|n| < j$, which give $4k^2$ coordinates on the two sides. By Theorem 5.1(e) and $\dim W_k = 4k^2$, the map recording the masses of $w \in W_k$ and of $\mathcal{F} w$ at these points is bijective, so there is a unique $w \in W_k$ with the same masses there as T_0 and $\mathcal{F} T_0$; it depends linearly on these masses, which are bounded by $C \|T_0\|_{-(p+2)}$. Since $A_k w = B_k w = 0$, the distribution $T = T_0 - w$ belongs to $\mathcal{M}_{L_{\gamma'}} \cap H_{-(p+2)}$, $\mathcal{R} T = H$, and $\|T\|_{-(p+2)} \leq C \|H\|_{-p}$. □

Appendix C. Proof of Lemma 6.5

Proof. Conjugating by $e^{-\pi i x/\varpi}$ turns the antiperiodic difference operator $\tau_\varpi + I$ into a nonzero multiple of the ordinary difference operator $\tau_\varpi - I$, so it suffices to treat $(\tau_\varpi - I)^{\Delta+1} U = 0$ on the ϖ-periodic open set $\mathbb{R} \setminus \Pi$. There, the distribution $Y = (\tau_\varpi - I)^\Delta U / (\Delta!\, \varpi^\Delta)$ is ϖ-periodic. Choose $\chi_\varpi \in C_c^\infty$ with $\sum_{n \in \mathbb{Z}} \chi_\varpi(x + n\varpi) = 1$, and let v_Δ be the periodization $\sum_n \tau_{n\varpi}$ of $\chi_\varpi\, (\tau_\varpi - I)^\Delta U / (\Delta!\, \varpi^\Delta)$. It is a ϖ-periodic tempered distribution which agrees with Y on $\mathbb{R} \setminus \Pi$. Since $(\tau_\varpi - I)^\Delta (x^\Delta v) = \Delta!\, \varpi^\Delta v$ for ϖ-periodic v, the distribution $U - x^\Delta v_\Delta$ satisfies $(\tau_\varpi - I)^\Delta (U - x^\Delta v_\Delta) = 0$ on $\mathbb{R} \setminus \Pi$. We descend in the degree, undo the conjugation, and put $W = U - \sum_r x^r v_r$, which vanishes on $\mathbb{R} \setminus \Pi$.

All operations used are translations by at most $24p$, multiplication by a compactly supported smooth function, periodization, modulation, and multiplication by x^r. The transpose of periodization is $\varphi \mapsto \chi_\varpi(x) \sum_n \varphi(x + n\varpi)$, whose local derivatives are bounded by Schwartz seminorms with weight 2, uniformly for $\varpi \in [\frac{1}{2}, 2]$; by the norm equivalence of Section 3, periodization loses at most three Hermite orders, and multiplication by x^r loses at most $r + 1$. The total loss is at most $6p + \sum_{r=0}^{12p} (r + 4) = 72p^2 + 60p + 4 < m_p$. The constants are independent of Π and of the spacing of its points; in particular no division by a flat annihilating function occurs. □

Appendix D. Proofs of the technical results of Section 6

D.1. **Separation and selectors.**

Proof of Lemma 6.2. Pairs with $|x - y| \geq 1$ satisfy (13) with $c_s = 1$. For $K \geq 0$ consider the pairs with $\ell \neq \ell'$, $2^K \leq 1 + |x| < 2^{K+1}$ and $|x - y| < 1$, and write $x = t_\ell u$, $y = t_{\ell'} v$ with $u \in C_i$, $v \in C_{i'}$. Then $|u|, |v| \leq 2^{K+3}$, so $i, i' \leq K + 3$, and by Proposition 5.3(iv) the number of possible quadruples (ℓ, u, ℓ', v) is at most $CK^2 16^K$. For each of them we bound the probability that $|t_\ell u - t_{\ell'} v| < \delta$. If $i \neq i'$, condition on all scales except s_i; the map $s_i \mapsto t_\ell u$ has derivative of modulus at least $|u|/4 \geq \frac{1}{8}$ on $[1, 2]$, so the probability is at most 16δ. If $i = i'$, then $\ell' = \ell^*$ and $t_\ell u - t_{\ell'} v = \pm(s_i u' - v'/s_i)$ with $\{u', v'\} = \{u, v\}$; if u' and v' have opposite signs this has modulus at least $|u'| \geq \frac{1}{2}$, and otherwise its derivative in s_i has modulus at least $|u'| \geq \frac{1}{2}$, so the probability is at most 4δ for $\delta < \frac{1}{4}$. With $\delta_K = 2^{-6K}$ the probability that some pair in the K-th window violates $|x - y| \geq \delta_K$ is at most $CK^2 16^K 2^{-6K}$, which is summable. By the Borel–Cantelli lemma, almost surely only finitely many windows contain a violation; in the remaining finitely many windows there are finitely many pairs, and each exact coincidence $t_\ell u = t_{\ell'} v$ has probability zero by the same derivative bounds. Since $2^{-6K} \geq (1 + |x|)^{-6}$ in the K-th window, (13) follows. □

Proof of Lemma 6.3. Let $X = \bigcup_{\ell\in E}\Gamma_\ell$ and $Y = \Lambda\setminus X$. Let $(\psi_K)_{K\ge 0}$ be a smooth partition of unity with ψ_K supported in $\{2^{K-1}\le 1+|x|\le 2^{K+1}\}$ ($K\ge 1$) and $|\psi_K^{(j)}|\le C_j 2^{-jK}$. By (13), points of X and of Y in $\{1+|x|\le 2^{K+2}\}$ are at distance at least $c_s 2^{-6(K+3)} = 8r_K$, where $r_K = c_s 2^{-6(K+3)}/8$. Let ϕ be a nonnegative mollifier supported in $[-1,1]$ with $\int\phi = 1$, $\phi_r = r^{-1}\phi(\cdot/r)$, and let f_K be the convolution of ϕ_{r_K} with the indicator function of the $2r_K$-neighbourhood of X. Then, in $\{1+|x|\le 2^{K+1}\}$, $f_K = 1$ near X and $f_K = 0$ near Y, and $|f_K^{(j)}|\le \left\|\phi_{r_K}^{(j)}\right\|_{L^1}\le C_j r_K^{-j}$; this bound does not depend on the number of points. Put $\chi_E = \sum_K \psi_K f_K$. Near a point of X every term with $\psi_K\neq 0$ has $f_K = 1$, so $\chi_E = 1$; similarly $\chi_E = 0$ near Y. The Leibniz rule gives the derivative bounds. For (14), by duality it suffices to show $\|\chi_E\varphi\|_p\le C_{p,s}\|\varphi\|_{6p}$. For $a+b\le 2p$, $\left\|x^a\partial^b(\chi_E\varphi)\right\|_{L^2}\le C\sum_{j\le b}\left\|(1+|x|)^{a+6j}\partial^{b-j}\varphi\right\|_{L^2}$, and $a+6j+(b-j)\le a+b+5b\le 12p$; by the norm equivalence of Section 3 this is at most $C\|\varphi\|_{6p}$. □

D.2. **Sublevel sets and nonresonance.**

Lemma D.1. *Let $q\ge 1$ and let f be a complex polynomial in $(\sigma_1,\dots,\sigma_q)$ of degree at most 2 in each variable, one of whose coefficients has modulus at least $\delta>0$. Then, for $u>0$,*

$$\left|\{\sigma\in[1,2]^q:\ |f(\sigma)|\le u\}\right|\le C_q\min\{1,(u/\delta)^{1/(2q)}\},$$

with C_q depending only on q.

Proof. Replacing f by its real or imaginary part and δ by $\delta/\sqrt{2}$, we may assume f real. Let $q=1$, and let S be the sublevel set, of measure $\lambda>0$. Using the distribution function of S one finds $\sigma_0<\sigma_1<\sigma_2$ in S with $\sigma_{j+1}-\sigma_j\ge\lambda/4$. By Lagrange interpolation at these points, every coefficient of f is at most $192u/\lambda^2$ in modulus, so $\lambda\le(192u/\delta)^{1/2}$. For $q\ge 2$ write $f=\sum_{e=0}^2 f_e(\sigma')\sigma_q^e$ with $\sigma'=(\sigma_1,\dots,\sigma_{q-1})$; some f_{e_0} has a coefficient of modulus at least δ. For $0<\tau\le\delta$, the set where $|f_{e_0}(\sigma')|\le\tau$ has measure at most $C_{q-1}(\tau/\delta)^{1/(2q-2)}$ by induction, and for the other σ' the section of the sublevel set has measure at most $(192u/\tau)^{1/2}$. If $u<\delta$, the choice $\tau=\delta(u/\delta)^{(q-1)/q}$ and Fubini's theorem give the bound with $C_q = C_{q-1}+\sqrt{192}$; if $u\ge\delta$ there is nothing to prove. (Much sharper multidimensional sublevel estimates are known, see [CCW99]; this crude bound suffices here.) □

Proof of Lemma 6.4. Write $z = t_\ell(n+\theta)$ with $\ell=(i,\sigma)$, and $y = t_{\ell'}u$ with $u\in C_{i'}$. Multiplying $y+\omega_k-z$ by $\prod_{j\le M}s_j$, and by $s_{i'}$ if $i'>M$, gives a polynomial f in at most $M+1$ of the scales, of degree at most 2 in each. We claim that, if (y,k) is not expected, some coefficient of f has modulus at least $\min\{d_M,1\}$. The term $-t_\ell(n+\theta)$ and the term $k_{i,\sigma}t_\ell$ of ω_k produce the same monomial, and no other term does: the terms of ω_k attached to other coordinates, and the term y if $\ell'\neq\ell$, have a different degree in s_i or in $s_{i'}$. Hence if $\ell'\neq\ell$ this coefficient is $k_{i,\sigma}-n-\theta$, of modulus at least the distance from θ to $\mathbb{Z}$, which is at least d_M. If $\ell'=\ell$ and some coordinate of k other than $k_{i,\sigma}$ is nonzero, that integer is the coefficient of its own monomial. Otherwise $y+\omega_k-z = t_\ell(n'+k_{i,\sigma}-n+\theta'-\theta)$ with $u=n'+\theta'$; this is a nonzero integer multiple of t_ℓ if $\theta'=\theta$ (the pair being unexpected), and has coefficient of modulus at least d_M if $\theta'\neq\theta$. This proves the claim. Since $\prod s_j\le 2^{M+1}$, Lemma D.1 gives, for each unexpected tuple, $\mathbb{P}(|y+\omega_k-z|<\varepsilon)\le C_{M+1}(2^{M+1}\varepsilon/\min\{d_M,1\})^{1/(2M+2)}$.

Fix a dyadic L. Only base atoms u with $|t_{\ell'}u|\le|z|+|\omega_k|+1$ can give $|y+\omega_k-z|<1$; there are $O_M((1+L)^2\log(2+L))$ of them, uniformly over all continuations with $R_i\ge 2^i$, and there are $O_M((1+L)^{2M})$ vectors k and $O_M(1)$ targets. With $\varepsilon = c_M(1+L)^{-\beta_M}$ we have $\varepsilon^{1/(2M+2)} = c_M^{1/(2M+2)}(1+L)^{-(2M+6)}$, so the probability of a violation at level L is $O_M(c_M^{1/(2M+2)}(1+L)^{-4}\log(2+L))$. Summing over dyadic L and choosing c_M small enough gives the claim. □

For $L\ge 1$ put $\varrho_M(L) = \min\{1, c_M(3+2L)^{-\beta_M}\}/10$. On A_M, every unexpected distance with $\|k\|_\infty\le L$ is at least $10\varrho_M(L)$, since a dyadic L' with $L\le L'<2L+1$ may be used in (18).

The event A_M depends on the whole set Λ, but its probability bound and c_M do not depend on the radii chosen later.

D.3. **Translation and phase errors.**

Lemma D.2. *Let $m, \Delta, N \geq 1$ and $M' \geq 1$. For $\nu \leq N$ let $\varpi_\nu \in [\frac{1}{2}, 2]$ and let $v_{\nu,0}, \dots, v_{\nu,\Delta} \in H_{-m}$ satisfy $\tau_{\varpi_\nu} v_{\nu,r} = -v_{\nu,r}$; put $\mathcal{A} = \max_{\nu,r} \|v_{\nu,r}\|_{-m}$ and $V = \sum_\nu \sum_{r \leq \Delta} x^r v_{\nu,r}$. Let $\varphi \in C_c^\infty$ with $\operatorname{supp}\varphi \subset [-M', M']$. For $h \geq 1$ let $h_\nu \in [0, 2\varpi_\nu)$ with $h - h_\nu \in 2\varpi_\nu\mathbb{Z}$. Then*

$$\Big| h^{-\Delta} V(\varphi(\cdot - h)) - \sum_\nu v_{\nu,\Delta}(\varphi(\cdot - h_\nu)) \Big| \leq C h^{-1} \mathcal{A} \, \|\varphi\|_{C^{2m}} ,$$

and, for $v \in H_{-m}$ and $|a|, |a'| \leq 4$, $|v(\varphi(\cdot - a)) - v(\varphi(\cdot - a'))| \leq C|a - a'| \, \|v\|_{-m} \, \|\varphi\|_{C^{2m+1}}$, where C depends only on m, Δ, N, M'.

Proof. We have $v(x^r \varphi(x - h)) = (\tau_h v)((x + h)^r \varphi) = \sum_{i \leq r} \binom{r}{i} h^{r-i} (\tau_h v)(x^i \varphi)$, and $\tau_h v_{\nu,r} = \tau_{h_\nu} v_{\nu,r}$ because $v_{\nu,r}$ is $2\varpi_\nu$-periodic. Multiplying by $h^{-\Delta}$, the terms with $r = \Delta$, $i = 0$ give $\sum_\nu v_{\nu,\Delta}(\varphi(\cdot - h_\nu))$, and all other terms carry a factor $h^{r-i-\Delta} \leq h^{-1}$. Each of them is bounded by $\binom{r}{i} \|v_{\nu,r}\|_{-m} \, \|(x - h_\nu)^i \varphi(x - h_\nu)\|_m \leq C\mathcal{A} \, \|\varphi\|_{C^{2m}}$, since the norm $\|\cdot\|_m$ of a function supported in a fixed compact set is bounded by its C^{2m} norm. The second estimate follows from $\|\varphi(\cdot - a) - \varphi(\cdot - a')\|_m \leq |a - a'| \sup_{|t| \leq 4} \|\varphi'(\cdot - t)\|_m$. □

Parameter	depends on	used in
R_{M+1}	all parameters of stage M	(P4)
c_s, $C_{j,s}$, $C_{p,s}$	s (fixed at the end)	Lemmas 6.2, 6.3
d_M, c_M, β_M	$R_1, \dots, R_M$	Lemma 6.4
L_M, d_M^A	c_M, β_M, $g_1, \dots, g_M$	(P1), Step 1
γ_M, d_M^J	$R_1, \dots, R_M$	(P2), Step 3
$\kappa_{M,j}$, J_M	$k_1, \dots, k_M$	(P3), Step 2
m_p, N_p	p	Lemmas 6.5, D.2
$\underline{h}_M$, η_M	partition, k_M	(P3), Step 2
$\overline{h}_M$	$\underline{h}_M$, η_M, $R_1, \dots, R_M$	(P3)

TABLE 5. The parameters of the construction.

D.4. **Proof of Proposition 6.6.** The choices are made recursively in M; the choices at stage M depend only on $R_1, \dots, R_M$, never on an element of $\mathcal{M}_\Lambda$ or on radii chosen later.

Separation of the periodic sets. For every pair of labels the sets $\widetilde{K}_\ell$ and $\widetilde{K}_{\ell'}$ are disjoint for almost every s: a coincidence $x/t_\ell = x'/t_{\ell'}$ with $x \in K_i + \mathbb{Z}$, $x' \in K_{i'} + \mathbb{Z}$ (both nonzero) is a nontrivial equation in one scale, and there are countably many of them. Let $\mathcal{D}_M(s)$ be the minimal distance between the sets $\widetilde{K}_\ell \cap [-M-3, M+3]$, ℓ of index $\leq M$; it is a Borel function of $(s_1, \dots, s_M)$ and is almost surely positive. Choose $\gamma_M \in (0, 1/M]$ with $\mathbb{P}(\mathcal{D}_M < \gamma_M) \leq 2^{-M-4}$, let $\mathsf{B}_M = \{\mathcal{D}_M \geq \gamma_M\}$, and put $d_M^J = \gamma_M/10$.

Partition and translation budgets. Choose a smooth partition of unity $(\kappa_{M,j})_{j \leq J_M}$ of $[-M, M]$ by functions whose supports have diameter less than $\min\{1/M, \min_{i \leq M} 1/(100k_i)\}$. Then choose $\underline{h}_M \geq M + 1$ and $\eta_M \in (0, 1/(1000k_M)]$ so that, by Lemma D.2 with $m = m_p$, $N = 2M$ and $M' = M$, the left-hand side of (22) is at most $2^{-M}\mathcal{A}/(1 + J_M)$ whenever $h \geq \underline{h}_M$ and $h - a_\ell \in 2\varpi_\ell\mathbb{Z} + [-\eta_M, \eta_M]$, for every $p \leq M$; this is possible because $\|\kappa_{M,j}\psi\|_{C^{2m_p+1}} \leq C_M \, \|\psi\|_{C^{N_p}}$.

Kronecker nets. For almost every s the $2M$ numbers $t_\ell/2$, ℓ of index $\leq M$, are linearly independent over $\mathbb{Q}$, since an integer relation becomes a nontrivial polynomial equation after clearing denominators. Then, by Kronecker's theorem, the image of the half-line $[\underline{h}_M, \infty)$ under $h \mapsto (h \bmod 2\varpi_\ell)_\ell$ is dense in the torus $\prod_\ell \mathbb{R}/2\varpi_\ell\mathbb{Z}$. Let $\mathcal{H}_M(s, \overline{h})$ be the covering radius of the image of $[\underline{h}_M, \overline{h}]$; it is a Borel function, nonincreasing in $\overline{h}$, and tends to 0 almost surely. Choose $\overline{h}_M$ with $\mathbb{P}(\mathcal{H}_M(\cdot, \overline{h}_M) > \eta_M/4) \leq 2^{-M-4}$, and let C_M be the complementary event.

Nonresonance and the Fourier tail. Let c_M, β_M and A_M be as in Lemma 6.4. Choose $L_M \geq M$ so large that (21) holds for every $p \leq M$ and every bump f of radius $\varrho_M(L_M)$ centred at a

target, and put $d_M^A = \varrho_M(L_M)$. This is possible because the H_p norm of such a bump grows at most like a fixed power of L_M, while the coefficients $A_M(k)$ decay faster than any power.

The next radius. Choose an integer $R_{M+1} \geq \max\{2^{M+1}, 2R_M, 6(\overline{h}_M + M + 2)\}$ so large that (23)–(25) hold for every $p \leq M$. The families of test functions in (23) and (25) are bounded in $\mathcal{S}$ uniformly for $(s_1, \dots, s_M) \in [1,2]^M$; in (24), integration by parts with the fixed number N_p of derivatives gives uniform polynomial decay of the Fourier transforms, with enough margin for the H_{6p} norm. Since all parameters of stage M are already fixed, each left-hand side tends to zero uniformly as $R \to \infty$. Constants depending on M are paid for by a larger radius, never by more regularity of a fixed test function. We start with $R_1 = 2$.

The good set. Let Ω_0 be the event of full measure on which (13) holds and all the sets $\widetilde{K}_\ell$ are pairwise disjoint, and put $\Omega = \Omega_0 \cap \bigcap_{M \geq 1} (\mathsf{A}_M \cap \mathsf{B}_M \cap \mathsf{C}_M)$. Then $\mathbb{P}(\Omega) \geq 1 - 3\sum_M 2^{-M-4} = 13/16$. The bound for A_M applies to the finally chosen sequence of radii because it holds for every continuation. On A_M the first part of (P1) holds with $d_M^A = \varrho_M(L_M)$; (P2) holds on B_M; and (P3) holds on C_M, since a covering radius at most $\eta_M/4$ provides the required h. The constants c_s, $C_{j,s}$ and $C_{p,s}$ of Lemmas 6.2 and 6.3 depend on s; they enter only as fixed multiplicative factors in estimates that tend to zero as $M \to \infty$, and the deterministic choices above do not depend on them. □

Appendix E. The formal developments

The existence statements of Theorems A and B have formal counterparts in Lean 4 on top of Mathlib, with all dependencies pinned to fixed commits: the existence statement of Theorem A(i), for Fourier quasicrystals, in the development [Mar26], recorded in Palomar as [Pal26c]; and the existence statement of Theorem B(i), for all crystalline measures and without the symmetry of Λ, in the development [MS26], recorded in Palomar as [Pal26b]. A Palomar record certifies that the formal proof of the statement of record has been machine-checked; as the registry itself states, it does not certify novelty, nor that the formal statement matches an informal one. We therefore record the relevant data and describe each statement of record explicitly. According to the two records, the proofs were rechecked during registration by the kernel checkers `nanoda` and `con-ron`, and copies of the source and dependency repositories at the verified commits are retained in Palomar's archive.

The Fourier quasicrystals of Theorem A. The development [Mar26] was registered in Palomar on 8 October 2026 [Pal26c].

- Toolchain: `leanprover/lean4:v4.35.0-rc3`, with Mathlib and the other dependencies pinned in `lake-manifest.json`.
- Statement of record: the theorem `strongCardinalClaim` in the file `Challenge.lean`, proved in `Solution.lean` from the theorem `exists_stronglyTemperedMeyerSpace_rank_aleph0` of the module `OriginalClassicalStrongCountable`.
- Build and check: `lake exe cache get Challenge.lean Solution.lean`, then `lake build Challenge Solution` and `python3 scripts/check.py`. A configuration file restricts the axioms of the main theorem to `propext`, `Classical.choice` and `Quot.sound`.

In the notation of this paper the statement of record asserts: there is $\Lambda \subset \mathbb{R}$ meeting every bounded interval in a finite set such that $\dim_{\mathbb{C}} \mathcal{M}_\Lambda^{\mathrm{str}} = \aleph_0$. Here $\mathcal{M}_\Lambda^{\mathrm{str}}$ is defined as the linear span of the tempered distributions T such that T and $\mathcal{F}T$ are given on compactly supported Schwartz functions by local atomic formulas on Λ whose coefficients a_x satisfy $\sum_{x \in \Lambda} |a_x|(1+|x|)^{-N} < \infty$ for some N (possibly different for T and $\mathcal{F}T$), and the statement includes that this span consists exactly of such T. Since $X_N(\Lambda) \subseteq X_{N+1}(\Lambda)$, this is the space $\bigcup_N X_N(\Lambda)$ of Section 3.5. The Fourier transform is that of Mathlib, with kernel $e^{-2\pi i x\xi}$.

The statement of record concerns the rank only: it does not identify a basis, and the finite dimension of every layer $X_N(\Lambda)$, although proved in the development, is not part of it. The formal witness is a union of a coarse cone and of blocks of the kind used in Section 4, with the same slope β (Section 4.1), the same scale $c = 2^{1/4}$ and the same Lee–Yang factors p_a. It differs from the set of Theorem 4.12 in the choice of the remaining data: the sheet number

of a block at the prime m is the least s with $2m^4 \le s^3$, the parameters are taken in disjoint dyadic intervals, and the formal proof follows the corresponding, less simplified, route. The radii are obtained from a compactness statement, as in Lemma 4.11, and are then selected by classical choice (Remark 4.13). Thus the set Λ of Theorem 4.12, parts (ii) and (iii) of Theorem A and Proposition 4.9 as stated here are not part of this formal statement. The following table indicates, by module name (in the directory `MeyerGeneralProblem/Cardinal/Strong`), where the formal proof establishes analogues of some of the main steps; since the two routes differ in their parameters, the correspondence is only approximate.

Paper	Modules
Lemma 4.1	`ParityContinuants`, `ParityLimit`, `ParityBarrier`, `QuarterPhase`
Lemma 4.4	`ReturnMassGrowth`
Proposition 4.7	`OriginalStrongNoReturn`, `OriginalNoReturnGeometry`
Lemma 4.8	`ScheduledOriginalChartLocalSplitting`, `ScheduledOriginalGlobalSplitting`
Lemma 4.11, Theorem 4.12	`OriginalClassicalAdaptiveRadii`, `OriginalClassicalLayerExhaustion`, `ScheduledOriginalIndependentSources`, `OriginalClassicalStrongCountable`

The crystalline measures of Theorem B. The development [MS26] consists of roughly four hundred files. It was submitted to Palomar on 30 September 2026 [Pal26b].

- Toolchain: `leanprover/lean4:v4.35.0-rc3`.
- Statement of record: the theorem `exhaustiveCardinalClaim` in the file `Challenge.lean`, proved in `Solution.lean`.
- Build and check: as above, with the same restriction on the axioms of the main theorem.

In the notation of this paper the statement of record asserts: there is $\Lambda \subset \mathbb{R}$ meeting every bounded interval in a finite set such that $\dim_{\mathbb{C}} \mathcal{M}_\Lambda = \aleph_0$, such that membership in $\mathcal{M}_\Lambda$, defined as the annihilator of the Schwartz functions vanishing on Λ together with the analogous condition for the Fourier transform, is equivalent to the two local atomic formulas, and such that for some sequence (g_n) every element of $\mathcal{M}_\Lambda$ is a unique finite linear combination of the g_n. Tempered distributions and the Fourier transform are those of Mathlib.

Of the results on the spaces $\mathcal{M}_\Lambda$ in Sections 5–8, the symmetry of Λ, parts (ii)–(iii) of Theorem B, Theorem C, Section 7 and Section 8 are not part of this formal statement. The following table indicates, by module name (in the directory `MeyerGeneralProblem/Cardinal/Adaptive`), where the main steps of the proof for the profile $P_i = i$ are formalised. It is meant as a guide for the reader; the statement of record is the one described above.

Paper	Modules
Theorem 5.1(a)–(c)	`ActualSmallTailUniqueness`, `SeamCompleteKernel`, `EndpointUniformAdmission`, `RapidNativeAdmission`, `RapidNativeExclusion`, `RapidCompleteLine`
Theorem 5.1(d)	`CriticalHeadInverse`, `CriticalForwardMap`, `CriticalGreenInverse`, `BlockTailEquivalence`
Lemma 5.2, Theorem 5.1(e)	`FiniteHoleExchange`, `ActualHoleExchange`, `ConcreteHoleCarriers`, `FiniteHeadExhaustion`
Proposition 5.3	`HighOrderGapLines`
Lemmas 6.2, 6.3	`SectorSeparation`, `QuantitativeSectorSeparation`, `SmoothSelectors`, `ActualSourceSplitting`
Lemmas D.1, 6.4	`PolynomialSublevel`, `QuadraticSublevel`, `CatalogueNonresonance`, `CatalogueEvents`
Lemmas 6.5, D.2	`AntiperiodicLift`, `PeriodicLift`, `SinglePhaseLift`, `LeadingTranslationErrors`, `PhaseTranslationErrors`
Proposition 6.6	`StagePhaseParameters`, `TorusPhaseNets`, `FiniteSmoothPartition`, `ActualRecursion`, `ActualGoodScales`, `ProbabilityBudget`
Sections 6.4–6.6	`ActualConvolutionVanishing`, `PolynomialSupportDescent`, `ActualPolynomialSupport`, `ActualAtomRecovery`, `IsolatedMassRecovery`
Theorem 6.1, Corollary 6.7	`ConstructedFiniteExhaustion`, `ConstructedGenerators`, `SectorIndependentFamily`, `ConstructedAlgebraicBasis`, `ConstructedNativeUpperBound`

Provenance and use of AI tools

The result on all crystalline measures (Theorem B, Part II) was obtained by GPT-5.6 Sol, working in Codex, on 15 September 2026. It was found mostly autonomously, using a custom auto-research harness that the author developed to make autonomous discovery more efficient. The Lean formalisation of its existence assertion (Theorem B(i), without the symmetry of Λ) was updated five days later, on 20 September 2026, and was submitted to Palomar on 30 September 2026 [Pal26b]. The results on Fourier quasicrystals (Part I, the main result, Section 4) were obtained with the same harness on 3 October 2026, and the formalisation of the existence statement of Theorem A was registered in Palomar on 8 October 2026 [Pal26c]. The paper was written by Claude Opus 5.5.

The author is not a professional mathematician. The author has reviewed the paper as carefully as possible but, as a non-specialist in this area, has limited ability to check it beyond the verification tooling described in Appendix E. The author takes full responsibility for its content.

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