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Twisted conjugacy in spherical Artin groups and their crystallographic quotients

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Abstract

A group has property R_∞ if every automorphism has infinitely many twisted conjugacy classes. Let A be an irreducible spherical Artin group of Coxeter rank at least two, with pure subgroup P. We classify property R_∞ for the crystallographic quotient $\Gamma = A/[P,P]$. If A has Coxeter rank at least three, then Γ has R_∞. In type $I_2(m)$, this holds exactly when $4 \mid m$, or when m is odd and $3 \mid m$. Together with the known dihedral Artin-group cases, the positive quotient results imply that every irreducible spherical Artin group of rank at least two has R_∞.

The positive proofs combine the rational representation theory of the holonomy action with integral lattice constraints and explicit fixed-vector witnesses. In the negative dihedral cases, we construct automorphisms compatible with the crystallographic extension. For type $I_2(5)$, we exhibit an automorphism with Reidemeister number 18, showing concretely that the quotient construction does not preserve R_∞. This example, including the identification with the presented Artin quotient and the exact Reidemeister count, is formalised in Lean. An alternative proof for seven Artin groups is formalised under explicitly stated external hypotheses.

AI transparency statement. The mathematical arguments, programs, Lean formalisation and manuscript were developed by AI systems under the author's direction. The scope of the formal verification is stated in Appendix C; details of AI use are given at the end of the paper.

1 Introduction

Let G be a group and $\varphi \in \mathrm{Aut}(G)$. Elements $x, y \in G$ are *φ-twisted conjugate* if $y = zx\varphi(z)^{-1}$ for some $z \in G$. The number $R(\varphi) \in \{1, 2, \ldots, \infty\}$ of classes is the *Reidemeister number*, and G has *property* R_∞ if $R(\varphi) = \infty$ for every $\varphi \in \mathrm{Aut}(G)$ [14, 13].

For an Artin group $A = A(X)$ with Coxeter group $W = W(X)$, the question "which Artin groups have R_∞?" is Problem 8.6 of the AIM problem list on Artin groups [1]. Known cases include:

- braid groups [12] and pure braid groups [7];
- the spherical types A_n, B_n, D_4, $I_2(m)$ [3];
- D_n for $n \geq 6$ [27, Thm. 1];
- many non-spherical types [20, 27, 25].

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The case D_5 was left as a conjecture in [27, §1]. It follows from Paris–Soroko [24, Lemma 4]: the kernel of the natural epimorphism $A[D_5] \to A[A_4]$ is fully characteristic, so the known braid-group case [12] and Lemma 2.1(a) imply property R_∞ for $A[D_5]$. This is a consequence of their lemma; we give an independent proof as part of our uniform quotient argument. For spherical types the known proofs use an explicit description of $\mathrm{Aut}(A)$ or $\mathrm{Aut}(A/Z(A))$. For $I_2(m)$ they use instead the virtual freeness of $A/Z(A)$ [3].

Theorem A. *Every irreducible Artin group of spherical type and rank at least two has property R_∞.*

Theorem A completes the irreducible spherical case of AIM Problem 8.6. Our proof treats all ranks at least three through their crystallographic quotients; it gives explicit proofs for $D_5, E_6, E_7, E_8, F_4, H_3, H_4$. Rank one must be excluded: $A(A_1) \cong \mathbb{Z}$, and inversion has Reidemeister number 2.

Let $P = \ker(A \to W)$. Marin [23, Thm. 2.1] proves that $A/[P,P]$ is crystallographic with holonomy $W/Z(W)$. Theorem B gives a separate classification of property R_∞ for these crystallographic quotients. It concerns all automorphisms of the quotient, including those that need not lift to A.

Theorem B. *Let X be irreducible spherical of rank at least two, and let $\Gamma_X = A(X)/[P,P]$. If X has rank at least three, then Γ_X has property R_∞. In rank two, writing $X = I_2(m)$ with $m \geq 3$, we have*

$$\Gamma_X \text{ has } R_\infty \quad \Longleftrightarrow \quad 4 \mid m \ \text{ or } \ (m \text{ odd and } 3 \mid m).$$

The proof is completed in Section 6. For the exceptional types it uses the full holonomy-automorphism calculations in Appendix B; the classical and dihedral families have uniform proofs. Since P and hence $[P,P]$ are characteristic [5], the positive cases of Theorem B imply R_∞ for the corresponding Artin groups. Together with the known dihedral Artin-group result [3], this proves Theorem A. Appendix C explains the different dependencies of the accompanying Lean development. The next result makes the failure of preservation concrete.

Proposition C. *The crystallographic quotient $A(I_2(5))/[P,P]$ admits an automorphism φ with $R(\varphi) = 18$. In particular, it does not have property R_∞.*

Thus $A(I_2(5))$ has R_∞, whereas its natural crystallographic quotient does not. Section 6.6 gives the affine automorphism, checks the precise translation lattice and extension, and counts twisted classes directly. The count takes account of torsion. The positive results use absolutely irreducible multiplicity-one constituents to restrict commuting lattice automorphisms to signs. In the dihedral example the nontrivial rational constituent instead has an infinite-order unit in its commutant; a separate extension calculation shows that this unit yields an admissible automorphism.

A close precedent is the work of Dekimpe–Gonçalves–Ocampo [8, Thms. 28, 31 and proof of Cor. 32], which proves R_∞ for $VB_n/[VP_n, VP_n]$ for $n = 3, 4$. Those are virtual-braid quotients; the spherical Artin quotients in Theorem B have different holonomy modules and extensions. Recent work also studies torus bundles arising from Bieberbach subgroups of classical braid quotients [22] and crystallographic quotients of framed braid groups [21]. Ocampo points to Nielsen and Reidemeister numbers of fibre-preserving maps as further questions for the torus bundles [22, §1]. The smallest positive case A_2 is already implicit in earlier classifications: its quotient is space group IT 161 [17, §5], which is absent from the complete list of three-dimensional groups without R_∞ in [9, Table 2].

The proof uses the finite holonomy action rather than a description of $\mathrm{Aut}(A)$. Representation-theoretic approaches to R_∞ occur in [7] and in the multiplicity-one criterion of Dekimpe–De Rock–Penninckx [6, Thm. 4.7]. We combine this mechanism with the crystallographic determinant criterion [10, Cor. 3.10], which permits torsion, and with explicit witnesses on stable or exchanged

constituents. Section 6.3 explains an alternative using Hiss–Lutowski [18, 19] for D_5, E_6, E_7, E_8, H_4. That odd-dimensional criterion does not supply the F_4 and H_3 cases.

The resulting classification depends on two separate features: the rational holonomy representation and the integral normaliser compatible with the crystallographic extension. Our positive witnesses and negative affine maps make both features explicit. The witness argument avoids the classification of finite simple groups, which enters the Hiss–Lutowski theorem; it still uses results on finite Coxeter groups and exact exhaustive computations.

Organisation of the proof. Section 2 identifies the reflection lattice and the precise translation lattice of Γ. Section 3 states a fixed-vector criterion for crystallographic groups. Sections 4 and 5 construct the exceptional and classical witnesses; the latter also controls character-preserving holonomy automorphisms in all classical ranks. Section 6 combines these inputs and deduces Theorem A before proving the dihedral classification and counting the twisted classes in Proposition C. Appendix A specifies the computational inputs, Appendix B controls the exceptional holonomy automorphisms, and Appendix C specifies the scope and dependencies of the formalisation.

Verification. Exact computational certificates accompany the exceptional proofs. A conditional Lean 4/Mathlib formalisation proves seven direct Artin cases through Franzsen–Howlett, using the same fixed-vector mechanism and finite certificates; Appendix C specifies its four external hypotheses. Lean also proves Proposition C for the presented Artin quotient, including its identification with the affine model and the exact count $R(\varphi) = 18$, without external mathematical hypotheses. The full quotient classification remains outside the formalisation.

2 The characteristic reflection lattice

Lemma 2.1. *Let $\psi \in \mathrm{Aut}(G)$.*

(a) If $K \trianglelefteq G$ with $\psi(K) = K$, then $R(\psi) \geq R(\bar{\psi})$.

(b) If $G = \mathbb{Z}^m$ and ψ is given by B, then $R(\psi) = |\operatorname{coker}(I - B)|$. This is ∞ if B has eigenvalue 1.

(c) $R(\iota_g \circ \psi) = R(\psi)$, where $\iota_g(x) = gxg^{-1}$.

(d) If $[G : H] < \infty$ and $\psi(H) = H$, then $R(\psi|_H) \leq [G : H]\, R(\psi)$.

Proof. (a) The projection maps classes onto classes.
(b) $y \sim x$ iff $y - x \in (I - B)\mathbb{Z}^m$.
(c) $x \mapsto xg$ maps $\iota_g\psi$-classes bijectively to ψ-classes.
(d) Fix $x \in H$ and write $G = \bigsqcup_i Hg_i$. If $y = zx\psi(z)^{-1} \in H$ with $z = hg_i$, then y is $\psi|_H$-conjugate to $g_i x\psi(g_i)^{-1} \in H$. Thus each G-twisted class meeting H contains at most $[G : H]$ twisted classes for the restriction. □

Notation. For a Coxeter graph with vertex set S:

- $A = \langle S \mid \Pi(s,t;m_{st}) = \Pi(t,s;m_{st})\ (s \neq t,\ m_{st} < \infty)\rangle$, where $\Pi(s,t;m) = sts\cdots$ has m letters;
- $W = A/\langle\langle s^2 \rangle\rangle$, with quotient map $\mu\colon a \mapsto \bar{a}$, and $P = \ker \mu$;
- $\mathcal{T}$ is the set of reflections, $\rho(w)(r) = wrw^{-1}$, and $\rho(w)$ also denotes the permutation matrix on $\mathbb{Z}^{\mathcal{T}}$; its permutation character is $\chi_{\mathcal{T}}(g) = \#\{r : g(r) = r\}$.

In spherical type, write w_0 for the longest Coxeter element and Δ for its standard Garside lift, obtained by replacing the simple reflections in a reduced expression by Artin generators.

Since $\mathcal{T}$ generates W, $\ker\rho = Z(W)$. For finite irreducible W, in its Euclidean reflection representation, a central orthogonal operator preserves each simple-root line; commuting with adjacent simple reflections forces the corresponding signs to agree. Connectivity therefore gives $Z(W) \subseteq \{\pm 1\}$. In rank at least three, $-r$ is not a reflection, so $r \mapsto \rho(r)$ is injective on $\mathcal{T}$.

Theorem 2.2 ([5, Thm. 1.4])**.** *If A is irreducible of spherical type, then P is characteristic in A.*

This was extended to reducible types in [15]. The proof in [5] is computer-assisted in several cases:

- for H_4, F_4, E_6, E_7, E_8 it uses GAP calculations [5, Prop. 5.4, Cor. 5.5];
- for H_3 it uses one GAP calculation [5, Lemma 5.1];
- among the classical types it uses GAP for B_4, B_6, D_4, D_6 [5, Lemmas 3.5, 4.4].

We have not re-run these calculations.

Lemma 2.3. *For any Coxeter group, $s \mapsto (e_s, s)$ defines a homomorphism $f\colon A \to \mathbb{Z}^{\mathcal{T}} \rtimes W$, with $(v,w)(v',w') = (v + w\cdot v', ww')$ and $w \cdot e_r = e_{wrw^{-1}}$.*

Proof. For $\Pi(s,t;m)$, the $\mathbb{Z}^{\mathcal{T}}$-component is $\sum_{k=1}^{m} e_{r_k}$ with $r_k = (st)^{k-1}s$. For $\Pi(t,s;m)$ it is $\sum_k e_{r'_k}$ with $r'_k = (st)^{m-k}s$. Both sets equal $\{(st)^j s : 0 \le j < m\}$. Compare [5, Lemma 3.7]. □

Proposition 2.4. *Let A be of spherical type. Then $f(P) = 2\mathbb{Z}^{\mathcal{T}}$, and f induces an isomorphism $H_1(P;\mathbb{Z}) \cong 2\mathbb{Z}^{\mathcal{T}}$ under which conjugation by a acts as the permutation $\rho(\bar{a})$.*

Proof. We have $f(as^2a^{-1}) = (2e_{\bar{a}s\bar{a}^{-1}}, 1)$, and these elements generate $f(P)$. Moreover $P \cong \pi_1$ of the complexified reflection-arrangement complement [2], whose integral H_1 is free on one meridian per hyperplane [26, §2.3]. Indeed, choose a spanning disk in the ambient complex vector space transverse to the hyperplanes and disjoint from their pairwise intersections. Removing small disks around the intersection points expresses the loop's homology class as a sum of meridians; winding numbers of defining linear forms show that these meridians are independent. Thus $H_1(P;\mathbb{Z})$ has rank $|\mathcal{T}|$, and the surjection onto $2\mathbb{Z}^{\mathcal{T}}$ is an isomorphism. □

Proposition 2.5. *For irreducible spherical A, the group $\Gamma = A/[P,P]$ has a faithful crystallographic action on $\mathbb{R}^{\mathcal{T}}$, with holonomy $G = \rho(W) = W/Z(W)$ and characteristic translation lattice*

$$L = \begin{cases} 2\mathbb{Z}^{\mathcal{T}}, & Z(W) = 1, \\ 2\mathbb{Z}^{\mathcal{T}} + \mathbb{Z}\mathbf{1}, & Z(W) \neq 1. \end{cases}$$

Every permutation of $\mathcal{T}$ preserves L.

Proof. By Proposition 2.4, $f(A) \cong \Gamma$. Give it the action $(v,w)\cdot x = v + \rho(w)x$. *The translation lattice.* Let $\Lambda = \{(v,w) \in \Gamma : w \in Z(W)\}$, the preimage of $Z(W)$ [23, Thm. 2.1]. Its elements act as translations by v. This action is faithful: elements with W-part 1 have even coordinates, while those with W-part w_0 (if $w_0 = -1$) have odd coordinates, since a reduced expression for w_0 encounters each positive root once, giving $f(\Delta) = (\mathbf{1}, w_0)$. So $\Lambda \cong L$, where $L = 2\mathbb{Z}^{\mathcal{T}}$ if $Z(W) = 1$ and $L = 2\mathbb{Z}^{\mathcal{T}} + \mathbb{Z}\mathbf{1}$ otherwise. Every permutation matrix preserves L. Since $\ker\rho = Z(W)$, the resulting affine action is faithful. Its full translation lattice has finite index, so the action is discrete and cocompact.

Λ *is characteristic.* $\Gamma/\Lambda = \rho(W)$ acts faithfully on L. Let N' be an abelian normal subgroup of finite index. For $n \in N'$, the linear part of n fixes the finite-index sublattice $N' \cap \Lambda$ pointwise, so it is trivial and $n \in \Lambda$. Thus Λ is the unique maximal abelian normal subgroup of finite index, hence characteristic.

This identifies the quotient of Marin [23, Thm. 2.1, Cor. 2.2]; see also [17, 11]. □

3 The fixed-vector criterion

Let L be a lattice, $V = L \otimes \mathbb{Q}$, and $G \leq GL(L)$ finite. A rational G-submodule $U \subseteq V$ is *good* if $U \otimes \mathbb{C}$ is irreducible and occurs with multiplicity one in $V \otimes \mathbb{C}$. Such a U is a full isotypic component. Our applications have $V = \mathbb{Q}^{\mathcal{T}}$.

Lemma 3.1. *If U is good and $N \in GL(L)$ commutes with G, then $NU = U$ and $N|_U = \pm 1$.*

Proof. Multiplicity one gives $NU = U$, and absolute irreducibility gives $N|_U = c \in \mathbb{Q}^\times$. Both N and N^{-1} preserve $U \cap L$. Applying them to a primitive lattice vector gives $c, c^{-1} \in \mathbb{Z}$, hence $c = \pm 1$. □

Lemma 3.2. *A nonzero invariant $U \subseteq \mathbb{Q}^{\mathcal{T}}$ is good in each of the following cases.*

(i) Every orbital matrix of $\rho(W)$ on $\mathcal{T} \times \mathcal{T}$ acts on U by a scalar.

(ii) $\langle \chi_U, \chi_U \rangle = \langle \chi_U, \chi_{\mathcal{T}} \rangle = 1$.

(iii) $U \otimes \mathbb{C}$ is irreducible and $\sum_r \dim(U \otimes \mathbb{C})^{\mathrm{Stab}(r)} = 1$, where r runs over representatives of the $\rho(W)$-orbits on $\mathcal{T}$ (Frobenius reciprocity).

Proof. (i) The orbital matrices span the commutant $\mathcal{C} = \mathrm{End}_{\mathbb{C}W}(\mathbb{C}^{\mathcal{T}}) = \prod_i M_{m_i}(\mathbb{C})$, where $\mathbb{C}^{\mathcal{T}} = \bigoplus_i V_i \otimes \mathbb{C}^{m_i}$. By hypothesis $\mathcal{C}$ maps $U_{\mathbb{C}} = U \otimes \mathbb{C}$ into itself and acts on it by scalars. The central idempotents $e_i \in \mathcal{C}$ act on $U_{\mathbb{C}}$ by 0 or 1 and sum to 1, so exactly one e_i acts as 1. Hence $U_{\mathbb{C}} \subseteq V_i \otimes \mathbb{C}^{m_i}$ meets only one isotypic component. Being W-invariant, $U_{\mathbb{C}} = V_i \otimes E$ for some $0 \neq E \subseteq \mathbb{C}^{m_i}$, and $M_{m_i}(\mathbb{C})$ preserves E and acts on it by scalars. This forces $m_i = 1$ and $E = \mathbb{C}$. (ii) and (iii) are standard. □

Theorem 3.3 (Fixed-vector criterion)**.** *Let Γ be crystallographic with translation lattice L and holonomy G. Suppose that every linear part D of an automorphism of Γ admits $\pi \in GL(L)$ inducing the same automorphism of G, such that one of the following holds.*

(i) (stable) *There is a good U with $\pi U = U$ such that, for each $\varepsilon = \pm 1$, some $g\pi$, $g \in G$, has eigenvalue ε on U.*

(ii) (swap) *There are distinct good constituents U_1, U_2 with $\pi U_1 = U_2$ and $\pi U_2 = U_1$ such that, for each $\varepsilon = \pm 1$, some $(g\pi)^2$ has an eigenvector in U_1 of eigenvalue ε.*

Then Γ has property R_∞.

Proof. Fix $\varphi \in \mathrm{Aut}(\Gamma)$, and let $D = \varphi|_L$. The operator $N = \pi^{-1}D$ centralises G, so Lemma 3.1 applies. In case (i), write $N|_U = c \in \{\pm 1\}$ and choose $g\pi v = cv$ with $v \neq 0$. Then $gDv = v$. In case (ii), write $N|_{U_i} = c_i \in \{\pm 1\}$ and choose $(g\pi)^2 v = c_1 c_2 v$ with $0 \neq v \in U_1$. Since $g\pi$ exchanges the two constituents, $(gD)^2 v = v$. The vector $v + gDv$ is nonzero, because its two terms lie in distinct isotypic components, and is fixed by gD.

Lift g to $\gamma \in \Gamma$. The restriction of $\iota_\gamma \varphi$ to the characteristic finite-index subgroup L has a nonzero rational fixed vector, hence an integral one. Lemma 2.1(b),(d),(c) gives $R(\varphi) = \infty$. □

Remark 3.4 (Relation to the crystallographic criteria)**.** The stable case uses the multiplicity-one device of [6, Thm. 4.7]. The determinant criterion [10, Cor. 3.10], stated in [9, Thm. 3.3], applies also with torsion: $R(\varphi) = \infty$ exactly when $\det(I - gD) = 0$ for some $g \in G$. The swap case covers exchanged constituents excluded by the non-conjugacy hypothesis of the stable criterion. Only holonomy automorphisms induced by the actual extension need be considered. Their image satisfies

$$\Theta_\Gamma \subseteq \Theta_\chi := \{\theta \in \mathrm{Aut}(G) : \chi_V \circ \theta = \chi_V\},$$

since D intertwines the representations. Controlling all of $\mathrm{Aut}(G)$ is a sufficient stronger condition.

Remark 3.5.

- If π works for θ, then $g\pi$ works for $\iota_g\theta$, since $\{hg\pi\} = \{h\pi\}$. So one θ per coset of $\mathrm{Inn}(\rho(W))$ suffices.
- For the identity representative of the inner class, take $\pi = 1$. Then (i) holds for any good U on which some reflection acts nontrivially. This is automatic for a nontrivial good U when all reflections are conjugate: the kernel of the action is normal, so if it contained one reflection it would contain all of them.
- If $r \mapsto \rho(r)$ is injective on $\mathcal{T}$ (true for finite irreducible W in rank ≥ 3, and for odd dihedral types) and $\theta(\rho(\mathcal{T})) = \rho(\mathcal{T})$, then $\pi(r) = \rho^{-1}\theta\rho(r)$ realises θ.

4 The exceptional types

Write Θ_X for the representatives of holonomy automorphisms modulo inner automorphisms listed in the table below. Appendix B proves that these exhaust the possibilities. We construct the good constituents and witnesses needed for each representative. For a displayed root-line permutation π_F normalising $G = \rho(W)$, write $\theta_F(g) = \pi_F g \pi_F^{-1}$ for the induced automorphism of G. This convention applies to $F = \Sigma, L, \Psi$ below.

Proposition 4.1 (E_6, E_7, E_8)**.** *Let $A_\perp$ be the adjacency matrix of the orthogonality graph on root lines.*

(a) $\rho(W)$ has exactly three orbitals on $\mathcal{T} \times \mathcal{T}$: equality, orthogonality and non-orthogonality.

(b) The graph is strongly regular with parameters $(36, 15, 6, 6)$, $(63, 30, 13, 15)$, $(120, 63, 30, 36)$. *Its nontrivial eigenvalues are* $3^{15}, (-3)^{20}$; $3^{35}, (-5)^{27}$; $3^{84}, (-9)^{35}$ *(multiplicities as exponents).*

(c) The two nontrivial eigenspaces are good, and every reflection acts nontrivially on them.

Proof. By the pointwise-stabiliser theorem [28, Thm. 1.12(c),(d), pp. 22–23], the stabiliser of a root, or of a pair of roots, is the Weyl group of the perpendicular root subsystem. The perpendicular subsystems are listed below. The root counts and ranks are finite computations, certified exactly (exact rational ranks) as proof certificates in Appendix A. Each pair (number of roots, rank) in the table occurs for exactly one simply-laced root system, irreducible or not, of that rank, so the counts determine the types. These are the classical facts that the centraliser of a reflection in $W(E_8)$, $W(E_7)$, $W(E_6)$ involves $W(E_7)$, $W(D_6)$, $W(A_5)$.

	$\alpha^\perp$	$\{\alpha, \gamma\}^\perp$, $\alpha \perp \gamma$	$\{\alpha, \beta\}^\perp$, $(\alpha, \beta) = 1$
E_6	A_5 (30, rank 5)	A_3 (12, rank 3)	A_2^2 (12, rank 4)
E_7	D_6 (60, rank 6)	D_4A_1 (26, rank 5)	A_5 (30, rank 5)
E_8	E_7 (126, rank 7)	D_6 (60, rank 6)	E_6 (72, rank 6)

W is transitive on roots. Inside $W(\alpha^\perp)$, the orbits are as follows.

- The orbit of an orthogonal root has size $720/24 = 30$, $23040/384 = 60$, $2903040/23040 = 126$, i.e. all of $\alpha^\perp$.
- For $(\alpha, \beta) = 1$, the orbit sizes are $720/36 = 20$, $23040/720 = 32$ and $2903040/51840 = 56$. Each orbit consists of all such roots.

Passing to lines gives (a). Then k, λ, μ are half the root counts in the table, and (b) follows from the standard formulas for strongly regular graphs.

By (a) the commutant is spanned by I, $A_\perp$ and J. These act by scalars on the nontrivial eigenspaces, which are rational because the eigenvalues are integers. So Lemma 3.2(i) applies, and Remark 3.5 gives the last assertion. □

Proposition 4.2 (H_3)**.** *(a) $\rho(W) \cong A_5$. The 15 root lines form 5 orthogonal triples (*frames*), which correspond to the Klein four-subgroups.*

(b) The frame module U (functions constant on frames, with sum 0) is a good 4-dimensional constituent. More precisely, writing irreducible A_5-modules by their dimensions,

$$\mathbb{Q}^{\mathcal{T}} \cong 1 \oplus U \oplus 5 \oplus 5.$$

(c) For $\Sigma = \varsigma \circ \sigma$, where σ is Galois conjugation $\sqrt{5} \mapsto -\sqrt{5}$ on coordinates and ς swaps the last two coordinates, $\Sigma\Phi = \Phi$ and $\Sigma s_\alpha \Sigma^{-1} = s_{\Sigma\alpha}$. The induced π_Σ is not inner on $\rho(W)$.

(d) For every outer $\theta \in \mathrm{Aut}(\rho(W))$, let π be its induced permutation of root lines, identified with the involutions of $\rho(W)$. There is $h \in \rho(W)$ such that $h\pi$ acts on the frames as a transposition. It then has eigenvalues $+1$ (multiplicity 3) and -1 (multiplicity 1) on U.

Proof. (a) Commuting reflections correspond to orthogonal lines, and the 15 involutions of A_5 lie in 5 Klein four-subgroups, permuted as A_5 permutes 5 points.

(b) U is the augmentation module of a 2-transitive action, hence absolutely irreducible. Since $\mathbb{Q}^{\mathcal{T}} = \mathrm{Ind}_{V_4}^{A_5} 1$, the multiplicity of an irreducible χ is $\dim \chi^{V_4}$. This equals $1, 0, 0, 1, 2$ for χ of degree $1, 3, 3', 4, 5$, which gives the displayed decomposition.

(c) Σ satisfies $(\Sigma x, \Sigma y) = \sigma(x, y)$. θ_Σ is not inner: on the rotation subgroup $W^+ \cong \rho(W)$ it Galois-conjugates the character of $\mathbb{R}^3$ on elements of order 5, while inner automorphisms preserve characters. This is also checked by machine.

(d) An outer automorphism of A_5 is conjugation by an odd permutation, and π permutes the frames accordingly. So $\{h\pi\}$ contains a transposition. In fact π_Σ itself is a transposition on frames. □

Proposition 4.3 (F_4)**.** *Let $L(x) = (x_1+x_2, x_1-x_2, x_3+x_4, x_3-x_4)$ and let π_L be its permutation of root lines. Conjugation by L induces a non-inner holonomy automorphism θ_L. Each class of 12 root lines (long, short) splits into 3 orthogonal frames of 4 lines. Let $U_{\mathrm{long}}, U_{\mathrm{short}}$ be the corresponding frame modules (sum zero). Then:*

(a) both are good and 2-dimensional;

(b) long reflections act on U_{long} as transpositions of frames and trivially on U_{short}, and vice versa;

(c) π_L exchanges U_{long} and U_{short}, and $\pi_L^2 = 1$;

(d) for a long simple reflection s, $(s\pi_L)^2 = s \cdot \pi_L s \pi_L$ has eigenvalue -1 on U_{short}, and π_L^2 gives eigenvalue $+1$.

Hence case (ii) of Theorem 3.3 holds for θ_L.

Proof. The similarity L exchanges the long and short root lines and satisfies $Ls_\alpha L^{-1} = s_{L\alpha}$, so it normalises W. It exchanges the two reflection classes, whereas inner automorphisms preserve them. It remains non-inner on $\rho(W)$: equality of two reflection images there forces equality of the reflections, since $-s$ has a three-dimensional (-1)-eigenspace and is not a reflection in rank four.

The long roots $\pm e_i \pm e_j$ form a D_4, whose frames correspond to the pairings $ij|kl$. The short lines e_i form one frame. The lines $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ form two frames, according to the parity of the number of minus signs. Each frame system is permuted through all of S_3, so each frame module is the standard module of S_3.

Multiplicity one (Lemma 3.2(iii)):

- the stabiliser of the long line $e_1 + e_2$ fixes one long frame and contains $s_{e_1 - e_2}$, which swaps the other two, so $\dim U_{\text{long}}^{\text{Stab}} = 1$;
- the stabiliser of the short line e_1 contains $\mathrm{Sym}\{2,3,4\}$, which induces S_3 on pairings, so $\dim U_{\text{long}}^{\text{Stab}} = 0$;
- U_{short} follows by applying π_L.

(b) Check on $s_{e_1 - e_2}$ and s_{e_1}, and use conjugacy.
(c) L maps orthogonal frames to orthogonal frames and long to short lines, and $L^2 = 2I$.
(d) $\pi_L s \pi_L$ is a short reflection. □

Lemma 4.4 (Icosian model of H_4)**.** *Halve the 120 roots of H_4 (Appendix A) and read the first coordinate as the real part. These form a group I of unit quaternions (the binary icosahedral group), and $K = I/\{\pm 1\} \cong A_5$. Identify $\mathcal{T}$ with K. Here $K \cong A_5$ is certified by a finite computation (Appendix A): K has 60 elements and 15 involutions, which lie in 5 Klein four-subgroups; conjugation on these gives an injective homomorphism $K \to S_5$ whose image consists of even permutations. Then:*

(a) the reflection in a unit root r is $x \mapsto -r\bar{x}r$, and it acts on K by $x \mapsto rx^{-1}r$;

(b) $\rho(W) = \{x \mapsto ax^{\pm 1}b : a, b \in K\} \cong (K \times K) \rtimes C_2$.

Proof. (a) For $|r| = 1$, $r\bar{x}r = 2(x,r)r - x$, using $\bar{x}r + \bar{r}x = 2(x,r)$.
(b) By (a), $\rho(W)$ is contained in the right-hand group, which has order $7200 = |W|/2 = |\rho(W)|$. The closure of I under multiplication is a finite computation, certified exactly in Appendix A. □

Proposition 4.5 (H_4)**.** *Let V be the 4-dimensional augmentation module of $K \cong A_5$ acting on its five Klein four-subgroups, and let $U \subseteq \mathbb{Q}^K$ be the span of the matrix coefficients $x \mapsto \mathrm{tr}(X\lambda_V(x))$ for $X \in \mathrm{End}_{\mathbb{Q}}(V)$, so $U \cong \mathrm{End}(V)$ has dimension 16.*

(a) U is good.

(b) Let $\Psi = \Omega \circ \sigma$, where σ is Galois conjugation and Ω is the rotation $j \mapsto k$, $k \mapsto -j$. Then Ψ is a ring automorphism of $\mathbb{H}_{\mathbb{Q}(\sqrt{5})}$ with $\Psi(I) = I$. The induced permutation π_Ψ normalises $\rho(W)$, preserves reflections, and is not inner.

(c) Every outer automorphism of $\rho(W)$ is realised by a permutation π for which some $h\pi$, $h \in \rho(W)$, acts on U as conjugation by a transposition. This has eigenvalues $+1$ (multiplicity 10) and -1 (multiplicity 6).

Proof. (a) By Wedderburn, $\mathbb{C}[K] = \bigoplus_{V'} V' \otimes V'^*$ as a $K \times K$-module, each summand irreducible and occurring once. The summand for $V_{\mathbb{C}}$ is $U \otimes \mathbb{C} \cong V_{\mathbb{C}} \otimes V_{\mathbb{C}}^*$, which is preserved by inversion. Since V is defined over $\mathbb{Q}$, so is $U \cong \mathrm{End}_{\mathbb{Q}}(V)$. Hence U is good for $\rho(W)$. (We do not need the remaining summands. Over $\mathbb{C}$ they are 1, $3 \otimes 3$, $3' \otimes 3'$ and $5 \otimes 5$; over $\mathbb{Q}$ the two 9-dimensional ones combine into one 18-dimensional summand.)

(b) Ψ commutes with quaternion conjugation, so $\Psi s_r \Psi^{-1} = s_{\Psi r}$. The equality $\Psi(I) = I$ is a finite computation, certified exactly in Appendix A. If θ_Ψ were inner, then $\pi_\Psi \in \rho(W)$, because the centraliser of $\rho(W)$ in $\mathrm{Sym}(K)$ is trivial. Indeed, a permutation commuting with all left and right translations is a translation by an element of $Z(K) = 1$. Since π_Ψ fixes 1, it would be $x \mapsto ax^{\pm 1}a^{-1}$. The sign $-$ is impossible, since π_Ψ is an automorphism of the non-abelian group K. The sign $+$ is impossible, since conjugation preserves real parts up to sign, whereas Ψ maps a quaternion with real part $\tau/2$ (it has order 10 in I and represents an element of order five in K) to one with real part $-\tau^{-1}/2$ (here $\tau = \frac{1+\sqrt{5}}{2}$).

(c) We argue in three steps.

(i) By Appendix B and Remark 3.5, we may take $\pi \in \rho(W)\pi_\Psi$. So $\{h\pi : h \in \rho(W)\} = \{h\pi_\Psi\}$.

(ii) By (b), $\pi_\Psi = \psi$ is an outer automorphism of K.

(iii) Since $\mathrm{Aut}(A_5) \cong S_5$, acting on the five Klein four-subgroups, ψ is conjugation by an odd permutation δ'. For $a \in K = A_5$, the element $h = (x \mapsto axa^{-1})$ lies in $\rho(W)$, and $h\psi$ is conjugation by $a\delta'$. Choose a with $a\delta' = \delta$ a transposition.

The action of K on V extends to S_5, with $\lambda_V(\delta)$ having eigenvalues $1, 1, 1, -1$. Since $h\pi_\Psi$ acts on $U \cong \mathrm{End}(V)$ as $X \mapsto \lambda_V(\delta) X \lambda_V(\delta)^{-1}$, the eigenvalue $+1$ has multiplicity $9 + 1$ and -1 has multiplicity 6. □

type	$\lvert\mathcal{T}\rvert$	$\lvert\rho(W)\rvert$	Θ_X	good constituent used (dim)	outer witness
E_6	36	51840	1	**SRG** eigenspace (15; also 20)	—
E_7	63	1451520	1	**SRG** eigenspace (35; also 27)	—
E_8	120	348364800	1	**SRG** eigenspace (84; also 35)	—
F_4	24	576	$\{1, \theta_L\}$	$\mathbf{U_{long}}, \mathbf{U_{short}}$ (2+2)	swap: π_L^2, $(s\pi_L)^2$
H_3	15	60	$\{1, \theta_\Sigma\}$	**frame** module (4)	eigen-multiplicities 3, 1
H_4	60	7200	$\{1, \theta_\Psi\}$	End(**V**) (16)	eigen-multiplicities 10, 6

Table 1: The exceptional-type data used for Theorems A and B. The dimensions agree with independent computer certificates (Appendix A).

5 The classical quotients

We first construct the constituents, then control the character-preserving holonomy automorphisms.

Proposition 5.1 (A_n, $n \geq 2$)**.** *The standard constituent $S^{(n,1)}$ of $\mathbb{Q}^{\mathcal{T}}$ is good. Every character-preserving automorphism of $\rho(W) = S_{n+1}$ is inner.*

Proof. Here $\rho(W) = S_{n+1}$ acts on the transpositions, i.e. on the 2-subsets. By Young's rule $\mathbb{Q}^{\mathcal{T}} = S^{(n+1)} \oplus S^{(n,1)} \oplus S^{(n-1,2)}$ (for $n = 2$ the last summand is absent). So $S^{(n,1)}$ is good. For $n+1 \neq 6$, $\mathrm{Aut}(S_{n+1}) = \mathrm{Inn}$. For $n+1 = 6$, an outer automorphism maps $(12) \mapsto (12)(34)(56)$, but $\chi_{\mathcal{T}}$ takes the values 7 and 3 there, so it cannot preserve $\chi_{\mathcal{T}}$. □

For B_n and D_n let $u_{ij} = e_{[e_i+e_j]} - e_{[e_i-e_j]}$ $(i < j)$ and $U^- = \mathrm{span}\{u_{ij}\}$. Let $E \leq W$ be the group of sign changes: all of them for B_n, the even ones for D_n.

Proposition 5.2 (B_n, $n \geq 3$; D_n, $n \geq 5$)**.** *U^- is good. It has witnesses for both signs in the inner class and, for D_n, in the class of the diagram involution.*

Proof. Weights. E acts on u_{ij} by $\chi_{ij}(\varepsilon) = \varepsilon_i\varepsilon_j$. These characters are nontrivial and pairwise distinct on E. For D_n this needs $n \geq 5$: on even sign changes $\chi_{ij} = \chi_{kl}$ iff $\varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_l$ is trivial or the product of all signs.

Goodness. E fixes $e_{[e_i+e_j]} + e_{[e_i-e_j]}$ and the short lines. So each χ_{ij} occurs once in $\mathbb{Q}^{\mathcal{T}}$. Since $S_n \leq W$ permutes the weights transitively, $U^- \otimes \mathbb{C}$ is irreducible.

Witnesses. For the inner class, $\pi = 1$ works: the reflection s_{e_n} (B_n), resp. $s_{e_{n-1}-e_n}$ (D_n), acts nontrivially on U^-. The diagram involution is realised by negating e_n, which acts on u_{ij} by -1 if $n \in \{i, j\}$ and by $+1$ otherwise. So case (i) holds with $g = 1$. □

Proposition 5.3 (D_4)**.** *In type D_4:*

- *U^- is reducible: $J(u_{ij}) = u_{kl}$ ($\{k, l\}$ complementary) is a non-scalar operator on U^- commuting with $W(D_4)$;*
- *the frame module U on the 3 orthogonal frames $\{12|34, 13|24, 14|23\}$ is good of dimension 2.*

Proof. For J: a permutation $\sigma \in S_4$ maps u_{ij} to $u_{\sigma i\, \sigma j}$ and commutes with taking complements; an even sign change ε acts by $\varepsilon_i\varepsilon_j$ on u_{ij}, and $\varepsilon_i\varepsilon_j = \varepsilon_k\varepsilon_l$ because $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1$. Hence J commutes with $W(D_4) = E \rtimes S_4$, and its ±1-eigenspaces are proper invariant subspaces of U^-. The E-fixed part of $\mathbb{Q}^{\mathcal{T}}$ is the permutation module on the 6 pairs, namely $1 + 3 + 2$ for S_4. Its 2-dimensional summand is U. On U^- the three nontrivial sign characters are permuted transitively, so every irreducible constituent there has dimension divisible by three. Thus U is good and is the unique rational irreducible constituent of dimension two. The image of W on U is S_3; a transposition has eigenvalues $1, -1$. □

5.1 Character-preserving holonomy automorphisms

The distinction between all holonomy automorphisms and those preserving the reflection character is useful in the classical families. Write $\chi_{\mathcal{T}}$ for the permutation character on root lines.

Lemma 5.4. *For W of type B_n, $n \geq 3$, or D_n, $n \geq 5$, every automorphism of $W/Z(W)$ preserving $\chi_{\mathcal{T}}$ preserves the images of reflections. In type B_n it preserves the short and long classes separately.*

Proof. An involution in $W/Z(W)$ lifts to a signed permutation w with $w^2 = I$ or $w^2 = -I$. In the second case w fixes no root line, since a real invariant line would give a real eigenvalue whose square is -1. Thus its character value is zero, unlike that of a reflection in the indicated ranks.

For $w^2 = I$, conjugate the signed transposed pairs to positive pairs and write (a, b, c) for the numbers of positive fixed coordinates, negative fixed coordinates, and transposed pairs; $a+b+2c = n$. Counting fixed root lines gives

$$\chi_D(w) = a(a-1) + b(b-1) + 2c, \qquad \chi_B(w) = a^2 + b^2 + 2c.$$

These counts do not depend on the signs on a transposed pair. In type D_n we require b even.

For a long reflection of B_n, the value is $(n-2)^2+2$. Equality is equivalent to

$$ab = 2n(1-c) + 2c^2 + c - 3.$$

For $c \geq 2$, the right-hand side is at most $-(2c-3)(c-1) < 0$, using $n \geq 2c$. For $c = 1$ it forces $ab = 0$, giving a reflection or its negative. For $c = 0$ it gives $(a-2)(b-2) = 1$, whose only solution with $n \geq 3$ is $(a,b) = (3,3)$ in B_6. That class has size $\binom{6}{3}/2 = 10$ in $W/Z(W)$, whereas the long-reflection class has size 30. An automorphism cannot exchange them. For a short reflection the value is $n^2 - 2n + 2$, and the equation is

$$ab = n(1-2c) + 2c^2 + c - 1.$$

Its right-hand side is negative for $c \geq 1$, $n \geq 3$, $n \geq 2c$ (for $c = 1$ it is $2-n$). For $c = 0$ it reduces to $(a-1)(b-1) = 0$, giving the short reflections and their negatives. The two reflection character values differ when $n \geq 3$.

For D_n the reflection value is $(n-2)(n-3)+2$, and the equation becomes

$$ab = 2n(1-c) + 2c^2 + 2c - 4.$$

For $c = 2$ its right-hand side is $8 - 2n < 0$; for $c \geq 3$ it is at most $-2(c-1)(c-2) < 0$. For $c = 1$ we again get $ab = 0$. For $c = 0$ we get $(a-2)(b-2) = 0$. The last possibility is represented, up to the central sign when available, by negating two coordinates. Its class has size $\binom{n}{2}$, whereas the reflection class has size $n(n-1)$. These are also their class sizes in the even signed-permutation group: each representative has an unused fixed coordinate, whose sign change is an odd element of its centraliser, so neither signed-permutation conjugacy class splits upon restriction to $W(D_n)$. For $n \geq 5$ passing to $W/Z(W)$ does not halve either size: a lift and its negative have different signed cycle types whenever both belong to W. This excludes the extra class and proves the lemma. □

Proposition 5.5. *The quotients Γ_{A_n} $(n \geq 2)$, Γ_{B_n} $(n \geq 3)$, and Γ_{D_n} $(n \geq 4)$ have property R_∞.*

Proof. Let D be the linear part of a quotient automorphism and θ its holonomy automorphism. It preserves $\chi_{\mathcal{T}}$, and every permutation of $\mathcal{T}$ preserves the translation lattice of Proposition 2.5.

Type A_n. The proof of Proposition 5.1 applies directly: the exceptional outer automorphism of S_6 does not preserve $\chi_{\mathcal{T}}$, so θ is inner, and the standard constituent $S^{(n,1)}$ supplies the two signs.

Types B_n, D_n, $n \geq 5$ in type D. By Lemma 5.4, θ preserves reflections. For D_n, Lemma B.1 makes it inner up to a diagram automorphism. The same root argument works for B_n, as follows. Choose a root β_i for each image of a simple reflection, with the same length as the corresponding simple root α_i; length is preserved by the lemma. Orders of products of two distinct reflections agree in W and $W/Z(W)$, because their plane action fixes a nonzero complement and hence no nontrivial power can equal $-I$. The Coxeter product orders therefore determine $|(\beta_i, \beta_j)|$. Choose signs along the diagram tree to match all simple-root inner products. The isometry $g\colon \alpha_i \mapsto \beta_i$ satisfies $gWg^{-1} \leq W$ and

$$g\Phi = (gWg^{-1})\{\beta_i\} \subseteq \Phi.$$

Since g is injective and Φ is finite, equality holds. In type B_n every such isometry is a signed permutation (it permutes the short roots), so it belongs to W. Thus θ is inner in type B_n. The U^- witnesses of Proposition 5.2, including its diagram-involution case for D_n, now apply through Theorem 3.3. This includes B_3 and B_4.

Type D_4. By Proposition 5.3, the frame constituent U is the unique rational irreducible constituent of dimension two. Hence $DU = U$. The holonomy image on U is S_3. Every automorphism

of S_3 is inner, so $D|_U = c\rho_U(g)$ for some g and $c \in \mathbb{Q}^\times$. Both operators preserve $U \cap L$, forcing $c = \pm 1$. A reflection of this S_3 has eigenvalues $1, -1$, giving the required fixed vector after a holonomy twist. □

6 The quotient classification

6.1 The six exceptional quotients

Theorem 6.1 (Six exceptional quotients)**.** *For $X \in \{E_6, E_7, E_8, F_4, H_3, H_4\}$, the crystallographic group $\Gamma = A(X)/[P, P]$ has property R_∞.*

Proof. By Proposition 2.5, the translation lattice L is characteristic and every permutation of $\mathcal{T}$ preserves it. An automorphism of Γ therefore induces $D \in GL(L)$ normalising $G = \rho(W)$. Appendix B realises each holonomy automorphism by a permutation π consisting of an inner permutation followed by the appropriate representative 1, π_Σ, π_L or π_Ψ from Section 4. For E_6, E_7, E_8, the holonomy automorphism is inner, and Proposition 4.1 supplies a good constituent with both signs. For H_3, H_4, Propositions 4.2 and 4.5 give stable constituents and witnesses. For F_4, Proposition 4.3 gives the stable inner case and the exchanged outer case. Theorem 3.3 completes the proof. □

The proof uses all automorphisms of $\rho(W)$, not only those induced from $\mathrm{Aut}(W)$. The calculations in Appendix B are therefore necessary inputs to this explicit proof.

6.2 Deduction of Theorem A

Theorem 6.1 and Proposition 5.5 prove the rank-at-least-three part of Theorem B. This already suffices for the Artin-group classification.

Proof of Theorem A. For rank at least three, P and hence $[P, P]$ are characteristic in A (Theorem 2.2). Every $\varphi \in \mathrm{Aut}(A)$ therefore induces $\bar{\varphi} \in \mathrm{Aut}(A/[P, P])$. Projection is surjective on twisted conjugacy classes, so

$$R(\varphi) \geq R(\bar{\varphi}) = \infty$$

by the positive quotient results. In rank two, the Artin-group result is [3, Thm. 1]. No negative dihedral construction or exact Reidemeister count is needed for this deduction. □

6.3 The odd-dimensional alternative

The multiplicity-one argument of Dekimpe–De Rock–Penninckx [6, Thm. 4.7], combined with Hiss–Lutowski [18, Def. 1.1.3, Thm. 1.1.5], completed in [19], provides an alternative for five positive cases. We give the reduction explicitly, since the flat-manifold statement in [6] assumes torsion-freeness, whereas our quotients may have torsion.

Suppose V has a nontrivial good constituent U of odd dimension d, and no other irreducible complex constituent has dimension d. Every $D \in N_{GL(L)}(G)$ preserves U: it permutes isotypic components and preserves their irreducible dimensions. Moreover $N_{GL(U\cap L)}(G|_U)$ is finite. Indeed, conjugation maps this normaliser to the finite group $\mathrm{Aut}(G|_U)$, with kernel $\{\pm I\}$ by absolute irreducibility and integrality. Hence $D|_U$ has finite order. The Hiss–Lutowski theorem supplies $g \in G$ with $gD|_U$ having eigenvalue one. The determinant criterion [9, Thm. 3.3], valid with torsion, proves R_∞. The uniqueness of the constituent also verifies the non-conjugacy hypothesis in [6, Thm. 4.7].

type	odd dimension d	explicit witnesses	odd-dimensional route
D_5	5	applies	applies
E_6	15	applies	applies
E_7	27	applies	applies
E_8	35	applies	applies
H_4	25	applies	applies
F_4, H_3	—	applies	not supplied by this criterion

Here are the constituent checks. For E_n, Proposition 4.1 gives the complete decomposition. For D_5, the sums $e_{[e_i+e_j]} + e_{[e_i-e_j]}$ form the permutation module on the 2-subsets of five letters, with irreducible dimensions $1, 4, 5$ by Young's rule; its complement is the good 10-dimensional U^- of Proposition 5.2. For H_4, the 5-dimensional irreducible representation of A_5 is defined over $\mathbb{Q}$: it is the augmentation module of the action on the six Sylow 5-subgroups. This action is doubly transitive, so the augmentation module is absolutely irreducible. Its matrix coefficients give a rational good constituent of dimension 25, by the argument of Proposition 4.5(a); the remaining complex dimensions are $1, 9, 9, 16$. Thus the required odd constituent is unique in each case. For H_3 the odd nontrivial constituent has multiplicity two, and for F_4 the explicit proof uses the exchanged 2-dimensional constituents. The table concerns this particular odd-dimensional route, not every possible application of [18, 19].

6.4 Compatibility with the extension

An infinite integral normaliser alone does not decide R_∞. The extension determines which of its elements actually occur as linear parts. We record the standard compatibility test (cf. [9, Algorithm 1]) in notation useful for the counterexample.

Proposition 6.2. *Write $\Gamma = \bigcup_{g\in G}(a_g + L, g) \le V_{\mathbb{R}} \rtimes G$, with $a_1 = 0$, and let $D \in N_{GL(L)}(G)$ and $\theta(g) = DgD^{-1}$. Conjugation by (d, D) is an automorphism of Γ if and only if*

$$Da_g + (I - \theta(g))d - a_{\theta(g)} \in L \qquad (g \in G). \tag{1}$$

For such an automorphism, R is finite if and only if $\det(I - gD) \neq 0$ for every $g \in G$.

Proof. Affine conjugation sends (a_g, g) to $(Da_g + (I - \theta(g))d, \theta(g))$. Condition (1) is exactly membership in the required coset. Since $DL = L$ and θ is onto, these cosets map onto all of Γ. The last assertion is [9, Thm. 3.3]. It is enough to check membership on a set of generators of Γ, together with $DL = L$ and surjectivity on G. □

Equivalently, D must preserve the extension class with the indicated change of holonomy action. This condition is separate from avoiding eigenvalue one in all the matrices gD.

6.5 The dihedral classification

Theorem 6.3. *For $m \ge 3$, the quotient $\Gamma_{I_2(m)}$ has R_∞ if and only if $4 \mid m$, or m is odd and $3 \mid m$.*

Proof. On the basis e_j, $j \in \mathbb{Z}/m\mathbb{Z}$, put $Ce_j = e_{j+1}$ and $Se_j = e_{-j}$. The actual quotient is generated affinely by (e_0, S) and (e_1, C^2S), with holonomy $G = \langle C^2, S\rangle$. Its translation lattice is $2\mathbb{Z}^m$ for odd m and $2\mathbb{Z}^m + \mathbb{Z}\mathbf{1}$ for even m.

Odd m divisible by three. The permutation module on m vertices is the regular module for the cyclic rotation subgroup. Its rational decomposition is $\mathbb{Q}^m = \bigoplus_{d\mid m} U_d$, where $U_d \cong \mathbb{Q}(\zeta_d)$, rotation acts by multiplication by ζ_d, and reflection acts by complex conjugation. Each U_d is rationally

irreducible and has dimension $\phi(d)$. Among odd divisors d, the equality $\phi(d) = 2$ holds only for $d = 3$. Thus for odd m divisible by three, the unique 2-dimensional constituent is U_3, whose holonomy image is S_3. Every rational normaliser preserves this constituent. As in the D_4 case of Proposition 5.5, its restriction is a holonomy element times a rational scalar: all automorphisms of S_3 are inner and the remaining commuting operator is scalar. Preservation of the constituent lattice by the operator and its inverse forces this scalar to be ± 1. The identity or a reflection then gives a fixed vector after a holonomy twist, so R_∞ holds.

m divisible by four. Write $m = 2q$, with q even. The only nontrivial rational 1-dimensional constituents are the multiplicity-one lines spanned by

$$u_0 = \sum_{j=0}^{q-1} (-1)^j e_{2j}, \qquad u_1 = \sum_{j=0}^{q-1} (-1)^j e_{2j+1}.$$

Indeed, on each parity orbit C^2 is a q-cycle, so its 1- and (-1)-eigenspaces are lines; the 1-eigenspaces are trivial G-modules. Both displayed lines have character -1 on C^2, and their characters on S are 1 and -1, respectively. Any $D \in N_{GL(L)}(G)$ preserves or swaps these lines. If it preserves them, it acts on each by ± 1, and a holonomy twist gives eigenvalue one. If it swaps them, D^2 acts by a common scalar $c = \pm 1$, by preservation of the one-dimensional lattices. On their sum, $(gD)^2$ acts by $c\chi_0(g)\chi_1(g)$. Choose $g = 1$ or S to make this scalar 1. Then $v + gDv$ is a nonzero fixed vector for any nonzero v on the first line.

Odd m not divisible by three. Put $D = I - C - C^{-1}$ and $d = e_0 + e_1$. The identity $(I + C)(I - C + C^2) = I + C^3$ shows $\det(I - C + C^2) = 1$: both C and C^3 are m-cycles and $\det(I + C) = \det(I + C^3) = 2$ for odd m. Since $D = -C^{-1}(I - C + C^2)$, it follows that $\det D = -1$. Thus D preserves $L = 2\mathbb{Z}^m$ and commutes with G. Moreover,

$$(D - I)e_0 + (I - S)d = -2e_{m-1}, \qquad (D - I)e_1 + (I - C^2S)d = -2e_2.$$

Proposition 6.2 therefore gives an automorphism of the actual quotient. The eigenvalues of D are $1 - 2\cos(2\pi j/m)$. The only one of absolute value one is -1 on the constant line: an eigenvalue 1 would require $4 \mid m$. As D commutes with the finite-order operators $g \in G$, their restrictions to its eigenspaces have roots of unity as eigenvalues. Hence gD cannot have eigenvalue one off the constant line, and on that line it acts by -1. All the determinants $\det(I - gD)$ are nonzero, proving finite R.

$m \equiv 2 \pmod 4$. Write $m = 2q$ with q odd. The two parity orbits of reflections are G-equivariantly identified by $j \mapsto j + q$. In the ordered pairs (e_j, e_{j+q}), j even, let

$$D = I_q \otimes \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \qquad d = 0.$$

This operator commutes with G, is integral unimodular, and is congruent to I modulo two. The same is true of D^{-1}, so both preserve $L = 2\mathbb{Z}^m + \mathbb{Z}\mathbf{1}$. The generator corrections $(D-I)e_0, (D-I)e_1$ lie in $2\mathbb{Z}^m$, proving extension compatibility. The eigenvalues of D are $3 \pm 2\sqrt{2}$, neither of modulus one. As in the preceding case, no gD has eigenvalue one. This supplies an automorphism with finite R and completes the classification. □

Proof of Theorem B. Theorem 6.1 and Proposition 5.5 cover every irreducible spherical type of rank at least three. Theorem 6.3 covers rank two, including $A_2 = I_2(3)$ and $B_2 = I_2(4)$. □

6.6 A dihedral counterexample

Proposition 6.4 (Proposition C)**.** *The group* $\Gamma = A(I_2(5))/[P,P]$ *has an automorphism* φ *with* $R(\varphi) = 18$.

Proof. Index the reflections by $j \in \mathbb{Z}/5\mathbb{Z}$, with the two simple reflections indexed by $0, 1$. On the reflection lattice set

$$Ce_j = e_{j+1}, \qquad Se_j = e_{-j}, \qquad G = \langle C, S\rangle.$$

The two simple reflections act as S and C^2S. By Proposition 2.4, the actual quotient is

$$\Gamma = \langle (e_0, S), (e_1, C^2S)\rangle \leq \mathbb{Z}^5 \rtimes G, \qquad L = 2\mathbb{Z}^5.$$

Here $Z(W) = 1$, so L is the full translation lattice. Put

$$D = I - C - C^{-1}, \qquad d = e_0 + e_1, \qquad \varphi(v, g) = (Dv + (I - g)d, g).$$

We have $D^{-1} = I - C^2 - C^{-2}$ and D commutes with G, so $DL = L$. For the two generators,

$$(D - I)e_0 + (I - S)d = -2e_4, \qquad (D - I)e_1 + (I - C^2S)d = -2e_2.$$

These corrections lie in L. Thus conjugation by (d, D) preserves Γ and is onto, by Proposition 6.2. This checks the extension as well as the integral lattice.

The characteristic polynomial of D is $(x + 1)(x^2 - 3x + 1)^2$, and direct integer determinants give

g	1	C^k $(1 \leq k \leq 4)$	C^kS $(0 \leq k \leq 4)$
$\det(I - gD)$	2	22	-10.

In particular all determinants are nonzero, which already proves $R(\varphi) < \infty$.

For completeness we count the classes without a torsion-free assumption. Since φ induces the identity on G, twisted classes project to ordinary conjugacy classes of G, represented by $1, C, C^2, S$. Choose lifts $(a_g, g) \in \Gamma$. Twisted conjugation by translations identifies the fibre above g with

$$Q_g = \mathbb{Z}^5/(I - gD)\mathbb{Z}^5, \qquad x \longleftrightarrow (a_g + 2x, g).$$

Every twisted class can first be moved into one of these four representative fibres. Two elements of the same fibre can then be identified only by a conjugator whose holonomy centralises g. Thus the remaining identifications after translation-twisted conjugacy are precisely the action of $C_G(g)$. For a lift (b_h, h) with $hg = gh$, that action is

$$x \longmapsto hx + \tfrac{1}{2}(b_h + ha_g - gDb_h - g(I - h)d - a_g) \pmod{(I - gD)\mathbb{Z}^5}. \tag{2}$$

The following table gives lifts, cyclic coordinates k on Q_g, and the induced actions of centraliser generators. All vectors are ordered from 0 to 4.

g	a_g	Q_g	centraliser action	orbits
1	$(0,0,0,0,0)$	$\mathbb{Z}/2$	$C : k \mapsto k$, $S : k \mapsto k + 1$	1
C	$(1,0,1,1,1)$	$\mathbb{Z}/22$	$C : k \mapsto 5k + 6$	6
C^2	$(0,1,1,0,0)$	$\mathbb{Z}/22$	$C : k \mapsto 3k + 4$	6
S	$(1,0,0,0,0)$	$\mathbb{Z}/10$	$S : k \mapsto 1 - k$	5

Here $a_{C^2} = e_1 + e_2$ comes from the product of the second generator and the first, and a_C is the parity representative of its cube. The cyclic coordinates are $k = \ell_g x$ modulo the stated order, with

$$\ell_1 = (1,1,1,1,1), \quad \ell_C = (1,5,3,15,9), \quad \ell_{C^2} = (1,3,9,5,15), \quad \ell_S = (5,1,7,3,9).$$

Each functional is onto and annihilates $(I - gD)\mathbb{Z}^5$ modulo $|\det(I - gD)|$, so it identifies Q_g with the indicated cyclic group. Substitution in (2) gives the table. Each order-five action on $\mathbb{Z}/22$ has two fixed points and four orbits of length five; the involution $k \mapsto 1-k$ on $\mathbb{Z}/10$ has no fixed points. Consequently

$$R(\varphi) = 1 + 6 + 6 + 5 = 18.$$

□

Remark 6.5 (Torsion and the determinant average)**.** The element $((-1,0,-1,1,1), C)$ belongs to Γ and has order five: its translation vector differs from a_C by an element of $2\mathbb{Z}^5$, and its five translates sum to zero. Thus the Bieberbach averaging formula [9, Thm. 3.4] cannot be invoked here. In fact its determinant average is $(2 + 4 \cdot 22 + 5 \cdot 10)/10 = 14$, whereas $R(\varphi) = 18$. The exact checker `code/indep/dihedral_quotient.py` verifies the preceding calculation and independently enumerates all 140 translation-twisted classes: there are 13 holonomy orbits of size 10 and 5 of size 2. Equivalently, the identity fixes all 140 classes, each nonidentity rotation fixes 10, and each reflection fixes none. Burnside's lemma gives

$$R(\varphi) = \frac{140 + 4 \cdot 10 + 5 \cdot 0}{10} = 18.$$

The holonomy action is not free, which explains why dividing 140 by 10 gives the wrong answer.

Remark 6.6 (The representation-theoretic distinction)**.** For $I_2(5)$, $\mathbb{Q}^5 = \mathbb{Q}\mathbf{1} \oplus U$, where U is rationally irreducible but splits into two conjugate 2-dimensional constituents over $\mathbb{Q}(\sqrt{5})$. Its commuting algebra is $\mathbb{Q}(\sqrt{5})$. The operator D acts by -1 on the trivial line and by the conjugate units $(3 \pm \sqrt{5})/2$ on those two constituents. These units have infinite order. The displayed generator corrections are what make this particular unit compatible with the extension; the existence of units alone would not establish Proposition C.

Further question. For the negative dihedral cases in Theorem B, determine the Reidemeister spectrum $\{R(\varphi) : \varphi \in \mathrm{Aut}(\Gamma_{I_2(m)})\}$. Proposition C shows that 18 belongs to this spectrum for $m = 5$. The extension-compatible integral normaliser and the torsion-sensitive class count both matter for this more detailed question.

A Computational certificates and cross-checks

The repository for the computational sources and Lean development is

`https://github.com/jasondavies/spherical-artin-rinf`.

The Python certificates and Lean development used in this paper are fixed by commit `ab2165a10c66` of that repository. That revision also fixes their dependency versions and includes the presented-quotient theorem in `lean/ArtinRinf/Dihedral5/Quotient.lean`. The command `python3 scripts/verify.py --clean` runs the certificates and builds the paper and Lean development. Dependencies are pinned; each run records source, PDF and log hashes in its own directory under `verification/runs/`. The complete submitted snapshot, including the manuscript and PDF, is

identified separately by the SHA-256 checksum of the accompanying review archive. The command `python3 scripts/package_review.py` creates that archive only when its inputs, PDF and logs match the latest successful full verification run. Its manifest records the source-and-PDF snapshot digest, base commit and verification run identifier. Programs, commands and expected outputs are listed in `code/README.md` of the accompanying repository. We distinguish *proof certificates* from *corroboration*.

- Proof certificates are finite computations that the written proofs rely on. They use exact arithmetic over $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{5})$, including exact rational row reduction for ranks, and the programs assert the expected values (a nonzero exit status on any discrepancy).
- Corroboration consists of independent recomputations, some of them numerical.

Coordinates. For E_8 we use Bourbaki's simple roots in $\mathbb{R}^8$ with basis $e_1, \dots, e_8$:

$$\alpha_1 = \tfrac{1}{2}(e_1 + e_8) - \tfrac{1}{2}(e_2 + \cdots + e_7), \qquad \alpha_2 = e_1 + e_2, \qquad \alpha_k = e_{k-1} - e_{k-2} \quad (3 \le k \le 8).$$

E_7 and E_6 use $\alpha_1, \dots, \alpha_7$, resp. $\alpha_1, \dots, \alpha_6$, in $\mathbb{R}^8$. The other types:

- F_4: $e_2 - e_3,\ e_3 - e_4,\ e_4,\ \tfrac{1}{2}(e_1 - e_2 - e_3 - e_4)$;
- A_n, B_n, D_n: standard simple roots;
- H_3: roots $\pm 2e_i$ and the even permutations of $(\pm\tau, \pm 1, \pm\tau^{-1})$, with $\tau = \frac{1+\sqrt{5}}{2}$;
- H_4: roots $\pm 2e_i$, $(\pm 1, \pm 1, \pm 1, \pm 1)$ and the even permutations of $(\pm\tau, \pm 1, \pm\tau^{-1}, 0)$.

Proof certificates.

(1) Proposition 4.1: perpendicular root counts and their exact ranks (used to identify the subsystems), together with the SRG parameters, the three orbitals and the eigenspace dimensions.

(2) Propositions 4.2–4.5 and Lemma 4.4:

- frames, and character norms $\langle \chi, \chi \rangle = \langle \chi, \chi\tau \rangle = 1$;
- the images of W on frames (A_5, S_3);
- the $(+1, -1)$-eigenspace dimensions of π_Σ on the H_3 frame constituent: $(3, 1)$; of π_L^2 and $(s\pi_L)^2$ on U_{short} in type F_4, for a long simple reflection s: $(2, 0)$ and $(1, 1)$, respectively; and of $h\pi_\Psi$ on the 16-dimensional H_4 coefficient constituent, with the inner adjustment h from Proposition 4.5(c): $(10, 6)$;
- the closure of I, the reflection formula, and $\rho(W) = \{x \mapsto ax^{\pm 1}b\}$;
- $K \cong A_5$: $|K| = 60$, 15 involutions, 5 Klein four-subgroups, and conjugation is an injective homomorphism into A_5; also $Z(K) = 1$;
- $\Psi(I) = I$, Ψ multiplicative, and the real-part computation in Proposition 4.5(b).

(3) The exceptional cases of Theorem B use the full holonomy-automorphism certificates in Appendix B.

Corroboration.

(4) Finite checks supporting the algebraic proofs of Propositions 5.2 and 5.3:

- the weight argument for B_3–B_6 and D_4–D_7, including its failure for D_4;
- the operator J and the frame module of D_4;
- witnesses for all six graph automorphisms of D_4.

(5) Proposition C has the direct calculation of Section 6.6; its exact checker verifies the extension, all determinants, the cyclic quotient table, and the full twisted-class count.

(6) Independent certificates for the six exceptional types:

- commutant certificates;
- character certificates written as fresh code;
- witnesses for abstractly computed outer automorphisms.

They give the same dimensions $15, 20$; $27, 35$; $35, 84$; 4; $2 + 2$; $16, 25$.

(7) The criterion (for automorphisms preserving $\chi_{\mathcal{T}}$) checked uniformly for A_2–A_7, B_3–B_7, D_4–D_7, odd $I_2(m)$ with $m \leq 21$, and the exceptional types other than E_8.

(8) Additional independent checkers. The one for D_n, F_4, H_3 and parts of the one for H_4 use numerical linear algebra, so they are numerical corroboration only; the exact constituent and classical checks above provide their counterparts.

(9) The uniform arguments in Lemma 5.4 and Theorem 6.3 have an independent finite cross-check in `code/indep/quotient_families.py`. It counts fixed root lines directly, checks the exceptional character coincidences and conjugacy-class sizes, and verifies the positive constituents or the affine maps and all adjusted determinants in the dihedral cases. The unbounded statements are proved in the text, not inferred from these finite runs.

B Automorphisms of the holonomy groups

For the six exceptional cases of Theorem 6.1, we control all holonomy automorphisms, a sufficient strengthening of the character-preserving restriction. The following computations are proof certificates for these cases of Theorem B, as well as corroboration of [16, Thm. 31]:

- $\mathrm{Out}(\rho(W)) = 1$ for E_6, E_7, E_8;
- $|\mathrm{Out}(\rho(W))| = 2$ for H_3, H_4, F_4.

These statements were obtained as follows.

- *Involution classes and Coxeter tuples*, by exact enumeration, for H_3, H_4, F_4, E_6, E_7. No involution class other than the reflection classes has a reflection-class size, so every automorphism preserves $\rho(\mathcal{T})$; then the Inn-orbits of generating Coxeter tuples of reflections are counted.
- For H_3, F_4, an independent generator-image enumeration also counts all automorphisms without first assuming reflection preservation.
- For E_7 and E_8, the reflection-preserving stage was re-verified with Schreier–Sims stabiliser chains, and it also follows from Lemma B.1 below. For E_7 the reflection-preservation stage therefore rests on the single exact enumeration above.
- For E_8, Proposition B.2 below gives a proof by hand.

The search in `code/indep/out_check.py` enumerates the generated permutation group to closure and partitions all its involutions by conjugation with the simple generators. At each stage of the Coxeter-tuple search it considers every reflection satisfying the pair-product order constraints, up to the stabiliser of the preceding tuple; at a leaf it checks generation of the whole group. Thus no automorphism image tuple is omitted. Conversely, a generating tuple defines a surjection $W \to \rho(W)$ whose kernel has order $|Z(W)| \leq 2$. If this order is two, the kernel is central, so it equals $Z(W)$; the tuple therefore induces an automorphism of $\rho(W)$. The checked injectivity of the reflection images identifies tuple stabilisers with point stabilisers, so the resulting orbits are exactly the inner-automorphism orbits.

The separate computation $\mathrm{Out}(\rho(W(D_5))) = 1$ is optional corroboration. The main proof covers D_5 uniformly in Proposition 5.5 and does not use it. Write Gr for the diagram automorphism group, acting by isometries that permute the chosen simple roots.

Lemma B.1. *Let Φ be an irreducible simply-laced root system of rank ≥ 3 (so its diagram is a tree). Then every $\theta \in \mathrm{Aut}(\rho(W))$ with $\theta\rho(\mathcal{T}) = \rho(\mathcal{T})$ is inner up to a graph automorphism.*

Proof. Write $\theta\rho(s_i) = \rho(s_{\beta_i})$; this uses that ρ is injective on $\mathcal{T}$. Orders of products agree in W and in $\rho(W)$. Indeed, for roots $\beta \neq \pm\gamma$ the element $s_\beta s_\gamma$ is a plane rotation fixing a complement of dimension ≥ 1, so no nonidentity power of it lies in $Z(W) \subseteq \{\pm 1\}$; in particular $s_i s_j \neq -1$. Hence $(\beta_i, \beta_j) = 0$ or ± 1 according to the diagram. Choose the signs along the tree. Then $g\colon \alpha_i \mapsto \beta_i$ is an isometry, and $g\Phi = (gWg^{-1})\{\beta_i\} \subseteq \Phi$. So $g\{\alpha_i\}$ is a simple system. As W is simply transitive on simple systems, $g = w\gamma$ with $w \in W$ and γ an isometry permuting $\{\alpha_i\}$, i.e. $g \in W \rtimes \mathrm{Gr} = \mathrm{Aut}(\Phi)$ [28, §1.8, p. 15]. Compare [16, Thm. 18, Cor. 19]. □

Proposition B.2. *Every automorphism of $W(E_8)/\{\pm 1\}$ is inner.*

Proof. $G^+ = W^+/\{\pm 1\}$ is the commutator subgroup, of index 2. An involution outside G^+ lifts to w with $w^2 = 1$ and $\det w = -1$; if $w^2 = -1$ then $\det w = 1$. By [4, Lemma 5, p. 5] (provable via Steinberg's theorem), w is a product of $k = \dim\ker(w+1)$ reflections in mutually orthogonal roots, and k is odd. Replacing w by $-w$, we may take $k \in \{1, 3\}$. For $k = 3$, the (-1)-eigenspace of w is spanned by the three roots, and contains only the 6 roots $\pm\beta_i$: a root $\sum c_i\beta_i$ has $2c_i = (\cdot, \beta_i) \in \{0, \pm 1, \pm 2\}$ and $\sum c_i^2 = 1$. So w determines the triple, and $-w$ (with $k = 5$) is not of this form.

The products for $k = 3$ form one class of size $240 \cdot 126 \cdot 60/48 = 37800$, by transitivity along $E_8 \supset E_7 \supset D_6$. So the reflections form the unique involution conjugacy class of size 120 outside G^+. Hence every automorphism preserves $\rho(\mathcal{T})$, and Lemma B.1 applies. □

C Scope of the formalisation

The Lean development proves the direct Artin cases D_5, E_6, E_7, E_8, F_4, H_3 and H_4 using the same reflection-lattice and fixed-vector mechanism, with separately certified constituent bases and witnesses. Its automorphism reduction follows a different route: Franzsen–Howlett [16, Thm. 31] express Coxeter automorphisms as reflection-preserving automorphisms followed by central twists. Central twists disappear on $W/Z(W)$. Reflection-tuple search certificates then control the induced holonomy automorphisms. This route avoids the stronger full-holonomy computations required by Theorem B. The complete mathematical account is supplied with the Lean sources in `lean/DIRECT_PROOF.md`.

The Artin-group criterion used there applies the fixed-vector mechanism to the induced map on P^{ab}: a holonomy twist with eigenvalue one gives infinitely many twisted classes on P^{ab}, hence on P, and then on A by Lemma 2.1. The finite part of the induced-automorphism reduction is checked

by orbit–stabiliser search certificates for all seven types. The kernel also checks the presented Artin group, its affine map, the equivariant pure abelianisation, the scalar commutants and the sign/swap witnesses. The constituent bases are computed certificates; the geometric constructions in Section 4 are not formalised.

Each of the seven final Franzsen–Howlett theorems has four external hypotheses: finiteness of W; characteristicity of P [5]; the abstract group isomorphism $P^{ab} \cong \mathbb{Z}^{|\mathcal{T}|}$ [2]; and the Franzsen–Howlett decomposition [16, Thm. 31]. These cited results are hypotheses, not formalised theorems of the development. For H_3 and F_4 there are also theorems replacing the last hypothesis by kernel-checked full-holonomy enumerations. The source contains no `sorry`, custom `axiom` or `native_decide`; finite certificates are checked by the Lean kernel. The scalar and stable/swap arguments are separated into a general integral-lattice criterion. For Proposition C, the kernel checks the parity extension, its generation by the two affine Artin images, the automorphism and its inverse, and the reduction of all twisted classes to 140 finite states with exactly 18 orbits. A Schreier calculation using ten coset representatives proves that five explicit meridians generate the pure subgroup. Their affine images are the five standard basis translations, so the pure abelianisation maps isomorphically to $\mathbb{Z}^5$. This proves that the affine map has kernel $[P, P]$ and identifies the presented quotient with the affine model. The transferred Reidemeister-number theorem therefore has no external mathematical hypotheses; in particular, it does not assume the arrangement-homology rank statement. The full quotient classification remains outside the formalisation.

Declaration of AI-generated research and authorship

This paper was developed through an AI-driven research process directed by Jason Davies. Anthropic's Claude and OpenAI's Codex contributed to the mathematical arguments, computational verification, Lean formalisation, and preparation and revision of the manuscript.

The Lean kernel checks the seven direct Artin-group results under the external hypotheses stated in Appendix C, as well as Proposition C for the presented Artin quotient without external mathematical hypotheses. The full quotient classification remains outside the formalisation. Its written proof is accompanied by the exact computational checks described in Appendix A. These checks do not replace mathematical review of the written arguments. The author takes responsibility for the content of the paper and the accuracy of this disclosure.

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