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SMOOTH FUNCTIONS WITH ISOLATED CRITICAL VALUES AND SYSTEMS OF FIBERS

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Abstract. In the 2020s, Marzantowicz and Michalak have introduced the notion of system of smooth hypersurfaces and discussed Morse functions compatible with them: the hypersurfaces are regular level sets of the functions. This is a kind of fundamental theory on Morse functions and related differential topology and geometric topology.

We extend this to a certain case: we discuss systems of *regular* level sets and *critical* level sets (*regular fibers* and *singular* fibers) of smooth functions with isolated critical values.

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1. Introduction.

Theory of *Morse* functions is of classical and strong fundamental theory in mathematics, especially in geometry. Fundamental aspects of Morse functions are still immature and developing. We discuss related differential topological theory and geometric topological one. More explicitly, we discuss extending theory [11, The 4th section] of Marzantowicz and Michalak and [11, Theorem 4.15].

They have introduced the notion of system of hypersurfaces of manifolds and Morse function compatible with this. The hypersurfaces are unions of some connected components of *regular* level sets of the functions. They studied construction and classifications of Morse functions compatible with this. This is explicitly presented later, as Theorem 1.

For Morse functions, refer to the textbooks such as [14, 15] and for related singularity theory, refer to the textbook [6], for example. We also refer to the paper [9] of the author for notions related to Morse functions such as their Reeb (*di*)*graphs*, which are spaces realized as the quotient spaces of the manifolds of the domains, whose underlying sets consist of all connected coponents of all level sets of the functions, and which have been fundamental and strong tools in understanding the manifolds ([16]). For a level set of a smooth function, we say it is *critical* (*regular*) if it contains some critical points (resp. no critical point).

In the present paper, we extend Theorem 1. For this, we explain fundamental theory of *fibers* of smooth functions and maps. A *fiber* is, in short a local smooth map (germ) along a single point in the manifold of the target and if the single point is a singular point of the map, it is a *singular* fiber, where it is a *regular* fiber in the case the single point is not a singular point of the map. The notion is discussed systematically, recently, in [19], in the story of so-called singularity theory of smooth functions and maps, especially global theory and related differential topology and

geometric topology. Related global theory is first discussed in [3, 17, 18] and original higher dimensional studies of Morse functions have started in the 1950s, in [21, 22], for example.

We consider regular fibers of smooth functions, and singular ones of smooth functions with critical values being discrete, or *critical* fibers of them. Such a singular fiber is also said to be an *isolated critical fiber*. One of our main result, Theorem 2, is an extension for the case of systems of such fibers.

In the next section, we review [11] ([11, The 4th section]). We also review fundamental notions such as *Reeb space* and *Reeb (di)graph* of a smooth function. We explain fibers again and exhibit Theorem 2 with its proof, in the third section. The fourth section is for our additional remark.

2. The theory of Marzantowicz and Michalak, systems of hypersurfaces and Morse functions such that the hyper surfaces are their regular level sets.

We review [11] ([11, The 4th section]). Let M be a compact and connected manifold of dimension $m \geq 1$ whose boundary ∂M may be non-empty. We consider a finite family $\mathcal{F} := \{\overrightarrow{F_j}\}_{j=1}^{l}$ of $l \geq 0$ mutually disjoint hypersurfaces, smooth compact submanifolds of dimension $m-1$ with no boundary of $M - \partial M$ whose underlying manifolds are F_j, whose normal bundles $F_j \times \mathbb{R}$ are trivial, oriented, and oriented naturally by the products $F_j \times D^1 = F_j \times \{-1 \leq t \leq 1\}$ and D^1. We also identify F_j with $F_j \times D^1$ by the map mapping $x \in F_j$ to $(x, 0) \in F_j \times D^j$. The family $\mathcal{F}$ is called a *hypersurface system of* M and it is *normal* if each F_j is connected and $M - \sqcup_{j=1}^{l} F_j$ is also connected.

We review the *Reeb (di)graph* $\overrightarrow{\mathcal{R}_f}$ of a Morse function $f : M \to \mathbb{R}$ on a compact and connected manifold M of dimension $m \geq 1$ whose boundary ∂M may be non-empty, contains no critical point of f, and is a disjoint union of connected components of several level sets of f. More generally, we consider a smooth function with finitely many critical values and with the same properties on the boundary ∂M. We call such a smooth function an *admissible* function. Reeb (di)graphs are classical topological and combinatorial objects and they are also important tools in theory of Morse functions. See [16] as a pioneering paper. We have an equivalence relation $\sim_f$ by the following rule, on M: $p_1 \sim_f p_2$ if and only if p_1 and p_2 are in a same connected component of a same level set of f. The quotient space $\mathcal{R}_f := M/\sim_f$ is the *Reeb space* of f. We have the quotient map $q_f : M \to \mathcal{R}_f$ and the unique continuous function $\bar{f} : \mathcal{R}_f \to \mathbb{R}$ satisfying $f = \bar{f} \circ q_f$. According to [18], defining vertices by the following rule, this is the graph $\mathcal{R}_f$ and the *Reeb graph* of f: a point $v \in \mathcal{R}_f$ is its vertex if and only if either the following holds.

- ${q_f}^{-1}(v)$ contains some critical point of f.
- ${q_f}^{-1}(v)$ is a connected component of ∂M.

By the following rule, it is also a digraph: an edge e_{v_1,v_2} is oriented as an edge departing from v_1 and entering v_2 if and only if $\bar{f}(v_1) < \bar{f}(v_2)$. The digraph $\overrightarrow{\mathcal{R}_f}$ is the *Reeb digraph* of f.

If we consider a circle-valued function $f : M \to S^1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid {x_1}^2 + {x_2}^2 = 1\}$ whose singularities are as above and S^1 is oriented in the counter clockwise way, instead, then we can discuss samely and have the Reeb digraph $\mathcal{R}_f$.

We explain several terminologies for a (finite and connected) digraph $\overrightarrow{G}$. This is *acyclic* if it has no oriented cycle. A vertex v of $\overrightarrow{G}$ is a *source* (*sink*) if every edge incident to it departs from (enters) v. A vertex which is not a source or sink is a *transit* vertex. Two digraphs are *equivalent* if and only if there eixsts a homeomorphism mapping the vertex set of a digraph onto that of the other digraph and preserving the orientations. We call such a homeomorphism an *isomorphism* between the digraphs. Two digraphs are *weakly equivalent* if and only if there exists a homeomorphism mapping the vertex set of a digraph onto that of the other digraph and preserving the orientations. We call such a homeomorphism a *weak isomorphism* between the digraphs.

A vertex v of a digraph $\overrightarrow{G}$ incident to exactly $n_{v,1}$ edges entering v and $n_{v,2}$ edges departing from v is said to be an $(n_{v,1}, n_{v,2})$-*vertex* of $\overrightarrow{G}$.

Remark 1. For the graph structure of $\mathcal{R}_f$, see [16] and see also [8] for the Morse function case. In addition, see [10] for the *Morse-Bott* function case, where we do not consider Morse-Bott functions here: for Morse-Bott functions, see [1] for example.

Definition 1. We consider a quintuple $(\overrightarrow{G}, V_{\overrightarrow{G},1,1,1}, V_{\overrightarrow{G},1,1,2}, V_{\overrightarrow{G},1,2,1}, V_{\overrightarrow{G},1,2,2})$ of a finite and connected digraph $\overrightarrow{G}$ which has at least one edge and is not acyclic, and four disjoint sets of vertices of degree 1, satisfying the following.

- Every source and every sink of the digraph must be of degree 1.
- The disjoint union $V_{\overrightarrow{G},1,1,1} \sqcup V_{\overrightarrow{G},1,1,2} \sqcup V_{\overrightarrow{G},1,2,1} \sqcup V_{\overrightarrow{G},1,2,2}$ is the set of all vertices of degree 1.
- $V_{\overrightarrow{G},1,1,1} \sqcup V_{\overrightarrow{G},1,1,2}$ and $V_{\overrightarrow{G},1,2,1} \sqcup V_{\overrightarrow{G},1,2,2}$ are non-empty.

We call such a quintuple an *M-digraph*. Two M-digraphs $(\overrightarrow{G_1}, V_{\overrightarrow{G_1},1,1,1}, V_{\overrightarrow{G_1},1,1,2}, V_{\overrightarrow{G_1},1,2,1}, V_{\overrightarrow{G_1},1,2,2})$ and $(\overrightarrow{G_2}, V_{\overrightarrow{G_2},1,1,1}, V_{\overrightarrow{G_2},1,1,2}, V_{\overrightarrow{G_2},1,2,1}, V_{\overrightarrow{G_2},1,2,2})$ are *equivalent* (*weakly-equivalent*) if there exists an isomorphism (resp. weak isomorphism) between $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ and mapping $V_{\overrightarrow{G_1},1,i_1,i_2}$ onto $V_{\overrightarrow{G_2},1,i_1,i_2}$.

This is a fundamental theorem from [12]. See also [11], for example.

Proposition 1. (1) *For each M-digraph* $(\overrightarrow{G}, V_{\overrightarrow{G},1,1,1}, V_{\overrightarrow{G},1,1,2}, V_{\overrightarrow{G},1,2,1}, V_{\overrightarrow{G},1,2,2})$ *and any integer* $m \geq 2$*, there exists some admissible Morse function or circle-valued admissibe Morse function* $f : M \to N$ *with* $N := \mathbb{R}$ *or* $N := S^1$*, on some compact and connected manifold* M *of dimension* m *with the decomposition* $\partial_1 M \sqcup \partial_2 M$ *of* ∂M *as follows.*

(a) $\overrightarrow{\mathcal{R}_f}$ *is equivalent to* $\overrightarrow{G}$ *through an isomorphism* $\phi_{\overrightarrow{\mathcal{R}_f},\overrightarrow{G}} : \overrightarrow{\mathcal{R}_f} \to \overrightarrow{G}$.

(b) ${q_f}^{-1}({\phi_{\overrightarrow{\mathcal{R}_f},\overrightarrow{G}}}^{-1}(V_{\overrightarrow{G},1,1,1} \sqcup V_{\overrightarrow{G},1,1,2}))$ *is the set of all points yielding local minimum of* f. ${q_f}^{-1}({\phi_{\overrightarrow{\mathcal{R}_f},\overrightarrow{G}}}^{-1}(V_{\overrightarrow{G},1,2,1} \sqcup V_{\overrightarrow{G},1,2,2}))$ *is the set of all points yielding local maximum of* f.

(c) ${q_f}^{-1}({\phi_{\overrightarrow{\mathcal{R}_f},\overrightarrow{G}}}^{-1}(V_{\overrightarrow{G},1,i,2})) = \partial_i M$ *for* $i = 1, 2$.

(2) *Conversely, for each admissible Morse function or a circle-valued one* $f : M \to N$ *with* $N := \mathbb{R}$ *or* $\mathbb{S}^1$ *on an* m*-dimensional compact and connected manifold* M *with the decomposition* $\partial_1 M \sqcup \partial_2 M$ *of* ∂M *and with* $m \geq 2$*, we have an M-digraph* $(\overrightarrow{\mathcal{R}_f}, q_f(C_{\mathrm{l,m}}), q_f(\partial_1 M), q_f(C_{\mathrm{l,M}}), q_f(\partial_2 M))$*, where* $C_{\mathrm{l,m}}$ *and* $C_{\mathrm{l,M}}$ *are the sets of all critical points of index* 0 *and that of index* m.

We review a main theorem from [11] ([11, Theorem 4.15]).

Theorem 1. *Let M be a compact and connected manifold of dimension $m \geq 2$ whose boundary ∂M may be non-empty. Consider the situation as above and suppose that $\mathcal{F} := \{\overrightarrow{F_j}\}_{j=1}^{l}$ is a normal hypersurface system of M.*

Under this, for each M-digraph $(\overrightarrow{G}, V_{\overrightarrow{G},1,1,1}, V_{\overrightarrow{G},1,1,2}, V_{\overrightarrow{G},1,2,1}, V_{\overrightarrow{G},1,2,2})$ such that the 1st Betti number $b_1(G)$ of the underlying graph G is l, we have an admissible Morse function $f : M \to \mathbb{R}$ as in Proposition 1 (1) satisfying the following and with the condition (1a) replaced by the weaker condition $\phi_{\overrightarrow{\mathcal{R}_f}, \overrightarrow{G}} : \overrightarrow{\mathcal{R}_f} \to \overrightarrow{G}$ being a weak isomorphism.

1. *Each F_i is a connected component of a regular level set of f.*
2. *Each $\overrightarrow{F_i}$, equipped with the oriented normal bundle $F_i \times D^1$ as above, is mapped into the interior of an edge of $\overrightarrow{\mathcal{R}_f}$. The orientation of its image induced canonically from the orientations of the bundle $F_i \times D^1$ and D^1 is same as the orientation induced from the original orientation of $\overrightarrow{\mathcal{R}_f}$.*

Definition 2. Let $(\overrightarrow{G}, V_{\overrightarrow{G},1,1,1}, V_{\overrightarrow{G},1,1,2}, V_{\overrightarrow{G},1,2,1}, V_{\overrightarrow{G},1,2,2})$ be an M-digraph.

1. If we have an admissible smooth function or a circle-valued one $f : M \to N$ with $N := \mathbb{R}$ or $N := S^1$ as in Proposition 1 (1), then $(\overrightarrow{G}, V_{\overrightarrow{G},1,1,1}, V_{\overrightarrow{G},1,1,2}, V_{\overrightarrow{G},1,2,1}, V_{\overrightarrow{G},1,2,2})$ is a *realization* of f.
2. In (1) above, if we replace the isomorphism $\phi_{\overrightarrow{\mathcal{R}_f}, \overrightarrow{G}} : \overrightarrow{\mathcal{R}_f} \to \overrightarrow{G}$ in the condition (1a) of Proposition 1 by a weak homeomorphism, then $(\overrightarrow{G}, V_{\overrightarrow{G},1,1,1}, V_{\overrightarrow{G},1,1,2}, V_{\overrightarrow{G},1,2,1}, V_{\overrightarrow{G},1,2,2})$ is a *weak-realization* of f.

Hereafter, let us use $\overrightarrow{G}$ for $(\overrightarrow{G}, V_{\overrightarrow{G},1,1,1}, V_{\overrightarrow{G},1,1,2}, V_{\overrightarrow{G},1,2,1}, V_{\overrightarrow{G},1,2,2})$, unless we need to clarify the sets $V_{\overrightarrow{G},1,i_1,i_2}$.

3. Our main result, Theorem 2, and an application, Theorem 3.

For a smooth map $f : M \to N$, (its) *fiber* (around $p \in N$) means a local map (map germ) represented as the restriction of f to a small open nighborhood $U_{f^{-1}(p)}$ of $f^{-1}(p)$ in M. If p is a regular value, then it is a *regular* (*singular*) fiber. Here, we only consider cases of smooth functions whose critical values are isolated and for such a function, a singular fiber is also an *isolated critical fiber*.

For two smooth maps $f_1 : M_1 \to N_1$ and $f_2 : M_2 \to N_2$, a fiber of f_1 around $p_1 \in N_1$ and that of f_2 around $p_2 \in N$ are defined to be *equivalent* if there exists a local (smooth) diffeomorphism (a germ of a (smooth) diffeomorphism) Φ_{p_1,p_2} mapping ${f_1}^{-1}(p_1)$ onto ${f_2}^{-1}(p_2)$ and one ϕ_{p_1,p_2} mapping p_1 to p_2 with the local relation $\phi_{p_1,p_2} \circ f_1 = f_2 \circ \Phi_{p_1,p_2}$. If we can choose ϕ_{p_1,p_2} as a diffeomorphism preserving the orientations, then the two fibers are defined to be *oriented-equivalent*.

In Theorem 1, each element of $\mathcal{F}$ is identified with a regular fiber of f around a point of $\mathbb{R}$. In situations of Theorem 1, an element $\overrightarrow{F_j} := \overrightarrow{F_{1,j_1}}$ of $\mathcal{F} := \mathcal{F}_1$ and an element $\overrightarrow{F_j} := \overrightarrow{F_{2,j_2}}$ of $\mathcal{F} := \mathcal{F}_2$ are equivalent if and only if the underlying manifolds F_{1,j_1} and F_{2,j_2} are diffeomorphic. They are also oriented-equivalent if and only if the orientations of D^1 of the normal bundles $F_1 \times D^1$ and $F_2 \times D^1$ are same. This is, essentially, implicit function theorem.

Hereafter, we consider fibers for *proper* smooth functions $f : M \to \mathbb{R}$ or $f : M \to S^1$ on manifolds with no boundary around $p \in \mathbb{R}$ such that the level sets $f^{-1}(p)$ are connected. For the isolated critical fiber case, the fiber is identified with an admissible smooth function $f|_{\overrightarrow{V_{f^{-1}(p),p}}}$ on a small connected and compact neighborhood $\overrightarrow{V_{f^{-1}(p),p}} \subset M$ of $f^{-1}(p)$ in M which has exactly one critical value. Furthermore, the function is regarded to be an admissible smooth function whose Reeb digraph $\overrightarrow{R_{f|_{\overrightarrow{V_{f^{-1}(p),p}}}}}$ has at most one transit vertex.

Definition 3. We consider a finite family $\mathcal{F} := \{\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}\}_{j=1}^{l}$ of $l \geq 0$ mutually disjoint, smooth compact and connected submanifolds of dimension m of $M - \partial M$ each of whose underlying manifolds is $\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}$ and regarded to be a regular or an isolated critical fiber of a smooth function $f_j : \overrightarrow{V_{jf^{-1}(p_j),p_j}} \to \mathbb{R}$ around $p_j \in \mathbb{R}$ or $p_j \in S^1$. The family $\mathcal{F}$ is called a *regular or isolated critical fiber system of* M, or an *RICF system* of M. If for an RICF system $\mathcal{F} := \{\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}\}_{j=1}^{l}$ of M, $M - \sqcup_{j=1}^{l}\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}$ is connected, then $\mathcal{F}$ is *normal*.

The definition is regarded to be an extension compatible with the case all $\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}$ being regular fibers.

Theorem 2. *Let M be a compact and connected manifold of dimension $m \geq 2$ whose boundary ∂M may be non-empty. Suppose that $\mathcal{F} := \{\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}\}_{j=1}^{l}$ $(l > 0)$ is a normal RICF system of M as above, satisfying the following.*

- *Let $0 \leq l_{\mathrm{r}} \leq l$ be the number of all regular fibers $\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}$ (of f_j around p_j) in $\mathcal{F} := \{\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}\}_{j=1}^{l}$.*
- *$\overrightarrow{\mathcal{R}_{f_j|_{\overrightarrow{V_{f_j{}^{-1}(p),p}}}}}$ has one transit vertex v_j which is also an $(n_{v_j,1}, n_{v_j,2})$-vertex if and only if $\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}$ is not a regular fiber. In the case $\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}$ is a regular fiber, let $n_{v_j,1} = n_{v_j,2} = 1$.*

Under this, for each M-digraph $(\overrightarrow{G}, V_{\overrightarrow{G},1,1,1}, V_{\overrightarrow{G},1,1,2}, V_{\overrightarrow{G},1,2,1}, V_{\overrightarrow{G},1,2,2})$ such that the 1st Bett number $b_1(G)$ of the underlying graph G is $b_1(G) = \Sigma_{j=1}^{l}(n_{v_j,1} + n_{v_j,2} - 1)$ and that the following are satisfied, we have its weak-realization $f : M \to \mathbb{R}$ which is Morse outside $M - \sqcup_{j=1}^{l}\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}$.

(1) *There exists a family of $l - l_{\mathrm{r}}$ distinct vertices $V_{\overrightarrow{G},\mathcal{F},\mathrm{s}}$ of $\overrightarrow{G}$ each of which is an $(n_{v_j,1}, n_{v_j,2})$-vertex for an isolated critical fiber $\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}$. Furthermore, distinct members of the set $V_{\overrightarrow{G},\mathcal{F},\mathrm{s}}$ correspond to mutually distinct isolated critical fibers $\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}$ in $\mathcal{F}$.*
(2) *$G - V_{\overrightarrow{G},\mathcal{F},\mathrm{s}}$ is connected.*

We have Proposition 2 from the proof of [11, Theorem 4.15] or Theorem 1.

Proposition 2. *Let M be a compact and connected manifold of dimension $m \geq 2$ with the decomposition $\partial_1 M \sqcup \partial_2 M$ of ∂M each $\partial_i M$ of which consists of exactly n_i connected components. Let $(\overrightarrow{G}, V_{\overrightarrow{G},1,1,1}, V_{\overrightarrow{G},1,1,2}, V_{\overrightarrow{G},1,2,1}, V_{\overrightarrow{G},1,2,2})$ be an arbitrary M-digraph such that the size of $V_{\overrightarrow{G},1,i,2}$ is n_i and that the underlying graph G is a tree. Let $\{\partial_{i,j'} M\}_{j'=1}^{n_i}$ be an arbitrary label by the first n_i positive integers on*

the set of all connected components of $\partial_i M$ and an arbitrary one $\{v_{1,i,2,j'}\}_{j'=1}^{n_i}$ on the set of all vertices of $V_{\overrightarrow{G},1,i,2}$, we have its weak-realization $f : M \to \mathbb{R}$ which is Morse and satisfies $q_f(\partial_{i,j'} M) = \{v_{1,i,2,j'}\}$ for every i and j'.

A proof of Theorem 2. We first (re)construct a local map onto a small connected and compact neighborhood $N(v_{\overrightarrow{G},\mathcal{F},\mathrm{s},j})$ of each vertex $v_{\overrightarrow{G},\mathcal{F},\mathrm{s},j} \in V_{\overrightarrow{G},\mathcal{F},\mathrm{s}}$. More precisely, we reconstruct an isolated critical fiber oriented-equivalent to the fiber $\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}$ such that the quotient map onto the Reeb digraph is regarded to be a surjection onto the connected and compact neihborhood $N(v_{\overrightarrow{G},\mathcal{F},\mathrm{s},j})$ of the vertex $v_{\overrightarrow{G},\mathcal{F},\mathrm{s},j}$.

By the assumption that $G - V_{\overrightarrow{G},\mathcal{F},\mathrm{s}}$ is connected and that $b_1(G) = \Sigma_{j=1}^{l}(n_{v_j,1} + n_{v_j,2} - 1)$, from G, we can remove the interiors of small connected and compact neghborhoods $N(v_{\overrightarrow{G},\mathcal{F},\mathrm{s},j})$ of $v_{\overrightarrow{G},\mathcal{F},\mathrm{s},j} \in V_{\overrightarrow{G},\mathcal{F},\mathrm{s}}$, and we can choose and remove $l - l_{\mathrm{r}}$ additional distinct points which are not vertices of G and their small open neighborhoods, homeomorphic to $\{t \mid -1 < t < 1\}$. The resulting remaining set $G_{V_{\overrightarrow{G},\mathcal{F},\mathrm{s}},l-l_{\mathrm{r}}} \subset G - V_{\overrightarrow{G},\mathcal{F},\mathrm{s}}$ is compact and still connected. The resulting set is also a tree and canonically an M-digraph.

We reconstruct a regular fiber oriented-equivalent to the fiber $\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}$ such that the quotient map onto the Reeb digraph is regarded to be a surjection onto a small connected and compact neighborhood of each of the $l - l_{\mathrm{r}}$ additional points, which is homeomorphic to $\{t \mid -1 \leq t \leq 1\}$ and whose interior is regarded to be the open neighborhood removed above.

For the M-digraph $\overrightarrow{G_{V_{\overrightarrow{G},\mathcal{F},\mathrm{s}},l-l_{\mathrm{r}}}} := G_{V_{\overrightarrow{G},\mathcal{F},\mathrm{s}},l-l_{\mathrm{r}}}$ and the manifold $M - \sqcup_{j=1}^{l} \overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}$, we apply Proposition 2 and construct a weak-realization of the M-digraph $\overrightarrow{G_{V_{\overrightarrow{G},\mathcal{F},\mathrm{s}},l-l_{\mathrm{r}}}}$ on $M - \sqcup_{j=1}^{l} \overrightarrow{V_{f_j{}^{-1}(p_j),p_j}}$ being also Morse.

By gluing the regular fibers and the isolated critical fibers above and this weak-realization of the M-digraph $\overrightarrow{G_{V_{\overrightarrow{G},\mathcal{F},\mathrm{s}},l-l_{\mathrm{r}}}}$ together suitably and naturally, we have a desired function. This completes the proof. □

Definition 4. In Theorem 2, we call the resulting function $f : M \to \mathbb{R}$ an $\mathcal{F}$-*admissible* function. Each element $\overrightarrow{V_{f_j{}^{-1}(p_j),p_j}} \in \mathcal{F}$ is said to be an $(n_{v_j,1}, n_{v_j,2})$-*fiber*.

We present another application.

Let $G_{\mathrm{F},l}$ denote the free group of rank $l \geq 0$.

We discuss the epimorphism represented as the composition of $q_{f*} : \pi_1(M) \to \pi_1(\mathcal{R}_f)$ with some isomorphism $\phi_{M,\mathcal{F},q_f}$ for an admissible Morse function $f : M \to \mathbb{R}$ as in Theorem 1. This is first discussed in [11, The sections 2 and 4].

Let M be a compact and connected manifold of dimension $m \geq 2$ whose boundary ∂M may be non-empty. Suppose that $\mathcal{F} := \{\overrightarrow{F_j}\}_{j=1}^{l}$ $(l > 0)$ is a normal hypersurface system of M. We have an epimorphism $\phi_{M,\mathcal{F}} : \pi_1(M) \to G_{\mathrm{F},l}$ up to isomorphisms of the groups of the targets, by a certain rule and it is represented as the composition of $q_{f*} : \pi_1(M) \to \pi_1(\mathcal{R}_f)$ with some isomorphism $\phi_{M,\mathcal{F},q_f}$ for an admissible Morse function $f : M \to \mathbb{R}$ as in Theorem 1.

We consider a circle-valued case $f : M \to S^1$ instead and discuss $q_{f*} : \pi_1(M) \to \pi_1(\mathcal{R}_f)$. Related to Theorem 2,we also consider the case of a certain simple general RICF system $\mathcal{F}$.

Related to this, we review studies on Reeb digraphs of circle-valued Morse-Bott functions. [4, 5] are on Reeb digraphs of circle-valued Morse-Bott functions on closed surfaces and closed manifolds whose dimensions are greater than 2. In studies of existence and construction of *non-singular extensions* of smooth functions (Morse functions), in other words, studies on extending a smooth function to another smooth function with no critical point on a compact manifold whose boundary is the original manifold such as [2], Reeb graphs are fundamental and important tools and in [7], a certain circle-valued case is discussed.

We go back to our problem. We prove Theorem 3.

Theorem 3. *Let M be a compact and connected manifold of dimension $m \geq 2$ whose boundary ∂M may be non-empty. Suppose that $\{\overrightarrow{F}\}$ is a normal RICF system of M consisting of the single element $\overrightarrow{F}$ which is an $(n_{v,1}, n_{v,2})$-fiber. Suppose that there exists an $\mathcal{F}$-admissible function $f_0 : M \to \mathbb{R}$. For each positive integer l', we have a smooth admissible function $f_{0,l'} : M \to S^1$ satisfying the following.*

1. *The two M-digraphs $\overrightarrow{\mathcal{R}_{f_0}}$ and $\overrightarrow{\mathcal{R}_{f_{0,l'}}}$ are equivalent.*
2. *$q_{f_{0,l'}*} : \pi_1(M) \to \pi_1(\mathcal{R}_{f_{0,l'}})$ is an epimorphism onto $\pi_1(\mathcal{R}_{f_{0,l'}})$, isomorphic to $G_{\mathrm{F},n_{v,1}+n_{v,2}-1}$.*
3. *There exists a family $\{g_{M,j'}\}_{j'=1}^{n_{v,1}+n_{v,2}-1} \subset \pi_1(M)$ of exactly $n_{v,1}+n_{v,2}-1$ elements satisfying the following.*
   (a) *The subgroup generated by the set of these elements is mapped onto $\pi_1(\mathcal{R}_{f_{0,l'}})$.*
   (b) *Under a natural identification of $\pi_1(S^1)$ and the integer group $\mathbb{Z}$, the value of $f_{0,l'*} : \pi_1(M) \to \pi_1(S^1)$ at $g_{M,j'}$ is l'.*

Proof. A main ingredient is removing (the interior of) $\overrightarrow{F}$ from M and regluing this in a bit different way. From the construction, we have the presented properties.

We embed the target space $\mathbb{R}$ of the function $f_0 : M \to \mathbb{R}$ into S^1 first and we have a circle-valued smooth admissible function. The image of the function $f_0|_{\overrightarrow{F}}$ is represented by the form $\{(\cos u, \sin u) \mid u_0 \leq u \leq u_1\}$ with $0 \leq u_0 < u_1 < 2\pi$. We remove this. Roughly, we add a rotation of l' times along S^1 in the counter clockwise way and we reglue this revised map into S^1.

More precisely, we rescale the function in the following way. A point which is by the original function $f_0|_{\overrightarrow{F}}$ mapped to u $(u_0 \leq u \leq u_1)$ is changed to a point which is mapped to $u_0 + (l'+1)(u - u_0)$.

The construction and the theory [11, The sections 2 and 4] complete the proof. □

With a little effort, this theorem can be presented in a more general way and for simplicity, we present the theorem in this way.

4. Conflict of interest and Data availability.

The author is a researcher at Osaka Central Advanced Mathematical Institute (OCAMI researcher). this is supported by MEXT Promotion of Distinctive Joint Research Center Program JPMXP0723833165. He does not work there, where he thanks the members for such an opportunity.

Data availability.

We have generated no data other than the present file. For the mathematical content of the present paper, we do not use AI, essentially.

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