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Definability and Lifting in Model Subgroupoids

Zhenyu Sun

Abstract

Restriction to a wide subgroupoid of a model groupoid can introduce invariant relations, maps, and objects. For open topological groupoids, a map between restricted objects lifts exactly when its graph is invariant under the full groupoid, and a specified quotient presentation lifts exactly when its kernel relation is invariant. Under coherent restriction, these criteria recover definable maps and coherent quotient presentations. For finite continuous actions of a group with an open subgroup of finite index, coinduction identifies extensions with invariant subsets on which evaluation is bijective.

For a finite group G and a normal subgroup N, we classify extensions of finite N-sets by normalizer extensions of their orbit types. Multiplicities must be constant on conjugacy orbits of types; within each such block, the liftable multiplicities form the additive monoid generated by indices of subgroups over which the normalizer extension splits. The same subgroup data determine the groupoid of lifts and its isomorphism-class and mass generating functions. This classification also applies to finite continuous actions of profinite groups with an open normal subgroup. For a fixed action, a centralizer extension gives the splitting obstruction, with an H^2 obstruction and an H^1 classification when its kernel is abelian. The center of $\mathrm{SL}_2(\mathbb{F}_5)$ gives the monoid $\langle 12, 20\rangle$; Heisenberg groups over fields of odd prime order give unbounded lifting indices at fixed cohomological period. Finite torsor theories and infinite pure sets realize the definability criteria. For profinite extensions with finite kernel, one finite quotient detects all liftable multiplicities, though it need not carry every lift.

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1 Introduction

The groupoid of models of a coherent theory, equipped with an appropriate logical topology, represents its classifying topos. Its coherent objects recover the syntactic pretopos. This representation makes the role of isomorphisms especially concrete: an isomorphism transports every definable object, and a definable relation must be invariant under all such transports. If some isomorphisms are removed while every model is retained, additional invariant relations and maps can appear. The resulting semantics remains conservative, but it can admit objects that have no interpretation in the original syntax.

This paper studies restriction along a wide model subgroupoid. The main question is relative. Given a semantic object for the subgroupoid, when can it be equipped with the action of the original groupoid? For objects already defined by the original theory, when does a subgroupoid-equivariant map remain definable in the original theory? These questions require separate criteria. A conservative functor can fail both fullness and essential surjectivity.

We use syntactic pretoposes to retain finite sums and definable quotients. Thus $\mathcal{B}$ may be the pretopos $\mathrm{Def}(T^{\mathrm{eq}})$ of a coherent or classical first-order theory. At the level of general representing groupoids, the comparison is expressed in terms of invariant graphs and invariant equivalence relations. At the level of finite atomic semantics, it becomes an extension problem for finite group actions. The latter admits an explicit classification by orbit types, normalizer extensions, and multiplicities.

1.1 Principal results

Let $\mathcal{G}$ be an open topological groupoid with coherent equivariant topos $\mathcal{E} = \mathrm{Sh}(\mathcal{G})$, and let $\mathcal{H} \subseteq \mathcal{G}$ be a wide subgroupoid with the subspace topology on arrows. The object spaces and their topologies are the same. Assume that the source and target maps of $\mathcal{H}$ are open as well; the arrow space H_1 need not be open in G_1. Restriction defines

$$R : \mathcal{E} \longrightarrow \mathcal{F} = \mathrm{Sh}(\mathcal{H}).$$

When $\mathcal{F}$ is coherent and R preserves coherent objects, the graph criterion of Proposition 3.7 and the presentation criterion of Theorem 3.6 identify the maps and quotient presentations that lift. In the second criterion, the presentation is part of the data: an object can admit a lift even when a particular presentation does not lift.

For finite continuous actions of a topological group G and an open subgroup H of finite index, Proposition 5.1 identifies action extensions on a fixed H-set X with G-invariant subsets of $\mathrm{Coind}_H^G X$ on which evaluation is bijective. The full evaluation map has invariant kernel only when it is bijective, even if X itself admits an extension.

The main classification concerns $N \trianglelefteq G$ with G finite. Fix $K \leq N$ and put

$$I = NN_G(K), \qquad D = N_G(K), \qquad E_K = D/K, \qquad W_K = N_N(K)/K, \qquad Q_K = I/N. \tag{1.1}$$

There is an exact sequence

$$1 \longrightarrow W_K \longrightarrow E_K \longrightarrow Q_K \longrightarrow 1. \tag{1.2}$$

For $m \geq 0$, let $X_{K,m}$ contain m copies of each N-orbit type in the G-orbit of N/K. Theorem 8.1 proves that $X_{K,m}$ extends to a G-set exactly when

$$m \in S_K = \langle [Q_K : P] \ : \ P \leq Q_K,\ E_K|_P \longrightarrow P \text{ splits} \rangle_{\mathbb{N}}. \tag{1.3}$$

More generally, an arbitrary finite N-set first has to have constant multiplicity on each G-orbit of N-orbit types; the condition (1.3) then applies independently to those orbits.

The classification retains the full groupoid of lifts. If L runs over the E_K-conjugacy classes of subgroups satisfying $L \cap W_K = 1$, write

$$w_L = [Q_K : \pi(L)], \qquad a_L = |N_{E_K}(L)/L|.$$

The isomorphism-class and mass series are respectively

$$\prod_{[L]} \frac{1}{1 - z^{w_L}}, \qquad \exp\left(\sum_{[L]} \frac{z^{w_L}}{a_L}\right). \tag{1.4}$$

Thus the same subgroup data determine existence, the number of isomorphism classes, and automorphism groups.

The normalizer description can be compared with a centralizer extension attached to a fixed action $\rho : N \to \mathrm{Sym}(X)$. Section 6 proves the exact splitting criterion, gives the nonabelian factor equations, and identifies the abelian obstruction and the classification of splittings. Section 11 computes the multiplicity monoids and series in several families. For the central subgroup of $\mathrm{SL}_2(\mathbb{F}_5)$ the monoid is $\langle 12, 20\rangle$, so eventual divisibility leaves genuine small gaps. For a Heisenberg group of order p^{1+2r}, the monoid is $p^r\mathbb{N}$, and isotropic subspaces determine all lift types and their automorphisms.

1.2 Relation to existing work

The categorical syntax used here follows Makkai and Reyes [11]. The model-groupoid representation is due to Butz–Moerdijk, Awodey–Forssell, and Forssell [5, 1, 8]. Breiner's logical schemes describe local syntax by a sheaf of pretoposes, whose equivariant global sections recover the theory [3, Proposition 3.2.1 and Theorem 3.2.6]. These reconstruction results supply the ambient syntax; our question concerns its recovery after restriction to a wide subgroupoid.

Wrigley characterizes representing groupoids through conservativity and elimination of parameters [15, Theorem 15]. Restriction to a wide subgroupoid isolates the role of the arrows in this problem. It retains the conservative family of models and changes invariance. This differs from restricting to a strictly full subgroupoid by discarding models, as in the theory of subgroupoids and quotient theories [7].

The finite calculations use the classification of transitive G-sets by subgroups and the positive part of Burnside restriction; these are standard ingredients of Burnside ring theory [2]. The centralizer criterion uses classical group-extension theory [4]. The analogous extension problem for linear group representations is studied by Thévenaz [13]; our target automorphisms are permutations of a prescribed finite set. The cyclic examples are instances of the theory of roots of permutations [12]. The least degrees in the Heisenberg examples are related to the minimal faithful permutation degree studied by Johnson and Easdown–Praeger [9, 6]. Our results place these ingredients in a single relative definability problem and give a simultaneous description of lifting monoids, lift groupoids, and their enumerating series. The proofs keep track of the distinction between an action on a fixed set, its isomorphism class, and a quotient presentation.

1.3 Conventions

A pretopos is a small exact extensive category. The coherent topology J_{coh} has as covers the finite families whose images have union equal to the target. For a classical theory, $\mathrm{Def}(T^{\mathrm{eq}})$ denotes

its Boolean syntactic pretopos, including finite disjoint sums and definable quotients. An exact functor preserves finite limits, finite coproducts, and effective quotients. We write $\mathbb{N} = \{0, 1, 2, \ldots\}$. All group actions are left actions; S_m denotes the symmetric group on m letters. Finite continuous actions use the discrete topology on the finite set, and $\mathbf{Fin}^G$ denotes their category. For a finite group these are simply finite G-sets. A groupoid of lifts has equivariant bijections as morphisms. An action extension on a *fixed* H-set retains the given underlying set and the given H-action. Passing to the action groupoid under $\mathrm{Aut}_H(X)$ forgets that marking. This convention is used throughout the counting formulas.

2 Logical topology and coherent syntax

We record a model presentation that allows empty interpretations and all finite tuples. Its role is to specify the semantic setting of the restriction problem. Subsequent arguments may equally start from any open groupoid representing a coherent topos.

2.1 Partial indices and tuple conditions

Fix a small pretopos $\mathcal{B}$ and an infinite cardinal κ at least as large as its set of morphisms. For every $A \in \mathcal{B}$ choose a set K_A of cardinality κ. A partially indexed model consists of an exact functor $M : \mathcal{B} \to \mathbf{Set}$ with $|M(A)| \leq \kappa$ and surjections

$$e_{M,A} : D_{M,A} \twoheadrightarrow M(A), \qquad D_{M,A} \subseteq K_A.$$

The domain $D_{M,A}$ may be empty. In particular $M(0) = \varnothing$ has the empty enumeration. Each nonempty fibre of an enumeration is required to be infinite. Models can be represented by quotients of subsets of K_A so that the collection is a set.

For $U \hookrightarrow A_1 \times \cdots \times A_n$ and $i_j \in K_{A_j}$ let

$$\langle U; i_1, \ldots, i_n \rangle = \{(M, e) : i_j \in D_{M,A_j} \text{ for every } j, \\ (e_{M,A_1}(i_1), \ldots, e_{M,A_n}(i_n)) \in M(U)\}.$$

These sets generate the object topology. The case $n = 0$ includes sentence conditions, and $U = A$ records that a particular name is defined. Taking a diagonal gives equality between names. Taking the graph of $f : A \to B$ gives

$$\{(M, e) : M(f)e_{M,A}(i) = e_{M,B}(j)\},$$

with both names required to be defined.

The arrows are natural isomorphisms of the underlying models. They need not preserve the enumerations. The arrow topology is generated by source and target inverse images of object opens and by the transport conditions

$$\langle A; i \mapsto j \rangle = \{\alpha : (M, e) \cong (M', e') : i \in D_{M,A},\ j \in D_{M',A},\ \alpha_A(e_{M,A}(i)) = e_{M',A}(j)\}.$$

Proposition 2.1. *These topologies make the indexed model groupoid an open topological groupoid. For $A \in \mathcal{B}$, the evaluation family*

$$[\![A]\!] = \coprod_{(M,e)} M(A) \longrightarrow G_0$$

is an equivariant local homeomorphism, and every $f : A \to B$ induces a continuous equivariant map $[\![f]\!] : [\![A]\!] \to [\![B]\!]$.

Proof. Source and target are continuous by construction. The inverse image of a transport open under the identity section is the tuple equality open. Inversion exchanges $i \mapsto j$ with $j \mapsto i$. The inverse image of $i \mapsto k$ under composition is the union, over all defined intermediate names j, of the open sets imposing $i \mapsto j$ and $j \mapsto k$. Surjectivity of the intermediate partial enumeration proves this equality.

To check that source is open, consider a finite intersection of arrow subbasic opens. Introduce one variable for each distinct target index occurring in these conditions, with its prescribed sort. Transfer all target formulas to the source model, and impose the equalities required by the transport conditions. The source image is precisely the object condition obtained by existentially quantifying these new variables. Indeed, any arrow gives such witnesses. Conversely, witnesses in a source model give an arrow to the same underlying model with a new enumeration realizing the finitely many target names. A finite assignment of names extends to partial surjections with infinite fibres because κ is infinite and bounds every carrier. The existential condition is a logical open, since images exist in $\mathcal{B}$. Thus source is open; inversion gives the same assertion for target.

For each $i \in K_A$, the section $s_i(M, e) = e_{M,A}(i)$ is defined on the open set $D_{A,i} = \{(M, e) : i \in D_{M,A}\}$. Give the evaluation family the topology generated by the images $s_i(O)$, for $O \subseteq D_{A,i}$ open. Two such charts agree exactly on an equality open. Consequently each chart is homeomorphic to its domain, and the charts cover the family. On two charts, continuity of the groupoid action is the transport condition. Continuity of $[\![f]\!]$ is the graph condition displayed above. Equivariance follows from naturality of model isomorphisms. □

Finite limits, finite coproducts, images, and effective quotients are preserved by evaluation. Indeed, the comparison maps are continuous by Proposition 2.1 and are bijections on every fibre, since every model is exact. A continuous fibrewise bijection between local homeomorphisms is an isomorphism. The same argument treats complements when $\mathcal{B}$ is Boolean.

2.2 The representing hypothesis

The logical-groupoid representation theorem applies to a conservative family of small models equipped with all partial indexings with infinite fibres and all model isomorphisms. Including all such indexings gives closure under finite reindexing, a sufficient condition for parameter elimination. The representation theorem therefore applies [15, Proposition 73 and Corollary 74(2)]; see also [1, 8]. The partial surjections and infinite-fibre convention above agree with Definitions 7 and 72 of [15]. For a separately chosen model subgroupoid, representability must still be established: conservativity of its objects alone does not suffice.

In the rest of the paper a *representing groupoid for* $\mathcal{B}$ means an open topological groupoid $\mathcal{G}$ with a specified equivalence

$$\mathrm{Sh}(\mathcal{G}) \simeq \mathrm{Sh}(\mathcal{B}, J_{\mathrm{coh}}) \tag{2.1}$$

carrying evaluation to the coherent Yoneda embedding. Since $\mathcal{B}$ is already a pretopos, the coherent-topos correspondence gives

$$\mathcal{B} \simeq \mathrm{Coh}(\mathrm{Sh}(\mathcal{G})). \tag{2.2}$$

This is the pretopos-completion description of coherent objects in [1, arXiv version, Section 2.1.7]. Thus a representing topological groupoid recovers the coherent syntactic pretopos.

Local syntax over $A \in \mathcal{B}$ is the slice $\mathcal{B}/A$. On the coherent site, pullback between coherent bases preserves the pretopos structure, and regular epimorphisms are effective descent morphisms. Hence $A \mapsto \mathcal{B}/A$ is a stack for finite coherent covers [3, Proposition 3.2.1]. All slice descent used

here is over coherent bases with finite jointly regular-epimorphic covers. The restriction criteria below are formulated directly in $\mathcal{E}$ and its coherent objects.

3 Restriction and definable recognition

Let G be an open topological groupoid, and let $H \subseteq G$ be a wide topological subgroupoid with the same object space S and the subspace topology on arrows. Assume that H is also an open topological groupoid, meaning that its source and target maps are open. This assumption does not require H_1 to be open in G_1. Set

$$\mathcal{E} = \mathrm{Sh}(G), \qquad \mathcal{F} = \mathrm{Sh}(H), \qquad R = \mathrm{Res}^G_H : \mathcal{E} \longrightarrow \mathcal{F}.$$

Restriction retains the underlying sheaf on S and restricts its action to H.

Proposition 3.1. *Restriction R is faithful and conservative, preserves finite limits and all small colimits, and reflects epimorphisms and monomorphisms. It is the inverse-image part of a surjective geometric morphism $\mathcal{F} \to \mathcal{E}$.*

Proof. A morphism remains the same map of underlying sheaves, proving faithfulness. If the underlying map of a G-equivariant morphism is invertible, its inverse is G-equivariant, proving conservativity. The forgetful functors from equivariant sheaves to $\mathrm{Sh}(S)$ create finite limits and small colimits: the action on a limit or colimit is induced by the corresponding operations on sheaves, since inverse image along source and target preserves finite limits and colimits. Consequently restriction preserves these operations. Monomorphisms, images and epimorphisms are detected on the underlying sheaves, so restriction reflects them. The geometric morphism induced by a continuous functor of open topological groupoids has this restriction as inverse image [7, Section 2.1]; its conservativity makes that geometric morphism surjective. □

For a model groupoid this operation preserves every represented model. The retained family remains conservative, while fewer symmetry conditions allow additional invariant relations and maps. This differs from restriction to a full subgroupoid on a smaller collection of models, the operation governing quotient theories in [7].

Definition 3.2. An object C of a Grothendieck topos is *compact* if every jointly epimorphic family with codomain C has a finite jointly epimorphic subfamily. For a subobject $U \hookrightarrow RA$, say that U is *G-invariant* if its underlying open subset of the *given* G-sheaf A is invariant under every arrow of G.

Lemma 3.3. *An H-subobject $U \hookrightarrow RA$ is the restriction of a G-subobject of A if and only if U is G-invariant. The G-subobject, when it exists, is unique as a subobject of A. Moreover, if RX is compact, then X is compact.*

Proof. Subobjects of equivariant sheaves correspond to invariant open subsets of their total spaces. A G-invariant open subset inherits the restricted continuous action, and this is its only possible action making the inclusion equivariant. This proves the first assertion.

For the second, restrict any jointly epimorphic family over X. Compactness of RX gives a finite subfamily whose restriction is jointly epimorphic. Proposition 3.1 reflects the corresponding epimorphism from the finite coproduct, so this subfamily already covers X. □

3.1 Recognition by kernels and graphs

In this section $\mathcal{E}$ is coherent, and its full subcategory of coherent objects is identified with a small pretopos $\mathcal{B}$. For the model groupoid of Section 2 this is the identification in (2.2). We write A for both an object of $\mathcal{B}$ and its realization in $\mathcal{E}$.

Theorem 3.4 (Lifting a presented quotient)**.** *Let $A \in \mathcal{E}$, and let*

$$p : RA \twoheadrightarrow X$$

be an epimorphism in $\mathcal{F}$. Regard its kernel pair as the H-subobject

$$K_p = RA \times_X RA \hookrightarrow R(A \times A).$$

The following conditions are equivalent.

(i) *There are a G-sheaf $\overline{X}$, an isomorphism $\theta : R\overline{X} \xrightarrow{\sim} X$, and a G-equivariant epimorphism $\overline{p} : A \twoheadrightarrow \overline{X}$ such that $\theta R\overline{p} = p$.*

(ii) *K_p is G-invariant.*

Such a lift of the presented quotient is unique up to a unique isomorphism respecting $\overline{p}$ and θ. If A is coherent, then the lift $\overline{X}$ is coherent if and only if the lifted G-subobject $\overline{K}_p \hookrightarrow A \times A$ is compact.

Proof. A lifted quotient has G-invariant kernel pair, proving necessity. Conversely, Lemma 3.3 equips K_p with a unique G-action compatible with its inclusion in $A \times A$. Reflexivity, symmetry and transitivity can be checked on the underlying sheaves, where K_p is already an equivalence relation. Thus $\overline{K}_p$ is an equivalence relation in $\mathcal{E}$. Take its effective quotient

$$\overline{p} : A \twoheadrightarrow A/\overline{K}_p = \overline{X}.$$

Restriction preserves this quotient. Since p is also the effective quotient of K_p, it induces the required unique comparison $\theta : R\overline{X} \xrightarrow{\sim} X$.

Fibrewise, the action is forced by

$$g \cdot p(a) = p(g \cdot a).$$

Surjectivity of p and invariance of its kernel show respectively uniqueness and well-definedness. The topos quotient construction ensures continuity. The universal property of the quotient gives the stated uniqueness of the lifted presentation.

Suppose now that A is coherent. A compact subobject of a coherent object in a coherent topos is coherent: cover it by coherent subobjects, use compactness to choose a finite subcover, and take their finite union in the coherent pretopos. Hence if $\overline{K}_p$ is compact, it is coherent, and its effective quotient belongs to the pretopos $\mathcal{B}$. Conversely, if $\overline{X}$ is coherent, then $\overline{K}_p = A \times_{\overline{X}} A$ is coherent by closure under finite limits, and is therefore compact. □

Definition 3.5. We call the restriction *coherent* when $\mathcal{F}$ is a coherent topos and R sends coherent objects of $\mathcal{E}$ to coherent objects of $\mathcal{F}$. In this case put

$$\mathcal{C}_H = \mathrm{Coh}(\mathcal{F}).$$

A *finite source atlas* of $X \in \mathcal{C}_H$ is a finite jointly regular-epimorphic family $\{RA_i \to X\}_{i=1}^n$ with $A_i \in \mathcal{B}$. Equivalently, it is an epimorphism $RA \twoheadrightarrow X$ with $A = \coprod_i A_i \in \mathcal{B}$.

The definition of $\mathcal{C}_H$ uses the target topos alone. No assumption that its objects are restrictions of syntax is included.

Theorem 3.6 (Coherent image recognition)**.** *Assume that restriction is coherent. For $X \in \mathcal{C}_H$, the following are equivalent.*

(i) *$X \cong RB$ for some $B \in \mathcal{B}$.*

(ii) *X admits a finite source atlas $p : RA \twoheadrightarrow X$ whose kernel pair is G-invariant in $R(A \times A)$.*

For a specified finite source atlas, G-invariance of the kernel is equivalent to the existence of a coherent lift making that atlas G-equivariant.

Proof. For (i)⇒(ii), choose the one-chart atlas given by the specified isomorphism $RB \cong X$. Its kernel is the diagonal, which is G-invariant.

Conversely, K_p is coherent in $\mathcal{F}$, because RA and X are coherent and coherent objects are closed under finite limits. In particular it is compact. If K_p is G-invariant, it lifts to $\overline{K}_p \hookrightarrow A \times A$, and Lemma 3.3 makes $\overline{K}_p$ compact. Theorem 3.4 therefore gives a coherent quotient $B = A/\overline{K}_p$ and an isomorphism $RB \cong X$. The same argument proves the statement about a specified atlas. □

Proposition 3.7 (Subobjects and maps)**.** *Assume that restriction is coherent. Let $A, B \in \mathcal{B}$.*

(i) *A coherent H-subobject $U \hookrightarrow RA$ is the restriction of a subobject in $\mathcal{B}$ if and only if it is G-invariant.*

(ii) *An H-equivariant map $u : RA \to RB$ is the restriction of a morphism in $\mathcal{B}$ if and only if its graph is G-invariant in $R(A \times B)$.*

The assertion about maps does not require coherence of restriction.

Proof. For (i), lift the invariant open subset using Lemma 3.3. Its compactness in $\mathcal{F}$ implies compactness in $\mathcal{E}$, and a compact subobject of A is coherent. The converse follows from preservation of coherent objects and invariance of every G-subobject.

For (ii), graph invariance says exactly that $u(g \cdot a) = g \cdot u(a)$ for every arrow g and every suitable element a. Thus u is a G-equivariant map, and fullness of the coherent inclusion $\mathcal{B} \subseteq \mathcal{E}$ identifies it with a morphism of $\mathcal{B}$. Necessity is immediate. □

Theorem 3.8 (Maps between quotient presentations)**.** *Let $p : RA \twoheadrightarrow X$ and $q : RB \twoheadrightarrow Y$ be presentations with G-invariant kernels, and write $\overline{p} : A \twoheadrightarrow \overline{X}$ and $\overline{q} : B \twoheadrightarrow \overline{Y}$ for their lifts. For an H-equivariant map $u : X \to Y$, form*

$$T_u = \{(a, b) : u(p(a)) = q(b)\} \hookrightarrow R(A \times B).$$

Then u lifts to a map $\overline{X} \to \overline{Y}$ if and only if T_u is G-invariant. Such a lift is unique. If restriction is coherent, $A, B \in \mathcal{B}$, and $X, Y \in \mathcal{C}_H$, then a G-invariant T_u is itself the restriction of a coherent subobject of $A \times B$.

Proof. Necessity follows by pulling back the graph of a lifted map along $\overline{p} \times \overline{q}$. For sufficiency, let $x = p(a)$ and choose b with $q(b) = u(x)$. If g is any compatible G-arrow, invariance of T_u gives

$$u(g \cdot x) = u(p(g \cdot a)) = q(g \cdot b) = g \cdot u(x).$$

This is equivariance of the underlying sheaf map u for the lifted actions. It is the only possible lift because restriction is faithful. In the coherent situation, T_u is a finite limit of coherent objects in $\mathcal{F}$, hence is compact; its invariant lift is compact by Lemma 3.3 and coherent as a subobject of $A \times B$. □

Example 3.9 (The compactness condition)**.** Let S be the Cantor space and let $G = H$ be its identity groupoid. Then $\mathrm{Sh}(S)$ is coherent and its coherent objects form a Boolean pretopos. An open subset $U \subseteq S$ which is not closed gives an invariant subobject $U \hookrightarrow 1$ which is not coherent: compact open subsets of a compact Hausdorff space are closed. Thus invariance of an arbitrary subobject does not imply coherent definability.

There is a corresponding quotient example. In $2 = 1 \amalg 1$, identify the two sections precisely over U. The resulting open equivalence relation is invariant, and its quotient exists in $\mathrm{Sh}(S)$. Its off-diagonal component is U, so the relation is not coherent. The quotient is not coherent either, since otherwise its kernel pair on the coherent object 2 would be coherent. This is the finiteness obstruction isolated by Theorem 3.4.

3.2 Arrow density and semantic recognition

Recall that an object B of a topos is decidable if its diagonal is complemented. For an equivariant sheaf this means that the relative diagonal in its total space is both open and closed.

Lemma 3.10 (Density and decidable targets)**.** *Suppose that H_1 is dense in G_1. If $A, B \in \mathrm{Sh}(G)$ and B is decidable, every H-equivariant map $u : RA \to RB$ is G-equivariant. Consequently restriction is fully faithful on any full subcategory of decidable G-sheaves.*

Proof. On $G_1 \times_{s,S} A$, consider the continuous maps

$$(g, a) \longmapsto u(g \cdot a), \qquad (g, a) \longmapsto g \cdot u(a).$$

They have the same projection to S. Their equalizer is closed, because the relative diagonal of B is closed. It contains $H_1 \times_{s,S} A$ by H-equivariance. The projection $G_1 \times_{s,S} A \to G_1$ is a local homeomorphism, being a base change of $A \to S$, and is therefore open. The inverse image of a dense subset under an open map is dense: every nonempty open subset has nonempty open image meeting that dense subset. Thus $H_1 \times_{s,S} A$ is dense. The equalizer is all of $G_1 \times_{s,S} A$, proving G-equivariance. Faithfulness is Proposition 3.1. □

Corollary 3.11 (Recognition under dense restriction)**.** *Suppose that $\mathcal{B} = \mathrm{Coh}(\mathcal{E})$ is Boolean, restriction is coherent, and the following conditions hold:*

(i) *H_1 is dense in G_1;*

(ii) *$\mathcal{C}_H = \mathrm{Coh}(\mathcal{F})$ is a Boolean pretopos;*

(iii) *every object of $\mathcal{C}_H$ admits a finite source atlas.*

Then restriction induces an equivalence of pretoposes

$$R : \mathcal{B} \xrightarrow{\sim} \mathcal{C}_H.$$

Proof. Every object of the Boolean pretopos $\mathcal{B}$ is decidable, so Lemma 3.10 gives full faithfulness. For $X \in \mathcal{C}_H$, choose a finite source atlas $p : RA \twoheadrightarrow X$. Its kernel pair K_p is a coherent subobject of $R(A \times A)$. Booleanity of $\mathcal{C}_H$ supplies its complement as a coherent subobject; the coproduct decomposition is preserved by the inclusion into $\mathcal{F}$. Thus K_p has an H-equivariant characteristic map

$$\chi_{K_p} : R(A \times A) \longrightarrow 1 \amalg 1 = R(1 \amalg 1).$$

The codomain is decidable. Lemma 3.10 makes this map G-equivariant, so K_p is G-invariant. Coherent image recognition supplies $B \in \mathcal{B}$ with $RB \cong X$. Thus restriction is essentially surjective as well as fully faithful. Its exactness was proved in Proposition 3.1. □

Coherent restriction alone does not imply the finite-atlas condition: Example 12.15 gives a target coherent object with no source atlas.

4 Continuous group actions

For a topological group G, let $\mathrm{Cont}(G)$ be the topos of continuous actions on discrete sets. A transitive object is G/U for an open subgroup U. Call an action *finitary* if it has finitely many orbits. In this section G is Hausdorff and non-Archimedean, meaning that open subgroups form a neighbourhood basis at the identity, and satisfies

$$|U\backslash G/V| < \infty \quad \text{for all open subgroups } U, V \leq G. \tag{RP}$$

This is the usual Roelcke precompactness condition in the non-Archimedean setting; oligomorphic groups are examples [14].

Proposition 4.1. *The topos* $\mathrm{Cont}(G)$ *is coherent, and its coherent objects are exactly the finitary continuous* G*-sets. They form a Boolean pretopos, denoted* $\mathcal{B}_G$*.*

Proof. An action is compact precisely when it has finitely many orbits. Indeed, its orbits form an epimorphic covering family, and any cover of a transitive action has one member whose image is the whole orbit. Moreover,

$$(G/U) \times (G/V) \cong \coprod_{[g]\in U\backslash G/V} G/(U \cap gVg^{-1}).$$

Condition (RP) therefore makes finite products of compact objects compact. Subobjects of a finitary action are unions of some of its finitely many orbits and are again compact. In particular, a pullback of maps between compact objects is compact, so compact objects are coherent. Every action is a coproduct of transitive ones, which gives a set of coherent generators. This proves coherence of the topos. Finitary actions are closed under finite limits, finite coproducts and quotients of equivalence relations, and every invariant subset has an invariant complement. Hence they form the asserted Boolean pretopos. □

Theorem 4.2 (Double-coset criterion for coherent restriction)**.** *Let* $H \leq G$ *have the subspace topology. The following are equivalent:*

(i) *restriction* $\mathrm{Cont}(G) \to \mathrm{Cont}(H)$ *is coherent;*

(ii) $H\backslash G/U$ *is finite for every open subgroup* $U \leq G$*.*

Under these conditions H *also satisfies* (RP)*. For every open* $U \leq G$ *there is an orbit decomposition*

$$\mathrm{Res}^G_H(G/U) \cong \coprod_{[g]\in H\backslash G/U} H/(H \cap gUg^{-1}). \tag{*}$$

In particular, every open subgroup $H \leq G$ *gives a coherent restriction. For an open subgroup of finite index* d*, the number of summands in* (*) *is at most* d*.*

Proof. The decomposition follows by taking the H-orbits of the cosets gU. Coherent restriction sends the compact object G/U to a compact H-set, so the orbit count is finite, proving (i)⇒(ii).

Assume (ii). If V, W are open subgroups of H, choose open subgroups U, U' of G such that $H \cap U \subseteq V$ and $H \cap U' \subseteq W$. The product $(G/U) \times (G/U')$ is a finite union of transitive continuous G-sets. By (ii), its restriction has finitely many H-orbits. Its H-invariant subset

$$H/(H \cap U) \times H/(H \cap U')$$

therefore has finitely many H-orbits, as does its quotient $(H/V) \times (H/W)$. Thus $V\backslash H/W$ is finite, and H satisfies (RP). Proposition 4.1 applies to both groups. Equation (*) shows that restriction preserves finitary objects, proving (i).

If H is open, condition (ii) is a special case of (RP). The index bound follows from the surjection $H\backslash G \twoheadrightarrow H\backslash G/U$. □

For $d_U = |H\backslash G/U|$, the subobject counts are

$$|\operatorname{Sub}_G(G/U)| = 2, \qquad |\operatorname{Sub}_H(\operatorname{Res}^G_H(G/U))| = 2^{d_U}.$$

More generally, if $A = \coprod_{i=1}^r G/U_i$, the corresponding counts are 2^r and $2^{\sum_i d_{U_i}}$. These formulas count the additional invariant predicates made available by restriction.

Theorem 4.3 (Fullness and density)**.** *For a subgroup $H \leq G$ with its subspace topology, restriction on $\mathcal{B}_G$ is full if and only if H is dense in G. If H is dense, restriction induces an equivalence*

$$\operatorname{Cont}(G) \simeq \operatorname{Cont}(H)$$

and hence an equivalence of the finitary subcategories.

Proof. For a dense subgroup, an H-equivariant map between discrete continuous G-sets is G-equivariant: for each element x, the equation $u(gx) = gu(x)$ is closed in g and holds on the dense subgroup. This also follows from Lemma 3.10.

If H is not dense, choose $g \notin \overline{H}$ and an open subgroup $U \leq G$ with $gU \cap H = \varnothing$. The H-orbit HU/U is a nonempty proper subset of G/U. Its characteristic map

$$\operatorname{Res}^G_H(G/U) \longrightarrow 2$$

is H-equivariant and is not G-equivariant. Both G/U and the trivial two-point action are in $\mathcal{B}_G$, so restriction is not full.

For essential surjectivity when H is dense, let H/V be any transitive continuous H-set. Since V is open in H, its closure $L = \overline{V}$ is an open subgroup of G. Indeed, some identity neighbourhood $O \subseteq G$ satisfies $H \cap O \subseteq V$, and density implies $O \subseteq \overline{V}$. Also $H \cap L = V$, because V is closed in H. Density of H meets every open coset of L, so

$$H/V \xrightarrow{\sim} \operatorname{Res}^G_H(G/L), \qquad hV \longmapsto hL.$$

Taking coproducts lifts every continuous H-action. Fullness and faithfulness were already proved, so restriction is an equivalence. It preserves and reflects orbit decompositions, hence restricts to the finitary subcategories. □

For a profinite group, every transitive continuous action is finite. Thus $\mathcal{B}_G = \operatorname{FinCont}(G)$, and Theorem 4.3 is a finite-action criterion. For oligomorphic groups the finitary category includes infinite transitive actions; replacing it by finite underlying sets is not justified.

Proposition 4.4 (Finite quotients in the profinite case)**.** *Let G be profinite and $H \leq G$ have the subspace topology. Every finite continuous H-set has a finite source atlas from a finite continuous G-set. If H is dense, every such action extends uniquely to a continuous action of G on the same set. In general, an action $\rho : H \to \mathrm{Sym}(X)$ on a finite set extends to G if and only if there is an open normal subgroup $N \triangleleft G$ such that*

$$H \cap N \subseteq \ker \rho$$

and the resulting action of $HN/N \leq G/N$ on X extends to G/N.

Proof. Continuity of ρ and finiteness of X make its kernel open in H. Choose an open normal $N \triangleleft G$ with $H \cap N \subseteq \ker \rho$. Write $Q = HN/N \leq G/N$. Every H-orbit in X is a quotient of the regular Q-set. The restricted regular G/N-set is a nonempty finite union of regular Q-sets. For each orbit of X, one copy of this restricted G/N-set maps onto that orbit, by choosing the image of one point in each regular Q-orbit. Their finite coproduct gives the required atlas. The empty action has the empty atlas.

When H is dense, $Q = G/N$, so ρ factors through all of G/N and defines an extension. Two continuous extensions to the finite discrete group $\mathrm{Sym}(X)$ agree on a dense subgroup and are equal. Finally, an extension of ρ has an open normal kernel, which proves necessity of the finite-quotient criterion; composing an extension from G/N proves sufficiency. □

5 A finite coinduction criterion

For an open subgroup of finite index, coinduction places every extension of a finite action inside one finite G-set. The required copies of the original set are exactly the invariant subsets on which evaluation is bijective.

Let H be an open subgroup of finite index d in a topological group G, and let X be a finite continuous left H-set. Define

$$Z = \mathrm{Coind}_H^G(X) = \{f : G \to X : f(hg) = h \cdot f(g) \text{ for } h \in H, g \in G\},$$

with left G-action

$$(g \cdot f)(t) = f(tg).$$

The functions are continuous: on every open coset Ht their values are obtained from the continuous H-action on $f(t)$. Evaluation at the identity is the H-equivariant map

$$\varepsilon : \mathrm{Res}_H^G Z \longrightarrow X, \qquad f \longmapsto f(1).$$

Proposition 5.1. *The G-set Z is finite and continuous, with $|Z| = |X|^d$, and ε is surjective. Continuous G-actions on the* fixed *set X extending its given H-action correspond bijectively to G-invariant subsets $C \subseteq Z$ for which*

$$\varepsilon|_C : \mathrm{Res}_H^G C \xrightarrow{\sim} X$$

is an isomorphism. The action of $\mathrm{Aut}_H(X)$ by conjugation on extensions corresponds to its action on such subsets by

$$(a_* f)(t) = a(f(t)).$$

Consequently, isomorphism classes of extensions are the $\mathrm{Aut}_H(X)$-orbits of these subsets.

Proof. Choose representatives $t_1 = 1, t_2, \dots, t_d$ for the cosets Ht in $H\backslash G$. An equivariant function is determined freely by the d values $f(t_i)$, giving the cardinality formula and surjectivity of evaluation. This includes $X = \varnothing$, when both its domain and codomain are empty. For a fixed f, the open subgroup

$$\bigcap_{i=1}^{d} t_i^{-1} H_{f(t_i)} t_i$$

fixes f under right translation. Thus the discrete action on Z is continuous.

If ρ is an extension, define

$$s_\rho : X \longrightarrow Z, \qquad s_\rho(x)(t) = \rho(t)x.$$

This is well-defined, G-equivariant and a section of ε. Its image C_ρ is G-invariant and $\varepsilon|_{C_\rho}$ is an isomorphism of H-sets.

Conversely, transfer the action on such a subset C along $\varepsilon|_C$ to obtain a continuous G-action on X. Its restriction is the given H-action. If $f \in C$ and $x = f(1)$, the transferred action satisfies

$$\rho(t)x = \varepsilon(t \cdot f) = f(t).$$

Thus $s_\rho(x) = f$, so these constructions are inverse.

For $a \in \mathrm{Aut}_H(X)$, the map a_* preserves the defining equivariance of functions and commutes with the G-action. For the conjugate extension $\rho' = a\rho a^{-1}$,

$$s_{\rho'}(ax) = a_* s_\rho(x).$$

Hence $C_{\rho'} = a_* C_\rho$. This proves the orbit assertion. □

Proposition 5.2 (The kernel of evaluation)**.** *If $d > 1$ and $|X| > 1$, the kernel pair of the canonical evaluation $\varepsilon : \mathrm{Res}^G_H Z \twoheadrightarrow X$ is not G-invariant, even when X admits a G-extension. More generally, its kernel is G-invariant if and only if ε is bijective.*

Proof. If the kernel is invariant and $f(1) = f'(1)$, then for every $g \in G$ invariance under right translation gives $f(g) = f'(g)$. Thus $f = f'$, and the already surjective map ε is bijective. The converse holds because the diagonal is invariant. If $d > 1$ and $|X| > 1$, the formula $|Z| = |X|^d$ excludes bijectivity. A finite set with trivial H-action has a trivial G-extension, so this failure occurs even for extendible actions. □

An invariant subset with bijective evaluation gives a lift even when the full evaluation map has noninvariant kernel. This realizes the distinction in Theorem 3.6 between lifting an object and lifting its specified presentation.

6 Centralizer extensions and action obstructions

In this section and Sections 7–9, G is a profinite group and $N \trianglelefteq G$ is open. The quotient $Q = G/N$ is finite. All actions on finite sets are assumed continuous, and finite groups carry the discrete topology. The results therefore include the case of finite G.

Fix a finite N-set X, with action

$$\rho : N \longrightarrow \mathrm{Sym}(X).$$

An extension on the fixed set X is a continuous homomorphism $\alpha : G \to \mathrm{Sym}(X)$ whose restriction to N is ρ. The set of such homomorphisms is denoted $\mathrm{Ext}_G(\rho)$. Put

$$A_X = \mathrm{Aut}_N(X) = C_{\mathrm{Sym}(X)}(\rho(N)).$$

The group A_X acts on $\mathrm{Ext}_G(\rho)$ by conjugation. We denote the resulting action groupoid by

$$\mathrm{ExtAct}_G(X) = \mathrm{Ext}_G(\rho) /\!\!/ A_X.$$

Equivalently, $\mathrm{ExtAct}_G(X)$ is the groupoid of finite continuous G-sets whose restriction is isomorphic to X, with G-equivariant bijections as morphisms. This equivalence does not retain a specified identification of the restriction with X.

Remark 6.1 (Identifications and automorphisms). If an identification $\theta : \mathrm{Res}^G_N Y \xrightarrow{\sim} X$ is retained, and morphisms must commute with θ, the resulting groupoid is equivalent to the discrete set $\mathrm{Ext}_G(\rho)$. Indeed, transport by θ fixes the underlying set, and a morphism preserving that identification is the identity permutation. The action groupoid $\mathrm{ExtAct}_G(X)$ instead allows all N-equivariant changes of that identification. The counting formulas below concern the latter action groupoid.

6.1 The transporter extension

An extension must first preserve the isomorphism class of the given N-action under conjugation. Once this condition holds, the remaining problem is to choose the transporting permutations compatibly with multiplication in G. The following extension records that compatibility.

Define

$$\widetilde{E}_X = \{(g, s) \in G \times \mathrm{Sym}(X) : s\rho(n)s^{-1} = \rho(gng^{-1}) \text{ for every } n \in N\},$$

and let $I_X \subseteq G$ be its image under projection to G. Thus I_X is the stabilizer of the isomorphism class of the N-action under conjugation by G. Set

$$\Delta_\rho(N) = \{(n, \rho(n)) : n \in N\}, \qquad E_X = \widetilde{E}_X / \Delta_\rho(N).$$

Theorem 6.2 (Transporter criterion)**.** *The group I_X is an open subgroup containing N. The group E_X is finite, and there is a natural exact sequence*

$$1 \longrightarrow A_X \longrightarrow E_X \xrightarrow{p} I_X/N \longrightarrow 1, \qquad a \longmapsto [(1, a)]. \tag{6.1}$$

The action ρ extends to G if and only if $I_X = G$ and (6.1) splits. When $I_X = G$, there is a natural bijection

$$\mathrm{Ext}_G(\rho) \xrightarrow{\sim} \mathrm{Spl}(p), \qquad \alpha \longmapsto \big(gN \longmapsto [(g, \alpha(g))]\big). \tag{6.2}$$

It is equivariant for the conjugation action of A_X, and hence gives an isomorphism of action groupoids

$$\mathrm{ExtAct}_G(X) \cong \mathrm{Spl}(p) /\!\!/ A_X.$$

For a corresponding pair (α, σ), its automorphism group is

$$\mathrm{Aut}_G(X, \alpha) = \{a \in A_X : a\sigma(q)a^{-1} = \sigma(q) \text{ for all } q \in Q\}.$$

Proof. If $(g,s),(h,t) \in \widetilde{E}_X$, then for every $n \in N$,

$$st\rho(n)(st)^{-1} = s\rho(hnh^{-1})s^{-1} = \rho(ghn(gh)^{-1}).$$

The identity and inverses satisfy the same defining condition. Thus $\widetilde{E}_X$ is a subgroup of $G \times \mathrm{Sym}(X)$. It is closed, because each of its defining equalities is a closed condition. It is therefore profinite. Its image I_X is a subgroup containing N; as a union of N-cosets, it is open. The subgroup $\Delta_\rho(N)$ is closed, and

$$(g,s)(n,\rho(n))(g,s)^{-1} = (gng^{-1}, \rho(gng^{-1})),$$

so it is normal in $\widetilde{E}_X$.

Projection to I_X/N factors through E_X and is surjective. A pair (n,s) over N has $\rho(n)^{-1}s \in A_X$, and

$$[(n,s)] = [(1,\rho(n)^{-1}s)].$$

Moreover, $(1,a) \in \Delta_\rho(N)$ implies $a = 1$. This proves exactness. Since both A_X and I_X/N are finite, E_X is finite. Its quotient topology is Hausdorff, and hence discrete. More explicitly, over the open subgroup N the homeomorphism

$$N \times A_X \longrightarrow \widetilde{E}_X \cap (N \times \mathrm{Sym}(X)), \qquad (n,a) \longmapsto (n,\rho(n)a)$$

identifies $\Delta_\rho(N)$ with $N \times \{1\}$, which is open.

An extension α supplies an element $(g,\alpha(g))$ for every $g \in G$, so necessarily $I_X = G$. The formula in (6.2) is independent of the representative of gN, since

$$(gn,\alpha(gn)) = (g,\alpha(g))(n,\rho(n)).$$

It is a homomorphism and a section of p.

Conversely, suppose $I_X = G$ and let $\sigma : Q \to E_X$ be a section. For each $g \in G$ there is a unique $s_g \in \mathrm{Sym}(X)$ satisfying

$$[(g,s_g)] = \sigma(gN).$$

Existence follows by modifying a representative of $\sigma(gN)$ by an element of $\Delta_\rho(N)$. Uniqueness follows because two representatives with first coordinate g can differ by an element of $\Delta_\rho(N)$ only if that element has first coordinate 1. The homomorphism property of σ now implies $s_{gh} = s_g s_h$. For $n \in N$ the condition $[(n,s_n)] = 1$ implies $s_n = \rho(n)$. Hence $\alpha(g) = s_g$ extends ρ. It is continuous: on each of the finitely many open cosets gN it is given by $gn \mapsto s_g\rho(n)$. These constructions are inverse. Finally,

$$[(1,a)][(g,\alpha(g))][(1,a^{-1})] = [(g,a\alpha(g)a^{-1})],$$

which proves equivariance and the assertions about groupoids and automorphisms. □

Lemma 6.3 (Finite quotient reduction)**.** *If $I_X = G$, then $K_\rho = \ker(\rho)$ is open and normal in G. Every extension of ρ factors through the finite group G/K_ρ.*

Proof. The subgroup K_ρ is open in N and hence in G. If $n \in K_\rho$ and $g \in G$, choose s with $(g,s) \in \widetilde{E}_X$. Then $\rho(gng^{-1}) = s\rho(n)s^{-1} = 1$, proving normality. Any extension restricts to ρ on N, and therefore kills K_ρ. □

6.2 Factor systems and cohomological obstructions

Assume $I_X = G$ and write $A = A_X$. Choose a normalized set-theoretic section $e : Q \to E_X$, with $e(1) = 1$. Define

$$\theta_q(a) = e(q)ae(q)^{-1}, \qquad c(q,r) = e(q)e(r)e(qr)^{-1}.$$

Proposition 6.4 (Explicit nonabelian factor system)**.** *The functions (θ, c) satisfy*

$$\theta_q\theta_r = \mathrm{Ad}_{c(q,r)}\,\theta_{qr}, \tag{6.3}$$

$$c(q,r)c(qr,s) = \theta_q(c(r,s))c(q,rs), \tag{6.4}$$

with $\theta_1 = \mathrm{id}$ and $c(1,q) = c(q,1) = 1$. Changing the section to $e'(q) = b(q)e(q)$, where $b(1) = 1$, changes these data to

$$\theta'_q = \mathrm{Ad}_{b(q)}\,\theta_q, \tag{6.5}$$

$$c'(q,r) = b(q)\theta_q(b(r))c(q,r)b(qr)^{-1}. \tag{6.6}$$

An extension of ρ exists if and only if there is a function $b : Q \to A$, $b(1) = 1$, for which the right side of (6.6) is 1 for every q, r.

Proof. Conjugation by $e(q)e(r) = c(q,r)e(qr)$ gives (6.3). Computing $e(q)e(r)e(s)$ in its two associative orders gives (6.4). Direct substitution gives (6.5)–(6.6). Finally, e' is a homomorphism exactly when $c' = 1$, and Theorem 6.2 applies. □

Corollary 6.5 (Abelian centralizer)**.** *Suppose $I_X = G$ and A_X is abelian. Conjugation in E_X gives a well-defined Q-module structure on A_X, and the factor system determines a class*

$$o_X = [c] \in H^2(Q, A_X).$$

The action extends precisely when $o_X = 0$. When extensions exist, their isomorphism classes form a torsor under $H^1(Q, A_X)$; a chosen extension identifies this torsor with $H^1(Q, A_X)$. The automorphism group of every extension is A_X^Q.

Proof. Inner automorphisms of the abelian group A_X are trivial, so (6.3) makes θ an action independent of the chosen section. Equation (6.4) is the normalized 2-cocycle identity, and (6.6) changes c by a coboundary. This proves the vanishing criterion. Fix a splitting σ_0. Every other splitting is uniquely $\sigma(q) = b(q)\sigma_0(q)$, and it is a homomorphism exactly when

$$b(qr) = b(q)\theta_q(b(r)).$$

Conjugation by $a \in A_X$ replaces $b(q)$ by $ab(q)\theta_q(a)^{-1}$. This gives the stated H^1-classification. The elements stabilizing a splitting are exactly A_X^Q. □

Remark 6.6. For nonabelian A_X, Proposition 6.4 is the appropriate statement without extra choices of a band. If a splitting is fixed, the other splittings up to A_X-conjugacy are described by the pointed nonabelian cohomology set $H^1(Q, A_X)$ for the resulting action. The nonsplit case is described by the factor equations above. This is the classical extension mechanism [4], applied to the canonical transporter extension of a finite action.

7 Orbit types and normalizer reduction

The transporter criterion treats a prescribed action. To classify all liftable multiplicities, we first separate the G-orbits of N-orbit types. Within one such block, taking fixed points converts the extension problem into an action of a finite normalizer quotient with a free kernel action.

The group G acts by conjugation on the set of N-conjugacy classes of open subgroups of N. Since N acts trivially on this set, the action factors through Q. Such a class $[K]_N$ is the type of the transitive finite N-set N/K.

For an open subgroup $K \leq N$, put

$$D = N_G(K), \quad I = I_K = ND, \quad Q_K = I/N, \quad E = D/K, \quad W = (D \cap N)/K. \tag{7.1}$$

The groups E, W, and Q_K are finite: K is open in N, hence in G. There is an exact sequence

$$1 \longrightarrow W \longrightarrow E \xrightarrow{\pi} Q_K \longrightarrow 1. \tag{7.2}$$

Let Ω_K be the G-orbit of $[K]_N$, and choose one representative $K_t \leq N$ for each $t \in \Omega_K$. For $m \in \mathbb{N}$ define

$$X_{K,m} = \coprod_{t \in \Omega_K} (N/K_t)^{\amalg m}.$$

Its isomorphism class is independent of the choices.

Lemma 7.1 (Inertia and the normalizer extension)**.** *The stabilizer of $[K]_N$ in G is $I = NN_G(K)$. The transporter extension of the transitive N-set N/K, with ambient group I, is canonically isomorphic to (7.2). Under this isomorphism $aK \in W$ acts on N/K by*

$$nK \longmapsto na^{-1}K.$$

Proof. The class $[K]_N$ is fixed by g exactly when $gKg^{-1} = nKn^{-1}$ for some $n \in N$, equivalently $n^{-1}g \in N_G(K)$. This proves the formula for I. For $h \in N_G(K)$ let

$$\tau_h(nK) = hnh^{-1}K.$$

Then $h \mapsto [(h, \tau_h)]$ is a homomorphism $N_G(K) \to E_{N/K}$. For any transporter pair (g, s), write $s(K) = uK$ with $u \in N$. Equivariance gives

$$s(nK) = gng^{-1}uK.$$

Equality of stabilizers implies $u^{-1}g \in N_G(K)$. With $h = u^{-1}g$ we obtain $(g, s) = (u, \rho(u))(h, \tau_h)$, proving surjectivity after quotienting by $\Delta_\rho(N)$. Its kernel is K: if $[(h, \tau_h)] = 1$, then $h \in N$ and $\tau_h = \rho(h)$; evaluating at K gives $hK = K$. The converse is immediate. Finally, for $h \in N_N(K)$ the representative with first coordinate 1 is $(1, \rho(h)^{-1}\tau_h)$, which sends nK to $nh^{-1}K$. □

Theorem 7.2 (The category of a single orbit type)**.** *Let $\mathcal{C}_K(I)$ be the full subcategory of finite continuous I-sets whose N-orbits all have type N/K. Let $\mathcal{F}_W(E)$ be the full subcategory of finite E-sets on which W acts freely. Taking K-fixed points gives an equivalence of categories*

$$\mathcal{C}_K(I) \xrightarrow{\sim} \mathcal{F}_W(E), \qquad X \longmapsto X^K.$$

An inverse sends an E-set Y, inflated to $D = N_G(K)$, to $I \times_D Y$. Under this equivalence the N-orbits of X correspond to the W-orbits of X^K.

Proof. If $x \in X^K$, its stabilizer N_x contains K and is conjugate to K in N. These subgroups have the same finite index in N, so $N_x = K$. Consequently X^K is stable under D, the action factors through $E = D/K$, and $W = (D \cap N)/K$ acts freely. Every N-orbit meets X^K, because it is isomorphic to N/K. Within an N-orbit, two K-fixed points differ by an element of $N_N(K)$: an element of N carrying one to the other must conjugate their common stabilizer K to itself. Thus each intersection with X^K is one W-orbit.

Conversely, inflate a finite E-set Y free on W to D. Since $I = ND$, restriction to N identifies $I \times_D Y$ with $N \times_{D\cap N} Y$. Every stabilizer in $D \cap N$ of a point of Y is K, so all resulting N-orbits have type N/K. The natural map

$$I \times_D X^K \longrightarrow X, \qquad [i, x] \longmapsto ix$$

is surjective by the preceding orbit description. If $ix = jy$ with $x, y \in X^K$, then $j^{-1}i$ conjugates the stabilizer K of x in N to the stabilizer K of y. It therefore belongs to D, proving injectivity.

The natural map $Y \to (I \times_D Y)^K$, $y \mapsto [1, y]$, is also bijective. Indeed, if $[i, y]$ is fixed by K, then for each $k \in K$ the element $i^{-1}ki$ belongs to $D \cap N$ and fixes y. Freeness on W implies $i^{-1}Ki \subseteq K$. Equality follows by comparison of the finite indices in N, so $i \in D$ and $[i, y] = [1, iy]$. These bijections are natural for all equivariant maps, not just bijections, and prove the equivalence of categories. □

Remark 7.3. The preceding equivalence concerns full categories of finite actions. These categories need not be pretoposes. If $W \neq 1$, the category $\mathcal{F}_W(E)$ has no terminal object: the regular E-set belongs to it, and $\mathrm{Hom}_E(E, T) \cong T$ forces any terminal object T to have one element, on which W cannot act freely. The equivalence will be used to classify objects and equivariant bijections.

Proposition 7.4 (Reduction to an inertia group)**.** *Induction and extraction of the $[K]_N$-isotypic part give mutually inverse equivalences of groupoids*

$$\mathrm{ExtAct}_I((N/K)^{\amalg m}) \simeq \mathrm{ExtAct}_G(X_{K,m}).$$

More generally, a finite N-set X can extend to G only if its orbit multiplicities are constant on every G-orbit of N-orbit types. If this condition holds and m_Ω denotes the common multiplicity, its extension groupoid is the product of the extension groupoids of the blocks X_{K,m_Ω}.

Proof. For $m = 0$ both groupoids are terminal. Suppose $m \geq 1$. In a G-set whose restriction has types in Ω_K, let Y_0 be the union of those N-orbits of type $[K]_N$. Its setwise stabilizer in G is I, and its restriction to N is $(N/K)^{\amalg m}$. The map

$$G \times_I Y_0 \longrightarrow Y, \qquad [g, y] \longmapsto gy,$$

is a G-equivariant bijection: the translates of Y_0 are the disjoint isotypic parts indexed by G/I. Conversely, induction of an I-action on Y_0 has exactly these isotypic parts. These constructions commute with equivariant bijections. For a general X, the union of the isotypic parts in each G-orbit of types is G-stable in every extension. Distinct such blocks admit no N-equivariant bijection, so the extension groupoid decomposes as asserted. □

Proposition 7.5 (The extension for repeated orbits)**.** *For $m \geq 1$, the transporter extension for $(N/K)^{\amalg m}$ over $I/N = Q_K$ is isomorphic to*

$$1 \longrightarrow W \wr S_m \longrightarrow (E \times_{Q_K} \cdots \times_{Q_K} E) \rtimes S_m \longrightarrow Q_K \longrightarrow 1. \tag{7.3}$$

Here the fibre product has m factors, S_m permutes them, and the projection records their common image in Q_K.

Proof. Choose labels for the m copies. A semilinear bijection with first coordinate $g \in I$ permutes these copies. On each copy it gives a transporter pair for N/K with the same first coordinate g. After quotienting by the diagonal copy of N, these give m elements of E with common image gN. Conversely, such elements have unique transporter representatives with prescribed first coordinate g, and therefore assemble, together with the chosen permutation, into a semilinear bijection of the disjoint union. Composition is the usual wreath-product multiplication (indexing coordinates by the target copies). This identifies the groups and their maps to Q_K. The kernel is $W^m \rtimes S_m$. □

8 Multiplicity, classification, and enumeration

Let $\mathcal{L}_K$ be a set of representatives for the $N_G(K)$-conjugacy classes of subgroups

$$K \le L \le N_G(K), \qquad L \cap N = K.$$

It is finite, because all such subgroups contain K and hence correspond to subgroups of the finite group E. Define

$$w_L = [I : NL], \qquad a_L = |N_G(L)/L|,$$

and let $\mathcal{P}_K$ be the set of subgroups $P \le Q_K$ for which the restriction of (7.2) to P splits. The weight w_L will be the multiplicity of each orbit type in Ω_K within $\mathrm{Res}^G_N(G/L)$; a_L is the order of $\mathrm{Aut}_G(G/L)$. All generating functions below are formal power series.

Theorem 8.1 (Multiplicity, classification, and enumeration)**.** *The following assertions hold.*

1. *The finite N-set $X_{K,m}$ extends to G if and only if*

$$m \in S_K := \Big\langle [Q_K : P] : P \in \mathcal{P}_K \Big\rangle = \langle w_L : L \in \mathcal{L}_K \rangle \subseteq \mathbb{N}. \tag{8.1}$$

   In particular, $|Q_K|\mathbb{N} \subseteq S_K$.

2. *Every extension, up to G-equivariant isomorphism, is uniquely of the form*

$$Y \cong \coprod_{L \in \mathcal{L}_K} (G/L)^{\amalg b_L}, \qquad b_L \in \mathbb{N}, \qquad \sum_L w_L b_L = m. \tag{8.2}$$

   Its automorphism group is

$$\mathrm{Aut}_G(Y) \cong \prod_{L \in \mathcal{L}_K} (N_G(L)/L) \wr S_{b_L}. \tag{8.3}$$

3. *If c_m is the number of isomorphism classes of extensions, then*

$$\sum_{m \ge 0} c_m z^m = \prod_{L \in \mathcal{L}_K} (1 - z^{w_L})^{-1}. \tag{8.4}$$

   The groupoid cardinalities satisfy

$$\sum_{m \ge 0} \left(\sum_{[Y] \in \pi_0 \, \mathrm{ExtAct}_G(X_{K,m})} \frac{1}{|\,\mathrm{Aut}_G(Y)|} \right) z^m = \exp\left(\sum_{L \in \mathcal{L}_K} \frac{z^{w_L}}{a_L} \right). \tag{8.5}$$

Proof. Consider first a transitive finite continuous G-set $Y = G/L$ whose N-orbit types belong to Ω_K. Conjugating L in G, we may assume $L \cap N = K$. Since N is normal in G, the subgroup K is normal in L, and therefore $L \le N_G(K)$. Conversely, any subgroup in the definition of $\mathcal{L}_K$ is open in G and defines a finite continuous transitive G-set.

The N-orbits in G/L are indexed by $N\backslash G/L = G/NL$. The stabilizer in N of gL is gKg^{-1}. Its type is $[K]_N$ exactly when $g \in I$. Thus there are $[I : NL]$ orbits of type $[K]_N$. Translation by G gives the same number of every other type in Ω_K. Consequently

$$\mathrm{Res}^G_N(G/L) \cong X_{K,w_L}. \tag{8.6}$$

Two subgroups L, L' with intersection exactly K are conjugate in G only if the conjugating element normalizes K. Hence their G-conjugacy classes are precisely their $N_G(K)$-conjugacy classes. Every finite G-set has a unique decomposition into transitive G-sets, up to their order and isomorphism. Equation (8.6) now proves (8.2) and the second description of S_K.

To compare with the first description, let $P = NL/N \le Q_K$. The subgroup $L/K \le E$ intersects W trivially and maps isomorphically onto P. It is therefore the image of a splitting of $\pi^{-1}(P) \to P$. Conversely, the inverse image in $N_G(K)$ of the image of any such splitting is a subgroup L with $L \cap N = K$. In both directions $w_L = [Q_K : P]$. This proves (8.1). The subgroup $P = 1$ always belongs to $\mathcal{P}_K$, giving $|Q_K| \in S_K$.

The automorphism group of G/L is isomorphic to $N_G(L)/L$, using $aL : gL \mapsto ga^{-1}L$. Automorphisms can permute the b_L isomorphic copies, but cannot mix distinct transitive isomorphism types. This proves (8.3). Formula (8.4) follows by choosing each b_L independently. Since a wreath product with S_a has order $a_L^a a!$, the same argument gives

$$\prod_L \sum_{a\ge 0} \frac{z^{aw_L}}{a_L^a a!} = \exp\left(\sum_L \frac{z^{w_L}}{a_L}\right),$$

which is (8.5). □

Remark 8.2 (Automorphisms in the finite normalizer quotient). For every $L \in \mathcal{L}_K$, a normalizer of L in G also normalizes $L \cap N = K$. Hence $N_G(L) \le D = N_G(K)$, and quotienting by K gives

$$N_G(L)/L \cong N_E(L/K)/(L/K).$$

Thus all the stabilizer weights and automorphism groups in Theorem 8.1 are computed inside the single finite extension $1 \to W \to E \to Q_K \to 1$.

Corollary 8.3 (Number of actions on a fixed set)**.** *Write $a = |W|$ and $t = |\Omega_K| = [G : I]$. The number of extensions on a fixed N-set $X_{K,m}$ is*

$$(a^m m!)^t \sum_{\substack{(b_L)\in\mathbb{N}^{\mathcal{L}_K} \\ \sum_L w_L b_L = m}} \prod_{L\in\mathcal{L}_K} \frac{1}{a_L^{b_L} b_L!}.$$

Proof. There are t distinct N-orbit types, each repeated m times, so $|\operatorname{Aut}_N(X_{K,m})| = (a^m m!)^t$. For each isomorphism class of extensions, orbit–stabilizer for the conjugation action of $\operatorname{Aut}_N(X_{K,m})$ gives the quotient of this order by $|\operatorname{Aut}_G(Y)|$. Sum over (8.2). □

Remark 8.4 (Uniform finite quotient). All statements for $X_{K,m}$, $m \ge 1$, take place in the single finite quotient $G/\operatorname{core}_G(K)$. Indeed, every subgroup L in $\mathcal{L}_K$ contains $\operatorname{core}_G(K)$, and its core in G contains $\operatorname{core}_G(K)$. Thus every G/L above factors through that quotient. The profinite formulation records continuity and finite-quotient independence; it does not introduce an additional infinite obstruction.

9 Actual and virtual extension

9.1 The conductor of the lifting monoid

Let

$$\delta_K = \gcd\{w_L : L \in \mathcal{L}_K\}.$$

Put $S'_K = \delta_K^{-1} S_K \subseteq \mathbb{N}$ and

$$h_K = \min\{w_L/\delta_K : L \in \mathcal{L}_K\}.$$

Proposition 9.1 (Conductor formula)**.** *The monoid S'_K is a numerical semigroup. For each $r \in \{0, \dots, h_K - 1\}$ let*

$$u_r = \min\{u \in S'_K : u \equiv r \pmod{h_K}\}.$$

These minima exist, and its conductor is

$$c_K = \max_r u_r - h_K + 1.$$

Consequently, for every sufficiently large m, the set $X_{K,m}$ extends precisely when $\delta_K \mid m$. Every multiple of δ_K at least $\delta_K c_K$ is a liftable multiplicity. The values u_r are shortest-path distances from 0 in the directed graph on $\mathbb{Z}/h_K\mathbb{Z}$ with an edge of weight w_L/δ_K from r to $r + w_L/\delta_K$ for every L.

Proof. The normalized generators have gcd 1. The submonoid they generate in the finite additive group $\mathbb{Z}/h_K\mathbb{Z}$ is a subgroup: any element of a finite group has its inverse among its nonnegative powers. The gcd condition makes this subgroup all of $\mathbb{Z}/h_K\mathbb{Z}$. Thus every residue occurs, and the minima u_r exist. In residue r, the elements of S'_K are exactly $u_r + h_K\mathbb{N}$, because h_K itself is a generator. The last missing nonnegative integer, if there is one, is therefore $\max_r(u_r - h_K)$; the same formula gives -1 when $S'_K = \mathbb{N}$. This proves the conductor and eventual divisibility assertions. A path records a sum of normalized generators, so the shortest-path description is identical to the definition of u_r. □

9.2 The image of Burnside restriction

Proposition 9.2 (Actual and virtual extension)**.** *Let $B(G)$ and $B(N)$ be the Burnside rings of finite continuous sets with action, and put $u_K = [X_{K,1}] \in B(N)$. For an integer m, there is a virtual G-set $v \in B(G)$ with*

$$\mathrm{Res}^G_N(v) = m u_K$$

if and only if $\delta_K \mid m$. For $m \geq 0$, an actual finite G-set with this restriction exists if and only if $m \in S_K$. More generally, choose one K_Ω for every G-orbit Ω of N-orbit types. As additive subgroups of $B(N)^Q$,

$$\mathrm{im}(\mathrm{Res}^G_N : B(G) \to B(N)) = \bigoplus_\Omega \delta_{K_\Omega}\mathbb{Z}\, u_{K_\Omega}. \tag{9.1}$$

The image of the additive monoid of actual finite G-sets is $\bigoplus_\Omega S_{K_\Omega} u_{K_\Omega}$.

Proof. The additive group of $B(N)$ is free on the isomorphism classes of transitive finite N-sets. Since the conjugation action factors through the finite group Q, its invariant subgroup is free on the orbit sums u_{K_Ω}. The restriction of any transitive G-set has its N-orbit types in exactly one such orbit Ω, with equal multiplicity throughout that orbit. For the orbit of K, Theorem 8.1 shows that the possible coefficients contributed by transitive G-sets are exactly the weights w_L. Their integer span is $\delta_K\mathbb{Z}$, whereas their nonnegative integer span is S_K.

Contributions from distinct orbits of types are supported on disjoint subsets of the transitive N-set basis. They therefore cannot change each other's coefficients, even if virtual terms with negative coefficients are allowed. Taking the direct sum over the orbits proves (9.1) and its nonnegative analogue. The assertions on a single ray follow. □

Corollary 9.3. *The additive cokernel of Burnside restriction inside the invariant subgroup is*

$$B(N)^Q/\operatorname{im}(\operatorname{Res}_N^G) \cong \bigoplus_{\Omega} \mathbb{Z}/\delta_{K_\Omega}\mathbb{Z}.$$

Remark 9.4. The gaps of S_K distinguish actual extensions from virtual extensions. The eventual divisibility criterion computes their common group completion, while the conductor records the range in which the nonnegative condition still imposes an obstruction.

10 A finite quotient detecting all multiplicities

Let $1 \to W \to E \to Q \to 1$ be a profinite extension with finite kernel. For $m \geq 0$, write mW for the disjoint union of m regular W-orbits, and put

$$\Sigma(E, W) = \{m \in \mathbb{N} : mW \cong \operatorname{Res}_W^E Y \text{ for a finite continuous } E\text{-set } Y\}.$$

Write $\mathcal{E}_m(E, W)$ for the groupoid of these E-sets and equivariant bijections, without a fixed identification of the restriction with mW. For finite E these are the monoid and groupoid of Theorem 8.1, with $G = E$, $N = W$, and $K = 1$.

Proposition 10.1. *Let $1 \to W \to E \xrightarrow{\pi} Q \to 1$ be an extension of profinite groups with W finite. Define $\Sigma(E, W)$ using finite continuous E-sets. Then*

$$\Sigma(E, W) = \Big\langle [Q : S] : S \leq Q \text{ open, } \pi^{-1}(S) \to S \text{ has a continuous section} \Big\rangle_{\mathbb{N}}.$$

There is an open normal subgroup $J \trianglelefteq E$ with $J \cap W = 1$ such that

$$\Sigma(E, W) = \Sigma(E/J, WJ/J).$$

This assertion concerns the possible multiplicities; it does not assert that every extending action factors through this one quotient.

Proof. A transitive finite continuous E-set is E/L for an open subgroup L. It is free over W exactly when $L \cap W = 1$. Then $\pi|_L : L \to S = \pi(L)$ is a continuous isomorphism of profinite groups, S is open in Q, and the number of regular W-orbits is $[Q : S]$. Conversely, a continuous section over an open subgroup S has compact image L of finite index in E; such an L is open and disjoint from W. Decomposing into orbits proves the first assertion.

Choose an open normal subgroup of E avoiding every nonidentity element of the finite group W. Its finite coset action is free over W, so the spectrum contains a positive integer. Any additive submonoid $M \subseteq \mathbb{N}$ with a positive element is finitely generated: if d is its least positive element, choose the least element $m_a \in M$ in each residue class a mod d that occurs. Every element in that residue class is $m_a + td$ with $t \geq 0$. Thus d and the nonzero m_a's generate M.

Choose a finite continuous E-set witnessing each positive generator of $\Sigma(E, W)$, and let J be the intersection of their action kernels. These kernels are open and normal, and each is disjoint from W because the corresponding action is nonempty and free over W. All chosen witnesses factor through E/J, so $\Sigma(E/J, WJ/J)$ contains every generator. The opposite inclusion follows by inflation of finite actions. □

Remark 10.2 (Application to a profinite orbit type). If $N \trianglelefteq G$ is closed in a profinite group and $K \leq N$ is open, choose $U \trianglelefteq G$ open with $U \cap N \subseteq K$. For $u \in U$ and $k \in K$, the commutator $uku^{-1}k^{-1}$ lies in $U \cap N \subseteq K$. Hence U normalizes K, and $N_G(K)$ is open in G. The normalizer extension

$$1 \to N_N(K)/K \to N_G(K)/K \to NN_G(K)/N \to 1$$

has finite kernel. Theorem 7.2 remains valid for finite continuous actions: the equality of stabilizers uses equal finite indices in N in place of equal finite orders. Consequently Proposition 10.1 applies to this orbit type. For a general profinite group, the number of isomorphism classes at a fixed multiplicity can be infinite. The finite-product enumeration formulas apply when the relevant normalizer quotient E is finite, as in Sections 7–9.

Example 10.3. Let $E = C_2 \times \prod_{n\geq 1} C_2$ and let $W = C_2$ be the first factor. Every continuous character $\chi : \prod_{n\geq 1} C_2 \to C_2$ gives an extension of the regular W-action on two points by $(w, q) \cdot x = w + \chi(q) + x$. There are infinitely many such characters. The centralizer of the regular C_2-action is C_2, whose conjugation on these extensions is trivial. Thus $\mathcal{E}_1(E, W)$ has infinitely many isomorphism classes, although $\Sigma(E, W) = \mathbb{N}$ is already detected by the quotient $E \to W$.

11 Explicit lifting monoids and groupoids

The notation $\Sigma(E, W)$ and $\mathcal{E}_m(E, W)$ is as in Section 10. All groups in this section are finite. For a group A, BA denotes the groupoid with one object and automorphism group A.

11.1 Cyclic groups and roots of permutations

For cyclic groups the same problem is the classical problem of taking roots of a permutation; see [12, Section 2] and [10, Theorems 1–3]. Let $G = C_n = \langle g \rangle$, let $H = \langle g^q \rangle \cong C_d$, where $n = qd$, and let a_t be the multiplicity of the H-orbit of size t, for $t \mid d$. Set

$$q_t = \prod_{\substack{\ell \mid t \\ \ell \text{ prime}}} \ell^{v_\ell(q)}, \qquad R_t = q/q_t.$$

Then the H-set extends if and only if $q_t \mid a_t$ for all $t \mid d$. Its G-isomorphism classes of extensions are parametrized by nonnegative integers $b_{t,s}$, $s \mid R_t$, with

$$\sum_{s \mid R_t} s\, b_{t,s} = a_t/q_t.$$

The corresponding G-set has $b_{t,s}$ orbits of size tq_ts, and its automorphism group is

$$\prod_{t \mid d} \prod_{s \mid R_t} C_{tq_ts} \wr S_{b_{t,s}}.$$

Indeed a G-orbit of size u restricts to $\gcd(u, q)$ orbits of size $u/\gcd(u, q)$. To produce orbits of size t, write $u = tc$. The required condition is

$$\gcd(tc, q) = c,$$

equivalently $c \mid q$ and $\gcd(t, q/c) = 1$. Its solutions are precisely $c = q_ts$ with $s \mid R_t$. Each resulting orbit size $u = tc$ divides $n = qd$, since $t \mid d$ and $c \mid q$.

In particular, for $C_{p^2} \supset C_p$, m free C_p-orbits extend exactly when $m = ps$. At that multiplicity there is one isomorphism class, consisting of s regular C_{p^2}-orbits; its automorphism group is $C_{p^2} \wr S_s$. On one fixed C_p-set the number of extensions is

$$\frac{p^m m!}{p^{2s} s!}.$$

For $C_4 \supset C_2$ this becomes $(2s)!/s!$.

11.2 A monoid with gaps: $\mathrm{SL}_2(\mathbb{F}_5)$

Let $G = \mathrm{SL}_2(\mathbb{F}_5)$ and $N = \{I, -I\} \cong C_2$. The relevant normalizer quotients can be computed directly from two-dimensional matrices.

Theorem 11.1. *The lifting spectrum of the free N-sets is*

$$\Sigma(G, N) = \langle 12, 20 \rangle_{\mathbb{N}} = \{0, 12, 20, 24\} \ \cup \ \{4s : s \geq 8\}. \tag{11.1}$$

There is an equivalence

$$\mathcal{E}_m(G, N) \simeq \coprod_{\substack{a,b,c \geq 0 \\ 12a+20b+60c=m}} \mathrm{B}\left[(C_4 \wr S_a) \times (C_4 \wr S_b) \times (G \wr S_c)\right]. \tag{11.2}$$

In particular its isomorphism-class and groupoid-cardinality series are, respectively,

$$\frac{1}{(1-z^{12})(1-z^{20})(1-z^{60})}, \qquad \exp\left(\frac{z^{12}}{4} + \frac{z^{20}}{4} + \frac{z^{60}}{120}\right). \tag{11.3}$$

For one fixed action of N on mN, the number of extending G-actions is

$$\sum_{\substack{a,b,c \geq 0 \\ 12a+20b+60c=m}} \frac{2^m m!}{4^{a+b} a!b!\, 120^c c!}. \tag{11.4}$$

Proof. There are 24 choices for a nonzero first column of a determinant-one matrix and 5 choices for the second column, so $|G| = 120$. If $A^2 = I$, its minimal polynomial divides $(x-1)(x+1)$, which has distinct roots in $\mathbb{F}_5$. Thus A is diagonalizable. The determinant condition forces both eigenvalues to be 1 or both to be -1, so $A = I$ or $A = -I$. In particular, $-I$ is the unique involution.

If $L \leq G$ is disjoint from N, Cauchy's theorem implies that $|L|$ is odd. Its possible orders are $1, 3, 5, 15$. A group of order 15 is cyclic: the Sylow theorems make both its Sylow subgroups normal. There is no element of order 15 in G. To see this, a nonidentity element u of order 5 is unipotent, and, after choosing a basis, has the form $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Its centralizer in $\mathrm{GL}_2(\mathbb{F}_5)$ consists of $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$ with $a \neq 0$; the determinant-one condition gives $a = \pm 1$. Its centralizer in G therefore has order 10. An element of order 15 would centralize its third power, of order 5, which is impossible. Hence $|L|$ is $1, 3$, or 5.

Subgroups of orders 3 and 5 both exist. They are Sylow subgroups because 3^2 and 5^2 do not divide 120, and each order consequently gives one conjugacy class. We have therefore found exactly three conjugacy classes of subgroups disjoint from N. Their corresponding numbers of free N-orbits are $20, 12, 60$. This proves

$$\Sigma(G, N) = \langle 12, 20, 60 \rangle_{\mathbb{N}} = \langle 12, 20 \rangle_{\mathbb{N}}.$$

Since $\langle 3,5\rangle_{\mathbb{N}}$ has positive gaps $1,2,4,7$ and contains every integer at least 8, this is (11.1).

We next compute the normalizer quotients. Let

$$U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{F}_5 \right\}.$$

Every nonidentity element of U has the same unique fixed line. A matrix normalizing U must preserve that line, and conversely every determinant-one upper triangular matrix normalizes U. Consequently

$$N_G(U) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_5^\times,\ b \in \mathbb{F}_5 \right\}, \qquad N_G(U)/U \cong \mathbb{F}_5^\times \cong C_4.$$

For the subgroup of order 3, put

$$B = \begin{pmatrix} 0 & 4 \\ 1 & 4 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Direct multiplication gives

$$B^2 + B + I = 0, \qquad D^2 = -I, \qquad DBD^{-1} = B^{-1}.$$

The polynomial $x^2 + x + 1$ is irreducible over $\mathbb{F}_5$, so the centralizer of B in $\mathrm{GL}_2(\mathbb{F}_5)$ is $\mathbb{F}_{25}^\times$, acting by multiplication on the two-dimensional $\mathbb{F}_5$-space $\mathbb{F}_{25}$. The determinant is the field norm. Its kernel has order 6, so $|C_G(B)| = 6$. Conjugation maps $N_G(\langle B\rangle)$ into $\mathrm{Aut}(C_3) \cong C_2$, and D realizes the nontrivial automorphism. Thus the normalizer has order 12. The coset of D modulo $\langle B\rangle$ has order 4, because $D^2 = -I \notin \langle B\rangle$. It follows that

$$N_G(\langle B\rangle)/\langle B\rangle \cong C_4.$$

For the trivial subgroup the normalizer quotient is G. Now Theorem 8.1 gives (11.2)–(11.4). □

The greatest common divisor of the positive lifting multiplicities is 4, but multiplicities $4, 8, 16, 28$ do not lift. Thus a single divisibility condition does not describe restriction in general.

Corollary 11.2. *For the central extension $1 \to C_2 \to \mathrm{SL}_2(\mathbb{F}_5) \to G/N \to 1$, the extension class has order 2, the least positive liftable multiplicity is 12, and the greatest common divisor of the positive liftable multiplicities is 4. Every multiple of 4 at least 32 lifts.*

Proof. The extension is central, so its class belongs to $H^2(G/N, C_2)$ with trivial coefficients. This group has exponent 2. A splitting would give a subgroup of order 60 disjoint from N; Cauchy's theorem and the unique involution $-I$ exclude such a subgroup. The class is therefore nonzero and has order 2. The remaining assertions follow from $\Sigma(G,N) = \langle 12, 20\rangle$. □

11.3 Heisenberg groups and unbounded lifting indices

Fix an odd prime p, an integer $r \geq 1$, and a $2r$-dimensional symplectic vector space (V, ω) over $\mathbb{F}_p$. Define

$$G_r = \mathrm{H}_r(p) = V \times \mathbb{F}_p, \qquad (v,a)(w,b) = \left(v + w, a + b + \tfrac{1}{2}\omega(v,w)\right).$$

Its center is

$$N = \{0\} \times \mathbb{F}_p \cong C_p,$$

and $G_r/N = V$. The group G_r has order p^{1+2r} and exponent p, and

$$[(v,a),(w,b)] = (0,\omega(v,w)).$$

We use the same notation $\mathrm{H}_0(p) = C_p$. For $0 \le k \le r$, let $\mathcal{I}_{r,k}$ be the set of isotropic k-dimensional subspaces of V, and write

$$I_{r,k} = |\mathcal{I}_{r,k}| = \prod_{i=0}^{k-1} \frac{p^{2r-i} - p^i}{p^k - p^i}, \qquad I_{r,0} = 1. \tag{11.5}$$

The empty product convention applies throughout.

Theorem 11.3 (The extraspecial lifting groupoids)**.** *For G_r and N as above, the following hold.*

1. *The lifting spectrum is*
$$\Sigma(G_r, N) = p^r\mathbb{N}. \tag{11.6}$$
2. *The G_r-conjugacy classes of subgroups L with $L \cap N = 1$ are naturally indexed by all isotropic subspaces $W \le V$. If $k = \dim W$, the associated transitive G_r-set has*
$$p^{2r-k}$$
regular N-orbits and has automorphism group $\mathrm{H}_{r-k}(p)$.
3. *For every m, the groupoid $\mathcal{E}_m(G_r, N)$ is equivalent to*
$$\coprod_{\substack{(b_W)\in\mathbb{N}^{\coprod_{k=0}^r \mathcal{I}_{r,k}} \\ \sum_W p^{2r-\dim W} b_W = m}} \mathrm{B}\left(\prod_W \mathrm{H}_{r-\dim W}(p) \wr S_{b_W}\right). \tag{11.7}$$

In particular, if $c_{r,m}$ counts its isomorphism classes and $\mu_{r,m}$ denotes its groupoid cardinality, then

$$\sum_{m\ge 0} c_{r,m} z^m = \prod_{k=0}^{r} (1 - z^{p^{2r-k}})^{-I_{r,k}}, \tag{11.8}$$

$$\sum_{m\ge 0} \mu_{r,m} z^m = \exp\left(\sum_{k=0}^{r} \frac{I_{r,k}}{p^{1+2r-2k}} z^{p^{2r-k}}\right). \tag{11.9}$$

For one fixed action of N on mN, the number of extending G_r-actions is $p^m m!\,\mu_{r,m}$.

Proof. Suppose $L \cap N = 1$. Projection to V is injective on L. Every commutator of elements of L belongs to $L \cap N$, so it is trivial. Since G_r has exponent p, L is elementary abelian, and its image W is a vector subspace of V. The commutator formula shows that W is isotropic, so $\dim W = k \le r$.

Conversely, if $W \le V$ is isotropic, then $\pi^{-1}(W) = W \times \mathbb{F}_p$ is an elementary abelian subgroup. The subgroups of $\pi^{-1}(W)$ projecting isomorphically onto W are precisely

$$L_{W,f} = \{(w, f(w)) : w \in W\}, \qquad f \in \mathrm{Hom}_{\mathbb{F}_p}(W, \mathbb{F}_p).$$

Conjugation by (v,a) changes f to $f + \omega(v,-)|_W$. Nondegeneracy of ω implies that

$$V \longrightarrow W^*, \qquad v \longmapsto \omega(v,-)|_W$$

is surjective. Hence all these complements over a fixed W are conjugate. Conjugation acts trivially on $V = G_r/N$, so complements lying over different subspaces cannot be conjugate. This establishes the asserted parametrization.

We can choose $L_W = W \times \{0\}$ as a representative. The same conjugation formula gives

$$N_{G_r}(L_W) = W^{\perp} \times \mathbb{F}_p.$$

The form induced on $W^{\perp}/W$ is symplectic of dimension $2(r-k)$, so

$$N_{G_r}(L_W)/L_W \cong \mathrm{H}_{r-k}(p), \qquad |N_{G_r}(L_W)/L_W| = p^{1+2r-2k}. \tag{11.10}$$

The transitive set G_r/L_W has size p^{1+2r-k}, and its restriction to N has p^{2r-k} free orbits. Every such number is divisible by p^r, and a Lagrangian subspace W with $k = r$ realizes p^r. The spectrum formula follows from Theorem 8.1.

For completeness, count the ordered bases of an isotropic k-subspace. After i independent isotropic vectors have been chosen, their span U has dimension i, and the next vector can be any element of $U^{\perp} \setminus U$, giving $p^{2r-i} - p^i$ choices. Divide the product of these numbers by the number $\prod_{i=0}^{k-1}(p^k - p^i)$ of ordered bases of a fixed k-space. This proves (11.5). Now (11.7) and the two generating functions follow from Theorem 8.1. □

Corollary 11.4 (All minimal lifts)**.** *The least positive multiplicity for which mN extends to G_r is $m = p^r$. At this multiplicity, every extension is transitive, and there are exactly*

$$I_{r,r} = \prod_{j=1}^{r} (p^j + 1)$$

isomorphism classes, parametrized by the Lagrangian subspaces of V. Every one has automorphism group C_p. The number of actions extending one fixed N-action on $p^r N$ is

$$p^{p^r-1}(p^r)! \prod_{j=1}^{r} (p^j + 1).$$

Proof. Every transitive summand contributes at least p^r regular N-orbits. Equality forces one summand indexed by a Lagrangian subspace. Equation (11.5) at $k = r$ simplifies to the stated product, and (11.10) gives C_p. The last formula follows from Corollary 8.3. □

Corollary 11.5 (Cohomological period and lifting index)**.** *Let $\alpha_r \in H^2(V, \mathbb{F}_p)$ be the class of $1 \to N \to G_r \to V \to 1$. Its order is p, whereas the least positive element of the lifting spectrum is p^r. Thus, for fixed p, the ratio between this lifting index and the order of the extension class is unbounded.*

Proof. The cohomology group has exponent p, since its cochain complex consists of $\mathbb{F}_p$-vector spaces. The class α_r is nonzero: a split central extension of the abelian group V by $\mathbb{F}_p$ would be abelian, while G_r has nontrivial commutator form. Its order is therefore p. The lifting index was computed above. □

Here the lifting index means $\min(\Sigma(G_r, N) \setminus \{0\})$; it measures the smallest multiplicity of the regular central action admitting a permutation lift.

Remark 11.6 (Relation with minimal faithful permutation degree). The least cardinality p^{r+1} in this example is also the classical minimal faithful permutation degree of G_r. Indeed every nontrivial normal subgroup of a finite p-group meets its center nontrivially; when the center has order p, it therefore contains the center. Thus every nonempty N-free G_r-set is faithful. Conversely, any faithful action has an orbit on which N acts nontrivially, and such an orbit is N-free and has at least p^{r+1} points. A Lagrangian subgroup attains this bound.

The minimum belongs to the classical theory of faithful permutation representations [9, 6]. For a precise connection, the proof of [6, Theorem 2.1] invokes [9, Theorem 3] for the transitivity of minimal faithful representations of p-groups with cyclic center. The formulas above retain all multiplicities, all isomorphism classes, and their automorphism groups.

12 Logical realizations and parameter elimination

The restriction functors for finite groups occur directly in first-order semantics. The construction below also places the same finite imaginary categories inside a complete theory all of whose models are infinite. Unless a parameter is explicitly named, definability in this section is without parameters.

12.1 A finite torsor theory

Let G be a finite group. In a language with one sort U and a unary function $r_g : U \to U$ for each $g \in G$, let T_G have the following axioms, with free variables universally quantified:

$$\begin{aligned}
&\exists x\,(x = x),\\
&r_1(x) = x, \qquad r_h(r_g(x)) = r_{gh}(x) && (g, h \in G),\\
&r_g(x) \neq r_h(x) && (g, h \in G,\ g \neq h),\\
&\forall x\,\forall y \bigvee_{g \in G} y = r_g(x).
\end{aligned}$$

Thus U is a nonempty free transitive right G-set. The canonical model has $U = G$ and $r_g(x) = xg$. Write $\mathbf{Fin}^G$ for the category of finite left G-sets.

Lemma 12.1. *The theory T_G has a unique model up to isomorphism. Every model has $|G|$ elements, and the automorphism group of the canonical model is G acting by left multiplication. The theory is complete.*

Proof. For any $a \in U$, the function $g \mapsto r_g(a)$ is injective by freeness and surjective by transitivity. The composition axioms make it an isomorphism from the canonical right torsor. If an automorphism of the canonical model takes 1 to b, it takes $g = r_g(1)$ to $r_g(b) = bg$. Conversely, every left translation commutes with all right translations. This proves the assertions about models and automorphisms. A sentence has the same truth value in every model, and completeness of first-order logic makes T_G a complete theory. □

For $m \geq 1$ and $\mathbf{d} = (d_2, \ldots, d_m) \in G^{m-1}$, put

$$\theta_{\mathbf{d}}(x_1, \ldots, x_m) = \bigwedge_{i=2}^{m} x_i = r_{d_i}(x_1).$$

For $m = 1$ the conjunction is empty. For $m = 0$ there is a single empty-tuple pattern, denoted $\top$.

Lemma 12.2. *The formulas $\theta_{\mathbf{d}}$ define precisely the diagonal left G-orbits on U^m. Every invariant subset of a finite power of the canonical torsor is parameter-free definable.*

Proof. For a tuple $(a_1, \dots, a_m) \in G^m$, take $d_i = a_1^{-1} a_i$. Every tuple satisfying $\theta_{\mathbf{d}}$ is uniquely of the form $(b, bd_2, \dots, bd_m)$, hence is a left translate of the given tuple. There are $|G|^{m-1}$ such orbits when $m \geq 1$. An invariant subset is a union of these finitely many orbits, so a finite disjunction of their formulas defines it. The empty union is $\bot$; the empty-context case is immediate. □

Lemma 12.3. *Let $A_1, \dots, A_k$ be objects of $\mathrm{Def}(T_G^{\mathrm{eq}})$, interpreted in the canonical model. Every G-invariant subset of $\prod_i A_i(G)$ is parameter-free definable. In particular, every G-equivariant map between such objects is a definable map.*

Proof. An object of the syntactic pretopos has a presentation as the quotient of a finite disjoint sum of definable sets in real contexts. Choose such presentations for the A_i. Pulling the proposed relation back to the product of their presentations gives, on each summand, a G-invariant subset of a finite power of U. Lemma 12.2 defines each of these subsets. Their finite sum is a definable subobject of the product of the presentations. Its image under the quotient maps is a definable subobject of $\prod_i A_i$ with exactly the desired interpretation.

If the relation is the graph of a map, totality and functionality hold in the canonical model. These properties are first-order statements in the imaginary expansion and translate to first-order statements about representatives. Completeness from Lemma 12.1 therefore makes the relation provably functional and total. It defines a morphism of the syntactic pretopos. □

Theorem 12.4. *Evaluation in the canonical model, with its automorphism action, induces an equivalence of Boolean pretoposes*

$$\mathrm{Def}(T_G^{\mathrm{eq}}) \simeq \mathbf{Fin}^G.$$

For every subgroup $L \leq G$, the transitive action G/L is realized by the definable quotient U/E_L, where

$$E_L(x, y) = \bigvee_{\ell \in L} y = r_\ell(x).$$

Proof. Every real definable set is finite, and finite sums and quotients remain finite. Evaluation therefore takes values in $\mathbf{Fin}^G$. It is exact because interpretations of finite limits, finite sums and effective quotients have their usual set-theoretic meanings. Completeness makes it faithful: two definable maps with the same interpretation have graphs which are provably equivalent. Fullness is Lemma 12.3.

The relation E_L is an equivalence relation because L is a subgroup, and its classes in the canonical model are the cosets gL. Left translations induce the usual transitive left action on G/L. Every finite G-set is a finite disjoint sum of transitive actions G/L, so these quotients and finite coproducts prove essential surjectivity. Exactness of evaluation makes the resulting equivalence an equivalence of pretoposes. □

Proposition 12.5. *Let $H \leq G$, and retain only the automorphisms in H of the canonical model. The equivariant sheaf topos of this one-object groupoid is the category of H-sets, and its coherent part is $\mathbf{Fin}^H$. On coherent objects the induced restriction is*

$$\mathbf{Fin}^G \longrightarrow \mathbf{Fin}^H.$$

If 2 has the trivial action, then

$$|\operatorname{Hom}_H(U, 2)| = 2^{[G:H]}, \qquad |\operatorname{Hom}_G(U, 2)| = 2.$$

More generally, for every $m \geq 1$,

$$|\operatorname{Hom}_H(U^m, 2)| = 2^{[G:H]|G|^{m-1}}, \qquad |\operatorname{Hom}_G(U^m, 2)| = 2^{|G|^{m-1}}.$$

Proof. For a one-object discrete groupoid, an equivariant sheaf is a set with an action of its arrow group. Since H is finite, the compact, hence coherent, objects are exactly the finite H-sets. Restricting the automorphism action gives the stated functor.

Maps to the trivial two-point set correspond to invariant subsets. The left H-orbits on $U = G$ are the $[G : H]$ cosets Hg, whereas G is transitive. For $m \geq 1$, the diagonal H-action on U^m is free, since an element fixing a tuple fixes its first coordinate. It has $|G|^m/|H| = [G : H]|G|^{m-1}$ orbits. The analogous G-orbit count is $|G|^{m-1}$. Choosing an arbitrary union of orbits proves all four formulas. □

For a proper subgroup H, the extra $2^{[G:H]} - 2$ unary predicates are invariant in the restricted semantic groupoid. They are absent from the original parameter-free syntax, whose subobjects of U are just 0 and U. Thus retaining the same model while deleting some of its automorphisms gives a conservative semantic restriction with additional maps and predicates.

12.2 Adding an infinite equality sort

Enlarge the language of T_G by a disjoint sort V carrying only equality. There are no function or relation symbols connecting V and U. Add, for each $n \geq 1$, the axiom that V contains n distinct elements. Denote the resulting theory by $T_{G,\infty}$. Fix a countably infinite set Ω and the countable model

$$M = (G, \Omega).$$

If π is a partition of $\{1, \dots, n\}$, let

$$\eta_\pi(y_1, \dots, y_n) = \bigwedge_{i \sim_\pi j} y_i = y_j \ \wedge \bigwedge_{i \not\sim_\pi j} y_i \neq y_j.$$

The empty-context formula is again $\top$.

Lemma 12.6. *Every formula of $T_{G,\infty}$ with free variables $\mathbf{x} \in U^m$ and $\mathbf{y} \in V^n$ is equivalent, modulo $T_{G,\infty}$, to a finite disjunction of formulas*

$$\theta_{\mathbf{d}}(\mathbf{x}) \wedge \eta_\pi(\mathbf{y}). \tag{†}$$

The theory is complete and has a unique countable model up to isomorphism. Moreover,

$$\operatorname{Aut}(M) = G \times \operatorname{Sym}(\Omega),$$

with G discrete and $\operatorname{Sym}(\Omega)$ carrying the topology of pointwise convergence. The formulas (†) define exactly its orbits on real tuples.

Proof. All terms of sort U reduce to $r_g(x_i)$, and the truth of an equality between such terms is determined by a complete torsor pattern $\theta_{\mathbf{d}}$. All atoms of sort V are equalities, whose truth is determined by a partition pattern η_π. There are no mixed atomic relations. The finitely many patterns partition every real context, and each is realized in every model. Hence Boolean combinations of atoms are finite unions of these patterns.

It remains to check existential quantification. On the U-coordinates, forgetting one coordinate of a torsor orbit gives a torsor orbit on the remaining coordinates. If the first coordinate is forgotten,

the remaining relative labels are obtained by changing the anchor; for example the new labels are $d_2^{-1}d_i$. If the only U-coordinate is forgotten, nonemptiness of the torsor supplies a witness. On the V-coordinates, forgetting one coordinate restricts the partition to the remaining coordinates. When the deleted coordinate was a singleton block, infinitude of V supplies an element different from the finitely many remaining ones; otherwise its value is forced by an existing block. Thus the existential projection of each pattern is exactly a pattern. Quantification distributes over finite disjunction, and universal quantification follows from negation. Induction on formulas proves the normal form.

In an empty context there is one realized pattern. Every sentence is therefore equivalent to either $\top$ or $\bot$, which proves completeness. The finite torsor and countably infinite equality sort each have a unique countable interpretation, so the disjoint combination does as well. The automorphisms on the two sorts are independent, giving the asserted direct product and its topology. Left translations are transitive on each torsor pattern, and a permutation of Ω carries any finite tuple to another with the same equality pattern. This proves the orbit claim. □

Lemma 12.7. *For objects $A_1, \dots, A_k$ of $\mathrm{Def}(T^{\mathrm{eq}}_{G,\infty})$, every $\mathrm{Aut}(M)$-invariant subset of $\prod_i A_i(M)$ is parameter-free definable. Every imaginary object has a continuous automorphism action with finitely many orbits. Every equivariant map between imaginary objects is a parameter-free definable map.*

Proof. Choose a presentation of each imaginary object as the quotient of a finite disjoint sum of real definable sets. Every real context has finitely many automorphism orbits by Lemma 12.6, and its definable subsets are unions of these orbits. Pulling an invariant relation back to the product of the presentations gives, in each real context, a union of finitely many orbits. The explicit formulas (†) define it. Taking its image under the definable quotient maps gives the requested definable relation on the imaginary objects.

The presentations have finitely many orbits, so their quotients do as well. An imaginary element has a real tuple representative, whose pointwise stabilizer is open and fixes that imaginary element. Thus the action on each imaginary set is continuous when that set is discrete.

For an equivariant map, the preceding relation result applies to its graph. Totality and functionality hold in M and are expressible in the imaginary expansion. Translating these assertions through real representatives and applying the completeness in Lemma 12.6 makes them provable. The graph therefore defines a morphism. □

Proposition 12.8. *Let $K = G \times \mathrm{Sym}(\Omega)$. Evaluation gives an equivalence*

$$\mathrm{Def}(T^{\mathrm{eq}}_{G,\infty}) \simeq \mathrm{FinOrb}(K),$$

where $\mathrm{FinOrb}(K)$ denotes continuous actions on discrete sets with finitely many orbits.

Proof. Evaluation takes values in the stated category and is full by Lemma 12.7. Completeness makes it faithful, and interpretations preserve the pretopos operations.

For essential surjectivity, it suffices to realize a transitive continuous action K/L, where L is open. The topology of pointwise convergence supplies a finite real tuple $\mathbf{a}$ with pointwise stabilizer $K_{\mathbf{a}} \subseteq L$. Its orbit $D = K\mathbf{a}$ is definable by the mixed normal form and identifies with $K/K_{\mathbf{a}}$. The natural surjection

$$K/K_{\mathbf{a}} \twoheadrightarrow K/L$$

has a K-invariant kernel equivalence relation on D. Lemma 12.7, or directly the mixed normal form on pairs of tuples, makes this relation parameter-free definable. Its imaginary quotient realizes K/L. Finite disjoint sums realize every action with finitely many orbits. □

Lemma 12.9. *Every continuous action of* $\mathrm{Sym}(\Omega)$ *on a finite discrete set is trivial.*

Proof. Such an action is a continuous homomorphism $\rho : \mathrm{Sym}(\Omega) \to \mathrm{Sym}(X)$ with X finite. Its kernel N is open, closed and normal. Openness gives a finite subset $F \subseteq \Omega$ such that the pointwise stabilizer of F lies in N. A transposition of two elements outside F belongs to that stabilizer. Normality then puts every transposition in N, so N contains all finitely supported permutations.

These permutations are dense. Indeed, for a permutation σ and finite $E \subseteq \Omega$, its restriction to E extends to a permutation of the finite set $E \cup \sigma(E)$, and extending by the identity elsewhere gives a finitely supported permutation agreeing with σ on E. The closed subgroup N contains this dense subgroup, hence equals $\mathrm{Sym}(\Omega)$. □

Definition 12.10. Let $\mathcal{B}^{\mathrm{fin}}_{G,\infty}$ be the full subcategory of $\mathrm{Def}(T^{\mathrm{eq}}_{G,\infty})$ consisting of objects A for which $A(M)$ is finite. An equivalent condition is that $T^{\mathrm{eq}}_{G,\infty}$ proves $|A| = n$ for some nonnegative integer n.

The equivalence of the two conditions follows from completeness: the assertion of exactly n elements in an imaginary sort translates to a first-order assertion about real representatives. Consequently these objects have the same finite cardinality in every model. The subcategory is a Boolean pretopos, since finite limits, finite coproducts, subobjects and effective quotients preserve finiteness in M.

Theorem 12.11. *Evaluation induces an equivalence of Boolean pretoposes*

$$\mathcal{B}^{\mathrm{fin}}_{G,\infty} \simeq \mathbf{Fin}^G.$$

Every finite parameter-free imaginary object is parameter-free definably isomorphic to a finite disjoint sum of torsor quotients U/E_L*, with* $L \leq G$*. Under these equivalences the inclusion of the torsor language induces an equivalence*

$$\mathrm{Def}(T^{\mathrm{eq}}_{G}) \xrightarrow{\sim} \mathcal{B}^{\mathrm{fin}}_{G,\infty}.$$

Proof. For a finite imaginary object A, its automorphism action is continuous by Lemma 12.7. Lemma 12.9 makes the factor $\mathrm{Sym}(\Omega)$ act trivially. The action therefore factors through the finite group G.

Any G-equivariant map between two such finite objects is also equivariant for the full product $G \times \mathrm{Sym}(\Omega)$, since the second factor acts trivially on both. Its graph is parameter-free definable by Lemma 12.7. Thus evaluation into $\mathbf{Fin}^G$ is full; it is faithful by completeness and exact by the interpretation of pretopos operations.

Every finite G-set is a finite disjoint sum of orbits G/L. These are already interpreted by U/E_L in the torsor language, proving essential surjectivity. To see the claimed definable description for a given A, choose a G-equivariant bijection from such a finite sum onto $A(M)$. Both sides have trivial $\mathrm{Sym}(\Omega)$-action, so the bijection and its inverse are definable by the same invariant-graph argument. They give the required parameter-free definable isomorphism.

Finally, the inclusion of languages interprets the same torsor quotients and maps in the enlarged theory. Theorem 12.4 and the equivalence just proved identify this inclusion with the identity of $\mathbf{Fin}^G$. It is therefore an equivalence onto the finite imaginary subcategory. □

Remark 12.12. The finite imaginary qualification is essential. The whole pretopos of $T_{G,\infty}$ contains the infinite sort V and is not equivalent to $\mathbf{Fin}^G$. Indeed, for every $n \geq 1$ the definable object of pairwise distinct n-tuples in V^n is noninitial. An equivalence of Boolean pretoposes preserves

products, diagonals, their complements and the initial object. If it sent V to a finite G-set X, it would therefore send these objects to the sets of pairwise distinct n-tuples in X^n. The latter is empty for $n > |X|$, a contradiction.

Corollary 12.13. *In the countable model of $T_{G,\infty}$, replace the full automorphism group $G \times \mathrm{Sym}(\Omega)$ by $H \times \mathrm{Sym}(\Omega)$. On finite continuous actions the resulting semantic restriction is precisely*

$$\mathbf{Fin}^G \longrightarrow \mathbf{Fin}^H.$$

In particular, all predicate counts of Proposition 12.5 remain valid, although every model of the underlying complete theory is infinite.

Proof. Every finite continuous action of either product group has trivial $\mathrm{Sym}(\Omega)$-action by Lemma 12.9. The finite-action categories are therefore $\mathbf{Fin}^G$ and $\mathbf{Fin}^H$, and restriction acts through the inclusion $H \leq G$. The interpreted torsor and its powers have the same finite underlying actions as before. The count formulas follow directly. □

12.3 Naming one element in an infinite pure set

Let T_∞ be the complete theory of an infinite set in the language of equality, and let Ω be a countably infinite model. Give

$$G = \mathrm{Sym}(\Omega)$$

the topology of pointwise convergence. Fix $a \in \Omega$ and put $H = G_a$. Naming a replaces G by the open point stabilizer H. Both groups are oligomorphic. Their coherent continuous actions are therefore the actions with finitely many orbits, including actions on infinite underlying sets.

The parameter-free definable-imaginary category of T_∞ is equivalent to the category of finitary continuous G-sets. Here this identification can be seen directly. Formulas in finite tuples are Boolean combinations of equality patterns, so a definable set has finitely many G-orbits. Conversely, an open subgroup $U \leq G$ contains the pointwise stabilizer of a finite tuple $\bar{b}$. The transitive action G/U is a quotient of the orbit $G\bar{b}$. The kernel relation of this quotient is a G-invariant relation on a finite tuple space. Since there are only finitely many equality patterns on pairs of such tuples, this relation is parameter-free definable. Its quotient is an imaginary object. Finite coproducts give all finitary actions, and the same argument applied to graphs gives all equivariant maps. With the constant a added, equality patterns involving a give the corresponding description by finitary continuous H-sets.

Consider the transitive H-set

$$X = \Omega \setminus \{a\},$$

defined by the formula $x \neq a$. Its failure to descend is stronger than the failure of this particular subset to be parameter-free definable.

Proposition 12.14. *The following assertions hold.*

1. *The given H-action on X has no extension to an action of G, even if continuity of the proposed extension is omitted. In particular, X is not isomorphic to the restriction of any parameter-free definable-imaginary object of T_∞.*

2. *The H-set $1 \amalg X$ does extend to G: it is isomorphic to the restriction of the natural action on Ω.*

3. Let

$$A = \{(x, y) \in \Omega^2 : x \neq y\}.$$

The map

$$p : \mathrm{Res}_H^G A \longrightarrow X, \qquad p(x, y) = \begin{cases} x, & x \neq a, \\ y, & x = a, \end{cases}$$

is an H-equivariant epimorphism from a coherent source object. It is a one-chart finite source atlas, and its kernel relation is not G-invariant.

Proof. Suppose first that a homomorphism

$$\alpha : G \longrightarrow \mathrm{Sym}(X)$$

extends the natural action of H. Fix $b \in X$ and let

$$K = G_{a,b} = \mathrm{Sym}(\Omega \setminus \{a, b\}), \qquad t = (a\ b).$$

The action of K on X has the unique fixed point b and is the full symmetric action on $X \setminus \{b\}$. Its centralizer in $\mathrm{Sym}(X)$ is trivial. Indeed, a commuting permutation must fix the unique fixed point b, and its restriction to $X \setminus \{b\}$ centralizes the full symmetric group of that infinite set. This latter centralizer is trivial: if a permutation sends y to $z \neq y$, a transposition fixing y and moving z does not commute with it.

The transposition t commutes with every element of K. Therefore $\alpha(t)$ centralizes the given K-action on X and must be the identity. The same argument holds for every $b \neq a$. Choose distinct $b, c \in X$. Since

$$(b\ c) = (a\ b)(a\ c)(a\ b),$$

we obtain $\alpha((b\ c)) = 1$. But $(b\ c) \in H$, and its prescribed action on X is a nontrivial transposition. This contradiction proves the first assertion. Any isomorphism from a restricted G-set to X would transport its action to such an extension, so the same argument rules out lifting up to isomorphism.

For the second assertion, the bijection

$$1 \amalg X \longrightarrow \Omega, \qquad * \longmapsto a, \quad x \longmapsto x,$$

is H-equivariant. Transporting the natural continuous G-action across this bijection gives the extension. Elements of G moving a interchange points from the two H-orbits.

Finally, A is a transitive continuous G-set, and hence is coherent. Its restriction has three H-orbits, according as the first entry is a, the second entry is a, or neither entry is a. The map p is well-defined because $x = a$ implies $y \neq a$. It is H-equivariant because H fixes a, and it is surjective because $p(a, b) = b$ for every $b \in X$. Its graph is given by the formula

$$(x \neq y) \wedge \big((x \neq a \wedge z = x) \vee (x = a \wedge z = y)\big),$$

so the atlas is definable with the named parameter.

Its kernel fails invariance explicitly. Take pairwise distinct a, b, c. The points $(a, b), (b, c) \in A$ have the same image b. Applying $g = (a\ b)$ sends them to $(b, a), (a, c)$, whose images are b and c, respectively. Thus a pair belonging to the kernel is carried outside it by an arrow of the full group. □

This example exhibits an object defined after naming one parameter that has no parameter-free lift, despite having a finite source atlas. The obstruction lies in the kernel relation of that atlas. Adding one fixed point yields a liftable object. Here the point stabilizer H is not normal in G, so the two H-orbit types can occur together inside a single transitive G-action. The independent orbit-type conditions for a normal subgroup do not apply to this inclusion.

12.4 A coherent object without a source atlas

Example 12.15 (An infinite partition)**.** Let Ω be countably infinite, write $\Omega = P \amalg Q$ with both pieces infinite, and take

$$G = \mathrm{Sym}(\Omega), \qquad H = \mathrm{Sym}(P) \times \mathrm{Sym}(Q)$$

with pointwise-convergence topologies. Both groups are oligomorphic. The inclusion is closed, and restriction is coherent. Nevertheless, the transitive continuous H-set P admits no morphism from the restriction of any nonempty continuous G-set. In particular it admits no finite source atlas, although it is a coherent subobject of $\mathrm{Res}^G_H \Omega$.

Proof. Orbits of G on Ω^n are determined by equality patterns. Orbits of H are determined by equality patterns together with the choice of P or Q for each equivalence class of coordinates. There are finitely many such patterns for every n.

Every open subgroup $U \leq G$ contains the pointwise stabilizer $G_{(F)}$ of some finite subset $F \subseteq \Omega$. The set $G/G_{(F)}$ identifies with the injections $F \hookrightarrow \Omega$. The group H has finitely many orbits on this set, determined by the colour of each image coordinate. The quotient G/U therefore has finitely many H-orbits. Theorem 4.2 proves coherence of restriction.

Suppose that an H-equivariant map $u : \mathrm{Res}^G_H(G/U) \to P$ existed. Choose finite F with $G_{(F)} \subseteq U$, and choose $g \in G$ such that $gF \subseteq Q$. The H-stabilizer of gU contains

$$\mathrm{Sym}(P) \subseteq H \cap G_{(gF)} \subseteq H \cap gUg^{-1}.$$

Consequently $u(gU)$ would be fixed by all permutations of P. The full symmetric group of the infinite set P has no fixed point on P, a contradiction. Every nonempty continuous G-set contains a transitive orbit G/U, so no map from its restriction to P can exist. The last assertion follows because P is an H-invariant subset of Ω and is itself transitive, hence coherent. □

This example shows that coherent restriction and finite orbit decompositions do not imply the finite-atlas hypothesis of Corollary 3.11. It also separates new coherent subobjects from quotients presented by the original coherent objects.

13 Further questions

The results leave three concrete problems. First, for a coherent restriction of open groupoids, one can ask for semantic hypotheses ensuring that every target coherent object has a finite source atlas. Profinite actions satisfy this property by Proposition 4.4. Example 12.15 shows that coherent restriction between oligomorphic action toposes does not suffice. A criterion in terms of stabilizers and the topology of the object space would extend the presentation theorem beyond the cases treated here.

Second, for a nonnormal subgroup of a finite group, restriction is described by the full double-coset matrix rather than independent orbit-type blocks. The resulting existence problem is membership in the nonnegative integer image of that matrix. Determining useful structural conditions for saturation, minimal generators, or a bounded correction by additional orbit types would give a broader counterpart of the conductor calculation. The one-parameter example in Section 12.3 shows why interactions between different orbit types matter.

Third, Proposition 10.1 proves existence of a finite quotient detecting every multiplicity for a finite-kernel profinite extension. Its proof chooses witnesses for generators of an additive monoid.

An effective bound on the detecting quotient, for groups given by an explicit inverse system, would turn this existence result into an algorithm. Such a bound would concern multiplicities; Example 10.3 shows that the complete groupoid of actions can retain additional information.

A Exact finite calculations

The supplementary program `finite_lifting.py` uses only the Python standard library. It constructs finite multiplication tables, enumerates subgroups by adjoining generators until closure, computes conjugacy classes and normalizers, and records the restriction weights. All arithmetic is exact. The supplementary output `finite_results.json` contains the resulting subgroup counts, orbit data, and coefficients of the generating series. The proofs in the main text do not depend on these calculations.

For $\mathrm{SL}_2(\mathbb{F}_5)$ the program constructs all determinant-one matrices and finds 120 elements and 76 subgroups. The subgroups disjoint from the center give the following data.

$\lvert L\rvert$	conjugates of L	$\lvert N_G(L)\rvert$	w_L	$\lvert N_G(L)/L\rvert$
1	1	120	60	120
3	10	12	20	4
5	6	20	12	4

These values verify the three factors in (11.3). At multiplicity 60 there are three isomorphism classes, represented by one regular G-orbit, three G/C_3-orbits, or five G/C_5-orbits. The computed gaps among multiples of 4 are $4, 8, 16, 28$, as proved in Theorem 11.1.

For $\mathrm{H}_1(3)$ the program uses upper unitriangular coordinates, which are isomorphic to the symmetric coordinates of Section 11.3 by replacing the central coordinate z with $z - xy/2$. It finds 27 elements and 19 subgroups. The subgroups disjoint from the center consist of the trivial subgroup and four conjugacy classes of subgroups of order 3. Their weights are respectively 9 and 3, with automorphism-group orders 27 and 3. Hence

$$\sum_m c_{1,m} z^m = \frac{1}{(1-z^3)^4(1-z^9)}, \qquad \sum_m \mu_{1,m} z^m = \exp\left(\frac{4z^3}{3} + \frac{z^9}{27}\right).$$

The program also checks the cyclic cases $C_{p^2} \supset C_p$ for $p = 2, 3, 5$, and evaluates the isotropic-subspace counts in (11.5) using integer arithmetic.

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