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The Schur–Siegel–Smyth trace problem: the lower bound 1.8022246

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Abstract

We prove $\lambda_{\mathrm{SSS}} \geq 9011123/5000000 = 1.8022246$ for the smallest limiting trace-to-degree ratio of totally positive algebraic integers. The previous best bound is 1.8022225 (Tomar). The certificate consists of fourteen integer polynomials with rational multipliers and a continuous piecewise-linear density with 8 678 rational hat weights; its dual inequality is verified on $(0, \infty)$ in ball arithmetic. For the same family of polynomials the value of the underlying optimisation problem is 1.8022247 to within 10^{-9} (numerical), so the bound is within $1.1 \cdot 10^{-7}$ of what this family can give.

1 Statement

For a totally positive algebraic integer α of degree d write $\overline{\mathrm{tr}}(\alpha) = \mathrm{tr}(\alpha)/d$, and let $\lambda_{\mathrm{SSS}} = \liminf \overline{\mathrm{tr}}(\alpha)$.

Theorem 1 (PROVED, computer-assisted)**.** $\lambda_{\mathrm{SSS}} \geq \dfrac{9011123}{5000000} = 1.8022246.$

Previous lower bounds: 1.80203 by Orloski, Talebizadeh Sardari and Smith [2]; 1.80220 in [1]; 1.8022225 by Tomar [4]. Several upper bounds obtained from Smith's theorem [3] have been announced since October 2026, for instance 1.8070809 [4] and 1.80560940 [5].

2 The criterion

Lemma 1 ([2]; Proposition 2 of [1]; Lemma 2.1 of [4])**.** *Let $Q_1, \dots, Q_k \in \mathbb{Z}[x]$ be nonzero, $\lambda_Q \geq 0$, and let $g \geq 0$ be a bounded density with compact support and mass $s > 0$. Put $U_g(x) = \int \log|x-y|\, g(y)\, dy$ and $I(g) = \int U_g\, g$. If*

$$F(x) = x - \lambda - \sum_{Q} \lambda_Q \log|Q(x)| - 2U_g(x) + \frac{I(g)}{s} \;\geq\; 0$$

for every $x > 0$ that is not a root of one of the Q, then $\lambda_{\mathrm{SSS}} \geq \lambda$.

3 The certificate

The polynomials and multipliers are listed in Table 1; the polynomial $x^3 - 6x^2 + 9x - 1$ receives multiplier 0, so fourteen polynomials are used. The set is the one of [4]. The density is

$$g = \sum_{k=1}^{8678} \rho_k h_k,$$

where h_k is the hat function of mass one on $[e_{k-1}, e_{k+1}]$ with peak at e_k, for nodes $e_0 < \cdots < e_{8679}$ that are rationals with denominator 2^{44} in $[0.045, 5.369]$, and ρ_k are rationals with denominator 2^{64}. Exactly 6 067 weights are nonzero; the support of g is contained in $[0.04888, 5.35200]$, and

$$s = 0.535259605221817\ldots, \qquad I(g) = -8.42875692345304\ldots \cdot 10^{-8}.$$

The exact data are in the ancillary file `cert_SSS_1802224600.json` (format of [4]: keys `A`, `lamQ`, `nodes`, `rho`, `lambda`, `x_lo`$= 1/100$, `x_hi`$= 8$).

Q	λ_Q
x	0.503429120622
$x-1$	0.437992939258
$x-2$	0.042387356574
x^2-3x+1	0.156351499433
x^3-5x^2+6x-1	0.058154222706
$x^4-7x^3+13x^2-7x+1$	0.017297099538
$x^4-7x^3+14x^2-8x+1$	0.017604850341
$x^5-9x^4+28x^3-35x^2+15x-1$	0.002319719944
$x^5-9x^4+26x^3-29x^2+11x-1$	0.000660717275
$x^5-9x^4+27x^3-32x^2+13x-1$	0.000388680112
x^3-6x^2+9x-1	0
$x^5-9x^4+27x^3-31x^2+12x-1$	0.002585133289
$x^{10}-18x^9+134x^8-538x^7+1273x^6-1822x^5+1560x^4-766x^3+200x^2-24x+1$	0.000413411866
$x^{10}-18x^9+135x^8-549x^7+1320x^6-1920x^5+1662x^4-813x^3+206x^2-24x+1$	0.000395645259
$x^{10}-18x^9+134x^8-537x^7+1265x^6-1798x^5+1526x^4-743x^3+194x^2-24x+1$	0.000392591884

Table 1: Polynomials and multipliers (decimals rounded; exact rationals in the ancillary file).

4 Verification

Since g is continuous and piecewise linear, $g'' = \sum_n C_n \delta_{e_n}$ with exact rational C_n, and

$$U_g(x) = \sum_n C_n\, G(x - e_n), \qquad I(g) = \sum_{n,m} C_n C_m\, \Psi(e_n - e_m),$$

with $G(t) = \frac{t^2}{2}\log|t| - \frac{3}{4}t^2$ and $\Psi(t) = \frac{t^4}{24}\log|t| - \frac{25}{288}t^4$ ($G'' = \Psi'''' = \log|t|$). The program `check_sss.py` (ancillary) computes everything in arb ball arithmetic at 192 bits, starting from the exact rational data:

- the roots of the polynomials are certified real by Sturm sequences and enclosed in rational intervals of width 2^{-90};
- $I(g)$ is computed from the closed form above, its polynomial part exactly;
- on $(0, 1/100]$ and $[8, \infty)$, F is bounded below by monotonicity: the bounds are $0.48677\ldots$ and $0.53568\ldots$;
- $[1/100, 8]$ is cut at every node, at the root enclosures and into blocks; on each block the contribution of distant nodes is a Taylor polynomial of degree 24 with an explicit remainder; on each cell F is bounded below by a second-order bound, and by chords or tangents for the terms singular at the cell ends; failing cells are bisected.

Result: 98 710 cell checks, all positive; total time 4.6 minutes on two cores. Verdict: $F \geq 0$ on $(0, \infty)$ with $\lambda = 9011123/5000000$, which proves Theorem 1.

Controls (VERIFIED).

1. With $\lambda = 1.80222462$ (two units of 10^{-8} higher) the same program returns certified negative values, e.g. $F(0.049625) = -1.550 \cdot 10^{-8}$ and $F(0.0615) = -1.190 \cdot 10^{-8}$ (radii below 10^{-16}). The certificate therefore proves 1.8022246 and nothing above 1.80222462.

2. On the certificates of [4] the program reproduces the published values of $I(g)$ to all printed digits, returns VERIFIED for his bounds $1.8021892\ldots$ and 1.8022225, and returns certified negative values at $\lambda + 3 \cdot 10^{-6}$ for his first certificate at $x = 0.0996570746$ and $x = 0.1308260950$, the points reported by his own program.

3. The values of F at five points of $[0.05, 3.7]$ were recomputed by direct numerical integration of $\int \log|x-y| g(y)\, dy$ cell by cell (mpmath, 40 digits), without the closed forms; they agree with the ball values to twelve digits.

5 Remarks (numerical)

Remark. For the fourteen polynomials of Table 1, the best bound obtainable from Lemma 1 equals the value of the primal problem: minimise $\int x\, d\mu$ over probability measures μ on $[0, \infty)$ with $\int \log|Q|\, d\mu \geq 0$ and $I(\mu) \geq 0$. On a mesh of 8 680 nodes its value is 1.8022247019, stable to 10^{-9} under refinement. The bound of Theorem 1 is $1.02 \cdot 10^{-7}$ below it.

Remark. For the optimal measure μ of this problem, every irreducible monic $P \in \mathbb{Z}[x]$ with all roots in $[0, 5.3]$, degree at most 7 and trace at most twice the degree (883 polynomials), and every such P of degree 8 with roots in $[0, 5.4]$ (3 402 polynomials), satisfies $\int \log|P|\, d\mu \geq 0$. Outside Table 1 the smallest value is $+0.0407$ (degree 7) and $+0.160$ (degree 8). Since μ satisfies these constraints, adding any of these polynomials leaves the optimal value unchanged.

Acknowledgements

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References

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