IDEAL GENERATION FOR GENERAL REDUCED PROJECTIVE POINTS

VLADO VOJDANOVSKI

Abstract. We prove the Ideal Generation Conjecture for general reduced points in projective space over an algebraically closed field of characteristic zero. Equivalently, if the ideal of the points has initial degree D, multiplication by linear forms from degree D to degree $D+1$ has maximal rank. The proof combines hyperplane specialization with a construction using two transverse hyperplanes. In the latter construction, motion of their common trace gives the required rank. A uniform induction for all degrees $D \geq 4$ reduces the theorem to explicit quadratic and cubic constructions.

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1. Introduction

Let Z be a general set of p points in $\mathbb{P}^{n-1}$, and let $I_Z \subset S = k[x_1, \ldots, x_n]$ be its homogeneous ideal. The Hilbert function of S/I_Z is $\min\{p, \dim_k S_j\}$ in degree j. Determining the minimal generators of I_Z asks a further question: how much of its first nonzero component is needed to generate the next one? The Ideal Generation Conjecture predicts that multiplication by linear forms has the largest rank allowed by these dimensions.

The minimal generators of ideals of general points were studied by Geramita and Orecchia [1] and Geramita and Maroscia [2]. For a recent formulation of the conjecture and its relation to ideals generated by a single homogeneous component, see Gesmundo, Kayser, and Telen [3, Sections 2.2–2.3]. We prove the following statement.

Theorem 1.1. *Let k be an algebraically closed field of characteristic zero, let $n \geq 2$ and $D \geq 2$, and put*

$$N_j(n) = \binom{n+j-1}{j}.$$

For every integer p satisfying

$$N_{D-1}(n) \leq p < N_D(n),$$

a general reduced set Z of p points in $\mathbb{P}^{n-1}$ has maximal-rank multiplication

$$(I_Z)_D \otimes_k S_1 \longrightarrow (I_Z)_{D+1}.$$

In particular, put $r = N_D(n) - p$. The ideal has r minimal generators in degree D and

$$\max\left\{0, (n-1)\left(\frac{N_D(n)}{D+1} - r\right)\right\}$$

in degree $D+1$. There are no generators in higher degrees: the scalar postulation of general points gives $(D+1)$-regularity of their ideal sheaf. For a set of points spanning a proper linear subspace, its linear equations can first be split off and the theorem applied in its span.

The proof treats ideal generation as interpolation for $\Omega_{\mathbb{P}^{n-1}}(D+1)$ via the Euler sequence. This places the argument within the vectorial Horace method; compare Hirschowitz and Simpson [4] and Lauze [5, Section 3]. All the specialization arguments used here are developed explicitly.

For every $D \geq 4$, we deduce the degree-D statement from the statements in degrees $D-1$ and $D-2$. At fixed degree, hyperplane specialization, including a two-level step for fractional critical counts, covers $n \leq D(D-1)+1$. For larger n, we use two hyperplanes with common trace $\mathbb{P}^{n-3}$. Scalar interpolation identifies both the kernel and the cokernel of a specialized evaluation map.

Moving the trace in its two normal directions then supplies a maximal-rank first variation. When the trace itself has a kernel, one residual point with a free projection detects it.

The quadratic and cubic cases start this induction. Their explicit constructions and finite determinant calculations are given in Appendices A–C. Degree four is the first application of the same induction: the hyperplane steps reach thirteen variables, and the two-normal steps cover every larger dimension. Section 6 assembles the transfers into the proof.

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2. Ideal generation and cotangent interpolation

Throughout, k is an algebraically closed field of characteristic zero. We write

$$S = k[x_1, \dots, x_n], \qquad N_j(n) = \dim_k S_j = \binom{n+j-1}{j}, \qquad E(n,D) = \frac{N_D(n)}{D+1}.$$

If Z is a finite reduced set of points, I_Z denotes its homogeneous ideal and $\mathcal{I}_Z$ its ideal sheaf. A statement about general ordered points means that it holds on a nonempty Zariski open subset of the space of distinct ordered points. All the conditions considered below are unchanged by reordering.

2.1. **Scalar interpolation and the two critical counts.** General points impose independent conditions on forms: for every j,

$$\dim_k (I_Z)_j = \max\{0, N_j(n) - |Z|\}. \tag{2.1}$$

Indeed, evaluation functionals at points span S_j^*, since a form vanishing at every point is zero. Successively choosing independent functionals gives a configuration of the required rank, and the nonvanishing of the corresponding minor gives the assertion for general points. Finitely many such rank conditions hold simultaneously.

Suppose

$$N_{D-1}(n) \le p < N_D(n), \qquad r = N_D(n) - p. \tag{2.2}$$

For points with the scalar postulation in degrees $D-1, D, D+1$, the multiplication map

$$\mu_Z : (I_Z)_D \otimes_k S_1 \longrightarrow (I_Z)_{D+1} \tag{2.3}$$

has source dimension nr and target dimension $N_{D+1}(n) - p$. Their difference is

$$nr - \big(N_{D+1}(n) - p\big) = (n-1)\big(r - E(n,D)\big). \tag{2.4}$$

Thus the critical ideal dimensions are $\lfloor E(n,D) \rfloor$ and $\lceil E(n,D) \rceil$. They belong to the range in (2.2): the upper bound follows from

$$E(n,D) \le N_D(n) - N_{D-1}(n) = N_D(n-1),$$

which is equivalent to $D(n-2) \ge 0$.

Lemma 2.1 (Reduction to the critical counts)**.** *To prove maximal rank of (2.3) for general points throughout (2.2), it is enough to prove injectivity when $r = \lfloor E(n,D) \rfloor$ and surjectivity when $r = \lceil E(n,D) \rceil$.*

Proof. Adding points replaces $(I_Z)_D$ by a subspace, so an injective multiplication map remains injective. For surjectivity, let $Y \subset Z$ and assume that evaluation of degree-D forms on Z is onto. Choose a linear form ℓ nonzero at every point of Z and use the affine trivializations $\ell = 1$. Given $g \in (I_Y)_{D+1}$, scalar interpolation supplies $f \in S_D$ which is zero on Y and agrees with g on $Z \setminus Y$. Then $f \in (I_Y)_D$ and $g - \ell f \in (I_Z)_{D+1}$. If μ_Z is onto, this difference is a sum of linear multiples of elements of $(I_Z)_D \subset (I_Y)_D$, proving that μ_Y is onto. The relevant evaluation kernels form vector bundles on the scalar postulation open sets; maximal rank of multiplication is an open condition there. Hence each configuration constructed in this way proves the assertion for general points of its cardinality. □

We write $\mathrm{IGC}(n,D)$ for the assertion of this lemma for all the point counts in (2.2). We write $\mathrm{Inj}(n,D,r)$ for its injective assertion at ideal dimension r.

2.2. **The Euler sequence.** Let

$$\mathcal{E}_{n,D} = \Omega_{\mathbb{P}^{n-1}}(D+1).$$

The Euler sequence identifies its sections with polynomial relations:

$$H^0(\mathcal{E}_{n,D}) = \left\{ (F_1, \dots, F_n) \in S_D^n : \sum_{i=1}^{n} x_i F_i = 0 \right\}. \tag{2.5}$$

Its rank is $n-1$ and its space of sections has dimension $nN_D(n) - N_{D+1}(n)$.

Lemma 2.2 (Euler evaluation)**.** *Suppose Z imposes independent conditions in degrees D and $D+1$. Then*

$$\mathrm{ev}_Z : H^0(\mathcal{E}_{n,D}) \longrightarrow H^0(\mathcal{E}_{n,D}|_Z)$$

and μ_Z have naturally isomorphic kernels and cokernels. In particular, they have maximal rank simultaneously.

Proof. Compare the Euler sequence on global sections with its restriction to Z. The two rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathcal{E}_{n,D}) & \longrightarrow & S_D^n & \longrightarrow & S_{D+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(\mathcal{E}_{n,D}|_Z) & \longrightarrow & H^0(\mathcal{O}_Z(D))^n & \longrightarrow & H^0(\mathcal{O}_Z(D+1)) & \longrightarrow & 0 \end{array}$$

are exact. The last arrow of the bottom row is onto because the homogeneous coordinates do not all vanish at any point. The last two vertical arrows are onto by scalar postulation. The snake lemma gives

$$0 \longrightarrow \ker \mathrm{ev}_Z \longrightarrow (I_Z)_D^n \xrightarrow{\mu_Z} (I_Z)_{D+1} \longrightarrow \operatorname{coker} \mathrm{ev}_Z \longrightarrow 0,$$

as required. □

2.3. **Rank under deformation.** All configurations used below are actual point configurations, and all limiting calculations concern their evaluation maps. We record the elementary matrix principle used in these calculations.

Lemma 2.3 (First variation of rank)**.** *Let $A(\varepsilon) : V \to W$ be a matrix regular at $\varepsilon = 0$, and put $K = \ker A(0)$ and $Q = \operatorname{coker} A(0)$. The coefficient of ε induces a map $J : K \to Q$. If J has maximal rank, then $A(\varepsilon)$ has maximal rank for general ε.*

Proof. Choose complements to K and to $\operatorname{im} A(0)$. The block from the first complement to $\operatorname{im} A(0)$ is invertible at zero. Eliminating it leaves a block $\varepsilon J + O(\varepsilon^2)$ from K to Q. A nonzero maximal minor of J is the leading coefficient of the corresponding minor of this block. Together with the invertible block it gives a minor of size $\operatorname{rank} A(0) + \min\{\dim K, \dim Q\} = \min\{\dim V, \dim W\}$. □

When several open rank conditions are imposed on a single irreducible parameter space, it is enough to exhibit a point for each condition separately. Their nonempty open loci then intersect. In each application below, the parameter space and the projection to any auxiliary point configuration are specified explicitly.

3. Hyperplane specialization

We write $\mathrm{Inj}(n, D, r)$, respectively $\mathrm{Surj}(n, D, r)$, when multiplication in degrees $D, D+1$ is injective, respectively surjective, for general points whose degree-D ideal has dimension r. In this notation a point-count cell is

$$1 \le r \le N_D(n-1).$$

All point configurations below are ordered and have distinct points. Conditions are called independent when evaluation onto their stated condition space is surjective. Their parameter spaces, as well as the open subsets obtained by requiring points to avoid a fixed hyperplane, are irreducible.

Lemma 3.1 (Ordinary hyperplane specialization)**.** *Let $n \geq 3$, $D \geq 2$, and suppose the counts below belong to their point-count cells.*

(1) If $\mathrm{Inj}(n-1, D, r_2)$ *and* $\mathrm{Inj}(n, D-1, r_1)$ *hold, and*

$$(n-1)r_1 + r_2 \leq N_D(n-1),$$

then $\mathrm{Inj}(n, D, r_1 + r_2)$ *holds. The case $r_1 = 0$ is allowed, with no residual injectivity assumption.*

(2) If $\mathrm{Surj}(n-1, D, r_2)$ *and* $\mathrm{Inj}(n, D-1, r_1)$ *hold, where $r_1 \geq 1$, and*

$$(n-1)r_1 + r_2 \geq N_D(n-1),$$

then $\mathrm{Surj}(n, D, r_1 + r_2)$ *holds.*

Proof. Put $H = (h = 0)$ and $R = S/(h)$. Choose points Z_1 off H, independent in degree $D-1$, and points Z_2 on H, independent in degree D, so that

$$W = I_{Z_1}(D-1), \qquad V_H = I_{Z_2,H}(D), \qquad \dim W = r_1, \quad \dim V_H = r_2.$$

Let $\rho : S \to R$ be restriction, $V = I_{Z_1 \cup Z_2}(D)$, and $U = \rho(WS_1)$. A lift of a polynomial on H can be corrected at Z_1 by a multiple of h, since degree-$(D-1)$ evaluation on Z_1 is onto. Consequently

$$0 \longrightarrow hW \longrightarrow V \xrightarrow{\rho} V_H \longrightarrow 0, \qquad I_{Z_1}(D) \xrightarrow{\rho} R_D \longrightarrow 0.$$

The second map has kernel hW, and $WS_1 \cap hS_{D-1} = hW$: if hG vanishes on Z_1, then G vanishes there because h is nonzero at those points. Thus residual injectivity gives $\dim U = (n-1)r_1$.

Choose Z_1 first. Evaluation of the fixed space U at general points of H has maximal rank. Indeed, unless the current kernel is zero, a further point can be chosen where one of its nonzero polynomials does not vanish. This constructs a maximal-rank evaluation, and a nonzero minor makes the condition open. It follows that, simultaneously with the asserted trace multiplication property, we may arrange $U \cap V_H = 0$ in the first case and $U + V_H = R_D$ in the second.

Choose a basis w_j of W and lifts $\widetilde{g}_i$ of a basis g_i of V_H. In the injective case, a relation

$$\sum_j hw_j a_j + \sum_i \widetilde{g}_i b_i = 0, \qquad a_j, b_i \in S_1,$$

restricts to a trace relation. Trace injectivity gives $b_i = c_i h$, with $c_i \in k$. Divide by h and restrict once more. The resulting combination $\sum_i c_i g_i$ belongs to $U \cap V_H$, so every c_i vanishes. Residual injectivity now gives $a_j = 0$.

In the surjective case, let $\widetilde{G}$ denote the span of the lifts. Since $WS_1 + \widetilde{G}$ maps onto R_D and contains hW, it equals $I_{Z_1}(D)$. For $F \in I_{Z_1 \cup Z_2}(D+1)$, trace surjectivity allows us to subtract an element of $\widetilde{G}S_1$ with the same trace. The remainder is hG with $G \in I_{Z_1}(D) = WS_1 + \widetilde{G}$, and therefore belongs to VS_1. Finally, the exact sequence for V gives the required scalar postulation; openness passes the conclusion to general points. □

Corollary 3.2 (Integral trace)**.** *Suppose $E(n-1, D)$ is an integer, the ideal generation statement holds in degree D in $n-1$ variables, and*

$$\mathrm{Inj}\left(n, D-1, \left\lceil \frac{N_{D-1}(n)}{D+1} \right\rceil\right)$$

holds. Then both critical degree-D counts in n variables have maximal rank.

Proof. Set $r_2 = E(n-1, D)$ and $x = N_{D-1}(n)/(D+1)$. The identities

$$E(n, D) = E(n-1, D) + x, \qquad (n-1)x + E(n-1, D) = N_D(n-1)$$

show that the two cases of Lemma 3.1, with $r_1 = \lfloor x \rfloor$ and $r_1 = \lceil x \rceil$, give the injective and surjective critical counts. □

Lemma 3.3 (The projective plane)**.** *For every $D \geq 2$, the ideal generation statement holds for general points of $\mathbf{P}^2$.*

Proof. On $\mathbf{P}^1$, the ideal of D distinct points is generated by their product of linear equations. Its multiplication map from degree D to degree $D+1$ is an isomorphism. For $\mathbf{P}^2$ we prove the critical assertions with

$$r_{\text{inj}} = \left\lfloor \frac{D+2}{2} \right\rfloor, \qquad r_{\text{surj}} = \left\lceil \frac{D+2}{2} \right\rceil.$$

In degree one, one linear form gives injectivity, and the two linear equations of a point give surjectivity. For $D \geq 2$, apply Lemma 3.1 with $r_2 = 1$. On the injective side take $r_1 = \lfloor D/2 \rfloor$, and on the surjective side take $r_1 = \lceil D/2 \rceil$. Both residual counts are on the preceding degree's injective side, and

$$2\lfloor D/2 \rfloor + 1 \leq D + 1 \leq 2\lceil D/2 \rceil + 1.$$

These are precisely the two required inequalities. □

3.1. **A differential hyperplane step.** Write $H = (x_n = 0)$, $R = k[x_1, \ldots, x_{n-1}]$, and

$$\Sigma_t = \left\{ (f_1, \ldots, f_n) \in S_t^n : \sum_i x_i f_i = 0 \right\}, \qquad \mathcal{E}_{t-1} = S_{t-1}^{n-1}.$$

Use Σ_t' for the corresponding space over R. Thus $\Sigma_t = H^0(\mathbf{P}^{n-1}, \Omega_{\mathbf{P}^{n-1}}(t+1))$. For $g = (g_i) \in \mathcal{E}_{t-1}$ put $u_g = \sum_{i<n} x_i g_i$. There are two exact sequences

$$0 \longrightarrow \mathcal{E}_{t-1} \longrightarrow \Sigma_t \longrightarrow \Sigma_t' \longrightarrow 0, \qquad g \longmapsto (x_n g_1, \ldots, x_n g_{n-1}, -u_g), \tag{3.1}$$

$$0 \longrightarrow \Sigma_{t-1} \longrightarrow \mathcal{E}_{t-1} \longrightarrow R_t \longrightarrow 0, \qquad g \longmapsto \overline{u_g}. \tag{3.2}$$

In the first sequence the last map restricts the first $n-1$ components to H; a trace relation extends simply by using the same polynomials in S and adding a zero last component. In the second sequence, $\overline{u_g} = 0$ means $u_g = x_n \phi$, so $(g, -\phi) \in \Sigma_{t-1}$; surjectivity is the equality $R_1 R_{t-1} = R_t$.

At $y \in H$ write $T_y = \{v \in k^{n-1} : \sum_{i<n} y_i v_i = 0\}$. A *trace condition* at y asks that the first $n-1$ components of a section vanish there. It has $n-2$ scalar components. More generally a *partial trace condition* specified by $L \subset T_y$ asks that those components lie in L. These conditions will only be used as intermediate linear conditions; the final configurations consist of ordinary points.

Lemma 3.4 (Moving one point off the hyperplane)**.** *Impose on Σ_t full point conditions at Z_{off} off H and Z_{on} on H, together with trace conditions at $T \subset H$ and, if desired, a partial trace condition at a further point. Let $B \subset \Sigma_t'$ be the space satisfying the resulting trace conditions. Choose a further $y_0 \in H$ such that evaluation $B \to T_{y_0}$ is injective, and put $L = \text{ev}_{y_0}(B)$. On $\mathcal{E}_{t-1}$ consider the residual conditions*

$$g(Z_{\text{off}}) = 0, \qquad u_g(Z_{\text{on}}) = 0, \qquad g(y_0) \in L. \tag{*}$$

If these conditions have zero kernel, then adjoining a general point on the chosen line with a general nonzero parameter gives zero kernel on Σ_t. If the trace conditions and the residual conditions (*) are both independent, then all the resulting conditions on Σ_t are independent.

Proof. Take a vector v with $v_n \neq 0$ and move the point along $y_s = [y_0 + sv]$. In the injective case, a nonzero generic kernel would yield, after clearing denominators and dividing by a power of s, a polynomial relation $f(s) = f_0 + sf_1 + \cdots$ with $f_0 \neq 0$ and $f(s)(y_s) = 0$. The trace of f_0 belongs to B and vanishes at y_0, so it is zero. Sequence (3.1) gives $f_0 = (x_n g, -u_g)$, with g satisfying the first two residual conditions. The coefficient of s in the first $n-1$ components at y_s gives

$$v_n g(y_0) + \overline{f_1'}(y_0) = 0.$$

Since the trace of f_1 belongs to B, this says $g(y_0) \in L$. The residual hypothesis gives $g = 0$, contrary to $f_0 \neq 0$.

For independence, let A be the space satisfying the fixed conditions, before adjoining y_s, and let $M(s)$ be evaluation at y_s on A. The exact evaluation diagram for (3.1) shows that the fixed conditions are independent and $A \to B$ is onto. Put $b = \dim B = \dim L$. Residual independence implies that

$$K = \{g : g(Z_{\text{off}}) = 0,\ u_g(Z_{\text{on}}) = 0\} \longrightarrow k^{n-1}/L$$

is onto. Therefore $M(0)$ has image of dimension $b+1$: the b trace components in L, and the last component $-u_g(y_0)$. Its kernel consists of $(x_n g, -u_g)$ with $g \in K$ and $u_g(y_0) = 0$. The first derivative, modulo $\operatorname{im} M(0)$, is $g \mapsto v_n g(y_0) \bmod L$, and maps onto T_{y_0}/L. It has rank $n-2-b$.

For a matrix $M(s) = M_0 + sM_1 + \cdots$, the rank for general s is at least $\operatorname{rank} M_0 + \operatorname{rank}(\ker M_0 \to \operatorname{coker} M_0)$, where the second map is induced by M_1. To see this, put M_0 in block form with an identity block; the relevant larger minor has a nonzero leading coefficient. Here the lower bound is $(b+1) + (n-2-b) = n-1$, the full fiber dimension. □

Lemma 3.5 (Removing the scalar trace)**.** *In a problem on $\mathcal{E}_{t-1}$, allow full vector conditions, scalar conditions $u_g(y) = 0$ on H, and partial conditions $g(y) \in L \subset T_y$ on H. Let Y be the set of points of H occurring in any of these conditions. If $|Y| = N_t(n-1)$ and evaluation $R_t \to k^Y$ is an isomorphism, then the problem has the same kernel and cokernel as the following problem on Σ_{t-1}: keep full conditions off H, turn full vector conditions on H into trace conditions, retain partial trace conditions, and discard scalar conditions.*

Proof. Every condition at $y \in Y$ implies $\overline{u_g}(y) = 0$. In the evaluation diagram for (3.2), the quotient condition space is thus exactly k^Y. Its evaluation map from R_t is an isomorphism. On the kernel, the conditions are precisely those listed: off H, the first $n-1$ components determine the last component of a syzygy, whereas on H they give its trace. The kernel-cokernel exact sequence of this diagram proves the assertion. □

Lemma 3.6 (A fixed partial trace condition)**.** *Suppose ideal generation holds in degree t in $n-1$ variables. Fix $y \in H$ and $L \subset T_y$, with $\dim L = c$. For k general points $P \subset H$, chosen after (y, L), the space*

$$\{\beta \in \Sigma'_t : \beta(P) = 0,\ \beta(y) \in L\}$$

has dimension $\max\{0, \dim \Sigma'_t - (n-2)k - (n-2-c)\}$. The same assertion holds after one further general point is added, provided $k, k+1, k+2$ all lie in the degree-t point-count cell.

Proof. The ideal generation statements at k and $k+1$ determine the rank of evaluation at y on $\Sigma'_t(P)$. Its image has dimension $\min\{\dim \Sigma'_t(P), n-2\}$. For a general c-plane L in T_y, intersection with that image therefore has the expected dimension. The asserted dimension is the smallest possible one, so it defines a nonempty open condition. The stabilizer of y in $\mathrm{PGL}(H)$ acts on T_y through the full general linear group, up to scalar, and acts transitively on its c-planes. Consequently the good open has a nonempty fiber over every fixed L. Repeat the same argument with $k+1$ points, using the assertion at $k+2$. □

Lemma 3.7 (Selection of scalar conditions)**.** *Let $t \geq 2$, let Z be general points off H with $|Z| \geq N_{t-2}(n)$, and put $W = I_Z(t-1)$, $\dim W = \rho \geq 1$. Suppose multiplication $W \otimes S_1 \to S_t$ is injective. Choose general points $P \subset H$ after Z, with $|P| \geq (n-1)\rho$. Let $y_0 \in H$ satisfy $W(y_0) \neq 0$, and let $L \subset T_{y_0}$ be any c-dimensional subspace; L may depend on P. Set*

$$K^\infty = \{g \in W^{n-1} : g(y_0) \in L\}, \qquad k_0 = (n-1)\rho - (n-1-c).$$

Then $\dim K^\infty = k_0$, and scalar evaluation $g \mapsto (u_g(z))_{z \in P}$ is injective on K^∞. For $0 \leq q \leq |P|$, one may select q points of P whose scalar conditions kill K^∞ if $q \geq k_0$, or are independent on it if $q \leq k_0$.

Proof. Restriction $W \to \overline{W}$ is injective because its kernel is $x_n I_Z(t-2) = 0$. Moreover, $\overline{W} \otimes R_1 \to R_t$ is injective. Indeed, a trace relation lifts to $\sum_j w_j \ell_j = x_n G$ with $G \in W$. Expanding G in a basis w_j gives a linear relation on W; its injectivity forces every ℓ_j to be zero. Hence $U = \overline{W} R_1$ has dimension $(n-1)\rho$, and evaluation on general P is injective on all of U. It is therefore injective on the scalar polynomials u_g arising from K^∞. Since some member of W is nonzero at y_0, evaluation $W^{n-1} \to k^{n-1}$ is onto, which gives the stated dimension. Selecting independent rows of this injective evaluation proves both conclusions. □

3.2. **The two-level transfer.**

Proposition 3.8 (Fractional trace)**.** *Let $n \geq 4$, $D \geq 3$. Suppose ideal generation holds at $(n-1, D)$ and $(n-1, D-1)$, and $E(n-1, D)$ is not integral. Define*

$$\begin{aligned} q_0 &= N_D(n-1) - \lceil E(n-1,D) \rceil, & b_0 &= (n-2)(\lceil E(n-1,D) \rceil - E(n-1,D)), \\ \tau &= \lfloor E(n-1,D) \rfloor, & D_1 &= (n-1)N_{D-1}(n-1) - N_{D+1}(n-1), \\ q_1 &= \lfloor D_1/(n-2) \rfloor, & b_1 &= D_1 - (n-2)q_1. \end{aligned}$$

Let a and a' be the least nonnegative residues of $N_{D+1}(n-1)$ and $-N_{D+1}(n-1)$ modulo $n-1$, and put

$$\rho_I = \frac{q_1 + n - 1 - a - b_1}{n-1}, \qquad \rho_S = \frac{q_1 + n - 1 + a' - b_1}{n-1}.$$

Assume

$$D_1 \geq 0, \quad 1 \leq \rho_I \leq \rho_S \leq N_{D-2}(n-1), \quad \tau + q_1 \geq (n-1)\rho_S, \quad \mathrm{Inj}(n, D-2, \rho_S).$$

Assume also that $k, k+1, k+2$, where $k = \tau + q_1$, belong to the degree-$(D-1)$ point-count cell on H; equivalently,

$$N_{D-2}(n-1) \leq k, \qquad k+2 < N_{D-1}(n-1).$$

Then both critical degree-D counts in n variables have maximal rank.

Proof. Let $r = \lfloor E(n,D) \rfloor$ or $\lceil E(n,D) \rceil$ and $p = N_D(n) - r$. First fix q_0 general points $A \subset H$ and one further general $y^* \in H$. The degree-D trace space $B_D = \{\beta \in \Sigma'_D : \beta(A) = 0\}$ has dimension b_0, and evaluation at y^* is injective. These are exactly the surjective and injective critical assertions at $(n-1, D)$. Set $L^* = \mathrm{ev}_{y^*}(B_D)$. Lemma 3.4 reduces the problem to the residual vector conditions

$$g(Z_1) = 0, \qquad u_g(A) = 0, \qquad g(y^*) \in L^*, \qquad |Z_1| = p - q_0 - 1,$$

on $\mathcal{E}_{D-1}$.

Specialize τ of the points of Z_1 to a set $T \subset H$, leaving a set Z of $p - N_D(n-1)$ full points. Since $q_0 + 1 + \tau = N_D(n-1)$, the scalar evaluation on $A \cup \{y^*\} \cup T$ can be made an isomorphism. Lemma 3.5 then reduces the residual problem to Σ_{D-1} with full points Z, trace conditions at T, and the fixed partial trace condition (y^*, L^*).

We now construct this latter problem. Choose general points Z_{off} off H with

$$|Z_{\mathrm{off}}| = p - N_D(n-1) - q_1 - 1,$$

then general $P \subset H$ with $|P| = \tau + q_1$, and finally a general $y_0 \in H$. Put $W = I_{Z_{\mathrm{off}}}(D-2)$. Direct substitution gives

$$\dim W = r + q_1 + 1 - N_{D-1}(n-1) = \begin{cases} \rho_I, & r = \lfloor E(n,D) \rfloor, \\ \rho_S, & r = \lceil E(n,D) \rceil. \end{cases}$$

In particular these numbers are integers. The assumed cell bound gives $|Z_{\mathrm{off}}| \geq N_{D-3}(n)$, and the required injectivity of W follows from the hypothesis and point addition.

Write $\sigma'_t = \dim \Sigma'_t$. The identity

$$\sigma'_{D-1} - (n-2)\tau - (n-2-b_0) = D_1$$

and Lemma 3.6 show that

$$B = \{\beta \in \Sigma'_{D-1} : \beta(P) = 0,\ \beta(y^*) \in L^*\}$$

has dimension b_1, and evaluation at y_0 is injective. Let $L = \mathrm{ev}_{y_0}(B)$. The crucial point is that B and L depend on the whole set P, before it is partitioned. Lemma 3.7, applied with $t = D-1$, is valid for this L even though it depends on P. Its remaining dimension is

$$k_0 = (n-1)\dim W - (n-1-b_1) = \begin{cases} q_1 - a, & \text{injective side,} \\ q_1 + a', & \text{surjective side.} \end{cases}$$

Thus q_1 scalar conditions selected from P kill the residual kernel in the first case, and are independent in the second case. Call the selected set Z_{on} and put $T = P \setminus Z_{\mathrm{on}}$. Lemma 3.4 at degree $D-1$ gives the required rank after adjoining a point moving off y_0. This supplies Z with exactly the prescribed number of full points.

It remains to ensure that the two specializations use the same choices. Keep (A, y^*, L^*) fixed. The construction just given proves that the required rank is a nonempty open condition on the actual parameter space

$$H^\tau \times (\mathbf{P}^{n-1})^{|Z|}.$$

That space is irreducible. The condition that Z avoid H is dense open. Independently, the condition that $A \cup \{y^*\} \cup T$ interpolate R_D is nonempty open: the fixed points $A \cup \{y^*\}$ already impose independent conditions and can be completed by general points. Intersect these opens. Lemma 3.5 now applies to this same configuration, after which openness allows the points T in the vector problem to move off H. Finally apply Lemma 3.4 in degree D at y^*.

Scalar degree-D postulation can also be imposed in this intersection. The points off H are general and no more numerous than $N_{D-1}(n)$; the trace points $A \cup \{y^*\}$ impose independent degree-D conditions. Lifting a trace polynomial and correcting the off-hyperplane values by x_n times a degree-$(D-1)$ polynomial proves onto scalar evaluation at the specialized configuration. It remains onto after the final movement. The Euler evaluation criterion (Lemma 2.2) identifies the resulting maximal ranks with the required multiplication ranks. □

4. Two normal directions and an integral trace

The preceding hyperplane construction has a numerical limit. Beyond that limit we use a subspace of codimension two and move its points in the two independent normal directions. We first prove the transfer when the evaluation map on the trace is square. The same construction, with one additional free projection, will treat the nonsquare case.

4.1. **Three interpolation facts.**

We record the linear algebra used in both constructions. A finite dimensional space of scalar polynomials has maximal-rank evaluation at general points of an irreducible affine open set: choose the points successively, observing that a nonzero polynomial cannot vanish on that open set. The statement also holds for homogeneous polynomials after choosing an affine chart.

Lemma 4.1 (Two groups of scalar evaluations)**.** *Let V be a finite dimensional subspace of $V_A \oplus V_B$, where V_A, V_B are spaces of scalar polynomials on irreducible varieties. Write*

$$a = \dim V, \qquad h_A = \dim(V \cap (V_A \oplus 0)), \qquad h_B = \dim(V \cap (0 \oplus V_B)).$$

If $h_A, h_B \le b$, evaluation of the first component at b general points and of the second component at b independently general points has kernel of dimension $\max(0, a-2b)$. Both evaluations can simultaneously be chosen injective on their respective pure subspaces.

Proof. The first projection has dimension $a - h_B$. Its evaluation at b general points has maximal rank; hence the kernel remaining in V has dimension $\max(a-b, h_B)$. At the same time the evaluation is injective on the pure A-subspace. Consequently the remaining kernel projects injectively into V_B. Evaluation at the second group of points cuts this scalar polynomial space with maximal rank. Its final dimension is

$$\max\bigl(0, \max(a-b, h_B) - b\bigr) = \max(0, a - 2b).$$

Injectivity on the pure B-subspace is another nonempty open condition on the second group, and can be imposed at the same time. □

Fix $T = k[y_1, \dots, y_m]$, set $S = T[s,t]$, and identify $\Lambda = \{s = t = 0\}$ with $\mathbf{P}^{m-1}$.

Lemma 4.2 (Restriction of a point ideal)**.** *Let $e \geq 1$, $C = \dim S_e$, $M_e = \dim T_e$, and $q = C - g$, where $0 < g \leq M_e$. Suppose that the degree-e ideal component of $M_e - g$ general points of Λ has dimension g and injective multiplication by T_1. For q general points X with $s, t \neq 0$, the space $G = I_X(e)$ has dimension g; restriction $G \to T_e$ is injective, and its image $\overline{G}$ has injective multiplication*

$$\overline{G} \otimes T_1 \longrightarrow T_{e+1}.$$

Proof. Put $r_0 = C - M_e = \dim(s,t)_e$. Retain r_0 points with nonzero normal coordinates at which evaluation on $(s,t)_e$ is an isomorphism. Put the remaining $q - r_0 = M_e - g$ points on Λ, in general position. Given $f \in T_e$ vanishing at the latter points, a unique element of $(s,t)_e$ corrects its values at the retained points. Thus restriction identifies the degree-e ideal of this configuration with the ordinary point-ideal component specified in the hypothesis. This also proves independence of all q scalar conditions in degree e.

On the open set where those scalar conditions are independent, their kernels form a vector bundle. Injectivity of restriction and of its product map are open rank conditions on this bundle. They hold at the configuration just constructed, and therefore on a nonempty open set of configurations with both normal coordinates nonzero. □

Here is the interpolation statement needed for the attached projections. In particular, the projections of the points on a slice need not be independent of the ideal defined by those points.

Lemma 4.3 (Interpolation at attached projections)**.** *Let $H_A = \{t = 0\}$, let $D \geq 2$, and put*

$$L = N_{D-1}(m+1), \qquad M = N_{D-1}(m).$$

Let A' consist of b general points of $H_A \setminus \Lambda$, where $b \leq M$ and $b < L$. Suppose multiplication on $W_A = I_{A'}(D-1)$ by $k[H_A]_1$ is injective. Set

$$U_A = \sum_{i=1}^{m} y_i (W_A|_\Lambda) \subset T_D.$$

Then $\dim U_A = m(L-b)$. Let $Z_2 \subset \Lambda$ be a general ordered set containing the projections of A', with all its other points independently general relative to A'. Provided $|Z_2| \geq \dim U_A$, every subset of Z_2 of cardinality $\dim U_A$ is unisolvent for U_A. Consequently, in

$$Q_A = k^{Z_2} / \operatorname{ev}_{Z_2}(U_A),$$

every $\dim Q_A$ coordinate classes form a basis.

Proof. The kernel of restriction from $W_A\, k[H_A]_1$ to Λ is exactly sW_A. Indeed a polynomial in this product space which is divisible by s vanishes at A', and division by s, which does not vanish there, gives an element of W_A. The assumed product injection therefore gives $\dim U_A = (m+1)(L-b) - (L-b) = m(L-b)$.

We first show that U_A interpolates arbitrary values at the projections of A'. Choose representatives in one affine chart. For $v \in k[H_A]_{D-2}$, there exists $w_0 \in T_{D-1}$ such that

$$w_0(y(a')) = -s(a')v(a') \qquad (a' \in A'),$$

since $b \le M$. Hence $w_0 + sv \in W_A$. Using this separately for $v_1, \dots, v_m$, the values of U_A contain those of $-s\sum_i y_i v_i$ at A'. The polynomials $\sum_i y_i v_i$ constitute the degree-$(D-1)$ part of the ideal $(y_1, \dots, y_m)$ in $k[H_A]$, a space of dimension $L - 1 \ge b$. Its evaluation at general A' is onto, and multiplication by the nonzero values of s preserves this property.

For any chosen subset of Z_2 of size $\dim U_A$, first impose the rows which belong to these attached projections. They are independent by the preceding paragraph. The remaining labels are independently general relative to A', and their evaluations successively cut the remaining scalar polynomial space. This proves unisolvence for that subset. There are finitely many subsets, so all these open conditions hold simultaneously. The assertion about coordinate classes follows by taking complementary subsets. □

Lemma 4.4 (A first response in two normal directions)**.** *Let $A(\epsilon)$ be a matrix of regular functions near $\epsilon = 0$, with kernel K and cokernel Q at zero. Suppose $Q = Q_A \oplus Q_B$, and that its induced first derivative is*

$$J = \sum_z v_z e_z^A \otimes \alpha_z + u_z e_z^B \otimes \beta_z.$$

Assume that each of the families (α_z) and (β_z) spans K^*, and that in each target every subset of at most $\dim Q_A$, respectively $\dim Q_B$, of the indicated coordinate classes is independent. There are choices of u_z, v_z for which J has rank $\min(\dim K, \dim Q)$. For these choices $A(\epsilon)$ has maximal rank for general ϵ.

Proof. Write $h = \min(\dim K, \dim Q)$, and choose h_A, h_B with $h_A + h_B = h$, $h_A \le \dim Q_A$, and $h_B \le \dim Q_B$. Choose h_A independent members of the first source family. Extend them to h independent functionals using the second family. Set the corresponding velocities equal to one and all others to zero. The selected target vectors are independent, within each group by hypothesis and between groups by the direct sum. Thus J factors as a surjection onto k^h followed by an injection into Q, and has rank h.

Lemma 2.3 now gives maximal rank of $A(\epsilon)$ for general ϵ. □

4.2. **The integral-trace transfer.**

Proposition 4.5. *Let $D \ge 4$, $n = m + 2$, and $m \ge D(D-1)$. Suppose $E(m, D)$ is an integer and the degree-D ideal generation statement holds in m variables. If the ideal generation statements in degrees $D-1$ and $D-2$ hold in every number of variables, then the degree-D statement holds in n variables.*

Proof. It suffices to construct the two critical evaluation maps. Put $e = D - 2$, and use the following dimensions:

$$F = N_D(m), \quad M = N_{D-1}(m), \quad L = N_{D-1}(m+1),$$
$$C = N_e(n), \quad R = N_e(m+1), \quad M_e = N_e(m).$$

Pascal's identity and the consecutive binomial ratios give

$$\begin{aligned} N_D(n) - F &= 2L + C, \qquad N_{D-1}(n) = L + C, \\ DF &= mL, \qquad (m+1)C = (D-1)L, \\ \frac{L}{M} &= \frac{m+D-1}{m}, \qquad \frac{L}{R} = \frac{m+D-1}{D-1}. \end{aligned} \tag{4.1}$$

Take $r = \lfloor E(n,D) \rfloor$ or $\lceil E(n,D) \rceil$. Set

$$\begin{aligned} \Delta &= r - E(n,D), & p &= N_D(n) - r, \\ p_2 &= \frac{DF}{D+1}, & p_1 &= \frac{D(2L+C)}{D+1} - \Delta, \\ b &= \left\lceil \frac{DL}{D+1} \right\rceil, & \theta &= b - \frac{DL}{D+1}, \\ q &= p_1 - 2b = \frac{DC}{D+1} - \Delta - 2\theta, & g &= C - q = \frac{C}{D+1} + \Delta + 2\theta. \end{aligned} \tag{4.2}$$

All point counts are integers, $|\Delta| < 1$, and $0 \le \theta < 1$.

The numerical margins. We need the degree-$(D-1)$ injection for b points in $m+1$ variables and the degree-e injection for $M_e - g$ points in m variables. The inequalities

$$R \le b \le M < L, \qquad L - b < \frac{L}{D}$$

follow from $m \ge D(D-1)$ and (4.1); indeed $b \ge DR$. They put the first ideal component in its initial-degree cell and on the injective side.

For the second component, direct cancellation of the binomial coefficients gives

$$\frac{M_e}{D-1} - \frac{C}{D+1} = \frac{\prod_{j=2}^{e-1}(m+j)}{e!} \left(\frac{m(m+1)}{D-1} - \frac{(m+D-2)(m+D-1)}{D+1} \right). \tag{4.3}$$

The empty product is one. The numerator of the expression in parentheses, over D^2-1, is

$$2m(m-(D-1)(D-2)+1) - (D-1)^2(D-2).$$

It is increasing for $m \ge D(D-1)$, and its value at that endpoint is $(D^2-1)(3D-2)$. The prefactor in (4.3) is $M_e/(m(m+1)) \ge 1/2$. Consequently

$$\frac{M_e}{D-1} - \frac{C}{D+1} \ge \frac{3D-2}{2} \ge 5, \qquad 0 < g < \frac{C}{D+1} + 3 < \frac{M_e}{D-1}.$$

The consecutive binomial ratio then gives $M_e - g \ge N_{e-1}(m)$. Thus the second injection is also available in its initial-degree cell. In addition,

$$(m+1)g < b. \tag{4.4}$$

In fact the difference is greater than $L/(D+1) - 3(m+1)$, which is positive because

$$C \ge N_2(n) > \frac{m^2}{2} \ge \frac{D^2(D-1)^2}{2} > 3(D^2-1).$$

We also have $0 < q < C$, $q > C - M_e$, and $p_2 > p_1$. For the last assertion,

$$p_2 - p_1 = \frac{(m-2D)L - DC}{D+1} + \Delta > 0,$$

since $m - 2D \ge 4$, $DC/L = D(D-1)/(m+1) < 1$, and $L > D+1$. Finally

$$p_2 \ge M, \qquad \frac{p_2}{M} = \frac{m+D-1}{D+1} \ge D-1. \tag{4.5}$$

The marked family. Choose b points A' on $H_A = \{t = 0\}$ with $s \ne 0$, b points B' on $H_B = \{s = 0\}$ with $t \ne 0$, and q points X with $s, t \ne 0$. Their projections to Λ are required to be distinct. Let Z_2 consist of these p_1 projections and $p_2 - p_1$ further general points of Λ. Put

$$Z = A' \cup B' \cup X \cup Z_2.$$

The parameter space is irreducible and dominates the space of ordered p_2-tuples on Λ: prescribe the projections first and choose the normal coordinates freely. Every rank condition below is a nonempty open condition on this same parameter space. They can therefore be imposed

simultaneously. In proving that an individual condition is nonempty, we are free to fix one group of points and vary the other groups.

Scalar evaluation at Z is onto in degree D. Interpolate first on Z_2 using T_D, then on A' using $s\,k[H_A]_{D-1}$, and on B' using $t\,k[H_B]_{D-1}$. These corrections are possible since $b < L$ and vanish on the preceding groups. Finally stS_e corrects the values on X, since $q < C$. A linear form nonzero at all the points gives the same assertion in degree $D+1$.

The two slices. Apply Lemma 4.3 to A' and B'. The spaces $U_A, U_B \subset T_D$ both have dimension $m(L-b)$, and their evaluations on Z_2 are injective. Their quotient codes

$$Q_A = k^{Z_2}/\operatorname{ev}(U_A), \qquad Q_B = k^{Z_2}/\operatorname{ev}(U_B)$$

have the common dimension

$$c = p_2 - m(L-b) = m\theta, \tag{4.6}$$

and every c coordinate classes form a basis.

The trace evaluation for $\Omega_\Lambda(D+1)$ is an isomorphism by the hypothesis and the integral count. We claim that the slice evaluation for $\Omega_{H_A}(D+1)$ at $A' \cup Z_2$ is injective. Write a section as $(F_1, \dots, F_m, F_s)$. Its tangential trace is zero, so $F_i = sw_i$; vanishing at A' gives $w_i \in W_A$. Euler's relation gives $F_s = -\sum y_i w_i$. The normal trace values vanish on Z_2, and injectivity of evaluation on U_A implies $\sum y_i w_i \in sW_A$. The resulting linear relation on W_A is zero by its product injection. Thus the section is zero.

The cokernel of this slice evaluation is precisely Q_A. Indeed first remove arbitrary tangential trace values by a trace Euler section. Then remove arbitrary values at A' by sections $(sw_1, \dots, sw_m, -\sum y_i w_i)$, interpolating the w_i there. The remaining representatives are scalar normal values on Z_2; those arising from sections zero at A' are exactly $\operatorname{ev}(U_A)$, up to sign. The same argument identifies the other slice cokernel with Q_B.

The ambient kernel. By Lemma 4.2, the space $G = I_X(e)$ has dimension g, restricts injectively to T_e, and its restricted image has injective multiplication by T_1. Scalar independence also gives $\dim I_X(D-1) = L + C - q = L + g$.

Let A_0 be ambient Euler evaluation on Z. The restriction of any section in its kernel to either slice vanishes tangentially, by the slice injections just proved. Its components therefore have the form

$$F_i = sta_i, \qquad F_s = ta_s, \qquad F_t = sa_t, \qquad \sum_i y_i a_i + a_s + a_t = 0. \tag{4.7}$$

At X, this means $a_i \in G$ and $a_s \in I_X(D-1)$; the Euler relation then gives $a_t \in I_X(D-1)$. The remaining conditions are $a_t|_{A'} = 0$ and $a_s|_{B'} = 0$.

Consider the space of pairs of scalar polynomials obtained from

$$G^m \oplus I_X(D-1) \longrightarrow k[H_A]_{D-1} \oplus k[H_B]_{D-1}, \qquad (a_i, a_s) \longmapsto \left((a_s + \sum y_i a_i)|_{H_A},\, a_s|_{H_B}\right). \tag{4.8}$$

This map is injective. If both outputs vanish, restriction to Λ and the product injection on $\overline{G}$ give $a_i = 0$ for every i. Then $a_s = stc_0$, with c_0 vanishing at X. The polynomial $sc_0 \in G$ restricts to zero, so it too is zero.

The pair space has dimension $L + (m+1)g$. Its pure subspaces both have dimension $(m+1)g$: zero second component means $a_s = sf$, $f \in G$, while zero first component means $a_s + \sum y_i a_i = tf$, $f \in G$. By (4.4) and Lemma 4.1, imposing the A' and B' evaluations leaves a kernel of dimension

$$\dim \ker A_0 = \max(0, d), \qquad d = L + (m+1)g - 2b = 2m\theta + (m+1)\Delta. \tag{4.9}$$

We impose also the injectivity of the two pure evaluations.

The first response. The source dimension minus target dimension of A_0 is $(m+1)\Delta$. If $d \le 0$, it is already injective. If $d > 0$ and $c = 0$, it is already surjective. It remains to consider $d, c > 0$.

In that case (4.9) gives $\dim \operatorname{coker} A_0 = 2c$. The natural map

$$\operatorname{coker} A_0 \longrightarrow Q_A \oplus Q_B$$

is onto. To lift a pair of slice representatives, use the trace Euler isomorphism to make their tangential trace values agree. Compatible slice values lift to ambient fiber values; the two normal trace values are independent, and the values at X can be chosen freely. Since both spaces have dimension $2c$, this map is an isomorphism.

Write $K = \ker A_0$, and use a superscript 0 for restriction to Λ. Each map

$$K \longrightarrow T_{D-1}, \qquad F \longmapsto a_s^0, \quad F \longmapsto a_t^0$$

is injective. For if $a_s^0 = 0$, then $a_s|_{H_B} = th$ with $h \in k[H_B]_e$. It vanishes on B', and $b \geq R$, so generality of B' gives $h = 0$. The pair (4.8) is therefore pure A, and injectivity of its A' evaluation gives $F = 0$. The other map is treated symmetrically. Since $p_2 \geq M$, scalar degree-$(D-1)$ evaluation on Z_2 is injective. Thus each family of functionals $(a_s^0(z))_{z\in Z_2}$ and $(a_t^0(z))_{z\in Z_2}$ spans K^*.

Move a trace point z by $s(z) = \epsilon u_z$, $t(z) = \epsilon v_z$, keeping its trace coordinates fixed, and leave A', B', X fixed. Formula (4.7) shows that the first derivative on kernel and cokernel is

$$J = \sum_{z\in Z_2} v_z e_z^A \otimes [F \mapsto a_s^0(z)] + u_z e_z^B \otimes [F \mapsto a_t^0(z)].$$

The tangential components start in order ϵ^2. The coordinate-code property and Lemma 4.4 give a motion for which nearby Euler evaluation has maximal rank. Distinctness and scalar postulation persist on an open set. The Euler identification then gives the required maximal-rank multiplication map at the chosen critical count. Both choices of r are covered, so the point-count reduction proves the proposition. □

5. The two-normal transfer with a fractional trace

When the trace has a nonzero Euler kernel, attaching every residual point to a trace point preserves that kernel. We therefore leave one residual projection free. The extra point detects the trace kernel, while the lower-degree interpolation estimates still control the normal part of every relation.

Proposition 5.1. *Let $D \geq 4$, $m \geq D(D-1)$, and $n = m+2$. Assume ideal generation in degrees $D-2$ and $D-1$ in every dimension, and in degree D for general points of $\mathbf{P}^{m-1}$. If $E(m,D) \notin \mathbf{Z}$, then ideal generation in degree D holds for general points of $\mathbf{P}^{n-1}$.*

Proof. We prove maximal rank at each of the two critical counts. Retain the notation

$$T = k[y_1, \dots, y_m], \qquad S = T[s,t], \qquad e = D-2,$$
$$F = N_D(m), \quad M = N_{D-1}(m), \quad L = N_{D-1}(m+1),$$
$$C = N_e(n), \qquad R = N_e(m+1), \qquad M_e = N_e(m).$$

In particular,

$$N_D(n) - F = 2L + C, \qquad DF = mL, \qquad (m+1)C = (D-1)L. \tag{5.1}$$

Write

$$r_2 = \lceil E(m,D) \rceil, \quad \delta = r_2 - E(m,D), \quad p_2 = F - r_2.$$

Thus $0 < \delta < 1$. The Euler evaluation on p_2 general trace points is surjective and its kernel has dimension $\kappa = (m-1)\delta$. For either $r = \lfloor E(n,D) \rfloor$ or $r = \lceil E(n,D) \rceil$, put

$$\Delta = r - E(n,D), \qquad p = N_D(n) - r, \qquad p_1 = p - p_2, \qquad \sigma = (m+1)\Delta.$$

The integer σ is the source dimension minus the target dimension of the ambient Euler evaluation.

The numbers of residual points. Choose

$$\begin{aligned} b &= \left\lceil \frac{DL}{D+1} + \frac{\delta}{m} \right\rceil + 1, \qquad \theta = b - \frac{DL}{D+1}, \qquad a = m\theta - \delta, \\ q &= p_1 - 2b, \qquad g = C - q = \frac{C}{D+1} - \delta + \Delta + 2\theta. \end{aligned} \tag{5.2}$$

If $\eta \in [0,1)$ denotes the rounding error in the ceiling, then

$$\theta = 1 + \frac{\delta}{m} + \eta, \qquad a = m(1+\eta), \qquad m \le a < 2m.$$

Consequently

$$d := L + (m+1)g - 2b = 2a + \sigma - \kappa > 0, \tag{5.3}$$

because $\sigma > -(m+1)$ and $\kappa < m-1$. The additional one in the definition of b ensures this positivity.

Here are the bounds needed below:

$$\begin{aligned} R \le b < M, \qquad L - b < \frac{L}{D}, \qquad 0 < g+1 < \frac{M_e}{D-1}, \\ b > (m+1)(g+1), \qquad 1 < q < C, \\ p_1 - 1 \le p_2, \qquad p_2 \ge M. \end{aligned} \tag{5.4}$$

We verify them explicitly. First, the endpoint $m_0 = D(D-1)$ has integral $E(m_0, D)$. Indeed,

$$D\binom{D^2-1}{D} = (D^2-1)\binom{D^2-2}{D-1},$$

and D is coprime to $D+1$. We may therefore assume $m \ge m_0 + 1 \ge 13$; in particular $D+1 \le m/2$. Since $\theta < 2 + \delta/m$ and $|\Delta| < 1$,

$$0 < g < \frac{C}{D+1} + 5, \qquad g+1 < \frac{C}{D+1} + 6.$$

The margin for the preceding degree is

$$\frac{M_e}{D-1} - \frac{C}{D+1} = \frac{M_e}{m(m+1)} \left\{ \frac{m(m+1)}{D-1} - \frac{(m+D-2)(m+D-1)}{D+1} \right\}. \tag{5.5}$$

The expression in braces is increasing for $m \ge m_0$, and equals $3D-2$ at $m = m_0$; the factor preceding it is nondecreasing and at least $1/2$. For $D \ge 5$ the margin is therefore at least $(3D-2)/2 > 6$. For $D = 4$ it is exactly

$$\frac{m^2 - 5m - 9}{15} \ge \frac{19}{3} > 6 \qquad (m \ge 13).$$

This proves the bound for $g+1$. Its associated trace point count is in the degree-e initial cell: for $m \ge 2$,

$$\frac{M_e}{e+1} \le M_e - N_{e-1}(m).$$

Next,

$$M - \frac{DL}{D+1} = \frac{M(m-m_0)}{m(D+1)} \ge \frac{(m+1)(m+2)}{6(D+1)} \ge \frac{(m+1)(m+2)}{3m} > 4.$$

Here $M \ge N_3(m)$, while $\theta < 3$. Hence $b < M$. The ratio $L/R = (m+D-1)/(D-1)$ also gives $b \ge DR \ge R$. Thus b belongs to the degree-$(D-1)$ initial cell and $L - b \le L/(D+1) < L/D$, as required for the lower-degree injectivity hypothesis.

For the pure normal spaces, substitution in (5.2) gives

$$b-(m+1)(g+1)>\frac{L}{D+1}-6m-4>0.$$

For the second inequality use $L\geq N_3(m+1)$ and $D+1\leq m/2$. After multiplication by $3m$, the resulting lower bound has numerator m^3-12m^2-m+6, which is positive for $m\geq 13$. Moreover,

$$q>\frac{DC}{D+1}-5\geq\frac{4C}{5}-5>1,$$

since $C\geq N_2(n)\geq 120$; positivity of g gives $q<C$.

Finally,

$$\begin{aligned}p_2-p_1+1&=\frac{(m-2D)L-DC}{D+1}-2\delta+\Delta+1\\&>\frac{L}{D+1}\left(m-2D-\frac{D(D-1)}{m+1}\right)-2>0.\end{aligned}$$

Indeed, $m-2D\geq D(D-3)\geq 4$, $D(D-1)/(m+1)<1$, and $L>D+1$. Also

$$p_2=\frac{mL}{D+1}-\delta\geq(D-1)M-\delta\geq M.$$

This completes (5.4). Both p_2 and p_2+1 lie in the degree-D trace cell; the corresponding ideal dimensions are respectively the upper and lower critical integers. The same binomial ratio places the two ambient critical counts in their degree-D cell.

The marked configuration and its slice evaluations. Set

$$\Lambda=\{s=t=0\},\qquad H_A=\{t=0\},\qquad H_B=\{s=0\}.$$

Choose b general points A on $H_A\setminus\Lambda$, b general points B on $H_B\setminus\Lambda$, and q general points X with both s and t nonzero. Every point has nonzero y-coordinates. Designate one point $x_*\in X$, and let $y_*\in\Lambda$ be its projection. The trace set Z_2 contains the distinct projections of A, B, and $X\setminus\{x_*\}$, together with enough additional general points to have cardinality p_2. The inequality $p_1-1\leq p_2$ makes this possible. The projection y_* is general independently of the ordered trace set. Write

$$Z=Z_2\cup A\cup B\cup X.$$

These choices form an irreducible parameter space. Its map to the ordered pairs (Z_2,y_*) is dominant: the trace projections may be prescribed first, and all normal coordinates then vary freely. We impose below finitely many nonempty rank-open conditions on this space. In particular, all uses of general points refer to a common choice of Z.

Let

$$\mathcal{T}=\ker\Big(H^0(\Lambda,\Omega_\Lambda(D+1))\longrightarrow H^0(Z_2,\Omega_\Lambda(D+1)|_{Z_2})\Big).$$

The trace hypothesis gives $\dim\mathcal{T}=\kappa$ and surjectivity of this evaluation. A section $\tau=(\tau_i(y))\in\mathcal{T}$ extends to $(\tau_i(y),0,0)$ in the ambient Euler bundle. This extension vanishes on both A and B, because their projections belong to Z_2.

On H_A, put $W_A=I_A(D-1)$ and

$$U_A=\sum_{i=1}^{m}y_i(W_A|_\Lambda)\subset T_D;$$

define W_B,U_B similarly. The preceding degree gives injective multiplication on W_A and W_B. Lemma 4.3 applies, since $b<M$. It gives $\dim U_A=\dim U_B=m(L-b)$, and evaluation on every subset of $m(L-b)$ trace labels is an isomorphism.

The kernel of the slice Euler evaluation on $Z_2\cup A$ is exactly the extended space $\mathcal{T}$. Indeed, restrict a slice kernel section to its tangent trace component and subtract the corresponding extension of

τ. Its tangent trace components now vanish as polynomials. The remaining calculation is the one used in the integral case: divide those components by s, use injectivity of U_A on Z_2, and then use injectivity of linear multiplication on W_A. The remainder is zero. The same argument holds on H_B.

The scalar descriptions of the slice cokernels are consequently

$$Q_A = k^{Z_2}/\operatorname{ev}_{Z_2}(U_A), \qquad Q_B = k^{Z_2}/\operatorname{ev}_{Z_2}(U_B), \qquad \dim Q_A = \dim Q_B = a. \tag{5.6}$$

To see the cokernel description, first remove the tangent trace values by a trace Euler section, and then remove the values at A by a section whose tangent components are divisible by s. The remaining scalar trace values are taken modulo U_A. The possible ambiguity from $\mathcal{T}$ contributes zero, because its displayed extension vanishes at A. This proves (5.6); the argument on B is identical. The interpolation statement for U_A, U_B says that every a coordinate classes form a basis of the corresponding Q_A, Q_B.

Surjectivity of the normal part at X. Consider the space V of Euler sections

$$(sta_1, \dots, sta_m, ta_s, sa_t), \qquad \sum_i y_i a_i + a_s + a_t = 0, \tag{5.7}$$

that vanish at $A \cup B$. Here $a_i \in S_e$ and $a_s, a_t \in S_{D-1}$. Before imposing A, B, the free parameters are $(a_1, \dots, a_m, a_s)$, so their dimension is $(m+1)C + L$. The remaining conditions are

$$(a_s + \textstyle\sum_i y_i a_i)|_A = 0, \qquad a_s|_B = 0.$$

The map to the two complete slice polynomial spaces is onto. For any prescribed pair, choose a_s with the desired restriction on H_B; the difference of the desired restriction on H_A and $a_s|_{H_A}$ lies in the ideal $(y_1, \dots, y_m)$, apart from its possible pure s-term, which can first be incorporated in a_s. The remaining difference is $\sum y_i a_i$. Thus the $2b$ scalar conditions are independent, and

$$\dim V = (m+1)C + L - 2b.$$

Put $G = I_X(e)$. The height degeneration of Lemma 4.2, applied also to $\widetilde{G} = I_{X\setminus\{x_*\}}(e)$, gives

$$\begin{gathered} \dim G = g, \qquad \dim \widetilde{G} = g+1, \\ \widetilde{G} \longrightarrow T_e \text{ injective}, \qquad (\widetilde{G}|_\Lambda) \otimes T_1 \longrightarrow T_{e+1} \text{ injective}. \end{gathered} \tag{5.8}$$

The numerical hypothesis is $g + 1 < M_e/(e+1)$, already proved above, and the required ordinary point ideal is in its initial cell. The same restriction and multiplication injections hold for the subspace G.

The kernel of evaluation $V \to \bigoplus_{x\in X} \Omega_{\mathbf{P}^{n-1}}(D+1)|_x$ is described by

$$(a_i) \in G^m, \qquad a_s \in I_X(D-1),$$

subject to the two scalar conditions at A, B. As in the integral case, the associated polynomial-pair space

$$(a_i, a_s) \longmapsto ((a_s + \textstyle\sum_i y_i a_i)|_{H_A},\ a_s|_{H_B})$$

has dimension $L + (m+1)g$, and each of its pure-side subspaces has dimension $(m+1)g$. Its parametrization is injective by (5.8): vanishing of both outputs first forces all $a_i = 0$ on restriction to Λ, then makes a_s divisible by st, and restriction injectivity for G kills the remaining factor. Lemma 4.1 and $b > (m+1)g$ now give kernel dimension

$$L + (m+1)g - 2b = d > 0.$$

It follows that this evaluation on X is surjective, because

$$\dim V - d = (m+1)(C-g) = (m+1)q$$

is its target dimension.

Every ambient kernel section has a unique decomposition into an extended $\tau \in \mathcal{T}$ and a section of the form (5.7). Surjectivity just proved allows us to compensate the values of every τ at X. Thus, if K is the full Euler kernel on Z,

$$\dim K = \kappa + d = 2a + \sigma.$$

Its cokernel has dimension $2a$. Restriction to the two slices gives a surjection from this cokernel to $Q_A \oplus Q_B$: tangent trace representatives can be aligned using surjectivity of the trace evaluation, and the two normal trace values are independent. The dimensions agree, so

$$\operatorname{coker}(\mathrm{ev}_Z) \simeq Q_A \oplus Q_B. \tag{5.9}$$

The free projection detects the trace kernel. We show that each of the two maps

$$K \longrightarrow T_{D-1}, \qquad F \longmapsto a_s|_\Lambda, \quad \text{respectively } F \longmapsto a_t|_\Lambda, \tag{5.10}$$

is injective. At every point of $X \setminus \{x_*\}$, the extension of τ vanishes by attachment. Hence all a_i lie in $\widetilde{G}$. Its normal components vanish everywhere, so a_s and a_t vanish at all of X.

Suppose $a_s|_\Lambda = 0$. On H_B, write $a_s = th_B$, with $\deg h_B = e$. Since $t \neq 0$ on B, the polynomial h_B vanishes there. The bound $b \geq R$ and general scalar interpolation force $h_B = 0$. Thus a_s vanishes identically on H_B, so $a_s = sh$ in S. As $s \neq 0$ on X, we have $h \in G \subset \widetilde{G}$.

The condition at A is now

$$\left(\sum_i y_i a_i + sh \right) |_A = 0.$$

The polynomial space on H_A parametrized by $(a_1, \ldots, a_m, h) \in \widetilde{G}^{m+1}$ has dimension $(m+1)(g+1)$. Indeed, if its polynomial is zero, restriction to Λ and (5.8) give all $a_i = 0$. Then $h|_{H_A} = 0$, hence $h|_\Lambda = 0$; restriction injectivity for $\widetilde{G}$ gives $h = 0$. General scalar evaluation at the $b > (m+1)(g+1)$ points of A is injective on this space. It follows that all a_i, h, and hence the whole normal section, vanish.

Only τ remains. Its vanishing at x_* says that $\tau(y_*) = 0$. The trace set $Z_2 \cup \{y_*\}$ has ideal dimension

$$r_2 - 1 = \lfloor E(m, D) \rfloor.$$

The injective trace edge therefore gives $\tau = 0$. This proves the first injection in (5.10); interchanging A, s with B, t proves the second.

The scalar rank conditions in this argument can all be imposed on the same marked family. For example, fix $X \setminus \{x_*\}$ satisfying (5.8), and then choose A and B to give the required scalar injections. These are nonempty opens on the full parameter space. The trace conditions for Z_2 and $Z_2 \cup \{y_*\}$ are nonempty opens under its dominant map to (Z_2, y_*). Their intersection with the other rank opens is nonempty. In particular, no independence of an attached point from its own projection is required.

Normal motion and maximal rank. Move the original trace points, keeping their y-coordinates fixed, by

$$s(z) = \varepsilon u_z, \qquad t(z) = \varepsilon v_z \qquad (z \in Z_2),$$

and keep A, B, X fixed. The extension of τ has zero derivative in these directions. The first derivative of (5.7), followed by (5.9), is therefore

$$J = \sum_{z \in Z_2} v_z e_z^A \otimes [F \mapsto a_s|_\Lambda(z)] + \sum_{z \in Z_2} u_z e_z^B \otimes [F \mapsto a_t|_\Lambda(z)]. \tag{5.11}$$

Each family of source functionals spans K^*, by (5.10) and $p_2 \geq M$. Every a coordinate classes in either target form a basis. Lemma 4.4 therefore gives a choice of normal velocities for which J has maximal rank, and the deformed Euler evaluation has the required rank at the chosen critical count.

For completeness, Z has independent degree-D scalar evaluation. Interpolate its trace values in T_D, then correct the values at A and B separately with s and t times slice forms of degree $D-1$. Finally correct the values at X with stS_e; evaluation of S_e at X is onto because $q < C$ and X is general. A linear form avoiding Z gives degree-$(D+1)$ independence as well. These properties and distinctness persist under the deformation. The Euler criterion identifies the resulting maximal rank with that of the point-ideal multiplication map. Doing this at the two critical counts, followed by point addition and deletion, proves the proposition. □

6. The degree and dimension induction

6.1. **The range covered by one hyperplane.**

Proposition 6.1. *Let $D \geq 4$, and suppose that the ideal generation statement holds in degrees $D-1$ and $D-2$ for every number of variables. Then it holds in degree D for*

$$2 \leq n \leq D(D-1)+1.$$

Proof. We use induction on n at fixed degree D. The case $n = 2$ follows from the principal ideal description on $\mathbf{P}^1$, and $n = 3$ from Lemma 3.3. Thus the same-degree trace statement is available whenever an integral or fractional transfer is applied below.

When $E(n-1, D)$ is integral, the integral transfer asks for injectivity at residual dimension

$$\left\lceil \frac{N_{D-1}(n)}{D+1} \right\rceil.$$

For $n \geq 4$,

$$\frac{N_{D-1}(n)}{D(D+1)} \geq \frac{N_{D-1}(4)}{D(D+1)} = \frac{D+2}{6} \geq 1.$$

Consequently the required integer is at most $N_{D-1}(n)/D = E(n, D-1)$. The preceding-degree hypothesis gives injectivity. This residual lies in the ordinary scalar postulation cell, since $E(n, D-1) \leq N_{D-1}(n-1)$.

For $D = 4$, the only fractional trace edges in the range $4 \leq n \leq 13$ are $n = 7, 12$: the integer $\binom{n+2}{4}$ is divisible by 5 unless $n \equiv 2 \pmod 5$. The parameters of Proposition 3.8 are

n	τ	D_1	q_1	b_1	b_0	$a = a'$	$\rho_I = \rho_S$	$k = \tau + q_1$
7	25	84	16	4	4	0	3	41
12	200	143	14	3	8	0	2	214

Both rows have $D_1 \geq 0$ and $k \geq (n-1)\rho_S$. The three auxiliary counts are $41, 42, 43$ in the cubic point interval $[N_2(6), N_3(6)-1] = [21, 55]$, and $214, 215, 216$ in $[N_2(11), N_3(11)-1] = [66, 285]$, respectively. The residual quadratic ideal dimensions are 3 and 2, below $E(7, 2) = 28/3$ and $E(12, 2) = 26$. Their point counts are 25 and 76, in the quadratic initial-degree intervals; in particular $1 \leq \rho_I = \rho_S \leq N_2(n-1)$. Thus the two preceding-degree hypotheses and the smaller same-degree trace supply every hypothesis of the fractional transfer. Together with the integral steps, this proves the proposition for $D = 4$.

Assume henceforth that $D \geq 5$. We first treat fractional trace edges with $n \geq 5$. Use the notation $b_0, \tau, D_1, q_1, b_1, a, a', \rho_I, \rho_S$ of the fractional transfer, and put

$$\kappa = \frac{(n+D-1)(n+D-2)}{D(D+1)}, \qquad c = D(D-1)+1-n.$$

The identity

$$D_1 = N_{D-1}(n-1)(n-1-\kappa) = \frac{N_{D-1}(n-1)(n-2)c}{D(D+1)} \tag{6.1}$$

shows that $D_1 \geq 0$ throughout the asserted range.

The scalar-row budget follows from

$$(n-1)\rho_S = q_1 + n - 1 - b_1 + a' \le q_1 + 2n - 3 \quad \text{and} \quad \tau \ge 2n - 3.$$

For the latter inequality, monotonicity in D gives

$$E(n-1, D) \ge E(n-1, 4) = \frac{1}{5}\binom{n+2}{4}.$$

At $n = 5$ this equals $2n - 3 = 7$, and its increment as n increases is $\binom{n+2}{3}/5 > 2$.

The residual dimension is strictly below its critical value. Indeed, the definitions and $0 \le a' \le n - 2$ give

$$\begin{aligned}\rho_S &< E(n, D-2)\frac{n-1-\kappa}{n-2} + 2 \\ &= E(n, D-2) - E(n, D-2)\frac{\kappa - 1}{n-2} + 2.\end{aligned}$$

Here

$$\frac{\kappa - 1}{n-2} = \frac{2D + n - 1}{D(D+1)} \ge \frac{2}{D+1}, \qquad E(n, D-2) \ge E(5, D-2) = \frac{D(D+1)(D+2)}{24} \ge D + 1.$$

It follows that $\rho_S < E(n, D-2)$. The degree-$(D-2)$ hypothesis therefore supplies residual injectivity, once positivity is checked.

At the upper endpoint $n = D(D-1) + 1$, the trace edge is integral. To see this, write

$$D\binom{D^2 - 1}{D} = (D^2 - 1)\binom{D^2 - 2}{D - 1}.$$

The right side is divisible by $D + 1$, and $\gcd(D, D+1) = 1$. Thus a fractional edge has $c \ge 1$. We claim that

$$N_{D-1}(n-1)c \ge D(D+1)(n-2), \qquad q_1 \ge n - 2. \tag{6.2}$$

The second assertion follows from the first and (6.1). If $5 \le n \le D+2$, the ratio $N_{D-1}(n-1)/(n-2)$ increases with n, so

$$\frac{N_{D-1}(n-1)c}{n-2} \ge \frac{D(D+1)(D+2)}{18}(D^2 - 2D - 1) \ge D(D+1).$$

If $n \ge D + 3$, use $c \ge 1$ and $N_{D-1}(n-1) \ge \binom{n+2}{4}$. The ratio $\binom{n+2}{4}/(n-2)$ increases with n. At $n = D+3$ the required inequality is $\binom{D+5}{4} \ge D(D+1)^2$; after multiplication by 24, its difference is

$$(D-5)^4 + 10(D-5)^3 + 23(D-5)^2 + 110(D-5) + 720 > 0.$$

This proves (6.2). Since $a \le n - 2$ and $b_1 \le n - 3$,

$$(n-1)\rho_I = q_1 + n - 1 - a - b_1 \ge 2.$$

Both residual dimensions are therefore positive integers, and $\rho_I \le \rho_S < E(n, D-2)$. Also $E(n, D-2) \le N_{D-2}(n-1)$, which gives the required scalar cell and the vanishing in the preceding degree.

It remains to check the three auxiliary trace counts. Set $k = \tau + q_1$ and $r' = N_{D-1}(n-1) - k$. The exact identity

$$r' - E(n-1, D-1) = 1 + \frac{b_1 - b_0}{n-2}$$

places r' strictly between $E(n-1, D-1)$ and $E(n-1, D-1) + 2$. Moreover,

$$E(n-1, D-1) \ge E(4, 4) = 7, \qquad E(n-1, D-1) \le \tfrac{1}{2}N_{D-1}(n-2).$$

The last inequality is equivalent to $(D-2)(n-4) \geq 2$. Since $N_{D-1}(n-2) \geq 15$, all three integers $r', r'-1, r'-2$ lie between 1 and $N_{D-1}(n-2)$. Equivalently, $k, k+1, k+2$ lie in the required point cell. Every hypothesis of the fractional transfer is now satisfied.

For completeness, consider $n = 4$. A fractional trace edge occurs exactly when D is odd. For odd $D \geq 7$, direct substitution gives

$$b_0 = b_1 = 1, \qquad \tau = \frac{D+1}{2}, \qquad D_1 = D^2 - D - 3, \qquad q_1 = \frac{D^2 - D - 4}{2},$$
$$\rho_I = \frac{D(D-1)}{6} - \frac{a}{3}, \qquad \rho_S = \frac{D(D-1)}{6} + \frac{a'}{3},$$

where (a, a') is either $(0,0)$ or $(1,2)$. Thus $\rho_I \geq 1$, and

$$\rho_S < E(4, D-2) = \frac{D(D+1)}{6}.$$

The scalar-row budget reduces to $\tau \geq 2 + a'$, which holds for $D \geq 7$. The three auxiliary ideal dimensions are $(D+3)/2, (D+1)/2, (D-1)/2$; they all lie in $[1, D]$. The fractional transfer applies. The only remaining case is $(n, D) = (4, 5)$. Apply the ordinary splitting lemmas with residual dimension $r_1 = 6$ and trace dimensions $r_2 = 3$ and $r_2 = 4$, respectively. The residual is injective because $6 \leq E(4,4) = 7$, while

$$3 \cdot 6 + 3 = 21 = N_5(3), \qquad 3 \cdot 6 + 4 \geq N_5(3).$$

The resulting dimensions 9 and 10 are precisely the two critical dimensions in degree five. This completes the proof for $D \geq 5$.

□

6.2. **Completion of the proof.**

Proof of Theorem 1.1. The quadratic and cubic cases are Theorems A.1 and B.1. For $D \geq 4$, assume that the statement holds in degrees $D-1$ and $D-2$ for every number of variables. Proposition 6.1 proves the degree-D statement for $n \leq D(D-1)+1$.

Proceed by induction on n beyond this range. Put $m = n-2$; then $m \geq D(D-1)$, and the degree-D statement in m variables is already known. If $E(m, D)$ is integral, apply Proposition 4.5; otherwise apply Proposition 5.1. Their two lower-degree hypotheses are precisely the induction assumptions. In both cases the two critical counts in n variables have maximal rank. Lemma 2.1 gives every point count in the initial-degree interval. This completes the induction on n and then on D. □

Appendix A. The quadratic case

We prove the initial degree needed for the induction. The only previously established finite-dimensional input is the theorem of Fröberg and Löfwall [6, Theorem 6.2]: the Ideal Generation Conjecture holds for quadrics through general points in projective dimension at most 20. We use it only when $n \leq 12$.

Theorem A.1. *For every $n \geq 2$ and $n \leq p < N_2(n)$, the multiplication map*

$$I_Z(2) \otimes k[x_1, \ldots, x_n]_1 \longrightarrow I_Z(3)$$

has maximal rank for a general set of p points in $\mathbf{P}^{n-1}$.

We use two elementary constructions, according to the residue of n modulo three. Throughout this section $E(n,2) = n(n+1)/6$.

A.1. **Cones over a trace.** Write $S = T[s_1, \dots, s_c]$, where $T = k[y_1, \dots, y_m]$, $n = m + c$, and put $P = (s_1, \dots, s_c)$. Let $Z_2 \subset \mathbf{P}^{m-1} = V(P)$ be a trace set, and let Z_1 consist of points $(\sigma_z; \mu_z y_z)$, where $\sigma_z \neq 0$, $\mu_z \neq 0$, and $[y_z] \in Z_2$. Repeated projected supports are allowed. Set

$$K = P_2 \cap I_{Z_1}(2), \qquad V = I_{Z_2,T}(2), \qquad M = P_3 \cap I_{Z_1}(3).$$

Assume that the evaluations on P_2 and T_2 have full row rank. Every pure trace form vanishing on Z_2 also vanishes on Z_1. Separating the pure trace part of a polynomial therefore gives

$$I_{Z_1 \cup Z_2}(j) = I_{Z_2,T}(j) \oplus (P_j \cap I_{Z_1}(j)), \qquad j = 2, 3.$$

In these decompositions multiplication is the direct sum of trace multiplication and

$$\mu_M : (K \otimes S_1) \oplus \big(V \otimes \langle s_1, \dots, s_c \rangle\big) \longrightarrow M. \tag{A.1}$$

It suffices to construct a trace configuration for which the second map has the desired rank. Indeed the space of marked cone configurations, with a fixed assignment of projected supports to trace labels, is irreducible and dominates the space of ordered trace configurations. Full evaluation rank and the desired rank of μ_M are open conditions. Their nonempty open set consequently meets the pullback of the nonempty open set where the inductive trace theorem holds. This argument allows the trace used in an explicit calculation to be special.

A.2. **A two-normal interpolation construction.** Let $c = 2$, write the normal coordinates as s, t, and split the trace variables into $y_1, \dots, y_h, z_1, \dots, z_l$, where $h \geq 2$ is even, $l \geq 2$, and $m = h + l$. Put $\pi_a = e_{2a-1} + e_{2a}$. Choose $N_2(h)$ base points imposing independent conditions on quadrics in $\mathbf{P}\langle y_1, \dots, y_h \rangle$, including the coordinate points and all π_a. For example, the coordinate points and the points $e_i + e_j$, $i < j$, have this property. In each layer add

$$e_{z_j}, \qquad \pi_a + e_{z_j} \quad (1 \leq a \leq h/2).$$

Call the resulting set Z_f; thus

$$p_f = |Z_f| = N_2(h) + l(h/2 + 1).$$

The designated cone supports are the h base coordinate points and the two points $e_{z_j}, \pi_1 + e_{z_j}$ in each layer. There are $p_1 = h + 2l = 2m - h$ supports.

Over e_i choose normal direction $(c_i, -1)$, with the c_i distinct and nonzero. In each layer choose two independent normal directions. Choose nonzero cone heights. The mixed evaluation equations have independent rows: the base equations use separate coordinates, and each new layer adds an invertible two-by-two system. Hence

$$(*) \qquad K \longrightarrow \mathrm{Sym}^2\langle s, t \rangle \quad \text{is onto}, \qquad K_{00} := \ker(K \longrightarrow \mathrm{Sym}^2\langle s, t \rangle) \quad \text{has dimension } h.$$

Write a basis of K_{00} as

$$H_i = s\kappa_i^1 + t\kappa_i^2, \qquad \kappa_i = (\kappa_i^1, \kappa_i^2) = y_i(1, c_i) + \text{layer terms}.$$

Since the noncoordinate layer supports use only π_1, the layer terms vanish for $i \geq 3$.

Let

$$\mathcal{A} = \{(A, B) \in T_2^{\oplus 2} : \sigma_z \cdot (A, B)(y_z) = 0 \text{ at every designated support}\}.$$

Put

$$\mathcal{U} = \sum_{i=1}^{h} \kappa_i T_1, \qquad I_f = I_{Z_f,T}(2).$$

We claim that

$$\mathcal{A} = \mathcal{U} \oplus I_f^{\oplus 2}, \qquad \dim \mathcal{U} = hm. \tag{A.2}$$

On the base, the map

$$(\ell_i) \longmapsto \Big(\sum y_i \ell_i, \sum c_i y_i \ell_i\Big)$$

is injective: the coefficients of y_i^2 vanish, and the two equations for y_iy_j force both coefficients to vanish because $c_i \neq c_j$. Its image is exactly the space satisfying the base support equations, by dimension.

Inductively add a layer. Restriction, setting the new variable z_j equal to zero, identifies the new mixed residual space with the old one. At the layer points the allowed values of $\sum \lambda_i\kappa_i$ have total dimension h, and evaluation is injective. Here the two designated supports have one-dimensional fibres cut out by their normal equations, and the $h/2-1$ remaining points have unrestricted two-dimensional fibres. In fact, vanishing at e_{z_j} first kills the layer correction; vanishing at every $\pi_a + e_{z_j}$ then gives

$$\lambda_{2a-1} + \lambda_{2a} = c_{2a-1}\lambda_{2a-1} + c_{2a}\lambda_{2a} = 0,$$

so all λ_i vanish. This evaluation is therefore bijective. After lifting a decomposition from the preceding layers, its error is divisible by z_j; the preceding bijectivity supplies the required mixed correction. The layer points impose independent scalar conditions because their underlying vectors $e_{z_j}, \pi_1, \dots, \pi_{h/2}$ are independent. This proves the sum assertion. Injectivity of $K_{00} \otimes T_1 \to T_2^2$ follows by restriction and induction: the remaining coefficient forms are multiples of z_j, and linear independence of the κ_i kills them. Finally,

$$hm + 2(N_2(m) - p_f) = 2N_2(m) - p_1 = \dim \mathcal{A},$$

so the sum is direct.

Choose lifts

$$K_1 = s^2 + sU_1 + tV_1, \quad K_2 = st + sU_2 + tV_2, \quad K_3 = t^2 + sU_3 + tV_3$$

in K. They may be normalized so that the U_j contain only layer variables. Define

$$\mathcal{W} = \{(A,B) : sA+tB \in KS_1\}, \qquad \mathcal{W}' = \mathcal{W} \cap I_f^{\oplus 2}, \qquad \Omega = \{(k_s,k_t) \in K^2 : s\mathfrak{q}(k_s)+t\mathfrak{q}(k_t) = 0\},$$

where $\mathfrak{q}$ takes the pure normal quadratic part. Since $\mathfrak{q}$ is onto, $\dim \Omega = 2h+2$. If $k_s = Q_s + su_s + tv_s$ and $k_t = Q_t + su_t + tv_t$, the polynomial

$$-K_1u_s - K_2(v_s+u_t) - K_3v_t + sk_s + tk_t \tag{A.3}$$

has only normal degree one. Let $\phi(k_s,k_t)$ be its coefficient pair, and let $\pi : \mathcal{A} \to I_f^{\oplus 2}$ be the projection in (A.2). Eliminating normal degrees three and two shows that

$$\mathcal{W} = \mathcal{U} + \phi(\Omega), \qquad \mathcal{W}' = \pi\phi(\Omega), \qquad \dim \mathcal{W}' \leq 2h+2.$$

Lemma A.2. *The cone parameters can be chosen so that* $\dim \mathcal{W}' = 2h+2$ *and evaluation* $\mathcal{W}' \longrightarrow (k^2)^{h+1}$ *is an isomorphism at the* $h+1$ *points*

$$e_{y_1} + e_{z_1}, \quad e_{y_1} + e_{z_2}, \quad e_{z_1} + e_{z_2}, \quad e_{y_{2a-1}} + e_{z_j} \quad (2 \leq a \leq h/2,\ j = 1,2). \tag{A.4}$$

These points impose independent scalar conditions on I_f.

Proof. First restrict to y_1, y_2, z_1, z_2, s, t. The seven trace points in this subspace have quadratic ideal

$$I_{f,c} = \langle (y_1-y_2)z_1,\ (y_1-y_2)z_2,\ z_1z_2 \rangle.$$

Restriction commutes with the projection π: it maps $\mathcal{U}$ onto its core counterpart, while the restriction of a form in I_f vanishes on the core flag. It also maps K and Ω onto the corresponding core spaces.

We use the following direct calculation. If N points are linearly independent in $\mathbf{P}^{N-1}$, with pairwise distinct normal directions $s : t$, then

$$((s,t)_2 \cap I_Z) \otimes S_1 \longrightarrow (s,t)_3 \cap I_Z$$

is bijective. Put the points at the coordinate vectors, and write $s = \sum a_ix_i$, $t = \sum b_ix_i$. The quadrics $w_i = x_i(b_is - a_it)$ span the residual space, with the single relation $\sum w_i = 0$. If $\sum \ell_iw_i = 0$,

the coefficient of $x_i^2x_j$ shows that the i-th coefficients of ℓ_i and ℓ_j agree, since $b_ia_j - a_ib_j \neq 0$. Thus all ℓ_i agree, and the relation is the indicated redundant-generator relation. The dimensions of source and target are both $N(N-1)$.

For the six core cone points use, in order, the trace supports

$$Y_1 = e_{y_1}, \quad Y_2 = e_{y_2}, \quad Y_3 = e_{z_1}, \quad Y_4 = e_{y_1} + e_{y_2} + e_{z_1}, \quad Y_5 = e_{z_2}, \quad Y_6 = e_{y_1} + e_{y_2} + e_{z_2}.$$

Take normal directions

$$(1,-1), (2,-1), (1,1), (1,2), (1,3), (1,4)$$

and normal heights $(\theta_1, \dots, \theta_6) = (1,1,1,2,1,3)$, respectively. The actual six projective points are $P_i = (\theta_i\sigma_i; Y_i)$, where σ_i denotes the corresponding displayed direction. Eliminating four support-coordinate columns in their coordinate matrix leaves determinant -17; the six points are independent. The preceding calculation implies $\mathcal{W}'_c = (I_{f,c})^{\oplus 2}$. The first three points in (A.4) evaluate the displayed basis of $I_{f,c}$ as the identity matrix.

For an untouched pair $i = 2a-1, i' = 2a$, $a \geq 2$, put $\beta_i = (1, c_i)$. The projected products

$$sH_i - y_iK_1 - c_iy_iK_2, \qquad tH_i - y_iK_2 - c_iy_iK_3$$

give elements

$$\beta_{i'}(y_i - y_{i'})L_{i,s}(z), \quad \beta_{i'}(y_i - y_{i'})L_{i,t}(z), \tag{A.5}$$

and the analogous two elements with i, i' interchanged. Here the projection can be calculated by first subtracting the base-quadratic terms in $\mathcal{U}$, and then subtracting linear combinations of $\kappa_iz_j, \kappa_{i'}z_j$. If a remaining coefficient in the j-th layer is $a\beta_i + \gamma\beta_{i'}$, this last subtraction replaces $y_iz_j(a\beta_i + \gamma\beta_{i'})$ by $\gamma\beta_{i'}(y_i - y_{i'})z_j$, which proves the asserted form.

We check that the two forms for each fixed index are independent on $\langle z_1, z_2\rangle$. Let the normal directions of layer j be $\sigma_\nu = (a_\nu, b_\nu)$, $\nu = 1,2$, and let its independently variable normal heights be θ_1, θ_2. Write $M = (\sigma_1; \sigma_2)$ and $d_\nu = M^{-1}e_\nu$. Solving the support equations for the three lifts and differentiating with respect to θ_ν, the two coefficients of $(L_{i,s}, L_{i,t})$ in layer j have Jacobian columns

$$\lambda_i(d_\nu)(a_\nu + c_ib_\nu)\begin{pmatrix} a_\nu \\ b_\nu \end{pmatrix}, \qquad \lambda_i(v) = \frac{\det(\beta_i, v)}{\det(\beta_i, \beta_{i'})}.$$

Both scalar factors are nonzero if $\sigma_\nu \cdot \beta_i \neq 0$ for both ν. The Jacobian is then invertible. Thus general heights in the first two layers make both forms independent; the finitely many such conditions hold simultaneously. They are compatible with the core rank condition, since all are nonempty opens in the same irreducible parameter space.

For each pair, (A.5) and its two companions span a four-dimensional space $\mathcal{V}_a$. These spaces are independent, have zero core restriction, and have total dimension $2h-4$. Together with the six-dimensional core quotient they reach the upper bound $2h+2$. At the last two points assigned to pair a in (A.4), its four columns give an invertible block: the two directions $\beta_i, \beta_{i'}$ are independent, as are the two layer forms in each direction. The other pair spaces vanish there and all pair spaces vanish at the core points. The evaluation matrix is block triangular with invertible diagonal blocks. For scalar independence use the three displayed core quadrics, followed by $(y_i - y_{i'})z_1, (y_i - y_{i'})z_2$; their evaluation matrix is the identity. □

Take q additional trace points, first among (A.4) and then general if necessary. As long as these impose independent scalar conditions on I_f, their vector evaluation on $\mathcal{W}'$ has maximal rank. Normal-degree elimination in (A.3) identifies the kernel and cokernel of μ_M with those of this evaluation. For clarity, $K \otimes S_1 \to KS_1$ is injective: a relation first gives an element of Ω; its projection in $\mathcal{W}'$ vanishes, so that element is zero because $\Omega \to \mathcal{W}'$ is an isomorphism; then (A.2) kills the remaining mixed coefficients. The pure normal parts of degrees two and three are covered

by the three lifts and their normal multiples. The remaining cokernel is precisely the evaluation cokernel. Consequently

$$\dim\ker\mu_M=\max\{2h+2-2q,0\},\qquad \dim\operatorname{coker}\mu_M=\max\{2q-2h-2,0\}. \tag{A.6}$$

For $n=3\nu+2\ge 8$, choose

$$h=2\nu-2,\quad l=\nu+2,\quad q=h+1,\quad \dim K=2\nu+1,\quad \dim V=E(3\nu,2).$$

The trace count is $p_f+q=\nu(3\nu+1)$, and $\dim K+\dim V=E(n,2)$. Thus μ_M is bijective. For $n=3\nu+1\ge 7$, choose

$$h=2\nu-2,\quad l=\nu+1,\quad q=\nu-1,\quad \dim V=E(3\nu-1,2).$$

Now $p_f+q=\nu(3\nu-1)$, and $\dim K+\dim V=\lceil E(n,2)\rceil$. Equation (A.6) gives surjectivity. In both cases the trace endpoint is integral, so the cone splitting and the open-set argument prove the required full endpoint from the quadratic theorem in $n-2$ variables.

A.3. **The injective endpoint in the remaining residue class.** Keep the latter parameters, but choose the $q=h/2$ extra trace points

$$E_0=e_{y_1}+e_{z_1},\qquad E_a=e_{y_{2a-1}}+e_{z_1}\quad(2\le a\le h/2).$$

An element of Ω has a unique expression

$$k_s=\sum\alpha_iH_i+aK_2+bK_3,\qquad k_t=\sum\beta_iH_i-aK_1-bK_2.$$

Write $w_\omega=\pi\phi(\omega)$. The old middle kernel is $\{\omega : w_\omega(E_a)=0 \text{ for every } a\}$. Add the point e_s to Z_1. Since $K_1(e_s)=1$ and the other basis quadrics vanish there, the new residual space omits K_1. The new middle kernel is therefore the subspace of the old kernel on which the coefficient of K_1 in the relation is zero. Writing $H_i=su_i+tv_i$, this coefficient is

$$C(\omega)=-\sum_i\alpha_iu_i-a(t+U_2)-bU_3. \tag{A.7}$$

The corrections in $\mathcal{U}$ and $V\otimes\langle s,t\rangle$ do not alter it.

Here is an explicit core calculation for the parameters used in the preceding proof:

$$\begin{aligned}
u_1&=y_1+3z_1+15z_2, & v_1&=y_1-3z_1-5z_2,\\
u_2&=y_2+5z_1+27z_2, & v_2&=2y_2-5z_1-9z_2,\\
U_2&=-4z_1-12z_2, & V_2&=-y_1-2y_2+3z_1+3z_2,\\
U_3&=10z_1+132z_2, & V_3&=y_1+y_2-11z_1-47z_2.
\end{aligned}$$

The coefficients of y_1,y_2,t,z_1 in $C=0$ force $\alpha_1=\alpha_2=a=b=0$. The remaining two core directions, $(0,H_1),(0,H_2)$, have projected values $(30,60),(75,75)$ at E_0, with determinant -2250. One can check these values without solving the full core system. Set $x=y_1,y=y_2,z=z_1,z_2=0$. The coefficient pair for $(0,H_i)$ is $-u_i(U_2,V_2)-v_i(U_3,V_3)$. Subtracting $\kappa_1(-y-41z)+\kappa_2(x+33z)$ for $i=1$, and $\kappa_1(y-70z)+\kappa_2(-x+56z)$ for $i=2$, leaves respectively $30(1,2)(x-y)z$ and $75(1,1)(x-y)z$.

Thus the joint core map $\omega\mapsto(C(\omega),w_\omega(E_0))$ is injective. For each untouched pair, the two projected directions $(0,H_i),(0,H_{i'})$ have the forms

$$\beta_{i'}(y_i-y_{i'})L_{i,t},\qquad \beta_i(y_{i'}-y_i)L_{i',t}.$$

The Jacobian calculation above allows both z_1-coefficients to be nonzero simultaneously with the core condition. If $C(\omega)=0$, its y_i-coefficient first gives $\alpha_i=0$ for every $i\ge 3$. Restriction to the core, together with $w_\omega(E_0)=0$, then kills all core parameters. Finally the evaluation at E_a kills $\beta_i,\beta_{i'}$ for its pair, since the displayed two vectors are independent. Hence the new middle map is injective.

The new residual dimension is 2ν, and

$$2\nu + E(3\nu - 1, 2) = \frac{3\nu(\nu+1)}{2} = \lfloor E(n,2) \rfloor.$$

Scalar independence persists because K_1 separates the added point from all previous points. Pure trace forms vanish at e_s, so the cone splitting remains valid. Keeping e_s fixed, the same marked cone family still dominates the trace configuration space; intersecting the middle-rank open with the trace-isomorphism open proves the full injective endpoint.

A.4. **The quadratic step in three normal directions.** We prove the remaining quadratic induction step. Throughout this subsection, $h \geq 8$ and $h \equiv 2 \pmod 3$, and we put

$$m = h+4, \qquad n = h+7, \qquad T = k[y_1, \ldots, y_h, z_1, \ldots, z_4],$$
$$S = T[s_1, s_2, s_3], \qquad P = (s_1, s_2, s_3).$$

The construction below gives a set Z_1 of $2h+12$ points outside the trace $\mathbf{P}^{m-1}$ and a set Z_2 of $m(m+1)/3$ trace points, with all projections of Z_1 contained in Z_2, such that

$$(K \otimes S_1) \oplus (V \otimes \langle s_1, s_2, s_3 \rangle) \longrightarrow P_3 \cap I_{Z_1}(3) \tag{A.8}$$

is an isomorphism. Here $K = P_2 \cap I_{Z_1}(2)$ and $V = I_{Z_2,T}(2)$. Their dimensions will be $h+6 = n-1$ and $m(m+1)/6$, respectively. Consequently, quadratic ideal generation in $m = n-3$ variables implies quadratic ideal generation in n variables: the two dimensions add to $n(n+1)/6$, and the trace multiplication is an isomorphism at its integral critical count.

The cone family and its coefficient map. Choose distinct nonzero parameters t_i, and a primitive cube root ω. Set

$$v_i = (1, t_i, t_i^2), \qquad \ell_i(s) = s \cdot v_i,$$
$$\sigma_i^+ = (\omega^2 t_i^2, \omega t_i, 1), \qquad \sigma_i^- = (\omega t_i^2, \omega^2 t_i, 1),$$
$$\theta_i^+ = -\omega^2, \qquad \theta_i^- = \omega.$$

Over the trace coordinate point e_i place the two cone points $(\theta_i^+ \sigma_i^+; e_i)$ and $(\theta_i^- \sigma_i^-; e_i)$. The vectors $\sigma_i^\pm$ are independent and annihilate v_i. Moreover,

$$\theta_i^+ \sigma_i^+ - \theta_i^- \sigma_i^- = (t_i^2, -2t_i, 1), \qquad (\sigma_i^+ \cdot v_j)(\sigma_i^- \cdot v_j) = (t_i - t_j)^2 (t_i^2 + t_i t_j + t_j^2). \tag{A.9}$$

We choose the t_i so that the last expression is nonzero whenever $i \neq j$.

Partition the first six indices into two triples, leave $\{7, 8\}$ as a pair, and partition any remaining indices into triples. Write $\pi_B = \sum_{i \in B} e_i$ for a part B. In each layer z_j, put trace points at e_{z_j} and at $\pi_B + \zeta_{j,B} e_{z_j}$ for all parts B, where the $\zeta_{j,B}$ are nonzero. Lift the first three of these points, namely e_{z_j} and the points belonging to the first two triples, to cone points whose three normal-coordinate vectors are independent. There are three cone points and $(h+4)/3$ trace points in each layer. In addition to the h trace coordinate points, choose

$$f_h = \frac{h^2 + 2h - 8}{3}$$

free trace points in the base $z = 0$, and four further trace points. The latter four will remain in the fixed subspace on $y_1, \ldots, y_8, z_1, \ldots, z_4$. The total trace count is

$$h + f_h + 4\frac{h+4}{3} + 4 = \frac{(h+4)(h+5)}{3}.$$

The mixed equations at the cone points are independent. Thus $K \to k[s_1, s_2, s_3]_2$, given by restriction to $y = z = 0$, is onto, and its kernel has basis

$$H_i = \ell_i(s) y_i + \text{terms involving the } z_j.$$

Only $H_1, \dots, H_6$ have layer corrections. For each normal quadratic Q, choose a lift $K_Q \in K$, linearly in Q. At each base point its mixed coefficient $A_i(Q) \in k^3$ satisfies

$$\sigma_i^{\pm} \cdot A_i(Q) = -\theta_i^{\pm} Q(\sigma_i^{\pm}). \tag{A.10}$$

The three equations in a layer then determine its coefficient uniquely. These choices can be made compatibly whenever further base indices are added.

Put $U = \langle s_1, s_2, s_3 \rangle$ and

$$\Omega_h = \ker(U \otimes K \longrightarrow k[s_1, s_2, s_3]_3).$$

Since restriction $K \to k[s]_2$ is onto, $\dim \Omega_h = 3h + 8$. For $\eta = (k_1, k_2, k_3) \in \Omega_h$, write

$$\sum_{a=1}^{3} s_a k_a = \sum_b x_b R_b(s), \qquad x_b \in \{y_1, \dots, y_h, z_1, \dots, z_4\},$$

and subtract $\sum_b x_b K_{R_b}$. The result has normal degree one, so it is $s \cdot P_h(\eta)$ for a vector-valued quadric $P_h(\eta)$ on the trace. Let Φ_h denote the vector-valued quadrics arising from $K_{00} \otimes T_1$, where $K_{00} = \langle H_1, \dots, H_h \rangle$.

At a trace point supporting a cone point with normal vector a, the value of such a vector quadric satisfies $a \cdot F = 0$. At e_i there are two such conditions. Denote the direct sum of these allowed value spaces, with unrestricted k^3 at all other trace points, by $A(Z_2)$. Consider the square map

$$D_h : \Omega_h \oplus (K_{00} \otimes T_1) \longrightarrow A(Z_2), \qquad (\eta, \phi) \longmapsto \mathrm{ev}_{Z_2}(P_h(\eta) + \phi). \tag{A.11}$$

Indeed, both dimensions equal

$$3h + 8 + h(h+4) = 3|Z_2| - (2h + 12).$$

The kernels of D_h and (A.8) are naturally isomorphic. In particular, one map is invertible exactly when the other is. To see this, start with a relation in (A.8). Its normal-degree-three part gives an element of Ω_h; its normal-degree-two part uniquely determines the coefficients of the $K_Q T_1$ terms. The remaining relation has normal degree one and says that $P_h(\eta) + \phi$ belongs to $V \otimes U$, or equivalently has zero values on Z_2. Conversely, a pair in the kernel of D_h uniquely supplies the remaining $V \otimes U$ coefficients and hence an original relation. The dimensions of (A.8) agree. This argument is ordinary coefficient elimination and uses no ideal-generation assertion for the chosen special trace.

Enlargement by three base variables. We next show that invertibility of D_h implies invertibility of D_{h+3} for a suitable enlargement. Keep every old point fixed, add three new base indices B, their coordinate trace points and their two cone points each, and choose their t_b generally. All old lifts restrict to their original values, while $H_b = \ell_b(s) y_b$ for $b \in B$. The kernel of restriction on the source of (A.11) is the direct sum of

(1) the nine vectors $U \otimes \langle H_b : b \in B \rangle$;
(2) the $6h + 9$ products consisting of old H_i times new y_b, and new H_b times arbitrary old or new base variables;
(3) the twelve products $H_b z_j$.

Every vector in this kernel has zero value at all old trace points. The first two groups, restricted to the base, form a vector-quadric space of dimension $6h + 18$, as we now verify.

On the base, the products $H_i y_j$ have coefficient v_i on $y_i y_j$. They are independent: diagonal coefficients involve v_i, and an edge ij involves the independent vectors v_i, v_j. Write $\lambda_{ij} = v_i \times v_j$, $\sigma_i^+ \times \sigma_i^- = c_i v_i$, and $L_i = (t_i^2, -2t_i, 1)$. Modulo the product space, measure an edge coefficient b_{ij} by $-\lambda_{ij} \cdot b_{ij}$. Put $q_{ij} = (t_i - t_j)^2 (t_i^2 + t_i t_j + t_j^2)$. The vector identity

$$\lambda_{ij} = \frac{\sigma_j^+(\sigma_j^- \cdot v_i) - \sigma_j^-(\sigma_j^+ \cdot v_i)}{c_j}$$

and (A.10) give

$$\lambda_{ij}\cdot A_j((s\cdot u_i)\ell_i) = -\frac{q_{ij}}{c_j}\,L_j\cdot u_i.$$

Consequently the edge coordinate of the sum of the products

$$(s\cdot u_i)H_i - y_iK_{(s\cdot u_i)\ell_i}$$

is

$$-\frac{q_{ij}}{c_j}L_j\cdot u_i + \frac{q_{ij}}{c_i}L_i\cdot u_j.$$

Multiplication of this row by c_ic_j/q_{ij} and the invertible change of variables $w_i = c_iu_i$ therefore give

$$L_i\cdot w_j - L_j\cdot w_i. \tag{A.12}$$

Choose three old indices as roots. With all old $w_i = 0$, the nine root–new edge coefficients determine the three new vectors w_b, since the three root vectors L_i are independent. Hence the nine new motions are independent modulo the product space.

It remains to interpolate this space by actual points. After removing its three new diagonal columns, its coefficient description has two columns spanning $\langle v_i, v_j\rangle$ on every edge between an old nonroot and a new vertex and on the three new–new edges; on each of the nine root–new edges it has three columns spanning k^3. The latter columns may also have coefficients on the former edges. The number of two-column edges is $3h-6$, so the off-diagonal dimension is

$$2(3h-6)+27 = 6h+15 = 3(2h+5).$$

Here is an explicit interpolation argument. Assign positive integer exponents a_i close to $1,\dots,h-3$ on old nonroots, exponents close to $M, M+1, M+2$ on new vertices, and much larger exponents on the three roots. Choose M large, then perturb and clear denominators so that all relevant edge sums are distinct. The old-nonroot–new edges occur in matching groups of sizes $1,2,3,\dots,3,2,1$; no three consecutive edges form a star. Their number is divisible by three, and the final three new–new edges form a triangle. By a general choice of the three new conic points, the normals of every such triple of chords are independent. This choice is possible even with the old conic points fixed: two chords with a common endpoint meet there, and a third chord not containing that endpoint does not pass through it; for three disjoint chords, varying a new endpoint prevents persistent concurrence.

Evaluate at $2h+5$ points

$$y(\tau_j) = (\tau_j^{a_1},\dots,\tau_j^{a_{h+3}}), \qquad \tau_j = T^{b_j}, \qquad 0<b_1<\cdots<b_{2h+5}.$$

In the determinant, the maximal power of T groups each three successive two-column edges into two successive point blocks. The corresponding six-by-six leading coefficient is, up to a nonzero factor, the determinant of the three chord normals. For completeness, if the three coefficient planes have bases (A_1,A_2), (B_1,B_2), (C_1,C_2), this coefficient is

$$\det(A_1,A_2,B_1)\det(B_2,C_1,C_2) - \det(A_1,A_2,B_2)\det(B_1,C_1,C_2),$$

which equals the determinant of their plane normals up to sign. Each root–new edge subsequently contributes an invertible three-by-three block. All extra coefficients of those columns have strictly smaller edge degrees and therefore cannot cancel this leading term. Adding the three new coordinate points handles the three diagonal columns. We have obtained an invertible $(6h+18)$-square evaluation block at $2h+8$ new base points.

Finally add, in each layer, the point $\pi_B + \zeta_j e_{z_j}$. On the twelve columns H_bz_j, the four new point blocks are

$$\zeta_j[\,v_b : b\in B\,], \qquad 1\le j\le 4,$$

which are invertible. These twelve columns vanish on all new base points. Ordering old rows first, then new base rows, then new layer rows, the enlarged map is therefore

$$\begin{pmatrix} D_h & 0 & 0 \\ * & A & 0 \\ * & * & B \end{pmatrix}, \qquad \det A \neq 0, \quad \det B \neq 0.$$

This proves the enlargement. Notice that $f_{h+3} - f_h = 2h + 5$, exactly the number of new free base points used. Scalar quadratic independence can be retained simultaneously: quadrics involving the three new base variables have dimension $3h + 6 \geq 2h + 8$ and impose independent conditions at the new coordinate and general base points. The four new layer rows are distinguished by the monomials $y_b z_j$ for a fixed new index b. Their nonempty scalar-rank opens intersect the nonempty determinant opens just constructed.

An explicit initial configuration. We give the finite initial calculation in full reproducible form. Take $h = 8$, $t_i = i$, and work first over $\mathbf{Q}(\omega)$. The eight coordinate trace points and sixteen base cone points are as above. The twenty-four free base trace points, in the coordinates $(y_1, \ldots, y_8)$, are the rows of Table 1; their four z coordinates are zero. The four layer records are in Table 2: the trace points of layer j are

$$e_{z_j}, \quad (e_1 + e_2 + e_3) + \zeta_{j,1} e_{z_j}, \quad (e_4 + e_5 + e_6) + \zeta_{j,2} e_{z_j}, \quad (e_7 + e_8) + \zeta_{j,3} e_{z_j},$$

and the first three are lifted using the three listed normal vectors. The four final trace points are the rows of Table 3. All entries in these tables are rational integers, not algebraic parameters.

138	583	868	822	783	65	262	121
508	780	461	484	668	389	808	215
97	500	30	915	856	400	444	623
781	786	3	713	457	273	739	822
235	606	968	105	924	326	32	23
27	666	555	10	962	903	391	703
222	993	433	744	30	541	228	783
449	962	508	567	239	354	237	694
225	780	471	976	297	949	23	427
858	939	570	945	658	103	191	645
742	881	304	124	761	341	918	739
997	729	513	959	991	433	520	850
933	687	195	311	291	602	997	904
512	867	964	518	403	604	874	36
492	249	762	817	414	425	681	178
376	562	904	720	795	691	756	384
89	450	680	521	111	798	168	534
861	403	380	502	751	31	481	45
316	721	869	1007	630	608	593	404
663	175	173	515	233	1006	13	790
205	553	943	881	562	238	415	527
353	976	868	592	362	471	932	276
676	562	624	981	747	6	393	803
878	841	978	908	961	759	525	829

TABLE 1. Free base points for the three-normal initial configuration.

Here are precise instructions and intermediate values for the calculation, so that it requires no further data. Reduce modulo 1009 and set $\omega = 374$; then $374^2 + 374 + 1 \equiv 0 \pmod{1009}$. Order variables as $(s_1, s_2, s_3, y_1, \ldots, y_8, z_1, \ldots, z_4)$ and monomials $x_{i_1} \cdots x_{i_d}$ lexicographically by their

j	$(\zeta_{j,1},\zeta_{j,2},\zeta_{j,3})$	$a_{j,0}$	$a_{j,1}$	$a_{j,2}$
1	$(133,532,797)$	$(575,211,437)$	$(973,58,493)$	$(891,374,584)$
2	$(568,205,964)$	$(517,424,497)$	$(833,366,425)$	$(355,2,552)$
3	$(554,639,806)$	$(628,340,470)$	$(615,29,824)$	$(236,651,182)$
4	$(564,599,186)$	$(882,94,818)$	$(565,817,872)$	$(837,954,262)$

TABLE 2. Layer parameters and normal-coordinate vectors.

34	862	967	690	73	86	889	18	464	15	773	774
288	256	276	113	817	640	190	353	298	72	172	164
262	541	975	173	673	280	664	729	302	466	720	330
509	486	117	25	320	396	352	432	816	193	265	112

TABLE 3. The final four trace points, with the y coordinates first.

nondecreasing index tuples. Form the evaluation matrices E_K on the 42 quadratic monomials in P_2 at the 28 cone points, and E_V on the 78 trace quadratic monomials at the 52 trace points. Order points as in their construction: base coordinates or base cone pairs first, followed by free base points where applicable, successive layers, and final points. Row reduction, scanning columns from left to right and using the first available nonzero pivot, gives

$$\operatorname{rank} E_K = 28, \qquad \operatorname{rank} E_V = 52.$$

The determinants on their pivot columns are respectively 948 and 604 modulo 1009. The four layer-normal determinants are $832,164,963,296$.

For each evaluation matrix, use its reduced row-echelon form to make a kernel basis: in increasing order of free columns, set that free coordinate to one and all other free coordinates to zero. Multiply each of the 14 resulting K basis vectors by each of the 15 variables, then multiply each of the 26 resulting V basis vectors by s_1, s_2, s_3. The resulting cubic coefficient matrix has 680 rows and 288 columns. Forward elimination with the same first-pivot rule has 288 pivots. If the original row labels are carried along during elimination, the square submatrix on those pivot rows, in pivot order, has determinant

$$280 \pmod{1009}.$$

These assertions involve only the displayed integer tables, the sixteen explicit algebraic cone points, and addition, multiplication and inversion in $\mathbf{F}_{1009}$.

Every column of this cubic matrix lies in $P_3 \cap I_{Z_1}(3)$. Evaluation of P_3 on Z_1 has rank 28, since multiplication by s_1, which is nonzero at all cone points, carries the full-rank quadratic evaluation to cubic evaluation. Therefore

$$\dim(P_3 \cap I_{Z_1}(3)) = (680 - 364) - 28 = 288,$$

and the initial middle map is an isomorphism. The two nonzero evaluation minors ensure that the kernel bases are defined over a localization of $\mathbf{Z}[\omega]$ at the prime $(1009, \omega - 374)$. Thus the nonzero reduced cubic determinant proves that its characteristic-zero determinant is nonzero as well. This uses the displayed algebraic formulas for the first sixteen cone points; the other coordinates are the literal integers in the tables. The initial D_8 is consequently invertible.

The enlargement now proves invertibility for every $h = 8 + 3a$. Finally, the marked cone family used here is irreducible and dominates the space of ordered trace configurations: normal coordinates vary independently over each designated trace label. Middle-map invertibility and scalar independence are nonempty open conditions by the construction. If quadratic ideal generation holds in $m = n - 3$ variables, trace bijectivity is another nonempty open in this same family. Intersecting these opens and applying the exact cone splitting proves the quadratic step for every $n \equiv 0 \pmod 3$, $n \geq 15$.

Proof of Theorem A.1. The theorem of Fröberg and Löfwall [6, Theorem 6.2] supplies the cases $n \le 12$. Assume $n > 12$, and induct on n. If $n \equiv 2 \pmod 3$, the two-normal construction gives the integral endpoint from the theorem in $n-2$ variables. If $n \equiv 1 \pmod 3$, that construction and its one-point enlargement give the surjective and injective endpoints, respectively, again from $n-2$ variables. If $n \equiv 0 \pmod 3$, then $n \ge 15$, and the three-normal construction gives the integral endpoint from $n-3$ variables. Every predecessor is smaller than n. The reduction to critical point counts therefore proves the assertion for every $n \le p < N_2(n)$. □

Appendix B. The cubic case

We prove the degree-three case by adding two variables at a time. The construction used below retains all three normal degrees of a cubic. This is essential: the relations in the two upper normal degrees contribute to the final interpolation problem.

Theorem B.1. *For every $n \ge 2$, general points in $\mathbf{P}^{n-1}$ satisfy the ideal generation conjecture in initial degree three.*

B.1. **Reduction to a cone construction.** The cases $n \le 11$ follow from Gesmundo–Kayser–Telen [3, Theorem 5.1]. Their table uses projective dimension: the two largest cases needed here are 165 points in $\mathbf{P}^9$ and 214, 215 points in $\mathbf{P}^{10}$, within the respective bounds 207 and 267. The elementary case of $\mathbf{P}^1$ can also be treated directly.

Observe that

$$E(m,3) = \frac{m(m+1)(m+2)}{24}$$

is integral exactly when $m \not\equiv 1, 3, 5 \pmod 8$. Indeed, divisibility by 3 is automatic. If m is even, the two even factors have product divisible by 8; if m is odd, divisibility by 8 is equivalent to $8 \mid m+1$. Consequently Proposition 3.2 applies from $n-1$ to n unless $n \equiv 2, 4, 6 \pmod 8$. Its residual hypothesis is quadratic injectivity at

$$\left\lceil \frac{n(n+1)}{8} \right\rceil.$$

This is supplied by Theorem A.1, since for $n \ge 10$

$$\left\lceil \frac{n(n+1)}{8} \right\rceil \le \left\lfloor \frac{n(n+1)}{6} \right\rfloor.$$

It remains to provide a two-variable transfer for every even $n \ge 12$.

Write $n = m+2$, put $T = k[y_0, \dots, y_{m-1}]$, and set

$$S = T[s,t], \qquad U = \langle s,t \rangle, \qquad P = (s,t).$$

Choose $p_2 = 3N_3(m)/4$ points Z_2 in $\Lambda = \{s = t = 0\}$. A point attached to $y \in Z_2$ has a representative

$$z = (\sigma, \mu y), \qquad \sigma \in k^2 \setminus \{0\}, \quad \mu \ne 0.$$

Let Z_1 consist of $p_1 = 3(m+2)^2/4$ such points, and set

$$K = P_3 \cap I_{Z_1}(3), \quad V = I_{Z_2,T}(3), \quad M = P_4 \cap I_{Z_1}(4).$$

Every member of V, viewed as a polynomial in S, vanishes on the attached points. Separating the pure y-part of a polynomial therefore gives

$$I_{Z_1 \cup Z_2}(3) = V \oplus K.$$

Multiplication splits into the trace multiplication and the map

$$K \otimes S_1 \ \oplus \ V \otimes U \longrightarrow M. \tag{B.1}$$

When the scalar evaluations have full rank,

$$\dim K = \frac{(m+2)^2}{4}, \qquad \dim V = \frac{N_3(m)}{4}.$$

Thus (B.1) is square. Its size, in terms of $n = m + 2$, is $(4n^3 - 3n^2 + 2n)/12$.

The attachment pattern specifies an irreducible parameter space: the ordered points of Z_2 are free, and their normal directions and heights are free. This space dominates the ordered trace configuration space. Hence one full-rank instance of (B.1) gives a nonempty open set that meets the pullback of the inductive trace open set. We may therefore use specially supported trace points to construct the middle map, without requiring that special trace to satisfy ideal generation.

B.2. **Eliminating the upper normal degrees.** Write a residual cubic as

$$F = C + \sum_c B_c y_c + A, \qquad C \in \operatorname{Sym}^3 U, \quad B_c \in \operatorname{Sym}^2 U, \quad A \in U \otimes T_2.$$

Suppose the projection of K onto its C, B coefficients is onto. Let $K_{00} \subset U \otimes T_2$ be its kernel, and choose linear lifts

$$K_R = R + a(R), \qquad K_{Q,c} = Qy_c + a(Q, c)$$

for $R \in \operatorname{Sym}^3 U$ and $Q \in \operatorname{Sym}^2 U$. Define

$$\Omega = \{(k_s, k_t) \in K^2 : sC(k_s) + tC(k_t) = 0\}, \qquad \mathcal{X} = \Omega \oplus \left(\operatorname{Sym}^2 U \otimes \textstyle\bigwedge^2 T_1\right) \oplus (K_{00} \otimes T_1).$$

Lemma B.2. *There is a linear map $\Phi : \mathcal{X} \longrightarrow U \otimes T_3$ with the following properties. Its image is $(KS_1) \cap (U \otimes T_3)$. At every attached point $(\sigma, \mu y)$, it satisfies $\sigma \cdot \Phi(y) = 0$. If*

$$\mathcal{A}(Z_2) = \{(v_y) \in (k^2)^{Z_2} : \sigma \cdot v_y = 0 \text{ for every point attached to } y\},$$

then the kernel of (B.1) is naturally the kernel of

$$D : \mathcal{X} \longrightarrow \mathcal{A}(Z_2), \qquad D = \operatorname{ev}_{Z_2} \Phi. \tag{B.2}$$

With at most two independent normal directions above each support, $\dim \mathcal{A}(Z_2) = 2p_2 - p_1$.

Proof. Consider a tensor

$$k_s \otimes s + k_t \otimes t + \sum_b k_b \otimes y_b + V_s \otimes s + V_t \otimes t$$

in the source of (B.1). Its normal-degree-four equation says $(k_s, k_t) \in \Omega$. The normal-degree-three equation uniquely sets

$$C(k_b) = -sB_b(k_s) - tB_b(k_t) =: R_b.$$

Write $B(k_b) = \sum_c Q_{bc} y_c$. The normal-degree-two equation determines the symmetric part of the array Q_{bc} through

$$\sum_{b,c} Q_{bc} y_b y_c = -sA(k_s) - tA(k_t).$$

Its antisymmetric part is arbitrary and is precisely an element of $\operatorname{Sym}^2 U \otimes \bigwedge^2 T_1$. Thus

$$k_b = K_{R_b} + \sum_c K_{Q_{bc},c} + \kappa_b, \qquad \kappa_b \in K_{00}.$$

These equations define $\Phi = \sum_b A(k_b) y_b$. The remaining equation is $\Phi + sV_s + tV_t = 0$. It has a solution with $V_s, V_t \in V$ exactly when Φ vanishes on Z_2; then that solution is unique. Every preceding elimination was reversible. Moreover, $\Phi \in KS_1$ vanishes at each attached point; since it has normal degree one, this is equivalent to $\sigma \cdot \Phi(y) = 0$. Finally each support contributes two scalar coordinates minus the number of its independent normal directions. □

B.3. **The supported configurations and their lifts.** Let H be a graph on the m trace coordinates. Include in Z_2 the vertices e_i, the points $\pi_{ij} = e_i + \lambda_{ij} e_j$ for $i < j$, and

$$\frac{m(m+1)(m-2)}{8}$$

additional free points. Attach two points above every vertex; above π_{ij}, attach one point if $ij \in H$ and two otherwise. Directions at a double support are independent. The required number of attached points is equivalent to

$$|H| = \frac{m^2 - 8m - 12}{4}. \tag{B.3}$$

If the direction at a single support is σ_{ij}, write $\ell_{ij} = \sigma_{ij,t} s - \sigma_{ij,s} t$.

Lemma B.3. *For these configurations,*

$$K_{00} = \bigoplus_{ij \in H} k\, \ell_{ij} y_i y_j, \qquad \dim K = 4 + 3m + |H| = \frac{(m+2)^2}{4},$$

and the projection onto the C, B coefficients is onto. The lifts can be chosen with $a(Q, c) = y_c L_c(Q)$, where $L_c(Q)$ is a vector-valued linear form.

Proof. For a member of $U \otimes T_2$, the two vertex equations kill the coefficient of each y_i^2. A double-edge support then kills the coefficient of $y_i y_j$, whereas a single-edge support restricts it to $k\ell_{ij}$. This proves the formula for K_{00}. For arbitrary C, B, first solve the two equations for each coefficient of y_i^2; then solve the one or two equations for each coefficient of $y_i y_j$. The same independent normal directions make each solve possible.

Here are the explicit equations used below. Write $L_c(Q) = \sum_j \alpha_{cj}(Q) y_j$. For a cone point $(\sigma, \mu e_c)$, and for a point above the edge joining c and j, respectively, they are

$$\begin{aligned} \sigma \cdot \alpha_{cc}(Q) &= -Q(\sigma)/\mu, \\ \sigma \cdot \alpha_{cj}(Q) &= \begin{cases} -\big(Q(\sigma)/\mu + \sigma \cdot \alpha_{cc}(Q)\big)/\lambda_{cj}, & c < j, \\ -Q(\sigma)/\mu - \lambda_{jc} \sigma \cdot \alpha_{cc}(Q), & j < c. \end{cases} \end{aligned} \tag{B.4}$$

At a single support choose any fixed linear solution of its one equation. Likewise the coefficients $\beta_{ij}(R)$ of $a(R)$ are obtained from

$$\sigma \cdot \beta_{ii}(R) = -R(\sigma)/\mu^2, \qquad \sigma \cdot \big(\beta_{ii} + \lambda_{ij}\beta_{ij} + \lambda_{ij}^2 \beta_{jj}\big) = -R(\sigma)/\mu^2.$$

These formulas follow by evaluating the cubic at its cone point. In particular $L_c(Q)$ uses only the data at c and its incident edges, not the vertex data at their other endpoints. □

We use the following graph for every even $m \geq 10$. Start with

$$A_0 = \{0, 1, 2, 3, 4\}, \qquad B_0 = \{5, 6, 7, 8, 9\}, \qquad H_0 = \{17, 28\}.$$

Successively add $b_j = 8 + 2j$ to B and $a_j = 9 + 2j$ to A, with

$$N_H(b_j) = A_{\text{old}} \setminus \{0\}, \qquad N_H(a_j) = B_{\text{old}} \setminus \{5, 6\}. \tag{B.5}$$

There is no edge $a_j b_j$ in H. The graph remains bipartite, and each step adds $m - 3$ edges, which preserves (B.3). Thus triangles never occur. For general directions, multiplication $K_{00} \otimes T_1 \to U \otimes T_3$ is injective: a cubic monomial is supported by at most two graph edges, whose normal linear forms are independent when both occur.

B.4. **Adding a pair of variables.** At a step from m to $m+2$, denote the new variables by $w_1 = b_{j+1}$ and $w_2 = a_{j+1}$, and retain every old point. Put $N_1 = A_{\mathrm{old}} \setminus \{0\}$ and $N_2 = B_{\mathrm{old}} \setminus \{5,6\}$. Restriction $w_1 = w_2 = 0$ maps the new space $\mathcal{X}$ onto the old one. This follows either from the elimination equations, using compatible local lifts, or directly by lifting its Ω, antisymmetric, and K_{00} coordinates separately. Let $\mathcal{X}_{\mathrm{new}}$ be its kernel. All products from this kernel vanish on the old trace space, so in compatible bases

$$D_{m+2} = \begin{pmatrix} D_m & 0 \\ * & D_{\mathrm{new}} \end{pmatrix}, \qquad \dim \mathcal{X}_{\mathrm{new}} = \frac{3m^2}{2} + 3m - 3. \tag{B.6}$$

The latter count follows by taking differences in

$$\dim \mathcal{X} = (2 \dim K - 5) + 3\binom{m}{2} + m|H|.$$

We describe explicit coordinates on $\mathcal{X}_{\mathrm{new}}$. There are $2m-3$ coordinates whose products have no term linear in w_1, w_2: they are

$$\ell_{iw_b} y_i w_b \otimes w_{b'}, \quad i \in N_b,\ b, b' \in \{1,2\}, \qquad Q \otimes (w_1 \wedge w_2), \quad Q \in \operatorname{Sym}^2 U.$$

The remaining coordinates number $3m^2/2 + m$. Their linear parts are pure in w_1 or w_2. For $w = w_b$, after invertible column changes they are the following vector-valued quadrics on the old trace space:

$$\begin{array}{ll}
\ell_{ij} y_i y_j, & ij \in H, \\
\ell_{iw} y_i y_c, & i \in N_b, \quad 0 \le c < m, \\
y_i (L_w(Q) - L_i(Q))|_{w=0}, & 0 \le i < m, \quad Q \in \{s^2, st, t^2\}, \\
y_i L_w(u\ell_{iw})|_{w=0}, & i \in N_b, \quad u \in \{s,t\}, \\
a(uQ)|_{w=0} + \displaystyle\sum_j y_j L_w(u\alpha_{wj}(Q))|_{w=0}, & u \in \{s,t\}, \quad Q \in \{s^2, st, t^2\}.
\end{array} \tag{B.7}$$

Here $w = 0$ means both new coordinates are set to zero. To check the list, substitute the local lifts into Lemma B.2. The first two families come from $K_{00} \otimes T_1$, and the third from $Q \otimes (y_i \wedge w)$. For the fourth, the product associated to $(\ell_{iw} y_i w, 0) \in \Omega$ has linear part $-y_i(L_i(u\ell_{iw}) + L_w(u\ell_{iw}))/2$. Adding half of the corresponding third-family column gives the fourth family, up to sign. The same substitution for $(K_{Q,w}, 0)$ gives the last family after adding half of the appropriate third-family columns. This also accounts for every coordinate: their total, together with the $2m-3$ higher coordinates, equals $\dim \mathcal{X}_{\mathrm{new}}$.

Choose

$$\frac{3m^2}{4} + \frac{m}{2}$$

of the new free trace points in the form $p + \varepsilon(c_1 w_1 + c_2 w_2)$, with p in the old space. Keep another $m/2$ new free points away from the old space. There are also $m-3$ new single-edge supports, each contributing one allowed evaluation coordinate. Let B_{11} evaluate (B.7) at the first group, with weights c_1, c_2, and let B_{22} evaluate the higher coordinates at the single-edge supports and the second group. Dividing the first $3m^2/2 + m$ evaluation rows by ε gives at $\varepsilon = 0$

$$\begin{pmatrix} B_{11} & 0 \\ * & B_{22} \end{pmatrix}.$$

Thus

$$\det D_{\mathrm{new}}(\varepsilon) = \varepsilon^{3m^2/2+m} \det B_{11} \det B_{22} + O(\varepsilon^{3m^2/2+m+1}). \tag{B.8}$$

It remains to make these two fixed-type blocks nonsingular.

B.5. **The two evaluation blocks.** First consider B_{22}. At the support of iw_b, the same-index column $\ell_{iw_b} y_i w_b \otimes w_b$ has a nonzero scalar evaluation, whereas all the other higher columns vanish there. After these $m-3$ pivots, pair the remaining $m-3$ cross columns, leaving one, with index $i_0 = 1$, unpaired. For each pair use a point supported on its two old vertices and w_1, w_2. Its two rows give the two independent normal lines. Two final points supported on $\{i_0, w_1, w_2\}$ see the leftover cross column and the three columns

$$w_1 w_2 (L_{w_2}(Q) - L_{w_1}(Q)), \qquad Q = s^2, st, t^2.$$

The resulting last block is 4×4. Proposition C.1 provides parameters for which it is nonsingular. Its coefficients depend only on the two new vertices and their incident edges, so this is the same nonzero polynomial at every step.

For B_{11}, call the first two families in (B.7) the local columns. A face is a vertex or an edge of the old coordinate simplex. On a face F, the number of local columns in direction b is

$$t_b(F) = \mathbf{1}_{\{F \in H\}} + |F \cap N_b| \le 2.$$

Call F full or half according as this number is two or one. The inequality follows from bipartiteness: three local columns on an edge would give a triangle after adding w_b. Call the third and fourth families the star columns, and the last family the six remaining columns.

Take the following rows on the ten-coordinate initial pool:

$$\text{one point on every pool face with } t_b(F) > 0, \text{ in pure direction } w_b;$$

$$O_1 = \{\{5\},\{6\},\{7\},\{8\},\{9\},\{0\},\{7,8\},\{7,9\},\{8,9\}\}, \tag{B.9}$$

$$O_2 = \{\{0\},\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{1,2\},\{1,3\},\{2,3\}\}.$$

Use one additional point on each listed optional face in its specified pure direction. The two initial-pool blocks have sizes 86 and 74, respectively.

For an old pair (a_j, b_j), assign a face involving two old pairs to the earlier pair. Use one pure-direction point on every full face assigned to j, and thirteen points with general direction weights on

$$S_j = \{a_j, b_j, 0, 5, 6\}.$$

In direction w_1, the half faces involving the pair are

$$\{a_j\},\ \{a_j, b_j\},\ \{a_j, 0\},\ \{a_j, 5\},\ \{a_j, 6\};$$

in direction w_2, they are

$$\{b_j\},\ \{a_j, b_j\},\ \{b_j, 0\},\ \{b_j, 5\},\ \{b_j, 6\}.$$

There are no full faces of the pair inside S_j. There are sixteen star columns for the pair: five from a_j and three from b_j in the first direction, and three and five in the second. Consequently the thirteen points give a square 26×26 block consisting of these sixteen star columns and the ten half-face columns.

These rows and columns factor the determinant of B_{11}. Indeed, every pair column has a factor y_{a_j} or y_{b_j}. Pool rows annihilate all pair columns, and the rows assigned to a later pair annihilate earlier pair columns. Within a pair block, each full-face row sees precisely its two local columns; the thirteen points on S_j annihilate these full-face columns. Therefore, up to nonzero coordinate factors and a sign,

$$\det B_{11} = \det R \prod_j \left(\det E_j \prod_{\substack{F \text{ full} \\ F \text{ assigned to } j}} \det(\ell_F, \ell'_F) \right). \tag{B.10}$$

Here R is the 160×160 initial-pool block and every E_j is the same 26×26 polynomial construction, with its coordinate names changed. Their nonsingularity for some parameters is established in Proposition C.1.

B.6. **Compatible parameters and conclusion.** We must ensure that the successive nonvanishing conditions can be imposed on one configuration. The locality in (B.4) gives this directly. The determinant of R uses only the old pool data and the new vertices and their edges to the pool. The determinant of E_j uses the vertex data at a_j, b_j, their edges to $0, 5, 6$, and the new data on these five coordinates. In particular it does not use the vertex-cone data at $0, 5, 6$. The 4×4 determinant for B_{22} likewise uses only new vertex and incident-edge data.

A nonzero polynomial $f(x, y)$ remains a nonzero polynomial in y for x in a nonempty open set: exclude the common zero set of its coefficients. At the initial level $m = 10$, impose both the nonvanishing of the middle map and this coefficient-open condition for R; both are nonempty by Proposition C.1. For every subsequent pair, retain the coefficient-open condition for its future 26×26 block. When adding the next pair, each existing block is a nonzero polynomial in the shared new parameters and its own evaluation points. Their finite product is nonzero. The future condition for the new pair remains a nonempty open condition with the pool fixed, because its old-side parameters exclude the vertex data at the three fixed roots. Intersect these conditions with the nonzero normal determinants and the nonzero 4×4 determinant. The parameter space is irreducible, so the intersection is nonempty.

Equations (B.8) and (B.6) now give an invertible map D at every even level $m \geq 10$. This also proves scalar independence of the trace if it has not already been imposed. Indeed, Lemma B.2 gives injectivity of (B.1); if $\dim V > N_3(m)/4$, its source would be larger than its target, contrary to that injectivity. Scalar independence of the cone points in degree three follows from Lemma B.3, and independence in degree four follows by multiplication by a linear form nonzero at the points. Thus the middle map is an isomorphism with the asserted dimensions.

Finally restore a general trace in the irreducible attachment family and apply induction in n. The two-variable construction handles every even exceptional level $n \geq 12$; the integral hyperplane transfer handles every other level. Together with the published cases $n \leq 11$ and the quadratic residual theorem, this proves the two cubic critical counts for all n. Point-count monotonicity proves Theorem B.1.

Appendix C. Finite determinants for the cubic construction

The four determinants used in the cubic construction can be specified without listing their entries. We give the point coordinates by an integer recurrence, construct the matrices by polynomial evaluation, and record the elimination results. All the arithmetic in the first three constructions takes place in $\mathbb{F}_{1009}$.

Proposition C.1. *The cubic construction has the following nonvanishing properties.*

(1) For $m = 10$, the middle multiplication map is an isomorphism.
(2) The 160 root rows in the passage from $m = 10$ to $m = 12$ are independent on the new source.
(3) The 26-dimensional pair block is invertible.
(4) The final four-dimensional block is invertible.

Each assertion holds on a nonempty open subset of the corresponding space of cone and trace parameters over every characteristic-zero field.

C.1. **Conventions for the computations.** For $h \in \mathbb{Z}/2^{64}\mathbb{Z}$, replace h by

$$h \longleftarrow 6364136223846793005h + 1442695040888963407 \pmod{2^{64}}.$$

After this replacement define a draw from $[a, b)$ by

$$a + (\lfloor h/2^{32} \rfloor \bmod (b - a)). \tag{C.1}$$

Every draw advances the state once, including a draw whose value is subsequently discarded. Residues in $\mathbb{F}_{1009}$ are represented by $0, \dots, 1008$.

The degree-e monomials in $z_1, \dots, z_v$ are ordered lexicographically by decreasing exponent: first those with exponent e on z_1, then those with exponent $e - 1$, and so on, applying the same rule recursively to the remaining variables. A subcollection inherits this order. Polynomial coefficients are column vectors. For a matrix A, its canonical kernel matrix $K(A)$ is obtained as follows. Scan columns from left to right; in each column use the first available nonzero row as pivot, normalize it, and eliminate below it. The free columns, in increasing order, determine the kernel basis: set the corresponding free coordinate to one, the other free coordinates to zero, and solve backwards for the pivot coordinates. Thus $AK(A) = 0$.

The determinant recorded below is that of the submatrix on the selected pivot columns and the original pivot rows, the latter listed in their selection order. It is the product of the unnormalized pivots. These conventions specify both the intermediate kernel bases and each reported minor uniquely.

C.2. **The initial middle map and the root rows.** Begin with state $h = 20261108$. Use trace indices $0, \dots, 9$ and single-edge set $\{\{1, 7\}, \{2, 8\}\}$. A cone datum is a pair (σ, μ), with $\sigma = (1, t)$ and $\mu \neq 0$. For a request for $c \in \{1, 2\}$ cone data, first draw c distinct slopes from $[0, 1009)$, rejecting repeated draws, and then draw their c heights from $[1, 1009)$ in the same order.

For each vertex, in increasing order, request two cone data. Then visit all pairs $i < j$ in lexicographic order. For each pair draw λ_{ij} from $[1, 1009)$, then request one cone datum on a single edge and two otherwise. The trace supports are e_i and $e_i + \lambda_{ij} e_j$. A datum (σ, μ) over support y means the actual point $(\sigma, \mu y)$ in coordinates $(s, t, y_0, \dots, y_9)$. Finally draw 110 free trace points. Each has first coordinate one and its other nine coordinates are successive draws from $[0, 1009)$. List the trace points as vertices, edge supports, and free points, in these respective orders.

Let E_P evaluate the monomials of $(s, t)_3$ at the 108 cone points, and let E_T evaluate the degree-three trace monomials at the 165 trace points. Put

$$K = K(E_P), \qquad V = K(E_T).$$

Form M by multiplying the columns of K successively by $s, t, y_0, \dots, y_9$, and then the columns of V by s, t. For each multiplier retain the order of the kernel basis. The rows are the ordered monomials of $(s, t)_4$. This is the actual middle multiplication matrix.

Continue the same sequence of draws. Add trace indices $10, 11$ and single edges

$$\{i, 10\}\ (1 \le i \le 4), \qquad \{i, 11\}\ (7 \le i \le 9).$$

Keep all old data. First generate the two new vertices' cone data, then visit all pairs $i < j$ in lexicographic order and generate data only for pairs with a new endpoint, using the preceding rule. Let E'_P be the new cone evaluation matrix and $K' = K(E'_P)$. Let R be the rows of K' belonging to monomials not involving y_{10}, y_{11}, and set $K^0 = K'K(R)$.

Form a matrix C whose columns are the products of K^0 by the twelve old variables, followed by the products of K' by y_{10}, y_{11}. It has 254 columns. Let $C_{\ge 2}$ comprise its rows of total (s, t)-degree at least two, and put

$$\Phi = CK(C_{\ge 2}).$$

Its 177 columns have (s, t)-degree one and restrict to zero when $y_{10} = y_{11} = 0$. For $b = 1, 2$, extract their coefficients of y_{9+b}; these are $\langle s, t \rangle$-valued quadrics in the ten old trace variables.

For the first direction put $N_1 = \{1, 2, 3, 4\}$, and for the second put $N_2 = \{7, 8, 9\}$. The ordered face lists are:

(1) the vertices in N_b, in increasing order;
(2) the pairs $i < j$, in lexicographic order, for which $\mathbf{1}_{\{i,j\}\in H} + |\{i, j\} \cap N_b| \geq 1$;
(3) for $b = 1$, the faces $\{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{0\}, \{7, 8\}, \{7, 9\}, \{8, 9\}$; for $b = 2$, the faces $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$.

For each face draw η from $[1, 1009)$, using support e_i for a vertex and $e_i + \eta e_j$ for a pair $i < j$. The draw is made even for a vertex. Complete the first direction before the second. Evaluate the corresponding vector-valued quadrics, listing the s-component before the t-component at each point. Stacking these rows gives the root matrix.

The resulting elimination is as follows:

Matrix	Size	Rank	Selected determinant
E_P	108×144	108	1008
E_T	165×220	165	274
M	650×542	542	995
E'_P	147×196	147	903
R	144×49	36	697
$C_{\geq 2}$	287×254	77	859
First root direction	86×177	86	217
Second root direction	74×177	74	588
Root matrix	160×177	160	103

All entries in the last column are in $\mathbb{F}_{1009}$. The first two ranks give $\dim K = 36$ and $\dim V = 55$. Every column of M vanishes at the cone points, and its target has dimension $650 - 108 = 542$. The nonzero determinant therefore proves assertion (1).

For assertion (2), restriction of K' lands in the old 36-dimensional kernel. Moreover, the high-normal image of C has at most eight coefficients of normal degree three and 69 of normal degree two: these are precisely the coefficients involving a new trace variable. Its rank is consequently at most 77 before specialization. The table proves equality. The 17 higher-order columns described in the cubic construction lie in the kernel of the root rows. The root rank 160 is therefore the required rank on their quotient.

C.3. **The pair block.** Restart the recurrence at $h = 851$ and use ordered indices

$$0 < 5 < 6 < 10 < 11 < 12 < 13, \qquad b = 10,\ a = 11,\ w_1 = 12,\ w_2 = 13.$$

Every draw in this paragraph is from $[1, 1009)$. At each vertex draw two distinct slopes, rejecting repeats, then their two heights. Visit all edges lexicographically. On an edge, first draw its slope or distinct slopes, then its λ, then its heights. The only single edges are $\{11, 12\}$ and $\{10, 13\}$.

Construct the linear lifts $L_c(Q)$ from the vertex and edge equations (B.4). At a single edge solve its one equation by setting the t-component of the coefficient vector to zero. Let Q run in the order s^2, st, t^2. In direction w_1, list the three motion columns for $i = a$, then the two extra columns for a, then the three motion columns for $i = b$, then the five half-face columns for a. In direction w_2, list the three motion columns for $i = a$, then the three motion columns for $i = b$, then the two extra columns for b, then the five half-face columns for b. Explicitly, these columns are

$$\begin{aligned} &\text{motion:} && y_i(L_{w_j}(Q) - L_i(Q)), \\ &\text{extra:} && y_i L_{w_j}(s\ell), \quad y_i L_{w_j}(t\ell), \\ &\text{half-face:} && \ell y_i y_c, \end{aligned}$$

where $i = a$ in direction one and $i = b$ in direction two for the last two types, $\ell = t_{iw_j} s - t$, and c runs over i, the other member of $\{a, b\}$, $0, 5, 6$, in that order. All columns are restricted to coordinates $0, 5, 6, 10, 11$.

For each of thirteen samples, draw five point coordinates in this order and then two direction coordinates (v_1, v_2). Multiply a column from direction j by v_j, evaluate it, and list its two components as consecutive rows. The resulting 26×26 matrix has determinant

$$527 \pmod{1009}.$$

This proves assertion (3). In particular the computation uses the same cone coefficients in every column; the entries are not chosen independently.

C.4. **The four-dimensional block.** Here it is simpler to give rational coordinates. Write x, u, v for the leftover old coordinate and the two new coordinates. All edge parameters λ equal one. Use the following cone data:

Trace support	(σ, μ)
e_u	$((1,0),1),\ ((0,1),2)$
e_v	$((1,0),3),\ ((0,1),4)$
$e_x + e_u$	$((1,2),2)$
$e_x + e_v$	$((1,1),3),\ ((2,1),5)$
$e_u + e_v$	$((1,1),2),\ ((1,-1),3)$

At the single edge xu, set the s-component of the lift coefficient to zero. The four columns are

$$(2,-1)xuv, \qquad uv(L_v(Q) - L_u(Q)), \quad Q = s^2, st, t^2.$$

Their evaluation at $(x,u,v) = (1,1,1), (1,2,3)$, in component order s, t, is

$$\begin{pmatrix} 2 & -2/15 & -1/15 & 2/15 \\ -1 & -7/60 & 7/30 & 17/60 \\ 12 & -63/10 & 1/10 & 33/10 \\ -6 & -1/5 & 39/10 & -23/10 \end{pmatrix}, \qquad \det = -\frac{603}{40}.$$

C.5. **Passage to characteristic zero.** Lift the primitive parameters in the first three constructions to their integer representatives, and form the cone points $(\sigma, \mu y)$ over $\mathbb{Q}$. The selected pivot minors are units at 1009, so the same kernel charts are defined over $\mathbb{Z}_{(1009)}$ and reduce to the displayed matrices. The structural rank bounds and the nonzero pivot minors preserve all intermediate dimensions. Thus each final nonzero minor lifts to a nonzero rational minor. Clearing denominators gives a nonzero polynomial in the parameters; the rational fourth determinant does so directly. These polynomials remain nonzero over any characteristic-zero field, proving Proposition C.1.

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