An exact moment criterion for survival under renewed spatial catastrophes

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Abstract

We prove an exact moment criterion for survival in a binary growth process subject to independently renewed spatial catastrophes. Occupied sites spread through a finite set of allowed increments on the integer lattice, and Poisson catastrophes with unbounded cube sizes empty all occupied sites in each affected cube. Under finite one-site hitting intensity, a positive finite-seed survival phase exists if and only if the cube-size measure has a finite moment of order one more than the spatial dimension. The divergent endpoint yields extinction at every positive intensity. The proof centres on a one-dimensional positive-phase construction for connected interval errors under a finite second moment. Extremal-path resampling gives a conditional correlation inequality, which controls exact interval insertion counts and a finite-history potential. Coordinate restriction and envelope covering then establish the criterion in every dimension. We also prove the bulk-to-seed reversal identity and extend the criterion to uniformly fat shapes. For the one-sided one-dimensional rule, the stationary hole density has a quadratic dilute expansion under the same second-moment condition. Its excess over independently eroded Poisson events is strictly positive at every positive intensity, with an explicit quadratic coefficient in the dilute limit.

Keywords: interacting particle systems; spatial catastrophes; survival criterion; dependent percolation; probabilistic cellular automata

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1. Introduction

Spatially extended catastrophes couple the loss of sites that could otherwise sustain one another. We study how their size distribution controls survival in a binary growth process. At each integer time, occupied sites spread by a finite additive recovery rule. An independent Poisson field of cube-shaped catastrophes then empties the affected sites. This update combines finite-speed recovery with the spatial dependence created by a shared catastrophe.

Destructive geometry affects survival in contact processes with cube and hyperplane catastrophes [8]. Poisson hail connects spatial extent to stability through jobs that occupy regions and retain service requirements [1]. Foss, Konstantopoulos and Mountford establish small-arrival-rate stability under moments of order dimension plus one plus a positive margin, together with lower-moment instability

examples [3]. Mountford and Wang further analyse power-law stability regions [10]. These results supply the space-time moment perspective for the binary survival problem studied here. For the binary survival problem, we obtain the exact moment order $d+1$: finiteness gives a low-intensity survival phase, and divergence gives extinction at every positive intensity.

We prove a necessary-and-sufficient moment condition that includes the divergent endpoint. For cubes in dimension d, the standing local-finiteness assumption is a finite dth size moment, while a nonempty low-intensity survival phase exists exactly when the $(d+1)$st size moment is finite. A covering catastrophe can kill every possible descendant of a finite seed, giving necessity. Sufficiency reduces to the one-dimensional process through a coordinate line. The central estimate establishes a positive phase for unbounded interval errors under a finite second moment. A bounded law and a borderline heavy-tailed law can have the same one-site hitting probabilities while lying on opposite sides of this existence criterion.

Stability under local noise supplies the methodological starting point. Toom constructs stable local automata under small independent update noise [12]. Gács and Törmä treat errors satisfying an exponential bound on every prescribed finite set of error sites [4]. Hartarsky reproves the one-dimensional eroder criterion under sufficiently small independent deaths (Theorem 1.9) [5]. Hartarsky and Toninelli give exponential tail bounds for space-time zero-state cluster diameters in the upper invariant trajectory under small independent deaths (Corollary 9.5) [6], and Swart, Szabó and Toninelli develop Toom-contour estimates for local noisy dynamics [11]. Polynomial interval tails can violate this exponential error bound. For example, in one dimension take size masses $\nu_r = Cr^{-4}$ with $C > 0$. The second size moment is finite, yet at any positive intensity ρ a specified interval of integer length $L \geq 2$ is emptied with probability at least $1 - e^{-\rho C L^{-4}}$ (a single anchor gives this bound; summing over all anchors gives a covering intensity of order L^{-2}), which eventually exceeds ϵ^L for every fixed $0 < \epsilon < 1$. The construction below supplies a positive-phase estimate uniform in the interval-length cutoff.

Hilário, Sá, Sanchis and Teixeira establish directed survival in a layered environment with geometric defects and a common random edge-opening propensity within each forward layer, for sufficiently weak defects and sufficiently high base opening probability (Theorem 8.2) [7]. In our process, spatial dependence is generated by connected Poisson deletion events of unbounded sizes. To handle those events, we adapt the alternating leftmost/rightmost-path resampling argument of van den Berg, Häggström and Kahn [2]. Connectedness permits a left/right classification relative to a surviving path. The resulting conditional correlation bound controls an exact insertion count, and a finite-history potential gives the positive phase before a stationary limit is used.

Section 2 states the model, the moment criterion and the reversal identity. Section 3 develops the one-dimensional positive-phase proof; Section 4 combines it with covering and path counting to prove the cube criterion. Sections 5 and 6 give the stationary density expansion and the shape extension.

2. Model and main results

2.1. Space, size measure and catastrophe field

Let d be a positive integer. Sites are the points of the lattice $\mathbb{Z}^d$, and $0 = (0, \ldots, 0)$ denotes its origin. Write $e_1 = (1, 0, \ldots, 0)$ for the first coordinate unit vector. Fix a finite set $G \subset \mathbb{Z}^d$ containing 0 and e_1. The elements of G are the allowed displacement vectors in one growth step. In particular, the displacement 0 allows occupancy to persist at its present site before the next catastrophe.

Let $\mathcal{R} = \{2, 3, \ldots\}$, equipped with the power-set sigma-algebra, be the space of catastrophe sizes. Fix a nonnegative measure ν on $\mathcal{R}$ and write $\nu_r = \nu(\{r\})$. The measure is not required to be a probability measure. Define its moments, allowing the value $+\infty$, by

$$S_j = \sum_{r \in \mathcal{R}} r^j \nu_r, \qquad j \in \mathbb{Z}_{\geq 0}. \tag{1}$$

Our standing model assumption is $0 < S_d < \infty$. It implies that every ν_r is finite and that ν is a finite, nonzero measure. The higher moment S_{d+1} may be infinite. The intensity parameter ρ ranges over $[0, \infty)$; the measure ν and the growth rule G remain fixed when ρ varies.

For each time $t \in \mathbb{Z}$, anchor $l \in \mathbb{Z}^d$ and size $r \in \mathcal{R}$, let $N_{t,l,r}$ be a Poisson random variable with mean $\rho\nu_r$. These variables are mutually independent. A mark is one occurrence counted by $N_{t,l,r}$. The associated lattice cube is $B(l, r) = l + \{0, \dots, r-1\}^d$. The catastrophe set at time t is

$$D_t = \bigcup_{l \in \mathbb{Z}^d,\ r \in \mathcal{R}: N_{t,l,r} \geq 1} B(l, r).$$

A catastrophe empties all occupied sites in its cube. It leaves the lattice in place, and those sites can become occupied again through later growth. Multiple marks of the same type have the same effect as one mark. The fields D_t are independent and identically distributed in time. A *time layer* is $\mathbb{Z}^d \times \{t\}$; "row" below is an abbreviation for one such time layer, not for a spatial line within $\mathbb{Z}^d$.

For any fixed site x, exactly r^d anchors l satisfy $x \in B(l, r)$: in each coordinate, l_i can take the r values $x_i - r + 1, \dots, x_i$. Thus the total number of marks whose cubes contain x at one time has Poisson mean ρS_d. It is finite almost surely, and the probability that x avoids that catastrophe layer is $p_0 = \exp(-\rho S_d)$. This local finiteness also holds simultaneously on every finite space-time set.

2.2. The occupied-set Markov chain and its paths

For a deterministic initial set $B \subseteq \mathbb{Z}^d$, define the occupied-set process $(A_t^{B,G})_{t \in \mathbb{Z}_{\geq 0}}$ by $A_0^{B,G} = B$ and

$$A_{t+1}^{B,G} = (A_t^{B,G} + G) \setminus D_{t+1}, \qquad A + G = \{x + g : x \in A,\ g \in G\}. \tag{2}$$

The order of operations is growth first, followed by the catastrophe at the new time. The prescribed initial state at time 0 is not subjected to D_0. Under the indicator representation $A \leftrightarrow (\mathbf{1}_{\{x \in A\}})_{x \in \mathbb{Z}^d}$, the state space is $\{0,1\}^{\mathbb{Z}^d}$ with its product sigma-algebra. Independence of D_{t+1} from the preceding layers makes (2) a time-homogeneous discrete-time Markov chain. The empty set is absorbing.

We also use processes started at other integer times. For $m \leq t$, write $A_{m,t}^{B,G}$ for the same recursion, with state B prescribed at time m and catastrophe layers $m+1, \dots, t$ applied. Thus $A_t^{B,G} = A_{0,t}^{B,G}$. A *finite-seed process* has a finite, nonempty initial set B. The single-seed process has $B = \{0\}$; the corresponding prescribed space-time point $(0, 0)$ is called its root. The phrase "untested initial layer" means precisely that the recursion starts with the prescribed state and first applies D_1.

An allowed path from time m to time t is a sequence $(x_k)_{k=m}^t$ with $x_{k+1} - x_k \in G$. It is open if $x_k \notin D_k$ for $m < k \leq t$. Its initial point is not tested against D_m. Induction in (2) shows that $x \in A_{m,t}^{B,G}$ exactly when an open path starts at some $x_m \in B$ and ends at $x_t = x$. At a fixed time horizon, the possible predecessors of finitely many terminal sites form a finite set because G is finite. These predecessor sites, with their times, form the backward cones used below. Although infinitely many mark types can meet a cone, their total number of active marks is finite almost surely by local finiteness.

Let Pr denote probability over the catastrophe field. For the single-seed process define

$$s_t^G(\rho) = \Pr(A_t^{\{0\},G} \neq \varnothing), \qquad s_\infty^G(\rho) = \lim_{t \to \infty} s_t^G(\rho), \qquad \rho_c^G = \sup\{\rho > 0 : s_\infty^G(\rho) > 0\}. \tag{3}$$

The limit exists because extinction is absorbing, so the survival events decrease with t. We set the supremum of the empty set equal to zero. Poisson thinning couples all intensities so that increasing ρ adds catastrophes and decreases occupancy; consequently $s_\infty^G(\rho)$ is nonincreasing. Translation invariance gives the same single-seed survival probability from any site.

2.3. The upper stationary field

Use the two-sided time environment already defined. At a fixed observation time t, the sets $A_{m,t}^{\mathbb{Z}^d,G}$ decrease as m tends to $-\infty$. Indeed, the process started at an earlier time has a subset of $\mathbb{Z}^d$ when it reaches the later initial time, and the update map preserves set inclusion. Define

$$A_t^{*,G} = \bigcap_{m<t} A_{m,t}^{\mathbb{Z}^d,G}, \qquad \theta_d^G(\rho) = \Pr(0 \in A_0^{*,G}).$$

Each site's update depends on finitely many predecessors. For decreasing sets, this finite dependence allows the update to pass through the intersection, so $(A_t^{*,G})_{t\in\mathbb{Z}}$ satisfies (2). Its law is stationary in time and invariant under spatial translations, since its construction commutes with those shifts of the independent environment. We call its one-time law the *upper invariant distribution*. In the one-dimensional hole representation, we write $H_t^* = \mathbb{Z} \setminus A_t^{*,\{0,1\}}$ and use $H^* = H_0^*$ when only one time is under consideration.

Time-shift invariance of the environment gives $\Pr(0 \in A_T^{\mathbb{Z}^d,G}) = \Pr(0 \in A_{-T,0}^{\mathbb{Z}^d,G})$. Therefore these probabilities decrease to $\theta_d^G(\rho)$. A positive value of this density is what we mean by a positive upper occupied phase.

2.4. Moment criterion and reversal

Theorem 2.1 (exact survival moment criterion). For the Markov chain (2), under the standing assumption $0 < S_d < \infty$,

$$\rho_c^G > 0 \quad \Longleftrightarrow \quad S_{d+1} < \infty. \tag{4}$$

If $S_{d+1} < \infty$, set $A_*^{(d)} = (S_d + S_{d+1})/2$. Then

$$\frac{1}{32A_*^{(d)}} \le \rho_c^G \le \frac{\log|G|}{S_d}. \tag{5}$$

If $S_{d+1} = \infty$, then for every $\rho > 0$ and every deterministic finite initial set B, the process $A_t^{B,G}$ reaches the empty set in finite time almost surely.

Lemma 2.2 (reversal). For every $\rho \ge 0$ and integer $T \ge 1$,

$$\Pr(0 \in A_T^{\mathbb{Z}^d,G}) = e^{-\rho S_d} s_{T-1}^G(\rho), \qquad \theta_d^G(\rho) = e^{-\rho S_d} s_\infty^G(\rho). \tag{6}$$

Proof. The terminal site 0 must avoid D_T. This event has probability $e^{-\rho S_d}$ and is independent of all earlier layers. After removing this terminal test, reverse and reflect a path $(x_0, \dots, x_T)$ ending at $x_T = 0$ by putting $y_k = -x_{T-k}$ for $0 \le k \le T-1$. Its initial site is $y_0 = 0$, and its increments remain in G, because $y_k - y_{k-1} = x_{T-k+1} - x_{T-k}$. Its tested layers are the reflected layers $-D_{T-1}, \dots, -D_1$, in that order. Reflection sends $B(l,r)$ to $B(-l-(r-1)\mathbf{1}, r)$, where $\mathbf{1} = (1, \dots, 1)$. This is a bijection of anchors at each size. Reflection and reversal of the independent layers therefore preserve their joint law.

Conversely, a single-seed path with $T-1$ tests determines $x_1, \dots, x_T$; taking $x_0 = x_1$ supplies the remaining initial step because $0 \in G$ and the initial state is fully occupied. Thus the two path-existence events agree under the transformation, proving the finite-time identity. The limits in (3) and Section 2.3 give the density identity.

For completeness, survival at every finite time is equivalent to an infinite open path from the seed. The tree of finite open paths is finitely branching because G is finite. If it has a vertex at every depth, König's lemma gives an infinite branch. □

Corollary 2.3 (power-law boundary and equal local exposure). If $\nu_r = Cr^{-p}$ with $C > 0$, the standing assumption holds exactly when $p > d + 1$. Within that domain,

$$\rho_c^G = 0 \quad (d + 1 < p \le d + 2), \qquad \rho_c^G > 0 \quad (p > d + 2). \tag{7}$$

There are two size measures with the same value of S_d, and hence the same one-site no-hit probability at every intensity, for which the answers to whether $\rho_c^G > 0$ differ.

Proof. The series defining S_j converges exactly when $p > j + 1$. At $p = d + 2$, the series for S_{d+1} is harmonic. To obtain equal exposure, rescale a nonzero bounded-support measure and the measure with masses $r^{-(d+2)}$ to give them the same S_d. Theorem 2.1 yields a positive critical intensity for the former and zero for the latter. □

3. Positive phase for interval catastrophes

3.1. Hole process and finite histories

For this section, $d = 1$ and $G = \{0, 1\}$. All spatial intervals have integer endpoints: $[a, b]$ means $\{a, a + 1, \dots, b\}$ when $a \le b$, and the empty set otherwise. Starting from full occupancy, define the hole process by $H_t = \mathbb{Z} \setminus A_t^{\mathbb{Z},\{0,1\}}$ for integers $t \ge 0$. A site is called closed if it is a hole and open if it is occupied. For a set H, put $H + 1 = \{x + 1 : x \in H\}$. Taking complements in (2) gives $H_0 = \varnothing$ and

$$H_{t+1} = E(H_t) \cup D_{t+1}, \qquad E(H) = H \cap (H + 1). \tag{8}$$

Replacing the interval field by independent single-site errors gives the classical Stavskaya rule [13,14]. Here the independent error coordinates are interval types.

Let $\beta_t = \Pr(0 \in H_t)$, $u_t = 1 - \beta_t$, and $h_{j,t} = \Pr([0, j - 1] \subset H_t)$. The one-time laws of H_t increase with t: they can be represented by the nested fields $\mathbb{Z} \setminus A_{-t,0}^{\mathbb{Z},\{0,1\}}$ of Section 2.3. This is a coupling of different initial times at one observation time; successive fields along a forward sample path need not be nested. Their limit is H^*; write $\beta = \Pr(0 \in H^*) = \lim_t \beta_t$ and $u = 1 - \beta$. Define

$$V(r) = \frac{r(r + 1)}{2}, \qquad A_* = \sum_r \nu_r V(r) = \frac{S_2 + S_1}{2}, \qquad c = \rho A_*.$$

Proposition 3.1 (uniform finite-history estimate). If $S_2 < \infty$ and $0 < c \le 1/32$, then

$$\sup_{t \ge 0} \beta_t \le \frac{1 - \sqrt{1 - 8c}}{2} < \frac{1}{2}. \tag{9}$$

Consequently the upper occupied phase has positive density. The case $\rho = 0$ is immediate. We prove the proposition after establishing its correlation and insertion tools.

3.2. Conditional correlation

Association of independent ordered coordinates provides the correlation tool for the construction. For one ordered coordinate U and an independent copy U', bounded increasing functions satisfy

$$\operatorname{Cov}(f(U), g(U)) = \tfrac{1}{2}\mathbb{E}[(f(U) - f(U'))(g(U) - g(U'))] \ge 0.$$

For finitely many independent ordered coordinates, condition on the last coordinate. The conditional covariance is nonnegative by induction, and the covariance of the two conditional means is nonnegative by the one-coordinate identity. This proves association by the law of total covariance. The conclusion covers both Poisson counts and uniform or Bernoulli update variables. For countably many coordinates, conditional expectations on the first n coordinates retain monotonicity and converge in L^2 for bounded functions, so the inequality passes to the limit. Oppositely monotone functions have nonpositive covariance by applying association to f and $-g$.

Lemma 3.2 (conditional left–right correlation). Let F be the hole set after a finite number of layers started from full occupancy. For bounded functions f and g that are increasing under addition of holes and depend on finitely many sites in $(-\infty, 0)$ and $(0, \infty)$, respectively,

$$\mathbb{E}[f(F)g(F) \mid 0 \notin F] \le \mathbb{E}[f(F) \mid 0 \notin F]\,\mathbb{E}[g(F) \mid 0 \notin F].$$

The same inequality holds with any site z in place of 0 after translating the two sets of coordinates. In a finite model, the statement allows arbitrary independent type-dependent probabilities whenever the conditioning event has positive probability. In the infinite model used here, the type means in a finite history are bounded by $\rho\nu_r$, including a final layer with type means $\theta\nu_r$ with $0 \le \theta \le \rho$.

Proof. The argument has three steps. Step 1 shows that alternating leftmost/rightmost-path resampling is an ergodic chain with the conditioned product law as its limit. Step 2 shows that this chain moves monotonically in a left/right comparison relation. Step 3 applies product association and passes to the infinite model. Connectedness of the interval supports is used in three places: classifying each type as left of, crossing, or right of a path; preserving the leftmost path under a right update; and the common-update comparison in Step 2. At one row ($T = 1$) the statement is exact conditional independence, since a mark that misses the origin lies wholly on one side.

Step 1: conditional product law and resampling. Work first with a finite set $\mathcal{E}$ of interval types in layers $1, \dots, T$. A type $e = (t, l, r)$ acts on the spatial interval $I_e = [l, l + r - 1]$ in layer t. Take a finite slab containing the backward cones of all observed terminal sites, including 0, and prescribe full occupancy on its initial layer. Paths and their catastrophe tests have the convention of Section 2.2. For each type e, let $Y_e = 1$ mean that the disaster is absent; in the Poisson model this probability is $p_e = e^{-\rho\nu_r}$. These variables are independent; initially assume that their parameters lie strictly between zero and one. Let Γ be the event that an open path reaches terminal site 0, and let μ be the product law conditioned on Γ.

Allowed paths have increments in $\{0, 1\}$. Their pointwise minimum and maximum are allowed paths: both are nondecreasing with one-step increments at most 1, and at each time they choose an open site of one of the two original paths. Iterating over the finite collection of paths gives its extreme elements. Thus each configuration in Γ has a leftmost path L and a rightmost path R to 0. Relative to a path π, each interval type lies strictly to its left, intersects it, or lies strictly to its right in the interval's row. Write I_e for the interval of type e. Write these three type sets as $\mathcal{L}_\pi$, $\mathcal{C}_\pi$ and $\mathcal{R}_\pi$. Connectedness gives the useful implication

$$x > \pi(t), \quad x \in I_e \quad \Longrightarrow \quad e \in \mathcal{C}_\pi \text{ or } e \in \mathcal{R}_\pi,$$

with the corresponding left-hand implication. An interval in $\mathcal{C}_\pi$ is absent whenever π is open.

A right update resamples all variables in $\mathcal{R}_L$ from their original independent laws. It preserves L. Indeed, if an updated open path p went strictly left of L in any row, then $\min(p, L)$ would be an open path everywhere at or left of L. Its sites are unaffected by the resampling, so it would have been open before the update, contradicting leftmostness. Also, $\{L = \pi\}$ depends only on the variables in $\mathcal{L}_\pi \cup \mathcal{C}_\pi$. Any path extending left of π can be replaced by its minimum with π, so its existence can be tested using just these variables. Conditional on these variables and $L = \pi$, the

variables in $\mathcal{R}_\pi$ therefore retain their product law. Resampling them preserves μ. A left update resamples $\mathcal{L}_R$; the pointwise maximum argument preserves R and gives the same invariance.

Alternating right and left updates gives a finite Markov chain with invariant law μ. Each state can reach the all-1 configuration in one complete right-then-left step with positive probability. For this event, set every resampled absence variable to 1. Every allowed path to 0 lies at or left of the vertical path at 0. After the right step, an interval hitting that vertical path either intersects the old L and was already absent, or is strictly right of L and has just been removed. The vertical path is consequently open and is the rightmost path. The subsequent left step removes every type strictly left of it; types intersecting it are absent and types to its right were resampled in the first step. All types are now absent. Since μ is invariant and positive on every state of Γ, all states are recurrent. Their common access to the all-1 state makes the chain irreducible. Each state has a positive self-loop probability, so the chain is aperiodic and converges to μ.

Step 2: comparison, common updates and coordinate monotonicity. Compare configurations in Γ by writing $\widehat{\omega} \preceq \omega$ when

$$\widehat{L} \le L, \quad \widehat{R} \le R \quad \text{rowwise},$$
$$\widehat{Y}_e \ge Y_e \quad (e \in \mathcal{L}_{\widehat{L}}), \qquad \widehat{Y}_e \le Y_e \quad (e \in \mathcal{R}_R).$$

The remaining type variables are unrestricted. This is a comparison relation between two configurations; no order on the unrestricted coordinates is imposed. To define all random updates on a fixed product space, draw independent Bernoulli variables $\xi_{n,e}$ for every update number n and every type $e \in \mathcal{E}$, with $\Pr(\xi_{n,e} = 1) = p_e$. At a right update, use $\xi_{n,e}$ only for $e \in \mathcal{R}_L$; at a left update, use it only for $e \in \mathcal{L}_R$. Ignore the other entries of that row of the array. Thus the set of entries used can depend on the current state, but the underlying independent array is fixed in advance. In two chains driven by a common array, any type resampled in both chains receives the same value.

For a common right update, put $\alpha = \widehat{L} \le L = \gamma$. The two leftmost paths remain fixed. If p is open in updated $\widehat{\omega}$, then $\max(p, \gamma)$ is open in updated ω. At a row where $p(t) > \gamma(t)$, an interval covering that site either intersects γ and is absent, or lies strictly right of γ. In the latter case it lies right of both α and γ and has the same resampled variable in both configurations. At every other row the maximum follows the open path γ. Hence $\widehat{R}_{\text{new}} \le R_{\text{new}}$. Variables strictly left of α are untouched. Every type strictly right of R_{new} is right of γ, so its variables were identically resampled. These observations preserve all three comparison conditions.

For a common left update, put $\alpha = \widehat{R} \le R = \gamma$. The two rightmost paths remain fixed. If p is open in updated ω, then $\min(p, \alpha)$ is open in updated $\widehat{\omega}$. At a row where $p(t) < \alpha(t)$, an interval covering that site either intersects α and is absent, or is strictly left of both α and γ and has a common resampled variable. At other rows the minimum follows α. Thus $\widehat{L}_{\text{new}} \le L_{\text{new}}$. Every type strictly left of $\widehat{L}_{\text{new}}$ is left of α and was identically resampled. Types strictly right of γ are untouched. The comparison again persists.

Changing a single fresh right-update variable from 1 to 0, while keeping the previous state and all other fresh variables fixed, produces a configuration further left in this comparison. The leftmost path is preserved, the rightmost path can only move left, and all changed absence variables have decreased. Changing a single fresh left-update variable from 0 to 1 likewise moves the configuration left: the rightmost path is preserved, the leftmost path can only move left, and the changed absence variable has increased. Common subsequent updates preserve these comparisons. More explicitly, change one entry $\xi_{n,e}$ in a fixed finite array of updates. The two histories agree through update $n-1$, so their resampling sets at update n are identical. If e is outside that set, the change has no effect. If it is inside, the single-update comparison just proved applies. Inducting over subsequent common updates proves the stated monotonicity of the terminal configuration, even though their later resampling sets can differ.

The comparison orders terminal reachability on each side of 0. If a path to $x < 0$ is open in ω, its minimum with $\widehat{L}$ ends at x and is open in $\widehat{\omega}$. At a site strictly left of $\widehat{L}$, a covering interval either intersects $\widehat{L}$ and is absent, or belongs to $\mathcal{L}_{\widehat{L}}$, where the absence variables satisfy the required inequality. Thus left reachability increases as a configuration moves left. For $x > 0$, the maximum of a path in $\widehat{\omega}$ with R gives an open path in ω, using $\mathcal{R}_R$ in the same way. Right reachability therefore decreases as a configuration moves left.

Step 3: association and limits. Start the chain at the deterministic all-1 configuration. After any finite number of updates, a left increasing hole function decreases with each left-update absence variable and increases with each right-update absence variable. A right increasing hole function has the opposite monotonicities. Replace every right-update coordinate $\xi_{n,e}$ by $1 - \xi_{n,e}$, including the entries that a given trajectory ignores. In the resulting independent ordered coordinates, the left function is decreasing and the right function is increasing. The product association inequality therefore gives nonpositive covariance. Convergence of the finite chain to μ proves the conditional inequality for the finite model.

For the infinite model, fix the time horizon T and the finitely many terminal coordinates on which the functions depend. Their backward cones contain finitely many sites. Retain types of length at most K that intersect these cones; only finitely many types remain. The total mean of omitted types hitting the cones is bounded by

$$C\rho T \sum_{r>K} \nu_r(r + C) \longrightarrow 0,$$

for a finite constant C depending on the cones, by $S_1 < \infty$. One may take C to be the width of the union of the cones, at most $w + T$ for observation sites and the conditioning origin in a window of width w, since a type of length r meets the window for at most $r + C$ anchors in each of the T rows. Under the common Poisson coupling, the truncated and full fields therefore agree on the observations with probability tending to one. The conditioning event has probability at least that of the open vertical path, $e^{-\rho T S_1} > 0$, uniformly in the cutoff. Expectations of the bounded side functions and their product, and the conditioning probabilities, converge. Dividing by the conditioning probabilities passes the conditional inequality to the limit. Deterministic absence variables follow by parameter limits whenever the conditioning event has positive probability. Every step permits type-dependent parameters, including the final interpolation row. □

Connected interval supports make the extremal-path resampling strategy of [2] applicable here. For a disconnected error closing $\{-1, 1\}$ with probability $1/2$ while leaving 0 open, the two side-hole indicators have joint probability $1/2$ and product of marginals $1/4$. The conclusion is therefore false for arbitrary spatially correlated row errors.

3.3. Full interval insertion and phase selection

Lemma 3.3 (full insertion count). Let F be a translation-invariant random subset of $\mathbb{Z}$, interpreted as the holes. Inserting an interval means replacing F by $F \cup [0, r - 1]$. Define the run lengths

$$L_z = \sup\{k \in \mathbb{Z}_{\geq 0} : [z - k, z - 1] \subseteq F\}, \qquad R_a = \sup\{k \in \mathbb{Z}_{\geq 0} : [a, a + k - 1] \subseteq F\},$$

with values in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$. The empty-interval convention makes $k = 0$ admissible. In particular, L_z counts holes immediately left of z, and R_r counts holes starting at r. Here $z \notin F$ means that z is open. Each summand below is defined to be zero when $z \in F$, including configurations with infinite closed runs. The number of newly closed, nonempty finite intervals is

$$\Delta_r(F) = \sum_{z=0}^{r-1} \mathbf{1}_{\{z \notin F\}}(L_z + 1)(r - z + R_r). \tag{10}$$

A newly closed interval is a nonempty finite interval contained in $F \cup [0, r-1]$ but not in F. It has a unique leftmost site z that was previously open; this site belongs to $[0, r-1]$. Its left endpoint has $L_z + 1$ choices and its right endpoint has $r - z + R_r$ choices. This decomposition counts each newly closed interval exactly once. Write $\Delta_{r,j}(F)$ for the number of newly closed intervals of length j, and $\Delta_{r,\le M}(F) = \sum_{j=1}^{M} \Delta_{r,j}(F)$ for the count up to length M.

Set $u_F = \Pr(0 \notin F) > 0$, $\beta_F = 1 - u_F$, $h_0(F) = 1$ and $h_k(F) = \Pr([0, k-1] \subseteq F)$ for $k \ge 1$. Let $\zeta = \lim_{k\to\infty} h_k(F)$. Assume that the conditional inequality of Lemma 3.2 holds for F. For $k \ge 1$, translation invariance gives

$$\Pr(z \notin F, L_z \ge k) = h_k(F) - h_{k+1}(F).$$

Summing this identity, with truncation before passage to the limit, gives

$$\mathbb{E}[\mathbf{1}_{\{z \notin F\}}(L_z + 1)] = u_F + \beta_F - \zeta \le 1.$$

For $m = r - z$, let w be the last open site in $[z, r-1]$. For each $b \ge 1$, the event $\{z \notin F, [r, r+b-1] \subset F\}$ implies that w is open and $[w+1, r+b-1]$ is closed. Dropping the other restrictions gives

$$\Pr(z \notin F, R_r \ge b) \le \sum_{w=z}^{r-1} \big(h_{r+b-w-1}(F) - h_{r+b-w}(F)\big).$$

Sum over $b \ge 1$. For each w, the telescoping sum is $h_{r-w}(F) - \zeta \le \beta_F - \zeta$. There are m possible sites w, so

$$\mathbb{E}[\mathbf{1}_{\{z \notin F\}} R_r] \le m\beta_F, \qquad \mathbb{E}[\mathbf{1}_{\{z \notin F\}}(m + R_r)] \le m.$$

Conditional on $z \notin F$, the variables $L_z + 1$ and $m + R_r$ are increasing functions of holes strictly to the left and right of z. Apply Lemma 3.2 to $\min\{L_z, n\} + 1$ and $m + \min\{R_r, n\}$, which are bounded functions of finitely many sites, and then let $n \to \infty$ by monotone convergence:

$$\mathbb{E}[\mathbf{1}_{\{z \notin F\}}(L_z + 1)(m + R_r)] \le \frac{\mathbb{E}[\mathbf{1}_{\{z \notin F\}}(L_z + 1)]\mathbb{E}[\mathbf{1}_{\{z \notin F\}}(m + R_r)]}{u_F} \le \frac{m}{u_F}.$$

Summing over the first old open point z gives

$$\mathbb{E}\Delta_r(F) \le V(r)/u_F. \tag{11}$$

Proof of Proposition 3.1. For each anchor and size, take a Poisson process in the parameter $\theta \ge 0$ with rate ν_r, independently over types and independently of the first t catastrophe layers. Let $D^{(\theta)}$ be the union of intervals with at least one mark by parameter time θ. Thus these fresh catastrophe sets increase with θ. The last-layer interpolation field is

$$F_{t,\theta} = E(H_t) \cup D^{(\theta)}, \qquad 0 \le \theta \le \rho,$$

where the first t rows have intensities $\rho\nu_r$ and the last row has intensities $\theta\nu_r$. It is exactly a finite-history process of Lemma 3.2 with non-identical type parameters; its open density is at least $u_{t+1} \geq e^{-\rho(t+1)S_1} > 0$, using the disaster-free vertical path.

Work first on the infinite integer line with only lengths $2 \leq r \leq K$ retained. Write $H_t^{(K)}$ for that truncated process and $F_{t,\theta}^{(K)} = E(H_t^{(K)}) \cup D^{(\theta,K)}$ for its interpolation, where $D^{(\theta,K)}$ retains the same sizes. The law remains translation invariant on the full integer line. The density u_{t+1} in the bound below refers to the untruncated process. Each fixed finite observation and its derivative depend on finitely many types in a backward cone. Write $h_{j,t} = \Pr([0, j-1] \subset H_t)$. Define $\phi_M(F, x) = \sum_{j=1}^{M} \mathbf{1}_{\{[x,x+j-1] \subset F\}}$. For a finite family of Poisson variables with means $\theta\nu_r$, differentiating its product law gives the insertion formula: the derivative of a bounded functional is the sum, over types, of its expected change when one extra mark of that type is added, weighted by ν_r. Here only finitely many types affect ϕ_M when lengths are bounded. To identify the sum over anchors, fix r and let $C_{l,x,j}(F)$ be the indicator that inserting $[l, l+r-1]$ newly closes $[x, x+j-1]$. Translation invariance gives $\mathbb{E}C_{l,0,j} = \mathbb{E}C_{0,-l,j}$. Summing over l converts the insertion effect at the fixed observation window to the number of newly closed windows caused by an insertion at anchor 0. Hence Poisson differentiation and translation invariance give

$$\frac{d}{d\theta}\mathbb{E}\phi_M(F_{t,\theta}^{(K)}, 0) = \sum_{r=2}^{K} \nu_r \, \mathbb{E}\Delta_{r,\leq M}(F_{t,\theta}^{(K)}) \leq A_*/u_{t+1}. \tag{12}$$

Here $\Delta_{r,\leq M}$ counts new intervals of length at most M. The upper bound is independent of K: the truncated sum of $\nu_r V(r)$ is at most A_*, and the truncated process has open density at least the full-process value u_{t+1}. Integrate over θ and pass to $K \to \infty$ for each fixed M. The omitted-type estimate from Lemma 3.2 gives local convergence, uniformly in $0 \leq \theta \leq \rho$; the integrated bound is uniform. Erosion sends the closure probability of a j-interval to $h_{j+1,t}$, so the resulting finite-M inequality is

$$\sum_{j=1}^{M} h_{j,t+1} \leq \sum_{j=1}^{M} h_{j+1,t} + c/u_{t+1}.$$

Define $\Phi_t = \sum_{j \geq 1} h_{j,t}$. Since $\Phi_0 = 0$, induction and monotone convergence in the two nonnegative sums show that $\Phi_t < \infty$ at each finite t. Only after obtaining this finiteness do we subtract the two potentials. It follows that

$$\Phi_{t+1} - \Phi_t \leq c/u_{t+1} - \beta_t. \tag{13}$$

Adding past rows makes the hole marginals larger, so β_t is nondecreasing. A single new row gives $\beta_{t+1} \leq \beta_t + q$ with $q = 1 - e^{-\rho S_1} \leq c$. Independent time-block concatenation gives $u_{a+b} \geq u_a u_b$: choose, by a fixed rule using only the last a rows, an open path to 0 and its initial site z when one exists. The first b rows reach that site with probability u_b, independently. The shared boundary row is checked by the first block only. Thus $\beta_{a+b} \leq \beta_a + \beta_b$.

Sum $\beta_T \leq \beta_t + \beta_{T-t}$ over $1 \leq t \leq T-1$. Next sum (13) over $t = 0, \ldots, T-1$, use $\Phi_0 = 0$, $\Phi_T \geq 0$ and $u_{t+1} \geq u_T$. These two sums give

$$\frac{T-1}{2}\beta_T \leq \sum_{t<T} \beta_t \leq Tc/u_T, \qquad \beta_T(1-\beta_T) \leq \frac{2T}{T-1}c \leq 4c \quad (T \geq 2). \tag{14}$$

Since $\beta_1 = 1 - e^{-\rho S_1} \leq c < 1/2$, a first crossing time T of $1/2$ must satisfy $T \geq 2$. At such a time, $1/2 \leq \beta_T \leq 1/2 + c$. Its product with $1 - \beta_T$ is at least $1/4 - c^2 \geq 255/1024 > 1/8 \geq 4c$,

contradicting (14). Hence the crossing never occurs. Taking $T \to \infty$ in the sharper version of (14) gives $\beta(1-\beta) \le 2c$ on the small branch, proving (9). This first-crossing argument gives the sufficient constant $1/32$, without optimization. $\square$

Lemma 2.2 gives positive survival from one prescribed root whenever $\rho A_* \le 1/32$. The construction establishes the positive phase through finite-history estimates.

4. Proof of the cube criterion

4.1. Extinction from envelope-covering catastrophes

Let B be a nonempty finite seed and let $w_0 = 1 + \max_i(\max_{x\in B} x_i - \min_{x\in B} x_i)$ be the largest number of lattice sites in a coordinate projection of its smallest axis-aligned enclosing box. Put $K = \max_{g\in G} \|g\|_\infty \ge 1$, where $\|x\|_\infty = \max_i |x_i|$. At time t, every possible descendant lies in the deterministic box obtained by extending each coordinate interval by Kt sites on both sides. Each side therefore contains at most $w_0 + 2Kt$ lattice sites. Writing $(v)_+ = \max\{v, 0\}$, the number of anchors of side-r cubes that contain this box is at least

$$n_{t,r} = (r - w_0 - 2Kt + 1)_+^d.$$

The corresponding Poisson mean in row t is

$$\lambda_t \ge \rho \sum_r \nu_r n_{t,r}. \tag{15}$$

For all sufficiently large r, there are a positive constant times r integer times $t \ge 1$ with $w_0 + 2Kt \le r/2$. At each such time, $n_{t,r} \ge (r/2)^d$. Hence

$$\sum_{t\ge 1} \lambda_t \ge C_{d,K,w_0} \rho \sum_{r\ge r_0} r^{d+1} \nu_r.$$

If $S_{d+1} = \infty$, this sum diverges. The covering events use independent catastrophe rows, so the probability that no covering mark ever occurs is $\exp(-\sum_t \lambda_t) = 0$. The first such mark removes every possible descendant. This proves almost-sure extinction from each finite seed at every positive intensity.

4.2. Sufficiency by a coordinate restriction

Choose a seed site x and retain only paths on the coordinate line $L_x = x + \mathbb{Z}e_1$, with increments $0, e_1$. For fixed longitudinal anchor and side r, precisely r^{d-1} transverse anchors yield cubes intersecting this line: for each coordinate $i = 2, \dots, d$, the anchor lies in $\{x_i - r + 1, \dots, x_i\}$. Their projections give the same length-r interval. Summing these independent Poisson types gives an interval model with

$$\widetilde{\nu}_r = r^{d-1}\nu_r, \qquad \widetilde{S}_1 = S_d, \quad \widetilde{S}_2 = S_{d+1}, \quad \widetilde{A}_* = (S_d + S_{d+1})/2. \tag{16}$$

For distinct projected time-anchor-size types, the source type sets are disjoint; the projected types are therefore independent. Proposition 3.1 and Lemma 2.2 show survival of this restricted process when $\rho\widetilde{A}_* \le 1/32$. The unrestricted process contains it under the same catastrophe field. This proves sufficiency and the lower bound in (5).

4.3. Upper bound and finite seeds

A fixed increment path through t tested rows avoids the catastrophe at its tested site with probability $e^{-\rho S_d t}$. There are at most $|G|^t$ such paths from one root, so

$$s_t^G(\rho) \le \left(|G|e^{-\rho S_d}\right)^t. \tag{17}$$

For $\rho > \log|G|/S_d$, the right side tends to zero. Since the survival events decrease, their limiting probability is zero, proving the upper bound; no conclusion at equality is needed for this bound on the supremum. The process is additive in its initial set: at each finite time the descendants of a finite union are the union of the descendants of its individual sites. If every single-site process dies almost surely, the extinction times of finitely many such processes have an almost surely finite maximum. Conversely, survival from one site implies survival from any finite seed containing that site. This completes Theorem 2.1. By Lemma 2.2, the bulk and finite-seed positivity sets and critical intensities coincide.

The criterion applies to bidirectional and other finite recovery rules containing $0, e_1$. Their numerical upper bounds depend on $|G|$; their existence boundary is the same moment condition.

5. Stationary density in one dimension

Return to $d = 1$, $G = \{0, 1\}$, and assume $S_2 < \infty$. Use the observation-time field $H^* = H_0^*$ of Section 2.3 and β, u, A_*, c as in Section 3. Set $h_j = \Pr([0, j-1] \subset H^*)$, $h_0 = 1$. The local flux identity below yields a quadratic refinement of the stationary density.

Proposition 5.1 (quadratic density response). For a fixed nonzero measure ν with $S_2 < \infty$,

$$\beta = c + \tfrac{1}{2}c^2 + o(c^2) \qquad (\rho \downarrow 0), \qquad c = \rho A_*. \tag{18}$$

The expansion holds for each fixed measure under the stated second-moment condition.

5.1. A local flux identity

Lemma 5.2. At every intensity for which $u > 0$, let $D^{(\theta)}$ be a fresh catastrophe row independent of H^*, coupled increasingly for $0 \le \theta \le \rho$, and put $F_\theta = E(H^*) \cup D^{(\theta)}$. Then

$$h_j - h_{j+1} = \int_0^\rho \sum_r \nu_r \mathbb{E}\Delta_{r,j}(F_\theta)\, d\theta, \qquad j \ge 1, \tag{19}$$

Here $\Delta_{r,j}$ is the count defined in Lemma 3.3 for insertion at anchor 0. For a general hole field F, we write $h_j(F) = \Pr([0, j-1] \subseteq F)$ and $u_F = \Pr(0 \notin F)$. Moreover,

$$\beta = \int_0^\rho \sum_r \nu_r \mathbb{E}\Delta_r(F_\theta)\, d\theta, \qquad c \le \beta \le \frac{c}{u}. \tag{20}$$

Proof. Group the Poisson environment by spatial anchor: $\mathcal{X}_l = (N_{t,l,r})_{t \le 0, r \ge 2}$ for $l \in \mathbb{Z}$. These columns are independent and identically distributed. The construction of H^* is a measurable function of these columns and commutes with spatial translation. The shift of an i.i.d. sequence is ergodic, and a translation-equivariant function preserves ergodicity; hence H^* is spatially ergodic. When $u > 0$, the spatial ergodic theorem gives open-site density u in each direction almost surely. Thus open sites occur arbitrarily far to the left and right. In particular all closed runs are finite almost surely, and $h_j \to 0$ by continuity of probability for decreasing events. Since $F_\theta \subset F_\rho$ and F_ρ has the law of H^*, the same properties hold for the interpolation fields; their open densities satisfy $u_\theta \ge u$.

For fixed j, an inserted length-r interval affects at most $r+j-1$ windows. This is integrable against ν under $S_1<\infty$. Poisson differentiation in finite histories and finite length cutoffs therefore passes to the stationary local insertion difference by dominated convergence. The integrable window bound controls the insertion differences uniformly. At zero last-row intensity the closure probability is h_{j+1}; at intensity ρ it is h_j. This proves (19). For bounded side functions, conditional correlation passes from finite histories and size cutoffs to F_θ by local convergence; the conditioning probabilities are at least $u>0$. For the run functions used below, first make this passage with their finite-coordinate truncations, and only then remove those truncations by monotone convergence.

For any such interpolation field F, inclusion in F_ρ implies $h_k(F)\to 0$. The boundary-tail calculation of Lemma 3.3, now with $\zeta=0$, gives $\mathbb{E}[\mathbf{1}_{\{z\notin F\}}(L_z+1)]=1$. Lemma 3.3 consequently yields

$$\mathbb{E}\Delta_r(F)=V(r)+\sum_{z=0}^{r-1}\mathbb{E}[\mathbf{1}_{\{z\notin F\}}(L_z+1)R_r]\geq V(r).$$

Sum (19) over j by Tonelli. Its left side telescopes to β, and the nonnegative insertion counts sum to Δ_r. This gives the identity and lower bound in (20). Lemma 3.3 gives $\mathbb{E}\Delta_r(F_\theta)\leq V(r)/u_\theta\leq V(r)/u$, proving the upper bound. The insertion decomposition gives the lower bound directly. □

5.2. The quadratic coefficient

Proof of Proposition 5.1. Proposition 3.1 gives $\beta=O(\rho)$ and $u\to 1$. Lemma 5.2 then implies $\beta/\rho\to A_*$. Put

$$b_i=\sum_r \nu_r(r-i)_+,\qquad a_j=\sum_{i\geq j-1}b_i\quad(j\geq 1),\qquad a_1=\sum_{i\geq 0}b_i=A_*.$$

For $0\leq s\leq 1$, write $F_s=E(H^*)\cup D^{(s\rho)}$. Its finite-window hole probabilities vanish uniformly in s because $F_s\subset F_1$ and $\beta=O(\rho)$. For fixed r,j, an insertion at anchor 0 can affect only length-j windows whose left endpoints are in $[-j+1,r-1]$, so $\Delta_{r,j}$ depends on finitely many sites and is at most $r+j-1$. Consequently $\mathbb{E}\Delta_{r,j}(F_s)\to(r-j+1)_+$ uniformly in s. The bound $r+j-1$ in (19) gives $q_j/\rho\to b_{j-1}$, where $q_j=h_j-h_{j+1}$. Also

$$\sum_{j\geq 1}\frac{q_j}{\rho}=\frac{\beta}{\rho}\to A_*=\sum_{j\geq 1}b_{j-1}.$$

For clarity, write $x_j^{(\rho)}=q_j/\rho$ and $x_j=b_{j-1}$. For each finite J, the first J entries converge. Also $\sum_j x_j^{(\rho)}\to\sum_j x_j=A_*$. Given $\varepsilon>0$, choose J with $\sum_{j>J}x_j<\varepsilon$; the convergence of the total mass and of the first J entries then bounds $\sum_{j>J}x_j^{(\rho)}$ by 2ε for small ρ. This proves $\sum_j|x_j^{(\rho)}-x_j|\to 0$. Since $h_j=\sum_{k\geq j}q_k$, the same conclusion gives the required tail limits. Summing the tails and using (19) up to intensity $s\rho$ gives

$$\frac{h_j}{\rho}\to a_j,\qquad\frac{h_j(F_s)}{\rho}\to a_{j+1}+sb_{j-1},\tag{21}$$

for each fixed j, with the second convergence uniform in s.

Fix an insertion length r and $z\in[0,r-1]$. For a field $F=F_s$, use the run length R_a from Lemma 3.3, and put $T_{z,r}=\mathbb{E}[\mathbf{1}_{\{z\notin F\}}R_r]$ and $m=r-z$. Partition by the last old open site $w\in[z,r-1]$, using

$$J_{w,r}=\mathbf{1}_{\{w\notin F\}}(R_{w+1}-(r-w-1))_+.$$

Put $m_w = r - w \geq 1$. For $b \geq 1$,

$$\Pr(w \notin F, R_{w+1} \geq m_w - 1 + b) = h_{m_w-1+b}(F) - h_{m_w+b}(F).$$

The nonnegative tail sum telescopes to $\mathbb{E}J_{w,r} = h_{m_w}(F)$ because $h_j(F) \to 0$. The exact decomposition is $\mathbf{1}_{\{z\notin F\}}R_r = \sum_{w=z}^{r-1} \mathbf{1}_{\{z\notin F\}}J_{w,r}$. For $w > z$, condition on $w \notin F$ and use Lemma 3.2 to bound the omitted term by

$$\mathbb{E}[\mathbf{1}_{\{z\in F\}}J_{w,r}] \leq \frac{\beta_F h_{r-w}(F)}{u_F} \leq \frac{\beta_F^2}{u_F}.$$

Hence $0 \leq \sum_{j=1}^{m} h_j(F) - T_{z,r} \leq (m-1)\beta_F^2/u_F$. Conditional correlation at z, together with $\mathbb{E}[\mathbf{1}_{\{z\notin F\}}L_z] = \beta_F$, also gives $\mathbb{E}[\mathbf{1}_{\{z\notin F\}}L_z R_r] \leq m\beta_F^2/u_F$. Substitute these estimates into (10): for fixed r,

$$\mathbb{E}\Delta_r(F_s) - V(r) = \sum_{j=1}^{r}(r-j+1)h_j(F_s) + O_r(\beta^2/u).$$

Integration over r uses the separate upper and lower insertion bounds, which give

$$0 \leq \frac{\mathbb{E}\Delta_r(F_s) - V(r)}{\rho} \leq V(r)\frac{\beta}{\rho u} \leq C_\nu V(r) \tag{22}$$

for all sufficiently small ρ, uniformly in s. The final bound is integrable against ν by $S_2 < \infty$. Dominated convergence in (20), using (21), now gives

$$\lim_{\rho\downarrow 0}\frac{\beta - \rho A_*}{\rho^2} = \sum_{i\geq 0} b_i\left(\sum_{k>i} b_k + \frac{1}{2}b_i\right) = \frac{1}{2}A_*^2.$$

To see the first equality, use $\sum_r \nu_r \sum_{j=1}^{r}(r-j+1)f_j = \sum_{i\geq 0} b_i f_{i+1}$ for nonnegative f_j, and insert the limits (21), integrated over $s \in [0,1]$. For the second equality, all series are nonnegative and $\sum_i b_i = A_* < \infty$, so

$$\sum_{i<k} b_i b_k + \frac{1}{2}\sum_i b_i^2 = \frac{1}{2}\Big(\sum_i b_i\Big)^2.$$

This proves (18). □

5.3. Comparison with separately eroded events

Use the two-sided environment and observe at time 0. A catastrophe at time $-j$ has age $j \geq 0$. Erode each of its intervals separately, without first merging it with other intervals. Since $E^j([l, l+r-1]) = [l+j, l+r-1]$ for $0 \leq j < r$ and is empty for $j \geq r$, define its surviving-shadow union by

$$K^* = \bigcup_{j\geq 0}\ \bigcup_{l\in\mathbb{Z},\ r\geq 2: N_{-j,l,r}\geq 1} E^j([l, l+r-1]).$$

For fixed age j and size $r > j$, exactly $r - j$ anchors yield a residual interval containing 0. The total intensity of residual shadows hitting a site is therefore

$$\rho \sum_r \nu_r \sum_{j=0}^{r-1} (r-j) = c.$$

The Poisson void calculation gives $\Pr(0 \notin K^*) = e^{-c}$ [9]. Since E of a union contains the union of the individual erosions, $K^* \subset H^*$ under the common construction.

Corollary 5.3 (density excess). Under $0 < S_1$ and $S_2 < \infty$,

$$\Gamma(\rho) := \Pr(0 \in H^* \setminus K^*) = \beta - 1 + e^{-c} \geq \min\{c, 1\} - 1 + e^{-c} > 0 \quad (\rho > 0). \tag{23}$$

For the fixed measure, $\Gamma(\rho) = c^2 + o(c^2)$ as $\rho \downarrow 0$.

Proof. If $u > 0$, Lemma 5.2 gives $\beta \geq c$; if $u = 0$, then $\beta = 1$. Thus $\beta \geq \min\{c, 1\}$. Strict positivity follows from $e^{-c} > 1 - c$ for $0 < c < 1$ and from $e^{-c} > 0$ for $c \geq 1$. Finally subtract $1 - e^{-c} = c - c^2/2 + o(c^2)$ from (18). □

The same argument shows that $c \geq 1$ forces $u = 0$, so Lemma 2.2 gives the additional one-dimensional bound $\rho_c^{\{0,1\}} \leq 1/A_*$. Since $S_2 \geq 2S_1$ for sizes $r \geq 2$, one has $A_* \geq \frac{3}{2}S_1$, so $1/A_* \leq \frac{2}{3S_1} < \log 2/S_1$: in dimension one this bound is always sharper than the path-count bound in (5) for $|G| = 2$. The bound and density expansion hold for the one-sided rule.

6. Uniformly fat catastrophe shapes

The cube result extends by comparison. Let each supported shape $\mathcal{B}_r \subset \mathbb{Z}^d$ be finite and nonempty. Assume that, for every sufficiently large r, it contains a translated cube of side at least ar and is contained in a translated cube of side at most κr, where $a, \kappa > 0$. The finitely many shapes $\mathcal{B}_r$ at sizes below the threshold have finite diameters, so the outer bound extends to all sizes by enlarging κ. Choose the outer constant κ once and for all large enough that, for every supported size r, an integer side length $b_r \geq 2$ and a translation vector v_r satisfy $\mathcal{B}_r \subseteq v_r + \{0, \ldots, b_r - 1\}^d$ and $b_r \leq \kappa r$. All occurrences of κ in the quantitative bound below refer to this uniform choice, including the finitely many small-size exceptions. A mark with anchor l empties $l + \mathcal{B}_r$. For every $t \in \mathbb{Z}$, $l \in \mathbb{Z}^d$ and $r \geq 2$, the mark counts are independent Poisson variables with means $\rho\nu_r$; the chain has the update and initial-layer convention of Section 2.2. Write $\Lambda = \sum_r |\mathcal{B}_r|\nu_r$ and assume $0 < S_d < \infty$.

Proposition 6.1 (shape comparison). For every finite G containing $0, e_1$, the shape model has a positive survival threshold if and only if $S_{d+1} < \infty$. If $S_{d+1} < \infty$, a sufficient range is

$$\rho \frac{\kappa^d S_d + \kappa^{d+1} S_{d+1}}{2} \leq \frac{1}{32}, \qquad \rho_c^G \leq \frac{\log |G|}{\Lambda}. \tag{24}$$

Let $\theta_{\mathcal{B}}^G$ denote the upper stationary density of the shape process. If additionally $-\mathcal{B}_r$ is a translate of $\mathcal{B}_r$ for each supported size, then the same-law reversal identity is $\theta_{\mathcal{B}}^G = e^{-\rho\Lambda} s_\infty^G$, with s_∞^G here computed for that shape process.

Proof. Use the integer outer cube sides b_r and the uniform constant κ fixed before the proposition. Replace each marked shape by its chosen outer cube at the corresponding translated anchor. The anchor shift is deterministic for each size and preserves the independent mark law. The enlarged field is more destructive. Its projection onto a coordinate line has weights obtained by adding $b_r^{d-1}\nu_r$ to length b_r. Therefore its first two moments are bounded by $\kappa^d S_d$ and $\kappa^{d+1} S_{d+1}$. The one-dimensional result proves the sufficient range in (24).

For necessity, use a fixed inner cube of side $a_r \geq ar$ inside each sufficiently large shape. At row t, at least $(a_r - w_0 - 2Kt + 1)_+^d$ translations of that inner cube cover the deterministic descendant

envelope. Summing over t gives a lower bound proportional to r^{d+1}. A divergent S_{d+1} therefore gives the independent-row covering argument of Section 4.1.

The per-site hitting mean is $\rho\Lambda$, which is finite by the outer bound and positive because the supported shapes are nonempty. The path count gives the upper bound in (24). Write $\theta_{\mathcal{B}}^{G} = \Pr(0 \in A_0^{*,G})$ for the upper stationary density of the shape process, constructed as in Section 2.3. Under the additional reflection condition, reflection permutes the translated mark types at each size. The proof of Lemma 2.2 applies with terminal no-hit probability $e^{-\rho\Lambda}$. □

7. Concluding remarks

The moment criterion gives the exact boundary between a positive-intensity survival phase and extinction at every positive intensity. A bounded law and a borderline heavy-tailed law can produce the same one-site exposure and opposite survival outcomes. The proof obtains the dimensional boundary from a one-dimensional positive phase, coordinate restriction and envelope covering.

The one-dimensional analysis also determines the quadratic response of the stationary hole density. An integrable insertion bound controls the expansion under the same finite second moment as the positive-phase theorem. The resulting flux formula gives a strictly positive density excess over independently eroded events.

Connected interval supports enable the left/right resampling argument, and independent time rows drive the covering argument. Extending the criterion to temporally dependent environments or shapes with different diameter and volume scales requires corresponding correlation and covering estimates. A further question is the quantitative behaviour near the critical intensity.

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Data availability. This article is analytical and reports no empirical dataset. The companion archive contains the finite diagnostics and their exact inputs and outputs.

Code availability. A companion Python script implements exact finite checks of the conditional-correlation examples, the interval-insertion count and the coefficient identities. Its inputs, configurations and outputs are supplied with the manuscript.

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