hexagon:2611.00271v1

Spectral Sequences in Supermanifold Geometry

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November 10, 2026

Abstract

We study spectral sequences in supergeometry and construct the Gysin sequence within the supermanifold framework. First, the foundational definitions are established and utilized to develop differential forms and de Rham cohomology on supermanifolds. We then apply a standard de Rham-to-Čech cohomology isomorphism to assemble a spectral sequence that accounts for the $\mathbb{Z}_2$-graded nature of supermanifolds. Finally, we explore additional properties arising from this construction, leading directly into Gysin sequences.

1 Introduction

1.1 Supermanifold Theory

Manifolds can be equipped with many different structures like a Riemannian, Symplectic and Complex structures. With these structures they all have commutative coordinates in common. However we can equip a manifold M with Grassmann algebras to be able to consider both non and commutative coordinates, making the manifold a supermanifold. Supermanifolds are defined locally as $\mathcal{M}$ of dimension $n|m$ is given by

$$C^\infty(\mathbb{R}^n) \otimes \bigwedge \mathbb{R}^m$$

With $n|m$ being defined as the superdimension. With n being the commutative dimension and m being the anticommutative dimension. With these manifolds characterized by sheaves of $\mathbb{Z}_2$-graded (commutative) algebras, called superalgebras which might commute or anticommute depending on their odd or even $\mathbb{Z}_2$ degree. Supermanifolds are regularly used in both differential geometry and physics. Usually, the recent literature would combine these together to deal with problems in mathematical physics like supergravity. With this we now outline what the paper would contain along with the main components of each section.

1.2 Outline

We begin this paper with the basic definitions of supergeometry, which can be found in relevant preprint papers, to build the necessary structure needed throughout the paper. Next, the concept of a supermanifold is established alongside the algebra required for differential forms. From this foundation, differential forms in supergeometry are defined, along with the appropriate sheaf of $\mathcal{O}_\mathcal{M}$-superalgebras, denoted as:

$$\Omega^\bullet_\mathcal{M} := \bigoplus_{k=0}^{\infty} S^k_{\mathcal{O}_\mathcal{M}} \Omega^1_\mathcal{M}$$

With this framework in place, the de Rham complex is constructed using the isomorphism from Definition 3.11. [6] is used to establish the relationship between de Rham and Čech cohomology. This

relation proves useful as the Čech cochain complex,

$$C^\bullet(\mathfrak{U}, \mathcal{E}) = \bigoplus_{p \geq 0} C^p(\mathfrak{U}, \mathcal{E})$$

is used in conjunction with a Stein cover in hypercohomology $\mathfrak{U} = (U_i)_{i \in I}$ to construct the spectral sequence under a $\mathbb{Z}_2$ grading. When constructing the spectral sequence, the filtrations of $\mathcal{E}$ are used alongside the subcomplexes to yield a spectral sequence of bigraded groups E_r that converge to $E_\infty \simeq \operatorname{gr} H^\bullet(M, \mathcal{E})$. Defining the spectral sequence allows us to apply two propositions from [7] to establish the bigraded spaces/algebras. With the filtered spectral sequence established along with its identifications, the Gysin sequence in supergeometry is considered as a spectral fiber sequence. Its construction is similar to that in [1], with major adjustments required due to the $\mathbb{Z}_2$-grading nature of supergeometry. This construction incorporates the Berezin fiber integration used for π_* and the natural pullback π^*, with $\wedge e$ coming from the super Euler class of degree $k + 1$.

1.3 Acknowledgements

The author wants to gratefully acknowledge Simone Noja at the University of Bari Aldo Moro for their very useful suggestions on different sections of the paper.

2 Preliminary Defintions

In order to build up to gysin sequences in supergeometry, we need to first define the basic structures that concern supergeometry. We begin with superspaces.

2.1 Superspaces

Definition 2.1. A super space is defined as a pair $\mathcal{M} = (|\mathcal{M}|, \mathcal{O}_\mathcal{M})$. With $|\mathcal{M}|$ being a topological space and $\mathcal{O}_\mathcal{M}$ being the sheaf of $\mathbb{Z}_2$ graded supercommutative rings over $|\mathcal{M}|$.

From this we define the morphisms of superspaces.

Definition 2.2. Given two superspaces $\mathcal{M}$ and $\mathcal{N}$, we define a morphism $\psi : \mathcal{M} \to \mathcal{N}$ as a pair $\psi = (\phi, \phi^\sharp)$ with $\phi : |\mathcal{M}| \to |\mathcal{N}|$ is a continuous morphism of topological spaces;; $\phi^\sharp : \mathcal{O}_\mathcal{N} \to \phi_* \mathcal{O}_\mathcal{M}$ is a morphism of sheaves of $\mathbb{Z}_2$ graded rings. Additionally with it preserving the $\mathbb{Z}_2$ grading, along with the homomorphism $\phi_x^\sharp : \mathcal{O}_{\mathcal{N}, \phi(x)} \to \mathcal{O}_{\mathcal{M}, x}$ being local with it preserving the unique maximal ideal.

With this, we next move onto the defintions concerning supermanifolds.

2.2 Supermanifolds

We start with some preliminary results from [6] that concern supermanifolds

Definition 2.3 (Real / Complex Supermanifold)**.** (Definition 2.6 [6]) A real (complex) supermanifold $\mathcal{M}$ of dimension $n|m$ is a superspace that is *locally* isomorphic to some smooth (holomorphic) local model $\mathfrak{S}(\mathcal{M}_{red}, \mathcal{E})$, where $\mathcal{M}_{red}$ is a smooth (complex) manifold of dimension n and $\mathcal{E}$ is a smooth (holomorphic) vector bundle on $\mathcal{M}_{red}$ of rank m.

Definition 2.4. (Split Supermanifold) Definition 2.7 [6]. We say that a supermanifold $\mathcal{M}$ is a split supermanifold if it is globally isomorphic to its local model. Analogously, $\mathcal{M}$ is split if there exists a sheaf isomorphism $\mathcal{O}_\mathcal{M} \cong \bigwedge^\bullet \mathcal{E}^*$.

Definition 2.5. Definition 2.9 [6] Let $\mathcal{M}$ be a real (complex) supermanifold with structure sheaf $\mathcal{O}_\mathcal{M}$. We call the reduced space of $\mathcal{M}$ the smooth (complex) manifold $\mathcal{M}_{\text{red}}$ with structure sheaf given by the quotient $\mathcal{O}_{\mathcal{M}_{\text{red}}} := \mathcal{O}_\mathcal{M} / \mathcal{J}_\mathcal{M}$.

Definition 2.6. Definition 2.10 [6] ($\operatorname{Gr}\mathcal{O}_{\mathcal{M}}$ and $\operatorname{Gr}\mathcal{M}$). Let $\mathcal{M}$ be a supermanifold having odd dimension m together with the filtration of its structure sheaf $\mathcal{O}_{\mathcal{M}}$ as in (2.3). We define the following sheaf of supercommutative algebras

$$\operatorname{Gr}\mathcal{O}_{\mathcal{M}} := \bigoplus_{i=0}^{m} \operatorname{Gr}^{(i)}\mathcal{O}_{\mathcal{M}} = \mathcal{O}_{\mathcal{M}_{\mathrm{red}}} \oplus \mathcal{J}_{\mathcal{M}}/\mathcal{J}_{\mathcal{M}}^2 \oplus \cdots \oplus \mathcal{J}_{\mathcal{M}}^{m-1}/\mathcal{J}_{\mathcal{M}}^m \oplus \mathcal{J}_{\mathcal{M}}^m.$$

Definition 2.7. Definition 2.15 [6] (The Sheaf $\mathcal{D}_{\mathcal{M}}$). Let $\mathcal{M}$ be a complex supermanifold. We define the sheaf of differential operators of $\mathcal{M}$ to be the subsheaf of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_{\mathcal{M}})$ generated by $\mathcal{O}_{\mathcal{M}}$ and $\mathcal{T}_{\mathcal{M}}$, and we denote it by $\mathcal{D}_{\mathcal{M}}$.

3 Differential and Integral Forms on Supermanifolds

Building from the previous section, we now consider super differential forms on a supermanifold beginning with defining the de Rham algebra. From this we define the de Rham differential with it leading into the super de Rham complex. Next we consider the relation between de Rham and cech cohomology to build up to the spectral sequence construction. Naturally with differential forms, it arises the question of integral forms in supergeometry. We formally define integral forms and use it later in the paper.

3.1 Differential Forms and de Rham Cohomology of Supermanifolds

Since a supermanifold $\mathcal{M}$ has both commuting and anticommuting components, we need to adjust the defintions of differential forms in order to deal with the commuting problem. With this adjustment, everything defined on differential forms such as the de Rham complex need to be changed to work with supermanifolds. To begin, we define a superalgebra.

Definition 3.1. Let $\mathcal{M}$ be a real supermanifold of dimension $n|m$ with structure sheaf $\mathcal{O}_{\mathcal{M}}$. The de Rham algebra of $\mathcal{M}$ is the sheaf of $\mathcal{O}_{\mathcal{M}}$-superalgebras defined by

$$\Omega_{\mathcal{M}}^{\bullet} := \bigoplus_{k=0}^{\infty} \mathrm{S}^k_{\mathcal{O}_{\mathcal{M}}} \Omega^1_{\mathcal{M}}$$

where $\mathbf{S}^k : \mathbf{Sh}_{\mathcal{O}_{\mathcal{M}}} \to \mathbf{Sh}_{\mathcal{O}_{\mathcal{M}}}$ is the k-supersymmetric functor and $\Omega^1_{\mathcal{M}}$ is the sheaf of 1-forms of $\mathcal{M}$ defined as above, where the $\mathbb{Z}_2$-grading is induced by that of $\Omega^1_{\mathcal{M}}$. Next we consider the de Rham differential from [6].

Definition 3.2. Definition 3.4 (de Rham differential) [6]. Let $\mathcal{M}$ be a real supermanifold of dimension $n|m$ with structure sheaf $\mathcal{O}_{\mathcal{M}}$. Given a section ω of the de Rham superalgebra $\Omega_{\mathcal{M}}^{\bullet}$ represented as $\omega = dy^I \otimes f_I$ in a certain trivialization $y_a = x_i|\theta_\alpha$ for some multi-index I with $f_I \in \mathcal{O}_{\mathcal{M}}$, we define the de Rham differential $d : \Omega_{\mathcal{M}}^{\bullet} \to \Omega_{\mathcal{M}}^{\bullet}$ to be the odd derivation $d := \sum_a dy_a \otimes \partial_{y_a}$. More precisely we have

$$d : \Omega_{\mathcal{M}}^{\bullet} \longrightarrow \Omega_{\mathcal{M}}^{\bullet} \quad dy^I \otimes f_I \longmapsto d(dy^I \otimes f_I) = \sum_a (-1)^{|y_a||dy^I|} dy_a dy^I \otimes \partial_{y_a}(f_I)$$

From this we can establish a very important property, which would lead into de Rham cohomology.

Lemma 3.3. *On superdifferential forms,* $d^2 = 0$

Proof. This follows from [6] , consider the leibniz rule with $dy_a \otimes \partial_{y_a}$ and $dy_b \otimes \partial_{y_b}$

$$\begin{aligned} d^2 &= \sum_{a,b}(dy_a \otimes \partial_{y_a})(dy_b \otimes \partial_{y_b}) + \sum_{a,b}(dy_b \otimes \partial_{y_b})(dy_a \otimes \partial_{y_a}) \\ &= \sum_{a,b}(-1)^{|y_a||dx_b|}dy_a dy_b \otimes \partial_{y_a}\partial_{y_b} + \sum_{i,j}(-1)^{|y_b||dx_a|}(dy_b dy_a \otimes \partial_{y_b}\partial_{y_a}) \\ &= \sum_{a,b}\left((-1)^{|y_a||y_b|+|y_a|} + (-1)^{|y_a||y_b|+|y_a|+1}\right) dy_a dy_b \otimes \partial_{y_a}\partial_{y_b} \\ &= 0, \end{aligned} \tag{1}$$

□

From this we get the de Rham complex

Definition 3.4. We call the differential graded superalgebra $(\Omega^\bullet_{\mathcal{M}}, d)$ the de Rham complex of $\mathcal{M}$.

$$0 \longrightarrow \mathcal{O}_{\mathcal{M}} \xrightarrow{d} \Omega^1_{\mathcal{M}} \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n_{\mathcal{M}} \xrightarrow{d} \ldots$$

Now we consider the cohomology arising from the de Rham complex.

Definition 3.5. Definition 3.11 [6] (de Rham Cohomology of $\mathcal{M}$). Let $\mathcal{M}$ be a real supermanifold. We define the de Rham cohomology of the global sections of the (sheaf of) differential forms on $\mathcal{M}$ as $(\Omega^\bullet_{\mathcal{M}}, d)$, i.e., $H^k_{\mathrm{dR}}(\mathcal{M}) := H^k_d(\check{H}^0_\delta(\Omega^\bullet_{\mathcal{M}}))$, where $\check{H}^0_\delta(\Omega^\bullet_{\mathcal{M}})$ is the 0-Čech cohomology group of $\Omega^\bullet_{\mathcal{M}}$, i.e., the global sections of $\Omega^\bullet_{\mathcal{M}}$.

3.2 Integral Forms

To begin, we define the Berezinian Sheaf denoted as $\mathcal{B}er(\mathcal{M})$ then construct integral forms in supergeometry.

Definition 3.6. Definition 4.1 [6] (Berezinian - via Transition Functions). Let $\mathcal{M}$ be a real or complex supermanifold of dimension $n|m$ and let $\{(U_i, x_{U_i}|\theta_{U_i})\}_{i\in I}$ be an atlas of charts covering $\mathcal{M}$. Then we define the Berezinian sheaf $Ber(\mathcal{M})$ of $\mathcal{M}$ as the locally-free sheaf of right $\mathcal{O}_{\mathcal{M}}$-modules of rank $\delta_{0,n+m}|\delta_{0,n+m+1}$, whose local generators $\{\mathcal{D}_{U_i}(x_{U_i}|\theta_{U_i})\}_{i\in I}$ transforms as

$$\mathcal{D}_{U_j}|_{U_i\cap U_j}(x_{U_j}|\theta_{U_j}) = \mathcal{D}_{U_i}|_{U_i\cap U_j}(x_{U_i}|\theta_{U_i})Ber(\mathcal{J}ac(\varphi_{ij})) \tag{2}$$

with

$$Ber(\mathcal{J}ac(\varphi_{ij})) = \det(A - BD^{-1}C)\det(D)^{-1}, \tag{3}$$

where $\mathcal{J}ac(\varphi_{ij})$ is the super Jacobian of the change of coordinates

$$\begin{aligned} \varphi_{ij} : U_i|_{U_i\cap U_j} &\longrightarrow U_j|_{U_i\cap U_j} \\ x_{U_i}|\theta_{U_i} &\longmapsto \varphi_{ij,0}(x|\theta) = x_{U_j}|\varphi_{ij,1}(x|\theta) = \theta_{U_j}, \end{aligned}$$

and where we have posed

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := \left(\begin{array}{c|c} \partial_x\varphi_{ij,0} & \partial_\theta\varphi_{ij,0} \\ \hline \partial_x\varphi_{ij,1} & \partial_\theta\varphi_{ij,1} \end{array}\right) = \mathcal{J}ac(\varphi_{ij}). \tag{4}$$

With the Berezinian sheaf defined, we now consider the isomorphism between it and de Rham cohomology.

Definition 3.7. Let $\mathcal{M}$ be a supermanifold with dimension $n|m$. Then the homology of the de Rham complex is naturally isomorphic to the Berezinian sheaf of $\mathcal{M}$,

$$\mathcal{H}^p((\mathcal{DR}^\bullet_{\mathcal{M}}, \mathcal{D})) \cong \mathcal{B}er(\mathcal{M})$$

with the de Rham complex being isomorphic to zero when not equal to p

Proof. Theorem 4.19 [6] □

With these defintions established, we now move into integral forms.

Definition 3.8. Let $\mathcal{M}$ be a real or complex supermanifold of dimension $p|q$. We call integral forms of degree $p-i$ the sections of the sheaves

$$\Sigma^{p-i}_{\mathcal{M}} := \mathcal{B}er(\mathcal{M}) \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{S}^i \Pi \mathcal{T}_{\mathcal{M}} \tag{5}$$

With both differential and integral forms defined, we next need to define integration that works on both odd and even coordinates. This is where Berezinian integration is used in the next section to be able to compute both pushforwards and pullbacks in later sections.

4 Berezin Integration

Building up from both differential and integral forms, we now we formally define the Berezin integral. If $\mathcal{B}^m_c(X)$ denotes the R-module of the compactly supported sections of the Berezinian sheaf, then the Berezin integral is the $\mathbb{R}$-linear map on a supermanfold $\mathcal{X}$ of dimension $(n|m)$ is

$$\int_X : \mathcal{B}^m_c(X) \longrightarrow \mathbb{R}, \tag{6}$$

which is defined in local coordinates $X^A := \{x^i, \theta^\alpha\}$ with $i = 1, \ldots, n$, $\alpha = 1, \ldots, m$ on a domain U by

$$\int_U \tilde{v}_i := \int_{U_{red}} dx_{1,red} \ldots dx_{n,red} f_i^{1\ldots1}(x_{1,red} \ldots x_{n,red}), \tag{7}$$

where $\tilde{v}_i(x, \theta) = \sum_\epsilon \sigma^c \theta^1 \ldots \theta^n f_i^\epsilon(x_1, \ldots, x_m)$ for $\epsilon_j = \{0, 1\}$ and σ^c a constant Berezinian. With this, we now define many of the key properties of Berezin integration. With Berezin integration defined, we next define Berezin fiber integration from defintion 5.12 of [4].

Definition 4.1. Definition 5.12 (Berezin fiber integration). Let $\varphi : X \to S$ be a family of supermanifolds of relative dimension $n|m$. The **fiber integration morphism** is the counit of the adjunction $\mathbf{R}\varphi_! \dashv \varphi^!$ evaluated at $\mathcal{O}_S$:

$$\int_{X/S} : \mathbf{R}\varphi_! \, \omega^\bullet_{X/S} \longrightarrow \mathcal{O}_S \tag{8}$$

in $\mathrm{D}(\mathcal{O}_S)$, where $\omega^\bullet_{X/S} \cong \mathcal{B}^n_{X/S}[n]$.
For X and S orientable, transporting $\int_{X/S}$ along the identification $\omega^\bullet_{X/S} \cong \mathcal{B}^\bullet_{X/S}[n]$, yields a (non-canonical) morphism in $\mathrm{D}(\mathcal{O}_S)$, which we call the **Berezin fiber integration**

$$\int_{X/S} : \mathbf{R}\varphi_! \, \mathcal{B}^\bullet_{X/S}[n] \longrightarrow \mathcal{O}_S. \tag{9}$$

With Berezin integration defined, we next move onto spectral sequences which allows for the computation of cohomology using local data. With it motivating the use of Čech cohomology and more powerfully, spectral sequences to compute global de Rham invariants from local covers.

5 Spectral Sequences

To construct a filtration spectral sequence for the cohomology of a general sheaf $\mathcal{E}$ on a supermanifold, we begin by choosing an appropriate open cover such as a Stein cover to compute the corresponding Čech cochain complex. With this, we need to hypercohomology because of the holomorphic nature of the stein cover. The hypercohomology of the holomorphic de Rham complex results in a double complex. By introducing a natural filtration on the resulting double complex, we obtain a spectral sequence defined in terms of its subcomplexes. This systematic approach allows us to explicitly analyze the deeper homological structure of $\mathcal{E}$.

5.1 Spectral Sequence

We begin with letting $\mathcal{E}$ be a vector superbundle on a general supermanifold $(M, \mathcal{O})$ with dimension $n|m$. Start with a open stein cover $\mathfrak{U} = (U_i)_{i\in I}$ of M with hypercohomology, then consider the corresponding Čech cochain complex

$$C^*(\mathfrak{U}, \mathcal{E}) = \bigoplus_{p\geq 0} C^p(\mathfrak{U}, \mathcal{E})$$

The natural $\mathbb{Z}_2$ grading of $\mathcal{E}$ gives $\mathbb{Z}_2$ gradings in $C^*(\mathfrak{U}, \mathcal{E})$ and $H^*(M, \mathcal{E})$ with these being given by

$$\begin{aligned}
C_{\bar{0}}(\mathfrak{U}, \mathcal{E}) &= \bigoplus_{q\geq 0} C^{2q}(\mathfrak{U}, \mathcal{E}_{\bar{0}}) \oplus \bigoplus_{q\geq 0} C^{2q+1}(\mathfrak{U}, \mathcal{E}_{\bar{1}}),\\
C_{\bar{1}}(\mathfrak{U}, \mathcal{E}) &= \bigoplus_{q\geq 0} C^{2q}(\mathfrak{U}, \mathcal{E}_{\bar{1}}) \oplus \bigoplus_{q\geq 0} C^{2q+1}(\mathfrak{U}, \mathcal{E}_{\bar{0}}).\\
H_{\bar{0}}(M, \mathcal{E}) &= \bigoplus_{q\geq 0} H^{2q}(M, \mathcal{E}_{\bar{0}}) \oplus \bigoplus_{q\geq 0} H^{2q+1}(M, \mathcal{E}_{\bar{1}}),\\
H_{\bar{1}}(M, \mathcal{E}) &= \bigoplus_{q\geq 0} H^{2q}(M, \mathcal{E}_{\bar{1}}) \oplus \bigoplus_{q\geq 0} H^{2q+1}(M, \mathcal{E}_{\bar{0}}).
\end{aligned} \tag{10}$$

From the filtration of $\mathcal{E}$, we get

$$C^*(\mathfrak{U}, \mathcal{E}) = C_{(0)} \supset \ldots \supset C_{(p)} \supset \ldots \supset C_{(m+1)} = 0$$

of this complex with the subcomplexes being

$$C_{(p)} = C^*(\mathfrak{U}, \mathcal{E}_{(p)})$$

With an additional filtration coming from the image of the natural mapping $H^*(M, \mathcal{E}_{(p)}) \to H^*(M, \mathcal{E})$

$$H^*(M, \mathcal{E}) = H(M, \mathcal{E})_{(0)} \supset \cdots \supset H(M, \mathcal{E})_{(p)} \supset \ldots$$

Next we denote the bigraded group associated with the filtration by $\operatorname{gr} H^*(M, \mathcal{E})$,

$$\operatorname{gr} H^*(M, \mathcal{E}) = \bigoplus_{p,q\geq 0} \operatorname{gr}_p H^q(M, \mathcal{E}).$$

With this, the filtration gives way to a spectral sequence of bigraded groups E_r converging to $E_\infty \simeq \operatorname{gr} H^*(M, \mathcal{E})$. The spectral sequence is constructed by first taking any $p, r \geq 0$ with the vector space being defined as

$$C_r^p = (c \in C_p \mid dc \in C_{p+r})$$

Then take a fixed p with

$$C_{(p)} = C_0^p \supset \ldots C_r^p \supset C_{r+1}^p \ldots$$

Then we define the r th term of the spectral sequence as

$$E_r = \bigoplus_{p=0}^{m} E_r^p, r \geq 0$$

where

$$E_r^p = C_r^p / C_{r-1}^{p+1} + dC_{r-1}^{p-r+1}$$

With d inducing a derivation d_r of E_r with degree r such that $d_r^2 = 0$. Then E_{r+1} is naturally isomorphic to the homology algebra of $H(E_r, d_r)$. From this, we state a proposition from [7]

Proposition 5.1. *The first two terms of the spectral sequence (E_r) can be identified with the following bigraded spaces:*

$$E_0 = C^*(\mathfrak{U}, gr\ \mathcal{E}),$$
$$E_1 = H^*(M, gr\ \mathcal{E}).$$

Here

$$E_0^{p,q} = C^{p+q}(\mathfrak{U}, (gr\ \mathcal{E})_p),$$
$$E_1^{p,q} = H^{p+q}(M, (gr\ \mathcal{E})_p).$$

Proof. This follows from Proposition 2. [7] □

Proposition 5.2. *There is the following identification of bigraded algebras:*

$$E_\infty = \mathrm{gr}\, H^*(M, \mathcal{E}),$$

where

$$E_\infty^{p,q} = \mathrm{gr}_p\, H^{p+q}(M, \mathcal{E}).$$

Proof. This follows from Proposition 3. [7] □

This leads to the Gysin Sequence.

6 Gysin Sequence

Firstly, I want to thank Simone Noja at the University of Bari Aldo Moro for their very insightful comments in this section. In certain special situations, the spectral sequence simplifies to a long exact sequence. One special case being the cohomology of a sphere bundle. The resulting sequence is called the Gysin Sequence which we will derive in supergeometry for a fibered spectral sequence.
Consider $\pi : \mathcal{E} \to \mathcal{M}$ to be a locally trivial smooth supermanifold bundle whose reduced bundle is an oriented $s^{\mathcal{K}}$ bundle with $k \geq 1$ working over $\mathbb{R}$. From previous sections, the E_2 term is $E_2^{p+q} = H^p(\mathcal{M}) \bigotimes H^q(s^{\mathcal{K}})$. From this, consider $E_2^{n-k,k}$ with $n \geq 0$ and that the Leray–Serre spectral sequence of the reduced bundle then gives the usual two-row argument. From this we can conclude an inclusion

$$0 \to E_\infty^{n-k,k} \to E_2^{n-k,k}$$

which can be extended to an exact sequence

$$0 \to E_\infty^{n-k,k} \to E_\infty^{n+1,0} \to^{d_{k+1}} E_2^{n-k,k} \to E_2^{n+1,0} \to 0$$

$$\text{With} \quad E_2^{n+1,0} = H^{n+1}(\mathcal{M}) \quad \text{and} \quad E_2^{n-k,k} = H^{n-k}(\mathcal{M})$$

Because of the shape of E_2, there exists a exact sequence

$$0 \to E_\infty^{n,0} \to H^n(\mathcal{E}) \to E_\infty^{n-k,k} \to 0$$

We can combine these two sequences to yield the following result.

$$\cdots \to H^n(\mathcal{E}) \to^{\alpha} H^{n-k}(\mathcal{M}) \to^{d_{k+1}} H^{n+1}(\mathcal{M}) \to^{\beta} H^{n+1}(\mathcal{E}) \to \dots$$

This is the Gysin sequence. From here, we need to identify the maps in the sequence this is done from considering projections and d_{k+1}, with α being the integration along the fibre and β being the natural pullback map π^*. Along with orientation trivializing the top cohomology local system. Resulting in, Let $\pi : \mathcal{E} \to \mathcal{M}$ to be an locally trivial smooth supermanifold bundle whose reduced bundle is an oriented $s^{\mathcal{K}}$ bundle with $k \geq 1$ working over $\mathbb{R}$. The Leray–Serre spectral sequence of the reduced bundle then gives the usual two-row argument with there being exact long sequence

$$\cdots \to H^n(\mathcal{E}) \to^{\pi_*} H^{n-k}(\mathcal{M}) \to^{\wedge e} H^{n+1}(\mathcal{M}) \to^{\pi^*} H^{n+1}(\mathcal{E}) \to \dots$$

With π_* being the berezin integration along the fiber, $\wedge e$ being the multiplication by the super Euler class with degree $k+1$ and π^* being the natural pullback on the supermanifold.

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