PALINDROMIC SECTIONS IN TWO KOLAKOSKI PHASE ACTIONS

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Abstract. We prove the two specified-phase orbit statements in Shen's Conjectures 3.4 and 3.7. The all-ones phase for the alphabet $\{1, n\}$, with n positive and even, and both alternating phases for $\{-1, n\}$, with n any even integer, have orbit length $2^{\lceil d/2 \rceil}$ at depth d. The proof uses unreduced signed words and a palindrome class preserved by taking the section of the square at two phase coordinates. It also gives the all-ones statement for nonpositive even n under Shen's displayed signed recursion. These are finite phase-orbit results; they do not imply the density conjecture for the classical Kolakoski sequence.

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1. Statements and conventions

Shen's phase maps record the starting symbols left after successive run-length expansions. We use their signed extension from [2, Definition 3.6], which can be specified without referring to run lengths of nonpositive size.

Let $A = \{a, b\}$ be an alphabet of two distinct symbols and write $\overline{a} = b$, $\overline{b} = a$. Fix integers m, n and put $v(a) = m$, $v(b) = n$. For each $d \geq 0$ define permutations $G_{a,d}, G_{b,d}$ of A^d recursively. Both are the identity on the single empty word at depth zero. For $x, s \in A$ and $u \in A^d$, set

$$G_{x,d+1}(su) = \overline{s}\, G_{s,d}^{v(x)}(u). \tag{1.1}$$

Here an integer power denotes a power of a permutation, including inverse powers. This is well-defined by induction: the two subsets with first symbol a and b are interchanged, and the map on each suffix is a permutation. More explicitly,

$$G_{x,d+1}^{-1}(su) = \overline{s}\, G_{\overline{s},d}^{-v(x)}(u). \tag{1.2}$$

These formulas commute with deleting the final coordinate. We therefore write G_x when the depth is understood. When $m \neq n$, relabeling a, b as m, n identifies G_x with Shen's map $C_{m,n}(v(x), -)$.

For a permutation g and a point t, let $\mathrm{per}_g(t)$ denote the least positive integer r with $g^r(t) = t$. For $d \geq 0$, let α_d and β_d be the length-d prefixes of $abab\cdots$ and $baba\cdots$, respectively.

Theorem 1.1. *Let n be an even integer and use the recursion* (1.1)*. For every $d \geq 0$ the following hold.*

(i) *If $m = 1$, then* $\mathrm{per}_{G_a}(a^d) = 2^{\lceil d/2 \rceil}$.

(ii) *If $m = -1$, then*

$$\mathrm{per}_{G_a}(\alpha_d) = \mathrm{per}_{G_a}(\beta_d) = 2^{\lceil d/2 \rceil}.$$

At depth $2j - 1$, part (i) for positive even n is [2, Conjecture 3.4]; part (ii) for every even integer n is [2, Conjecture 3.7]. The positivity in the first statement comes from the standing convention before Shen introduces the signed extension. Part (i) also records the extension to all even integers using (1.1).

Shen's upper bound is $2^{\lceil d/2 \rceil}$ for any phase orbit [2, Proposition 3.3]. We include a proof of this bound below. MacAulay studies the transformation groups of these automata and counts their maximal orbits; the existence result in [1, Section 4] does not specify the all-ones phase. Our

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argument identifies the particular phases in Theorem 1.1. The original observation for $n = 2$ is attributed by Shen to Yongyi Chen [2, Section 4].

Remark 1.2. When $n = 0$, formula (1.1) makes G_b interchange the first symbol and leave the suffix fixed. In particular, G_b is not the identity at positive depth. This follows from the displayed formula in [2, Definition 3.6]; the sentence at the beginning of its Section 4 that describes the zero-input map as the identity is inconsistent with that formula. Throughout this paper, including the case $n = 0$, the convention is (1.1).

2. Signed words and their sections

A signed letter is x^ε with $x \in A$ and $\varepsilon \in \{1, -1\}$. A signed word is a finite list of signed letters. We keep these words unreduced: their lengths refer to the number of letters as written, even if neighboring inverse letters occur. For a word W, write $|W|$ for its length and $\mathrm{rev}(W)$ for its reversal, which retains the sign of every letter. Its complement $\overline{W}$ exchanges a and b and also retains all signs. A signed palindrome satisfies $W = \mathrm{rev}(W)$.

The action G_W means that the letters of W act successively from left to right. Thus $G_{UV} = G_V \circ G_U$. Empty words act as the identity. For a phase symbol s and an integer k, the word $s^{[k]}$ means k copies of s if $k \geq 0$, and $-k$ copies of s^{-1} if $k < 0$. Brackets distinguish this formal word from a phase word or a permutation power. In particular, $s^{[0]}$ is empty.

Definition 2.1. For $s \in A$, define the signed word $D_s(W)$ by scanning W from left to right with current phase initially s. A letter x emits $s^{[v(x)]}$, whereas x^{-1} emits $\overline{s}^{[-v(x)]}$. After either letter, interchange the current phase. Concatenate the emitted words in their scanning order.

The use of the opposite phase for an inverse letter is necessary: it is exactly the inverse formula (1.2). Define $\sigma_s(W) = s$ if $|W|$ is even and $\sigma_s(W) = \overline{s}$ if $|W|$ is odd.

Lemma 2.2. *For all signed words U, V, W, all $s, t \in A$, and every suffix u,*

$$D_s(UV) = D_s(U) D_{\sigma_s(U)}(V), \tag{2.1}$$

$$G_W(su) = \sigma_s(W)\, G_{D_s(W)}(u). \tag{2.2}$$

The section of G_W at the two-symbol phase st is consequently $G_{D_t(D_s(W))}$. Moreover,

$$|D_s(W)| = \sum_{x^\varepsilon \text{ in } W} |v(x)|, \qquad D_{\overline{s}}(W) = \overline{D_s(W)}. \tag{2.3}$$

Proof. After scanning U the current phase is $\sigma_s(U)$; this gives (2.1). For a single signed letter, (2.2) is (1.1) or (1.2). Applying these formulas successively proves (2.2) for any word. Applying it once more to the suffix proves the two-symbol assertion. Each input letter emits $|v(x)|$ letters, independently of its sign and the incoming phase. Complementing the incoming phase complements every emitted letter while preserving its sign. These observations prove (2.3). □

Lemma 2.3 (Reversal)**.** *Let P be a signed palindrome.*

(i) *If $|P|$ is even, then $|D_s(P)|$ is even and*

$$D_{\overline{s}}(P) = \mathrm{rev}(D_s(P)).$$

(ii) *If $|P|$ is odd, then $D_s(P)$ is a signed palindrome. Its length has the parity of $|v(x)|$, where x^ε is the central letter of P.*

Proof. Number the input positions from 1 to $\ell = |P|$. The phase at position i is obtained from s by $i - 1$ interchanges. The reflected position is $\ell + 1 - i$, whose phase is obtained by $\ell - i$ interchanges. The two positions carry exactly the same signed letter. If ℓ is even, their phases are opposite; if ℓ is odd, their phases agree. Each emitted block is a constant signed word and is therefore equal to its reversal. Reversing the order of all these blocks proves the asserted word identities. Their

lengths occur in equal reflected pairs. In the even case this makes the total even; in the odd case only the central block contributes to its parity. The empty palindrome is covered by the same conclusions. □

3. Two levels at a time

Lemma 3.1. *For any signed word W, its squared action $G_W^2 = G_{WW}$ fixes every phase prefix of length two.*

Proof. The first phase is interchanged $2|W|$ times. By (2.3), the first section of WW has length $2\sum_{x^\varepsilon \text{ in } W} |v(x)|$, an even integer. Its action therefore fixes the second phase as well. □

Proposition 3.2. *Every orbit of every signed word action on A^d has length dividing $2^{\lceil d/2 \rceil}$.*

Proof. At depths zero and one every such action has order dividing 1 and 2, respectively. For $d \geq 2$, Lemma 3.1 allows us to restrict G_W^2 to each fixed two-symbol prefix. By Lemma 2.2, its action on the remaining $d-2$ symbols is another signed word action. Induction shows that raising this section to $2^{\lceil (d-2)/2 \rceil}$ gives the identity. Hence $G_W^{2^{\lceil d/2 \rceil}}$ is the identity on A^d, proving the divisibility assertion. □

Lemma 3.3 (Exact doubling)**.** *Suppose that $|W|$ is odd and put $W' = D_t(D_s(WW))$. For every finite suffix u,*

$$\mathrm{per}_{G_W}(stu) = 2\,\mathrm{per}_{G_{W'}}(u).$$

At depth zero the period is one; at depth one it is two.

Proof. An odd number of applications of G_W interchanges the first phase, so every return requires an even number, say $2r$, of applications. By Lemma 3.1 the square fixes st, and by Lemma 2.2 its restriction over that prefix is $G_{W'}$. Thus $G_W^{2r}(stu) = stu$ if and only if $G_{W'}^r(u) = u$. Taking least positive return times proves the formula. The two base cases follow directly from the action on the empty word and from the first-phase interchange. □

4. Palindromes preserved at the specified phases

Proposition 4.1. *Suppose $m = 1$ and n is even. If $W = aP$ with P an even-length signed palindrome, then*

$$D_a(D_a(WW)) = aP'$$

for an even-length signed palindrome P'.

Proof. Set $U = D_b(P)$. Since $|P|$ is even, Lemma 2.3 gives $D_a(P) = \mathrm{rev}(U)$ and $|U|$ even. Scanning the four pieces a, P, a, P and using $v(a) = 1$ gives the exact word identity

$$D_a(WW) = a\,U\,b\,\mathrm{rev}(U). \tag{4.1}$$

Indeed, the phases at their starts are a, b, b, a, respectively. The word $V = Ub\,\mathrm{rev}(U)$ is an odd-length signed palindrome with central letter b. Applying D_a to the first letter a in (4.1) emits a and leaves incoming phase b for V. Consequently

$$D_a(D_a(WW)) = aD_b(V).$$

By Lemma 2.3, $D_b(V)$ is a signed palindrome of even length, since $|v(b)| = |n|$ is even. This proves the claim, including negative n and $n = 0$. □

Proposition 4.2. *Suppose $m = -1$ and n is even. The following classes are preserved:*

$$W = aP \quad \Longrightarrow \quad D_b(D_a(WW)) = aP',$$
$$W = Pa \quad \Longrightarrow \quad D_a(D_b(WW)) = P'a,$$

where P and P' are even-length signed palindromes.

Proof. In both cases put $U = D_b(P)$, so that $D_a(P) = \mathrm{rev}(U)$ and $|U|$ is even. For the first class, scanning a, P, a, P gives

$$D_a(WW) = a^{-1}Ub^{-1}\,\mathrm{rev}(U). \tag{4.2}$$

Let $V = Ub^{-1}\,\mathrm{rev}(U)$. This is an odd-length signed palindrome whose central letter has even absolute value. Under D_b, the first letter a^{-1} in (4.2) emits $a^{[-v(a)]} = a$ and switches the phase to a. Thus

$$D_b(D_a(WW)) = aD_a(V).$$

Lemma 2.3 says that $D_a(V)$ is an even-length signed palindrome.

For the second class, scan P, a, P, a, starting with phase b. The corresponding identity is

$$D_b(WW) = Ub^{-1}\,\mathrm{rev}(U)a^{-1} = Va^{-1}. \tag{4.3}$$

The scan of V under D_a emits the even-length palindrome $D_a(V)$. Since the input length $|V|$ is odd, the incoming phase for the final a^{-1} is b. This last letter emits a, as before, so (4.3) gives

$$D_a(D_b(WW)) = D_a(V)a.$$

Notice that it is the odd *input* length of V, rather than the length of its emitted word, that determines the phase for this last letter. □

Proof of Theorem 1.1. We first prove part (i) in the stronger form that every word aP, with P an even-length signed palindrome, has period $2^{\lceil d/2\rceil}$ at a^d. The assertion is true at depths zero and one by Lemma 3.3. If $d \geq 2$, the two-symbol prefix is aa. Proposition 4.1 puts the section of the square in the same word class. Deleting this prefix leaves a^{d-2}. Lemma 3.3 and induction now give

$$\mathrm{per}_{G_{aP}}(a^d) = 2\,2^{\lceil (d-2)/2\rceil} = 2^{\lceil d/2\rceil}.$$

The initial word a belongs to the class by taking P empty.

For part (ii), prove simultaneously that every aP has this period at α_d, and every Pa has this period at β_d. The initial depths are again Lemma 3.3. When $d \geq 2$, the two-symbol prefixes are ab and ba; Proposition 4.2 preserves the corresponding word class. Deleting two symbols leaves α_{d-2} and β_{d-2}. Exact doubling and induction give the same displayed value. Taking P empty in both classes gives the result for G_a. □

5. The positive odd-expansion formulation

Assume in this section that $m = 1$ and $n > 0$ is even. All words constructed below have positive letters. For such a word W, $D_s(W)$ is ordinary run-length expansion starting with the symbol s. Write $E(W, s_1 \cdots s_r)$ for iterating these expansions in the order $s_1, \ldots, s_r$, and let $E(W, \varnothing) = W$. This is Shen's expansion notation, after relabeling a, b by $1, n$.

Lemma 5.1. *For positive words U, V and a phase word t,*

$$E(UV, t) = E(U, t)E(V, G_U(t)).$$

In particular, the section of G_U at t is $G_{E(U,t)}$.

Proof. For a one-symbol t, the identity is (2.1). Induct on $|t|$, applying that identity to the two words obtained at the previous expansion level. The outgoing next phase is the appropriate coordinate of $G_U(t)$ by (2.2), so the induction gives the formula as stated. Repeated application of (2.2) proves the section assertion. □

Corollary 5.2. *For every integer $j \geq 0$, the word*

$$E(a^{2^j}, a^{2j})$$

has odd length. For $j > 0$ this is the odd-expansion statement of [2, Proposition 3.5].

Proof. Define $W_0 = a$ and $W_{j+1} = D_a(D_a(W_jW_j))$. Proposition 4.1 implies inductively that $W_j = aP_j$ with P_j an even-length palindrome. In particular, $|W_j|$ is odd. We prove

$$W_j = E(a^{2^j}, a^{2j}). \tag{5.1}$$

The case $j = 0$ is immediate. Proposition 3.2 says that $G_a^{2^j}$ fixes every phase word of length $2j$. By Lemma 5.1, expansion of two consecutive copies of a^{2^j} through those $2j$ levels therefore gives W_jW_j. Expanding through the next two a-levels gives

$$E(a^{2^{j+1}}, a^{2j+2}) = D_a(D_a(W_jW_j)) = W_{j+1},$$

which proves (5.1) by induction. Its left side has odd length, giving the result. □

References

[1] Noah MacAulay, *Group theory of the Kolakoski sequence*, preprint, 2026, arXiv:2605.14234v1, `https://arxiv.org/abs/2605.14234v1`.

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