TWO EXACT FREQUENCY EXTREMA FOR SMOOTH SEQUENCES OVER $\{1,3\}$

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Abstract. Using the frequency coding of Jamet, Marcovici, Poirier, and de la Rue, we determine the largest frequency of 1 in a smooth sequence over $\{1,3\}$ and the largest such frequency below $1/2$. They are $1/2+1/(3+\sqrt{5})$ and $1/2-1/(13-\sqrt{5})$, respectively. The unique attaining type directives are 1^∞ and 0001^∞. The proof is an order comparison for two planar homographies. The first value corrects the numerical expression in the stated extremum question.

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1. Statement and frequency coding

For a one-sided sequence w over $\{1,3\}$, its derivative Δw is the sequence of lengths of its successive maximal constant blocks. A sequence is *smooth* if $\Delta^n w$ is defined and belongs to $\{1,3\}^{\mathbb{N}}$ for every $n \geq 0$. Its frequency of 1, when it exists, is

$$\operatorname{freq}_1(w) = \lim_{N\to\infty} \frac{\#\{0 \leq j < N : w_j = 1\}}{N}.$$

Write F for the set of these frequencies over all smooth sequences. The bi-infinite convention gives the same set, as justified at the end of Section 3. For such a sequence put $c_n = (\Delta^n w)_0$ and define its *type directive* $\tau \in \{0,1\}^{\mathbb{N}}$ by

$$\tau_n = 0 \quad \Longleftrightarrow \quad c_n = c_{n+1}.$$

This is the one-sided form of the type convention in [1, Proposition 3.11]; in particular, 0 and 1 are not interchanged in the assertions below.

Theorem 1.1. *Every smooth sequence over $\{1,3\}$ has a frequency of 1, and*

$$\max F = \frac{1}{2} + \frac{1}{3+\sqrt{5}}, \tag{1.1}$$

$$\max(F \cap [0,1/2]) = \frac{1}{2} - \frac{1}{13-\sqrt{5}}. \tag{1.2}$$

Equality in (1.1) *holds exactly for type directive 1^∞; equality in* (1.2) *holds exactly for type directive 0001^∞.*

The existence of frequencies and their coding are established results of Jamet, Marcovici, Poirier, and de la Rue. We state the needed letter-frequency version precisely and give its proof in Section 3, before optimizing its image.

Theorem 1.2 (Frequency coding; cf. [1])**.** *Let*

$$E = \{(a,b) : 0 \leq a, b \leq 1/2,\ a+b \leq 3/4\},$$

and define

$$h_0(a,b) = \frac{(1-a,\, 1/2-a)}{3-2(a+b)}, \qquad h_1(a,b) = \frac{(1/2-a,\, 1-a)}{3-2(a+b)}. \tag{1.3}$$

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For every $\tau \in \{0,1\}^{\mathbb{N}}$ the intersection

$$\bigcap_{n\geq 1} h_{\tau_0} \circ \cdots \circ h_{\tau_{n-1}}(E) = \{(a(\tau), b(\tau))\}$$

is a singleton. Every directive is realized by a smooth sequence; every smooth sequence with directive τ has frequency

$$f_\tau = \frac{3/2 - 2a(\tau)}{3 - 2(a(\tau) + b(\tau))}. \tag{1.4}$$

Consequently $F = \{f_\tau : \tau \in \{0,1\}^{\mathbb{N}}\}$.

The source ingredients are [1, Proposition 2.14, Lemmas 3.3–3.5, Corollary 3.6, Sections 3.2–3.3, and equation (25)]. The extremum question appears in [1, Section 5.1, Problem 1] and [2, Section 5, Problem 1]. The present result concerns its two numerical extrema only. It does not determine the Hausdorff dimension of F and does not address the frequency problem for the classical alphabet $\{1,2\}$.

2. A contracting planar model

Introduce coordinates

$$u = 2(a+b) - 1, \qquad v = 2(b-a), \qquad a = \frac{1+u-v}{4}, \quad b = \frac{1+u+v}{4}. \tag{2.1}$$

The image of E is the convex compact set

$$D = \{(u,v) : |u| + |v| \leq 1,\ u \leq 1/2\}. \tag{2.2}$$

With the sign convention $0 \leftrightarrow -$ and $1 \leftrightarrow +$, direct substitution in (1.3) gives

$$H_\sigma(u,v) = \left(\frac{v}{2-u}, \frac{\sigma}{2-u}\right), \qquad \sigma \in \{-1,1\}. \tag{2.3}$$

We write $H_- = H_{-1}$ and $H_+ = H_1$. The frequency function becomes

$$f(u,v) = \frac{1}{2} + \frac{v}{2(2-u)}. \tag{2.4}$$

All denominators in these formulas are positive on D.

For completeness, we record the contraction and the precise coding properties used in the comparison argument.

Lemma 2.1. *The maps H_- and H_+ preserve D and are contractions with constant $2/3$ for the norm $\|(u,v)\|_1 = |u| + |v|$. For $\sigma = (\sigma_0, \sigma_1, \ldots) \in \{-1,1\}^{\mathbb{N}}$ there is a unique coded point $p(\sigma) \in D$. It satisfies*

$$p(\sigma) = H_{\sigma_0}(p(S\sigma)) = \lim_{n\to\infty} H_{\sigma_0} \circ \cdots \circ H_{\sigma_{n-1}}(0,0), \tag{2.5}$$

where S deletes the first sign. The coding is continuous.

Proof. On the square $0 \leq a, b \leq 1/2$, let $d = 3 - 2(a+b) \geq 1$. For h_0, the two numerators are nonnegative, and

$$2(1-a) \leq d, \qquad 2(1/2 - a) \leq d.$$

The first inequality is equivalent to $b \leq 1/2$, and the second follows from $b \leq 1$. Moreover,

$$4(3/2 - 2a) \leq 3d \quad \Longleftrightarrow \quad 6b - 2a \leq 3,$$

which also holds. Thus h_0 maps the square into E; h_1 does so by exchanging its two output coordinates. In particular, E is invariant.

On E the Jacobian of h_0 is

$$J_{h_0}(a,b) = \frac{1}{d^2}\begin{pmatrix} 2b-1 & 2-2a \\ 2b-2 & 1-2a \end{pmatrix}.$$

Each absolute row sum equals $d/d^2 = 1/d \le 2/3$. The same bound holds for h_1. Integrating the derivative along a line segment in the convex set E proves the contraction in the supremum norm. For coordinate differences, (2.1) gives

$$|\delta u| + |\delta v| = 2\big(|\delta a + \delta b| + |\delta b - \delta a|\big) = 4\max\{|\delta a|, |\delta b|\}.$$

The conjugate maps therefore have the stated contraction constant in $\|\cdot\|_1$ and preserve D.

For a fixed sign sequence the nonempty compact prefix images of D are nested and have diameters at most $(2/3)^n \operatorname{diam}_1 D$. Their intersection is consequently one point, and the finite compositions applied to $(0,0) \in D$ converge to it. Continuity of H_{σ_0} gives the recurrence. If two sequences agree for their first n signs, their coded points lie in the same prefix image, which proves continuity of the coding. □

Let $\mathcal{T} = \{p(\sigma) : \sigma \in \{-1,1\}^{\mathbb{N}}\}$. Theorem 1.2 and the change of coordinates identify $F = f(\mathcal{T})$ with the indicated sign convention.

3. Derivation of the frequency coding

Here we supply the letter-frequency argument underlying Theorem 1.2. This also fixes the realization and limiting issues needed to pass from homographies to smooth sequences. Use four symbols to encode runs:

$$A = 1, \qquad B = 3, \qquad C = 111, \qquad D = 333.$$

Call a sequence over $\{A, B, C, D\}$ *alternating* if its symbols alternate between $\{A, C\}$ and $\{B, D\}$. Reading the displayed blocks decodes an alternating sequence into a unique sequence over $\{1, 3\}$ with run lengths in $\{1, 3\}$. Define non-erasing substitutions by

(3.1)

	A	B	C	D
φ_0	A	D	ABA	DCD
φ_1	B	C	BAB	CDC

Both substitutions preserve alternation.

Lemma 3.1. *If w is smooth and y^j encodes the runs of $\Delta^j w$, then*

$$y^j = \varphi_{\tau_j}(y^{j+1}) \qquad (j \ge 0). \tag{3.2}$$

Conversely, any infinite family of alternating sequences satisfying (3.2) *decodes to a smooth sequence with directive τ. Such a family exists for every $\tau \in \{0,1\}^{\mathbb{N}}$.*

Proof. A run of $\Delta^j w$ has odd length, so consecutive runs start at positions of opposite parity. A run in $\Delta^{j+1} w$ therefore groups either one or three consecutive runs of $\Delta^j w$. If the first symbols of these two derivative levels agree, their four possible run encodings give the first row of (3.1); if they disagree, they give the second row. The same row applies to every subsequent run: after a group of odd size the starting symbol flips at both levels, preserving agreement or disagreement. This proves (3.2).

For the converse, inspection of each of the eight entries of (3.1) shows that differentiating its decoded word gives the decoded input symbol. Alternation ensures that adjacent decoded runs do not merge. Thus the words w^j decoded from y^j satisfy $\Delta w^j = w^{j+1}$. They also have the prescribed equality or inequality of first symbols. Hence w^0 is smooth with directive τ.

For existence, let $\mathcal{A}$ be the nonempty compact space of alternating one-sided sequences over $\{A, B, C, D\}$. It is closed in the product of a finite discrete alphabet. Non-erasing substitutions are continuous in the product topology, since any finite output prefix is determined by a finite input

prefix. In the compact space $\mathcal{A}^{\mathbb{N}}$, each equation in (3.2) is a closed condition. Any finite collection of these conditions can be met: choose an arbitrary alternating sequence at a level beyond all of them, substitute downwards, and fill unused higher levels arbitrarily. Compactness gives a family satisfying all the equations. □

Proof of Theorem 1.2. Lemma 2.1 gives the singleton intersections by conjugacy. We prove their frequency interpretation.

Fix a smooth sequence and its tower from Lemma 3.1. For a prefix of length m of any y^j, denote its letter counts by A_m, B_m, C_m, D_m. Alternation gives

$$A_m + C_m = m/2 + O(1), \qquad B_m + D_m = m/2 + O(1), \tag{3.3}$$

where each error has absolute value at most $1/2$. If this prefix is substituted once, its output length is exactly $m + 2(C_m + D_m)$. For φ_0, the output counts of A and B are $A_m + 2C_m$ and C_m; for φ_1 these two counts are exchanged. Therefore, if the normalized input counts of A, B converge to (a, b), the normalized output counts converge to $h_0(a, b)$ or $h_1(a, b)$, respectively. This follows by inserting (3.3); the limiting denominator is $3 - 2(a + b) \geq 1$.

The same conclusion holds for arbitrary output prefixes. Indeed, such a prefix differs from the substitution of a complete input prefix by at most two symbols, and its input-prefix length tends to infinity. Now let (a_0, b_0) be any accumulation point of normalized A, B counts in prefixes of y^0. Fix $L \geq 1$ and follow the corresponding complete prefixes through levels $1, \ldots, L$. Their lengths tend to infinity, because each substitution expands length by a factor between 1 and 3. By successive subsequence extraction, their normalized counts converge to points (a_j, b_j) in the square $K = [0, 1/2]^2$ for every $j \leq L$. The preceding count calculation gives

$$(a_j, b_j) = h_{\tau_j}(a_{j+1}, b_{j+1}) \quad (0 \leq j < L).$$

Thus (a_0, b_0) belongs to every finite prefix image of K. Since $h_i(K) \subset E$, it belongs to every finite prefix image of E. Their intersection is the singleton $(a(\tau), b(\tau))$. All accumulation points coincide, so the normalized A, B counts converge to this point. Equation (3.3) gives the remaining limiting counts $c = 1/2 - a$ and $d = 1/2 - b$.

A prefix ending at a run boundary in the decoded word contains $A_m + 3C_m$ symbols equal to 1 and has total length $A_m + B_m + 3(C_m + D_m)$. Its frequency therefore converges to

$$\frac{a + 3c}{a + b + 3(c + d)} = \frac{3/2 - 2a}{3 - 2(a + b)}.$$

An arbitrary prefix differs from one ending at a run boundary by at most two symbols, so it has the same limit. Finally, Lemma 3.1 realizes every directive by a smooth sequence. This proves all the assertions for one-sided sequences.

For bi-infinite sequences, index each derivative so that its zeroth run contains the origin. The same odd-length argument gives the substitution tower up to a shift of at most two symbols at each level. At every fixed level these shifts change prefix counts by a bounded amount, so the preceding limit argument applies unchanged. Conversely, the compactness proof of Lemma 3.1, now for alternating sequences indexed by $\mathbb{Z}$, produces every directive with origins aligned at run boundaries. Decoding gives bi-infinite smooth sequences. Thus the bi-infinite frequency set is also exactly $\{f_\tau : \tau \in \{0, 1\}^{\mathbb{N}}\}$. □

4. The order comparison

Set

$$t = \frac{\sqrt{5} - 1}{2}, \qquad P = (t^2, t). \tag{4.1}$$

We will repeatedly use

$$0 < t < 1, \quad t^2 = 1 - t, \quad t > 1/2, \quad 2 - t^2 = 1 + t = 1/t. \tag{4.2}$$

In particular $P \in D$, since $t^2 + t = 1$ and $t^2 < 1/2$, and (2.3) gives $H_+(P) = P$. The all-plus point is therefore $p(+^\infty) = P$.

For points in $\mathbb{R}^2$ we use coordinatewise order.

Lemma 4.1. *Every $q \in \mathcal{T}$ satisfies $q \le P$. If $q = p(\sigma)$ and $\sigma \neq +^\infty$, then at least one of these coordinate inequalities is strict.*

Proof. Suppose $q = (u, v) \in D$ satisfies $q \le P$. If $v < 0$, the first coordinate of $H_\sigma(q)$ is negative and hence less than t^2. If $v \ge 0$, then

$$\frac{v}{2 - u} \le \frac{t}{2 - t^2} = t^2.$$

For the second coordinate, the minus sign gives a negative number, while the plus sign gives

$$\frac{1}{2 - u} \le \frac{1}{2 - t^2} = t.$$

Thus both sign maps carry $\{q \in D : q \le P\}$ into itself. Because $(0, 0) \le P$, the finite compositions in (2.5) are bounded above by P, and taking limits proves the first assertion.

We also need a strictness observation. If $q \le P$ and $u < t^2$, then the second coordinate of $H_+(q)$ is strictly less than t; the same is true of $H_-(q)$ because it is negative. If instead $u = t^2$ and $v < t$, the first coordinate of either image is strictly less than t^2. Consequently a strict inequality in at least one coordinate persists through every outer sign map.

If $\sigma_j = -1$ for some j, write $r = p(S^{j+1}\sigma)$. The second coordinate of $H_-(r)$ is negative, hence strictly less than t. Applying the first j maps and the strictness observation proves the second assertion. □

Corollary 4.2. *Every $(u, v) \in \mathcal{T}$ satisfies $|u| \le t^2$ and $|v| \le t$.*

Proof. Write $(u, v) = H_\sigma(r)$. Lemma 4.1 gives $r_u \le t^2$, and hence $|v| = 1/(2 - r_u) \le t$. Apply this bound also to the tail r. Then $|u| = |r_v|/(2 - r_u) \le t/(2 - t^2) = t^2$. □

The linear functional that controls the frequency after one sign is

$$X(u, v) = 2u + v. \tag{4.3}$$

Substitution into (2.4) yields the useful identity

$$f(H_\sigma(u, v)) = \frac{1}{2} + \frac{\sigma}{2(4 - X(u, v))}. \tag{4.4}$$

Here $4 - X(u, v) > 0$ on D, for instance from $u \le 1/2$ and $v \le 1$. Frequencies below $1/2$ correspond exactly to a first minus sign; none equals $1/2$.

Lemma 4.3. *On $\mathcal{T}$ the function X has extrema*

$$\max X = 2 - t, \qquad \min X = t - 2.$$

The maximizing directive is uniquely $+^\infty$ and the minimizing directive is uniquely $--+^\infty$.

Proof. The coefficients of X are positive, so Lemma 4.1 gives

$$X(q) \le X(P) = 2t^2 + t = 2 - t,$$

with equality exactly for the all-plus directive.

Corollary 4.2 also gives

$$X(q) \ge -2t^2 - t = t - 2.$$

Equality forces $q = -P$. To identify its directive, write $q = H_\sigma(r)$. The second coordinate $q_v = -t$ forces $\sigma = -1$ and $r_u = t^2$. The first coordinate $q_u = -t^2$ then forces $r_v = -t$, because $t^2(2-t^2) = t$. Next write $r = H_\eta(z)$. Its second coordinate again forces $\eta = -1$ and $z_u = t^2$; its first coordinate $r_u = t^2$ forces $z_v = t$. Thus $z = P$. By Lemma 4.1 its directive is $+^\infty$. Conversely, direct substitution gives $H_-H_-(P) = -P$. The minimum is therefore attained exactly by $--+^\infty$. □

5. Evaluation of the frequency extrema

Proof of Theorem 1.1. The existence of every frequency and the realization of every directive follow from Theorem 1.2. It remains to optimize (4.4) over its two branches.

On the plus branch, the right-hand side of (4.4) is a strictly increasing function of $X(u, v)$. Lemma 4.3 therefore gives

$$\max_{q\in\mathcal{T}} f(H_+(q)) = \frac{1}{2} + \frac{1}{2(4-(2-t))} = \frac{1}{2} + \frac{1}{2(2+t)} = \frac{1}{2} + \frac{1}{3+\sqrt{5}}.$$

Every minus-branch value is less than $1/2$, so this is the global maximum. Equality forces a first plus sign and the all-plus tail, which gives the unique directive 1^∞.

On the minus branch, (4.4) is a strictly decreasing function of $X(u, v)$. Thus

$$\max_{q\in\mathcal{T}} f(H_-(q)) = \frac{1}{2} - \frac{1}{2(4-(t-2))} = \frac{1}{2} - \frac{1}{2(6-t)} = \frac{1}{2} - \frac{1}{13-\sqrt{5}}.$$

This branch is exactly $F \cap [0, 1/2]$. Equality forces the tail $--+^\infty$, so the full directive is $---+^\infty$, or 0001^∞ in the binary convention. This proves both assertions. □

Corollary 5.1. *The open interval*

$$\left(\frac{1}{2} - \frac{1}{13-\sqrt{5}}, \frac{1}{2} + \frac{1}{13-\sqrt{5}}\right)$$

contains no point of F. Both endpoints belong to F. The upper endpoint has unique type directive 1001^∞.

Proof. At a fixed tail, (4.4) gives $f(H_+(q)) = 1 - f(H_-(q))$. Thus F is invariant under $x \mapsto 1 - x$, with this reflection obtained by changing only the first sign. Apply Theorem 1.1 to the lower half of F. The uniqueness assertion follows from the same change of first sign. □

Remark 5.2. The value printed for the global supremum in [1, Problem 1] and [2, Problem 1] is $1/2 + 2/(3+\sqrt{5})$. The maximizing directive stated there is correct, but its value is $1/2 + 1/(3+\sqrt{5})$, as follows already by evaluating (2.4) at the fixed point P. The lower-half value in those questions agrees with (1.2).

References

[1] D. Jamet, I. Marcovici, L. Poirier, and T. de la Rue, *Frequency of patterns in smooth sequences over the alphabet* $\{1, 3\}$, arXiv:2604.11387v3, June 29, 2026, `https://arxiv.org/abs/2604.11387v3`.

[2] D. Jamet, I. Marcovici, L. Poirier, and T. de la Rue, *Frequency of Patterns in Smooth Sequences over the Alphabet* $\{1, 3\}$, arXiv:2606.13846v1, June 11, 2026, `https://arxiv.org/abs/2606.13846v1`.

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