A COUNTEREXAMPLE TO THE SUBTRACTION-FREE CONJECTURE FOR BOUNDED RATIOS OF MINORS

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Abstract. We disprove the subtraction-free conjecture of Boocher and Froehle for bounded ratios of minors of totally positive matrices. We construct products P, M, Q of ordinary minors of a matrix of order 3207 such that $0 < P/(MQ) \le 1/64$ throughout the totally positive locus, whereas $MQ - P$ has a coefficient equal to -1 in the standard bidiagonal parameters.

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1. Introduction

A real square matrix is *totally positive* if all its minors are strictly positive, and *totally nonnegative* if all its minors are nonnegative. Write TP_n for the set of totally positive matrices of order n. For integers a, b, put $[a, b] = \{a, a+1, \dots, b\}$, with the interval empty when $a > b$. For equally sized subsets $I, J \subseteq [1, n]$, let $A[I, J]$ have its rows and columns in increasing order, and write

$$\Delta_{I,J}(A) = \det A[I, J], \qquad \Delta_{\varnothing,\varnothing}(A) = 1.$$

A minor is *principal* when $I = J$. We consider ratios

$$R(A) = \frac{\prod_{\nu=1}^{r} \Delta_{I_\nu, J_\nu}(A)}{\prod_{\nu=1}^{s} \Delta_{K_\nu, L_\nu}(A)}, \tag{1.1}$$

with repetitions allowed. Such a ratio is *bounded* if there is a constant $C > 0$ such that $R(A) \le C$ for every $A \in \mathrm{TP}_n$. Describing these ratios is a multiplicative form of the problem of classifying determinantal inequalities.

The coordinate methods for this problem originate in the structural theory of total positivity. The factorization theorem of Loewner and Whitney [11, 14] expresses every invertible totally nonnegative matrix as a product of a positive diagonal matrix and nonnegative elementary bidiagonal factors. Their parameters also have a planar-network interpretation. The determinant formula for disjoint paths has its origins in Karlin and McGregor [9] and Lindström [10], and was developed in enumerative form by Gessel and Viennot [7]. It expresses each minor as a sum of path weights with nonnegative coefficients. Fomin and Zelevinsky [5] organized these factorizations, positivity tests and chamber-minor coordinates in a common framework. These coordinates make it possible to compare polynomial coefficients as well as values on the positive locus.

Fallat and Johnson [4] posed the bounded-ratio classification problem in their 2000 account of determinantal inequalities. Fallat, Gekhtman and Johnson [3] then studied products of principal minors. They obtained a complete classification through order five and a characterization of ratios involving two principal minors in each product in arbitrary order. Their arguments use bidiagonal factorizations, index-balance conditions and operations preserving inequalities. In [3, §6] they proposed two conclusions suggested by these classifications: every bounded ratio

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should be bounded by 1, and the difference between its denominator and numerator should have nonnegative coefficients in the bidiagonal parameters.

Skandera [13, Theorem 4.2] extended the two-factor classification to ordinary minors, using planar paths and the order of the row and column indices. His Problem 5.3 asks for the corresponding classification with more factors. Boocher and Froehle [2] subsequently factored every bounded two-factor ratio into elementary bounded ratios arising from short Plücker relations. They also formulated the subtraction-free conjecture for arbitrary products of ordinary minors in a fixed standard network.

Precisely, let $\mathbf{t}$ be the independent positive edge parameters of that network, and let $p(\mathbf{t}), q(\mathbf{t})$ be the numerator and denominator of (1.1) after path expansion. The coefficient condition in [2, Conjecture 37] is

$$R \text{ is bounded on } \mathrm{TP}_n \quad \Longleftrightarrow \quad q - p \in \mathbb{R}_{\geq 0}[\mathbf{t}], \tag{1.2}$$

where $\mathbb{R}_{\geq 0}[\mathbf{t}]$ denotes the polynomials with nonnegative coefficients. The reverse implication gives the bound $R \leq 1$. The proposed forward implication is the subject of this paper.

The factorization problem has since been resolved negatively. Soskin and Gekhtman [12], in work first circulated in 2022 and published in 2025, proved that bounded exponent vectors form a polyhedral cone and produced ratios of minors of order-four matrices outside the cone generated by primitive ratios. Their counterexamples satisfy the subtraction-free condition. Thus the failure of primitive factorization left the coefficient condition as a separate problem.

Gekhtman, Greenberg and Soskin [6] studied the corresponding cones for full-rank cluster algebras of finite type. They identified their extreme rays with the u-variables and proved that every bounded cluster-variable ratio is bounded by 1. Their Theorem 1.11 establishes subtraction-freeness in face weights and primitive factorization for Plücker ratios on $\mathrm{Gr}(2, n)$, $\mathrm{Gr}(3, 6)$, $\mathrm{Gr}(3, 7)$ and $\mathrm{Gr}(3, 8)$. The same paper gives a type-C_2 counterexample to the analogous condition that a denominator-minus-numerator can be expressed as a polynomial with nonnegative coefficients in the full set of cluster variables [6, Example 3.20]. Its ordinary-minor statement appears separately as Conjecture 1.4. The construction below realizes a coefficient obstruction using ordinary minors on the entire matrix locus TP_{3207} and the standard bidiagonal coordinates.

Bounded ratios also admit a broader convex-geometric description. Huang, Huh, Soskin and Wang [8] identified the bounded-ratio cone of Lorentzian matrices with a dual cut cone. Baldi and Kummer [1, Theorem 1.1] proved that the bounded-ratio cone of a semialgebraic subset of a positive orthant is dual to its tropicalization. Such descriptions govern growth of monomials. The coefficient question additionally depends on the support of the denominator polynomial, namely the set of exponent vectors of its nonzero monomials.

Our main result is the following.

Theorem 1.1. *There are products P, M of ordinary minors and six ordinary minors $Q_1, \ldots, Q_6$ of a matrix of order $N = 3207$ such that, with $Q = Q_1 \cdots Q_6$,*

$$0 < \frac{P(A)}{M(A)Q(A)} \leq \frac{1}{64} \qquad (A \in \mathrm{TP}_N). \tag{1.3}$$

In the standard bidiagonal network parameters, the polynomial $MQ - P$ has a coefficient equal to -1.

Theorem 1.1 disproves the forward implication in (1.2). The products P, M are defined by the chamber-exponent formula in Section 7, and the six factors of Q are listed in Table 2.

The proof starts with six matching polynomials whose product has an integral midpoint hole in its support. Two disjoint triangles prevent a decomposition of this midpoint into six matching

vectors. A reference matching and adjacent boundary bridges convert the matchings to directed paths. We embed the resulting network into the standard matrix network while preserving vertex capacities and integral edge flows. The unweighted edges of the latter network form a forest, which makes the weighted-edge exponents determine the complete flow and preserves the missing monomial. Chamber-minor inversion and six applications of the arithmetic–geometric mean inequality then give the ratio in Theorem 1.1.

2. Standard coordinates and chamber minors

For $1 \le i < n$, put $x_i(t) = I_n + tE_{i,i+1}$, where I_n is the identity matrix and $E_{i,j}$ is the matrix unit. We use the upper factor

$$U = \prod_{s=1}^{n-1} \prod_{i=1}^{n-s} x_i(t_{s,i}), \tag{2.1}$$

where both products are ordered increasingly from left to right. Set

$$A = LDU, \qquad D = \operatorname{diag}(d_1, \dots, d_n). \tag{2.2}$$

Here

$$L = \left(\prod_{s=1}^{n-1} \prod_{i=1}^{n-s} x_i(\ell_{s,i}) \right)^{\mathsf{T}}.$$

The $\ell_{s,i}$, d_i, and $t_{s,i}$ are independent positive variables. We write $U(t)$ for (2.1) and $U(\ell)$ for the same product with $t_{s,i}$ replaced by $\ell_{s,i}$, so $L = U(\ell)^{\mathsf{T}}$. Proposition 2.2 below proves that these products parametrize TP_n. In [2, Figure 1], the upper parameter $t_{s,i}$ is $u_{(s-1)n-s(s-1)/2+i}$; interleaving commuting chips gives the same network. The coefficient constructed below has lower-parameter exponent zero. It is therefore a coefficient of the specialization $L = I_n$, common to all lower bidiagonal words with $L(0) = I_n$, including the word in that figure.

Every elementary upper factor has a network with horizontal edges of weight 1 and one edge from rail i to rail $i+1$ of weight t; a lower factor has the reversed rail step. Matrix multiplication concatenates these networks. The diagonal factor places weight d_i on rail i. A path has weight equal to the product of its edge weights. A path family is a collection of vertex-disjoint directed paths, and its weight is the product of their weights.

Expanding a minor of the path matrix gives signed families of paths. Fix a topological order on the vertices. For an intersecting family, choose its first common vertex and the two least source indices of paths through that vertex, and interchange their tails. The operation preserves weight, reverses the endpoint permutation sign and leaves the chosen intersection and pair unchanged. It is therefore an involution. The surviving disjoint families preserve boundary order and have positive sign. Every minor is consequently the sum of their weights and is a polynomial with nonnegative integer coefficients.

We write $\mathbf{t}$ for the complete parameter tuple in (2.2), including the lower and diagonal parameters. For a monomial B, let $\nu(B)$ be its exponent vector in the complete parameter tuple. For a polynomial F, $[\mathbf{t}^u]F$ denotes the coefficient of the monomial with exponent vector u.

Define the ordinary minors

$$\mathsf{D}_r = \Delta_{[1,r],[1,r]}, \quad \mathsf{D}_0 = 1, \qquad \mathsf{C}_{a,k} = \Delta_{[1,a+k-1],\,[1,a-1]\cup[n-k+1,n]}, \tag{2.3}$$

for $1 \le r \le n$ and $a, k \ge 1$ with $a + k - 1 \le n$. We also set $\mathsf{C}_{a,0} = \mathsf{D}_{a-1}$ for $1 \le a \le n+1$, and identify $\mathsf{C}_{a,k} = \mathsf{D}_n$ when $a + k - 1 = n$.

Lemma 2.1. *The minors in* (2.3) *satisfy*

$$\mathsf{D}_r = d_1 \cdots d_r, \qquad \mathsf{C}_{a,k} = \mathsf{D}_{a+k-1} \prod_{s=1}^{k} \prod_{i=a+k-s}^{n-s} t_{s,i}. \tag{2.4}$$

In particular they are monomials with coefficient 1*, and*

$$t_{s,i} = \frac{\mathsf{C}_{i,s}\mathsf{C}_{i+2,s-1}}{\mathsf{C}_{i+1,s}\mathsf{C}_{i+1,s-1}}, \qquad d_i = \frac{\mathsf{D}_i}{\mathsf{D}_{i-1}}. \tag{2.5}$$

Proof. Set $r = a + k - 1$. The first r rows of a lower triangular matrix have nonzero entries only in its first r columns. Cauchy–Binet therefore removes L from each initial-row minor and contributes D_r from the diagonal factor.

In the upper network there is a unique disjoint family from $[1, r]$ to $[1, a-1] \cup [n-k+1, n]$. The first $a-1$ paths remain horizontal. If $r = n$, all paths are horizontal and the upper-parameter product in (2.4) is 1. Suppose $r < n$. The path from source r must reach sink n. The only chip entering rail n occurs in the first sweep, forcing the consecutive chain from r to n in that sweep. Inductively, the path from source $r - s + 1$ cannot rise before sweep s, since its first upward step would meet the path from the next higher source before that path departs. The last chip entering its sink $n - s + 1$ occurs in sweep s, so its chain in that sweep is forced. These chains are disjoint and use exactly the chips $a + k - s \le i \le n - s$.

For (2.5), the diagonal contributions cancel as

$$\frac{\mathsf{D}_{i+s-1}\mathsf{D}_{i+s}}{\mathsf{D}_{i+s}\mathsf{D}_{i+s-1}} = 1.$$

For an upper parameter $t_{p,q}$, the exponent difference between $\mathsf{C}_{i,s}$ and $\mathsf{C}_{i+1,s}$ is 1 exactly when $p \le s$ and $q = i + s - p$. The difference between $\mathsf{C}_{i+2,s-1}$ and $\mathsf{C}_{i+1,s-1}$ is -1 exactly when $p \le s - 1$ and $q = i + s - p$. Their sum is 1 precisely for $(p, q) = (s, i)$, and is 0 otherwise. The formula for d_i follows from $\mathsf{D}_i = d_1 \cdots d_i$. □

Proposition 2.2. *The map* (2.2)*, with L and U as above, is a bijection from the positive parameter orthant onto* TP_n*.*

Proof. First consider a unit upper triangular matrix V for which

$$F_{a,k}(V) = \det V[[a, a+k-1], [n-k+1, n]] > 0 \quad (a, k \ge 1,\ a + k - 1 \le n).$$

Set $F_{a,0}(V) = 1$ for $1 \le a \le n + 1$. Triangularity gives $F_{a,k}(V) = 1$ when $a + k - 1 = n$. Define positive numbers

$$t_{s,i} = \frac{F_{i,s}(V)F_{i+2,s-1}(V)}{F_{i+1,s}(V)F_{i+1,s-1}(V)}.$$

Lemma 2.1, with all diagonal parameters equal to 1, gives the same ratios for $U(t)$. Starting with $F_{a,0} = 1$, recover all $F_{i,s}$ in the order $s = 1, \ldots, n-1$ and $i = n - s, \ldots, 1$ from

$$F_{i,s} = t_{s,i}\frac{F_{i+1,s}F_{i+1,s-1}}{F_{i+2,s-1}}, \qquad F_{n-s+1,s} = 1.$$

Thus V and $U(t)$ have the same F-values.

These values determine V. Process columns $j = n, n-1, \ldots, 2$ and, within column j, rows $a = j-1, j-2, \ldots, 1$. Put $k = n - j + 1$ and expand $F_{a,k}(V)$ along its first column, of index j. The coefficient of $V_{a,j}$ is $F_{a+1,k-1}(V) > 0$. Every other term uses entries in columns greater than j and entries of column j in rows greater than a. Those entries have already been determined, or are fixed by unit triangularity. Hence $V_{a,j}$ is determined. It follows that $V = U(t)$, and the displayed ratios give uniqueness of the parameters.

Now take $A \in \mathrm{TP}_n$. Nonzero leading minors give unit triangular Gaussian elimination, so $A = LDU$ with $d_r = \mathsf{D}_r(A)/\mathsf{D}_{r-1}(A) > 0$. Indeed, the first step factors off $A_{11} > 0$ and leaves its Schur complement, whose leading minor of order r is $\mathsf{D}_{r+1}(A)/A_{11} > 0$; induction proves the factorization and the diagonal formula. For $r = a + k - 1$, Cauchy–Binet and triangularity give

$$\mathsf{C}_{a,k}(A) = (d_1 \cdots d_r) F_{a,k}(U).$$

All these F-values are positive, so the preceding reconstruction gives the required positive upper parameters. Applying the same argument to $A^{\mathsf{T}} = U^{\mathsf{T}} D L^{\mathsf{T}}$ gives the positive parameters of L^{T}. Uniqueness follows from uniqueness of unit triangular Gaussian elimination and of the two reconstructions.

Conversely, take positive parameters. The path formula makes every minor of $U(t)$ and $U(\ell)$ nonnegative. For $1 \le k < n$ and $J = \{j_1 < \cdots < j_k\}$, route source r to sink j_r in sweep $k - r + 1$, for $r = k, \dots, 1$. The chips exist because $j_r \le n - k + r$. If $r' > r$, the path from r' moves first while the other path is still on rail $r < r'$. When the path from r moves, the first path is already on rail $j_{r'} > j_r$. Thus the paths are disjoint and $\Delta_{[1,k],J}(U) > 0$. For $k = n$ the determinant is 1. In the Cauchy–Binet expansion of $\Delta_{I,J}(LDU)$, all summands are nonnegative, and the summand with intermediate set $[1, k]$ is

$$\Delta_{[1,k],I}\big(U(\ell)\big)(d_1 \cdots d_k)\Delta_{[1,k],J}\big(U(t)\big) > 0.$$

Therefore $LDU \in \mathrm{TP}_n$. □

Lemma 2.3. *Let $Q = \prod_{j=1}^{r} Q_j$ be a product of ordinary minors in the standard chart. Suppose each Q_j contains two distinct path monomials B_j, T_j, and suppose*

$$u = \frac{1}{2} \sum_{j=1}^{r} \big(\nu(B_j) + \nu(T_j)\big) \in \mathbb{Z}^{n^2}_{\ge 0}, \qquad [\mathbf{t}^u]Q = 0. \tag{2.6}$$

Assume that every lower-parameter coordinate of u is zero. Then there are products P, M of the chamber minors in (2.3) *such that*

$$\frac{P}{M} = \mathbf{t}^u, \qquad 0 < \frac{P(A)}{M(A)Q(A)} \le 2^{-r} \quad (A \in \mathrm{TP}_n),$$

and $[\mathbf{t}^{\nu(P)}](MQ - P) = -1$.

Proof. Apply (2.5) to the monomial $\mathbf{t}^u$, collect exponents of equal chamber minors, and place positive exponents in P and negative exponents in M. These are finite products of ordinary minors, and $P = M\mathbf{t}^u$ as an identity in the chart. For positive parameters, the path expansion and the arithmetic–geometric mean inequality give

$$Q \ge \prod_{j=1}^{r} (B_j + T_j) \ge 2^r \sqrt{\prod_{j=1}^{r} B_j T_j} = 2^r \mathbf{t}^u.$$

This proves the bound on the whole of TP_n. Finally M is a coefficient-one monomial, so

$$[\mathbf{t}^{\nu(P)}](MQ - P) = [\mathbf{t}^u]Q - 1 = -1.$$ □

3. A matching hole from two odd cycles

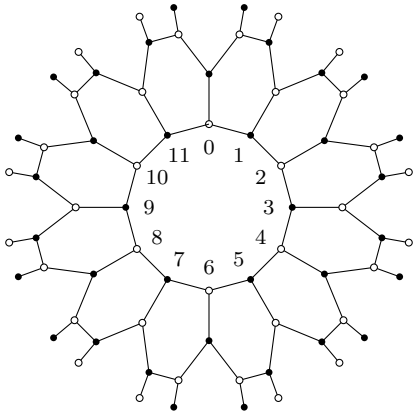
Sector subscripts on the vertices and edges are taken modulo 12 throughout the construction; boundary labels lie in $[0, 23]$. Form a bipartite graph in a disk with internal vertices

$$V_i, W_i, X_i, Y_i \quad (0 \le i < 12)$$

and boundary vertices $b_0, \dots, b_{23}$ in cyclic order. Its internal edges are

$$C_i = V_iV_{i+1}, \quad R_i = V_iW_i, \quad A_i = W_iX_i, \quad B_i = X_iY_i, \quad D_i = Y_iW_{i+1}. \tag{3.1}$$

The boundary edges are $\xi_i = X_ib_{2i}$ and $\eta_i = Y_ib_{2i+1}$. Embed the V_i on an inner circle, the W_i radially outside them, and the paths $W_iX_iY_iW_{i+1}$ in successive outer sectors. This describes a planar graph consisting of a central twelve-cycle and twelve surrounding hexagons. It has 48 internal vertices and 84 edges. Color V_i, X_i white for even i, and W_i, Y_i white for odd i; the other vertices have the opposite color, extended to the boundary leaves.



central vertices $V_0, \dots, V_{11}$

FIGURE 1. The common graph for the six matching polynomials. The boundary leaves are ordered $b_0, b_1, \dots, b_{23}$ around the outer circle. Filled and open vertices are black and white, respectively.

A matching is required to cover every internal vertex exactly once; boundary vertices may be covered or uncovered. For two distinct even sectors a, b, prescribe boundary bits 00 at sectors a, b, bits 11 at their predecessors, and bits 10 at all other sectors. The two bits at sector i record whether ξ_i, η_i are selected.

For any edge e let $\mathbf{e}_e$ be its unit coordinate in $\mathbb{Z}^{84}$. For even i, set

$$\alpha_i = \mathbf{e}_{A_i} + \mathbf{e}_{D_i} + \mathbf{e}_{C_i} - \mathbf{e}_{B_i} - \mathbf{e}_{R_i} - \mathbf{e}_{R_{i+1}}, \qquad \zeta = \sum_{i \text{ odd}} \mathbf{e}_{C_i} - \sum_{i \text{ even}} \mathbf{e}_{C_i}. \tag{3.2}$$

Let β_{ab} be the incidence vector of the following matching, where $S = \{a, b\}$:

$$\begin{aligned} &\bigcup_{i \in S} \{B_i, R_i, R_{i+1}, \xi_{i-1}, \eta_{i-1}\} \\ &\quad \cup \bigcup_{\substack{i \text{ even} \\ i \notin S}} \{D_{i-1}, D_i, \xi_{i-1}, \xi_i\} \quad \cup \{C_i : i \text{ even},\ i \notin S\}. \end{aligned} \tag{3.3}$$

This set has 30 edges.

Lemma 3.1. *For the prescribed boundary bits, the matching incidence vectors are exactly*

$$\beta_{ab}, \quad \beta_{ab} + \alpha_a, \quad \beta_{ab} + \alpha_b, \quad \beta_{ab} + \alpha_a + \alpha_b, \quad \beta_{ab} + \alpha_a + \alpha_b + \zeta. \tag{3.4}$$

Proof. For every unselected even sector i, sectors $i-1$ and i have boundary bits 10. Their X-vertices are matched to boundary leaves, which forces D_{i-1} and D_i. For a selected even sector i, the preceding sector has bits 11. After its occupied leaves are removed, the remaining outer path $W_iX_iY_iW_{i+1}$ has two possible matchings: use its middle edge B_i and the radial edges R_i, R_{i+1}, or use its outside edges A_i, D_i.

The first choice covers the central pair V_i, V_{i+1}, and the second leaves that pair to the central cycle. The selected pairs are disjoint. Removing any nonempty collection of them leaves paths with even numbers of vertices, uniquely matched by the remaining even-indexed C-edges. If neither pair is removed, the whole twelve-cycle remains and has two alternating perfect matchings. The four outer choices therefore give five matchings. Replacing the first choice at i by the second contributes α_i; exchanging the two central matchings contributes ζ. □

Use the ordered list of six pairs

$$\mathcal{E} = ((0,2),(2,4),(0,4),(6,8),(8,10),(6,10)). \tag{3.5}$$

They form two disjoint triangles on the six even sectors. For each pair choose the first and last vectors in (3.4) as its bottom and top endpoints. Their aggregate midpoint is

$$m = \sum_{(a,b)\in\mathcal{E}} \beta_{ab} + \sum_{i \text{ even}} \alpha_i + 3\zeta. \tag{3.6}$$

Every even sector occurs in two pairs, so twice this vector is the sum of the twelve endpoint incidence vectors. In particular, m is integral and nonnegative.

Lemma 3.2. *The vector m is not a sum of six matching incidence vectors, one with each of the prescribed boundary conditions.*

Proof. The six α_i and ζ are linearly independent: their coefficients are recovered first from the six coordinates A_i and then from C_1. A sum equal to (3.6) would therefore use the last state of (3.4) in exactly three factors. Each such state contributes both incident α-directions. All contributions to the α_i coefficients are nonnegative, and each total must be 1. The three selected pairs would therefore have to be pairwise disjoint. Each triangle contains at most one edge in such a matching, so their union contains at most two. □

4. From matchings to a network with separated terminals

4.1. **Acyclic orientation.** Choose the reference matching

$$\begin{aligned} M_0 = \{&C_2, C_4, C_6, C_8, C_{11}, R_1, R_{10}, A_0, D_1, B_2,\\ &A_3, D_3, B_4, A_5, D_5, B_6, A_7, D_7, B_8, A_9, B_{10}, A_{11},\\ &\eta_0, \xi_1, \eta_9, \eta_{11}\}. \end{aligned} \tag{4.1}$$

It is the unique matching occupying exactly the boundary vertices b_1, b_2, b_{19}, b_{23}. Remove these four occupied leaf edges and their endpoints and forbid all other leaf edges. The following forced deletions exhaust the remaining internal vertices; in each entry the vertex on the left has only the indicated available matching edge:

$$\begin{gathered} (W_1,R_1),(X_0,A_0),(V_0,C_{11}),(X_9,A_9),(X_{11},A_{11}),(Y_1,D_1),\\ (V_2,C_2),(X_2,B_2),(W_3,A_3),(Y_3,D_3),(V_4,C_4),(X_4,B_4),\\ (W_5,A_5),(Y_5,D_5),(V_6,C_6),(X_6,B_6),(W_7,A_7),(Y_7,D_7),\\ (V_8,C_8),(V_{10},R_{10}),(X_8,B_8),(X_{10},B_{10}). \end{gathered} \tag{4.2}$$

Orient the edges of M_0 from white to black and the other edges from black to white. A directed cycle would alternate between reference and nonreference edges; flipping it would contradict the uniqueness just proved. Thus the orientation is acyclic. Its boundary source labels are

$$I_0 = \{0,1,2,3,4,7,8,11,12,15,16,20\}. \tag{4.3}$$

Every internal vertex has both an incoming and an outgoing arc.

For a matching M, define $J(M) \subseteq [0,23]$ to consist of the labels j for which b_j is white with its leaf edge selected, or black with its leaf edge unselected. There are 24 internal vertices and 12 boundary vertices of each color. Since every internal vertex is matched, the numbers of occupied boundary vertices of the two colors are equal; hence $|J(M)| = 12$.

At an internal white vertex, the reference edge is the unique outgoing edge; at an internal black vertex it is the unique incoming edge. The symmetric difference $M \triangle M_0$ therefore has degree zero or two at each internal vertex, and its nontrivial components follow the orientation. Acyclicity makes them paths. Their source labels are $I_0 \setminus J(M)$ and their sink labels are $J(M) \setminus I_0$. Conversely, each directed path uses one reference and one nonreference edge at every internal vertex. Flipping its edges replaces one selected incident edge by one selected incident edge. This recovers a matching, and the two operations are inverse. On all 84 coordinates the correspondence is

$$f_e = \begin{cases} \chi_M(e), & e \notin M_0, \\ 1 - \chi_M(e), & e \in M_0, \end{cases} \tag{4.4}$$

where $\chi_M(e) \in \{0,1\}$ records whether $e \in M$. This is an invertible integral affine map.

For any of the directed graphs below, a *flow* is a nonnegative real vector on its arcs whose total incoming and outgoing values agree at each internal vertex. The flow of a path family records its arc use and has total incoming value at most 1 at every internal vertex. When the boundary source set is I and $J \subseteq [0,23]$ has size 12, a *path family for* J uses exactly the sources $I \setminus J$ and sinks $J \setminus I$.

4.2. **Boundary bridges.** Suppose that the current boundary labels have source set I. Choose $1 \le j \le 23$ with $j \in I$ and $p = j - 1 \notin I$. Replace the ports at labels p, j by new ports with the same boundary labels, and add

$$\text{new } p \longrightarrow \text{old } p \longrightarrow \text{old } j \longrightarrow \text{new } j. \tag{4.5}$$

The middle arc is called the bridge. Draw these arcs in a thin outer collar between the adjacent ports. The new source set is $I' = (I \setminus \{j\}) \cup \{p\}$, and the embedding remains planar.

Always choose the least possible j. If no such j exists, membership in I is closed under predecessors, so the 12-element set I equals $[0,11]$. Each step decreases the sum of its labels by one. The initial sum is 99, and the minimum is $0 + \cdots + 11 = 66$; thus the procedure terminates after exactly 33 steps. Each step adds two vertices and three arcs, giving a graph G with 138 vertices and 183 arcs.

Assign $\phi(b_j) = j$ and assign the internal values in Table 1. The edge definitions and (4.1) give $\phi(\text{head}\, e) \ge \phi(\text{tail}\, e)$ on every original arc. Give each new port its boundary-label value. The two short arcs in (4.5) have constant potential and the middle arc increases it by one, excluding a cycle containing a bridge. After all middle bridges are deleted, the new arcs only extend the boundary ports of the original acyclic graph by chains. Thus every stage is acyclic.

i	0	1	2	3	4	5	6	7	8	9	10	11
$\phi(V_i)$	19	6	6	6	10	10	14	14	18	18	19	19
$\phi(W_i)$	19	6	5	6	9	10	13	14	17	18	19	22
$\phi(X_i)$	19	5	5	6	9	10	13	14	17	18	21	22
$\phi(Y_i)$	6	5	5	9	9	13	13	17	17	19	21	23

TABLE 1. An integer potential for the original arcs and all boundary bridges.

Lemma 4.1. *For a fixed boundary set J, the operation* (4.5) *gives a bijection from the old path families for J to the new path families for J that use the middle bridge.*

Proof. Let $\mathbf{1}_J(i)$ equal 1 for $i \in J$ and 0 otherwise. Write $\varepsilon_p = \mathbf{1}_J(p)$ and $\varepsilon_j = \mathbf{1}_J(j)$. The old sink p receives flow ε_p, and the old source j emits flow $1 - \varepsilon_j$. Assign to the three new arcs, in the order in (4.5), the values

$$(1 - \varepsilon_p,\ 1,\ \varepsilon_j). \tag{4.6}$$

At the old port p, the incoming total becomes $\varepsilon_p + (1 - \varepsilon_p) = 1$; at the old port j, the outgoing total becomes $(1 - \varepsilon_j) + \varepsilon_j = 1$. Thus both converted ports have unit incoming and outgoing flow, and every other internal capacity is unchanged. The new boundary values are those for $I' \setminus J$ and $J \setminus I'$. Acyclicity and unit vertex capacities give a unique decomposition into vertex-disjoint paths.

Conversely, a new family using the middle bridge has value 1 there. Its boundary values force the two short-arc values in (4.6). Conservation then leaves incoming value ε_p at the old sink and outgoing value $1 - \varepsilon_j$ at the old source. Deleting the three new arcs and the two new ports recovers the old family. These operations are inverse. □

Let J_{ab} be the boundary set $J(M)$ common to the five matchings associated with the pair (a, b). Apply the lemma successively to each of these matchings for every pair in (3.5). Let $f^0_{ab}, f^1_{ab} \in \{0, 1\}^{183}$ be the flows obtained from its bottom and top matchings. Define

$$h = \frac{1}{2} \sum_{(a,b)\in\mathcal{E}} (f^0_{ab} + f^1_{ab}). \tag{4.7}$$

This is integral on all 183 arcs. On the original arcs it equals m_e off M_0 and $6 - m_e$ on M_0, with m from (3.6). On every bridge it equals 6. On the adjacent short arcs it equals, respectively,

$$\sum_{(a,b)\in\mathcal{E}} (1 - \mathbf{1}_{J_{ab}}(p)), \qquad \sum_{(a,b)\in\mathcal{E}} \mathbf{1}_{J_{ab}}(j).$$

Bottom and top in one factor have the same J_{ab}, so these are their midpoint values. Nonnegativity follows from (4.7).

Proposition 4.2. *There is no decomposition of h as a sum of six path-family incidence vectors in G, one with each boundary condition J_{ab}.*

Proof. An individual path family uses each bridge at most once. Since h has value 6 on every bridge, any putative decomposition must use every bridge in every factor. Apply Lemma 4.1 in reverse order. Then invert (4.4) on the six factors. This gives a decomposition of m by the six prescribed matchings, contradicting Lemma 3.2. □

5. A grid construction preserving vertex capacities

Construct a directed grid network from G as follows. Every edge points rightwards or upwards; a segment is the union of its unit lattice edges.

5.1. **Vertex numbers and arc labels.** Give V_i, W_i, X_i, Y_i, b_j the respective vertex numbers

$$i, \quad 12 + i, \quad 24 + i, \quad 36 + i, \quad 48 + j. \tag{5.1}$$

Number the original edges, starting with zero, in the following order:

$$C_0, \ldots, C_{11}; \quad R_0, \ldots, R_{11}; \quad (A_i, B_i, D_i)_{i=0}^{11}; \quad (\xi_i, \eta_i)_{i=0}^{11}. \tag{5.2}$$

Keep these labels after orientation. At bridge step k, starting with $k = 0$, give the new source and sink the vertex numbers $72 + 2k, 73 + 2k$ and give the three arcs in (4.5) the labels

$84+3k, 85+3k, 86+3k$ in that order. The new ports retain the boundary labels p, j. The least-j rule above determines all ports and incidences.

Process the 114 nonboundary vertices of G in the following order:

$$\begin{gathered}25,37,14,26,38,53,72,75,76,79,84,87,92,95,104,107,116,119,134,\\36,13,1,2,3,15,27,54,55,73,56,77,82,85,90,93,102,105,114,117,132,\\39,16,28,40,57,80,83,88,91,100,103,112,115,130,4,5,17,29,58,59,81,\\60,89,98,101,110,113,128,41,18,30,42,61,96,99,108,111,126,6,7,19,\\31,62,63,97,64,109,124,43,20,32,44,65,122,8,9,21,33,66,120,24,12,\\0,11,10,22,45,67,68,34,46,23,35,47.\end{gathered} \tag{5.3}$$

The initial ordered active arcs, one from each final source port, are

$$(60,61,62,63,64,87,93,105,117,135,153,180). \tag{5.4}$$

At a processed vertex list its outgoing arcs increasingly, except at vertices $11, 12, 22$, where the order is reversed. Its incoming arcs form a consecutive block in the active list. Replace that block by the outgoing list. Table 3 records all 114 replacements. Starting from (5.4), each row lists the consecutive incoming block at the stated position and its replacement. Thus every incoming arc is present when its vertex is processed, and the last active list is

$$(83,82,81,158,161,164,167,170,173,176,179,182). \tag{5.5}$$

Its heads are the final sink ports with boundary labels $23, 22, \dots, 12$. In particular, (5.3) is a topological ordering of the internal vertices.

5.2. **The tile rule.** At the beginning set the horizontal position $x = 0$ and put the twelve active arcs in rows $1, \dots, 12$, in their order. Suppose a vertex is to be processed. Replace its incoming block as above to form the new active list. If the largest old row is R, give the new list consecutive rows starting at $R + 2$.

Partition the transition into ordered components: each unchanged active arc is a component with one input and one output, and the processed vertex is the single component containing its incoming and outgoing blocks. Suppose there are c components. The new horizontal boundary is $x' = x + c + 1$. Number components $j = 1, \dots, c$ in active-list order, and place the vertical segment of component j in column

$$x + c - j + 1. \tag{5.6}$$

For this component let $a_1 < \cdots < a_r$ be its old rows and $b_1 < \cdots < b_s$ its new rows. Draw horizontal segments from the old boundary to the vertical column at the a_i, horizontal segments from the column to the new boundary at the b_i, and the vertical segment from a_1 to b_s. Subdivide the vertical segment at all a_i, b_i. Then continue with x' and the new active list.

For a flow f on G, the old and new horizontal segments carry the flow on their arc. The vertical segment between successive incoming rows carries the sum of the incoming flows at or below it; the segment from a_r to b_1 carries their total; the segment between b_i and b_{i+1} carries the sum of the outgoing flows at $b_{i+1}, \dots, b_s$. Thus each grid-edge flow is an integer linear function of f.

Lemma 5.1. *The tile construction gives a bijection between the vertex-disjoint path families in G and those in the constructed grid network with the corresponding terminals. The map on complete edge flows is integral linear and injective.*

Proof. First consider one component. There is a unique directed route from each incoming port to each outgoing port, and every such route contains the common vertical segment from a_r to b_1. Consequently two paths cannot traverse the processed vertex simultaneously. For an unchanged arc the route is unique as well.

Different components do not intersect. Their old ports and new ports occur in the same order, all new rows exceed all old rows, and their vertical columns occur in the reverse order. For two components in list order, the vertical column of the first lies to the right of that of the second. The old horizontal segments of the second start above the old ports of the first and stop before reaching its column; the new horizontal segments of the first start after the second column and lie below the new ports of the second. These inequalities exclude a crossing. Vertical segments have distinct columns. Consecutive tiles share only the corresponding boundary ports.

Every directed path crosses successive tile boundaries through active ports. Within a tile it stays in one component and follows the unique route between its incoming and outgoing ports. Hence contraction gives a path in G. The common vertical segments enforce the original vertex capacities, and the separation of components preserves disjointness. Conversely, replacing every passage through a vertex by its local route gives a disjoint grid family. The two operations are inverse. Every original arc has a private horizontal segment on which its flow can be read, proving injectivity. The displayed vertical-flow formulas prove integrality and linearity. □

For the data (5.3)–(5.4), the last row is $H = 1639$ and the last horizontal boundary is 1570. Thus the possible vertical columns are $1, \dots, T$, where $T = 1569$. These numbers can also be obtained without constructing segments. If m_ν is the active-list length after the ν-th processed vertex, and c_ν its number of components, then

$$H = 12 + \sum_{\nu=1}^{114} (m_\nu + 1) = 1639, \qquad T = \sum_{\nu=1}^{114} (c_\nu + 1) - 1 = 1569. \tag{5.7}$$

The m_ν values $12, 13, 14$ occur $13, 57, 44$ times, and the c_ν values $12, 13$ occur $26, 88$ times. Thus

$$H = 12 + 13 \cdot 13 + 57 \cdot 14 + 44 \cdot 15, \qquad T = 26 \cdot 13 + 88 \cdot 14 - 1.$$

Applying the integral map of Lemma 5.1 to (4.7) gives a nonnegative integral full-flow midpoint in the grid, with no decomposition into the six prescribed families.

6. Transfer to the fixed standard matrix network

Take

$$N = H + T - 1 = 3207 \tag{6.1}$$

in (2.1). Each of the first T sweeps contains all chips $1, \dots, H - 1$. Chip time k denotes the section after the first k elementary factors. For $1 \le s \le T$, represent the grid vertex in column s and row i on rail i at chip time

$$(s-1)N - \frac{(s-1)s}{2} + i - 1.$$

This is after chip $i - 1$ and before chip i of sweep s whenever those chips exist. In particular, the description includes the top rail at the last sweep, where chip H does not exist. Grid columns 0 and $T + 1$ are the terminal horizontal extensions. The vertical edge from row i to row $i + 1$ uses chip (s, i). A horizontal grid edge follows its rail between successive sweeps.

This identification preserves the capacity at a grid vertex: the incoming chip $i - 1$ ends on rail i immediately before the outgoing chip i begins. Any intervening rail segment is a subdivision common to the relevant routes. Other chips only subdivide the retained horizontal rails or add

weighted edges outside the selected grid. Extend source and sink paths along their rails to the boundaries of the full upper network, and extend them horizontally through L and through the diagonal edges. This specifies ordinary minors Q_{ab} in the full matrix A.

For a real vector f on directed edges, define its divergence at a vertex v by

$$\operatorname{div} f(v) = \sum_{e:v\to w} f_e - \sum_{e:w\to v} f_e.$$

Path-family flows have zero divergence at internal vertices.

Lemma 6.1 (Horizontal-forest lemma)**.** *In the full standard matrix network, the values on weighted edges and the divergence at every vertex determine a real edge vector uniquely, whenever it exists. If the specified values and divergences are integers, the complete vector is integral.*

Proof. Delete the weighted edges. What remains is a disjoint union of horizontal paths, hence an undirected forest. The difference of two vectors with the stated data is supported on this forest and has zero divergence. At a leaf its only incident edge must carry zero difference. Deleting leaves successively proves uniqueness.

For integrality, subtract the specified weighted-edge contributions from each vertex divergence. The resulting integer data are the divergences required on the forest. At a leaf the only remaining edge value equals that integer or its negative, according to its orientation. Delete that edge and adjust the divergence at its other endpoint. Induction proves that every edge value is integral. □

Let u be the weighted-edge projection of the complete midpoint of the twelve embedded endpoint families. On lower edges its coordinates are zero. On diagonal edges they are the source multiplicities in the six factors. On upper edges they are the integer grid flows just defined; all other upper coordinates are zero. The complete midpoint, including its horizontal edges, is integral: inside the grid this follows from Lemma 5.1; the terminal extensions have the fixed integral boundary counts.

Proposition 6.2. *With $Q = \prod_{(a,b)\in\mathcal{E}} Q_{ab}$ one has $[\mathbf{t}^u]Q = 0$. In each factor the embedded bottom and top paths give distinct monomials, and the sum of their twelve exponent vectors is $2u$.*

Proof. Suppose a term of Q had weighted exponent u. Its tuple of six families has the same aggregate boundary divergence as the embedded endpoints. Lemma 6.1 forces its complete edge flow to equal their midpoint. Every edge outside the retained grid and its terminal extensions has midpoint flow zero. Since path flows are nonnegative, no path in the tuple can use such an edge. In particular, no lower chip is used. Contract the terminal extensions and grid routes. Lemma 5.1 produces a decomposition of h in G, contrary to Proposition 4.2.

The endpoint identity follows from the definition of u. The bottom and top matching states are distinct, so their flows in G are distinct, and the grid map is injective. If their weighted monomials were equal in one factor, the horizontal-forest lemma with that factor's boundary divergence would make their full flows equal. This contradiction proves distinctness. □

7. The explicit ratio and the proof of the theorem

The final source with boundary label i occupies grid row $i+1$ for $0 \le i \le 11$. The final sink with boundary label j occupies row $H + 12 - j$ for $12 \le j \le 23$. For each pair (a, b), take the sources in $[0, 11] \setminus J_{ab}$ and the sinks in $J_{ab} \setminus [0, 11]$. The resulting minors $Q_1, \dots, Q_6$, in the order (3.5), are given in Table 2. The sets J_{ab} themselves can either be read from the boundary bits in Section 3 or calculated by the color rule in Section 4.

Factor	Row set	Column set
Q_1	$2,4,6,9,10$	$1629,1632,1633,1636,1637$
Q_2	$1,2,4,6,8,10$	$1628,1629,1632,1633,1636,1637$
Q_3	$2,5,6,8,10$	$1629,1632,1633,1636,1637$
Q_4	$1,2,5,6,9,10,12$	$1628,1629,1632,1633,1635,1637,1639$
Q_5	$1,2,5,6,9,10$	$1628,1629,1631,1633,1635,1637$
Q_6	$1,2,5,6,9,10,12$	$1628,1629,1631,1633,1636,1637,1639$

TABLE 2. The row and column sets of $Q_1,\dots,Q_6$.

Compute the integer flow h by (3.3), (3.2), (3.6), (4.4), and (4.6). Transport it through the tile rule. For each vertical segment in column s, from row l to row $r+1$, let v be its flow. It contributes the exponent v to $t_{s,i}$ for $l \le i \le r$. If adjacent segments occur in the same column their contributions are added. Let u_{d_i} be the source flow in row i, with value zero on rows not among the twelve initial rows.

In the free abelian group on the symbols D_r and $\mathsf{C}_{a,k}$, with the boundary identifications after (2.3), form

$$\begin{aligned}\mathcal{A} = &\sum_{i=1}^{N} u_{d_i}(\mathsf{D}_i - \mathsf{D}_{i-1}) \\ &+ \sum_{(s,l,r,v)} v\big(\mathsf{C}_{l,s} - \mathsf{C}_{r+1,s} + \mathsf{C}_{r+2,s-1} - \mathsf{C}_{l+1,s-1}\big).\end{aligned} \tag{7.1}$$

The second sum is over all vertical segments produced by the tile rule. Each chamber minor is a formal generator, and its coefficient records an exponent. Set $\mathsf{D}_0 = 0$ in this formal group, since the corresponding minor is the constant 1, and collect equal symbols. If the resulting coefficient of a chamber minor F is a_F, define

$$P = \prod_{a_F>0} F^{a_F}, \qquad M = \prod_{a_F<0} F^{-a_F}. \tag{7.2}$$

To see directly why (7.1) gives the required exponents, multiply (2.5) over $l \le i \le r$ in one sweep. The factors telescope to

$$\prod_{i=l}^{r} t_{s,i}^{v} = \left(\frac{\mathsf{C}_{l,s}\mathsf{C}_{r+2,s-1}}{\mathsf{C}_{r+1,s}\mathsf{C}_{l+1,s-1}}\right)^{v}. \tag{7.3}$$

Thus $P/M = \mathbf{t}^{u}$.

Proof of Theorem 1.1. Proposition 6.2 gives all hypotheses of Lemma 2.3 with $r = 6$. The products defined by (7.2) are exactly those obtained by its chamber clearing. Therefore

$$0 < P/(MQ) \le 2^{-6}, \qquad [\mathbf{t}^{\nu(P)}](MQ - P) = -1.$$

All factors in P, M, Q are ordinary minors of the same matrix of order 3207. This proves both assertions. □

Remark 7.1. The tile construction has 4596 segments, of which 1570 are vertical. Formula (7.1) gives 1642 distinct chamber minors in P and 1650 in M, with respectively 4842 and 4838 factors counted with multiplicity. Their degrees in the matrix entries are 7 675 713 and 7 675 677; the degree of Q is 36.

8. Cancellation and rational expressions

Proposition 8.1. *After cancellation of the common polynomial factors of P and MQ in the standard coordinates, the reduced denominator-minus-numerator has a negative coefficient. The function $1 - P/(MQ)$ is a quotient of polynomials with nonnegative coefficients.*

Proof. The polynomial P is a coefficient-one monomial. Every polynomial divisor of P is therefore a monomial up to a nonzero scalar. Choose the common monomial factor of P and MQ with coefficient 1. Dividing $MQ - P$ by it translates all exponents and preserves the coefficient -1.

Let B be the product of the six bottom endpoint monomials and T the product of the six top endpoint monomials. Both occur in Q with coefficient at least 1, and $BT = \mathbf{t}^{2u}$. If $B = T$, their equal exponent vectors would be u, contradicting $[\mathbf{t}^u]Q = 0$. Thus B, T are distinct. In Q^2 the two ordered cross-products BT and TB contribute at least 2 to the coefficient of $\mathbf{t}^{2u}$. All other coefficients are nonnegative. Consequently both numerator and denominator in

$$1 - \frac{P}{MQ} = 1 - \frac{\mathbf{t}^u}{Q} = \frac{Q^2 - \mathbf{t}^{2u}}{Q(Q + \mathbf{t}^u)}$$

have nonnegative coefficients. □

Appendix A. The active-list replacements

In Table 3, ν is the processing step, v is the vertex number, and p is the first position of its incoming block in the preceding active list, counted from 1. The columns "In" and "Out" give that block and its ordered replacement. Replacing the indicated block in each successive row, starting from (5.4), verifies the order and incidence conditions used in Section 5.

ν	v	p	In	Out	ν	v	p	In	Out	ν	v	p	In	Out
1	25	3	62	27,28	39	117	12	152,177	178	77	111	9	143,168	169
2	37	4	28,63	29	40	132	12	178	174,179	78	126	9	169	165,170
3	14	4	29	14,30	41	39	5	34,67	35	79	6	4	5,18	6
4	26	5	30,64	31	42	16	5	35	16,36	80	7	4	6	7,19
5	38	5	31	32,65	43	28	6	36,68	37	81	19	5	19,44	45
6	53	6	65,87	88	44	40	6	37	38,69	82	31	5	45	46,74
7	72	6	88	84,89	45	57	7	69,99	100	83	62	6	74,120	121
8	75	7	89,93	94	46	80	7	100	96,101	84	63	6	121	75,122
9	76	7	94	90,95	47	83	8	101,111	112	85	97	7	122,138	139
10	79	8	95,105	106	48	88	8	112	108,113	86	64	7	139	76,140
11	84	8	106	102,107	49	91	9	113,129	130	87	109	8	140,165	166
12	87	9	107,117	118	50	100	9	130	126,131	88	124	8	166	162,167
13	92	9	118	114,119	51	103	10	131,147	148	89	43	5	46,75	47
14	95	10	119,135	136	52	112	10	148	144,149	90	20	5	47	20,48
15	104	10	136	132,137	53	115	11	149,174	175	91	32	6	48,76	49
16	107	11	137,153	154	54	130	11	175	171,176	92	44	6	49	50,77
17	116	11	154	150,155	55	4	4	3,16	4	93	65	7	77,162	163
18	119	12	155,180	181	56	5	4	4	5,17	94	122	7	163	159,164
19	134	12	181	177,182	57	17	5	17,38	39	95	8	4	7,20	8
20	36	2	61	25,26	58	29	5	39	40,70	96	9	4	8	9,21
21	13	3	26,27	13	59	58	6	70,96	97	97	21	5	21,50	51
22	1	3	13	0,1	60	59	6	97	71,98	98	33	5	51	52,78
23	2	4	1,14	2	61	81	7	98,108	109	99	66	6	78,159	160
24	3	4	2	3,15	62	60	7	109	72,110	100	120	6	160	156,161
25	15	5	15,32	33	63	89	8	110,126	127	101	24	1	60,25	24
26	27	5	33	34,66	64	98	8	127	123,128	102	12	1	24	59,12
27	54	6	66,84	85	65	101	9	128,144	145	103	0	2	12,0	11
28	55	6	85	67,86	66	110	9	145	141,146	104	11	2	11	23,10
29	73	7	86,90	91	67	113	10	146,171	172	105	10	3	10,9	22
30	56	7	91	68,92	68	128	10	172	168,173	106	22	3	22	54,53
31	77	8	92,102	103	69	41	5	40,71	41	107	45	4	53,52	79
32	82	8	103	99,104	70	18	5	41	18,42	108	67	4	79,156	157
33	85	9	104,114	115	71	30	6	42,72	43	109	68	4	157	80,158
34	90	9	115	111,116	72	42	6	43	44,73	110	34	3	54,80	55
35	93	10	116,132	133	73	61	7	73,123	124	111	46	3	55	56,81
36	102	10	133	129,134	74	96	7	124	120,125	112	23	2	23,56	57
37	105	11	134,150	151	75	99	8	125,141	142	113	35	2	57	58,82
38	114	11	151	147,152	76	108	8	142	138,143	114	47	1	59,58	83

Table 3. All 114 consecutive-block replacements.

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