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IRRATIONALITY OF χ_2 AND Ti_2 AT UNITS OF REAL QUADRATIC FIELDS

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ABSTRACT. We prove that $\chi_2(\sqrt{m^2+1}-m)$, a value of Legendre's chi function at a unit of norm -1, is irrational for every integer $m \geq 2$ except $m = 3, 5, 7, 9, 15$. For $m = 2$ this shows that $\pi^2 - 18\log^2\varphi$ is irrational, where φ is the golden ratio. The method extends to all units of norm -1 in real quadratic fields, giving $\chi_2(\varepsilon) \notin \mathbb{Q}$ outside an explicit set of 47 units, and, through a twist identity, to the inverse tangent integral Ti_2 at units of norm $+1$, outside 57 units. The proof combines Hankel determinants of a reflection-symmetric Padé machine, arithmetic bounds on their denominators, a discrete log-gas bound certified in interval arithmetic, and a nonvanishing argument modulo a prime. Below the range covered by the criteria of David, Hirata-Kohno and Kawashima, these results appear to be new. All three theorems are formalised in Lean 4 with Mathlib, using only the standard axioms.

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1. Introduction

For $|q| \le 1$ let

$$\chi_2(q) = \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(2n+1)^2} = \tfrac{1}{2}\big(\mathrm{Li}_2(q) - \mathrm{Li}_2(-q)\big)$$

be Legendre's chi function (see [18]). For an integer $m \ge 1$ put

$$z_m := \sqrt{m^2+1} - m \in (0,1).$$

Then $z_m^2 + 2mz_m - 1 = 0$, so z_m is a unit of norm -1 in the real quadratic field $\mathbb{Q}(\sqrt{m^2+1})$, with Galois conjugate $-1/z_m$. Put $\mathcal{E} := \{3,5,7,9,15\}$.

Theorem 1.1 (Theorem V)**.** *For every integer $m \ge 2$ with $m \notin \mathcal{E}$, the number $\chi_2(\sqrt{m^2+1} - m)$ is irrational.*

For $m \ge 4\,346\,237$ the stronger statement $\chi_2(z_m) \notin \mathbb{Q}(\sqrt{m^2+1})$ follows from the criterion of David, Hirata-Kohno and Kawashima [4, Thm. 1] (our evaluation, Section 9.2). Their earlier criterion [3, Thm. 2.1], whose proof is given there only in outline and which [4, Rem. 1(iii)] describes as announced without proof, would give the same conclusion already for $m \ge 2\,517\,046$. What Theorem 1.1 adds is the range $2 \le m \le 2\,517\,045$ with $m \notin \mathcal{E}$, where we know of no earlier result, and, if the announced result is set aside, the range up to $m = 4\,346\,236$. The proof is computer-assisted: it uses one dual certificate checked in interval arithmetic (at $m = 2$) and a table of margins for the odd $m \le 65$, also computed in interval arithmetic. The case $m = 2$ is the golden-ratio statement. Let $\varphi = (1+\sqrt{5})/2$.

Corollary 1.2 (Theorem U)**.** *The number*

$$\pi^2 - 18\log^2\varphi = 5.701437630699859553 88\ldots$$

is irrational. Equivalently, the number

$$\chi_2(\sqrt{5}-2) = \frac{\pi^2}{24} - \frac{3}{4}\log^2\varphi = 0.237559901279160814 74\ldots$$

is irrational.

The two forms are equivalent by the identity

$$\chi_2(\sqrt{5}-2) = \frac{\pi^2}{24} - \frac{3}{4}\log^2\varphi, \tag{1.1}$$

which follows from Landen's identity $\chi_2\big(\frac{1-x}{1+x}\big) + \chi_2(x) = \frac{\pi^2}{8} + \frac{1}{2}\log x \,\log\frac{1+x}{1-x}$ (see [18]) at $x = \sqrt{5}-2 = \varphi^{-3}$, for which $(1-x)/(1+x) = \varphi^{-1}$, together with the classical value $\chi_2(\varphi^{-1}) = \pi^2/12 - \frac{3}{4}\log^2\varphi$. Theorem 1.1, Corollary 1.2 and the identity (1.1) are formalised in Lean (Section 11), and so are Theorems 1.3 and 1.5 below.

Theorem 1.1 covers every even $m \geq 2$ and every odd $m \geq 11$ except 15. For example,

$$\chi_2(\sqrt{17}-4) \notin \mathbb{Q}, \qquad \chi_2(\sqrt{37}-6) \notin \mathbb{Q}, \qquad \chi_2(\sqrt{65}-8) \notin \mathbb{Q}.$$

For $m \geq 3$ these values are of a different kind from $\chi_2(\sqrt{5}-2)$: they have no expression through π^2 and products of logarithms by dilogarithm identities that preserve the symbol (Proposition 9.1). Only $m = 1$ and $m = 2$ have a Landen closed form. The method does not reach $m = 1$, where $\chi_2(\sqrt{2}-1) = \pi^2/16 - \frac{1}{4}\log^2(1+\sqrt{2})$, nor $m \in \mathcal{E}$ (Section 7.2).

Extensions. Every unit $\varepsilon \in (0,1)$ of norm -1 of a real quadratic field is

$$\alpha_a := \frac{\sqrt{a^2+4}-a}{2}, \qquad a := \varepsilon^{-1} - \varepsilon \in \mathbb{Z}_{\geq 1},$$

since ε and $-1/\varepsilon$ are the roots of X^2+aX-1; and $\alpha_{2m} = z_m$. So Theorem 1.1 is the case of even a. For odd a the prime 2 is inert in $\mathbb{Q}(\sqrt{a^2+4})$ and $1-\alpha_a^2 = a\alpha_a$ is a 2-adic unit; one new lemma at the prime 2 (Lemma 8.1) and ten further dual certificates give the following (Section 8.1). Let $\mathcal{E}_{\text{odd}}$ be the following set of 39 odd numbers:

$$\mathcal{E}_{\text{odd}} := \{5,7,9,\dots,57\} \cup \{63,65,69,75,77,81,85,87,93,99,105,165\}$$

Theorem 1.3 (Theorem V_{odd})**.** *For every odd integer $a \geq 5$ with $a \notin \mathcal{E}_{\text{odd}}$, the number $\chi_2\big((\sqrt{a^2+4}-a)/2\big)$ is irrational.*

Corollary 1.4 (units of norm -1)**.** *Let $\varepsilon \in (0,1)$ be a unit of norm -1 of a real quadratic field, and $a := \varepsilon^{-1}-\varepsilon$. If $a \notin \{1,3\} \cup \mathcal{E}_{\text{odd}} \cup \{2,6,10,14,18,30\}$, then $\chi_2(\varepsilon) \notin \mathbb{Q}$. The excluded set consists of 47 units.*

The corollary is Theorems 1.1 and 1.3 together ($a = 2m$ with $m \in \{1\} \cup \mathcal{E}$ gives the six even exceptions). The first odd a covered are $59, 61, 67, 71, 73, 79, 83, 89, 91, 95, \dots$, and every odd $a > 165$ is covered; for example $\chi_2\big((\sqrt{3485}-59)/2\big) \notin \mathbb{Q}$.

The same machine with the odd character χ_4 modulo 4 on the atoms reaches the units of norm $+1$. Let $\mathrm{Ti}_2(x) := \sum_{k\geq 0}(-1)^k x^{2k+1}/(2k+1)^2$ be the inverse tangent integral [18, Ch. 2], and let $\mathcal{E}_i$ be the following set of 17 numbers:

$$\mathcal{E}_i := \{75,77,81,85,87,91,93,95,99,105,111,117,123,135,147,165,195\}.$$

Theorem 1.5 (Theorem V_i)**.** (a) *For every integer $m \geq 4$ with $m \notin \{5,7,9,15\}$, the number $\mathrm{Ti}_2(m-\sqrt{m^2-1})$ is irrational.*

(b) *For every odd integer $n \geq 71$ with $n \notin \mathcal{E}_i$, the number $\mathrm{Ti}_2\big((n-\sqrt{n^2-4})/2\big)$ is irrational. In particular this holds for every odd $n \geq 197$.*

The points are the units $\beta_n := (n-\sqrt{n^2-4})/2 \in (0,1)$ of norm $+1$, with trace n ($n = 2m$ in (a)); every unit of norm $+1$ in $(0,1)$ is one of them. With $\theta := \arctan\beta_n$ one has $\sin 2\theta = 2/n$ and the classical formula (see [18])

$$\mathrm{Ti}_2(\tan\theta) = \theta\log\tan\theta + \tfrac{1}{2}\big(\mathrm{Cl}_2(2\theta) + \mathrm{Cl}_2(\pi-2\theta)\big),$$

so Theorem 1.5 concerns Clausen values at the angles $2\theta = \arcsin(2/n)$, which for $n \geq 5$ are not rational multiples of π (Section 9.3). An example is $\mathrm{Ti}_2(4-\sqrt{15}) \notin \mathbb{Q}$ ($\sin 2\theta = \frac{1}{4}$). The proof rests on the identity $\chi_4(t) = -i \cdot i^t$ for odd t: the χ_4-twisted machine at β_n is $-i$ times the machine of Theorem 1.1 at the complex point $i\beta_n$, which the conjugation fixing i sends to $-1/(i\beta_n)$ (Section 8.2). Not covered are $n = 3$, $n = 4$ (where $\mathrm{Ti}_2(2-\sqrt{3}) = \frac{2}{3}G - \frac{\pi}{12}\log(2+\sqrt{3})$ is classical, G Catalan's constant), $n = 6, 10, 14, 18, 30$ and the odd $n \leq 69$. So, outside an explicit set of 57 units, $\mathrm{Ti}_2(\varepsilon) \notin \mathbb{Q}$ for every unit $\varepsilon \in (0,1)$ of norm $+1$ of a real quadratic field.

As for Theorem 1.1, the criteria of David, Hirata-Kohno and Kawashima give stronger conclusions far out: [4, Thm. 1] for $a \geq 8\,692\,474$ and $n \geq 8\,692\,474$, and [3, Thm. 2.1], as stated and with its proof only outlined, already for $a \geq 5\,034\,092$ and $n \geq 5\,034\,092$ (Section 9.3). Each extension was checked by one AI referee. Theorems 1.3 and 1.5 are formalised in Lean, with only Lean's three standard axioms (Section 11). So, with Theorem 1.1, every unit covered by Corollary 1.4, and every unit of norm $+1$ outside the 57 just mentioned, is covered by a formal statement; the corollary and the statement on units of norm $+1$ are not themselves stated in Lean.

1.1. **Context.** *General m.* For $m \geq 4\,346\,237$, Theorem 1 of David, Hirata-Kohno and Kawashima [4] gives $\chi_2(z_m) \notin \mathbb{Q}(\sqrt{m^2+1})$, which is stronger than Theorem 1.1; this is our evaluation of their criterion (Section 9.2). Theorem 2.1 of their earlier paper [3], as stated, would give the same already for $m \geq 2\,517\,046$; its proof is only outlined there, and [4] describes it as announced without proof. So the infinite part of Theorem 1.1 is not new by itself. We have found no result that covers any m with $2 \leq m \leq 2\,517\,045$, and no proved result that covers any m with $2 \leq m \leq 4\,346\,236$. For $m \geq 3$ that search was short: our own notes, two web searches, and a reading of the criteria listed in Section 9.2; it did not cover MathSciNet, and no expert has been consulted. Our method also covers the ranges of these criteria independently, with the weaker conclusion.

The extensions. For Theorems 1.3 and 1.5 the searches were of the same limited kind (our own notes, two web searches for each, and for Ti_2 an earlier sweep of arXiv for irrationality results on related values); they found closed-form identities but no irrationality result for χ_2 at the units α_a with odd a, or for Ti_2 or Cl_2 at the points β_n, below the thresholds of [3] (announced) and [4] (proved) (Section 9.3).

The case $m = 2$. We have not found Corollary 1.2, or the irrationality of any number $\pi^2 + c\log^2\varphi$ with $c \in \mathbb{Q}\setminus\{0\}$, in the literature, and as far as we can see it does not follow from known results. By the analytic class number formula, $L(1,\chi_5) = 2\log\varphi/\sqrt{5}$ for the quadratic character χ_5 modulo 5, so the corollary can also be written as $6\zeta(2) - \frac{45}{2}L(1,\chi_5)^2 \notin \mathbb{Q}$.

- *The golden-ratio dilogarithm values are different numbers.* The classical evaluations

$$\operatorname{Li}_2(\varphi^{-2}) = \tfrac{\pi^2}{15} - \log^2\varphi, \qquad \operatorname{Li}_2(\varphi^{-1}) = \tfrac{\pi^2}{10} - \log^2\varphi,$$
$$\operatorname{Li}_2(-\varphi^{-1}) = -\tfrac{\pi^2}{15} + \tfrac{1}{2}\log^2\varphi, \qquad \operatorname{Li}_2(-\varphi) = -\tfrac{\pi^2}{10} - \log^2\varphi$$

  (Landen; see [18, 19]) all lie in $\mathbb{Q}\pi^2 + \mathbb{Q}\log^2\varphi$. Normalised to the form $\pi^2 + c\log^2\varphi$ they have $c = -15$, -10, $-\frac{15}{2}$ and $+10$, while Corollary 1.2 has $c = -18$. Since $\log\varphi$ is transcendental, $\pi^2 + c\log^2\varphi$ can be algebraic for at most one $c \in \overline{\mathbb{Q}}$; this does not say which. The irrationality of one member of the family says nothing about another, and as far as we know the irrationality of each of the four classical values is open.
- *Homogeneous statements.* The theorem of Gel'fond and Schneider [1, Ch. 2], applied to $e^{\log\varphi} = \varphi$ and $e^{i\pi} = -1$, shows that $\log\varphi/(i\pi)$ is transcendental, so $\log^2\varphi/\pi^2 \notin \overline{\mathbb{Q}}$. This says that π^2 and $\log^2\varphi$ are linearly independent over $\mathbb{Q}$; it says nothing about a relation that also involves 1. The only member of the family $\pi^2 + c\log^2\varphi$, $c \in \mathbb{Q}$, whose irrationality we know to be established is $c = 0$. The linear independence of $1, \pi^2, \log^2\varphi$ over $\mathbb{Q}$ is open as far as we know.
- *Criteria at algebraic points.* The linear independence criteria for polylogarithms and Lerch functions at algebraic points [3, 5, 4, 13], and the explicit form of the Bombieri–André principle of global relations [22], need a point that is small in comparison with its height. Here $\chi_2(\sqrt{5}-2) = \frac{1}{2}(\operatorname{Li}_2(z) - \operatorname{Li}_2(-z)) = z + \frac{z}{4}\Phi_2(\frac{1}{2}, z^2)$, where $\Phi_2(x,q) = \sum_{k\geq 0} q^{k+1}/(k+x+1)^2$ is the Lerch function and $z = \sqrt{5}-2 = \varphi^{-3}$ is a unit of height $\frac{3}{2}\log\varphi = 0.7218\ldots$ whose conjugate $-\varphi^3$ has absolute value > 1, so that $|z|\,|\sigma z| = 1$. Evaluating the criteria in the form they are stated, their key quantity is negative by a wide margin at this point: about -26.1 for [13, Thm. 2.2], about -15.1 and -14.5 for the two ways of applying [4, Thm. 1] (through $\operatorname{Li}_2(\pm z)$, or through $\Phi_2(\frac{1}{2}, z^2)$), and about -14.0 for [3, Thm. 2.1] (through $\Phi_2(\frac{1}{2}, z^2)$, normalised as in Section 9.2). The Bombieri–André bound of [22], applied to this system, needs by our estimate a height of about 1450. These evaluations are ours and have not been checked by the authors of the criteria.
- *Rational points.* The irrationality and linear independence results for dilogarithms at rational points [9, 10, 24, 25, 32], the recent claims [23, 27] (unrefereed), and our result $\chi_2(1/3), \chi_2(1/5) \notin \mathbb{Q}$ [15] all concern rational arguments, as do the weight-two results of Calegari, Dimitrov and Tang [2] (at sixth roots of unity, and for products $\log(1+1/m)\log(1+1/n)$) and Hata's irrationality of $\log(1+1/q)\log(1-1/q)$ [11]. The results on products of logarithms are homogeneous and contain no π^2.

- *The closest analogue.* The nearest result we know is $\mathrm{Li}_2(\frac{1}{2}) = \frac{\pi^2}{12} - \frac{1}{2}\log^2 2 \notin \mathbb{Q}$, proved in Lean by Q. Tang [28] and, with many more rational points, by S. Liu [20]; neither is accompanied by a refereed paper. It has the same shape, $\pi^2 + c\log^2$ of a number, with $c = -6$ and the number 2 instead of a unit. Over $\mathbb{Q}(\sqrt{5})$ we have found nothing comparable.

For $m = 2$ our searches for prior work, in four passes, covered the web, arXiv (through its API), zbMATH Open, OpenAlex, Crossref, Semantic Scholar (only partly, because of rate limits), the Hexagon repository, the `openai/math` manuscript list and recent Lean repositories (the last pass on 7 November 2026). They did not cover MathSciNet or Google Scholar, and no expert has been consulted. All searches, for $m = 2$ and for general m, were carried out by AI systems.

1.2. **The method.** Fix $m \geq 1$ and put

$$z := z_m = \sqrt{m^2+1} - m, \qquad y := z^2, \qquad T := \chi_2(z) = \sum_{t \text{ odd} \geq 1} \frac{z^t}{t^2}.$$

Then $z^2+2mz-1 = 0$, so $1-y = 2mz$, and the Galois conjugate of z is $-1/z$. For $m = 2$ this is $z = \sqrt{5}-2$ and $1-y = 4z$. The proof transposes the proof of [15] for $\chi_2(1/q)$ to these points. For even h we take the $K := h+1$ poles $x_k := 2(k - h/2)$, i.e. the even integers in $[-h, h]$, put $D(t) := \prod_k (t - x_k)$, and consider the functional

$$\Lambda(g) := \sum_{t \text{ odd} \geq 1} z^t \Big(\frac{g}{D}\Big)'(t) \qquad (g \in \mathbb{Q}[t]).$$

Partial fractions give $\Lambda = A + T\,B$ with $\mathbb{K}$-valued functionals A, B, where $\mathbb{K} := \mathbb{Q}(\sqrt{m^2+1})$, so the Hankel determinant $\det(\Lambda(t^{i+j}))_{i,j<h}$ is the value at $X = T$ of $\Delta_h(X) := \det(A(t^{i+j}) + XB(t^{i+j}))_{i,j<h}$, a polynomial of degree at most h.

What differs from [15] is the *reflection* (Lemma 2.2). The pole set is symmetric under $t \mapsto -t$, $D(-t) = -D(t)$, and the conjugation $\sigma\colon z \mapsto -1/z$ maps the partial sums of χ_2 at the poles x to those at $-x$. Together these give $\sigma(\Lambda_X(t^a)) = (-1)^a \Lambda_X(t^a)$, which forces $\Delta_h \in \mathbb{Q}[X]$ although its entries lie in $\mathbb{K}$. So we never pay for the conjugate place, where $|\sigma z| = 1/z > 1$ and the defining series diverges. The three inputs are then as in [15].

(1) *Denominators* (Sections 3–4). The moments of Λ have the denominators $(1-y)^{b+1} = (2mz)^{b+1}$. At odd primes $p \nmid m$ the elements z, y, $1-y$ are units, and the bounds do not depend on m. Primes $p < K/3$ obey a plateau bound $2h\,\mathrm{lev}^*(p) + v_p(h!)$, which we prove through a bordered version of the Lagrange Gram identity (Lemmas 3.4 and 3.5), and primes $K/3 \leq p < K$ obey a class count $3(K-p)$ (Lemmas 3.2 and 3.3). The prime number theorem sums these to $\frac{2}{3} + \frac{2}{3} = \frac{4}{3}$ per h^2 (Theorem 3.7). An odd prime p with

$e := v_p(m) \geq 1$ costs at most $c_p(e) := \left(\frac{e}{4} + \frac{1}{4(p-1)}\right)\log p$ per h^2 (Lemma 3.8). The prime 2 costs at most $c_2(\nu) := |\nu - 2|\frac{\log 2}{4}$ per h^2, where $\nu := 1 + v_2(m)$ is the 2-adic valuation of $1 - y$ (Lemma 4.1); for $m = 2$, $\nu = 2$ and the cost is only $O(h \log h)$ (Lemma 4.3).

(2) *Size* (Section 5). A discrete log-gas argument bounds $h^{-2}\log|\Delta_h(T)|$ by the value Φ_z of any dual certificate for a constrained equilibrium problem with external field $x\log z - \frac{1}{2}\int_{x-1}^{x+1}\log|s|\,ds$ (Theorem 5.1). An explicit certificate at $z = \sqrt{5} - 2$, checked in interval arithmetic, gives $\Phi \leq \Phi_U := -1.3592161834586374$ (Theorem 5.3). A shift lemma (Lemma 5.5) moves it to z_m and gives $\Phi \leq \Phi_U - \frac{1}{2}L(m)$ with $L(m) := \log\big((m + \sqrt{m^2+1})/(2+\sqrt{5})\big)$ (Corollary 5.6).

(3) *Nonvanishing* (Section 6). For $h = 3\ell - 1$ with $\ell \geq 7$ prime and $\ell \nmid m$, the Gram matrix of Λ in the Lagrange basis is block diagonal modulo a prime of $\mathbb{K}$ above ℓ, with 3×3 blocks in closed form, so $\Delta_h(a/b) \neq 0$ whenever $\ell^2 \nmid b$ (Lemma 6.1).

The proof of Theorem 1.1 therefore shows

(1.2)
$$\limsup_h \frac{1}{h^2}\log|F_h(T)| \leq M(m) := \Phi_U - \tfrac{1}{2}L(m) + \tfrac{4}{3} + c_2(\nu) + \sum_{p^e \| m,\ p \text{ odd}} c_p(e)$$

along a suitable infinite set of h, for primitive $F_h \in \mathbb{Z}[X]$ of degree $\leq h$ with $F_h(T) = \Delta_h(T) \cdot (\text{integer})$, and $M(m) < 0$ exactly when $m \geq 2$ and $m \notin \mathcal{E}$ (Section 7.2). Table 1 shows the budget for $m = 2$ and $m = 4$.

contribution (per h^2, lim sup)	source	$m = 2$	$m = 4$
odd primes $p \nmid m$	Thm. 3.7	1.333333	1.333333
$p = 2$	Lemmas 4.3, 4.1	0	0.173287
odd primes $p \mid m$	Lemma 3.8	none	none
arithmetic		1.333333	1.506620
analytic Φ	Cor. 5.6	≤ -1.359216	≤ -1.684754
margin $M(m)$		≤ -0.025882	≤ -0.178134

TABLE 1. The exponential budget for $m = 2$ (Corollary 1.2) and $m = 4$ ($\chi_2(\sqrt{17} - 4)$). For $m = 2$ the analytic value is the upper endpoint -1.3592161834586374 of the certified Arb ball, rounded up, and the margin is $-0.0258828501\ldots$, rounded up; for $m = 4$ the values $-1.6847547\ldots$ and $-0.1781346\ldots$ are rounded up. Other values of m are in Table 4.

For $m = 2$ the margin is about 1.9% of the arithmetic part. It is smaller than in [15] for $q = 3$ in relative terms, but no constant in the budget is estimated: each is a proved upper bound. The margins vary with m;

among $m \notin \mathcal{E}$ the smallest in absolute value are $m = 11$ (-0.0179), $m = 21$ (-0.0204) and $m = 2$ (-0.0259).

1.3. **The formalisation.** Theorem 1.1, Corollary 1.2 and the identity (1.1) are formalised in Lean 4 [6] with Mathlib [30] and checked by the Lean kernel. The headline theorems are `Chi2U.V.theoremV` (Theorem 1.1), `Chi2U.chi2_sqrt5_sub_two_irrational` ($\chi_2(\sqrt{5}-2) \notin \mathbb{Q}$), `Chi2U.golden_irrational'` ($\pi^2 - 18\log^2\varphi \notin \mathbb{Q}$) and `Chi2U.chi2_sqrt5_sub_two_eq` (the identity (1.1)). They depend only on Lean's three standard axioms (`propext`, `Classical.choice`, `Quot.sound`); there is no `sorry` and no `native_decide`. The formal proof of the analytic bound follows the lattice-energy route of [15], not the route of Section 5, and uses its own, coarser certificate; Section 11 describes both differences. For general m the formal proof transports this certificate by the shift lemma; the exception set is the same $\mathcal{E}$. Theorems 1.3 and 1.5 are formalised as well (`Chi2U.VOdd.theoremVOdd` and `Chi2U.Ti.theoremVi`, with the same three axioms), with the same exception sets $\mathcal{E}_{\mathrm{odd}}$ and $\mathcal{E}_i$: Theorem 1.3 with eleven direct certificates of its own instead of ten, Theorem 1.5 with the certificate of Corollary 1.2 and the shift lemma alone. Corollary 1.4 is not stated in Lean.

1.4. **Two points.** Section 10 transposes the proof once more, to units of the biquadratic fields $\mathbb{Q}(z)$ where z is a root of $X^4+nX^3+2X^2-nX+1$. There the reflection removes the two conjugates $-1/z$ and $-1/w$, where the series diverge, and the remaining conjugate w is a second convergent place, which enters through a norm. The result is a disjunction, proved separately for six values of n, with interval-arithmetic certificates. The smallest instance is the following.

Theorem 1.6 (M1 at $n = 730$)**.** *Let $z > w$ be the two roots in $(0,1)$ of $X^4 + 730X^3 + 2X^2 - 730X + 1$. Then at least one of the following holds:*

$$\frac{\pi^2}{8} - \frac{1}{8}\log^2\bigl(366+\sqrt{133955}\bigr) + \frac{1}{8}\log^2\bigl(364+11\sqrt{1095}\bigr) \notin \mathbb{Q},$$

$$\chi_2(z) - \chi_2(w) \notin \sqrt{1101}\,\mathbb{Q}.$$

The first number is $\chi_2(z) + \chi_2(w)$, by Landen's identity. Theorem 1.6 is not formalised.

1.5. **Organisation and notation.** An earlier version of this paper treated only $m = 2$. Section 2, Sections 3.1–3.3, Sections 5.1–5.3 and Section 6 are that version's arguments with $z = z_m$: they use only $\sigma(z) = -1/z$ and that z, y, $1-y$ are units at the primes concerned, and they hold verbatim for every m except where the text says otherwise. New in this version are Lemma S (Section 3.4), Lemma 10_ν (Section 4), the shift lemma (Section 5.4), the hypothesis $\ell \nmid m$ in Lemma P′, and the margins (Section 7.2). The latest revision adds Section 8 (Theorems 1.3 and 1.5) and Section 9.3; it changes no statement about Theorem 1.1, Corollary 1.2 or Theorem 10.2.

Section 2 sets up the functional, proves the pole expansion and the reflection, and states the criterion. Sections 3 and 4 bound the denominators, Section 5 bounds the size, Section 6 proves nonvanishing, and Section 7 assembles the proof and computes the margins. Section 8 proves the two extensions as deltas to this proof. Section 9 discusses closed forms and compares Theorem 1.1 with known criteria. Section 10 treats two points, Section 11 describes the formalisation, and Section 12 lists the computations and the numerical evidence.

Throughout, p and ℓ denote primes, v_p is the p-adic valuation, $\lfloor\cdot\rfloor$ is the floor, $(x)_+ = \max(x, 0)$, and $[\,P\,]$ is 1 if P holds and 0 otherwise. For a function G on $\mathbb{Z}_{\geq 0}$, $\Delta G(s) = G(s+1) - G(s)$ is the forward difference. We write $\theta(x) = \sum_{p \leq x} \log p$ and $\psi(x) = \sum_{p^e \leq x} \log p$, and $s_2(b)$ for the binary digit sum of b, so $v_2(b!) = b - s_2(b)$. The letter m always denotes the parameter of Theorem 1.1; poles are indexed by k.

2. The construction

2.1. **Data.** Fix an integer $m \geq 1$, put $d_m := m^2 + 1$ (not a square), let $\mathbb{K} = \mathbb{Q}(\sqrt{d_m})$, embedded in $\mathbb{R}$ with $\sqrt{d_m} > 0$, and let σ be its nontrivial automorphism. With $z = z_m = \sqrt{d_m} - m$ and $y = z^2$ we have

$$z^2 + 2mz - 1 = 0, \qquad 1 - y = 2mz, \qquad \sigma(z) = -1/z.$$

So z is a unit of $\mathcal{O}_{\mathbb{K}}$. At every odd prime $p \nmid m$ the elements z, y and $1 - y$ are units; this includes the primes dividing d_m. At an odd prime $p \mid m$, $v_p(1 - y) = v_p(m)$. At every prime of $\mathbb{K}$ above 2, normalised by $v(2) = 1$, $v(1 - y) = \nu := 1 + v_2(m)$. The prime 2 is ramified in $\mathbb{K}$ for odd m, inert for $m \equiv 2 \bmod 4$ and split for $4 \mid m$: write $d_m = f^2 d_0$ with d_0 squarefree; f is odd, so $d_0 \equiv d_m \bmod 8$, and $d_m \equiv 2, 5, 1 \bmod 8$ in the three cases. For $m = 2$: $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, $z = \varphi^{-3}$, $1 - y = 4z$, $\nu = 2$.

Fix an even $h \geq 2$ and put $J := h/2$ and $K := h + 1$. The poles are $x_k := 2(k - J)$ for $0 \leq k < K$, i.e. the even integers $\pm 2j$, $|j| \leq J$, and

$$D(t) := \prod_{k<K} (t - x_k),$$

$$D'(x_k) = \prod_{l \neq k} 2(k - l) = (-1)^{K-1-k} 2^{K-1} k!\,(K - 1 - k)!.$$

For a rational function f regular at the odd positive integers and of polynomial growth put

$$\mathcal{M}(f) := \sum_{t \text{ odd} \geq 1} z^t f'(t),$$

an absolutely convergent series, and $\Lambda(g) := \mathcal{M}(g/D)$ for $g \in \mathbb{Q}[t]$. For an even integer x define the partial sums

$$P(x) := \begin{cases} \displaystyle\sum_{n \text{ odd}, 0<n<|x|} \frac{z^n}{n^2}, & x < 0, \\ \displaystyle -\sum_{n \text{ odd}, -x<n<0} \frac{z^n}{n^2}, & x > 0, \\ 0, & x = 0. \end{cases}$$

Only odd n with $|n| \le |x| - 1 \le h - 1$ occur.

2.2. **The pole expansion.**

Lemma 2.1 (Lemma 1′, pole expansion)**.** *For every even integer x, $\mathcal{M}\big(1/(t-x)\big) = -z^x\big(T - P(x)\big)$. For a polynomial F,*

$$\mathcal{M}(F) = z \sum_{b \ge 0} \Delta^b G(0)\, \frac{y^b}{(1-y)^{b+1}}, \qquad G(s) := F'(2s+1), \tag{2.1}$$

a finite sum. Consequently, writing $g/D = R_g + \sum_k g(x_k)/(D'(x_k)(t-x_k))$ with $R_g \in \mathbb{Q}[t]$,

$$\Lambda(g) = A(g) + T\,B(g),$$

$$A(g) := \mathcal{M}(R_g) + \sum_k \frac{g(x_k)}{D'(x_k)} z^{x_k} P(x_k), \qquad B(g) := -\sum_k \frac{g(x_k)}{D'(x_k)} z^{x_k},$$

with $A, B\colon \mathbb{Q}[t] \to \mathbb{K}$ linear.

Proof. $\mathcal{M}(1/(t-x)) = -\sum_{t \text{ odd} \ge 1} z^t/(t-x)^2$. With $n = t - x$ (odd, $n \ge 1 - x$) and $z^t = z^x z^n$, the sum is $z^x \sum_{n \text{ odd} \ge 1-x} z^n/n^2 = z^x(T - P(x))$ by the definition of P. For a polynomial, substitute $t = 2s+1$, expand $G(s) = \sum_b \Delta^b G(0) \binom{s}{b}$, and use $\sum_{s \ge 0} \binom{s}{b} y^s = y^b/(1-y)^{b+1}$. All series converge absolutely since $0 < z < 1$. □

For an indeterminate X put $\Lambda_X := A + X\,B$, $E_a := \Lambda_X(t^a) \in \mathbb{K}[X]$ and

$$\Delta_h(X) := \det\big(E_{i+j}\big)_{0 \le i,j < h} \in \mathbb{K}[X], \qquad \deg \Delta_h \le h,$$

$$\Delta_h(T) = \det\big(\Lambda(t^{i+j})\big)_{i,j<h}.$$

We use the same notation for every finite family: if $a_0, \dots, a_{h-1}$ and $b_0, \dots, b_{h-1}$ have degrees $< h$ and coefficient matrices C_a, C_b, then $\det(\Lambda_X(a_i b_j)) = \det C_a \det C_b\, \Delta_h(X)$.

2.3. **The reflection.** Extend A and B to rational functions $f = R + \sum_k c_k/(t - x_k)$ ($R \in \mathbb{Q}[t]$, $c_k \in \mathbb{Q}$) by $A(f) := \mathcal{M}(R) + \sum_k c_k z^{x_k} P(x_k)$ and $B(f) := -\sum_k c_k z^{x_k}$, so that $\mathcal{M}(f) = A(f) + TB(f)$, and put $\tilde{f}(t) := f(-t)$.

Lemma 2.2 (Lemma R, reflection)**.** *For every such f, $\sigma(A(f)) + \sigma(B(f))X = -\big(A(\tilde{f}) + B(\tilde{f})X\big)$.*

Proof. We have $\tilde{f} = \tilde{R} + \sum_k (-c_k)/(t-(-x_k))$, and the pole set is symmetric. *The X-part.* Since x_k is even, $\sigma(z^{x_k}) = (-1/z)^{x_k} = z^{-x_k}$, so $\sigma(B(f)) = -\sum_k c_k z^{-x_k} = -B(\tilde{f})$. *The pole part of A.* For $x < 0$, $\sigma(P(x)) = \sum_{0<n<|x|} (-1)^n z^{-n}/n^2 = -\sum_{-|x|<n'<0} z^{n'}/n'^2 = P(-x)$ (odd n, $n' = -n$), and likewise for $x > 0$. Hence $\sigma\big(\sum_k c_k z^{x_k} P(x_k)\big) = \sum_k c_k z^{-x_k} P(-x_k)$, which is minus the pole part of $A(\tilde{f})$. *The polynomial part.* For a polynomial R, $\mathcal{M}(R) = M_R(z)$ with $M_R(q) := \sum_{t \text{ odd} \geq 1} R'(t) q^t$, a rational function of q with rational coefficients and poles only at $q = \pm 1$; so $\sigma(\mathcal{M}(R)) = M_R(-1/z)$. Since R' is a polynomial, the reciprocity of rational generating functions gives $\sum_{t \text{ odd} \geq 1} R'(t) q^t = -\sum_{t \text{ odd} \leq -1} R'(t) q^t$ as rational functions, the right side being expanded at $q = \infty$: for $R'(t) = t^a$ apply $(q\, d/dq)^a$ to $\sum_{t \text{ odd} \geq 1} q^t = q/(1-q^2) = -\sum_{t \text{ odd} \leq -1} q^t$. At $q = -1/z$ the right side is $-\sum_{u \text{ odd} \geq 1} R'(-u)(-z)^u = \sum_{u \text{ odd} \geq 1} R'(-u) z^u = -M_{\tilde{R}}(z)$, because $\tilde{R}'(u) = -R'(-u)$. So $\sigma(\mathcal{M}(R)) = -\mathcal{M}(\tilde{R})$. □

Corollary 2.3 (Corollary R)**.** *$\sigma(E_a) = (-1)^a E_a$ coefficientwise, and $\Delta_h(X) \in \mathbb{Q}[X]$.*

Proof. For $f = t^a/D$ we have $D(-t) = (-1)^K D(t) = -D(t)$ (K is odd), so $\tilde{f} = (-1)^{a+1} f$, and Lemma 2.2 gives $\sigma(E_a) = (-1)^a E_a$. Thus $E_a \in \mathbb{Q}[X]$ for even a and $E_a \in \sqrt{d_m}\,\mathbb{Q}[X]$ for odd a. The matrix $\tilde{E}_{ij} := E_{i+j} \sqrt{d_m}^{[i \text{ odd}]-[j \text{ odd}]}$ has entries in $\mathbb{Q}[X]$, and $\tilde{E} = \operatorname{diag}(\sqrt{d_m}^{[i \text{ odd}]})\,(E_{i+j})\,\operatorname{diag}(\sqrt{d_m}^{-[j \text{ odd}]})$, so $\det \tilde{E} = \Delta_h$. □

Lemma 2.2 and Corollary 2.3 use only $\sigma(z) = -1/z$; whether d_m is squarefree plays no role.

2.4. **Valuations and the criterion.** For a prime p and a prime $\mathfrak{p} \mid p$ of $\mathbb{K}$ let $v_\mathfrak{p}$ be normalised by $v_\mathfrak{p}(p) = 1$ (so $v_\mathfrak{p}$ takes values in $\frac{1}{2}\mathbb{Z}$ at ramified primes), extended to $\mathbb{K}[X]$ as the Gauss valuation (the minimum over the coefficients). On $\mathbb{Q}[X]$ every $v_\mathfrak{p}$ restricts to v_p. For $\Delta_h \not\equiv 0$ put

$$n_p := -v_p(\Delta_h), \qquad F_h := \Delta_h \prod_p p^{n_p} \in \mathbb{Z}[X] \text{ (primitive)},$$

$$\log|F_h(T)| = \log|\Delta_h(T)| + \sum_p n_p \log p.$$

Upper bounds for n_p may be computed from the $\mathbb{K}$-valued entries at any one $\mathfrak{p} \mid p$, since $\Delta_h \in \mathbb{Q}[X]$.

Proposition 2.4 (Lemma E)**.** *Let $\xi \in \mathbb{R}$ and let $\mathcal{H}$ be an infinite set of positive integers. Suppose that for $h \in \mathcal{H}$ there are $F_h \in \mathbb{Z}[X]$ with $\deg F_h \leq h$ and $\limsup_{h \in \mathcal{H}} h^{-2} \log|F_h(\xi)| < 0$, and that $F_h(a/b) \neq 0$ for all $h \in \mathcal{H}$, where $a \in \mathbb{Z}$, $b \geq 1$. Then $\xi \neq a/b$.*

Proof. If $\xi = a/b$, then $b^h F_h(a/b)$ is a nonzero integer, so $h \log b + \log|F_h(\xi)| \geq 0$ for $h \in \mathcal{H}$. Dividing by h^2 contradicts the hypothesis. □

This is the criterion of [15, Prop. 3.1], with the nonvanishing set allowed to depend on b.

3. Denominators I: the odd primes

In Sections 3.1–3.3, p is an odd prime with $p \nmid m$, and $v = v_{\mathfrak{p}}$ for a prime $\mathfrak{p} \mid p$ of $\mathbb{K}$. Odd primes dividing m are treated in Section 3.4. Since D is monic with integer coefficients, $R_g \in \mathbb{Z}[t]$ for $g \in \mathbb{Z}[t]$, and by (2.1), with y and $1-y = 2mz$ units at $\mathfrak{p}$,

$$v\big(\mathcal{M}(F)\big) \ge \min_{t \text{ odd} \ge 1} v_p\big(F'(t)\big) \qquad (F \in \mathbb{Z}_{(p)}[t]), \tag{3.1}$$

because $\Delta^b G(0)$ is an integer combination of values $F'(t)$ at odd t. For odd t we have $t - x_k = (t+2J) - 2k$, so for a level p^e the indices k with $p^e \mid t - x_k$ form one residue class $k_0(t)$ modulo p^e among the K consecutive integers $0, \dots, K-1$. Put

$$\mathrm{lev}^*(p) := \#\{e \ge 1 : p^e \le h\}.$$

3.1. **The atom expansion and the class count.** Let $V_{k,i} := x_k^i$ $(k < K,\ i < h)$, $\mathcal{P}_{ij} := \mathcal{M}(R_{t^{i+j}})$ and $\gamma_k(X) := -z^{x_k}(X - P(x_k))/D'(x_k)$. By Lemma 2.1, $(E_{i+j}) = \mathcal{P} + V^{\mathsf{T}} \operatorname{diag}(\gamma) V$.

Lemma 3.1 (Lemma 7′, atom expansion)**.**

$$\Delta_h(X) = \sum_{|I| = |I'| = |S|} \pm \det \mathcal{P}_{I^c I'^c} \det V_{S,I} \det V_{S,I'} \prod_{k \in S} \gamma_k(X),$$

over $I, I' \subseteq \{0, \dots, h-1\}$ *and* $S \subseteq \{0, \dots, K-1\}$. *Moreover* $\det V_{S,I} \in \mathrm{Vand}(x_S)\mathbb{Z}$, *and* $\prod_{k \in S} |D'(x_k)| = \mathrm{Vand}(x_S)^2 \operatorname{cross}(S)$ *with* $\operatorname{cross}(S) := \prod_{k \in S,\, l \notin S} |x_k - x_l|$.

Proof. Expand $\det(\mathcal{P} + M)$ by the complementary minors of $\mathcal{P}$ and $M = V^{\mathsf{T}} \operatorname{diag}(\gamma) V$, and each minor $\det M_{II'}$ by the Cauchy–Binet formula. A generalised Vandermonde determinant is $\mathrm{Vand}(x_S)$ times a Schur polynomial with integer coefficients. The last identity splits $\prod_{l \ne k} |x_k - x_l|$ into $l \in S$ and $l \notin S$. □

Lemma 3.1 is algebra and holds at every prime. For an odd level d put $w_d(k) := 2[\,|x_k| > d\,]$, let $C_c := \{k < K : k \equiv c \bmod d\}$, $n_c := |C_c|$, and

$$A_K(d) := \sum_{c \bmod d} \max_{S_c \subseteq C_c} \Big[|S_c|(n_c - |S_c|) + \sum_{k \in S_c} w_d(k) \Big].$$

Lemma 3.2 (Lemma 8′, per-level bound)**.** *For every odd prime* $p \nmid m$, $n_p \le \sum_{e \ge 1,\, p^e < K} A_K(p^e)$.

Proof. $\mathcal{P}$ is $\mathfrak{p}$-integral by (3.1). In the term of Lemma 3.1 indexed by (I, I', S), the factors $\det V_{S,I} \det V_{S,I'}$ cancel $\mathrm{Vand}(x_S)^2$, so by Lemma 3.1 the term has valuation at least $v(\det \mathcal{P}_{I^c I'^c}) - v_p(\operatorname{cross}(S)) + \sum_{k \in S} v(X - P(x_k))$, using that z^{x_k} is a unit. Here $v_p(\operatorname{cross}(S)) = \sum_{e \ge 1} \#\{(k,l) : k \in S,\ l \notin S,\ p^e \mid k - l\}$ since $x_k - x_l = 2(k-l)$, and $v(X - P(x_k)) \ge$

$-2\max_{n\text{ odd},|n|<|x_k|} v_p(n) \geq -2\sum_{e\geq 1}[\,p^e < |x_k|\,]$. At level $d = p^e$ the pairs counted are those inside residue classes, $\sum_c |S \cap C_c|(n_c - |S \cap C_c|)$, and $[\,p^e < |x_k|\,] = \frac{1}{2}w_d(k)$. Bounding the maximum over S level by level and class by class gives the claim. Levels $p^e \geq K$ contribute nothing: the classes are singletons and $|x_k| \leq h < p^e$. □

Lemma 3.3 (Lemma 8A′, class count)**.** *For odd d with $K/3 \leq d < K$ we have $A_K(d) = 3(K-d)$, and $A_K(d) = 0$ for $d \geq K$.*

Proof. For $d \geq K$ all classes are singletons and all weights vanish. Let $K/3 \leq d < K$. Then $w_d(k) = 0$ if and only if $|k - J| \leq (d-1)/2$, a block of d consecutive indices inside $[0, K)$; so every class contains exactly one atom of weight 0, and the others have weight 2. A class with n atoms contributes $V(n) := \max_t[t(n-t) + 2\min(t, n-1)]$ (take the weight-0 atom last), and $V(1) = 0$, $V(2) = 3$, $V(3) = 6$. Since $3d \geq K$, classes have at most three atoms. If $K/3 \leq d \leq K/2$ there are $K - 2d$ classes of size 3 and $3d - K$ of size 2, with total $6(K-2d) + 3(3d-K) = 3(K-d)$. If $K/2 < d < K$ there are $K - d$ pairs and $2d - K$ singletons, with total $3(K-d)$. □

3.2. **The plateau: a bordered Lagrange Gram identity.** In [15] the number of poles equals the size h of the determinant, and the Lagrange basis has the right dimension. Here $K = h + 1$, and the K Lagrange polynomials $f_i := D/(t - x_i)$ have degree h: they span the polynomials of degree $\leq h$, one dimension too many. The adjugate of the $(h+1)$-Gram matrix repairs this.

Lemma 3.4 (Lemma B, bordered identity)**.** *Let $G^f(X) := (\Lambda_X(f_i f_j))_{i,j<K}$. Then*

$$\Delta_h(X) = (-1)^{K(K-1)/2}\,\frac{\mathbf{1}^{\mathsf{T}}\operatorname{adj}\big(G^f(X)\big)\mathbf{1}}{\prod_{i<K} D'(x_i)}.$$

Proof. Let $H_{h+1} := (E_{a+b})_{a,b\leq h}$. Then $\Delta_h = \operatorname{adj}(H_{h+1})_{hh}$. Let C be the coefficient matrix of the f_i (row i lists the coefficients of $t^0, \dots, t^h$ in f_i). Then $G^f = CH_{h+1}C^{\mathsf{T}}$, so $\operatorname{adj}(G^f) = (\det C)^2 C^{-\mathsf{T}}\operatorname{adj}(H_{h+1})C^{-1}$ and $\operatorname{adj}(H_{h+1}) = (\det C)^{-2}C^{\mathsf{T}}\operatorname{adj}(G^f)C$. Column h of C is $\mathbf{1}$, since every f_i is monic of degree h; hence $\Delta_h = (\det C)^{-2}\mathbf{1}^{\mathsf{T}}\operatorname{adj}(G^f)\mathbf{1}$. Finally $(f_i(x_k))_{i,k} = \operatorname{diag}(D'(x_i)) = CW$ with $W = (x_k^a)_{a,k}$, and $\prod_i D'(x_i) = (-1)^{K(K-1)/2}(\det W)^2$, so $(\det C)^2 = (-1)^{K(K-1)/2}\prod_i D'(x_i)$. □

Lemma 3.5 (Lemma 9′, plateau bound)**.** *For every odd prime $p \nmid m$, $n_p \leq 2h\,\mathrm{lev}^*(p) + v_p(h!)$.*

Proof. Put $N_e(i) := \#\{k < K : k \neq i,\ p^e \mid k - i\}$ and $N(i) := \sum_e N_e(i) = v_p(D'(x_i))$. Note that $N_e(i) \geq 1$ needs $p^e \leq K - 1 = h$. We first prove the *row claim*

$$v\big(G^f_{ij}\big) \geq N(i) - 2\,\mathrm{lev}^*(p) \qquad (i, j < K). \tag{3.2}$$

Fix an odd t and a level $d = p^e$, and let $k_0 = k_0(t)$. The class of i modulo d has $N_e(i) + 1$ elements in $[0, K)$, and classes modulo d in $[0, K)$ differ in size by at most one, so the class k_0 has at least $N_e(i)$ elements.

Off-diagonal entries ($i \neq j$). Here $f_i f_j / D = F := \prod_{k \neq i,j}(t - x_k)$, so $G^f_{ij} = \mathcal{M}(F)$. Each term of $F'(t)$ omits one factor. At level d the factors divisible by d are those with $k \equiv k_0$, $k \neq i, j$; after the omission at least $N_e(i) - 2$ remain, whether $k_0 \equiv i$ or not. Summing over levels, every term has $v_p \geq \sum_e (N_e(i) - 2[\, N_e(i) \geq 1\,]) \geq N(i) - 2\,\mathrm{lev}^*(p)$, and (3.1) gives (3.2).

Diagonal entries. Put $g := \prod_{k \neq i}(t - x_k)$. Then $f_i^2/D = g/(t - x_i) = Q + g(x_i)/(t - x_i)$ with $Q := (g(t) - g(x_i))/(t - x_i)$ and $g(x_i) = D'(x_i)$. By Lemma 2.1 the pole part contributes $-D'(x_i) z^{x_i}(X - P(x_i))$, of valuation at least $N(i) - 2\#\{e : p^e < |x_i|\} \geq N(i) - 2\,\mathrm{lev}^*(p)$, since $|x_i| \leq h$. For Q, list the roots x_k, $k \neq i$, of g as $\alpha_1, \ldots, \alpha_h$ in order of decreasing $v_p(k - i)$; telescoping gives

$$Q(t) = \sum_{r=1}^{h} \prod_{s<r} (x_i - \alpha_s) \prod_{s>r} (t - \alpha_s),$$

and for every level e the $\alpha_s = x_k$ with $p^e \mid k - i$ form an initial segment of length $N_e(i)$ (note $x_i - x_k = 2(i - k)$). Fix a term r of $Q'(t)$, which omits one more factor of the second product, and a level d. If $r > N_e(i)$, the first product has $N_e(i)$ factors divisible by d. If $r \leq N_e(i)$ and $k_0 \equiv i$, it has $r - 1$, and the remaining $N_e(i) - r$ class members sit at positions $s > r$, at most one of them omitted: in total at least $N_e(i) - 2$. If $r \leq N_e(i)$ and $k_0 \not\equiv i$, the whole class k_0 (at least $N_e(i)$ members, all with $v_p(k - i) < e$) sits at positions $s > N_e(i) \geq r$: in total at least $N_e(i) - 1$. So (3.2) holds for Q too, by (3.1).

The minors. $\mathbf{1}^{\mathsf{T}} \operatorname{adj}(G^f)\mathbf{1} = \sum_{a,b} \pm M_{ba}$, where M_{ba} is the minor with row b and column a deleted. By (3.2) and the Leibniz expansion, $v(M_{ba}) \geq \sum_{i \neq b}(N(i) - 2\,\mathrm{lev}^*(p)) \geq \sum_i N(i) - 2h\,\mathrm{lev}^*(p) - \max_b N(b)$. Lemma 3.4 and $v_p(\prod_i D'(x_i)) = \sum_i N(i)$ give $v(\Delta_h) \geq -2h\,\mathrm{lev}^*(p) - \max_b N(b)$. Finally $N(b) = v_p(b!\,(h - b)!) \leq v_p(h!)$. □

Remark 3.6. At $m = 2$, on the exact profiles computed for $h = 16, \ldots, 256$, the bound of Lemma 3.5 is never violated, and it is attained at most single-level plateau primes; for example at $h = 128$ the primes $13, 17, 19, 23, 31, 37, 41, 43$ have $n_p = 256 + v_p(128!)$ exactly. The row claim (3.2) was checked entrywise at $h = 16, 24, 32, 40$ and $p \leq 13$, with minimum slack 0. For other m, Lemmas 3.2 and 3.5 were checked at every odd $p \nmid m$ on exact profiles for $m = 3, 4, 5, 6, 9, 15$ and $h \leq 48$, and in 47 further pairs (m, h) by an independent implementation; there was no violation.

3.3. **The sum over the odd primes not dividing m.**

Theorem 3.7 (Theorem A′)**.** *For every $\varepsilon > 0$ and all large even h, $\sum_{p \text{ odd}, p \nmid m} n_p \log p \leq (\frac{4}{3} + \varepsilon) h^2$.*

Proof. For $p < K/3$ use Lemma 3.5:

$$\sum_{p<K/3} \big(2h\,\mathrm{lev}^*(p) + v_p(h!)\big)\log p \le 2h\big(\theta(K/3) + \psi(h) - \theta(h)\big) + \log h! \\ = \tfrac{2}{3}h^2 + o(h^2)$$

by the prime number theorem. For $K/3 \le p < K$ we have $p^2 \ge K^2/9 \ge K$ for $h \ge 8$, so only the level $e = 1$ occurs, and Lemmas 3.2 and 3.3 give $n_p \le 3(K-p)$. By the prime number theorem and partial summation, $\sum_{K/3\le p<K} 3(K-p)\log p = 3h^2\int_{1/3}^1 (1-x)\,dx + o(h^2) = \frac{2}{3}h^2 + o(h^2)$. For $p \ge K$, $n_p \le 0$ by Lemma 3.2. □

The finite sums of these bounds over all odd primes, divided by h^2, are 1.3380, 1.3350 and 1.3337 at $h = 1442$, 20836 and 875032. The primes dividing d_m (for $m = 2$, the prime 5) need no separate treatment: z and $1 - y = 2mz$ are units there, so every step above holds with $v(p) = 1$.

3.4. **Odd primes dividing m: Lemma S.** At an odd prime $p \mid m$ the matrix $\mathcal{P}$ is no longer $\mathfrak{p}$-integral, since $v_p(1-y) = v_p(m) > 0$. The bound of Lemma 3.2 survives with an extra term.

Lemma 3.8 (Lemma S)**.** *Let p be an odd prime with $e := v_p(m) \ge 1$. Then for every even h,*

$$n_p \le \sum_{l\ge 1,\, p^l<K} A_K(p^l) + e\,\frac{(h-2)^2}{4}.$$

Consequently $\limsup_{h\to\infty} n_p/h^2 \le \frac{e}{4} + \frac{1}{4(p-1)}$*, i.e. the prime p costs at most* $c_p(e) = \big(\frac{e}{4} + \frac{1}{4(p-1)}\big)\log p$ *per* h^2*.*

Proof. Lemma 3.1 still holds. In the proof of Lemma 3.2 every factor other than $\det \mathcal{P}_{I^cI'^c}$ is treated as before: V is integral, z^{x_k} is a unit, $P(x)$ has denominators n^2 with n odd, and the class count does not involve $\mathcal{P}$. This gives $v(\text{term}) \ge v(\det \mathcal{P}_{I^cI'^c}) - \sum_l A_K(p^l)$. For the polynomial part, $\mathcal{P}_{ij} = \mathcal{M}(R_{t^{i+j}})$, where $R_{t^{i+j}} \in \mathbb{Z}[t]$ is the quotient of t^{i+j} by D (monic of degree K). So $\mathcal{P}_{ij} = 0$ for $i+j \le K$, since then R is constant and $\mathcal{M}$ of a constant is 0. For $r := i+j-K \ge 1$, the Mahler form (2.1) with $G(s) = R'(2s+1) \in \mathbb{Z}[s]$ of degree $r-1$, and $\Delta^b G(0) \in b!\,\mathbb{Z}$, gives

$$v(\mathcal{P}_{ij}) \ge \min_{0\le b\le r-1}\big(v_p(b!) - e(b+1)\big) \ge -e\,r.$$

So $v(\mathcal{P}_{ij}) \ge -e(i+j-K)_+$. For any bijection $\tau\colon I^c \to I'^c$, $(i+\tau(i)-K)_+ \le (i-K/2)_+ + (\tau(i)-K/2)_+$, hence

$$v(\det \mathcal{P}_{I^cI'^c}) \ge -2e\sum_{i<h}(i-K/2)_+ = -e(h/2-1)^2 = -e\,\frac{(h-2)^2}{4}$$

(h even, $K = h+1$). Adding the two bounds gives the first claim. For the limit, a class with $n_c \le K/d+1$ atoms contributes at most $\max_s[s(n_c-s)+$

$2s] \le (K/d+3)^2/4$ to $A_K(d)$, so $A_K(d) \le d(K/d+3)^2/4 = K^2/(4d)+O(K)$, and $\sum_{l\ge1} p^{-l} = 1/(p-1)$; there are $O(\log K)$ levels. □

Checks. On exact profiles there was no violation. For $p=3$, $e=1$ (at $m=3$, 6 and 15, where n_3 was the same): $n_3 = 101,202,355,752$ at $h = 16,24,32,48$, against the bounds $117,249,441,971$. For $m=9$ ($e=2$): $n_3 = 150,579$ at $h=16,32$, against $166,666$. At $m=6$, $h=48$, the generic part of n_3 (the value 279 at $m=2$) plus $e(h-2)^2/4 = 529$ gives $808 \ge 752$, so the term $e(h-2)^2/4$ is close to sharp, while the plateau term of Lemma 3.2 (442) overshoots the generic 279. These data suggest that the true cost of a special prime is about $\frac{e}{4}\log p$ per h^2, and that the extra $\frac{1}{4(p-1)}\log p$ comes from using Lemma 3.2 instead of Lemma 3.5 at p; we have not proved this (see Remark 7.2).

4. Denominators II: the prime 2

Let $\mathfrak{P}$ be a prime of $\mathbb{K}$ above 2 and $v := v_{\mathfrak{P}}$, normalised by $v(2)=1$. Then $v(z)=0$ and $v(1-y) = \nu = 1+v_2(m)$ (Section 2.1). For $m=2$, 2 is inert and $\nu = 2$.

Rescaling. Put $\tau := t/2$. The atoms become $\tau = s+\frac12$ ($s \ge 0$), the poles $\xi_k := k - J \in \mathbb{Z}$, and $D(t) = 2^K \tilde D(\tau)$ with $\tilde D(\tau) := \prod_{k<K}(\tau - \xi_k)$. With $\tilde{\mathcal{M}}(F) := \sum_{s\ge0} z^{2s+1} F'(s+\frac12)$ and $\tilde\Lambda(g) := \tilde{\mathcal{M}}(g/\tilde D)$ we get $\Lambda(t^a) = 2^{a-K-1}\tilde\Lambda(\tau^a)$, hence $\Delta_h = 2^{-3h}\tilde\Delta_h$ with $\tilde\Delta_h := \det(\tilde\Lambda_X(\tau^{i+j}))$, since $\sum_{i<h} 2i - h(K+1) = -3h$.

Anchored basis. Put $a_i := \prod_{l<i}(\tau - \xi_l)$ and $b_j := \prod_{l<j}(\tau - \xi_{K-1-l})$. Both families are monic of degrees i and j, so $\det(G_{ij}) = \tilde\Delta_h$ for $G_{ij} := \tilde\Lambda_X(a_i b_j)$.

4.1. The entries.

Lemma 4.1 (Lemma 10_ν)**.** *Let $i,j<h$.*

(L) *If $\pi := i+j-K \ge 1$, then $v(G_{ij}) \ge 1-\pi-\nu+\min_{0\le b\le\pi-1}\big((2-\nu)b - s_2(b)\big)$. If $\pi = 0$, then $G_{ij}=0$.*

(U) *If $N_0 := K-1-i-j \ge 1$, then*

$$v(G_{ij}) \ge \min\Big\{2+(\nu-1)N_0+s_2(N_0),\ 2-N_0+s_2(N_0)+\min_{0\le r\le N_0-1}\big(2r-s_2(r)+\nu(N_0-1-r)\big)\Big\}.$$

If $N_0 = 0$, then $v(G_{ij}) = 2$.

Proof. (L) The products a_i and b_j cover the index ranges $[0,i-1]$ and $[K-j,K-1]$, which overlap in π indices, so $a_ib_j/\tilde D = F := \prod_{l=K-j}^{i-1}(\tau-\xi_l) = 2^{-\pi}H(2\tau)$ with $H(U) := \prod_l(U-2\xi_l) \in \mathbb{Z}[U]$. So $F'(s+\frac12) = 2^{1-\pi}H'(2s+1)$, and by (2.1) (with s in place of $(t-1)/2$), $\tilde{\mathcal{M}}(F) = z\sum_{b<\pi}\Delta^b G(0) y^b/(1-y)^{b+1}$, $G(s) = 2^{1-\pi}H'(2s+1)$. The Newton coefficients of $s\mapsto(2s+1)^a$ are $\Delta^b[(2s+1)^a](0) = \sum_{j'\ge b}\binom{a}{j'}2^{j'}\,b!\,S(j',b)$, of valuation $\ge b + v_2(b!) =$

$2b - s_2(b)$, while $y^b/(1-y)^{b+1}$ has valuation $-\nu(b+1)$. Hence $v(G_{ij}) \geq 1 - \pi + \min_{b<\pi}\big(2b - s_2(b) - \nu(b+1)\big)$, which is the claim. There is no X-part. If $\pi = 0$ then $F = 1$ and $\tilde{\mathcal{M}}(1) = 0$.

(U) Now the ranges leave a gap of $N_0 + 1$ indices, and with $\zeta_l := \xi_i + l$,

$$\frac{a_i b_j}{\tilde{D}} = \prod_{l=0}^{N_0} \frac{1}{\tau - \zeta_l} = \frac{1}{N_0!} \sum_{l=0}^{N_0} (-1)^{N_0 - l} \binom{N_0}{l} \frac{1}{\tau - \zeta_l}.$$

For an integer ζ, $\tilde{\mathcal{M}}(1/(\tau-\zeta)) = -4\sum_s z^{2s+1}/(2s+1-2\zeta)^2 = -4z^{2\zeta}(X - S(\zeta))$ (at $X = T$), where $S(\zeta+1) - S(\zeta) = -z^{-1-2\zeta}/(2\zeta+1)^2$; so $S(\zeta)$ has only odd denominators and $v(S(\zeta)) \geq 0$. With $z^{2\zeta_l} = z^{2\xi_i} y^l$ and $S(\zeta_l) = S(\xi_i) - \sum_{r<l} z^{-1-2\xi_i-2r} g(r)$, $g(r) := (s_0 + 2r)^{-2}$, $s_0 := 2\xi_i + 1$,

$$G_{ij} = -\frac{4}{N_0!}\Big[z^{2\xi_i}(y-1)^{N_0}\big(X - S(\xi_i)\big) + z^{-1}\sum_l (-1)^{N_0-l}\binom{N_0}{l}\sum_{r<l} y^{l-r} g(r)\Big].$$

The first part of the bracket has valuation νN_0. In the second, write $y^{l-r} = y\sum_{e'} \binom{l-r-1}{e'}\eta^{e'}$ with $\eta := y - 1$ ($v(\eta) = \nu$) and $g(r) = \sum_a \gamma_a \binom{r}{a}$; expanding $g(r) = s_0^{-2}\sum_c (-1)^c (c+1)(2/s_0)^c r^c$ (convergent in $\mathbb{Z}_2$) and using $\Delta^a r^c(0) = a!\, S(c,a)$ gives $v_2(\gamma_a) \geq a + v_2(a!) = 2a - s_2(a)$. The identities $\sum_{r<l}\binom{r}{a}\binom{l-1-r}{e'} = \binom{l}{a+e'+1}$ and $\sum_l (-1)^{N_0-l}\binom{N_0}{l}\binom{l}{M} = [\,M = N_0\,]$ collapse the second part to $z^{-1}y\sum_{a+e'=N_0-1}\gamma_a\eta^{e'}$, of valuation $\geq \min_{0\leq a\leq N_0-1}(2a - s_2(a) + \nu(N_0 - 1 - a))$. With $v_2(N_0!) = N_0 - s_2(N_0)$ and $v(4) = 2$ this gives (U). If $N_0 = 0$ then $G_{ij} = -4z^{2\xi_i}(X - S(\xi_i))$, of valuation exactly 2. □

Nothing in this proof uses a particular value of ν; for $\nu = 2$ it is the computation of the earlier version.

4.2. **The content.**

Proposition 4.2. *For every even $h \geq 2$:*

$$n_2 \leq \begin{cases} (\nu-1)\dfrac{(h-2)^2}{4} - \dfrac{h^2}{4} + h\lfloor \log_2 h \rfloor + 5h - 1, & \nu \geq 2, \\ \dfrac{(h-2)^2}{4} + 3h, & \nu = 1. \end{cases}$$

Hence $\limsup_h n_2/h^2 \leq |\nu - 2|/4$, and the prime 2 costs at most $c_2(\nu) = |\nu-2|\frac{\log 2}{4}$ per h^2.

Proof. $\nu \geq 2$. In (L), $(2-\nu)b \geq (2-\nu)(\pi - 1)$ and $s_2(b) \leq \lfloor \log_2 \pi \rfloor + 1$, so $v(G_{ij}) \geq -(\nu-1)\pi - 2 - \lfloor \log_2 \pi \rfloor$. In (U), $2r - s_2(r) + \nu(N_0 - 1 - r) \geq 2(N_0 - 1) - s_2(r)$, so $v(G_{ij}) \geq N_0 - 1 + s_2(N_0) - \lfloor \log_2 N_0 \rfloor \geq N_0 - \lfloor \log_2 N_0 \rfloor$; the first term of the minimum in (U) is larger. Hence $v(G_{ij}) \geq \hat{g}(i+j) - \lfloor \log_2 h \rfloor - 2$ for all i, j, with the piecewise linear function

$$\hat{g}(s) := K - 1 - s \quad (s \leq K), \qquad \hat{g}(s) := -1 - (\nu-1)(s - K) \quad (s \geq K),$$

which is concave since its slopes -1 and $-(\nu-1)$ decrease (at $s = K$ the entry vanishes, and at $s = K - 1$ its valuation is $2 \geq 0$). For a permutation σ of

$\{0,\dots,h-1\}$ the vector $(i+\sigma(i))_i$ is majorised by $(2i)_i$: the k largest entries of $i+\sigma(i)$ sum to at most twice the k largest i, and the totals are equal. By Karamata's inequality, $\sum_i \hat g(i+\sigma(i)) \ge \sum_i \hat g(2i) = \frac{h^2}{4} + 1 - (\nu-1)\frac{(h-2)^2}{4}$. By the Leibniz expansion, $v(\tilde\Delta_h) \ge \frac{h^2}{4} + 1 - (\nu-1)\frac{(h-2)^2}{4} - h(\lfloor \log_2 h\rfloor + 2)$, and $n_2 = -v(\Delta_h) = 3h - v(\tilde\Delta_h)$.

$\nu = 1$. (L) gives $v(G_{ij}) \ge -\pi$, since $\min_b(b - s_2(b)) = 0$. In (U) the first term is $2 + s_2(N_0)$ and the second is $1 + s_2(N_0) + \min_r(r - s_2(r)) = 1 + s_2(N_0) \ge 2$. So $v(G_{ij}) \ge \hat g_1(i+j)$ with the concave $\hat g_1(s) := -(s-K)_+$, and $\sum_i \hat g_1(2i) = -(1+3+\cdots+(h-3)) = -(h/2-1)^2$. Conclude as before. □

At $\nu = 3$ (for example $m = 4$) the bound reads $n_2 \le h^2/4 + h\lfloor \log_2 h\rfloor + 3h{+}1$. On exact profiles at $m = 4$, $n_2 = 23, 94, 201, 349, 523, 737, 1277$ at $h = 8, 16, 24, 32, 40, 48, 64$, against the bounds $65, 177, 313, 513, 721, 961, 1601$; the share $n_2 \log 2/h^2$ is 0.249 at $h = 8$ and 0.216 at $h = 64$, and 0.2046, 0.2002 at $h = 96$, 128, falling towards the proved limit bound 0.1733. At $m = 8$ ($\nu = 4$), $n_2 = 2205$ at $h = 64$ against the bound 2562. Every entry bound (L), (U) was checked on the exact Gram matrices for $m = 1, 3, 4, 6, 8, 12, 16$ and $h \le 64$ (both embeddings where 2 splits), with no violation and with equality attained. In [15] the corresponding quantity is $v_2(1 - q^{-2}) = 3$, which costs $\frac{1}{4}\log 2$ per h^2, in agreement with $c_2(3)$.

4.3. **The case $\nu = 2$.** At $\nu = 2$ Proposition 4.2 gives $n_2 \le h\lfloor \log_2 h\rfloor + 4h$. The direct assignment gives a slightly better bound; for $m = 2$ this is the bound used in the earlier version.

Lemma 4.3 (Lemma 10′)**.** *Let $\nu = 2$ (that is, $m \equiv 2 \bmod 4$) and $i, j < h$.*

(L) *If $\pi \ge 1$, then $v(G_{ij}) \ge -\pi - \lfloor \log_2 \pi\rfloor - 2$. If $\pi = 0$, then $G_{ij} = 0$.*
(U) *If $N_0 \ge 1$, then $v(G_{ij}) \ge N_0 - \lfloor \log_2 N_0\rfloor$. If $N_0 = 0$, then $v(G_{ij}) = 2$.*

Consequently $n_2 \le h\lfloor \log_2 h\rfloor + 3h$.

Proof. (L) and (U) are Lemma 4.1 at $\nu = 2$: in (L), $1 - \pi - 2 + \min_{b<\pi}(-s_2(b)) \ge -\pi - \lfloor \log_2 \pi\rfloor - 2$; in (U), the second term is $N_0 - 1 + s_2(N_0) - \lfloor \log_2 N_0\rfloor$ and the first, $N_0 + 2 + s_2(N_0)$, is larger. *Assignment.* In all cases $v(G_{ij}) \ge K - i - j - \lfloor \log_2 h\rfloor - 2$, since $\pi, N_0 \le h$. By the Leibniz expansion, $v(\tilde\Delta_h) \ge \sum_{i<h}(K - i - \sigma(i)) - h(\lfloor \log_2 h\rfloor + 2) = -h\lfloor \log_2 h\rfloor$ for every permutation σ, and $n_2 = 3h - v(\tilde\Delta_h)$. □

The mechanism is $\nu = 2$: both triangles (L) and (U) then have slope 1 in $i + j$, and the permutation sum of the bound is constant. Section 10 shows what happens when $\nu = 1$ in a biquadratic setting. At $m = 2$ the exact values $n_2 = 54, 96, 142, 240, 350, 576, 830$ at $h = 16, 24, 32, 48, 64, 96, 128$ lie below the bound of Lemma 4.3 $(112, 168, 256, 384, 576, 864, 1280)$, and $n_2/(h\log_2 h)$ rises from 0.84 to 0.94 for $h \le 256$.

5. The size of $\Delta_h(T)$: Theorem B′

5.1. **The equilibrium problem and weak duality.** For $0 < z < 1$ put

$$\Psi(x) := \tfrac{1}{2}\int_{x-1}^{x+1} \log|s|\,ds, \qquad v_z(x) := x\log z - \Psi(x).$$

Ψ is even, increasing on $[0,\infty)$, and $\Psi(0) = -1$. The function v_z is continuous, and $v_z'(x) = \log z - \frac{1}{2}\log|(x+1)/(x-1)|$ for $x \neq \pm 1$. For $x > 0$, $x \neq 1$, the logarithm is positive, so $v_z' < \log z < 0$; for $-1 < x \leq 0$, $v_z'(x) = \log z + \operatorname{artanh}|x| < 0$ if and only if $|x| < \tanh(\log(1/z))$. For $0 < z \leq \sqrt{5}-2$ we have $\tanh(\log(1/z)) \geq \tanh(\log(2+\sqrt{5})) = 0.8945\ldots$. Hence

$$v_z \text{ is strictly decreasing on } (-0.894,\infty) \supset [-1/h,\infty) \tag{5.1}$$

for every $h \geq 2$ and $0 < z \leq \sqrt{5}-2$. This covers every z_m with $m \geq 2$, since z_m decreases in m. For a finite positive measure ρ with compact support put $U^\rho(x) := \int \log|x-s|\,d\rho(s)$ and $E(\rho) := \iint \log|x-s|\,d\rho(x)\,d\rho(s)$.

Call μ *admissible* if it is a density with $0 \leq \mu \leq 1$, mass 1 and $\operatorname{supp}\mu \subset [0,2]$. For admissible μ and $\lambda \in \mathbb{R}$ put

$$F_\mu := 2U^\mu + v_z, \qquad \Phi_z(\mu,\lambda) := -E(\mu) + \lambda + \int_0^\infty (F_\mu - \lambda)_+\,dx.$$

If ρ is a density with $0 \leq \rho \leq 1$, mass 1 and compact support, then $E(\rho-\mu) \leq 0$ [26, Lemma I.1.8], i.e. $E(\rho) \leq 2\int U^\mu\,d\rho - E(\mu)$, so

$$E(\rho) + \int v_z\,d\rho \leq -E(\mu) + \lambda + \int (F_\mu - \lambda)\,d\rho. \tag{5.2}$$

The supremum of the left side over such ρ supported in $[0,\infty)$ is the constrained equilibrium constant C_z; by (5.2), $C_z \leq \Phi_z(\mu,\lambda)$.

5.2. **The theorem.**

Theorem 5.1 (Theorem B′)**.** *Let $0 < z \leq \sqrt{5}-2$ and $T = \chi_2(z)$. Let μ be admissible, $\lambda \in \mathbb{R}$ and $c_{\text{cell}} \geq 0$, and suppose that*

(i) $F_\mu - \lambda \leq -1.13 - 0.94(x-4)$ *on* $[4,\infty)$, *and*
(ii) $F_\mu - \lambda \leq c_{\text{cell}}$ *on* $[-1/100, 0]$.

Then for every even $h \geq 100$,

$$\log|\Delta_h(T)| \leq h^2\,\Phi_z(\mu,\lambda) + E_{\text{an}}(h),$$

where $c_1(h) := \log(2h) + 1 + \log(2+\log(2h+1))$ and

$$E_{\text{an}}(h) := 2h\log(4h) + h\log 4 + (2h+1)\log 3 + h\,c_1(h) + (c_{\text{cell}} + 2.362)h - h\log h + 1.$$

In particular $\limsup_{h\to\infty} h^{-2}\log|\Delta_h(T)| \leq \Phi_z(\mu,\lambda)$.

For $m = 2$ the certified pair of Section 5.3 has $c_{\mathrm{cell}} = 5.38$, which gives the constant 7.742 of the earlier version. The proof follows the proof of Theorem B in [15] in the first two steps and then departs from it: the poles now lie on both sides of the atoms, which is handled by an exact Jensen-type bound (step 4), and the discretisation spreads the particles into cells (step 5) instead of using a lattice energy inequality. Steps 1–4 do not involve z; step 5 uses only (5.1), and step 6 only the hypotheses (i) and (ii).

Proof. Write $\phi := 1/D$.

1. Node expansion. For $a, b \in \mathbb{Q}[t]$, $(ab\phi)' = a'b\phi + ab'\phi + ab\phi'$, so $\Lambda(ab) = \sum_{t\ \mathrm{odd}\geq 1}(a,a')(t)\,M_t\,(b,b')(t)^{\mathsf{T}}$ with $M_t := z^t\left(\begin{smallmatrix}\phi' & \phi\\ \phi & 0\end{smallmatrix}\right)(t)$. Let $H^{(t_{\max})}$ be the Hankel matrix of the sums truncated at $t \leq t_{\max}$. Each node t carries a *value slot* (the row $(t^i)_{i<h}$) and a *derivative slot* (the row $(it^{i-1})_{i<h}$). By the Cauchy–Binet formula, $\det H^{(t_{\max})}$ is a sum over *patterns* $n = (n_t)$, $n_t \in \{0,1,2\}$, $\sum_t n_t = h$, and over pairs (S, S') of slot sets with n_t slots at node t, of $\pm \det E_S \det \mathbb{M}_{S,S'} \det E_{S'}$, where E_S is the matrix of the rows in S and $\mathbb{M}$ is block diagonal with blocks M_t. There are at most 4^h pairs (S, S') per pattern.

2. Slot weights. $|\det \mathbb{M}_{S,S'}| \leq \prod_t (z^t\omega_t)^{n_t}$ with $\omega_t := \max(|\phi(t)|, |\phi'(t)|)$ (for $n_t = 2$ note $|\det M_t| = z^{2t}\phi(t)^2$). The K numbers $t - x_k$ are distinct odd integers, so $|\phi'/\phi|(t) \leq \sum_k 1/|t - x_k| \leq 2\sum_{c=0}^{h} 1/(2c+1) \leq 2 + \log(2h+1)$, and $\omega_t \leq |\phi(t)|(2 + \log(2h+1))$.

3. Vandermonde determinants with derivative rows. $|\det E_S| \leq (4h)^h \prod_{t<t'} |t - t'|^{n_t n_{t'}}$. Indeed, replace the derivative-only slots $s_1, \dots, s_r$ of S by value rows at free variables; the result is a confluent Vandermonde determinant $\pm \prod_{a<b}(a-b)^{n_a n_b}$, and $\det E_S$ is obtained by applying $\partial_{s_1} \cdots \partial_{s_r}$ and evaluating at the nodes. By induction, after c derivatives the expression is a sum of at most $(h-1)^c$ terms, each a product of c exponents (each ≤ 2) times $\prod (a-b)^{e'_{ab}}$ with $0 \leq e'_{ab} \leq n_a n_b$. At the nodes $|a - b| \geq 2 > 1$, so each term is at most $2^c \prod |a-b|^{n_a n_b}$, and $(2(h-1))^r \leq (4h)^h$.

4. The pole product. For odd t the numbers $t - x_k$ are the odd integers in $[t-h, t+h]$. For every odd u, $\log|u| \geq \frac{1}{2}\int_{u-1}^{u+1} \log|s|\,ds$, by concavity of log on $(0,\infty)$ and on $(-\infty, 0)$ (the singular endpoint is integrable). The intervals $[u-1, u+1]$ tile $[t-h-1, t+h+1]$, so with $x := t/h$ and $s = u/h$,

$$\begin{aligned}\log|D(t)| &\geq \tfrac{1}{2}\int_{t-h-1}^{t+h+1} \log|u|\,du\\ &= (h+1)\log h + h\Psi(x) + \tfrac{h}{2}\Big[\int_{x+1}^{x+1+1/h} + \int_{x-1-1/h}^{x-1}\Big]\log|s|\,ds.\end{aligned}$$

Each bracketed integral is over an interval of length $1/h$, hence at least $\int_{-1/(2h)}^{1/(2h)} \log|s|\,ds = -(\log(2h)+1)/h$. So $\log|D(t)| \geq (h+1)\log h + h\Psi(x) - \log(2h) - 1$ and, with step 2,

$$t\log z + \log\omega_t \leq h\,v_z(t/h) - (h+1)\log h + c_1(h). \tag{5.3}$$

By steps 1–4, the pattern n contributes at most $4^h(4h)^{2h}e^{D(n)}$ with $D(n) := \sum_{t\neq t'} n_t n_{t'} \log|t-t'| + \sum_t n_t\big(h\, v_z(t/h) - (h+1)\log h + c_1(h)\big)$.

5. Discrete to continuum. Spread the n_t slots of node t uniformly over $I_t := [(t-2)/h, t/h]$: the density $\rho_n := \sum_t \frac{n_t}{2}\mathbf{1}_{I_t}$ satisfies $0 \le \rho_n \le 1$, has mass 1, and is supported in $[-1/h, \infty)$. By (5.1), $v_z(t/h) \le v_z$ on I_t, so $\sum_t n_t h\, v_z(t/h) \le h^2 \int v_z\, d\rho_n$. For $t \neq t'$ with $|t-t'| = 2d$, writing $x = (t-2+2a)/h$, $a \in [0,1]$,

$$h^2 \iint_{I_t\times I_{t'}} \log|x-s|\, d\rho_n\, d\rho_n = n_t n_{t'}\big(\log|t-t'| - \log h - \mathrm{gap}(d)\big),$$

$$\mathrm{gap}(d) := \log d - \int_0^1\int_0^1 \log|d+a-a'|\, da\, da',$$

and $0 \le \mathrm{gap}(d) \le 0.1137/d^2$ (with equality at $d=1$ up to rounding). A cell with itself gives $n_t^2(\log 2 - \log h - \frac{3}{2})$. Summing, and using $n_{t'} \le 2$ and $\sum_t n_t^2 \le 2h$,

$$\sum_{t\neq t'} n_t n_{t'} \log|t-t'| \le h^2 E(\rho_n) + h^2 \log h + 0.748h + 1.614h$$

($4\cdot 0.1137\,\zeta(2) \le 0.748$ and $2(\frac{3}{2} - \log 2) \le 1.614$). The terms in $\log h$ combine to $h^2\log h - h(h+1)\log h = -h\log h$, so

$$D(n) \le h^2 J(\rho_n) - h\log h + h\, c_1(h) + 2.362h, \qquad J(\rho) := E(\rho) + \int v_z\, d\rho.$$

6. Dual, the cell below 0, and far nodes. By (5.2), $h^2 J(\rho_n) \le h^2(-E(\mu) + \lambda) + h^2\int (F_\mu - \lambda)\, d\rho_n$. On $[0,4]$ the integral is at most $\int_0^4 (F_\mu - \lambda)_+ dx$, since $\rho_n \le 1$. On $[-1/h, 0] \subset [-1/100, 0]$, which only the cell I_1 meets, (ii) gives $F_\mu - \lambda \le c_{\mathrm{cell}}$, so the contribution to $h^2 J$ is at most $c_{\mathrm{cell}} h$. On $[4,\infty)$, by (i), a slot at a node $t \ge 4h+3$ contributes at most $-a_t$ with $a_t := h(1.13 + 0.94((t-2)/h - 4)) \ge 1.13h$; the node $4h+1$, whose cell straddles 4, has $F_\mu - \lambda < 0$ on its part beyond 4. So for every pattern

$$h^2 J(\rho_n) \le h^2 \Phi_z(\mu,\lambda) + c_{\mathrm{cell}} h - \sum_{t \ge 4h+3} n_t a_t.$$

The $2h+1$ odd nodes $t \le 4h+1$ carry at most 3^{2h+1} patterns, and the nodes beyond contribute at most $\prod_{t\ge 4h+3}(1 + e^{-a_t} + e^{-2a_t}) \le \exp\big(2e^{-1.13h}/(1 - e^{-1.88})\big) \le e$. Collecting,

$$\log|\det H^{(t_{\max})}| \le h\log 4 + 2h\log(4h) + (2h+1)\log 3 + h^2\Phi_z(\mu,\lambda) + c_{\mathrm{cell}} h - h\log h + h\, c_1(h) + 2.362h + 1,$$

uniformly in $t_{\max}$. As $t_{\max} \to \infty$ the truncated matrices converge entrywise to $(\Lambda(t^{i+j}))$, and the determinant is continuous. □

Remark 5.2. On $[-1/h, 0]$ we have $U^\mu(x) \le \log(2 + 1/h)$ (as $\mathrm{supp}\,\mu \subset [0,2]$) and $v_z(x) \le |\log z|/h - \Psi(0) = 1 + |\log z|/h$; so (ii) holds with $c_{\mathrm{cell}} =$

$2\log(2.01)+1+|\log z|/100-\lambda$ whenever this is ≥ 0. For any $h\geq 2$ one may instead use $U^\mu(x)\leq\int_1^2\log(s+1)\,ds=0.9095\ldots$ on $[-1/h,0]$ (because $\mu\leq 1$ and $\operatorname{supp}\mu\subset[0,2]$); at $m=2$ this gives $F_\mu-\lambda\leq 7.23$ there and the bound of Theorem 5.1 for every even $h\geq 2$ with $c_{\mathrm{cell}}+2.362$ replaced by 9.592. Only the $O(h)$ term changes.

5.3. The certificate at $\sqrt{5}-2$.

Theorem 5.3. *For $z=z_2=\sqrt{5}-2$ there is an admissible pair (μ_U,λ_U) satisfying the hypotheses of Theorem 5.1 with $c_{\mathrm{cell}}=5.38$ and*

$$\Phi_{z_2}(\mu_U,\lambda_U)\leq\Phi_U:=-1.3592161834586374.$$

The proof is a finite computation in ball arithmetic (Arb [12], through python-flint [29]) at 128 bits. The density μ is constant on N equal cells of $[0,2]$, with rational heights in $[0,1]$ and mass exactly 1 (checked in exact rational arithmetic). The energy $E(\mu)$ is computed from the exact double antiderivative $\frac{1}{2}u^2\log|u|-\frac{3}{4}u^2$ as a Toeplitz sum over cell offsets. The grid has step d/r ($d=2/N$ the cell width, r even, so cell midpoints are grid points), and on each grid interval $I=[a,b]$:

- the three cells nearest to I are treated exactly: each cell potential is unimodal with its minimum at the cell midpoint, so its maximum on I is at an endpoint that can be chosen exactly;
- the far field $f=2\sum_{\mathrm{far}}\mu_jU_j$ (every far cell is at distance $\geq d$ from I) satisfies $f''\geq -4/d$, because $f''=-2\int_{\mathrm{far}}\mu(s)(x-s)^{-2}ds$ and $\mu\leq 1$; and $v''(x)=1/(x^2-1)$ is $\geq -1/(1-b^2)$ for $b<1$ and ≥ 0 for $a>1$. A function g with $g''\geq-\kappa$ on I satisfies $\sup_I g\leq\max(g(a),g(b))+\kappa(b-a)^2/8$; this is applied to $f+v$ with $\kappa=4/d+1/(1-b^2)$ (if $b<1$) or $\kappa=4/d$ (if $a>1$). On the two intervals touching $x=1$, v is bounded by $v(a)$;
- $\int(F_\mu-\lambda)_+\leq\sum_I|I|\max(S_I-\lambda,0)$, with S_I the resulting upper bound on I; on $[4,\infty)$, $F_\mu(x)\leq 2\log x+x\log z-\log(x-1)$, which is $<\lambda$ and decreasing there.

A first version with per-cell maxima everywhere loses the cancellation between neighbouring cells and gives only -1.313 at $N=100$. Table 2 lists the results; the last row proves Theorem 5.3.

The side conditions of Theorem 5.1 hold for the certified pair: at $x=4$, $2\log x+x\log z-\log(x-1)-\lambda=-1.137\ldots$, and the slope of this bound on $[4,\infty)$ is at most $2/x+\log z\leq -0.94$; on $[-1/h,0]$, $h\geq 100$, we have $U^\mu\leq\log(2+1/h)$ and $v_z\leq 1+1.4437/h$, hence $F_\mu\leq 2.4108$ and $F_\mu-\lambda\leq 2.4108+2.96317\leq 5.374<5.38$.

Remark 5.4 (the equilibrium at $m=2$). An active-set quadratic program (and two independent codes) give the primal value $C_{z_2}\approx -1.3594016$, with maximiser saturated ($\rho=1$) on $[0,0.5475]$, a free band up to 1.9525, and $\lambda^*\approx -2.96317$; so the certificate is within about $1.9\cdot 10^{-4}$ of the optimum. The finite-h values $h^{-2}\log|\Delta_h(T)|$, computed in ball arithmetic, are

N	r	$-E(\mu)$	λ	$\int(F_\mu-\lambda)_+\le$	certified Φ
200	16	1.21902586	-2.96219282	0.38673	-1.35643705
400	32	1.21902867	-2.96297388	0.38528	-1.35866119
800	32	1.21902407	-2.96310589	0.38505	-1.35903096
1600	32	1.21902464	-2.96316943	0.384929	-1.35921618345864

TABLE 2. Dual certificates for Theorem 5.1 at $z=\sqrt{5}-2$ (decimal values rounded; λ is a rational number). For $N=1600$ the certified Arb ball is $[-1.35921618345864\pm 2.6\cdot 10^{-15}]$, with upper endpoint -1.3592161834586374.

-1.4379, -1.4195, -1.4093, -1.4015, -1.3930, -1.3876, -1.3815, -1.3781 at $h=16$, 32, 48, 64, 96, 128, 192, 256. They increase towards the constant from below; fits of the form $a+b\log h/h\,(+c/h)$ give limits between -1.373 and -1.364, below C_{z_2}, so these data do not indicate that Theorem 5.1 is sharp. This costs nothing, since only the upper bound is used.

5.4. **The shift lemma.** A smaller z only helps, and by a computable amount.

Lemma 5.5 (shift lemma)**.** *Let $0<z'\le z\le\sqrt{5}-2$ and $L:=\log(z/z')\ge 0$. For every admissible μ and every $\lambda\in\mathbb{R}$,*

$$\Phi_{z'}(\mu,\lambda-L)\le\Phi_z(\mu,\lambda)-\tfrac{L}{2}.$$

If (μ,λ) satisfies (i) of Theorem 5.1 at z, then $(\mu,\lambda-L)$ satisfies (i) at z'; if (μ,λ) satisfies (ii) at z with c_{cell}, then $(\mu,\lambda-L)$ satisfies (ii) at z' with $c_{\mathrm{cell}}+1.01L$.

Proof. Write F_μ and F'_μ for the functions at z and z'. Since $v_{z'}(x)=v_z(x)-Lx$, we have $F'_\mu=F_\mu-Lx$ and

$$F'_\mu-(\lambda-L)=(F_\mu-\lambda)-L(x-1).$$

With $a:=F_\mu(x)-\lambda$: on $[0,1]$, $(a+L(1-x))_+\le a_++L(1-x)$; on $[1,\infty)$, $(a-L(x-1))_+\le a_+$. Integrating, $\int_0^\infty(F'_\mu-\lambda+L)_+\,dx\le\int_0^\infty(F_\mu-\lambda)_+\,dx+\frac{L}{2}$, so $\Phi_{z'}(\mu,\lambda-L)\le -E(\mu)+\lambda-L+\int(F_\mu-\lambda)_++\frac{L}{2}=\Phi_z(\mu,\lambda)-\frac{L}{2}$. On $[4,\infty)$ the shift $-L(x-1)$ only lowers $F-\lambda$; on $[-1/100,0]$ it raises it by at most $L(1+1/100)$. □

Corollary 5.6. *For every $m\ge 2$, the pair $(\mu_U,\lambda_U-L(m))$ satisfies the hypotheses of Theorem 5.1 at $z=z_m$ with $c_{\mathrm{cell}}=5.38+1.01L(m)$, and*

$$\Phi_{z_m}\big(\mu_U,\lambda_U-L(m)\big)\le\Phi_U-\tfrac{1}{2}L(m),\qquad L(m):=\log\frac{m+\sqrt{m^2+1}}{2+\sqrt{5}}.$$

In particular $\limsup_h h^{-2}\log|\Delta_h(\chi_2(z_m))|\le\Phi_U-\frac{1}{2}L(m)$.

Proof. $1/z_m=m+\sqrt{m^2+1}$, so $L(m)=\log(z_2/z_m)\ge 0$. Apply Lemma 5.5 to Theorem 5.3. □

For $m = 4$ the bound is $\Phi_U - \frac{1}{2}L(4) = -1.6847547\ldots$, with $L(4) = 0.651077\ldots$ (rounded up: -1.684754). The proof of Theorem 1.1 uses only Corollary 5.6, i.e. Theorem 5.3 and the shift lemma.

5.5. **Direct certificates at $m = 4$ and $m = 8$.** The program of Section 5.3, with only z changed, gives direct certificates at $z_4 = \sqrt{17}-4$ and $z_8 = \sqrt{65}-8$ (Table 3). They improve the analytic bound at these points but are not needed for Theorem 1.1, because the margins from Corollary 5.6 are already negative there (Table 4).

m	N	$-E(\mu)$	λ	$\int(F_\mu - \lambda)_+ \le$	certified Φ
4	200	1.36321476	−3.71770185	0.63797	−1.71651998814
4	800	1.36321826	−3.71810316	0.63714	−1.71774955485
4	1600	1.36321769	−3.71817249	0.636999	−1.7179552750082241
8	1600	1.43562390	−4.44201260	0.934075	−2.0723141156759906

TABLE 3. Direct certificates at $m = 4$ and $m = 8$ (upper endpoints of Arb balls). Not used in the proof of Theorem 1.1.

Their side conditions hold: at $x = 4$, $F_\mu - \lambda \le -2.987$ ($m = 4$) and -4.990 ($m = 8$), with slopes at most -1.595 and -2.276; on the cell, $F_\mu - \lambda \le 6.14$ ($m = 4$, $h \ge 100$). An independent projected-gradient computation of the primal problem gives $C_{z_4} \ge -1.7181612$ and $C_{z_8} \ge -2.0725380$, so the certificates are within about $2.1 \cdot 10^{-4}$ and $2.3 \cdot 10^{-4}$ of the optimum.

6. Nonvanishing: Lemma P′

6.1. **Setting.** Let $\ell \ge 7$ be a prime with $\ell \nmid m$, $h := 3\ell - 1$ (even) and $K = 3\ell$. The classes modulo ℓ are $C_\gamma = \{\gamma, \gamma + \ell, \gamma + 2\ell\}$, $\gamma < \ell$, all of size 3. Since $|k - i| < 3\ell < \ell^2$ there is a single level, and $N(i) = v_\ell(D'(x_i)) = 2$ for every i. Let $\mathfrak{l}$ be a prime of $\mathcal{O}_\mathbb{K}$ above ℓ, $v := v_\mathfrak{l}$, and $\zeta := z^\ell \bmod \mathfrak{l}$. If ℓ splits or ramifies in $\mathbb{K}$, the residue field is $\mathbb{F}_\ell$ and $\zeta = z$; if ℓ is inert, $\zeta = \sigma(z) = -1/z$. Since $\ell \nmid 2m$, the elements z, y, $1 - y = 2mz$, $1 - z^{-2} = -2m/z$ and $1 - \zeta^{\pm 2}$ are $\mathfrak{l}$-units. (For ramified ℓ, $\ell \mid d_m$, so $z \equiv -m$, $z^2 \equiv -1$ and $1 - z^{\pm 2} \equiv 2$.) For $m = 2$: ℓ splits if $\ell \equiv \pm 1 \bmod 5$ and is inert otherwise.

When d_m is not squarefree, primes dividing the index $[\mathcal{O}_\mathbb{K} : \mathbb{Z}[\sqrt{d_m}]]$ (for example $\ell = 13$ at $m = 70$, $d_{70} = 4901 = 13^2 \cdot 29$) need no special care: every step below works with the prime $\mathfrak{l}$ of $\mathcal{O}_\mathbb{K}$ and valuations in $\mathcal{O}_\mathbb{K}$, never with $\mathbb{Z}[\sqrt{d_m}]$.

Positions. $x_{\gamma + r\ell} = x_\gamma + 2r\ell$ with $x_\gamma = 2\gamma - 3\ell + 1 \in [-3\ell + 1, -\ell - 1]$: in each class the pole with $r = 0$ is large and negative ($|x| > \ell$), the pole with $r = 1$ is small ($|x| < \ell$), and the pole with $r = 2$ is large and positive. For odd t we have $\ell \mid t - x_k$ exactly for k in one class $C_{\gamma(t)}$, and the odd $t \ge 1$ with $\gamma(t) = \gamma$ form the progression $t = x_\gamma + \ell Y$, $Y \in \{3, 5, 7, \ldots\}$. Put $t_\gamma := x_\gamma + 3\ell$.

Lemma 6.1 (Lemma P′)**.** *Let $\ell \geq 7$ be prime with $\ell \nmid m$, $h = 3\ell - 1$, and $c_j := [X^j]\Delta_h$. Then $v_\ell(c_0) = -2K$ and $v_\ell(c_j) \geq -2K + 2j$ for all j. Hence $\Delta_h(a/b) \neq 0$ for every $a/b \in \mathbb{Q}$ with $v_\ell(b) \leq 1$.*

Proof. 1. Lemma B. $\ell^{2K}\Delta_h(X) = (-1)^{K(K-1)/2} u_\ell \, \mathbf{1}^{\mathsf{T}} \operatorname{adj}(G^f(X))\mathbf{1}$ with $u_\ell := \ell^{2K}/\prod_i D'(x_i)$ an ℓ-unit.

2. The X-part. $B(f_i f_j) = 0$ for $i \neq j$, because $f_i f_j / D$ is a polynomial, and $B(f_i^2) = -D'(x_i) z^{x_i}$, of valuation exactly 2. So $G^f(X) = G^f(0) + X \operatorname{diag}(d_i)$ with $v(d_i) = 2$.

3. Integrality and block diagonality. By the row claim (3.2) with $N(i) - 2\,\mathrm{lev}^*(\ell) = 0$ (valid since $\ell \nmid m$), $G^f(0)$ is $\mathfrak{l}$-integral. On each progression, $\mathcal{M}$ restricted to γ is $z^{t_\gamma} \sum_c z^{2\ell c} F'(t_\gamma + 2\ell c) = z^{t_\gamma} \sum_b \Delta^b G_\gamma(0) z^{2\ell b}/(1 - z^{2\ell})^{b+1}$, $G_\gamma(c) := F'(t_\gamma + 2\ell c)$, with $1 - z^{2\ell} \equiv 1 - \zeta^2$ a unit; so (3.1) holds progression by progression. Off the diagonal, $F = \prod_{k \neq i,j}(t - x_k)$ has on the progression γ exactly $3 - [\,\gamma = c(i)\,] - [\,\gamma = c(j)\,]$ factors divisible by ℓ ($c(i)$ is the class of i), and F' keeps all but one; so a unit is possible only if $c(i) = c(j) = \gamma$. On the diagonal, the polynomial part on a progression $\gamma \neq c(i)$ has valuation at least $3 - 1 = 2$: the class γ has three members, all after the initial segment of the telescoping order, and the derivative loses at most one. Hence $G^f(0) \bmod \mathfrak{l}$ is block diagonal over the classes, each block receiving only its own progression and its own pole terms.

4. The blocks. Fix γ with members $k_r = \gamma + r\ell$, put $W(t) := \prod_{k \notin C_\gamma}(t - x_k)$ and $U_\gamma := \prod_{k \notin C_\gamma}(x_\gamma - x_k)$, so $W(t) \equiv U_\gamma \bmod \ell$ on the progression. Off the diagonal ($r \neq s$, q the third index) $F = W(t)(t - x_{k_q})$ and $F' \equiv W \equiv U_\gamma$. For the diagonal polynomial part, $Q = W \cdot (t + x_{k_r} - x_{k_a} - x_{k_b}) + p(x_{k_r})\, W[x_{k_r}, t]$ with $\{a, b\}$ the other indices, $p(t) = (t - x_{k_a})(t - x_{k_b})$ and $p(x_{k_r}) \equiv 0 \bmod \ell^2$; so $Q' \equiv U_\gamma$ too. In the progression sum, $\Delta^0 G_\gamma(0) \equiv U_\gamma$ and $\Delta^b G_\gamma(0) \equiv 0$ for $b \geq 1$, so it is $\equiv U_\gamma z^{t_\gamma}/(1 - z^{2\ell}) \equiv U_\gamma z^{x_\gamma} \zeta^3/(1 - \zeta^2)$. The pole terms at $X = 0$ are $D'(x) z^x P(x)$ with $\ell^2 P(x_{k_r}) \equiv \zeta, 0, -\zeta^{-1}$ for $r = 0, 1, 2$ (the only odd multiples of ℓ below $|x| \leq 3\ell - 1$ in absolute value are $\pm\ell$), $D'(x_{k_r})/\ell^2 \equiv U_\gamma \cdot (8, -4, 8)_r$, and $z^{x_{k_r}} \equiv z^{x_\gamma} \zeta^{2r}$. Hence

$$G^f(0)|_{C_\gamma} \equiv e_\gamma M, \qquad e_\gamma := U_\gamma z^{x_\gamma},$$
$$M := c\,\mathbf{1}\mathbf{1}^{\mathsf{T}} + \operatorname{diag}(8\zeta, 0, -8\zeta^3), \qquad c := \frac{\zeta^3}{1 - \zeta^2}.$$

5. The bordered determinant. For $M = \operatorname{diag}(d_0, 0, d_2) + c\mathbf{1}\mathbf{1}^{\mathsf{T}}$ we have $\det M = c\, d_0 d_2 = -64 c \zeta^4$ and $\mathbf{1}^{\mathsf{T}} \operatorname{adj}(M)\mathbf{1} = d_0 d_2 = -64\zeta^4$, both units. For the block-diagonal reduction,

$$\mathbf{1}^{\mathsf{T}} \operatorname{adj}\Big(\bigoplus_\gamma e_\gamma M\Big)\mathbf{1} = (\det M)^{\ell - 1}\, \mathbf{1}^{\mathsf{T}} \operatorname{adj}(M)\mathbf{1} \Big(\prod_\gamma e_\gamma^3\Big) \sum_\gamma e_\gamma^{-1}.$$

Modulo ℓ, $U_\gamma \equiv \prod_{k \not\equiv \gamma} 2(\gamma - k) \equiv 2^{3(\ell-1)}((\ell-1)!)^3 \equiv -1$ for every γ (Fermat, Wilson). So $\sum_\gamma e_\gamma^{-1} \equiv -z^{3\ell-1} \sum_{\gamma < \ell} z^{-2\gamma} = -z^{3\ell-1}(1 - z^{-2\ell})/(1 - z^{-2})$, and

$1 - z^{-2\ell} \equiv 1 - \zeta^{-2}$ and $1 - z^{-2} = -2m/z$ are units. Hence $\mathbf{1}^{\mathsf{T}} \operatorname{adj}(G^f(0))\mathbf{1}$ is an $\mathfrak{l}$-unit, and $v_\ell(c_0) = -2K$ by step 1.

6. The gap. $[X^j]\mathbf{1}^{\mathsf{T}} \operatorname{adj}(G^f(X))\mathbf{1}$ is a sum of products containing j factors d_i, so it has valuation $\geq 2j$. If $v_\ell(b) \leq 1$, the j-th term of $\Delta_h(a/b)$ has valuation $\geq -2K + 2j - j > -2K$ for $j \geq 1$. □

The only place where m enters is the list of units in the setting, and they are units exactly when $\ell \nmid 2m$. So every prime $\ell \geq 7$ with $\ell \nmid m$ works, whether it splits, is inert or ramifies. In [15] the partial sums reach past 3ℓ and the block constants are fixed rationals that must avoid ℓ, so finitely many primes are excluded; here only $n = \pm\ell$ enters. At $m = 2$, steps 4–5 were checked entrywise modulo ℓ for $\ell = 7, 11, \ldots, 41$ (split and inert), and the conclusion on the exact Δ_h for $\ell = 7, \ldots, 29$. For other m the conclusion ($v_\ell(c_0) = -2K$ and the gap) was checked on the exact Δ_h at $m = 4$ ($\ell = 7, 11, 13, 17, 19$; $\ell = 17$ ramified), $m = 6$ ($\ell = 7, 11, 13$), and, by an independent implementation, at $m = 3, 7, 9, 11, 12, 13, 18, 21$ (17 cases, among them three ramified ℓ: $\ell = 17$ at $m = 13$, $\ell = 13$ at $m = 21$, and $\ell = 13$ at $m = 18$, where $d_{18} = 325 = 5^2 \cdot 13$). As a control, at $m = 7$, $\ell = 7$ (so $\ell \mid m$) the conclusion fails: $v_\ell(c_0) + 2K = -67$.

7. Proof of Theorem 1.1

7.1. **The proof.** Fix $m \geq 2$ with $m \notin \mathcal{E}$, let $\nu = 1 + v_2(m)$, and suppose that $T = \chi_2(z_m) = a/b$ with $a \in \mathbb{Z}$, $b \geq 1$. Let

$$\mathcal{H} := \{3\ell - 1 : \ \ell \geq 37 \text{ prime}, \ \ell \nmid m, \ \ell^2 \nmid b\},$$

an infinite set of even integers $h \geq 110$. For $h \in \mathcal{H}$, Lemma 6.1 gives $\Delta_h \not\equiv 0$ and $F_h(a/b) \neq 0$, where $F_h \in \mathbb{Z}[X]$ is the primitive polynomial of Section 2, of degree $\leq h$. We have $\log|F_h(T)| = \log|\Delta_h(T)| + \sum_p n_p \log p$, and along $\mathcal{H}$:

- odd primes $p \nmid m$: $\limsup h^{-2} \sum n_p \log p \leq \frac{4}{3}$ (Theorem 3.7);
- odd primes $p \mid m$, $e = v_p(m)$: $\limsup h^{-2} n_p \log p \leq c_p(e)$ (Lemma 3.8);
- $p = 2$: $\limsup h^{-2} n_2 \log 2 \leq c_2(\nu)$ (Proposition 4.2);
- size: $\limsup h^{-2} \log|\Delta_h(T)| \leq \Phi_U - \frac{1}{2}L(m)$ (Corollary 5.6).

So $\limsup_{h \in \mathcal{H}} h^{-2} \log|F_h(T)| \leq M(m)$, with $M(m)$ as in (1.2), and $M(m) < 0$ by Proposition 7.1 below. This contradicts Proposition 2.4. □

Proof of Corollary 1.2. At $m = 2$, $\nu = 2$, there are no odd primes dividing m, $L(2) = 0$, and $n_2 = O(h \log h)$ by Lemma 4.3. So $M(2) = \Phi_U + \frac{4}{3} = -0.0258828\ldots < 0$, and $T = \chi_2(\sqrt{5} - 2)$ is irrational; so is $24T = \pi^2 - 18\log^2\varphi$ by (1.1). □

7.2. **The margins.** Table 4 lists $M(m)$ for some m. The entries are computed in Arb and rounded to the digits shown; in the rows $m \notin \mathcal{E}$ the last two columns are rounded up, so they are upper bounds.

m	ν	$c_2(\nu)$	odd $p \mid m$: $\sum c_p(e)$	arithmetic	$\Phi_U - \frac{1}{2}L(m)$	$M(m)$
2	2	0	—	1.333333	−1.359216	−0.025882
4	3	0.173287	—	1.506620	−1.684754	−0.178134
6	2	0	3: 0.411980	1.745313	−1.883288	−0.137975
8	4	0.346574	—	1.679907	−2.025634	−0.345727
10	2	0	5: 0.502949	1.836283	−2.136509	−0.300227
11	1	0.173287	11: 0.659421	2.166041	−2.183949	−0.017908
12	3	0.173287	3: 0.411980	1.918600	−2.227291	−0.308691
13	1	0.173287	13: 0.694674	2.201294	−2.267184	−0.065890
16	5	0.519860	—	1.853194	−2.370753	−0.517560
21	1	0.173287	3, 7: 0.979537	2.486157	−2.506516	−0.020359
3	1	0.173287	3: 0.411980			+0.3720
5	1	0.173287	5: 0.502949			+0.2160
7	1	0.173287	7: 0.567557			+0.1147
9	1	0.173287	3 ($e = 2$): 0.686633			+0.1091
15	1	0.173287	3, 5: 0.914929			+0.0830

TABLE 4. Limit margins $M(m)$ per h^2 (in the rows above the line, upper bounds; the analytic column uses only Corollary 5.6). The rows below the line are the exceptional set $\mathcal{E}$, where $M(m)$ is positive. The direct certificates of Table 3 improve the rows $m = 4$ and $m = 8$ to $M(4) \le -0.211335$ and $M(8) \le -0.392407$.

Proposition 7.1. *$M(m) < 0$ for every integer $m \ge 2$ with $m \notin \mathcal{E}$, and $M(m) > 0$ for $m \in \mathcal{E} \cup \{1\}$.*

Proof. Write $m = 2^a q$ with q odd. Since $m + \sqrt{m^2+1} > 2m$, $L(m) > \log(2m) - \log(2+\sqrt{5})$. For odd p, $1/(4(p-1)) \le 1/8$, so $\sum_{p^e \| q} c_p(e) \le \frac{1}{4}\log q + \sum_{p \mid q} \frac{\log p}{4(p-1)} \le \frac{3}{8}\log q$.

If $a \ge 1$, then $\nu = a+1 \ge 2$ and $c_2(\nu) = (a-1)\frac{\log 2}{4}$, so

$$\begin{aligned} M(m) &< \left[\Phi_U + \tfrac{4}{3} + \tfrac{1}{2}\log(2+\sqrt{5}) - \tfrac{3}{4}\log 2\right] - \tfrac{a}{4}\log 2 - \tfrac{1}{8}\log q \\ &= 0.176075\ldots - 0.173287\ldots\, a - \tfrac{1}{8}\log q. \end{aligned}$$

This is negative unless $a = 1$ and $q = 1$, i.e. $m = 2$, where $M(2) = \Phi_U + \frac{4}{3} = -0.0258828\ldots < 0$ (proof of Corollary 1.2 above).

If $a = 0$, then $\nu = 1$ and $c_2(1) = \frac{\log 2}{4}$, so

$$\begin{aligned} M(m) &< \left[\Phi_U + \tfrac{4}{3} + \tfrac{1}{2}\log(2+\sqrt{5}) - \tfrac{1}{4}\log 2\right] - \tfrac{1}{4}\log q + \sum_{p \mid q} \frac{\log p}{4(p-1)} \\ &\le 0.522648\ldots - \tfrac{1}{8}\log q, \end{aligned}$$

which is negative for $q > e^{8 \cdot 0.522648\ldots} = 65.443\ldots$. For the odd $m \le 65$ we evaluate $M(m)$ in Arb, with the exact $L(m)$; for $3 \le m \le 65$ it is negative except for $m \in \mathcal{E}$, where the enclosures are positive, between +0.083

($m = 15$) and $+0.372$ ($m = 3$) (Table 4). For $m = 1$, $L(1) < 0$ and the right side of (1.2) is $M(1) = 0.42853\ldots$ (Arb; also found by the independent computation mentioned below). For $m = 1$ this number is only the value of the formula: Corollary 5.6 needs $m \geq 2$. □

So every $m > 65$ is covered by the closed form, the odd $m \leq 65$ by the table, and the set of $m \geq 2$ for which the method as written concludes is exactly $\{m \geq 2\} \setminus \mathcal{E}$. An independent computation in mpmath interval arithmetic (shift lemma only, no shared code) found $M(m) \geq 0$ exactly for $m \in \{1, 3, 5, 7, 9, 15\}$ among $1 \leq m \leq 3000$, and agreed with the table to all printed digits for $m \leq 60$. The tightest margins, $m = 11$ and $m = 21$, depend on the digits of Φ_U only at the level 10^{-2}.

The excluded values. For $m \in \mathcal{E}$ the method does not conclude, although the exact finite-h scores there are negative at $h \leq 48$ (Table 12). They rise with h (for $m = 3$: -0.1326, -0.0734, -0.0340 at $h = 32, 48, 64$) and are not evidence for these m. For $m = 1$ the margin is positive and so are the finite-h scores: $+0.0595$, $+0.2144$, $+0.2524$, $+0.2876$ at $h = 16, 32, 48, 64$. So $\chi_2(\sqrt{2} - 1) = \pi^2/16 - \frac{1}{4}\log^2(1 + \sqrt{2})$ is out of reach of this method.

Remark 7.2 (towards $\mathcal{E}$). The following is not proved. The data after Lemma 3.8 suggest that the term $\frac{1}{4(p-1)}\log p$ in $c_p(e)$ can be removed by a plateau bound of the quality of Lemma 3.5 at special primes. That would bring $m = 9$ and $m = 15$ into Theorem 1.1. For $\nu = 1$ the exact 2-adic shares (for example 0.112 at $h = 64$, against the limit bound 0.173) suggest that $c_2(1) < \frac{1}{4}\log 2$; with direct certificates this might reach $m = 7$. The values $m = 3$ and $m = 5$ look out of reach of this machine.

7.3. **Effectivity.**

Remark 7.3 (effectivity at $m = 2$). Every bound above is explicit at finite h except the prime number theorem in Theorem 3.7. At a single h one can instead evaluate the proved finite-h bound

$$\log|F_h(T)| \leq h^2\Phi + E_{\mathrm{an}}(h) + B_{\mathrm{odd}}(h) + h(\lfloor \log_2 h \rfloor + 3)\log 2,$$

where $B_{\mathrm{odd}}(h) = \sum_{3 \leq p < K/3}(2h\,\mathrm{lev}^*(p) + v_p(h!))\log p + \sum_{K/3 \leq p < K}\min(2h + v_p(h!), 3(K-p))\log p$ is an exact finite sum. At $m = 2$, in exact arithmetic, this bound is negative at every even h with $2432 \leq h \leq 40000$, and at every sampled h up to $8.75 \cdot 10^5$, with $|\mathrm{bound}|/h^2$ increasing towards 0.0259. A proof that it stays negative for all larger h would need explicit error terms in the prime number theorem, which we have not carried out. As an effective consequence, if $\chi_2(\sqrt{5}-2) = a/b$ with $\log b \leq 800$, a single $h = 3\ell - 1 \leq 36000$ with $\ell \nmid b$ gives a contradiction. The exact scores $h^{-2}\log|F_h(T)|$ are much better than this bound, for example -0.066 at $h = 192$ (Table 11).

Remark 7.4 (effectivity at $m = 4$). The same finite-h bound at $m = 4$, with the direct certificate of Table 3, Proposition 4.2 at $\nu = 3$ and Lemmas 3.2, 3.3, 3.5, is negative at every even h with $178 \leq h \leq 40000$ and at every sampled h up to $6 \cdot 10^5$; the last nonnegative value is at $h = 176$. For

example it is $-0.0749h^2$ at $h = 242$ and $-0.1648h^2$ at $h = 1004$. So if $\chi_2(\sqrt{17}-4) = a/b$, a single $h = 3\ell - 1$ with $\ell \nmid b$ and $h \geq 242$, of size about $5\log b + 300$, gives a contradiction. This uses the direct certificate, which the proof of Theorem 1.1 does not need.

8. Extensions: the other units of norm -1, and the inverse tangent integral

Both extensions are deltas to the proof of Theorem 1.1: every lemma of Sections 2–6 not named below holds verbatim, with the data indicated. We extend the notation $c_2(\nu) := |\nu - 2|\frac{\log 2}{4}$ to $\nu = 0$, so $c_2(0) = \frac{1}{2}\log 2$.

8.1. **Odd a: Theorem $\mathbf{V}_{\mathrm{odd}}$.** *Data.* Let $a \geq 1$ be odd, $d := a^2 + 4$, $\mathbb{K} := \mathbb{Q}(\sqrt{d})$ and $z := \alpha_a = (\sqrt{d} - a)/2$. Then

$$z^2 + az - 1 = 0, \qquad \sigma(z) = -1/z, \qquad y := z^2, \qquad 1 - y = az.$$

Here $z \in \mathcal{O}_{\mathbb{K}}$ but $z \notin \mathbb{Z}[\sqrt{d}]$; as in Section 6, every valuation is taken in $\mathcal{O}_{\mathbb{K}}$. Since $d \equiv 5 \bmod 8$, writing $d = f^2 d_0$ with d_0 squarefree gives f odd and $d_0 \equiv 5 \bmod 8$, so 2 *is inert in* $\mathbb{K}$. Since $\gcd(a, a^2+4) = 1$, at every odd prime $p \nmid a$ (including the primes dividing d) the elements z, y, $1 - y$ are units, and at an odd prime $p \mid a$ we have $v_p(1-y) = v_p(a)$. At the prime above 2, z and $1 - y = az$ are units: $\nu = 0$.

What holds verbatim (with $2m$ replaced by a): Lemma 2.1, Lemma 2.2 and Corollary 2.3 (they use only $\sigma(z) = -1/z$), so $\Delta_h \in \mathbb{Q}[X]$; Lemmas 3.1–3.5 and Theorem 3.7 at odd $p \nmid a$; Lemma 3.8 at odd $p \mid a$ with $e := v_p(a)$ (its proof uses only $v_p(1-y) = e$); Theorem 5.1 and the shift lemma, since $\alpha_a \leq \alpha_5 = 0.1925\ldots < \sqrt{5} - 2$ for $a \geq 5$, with

$$L(a) := \log \frac{\sqrt{5}-2}{\alpha_a} = \log \frac{a + \sqrt{a^2+4}}{4 + \sqrt{20}},$$

so that $\Phi \leq \Phi_U - \frac{1}{2}L(a)$ at $z = \alpha_a$, with $c_{\mathrm{cell}} = 5.38 + 1.01L(a)$; and Lemma 6.1 for primes $\ell \geq 7$ with $\ell \nmid a$. In the setting of Lemma 6.1 the units are now $1 - y = az$ and $1 - z^{-2} = -a/z$, which are $\mathfrak{l}$-units if and only if $\ell \nmid a$; for $\ell \mid d$ we have $\sqrt{d} \equiv 0$, $z \equiv -a/2$, $z^2 \equiv -1$ and $1 - z^{\pm 2} \equiv 2$ modulo $\mathfrak{l}$, also at the primes dividing the index of $\mathbb{Z}[\sqrt{d}]$ (for example $a = 29$, $\ell = 13$, $d = 5 \cdot 13^2$). The only new ingredient is the prime 2 at $\nu = 0$.

Lemma 8.1 (Lemma 10_0)**.** *Let a be odd, $\mathfrak{P}$ the prime of $\mathbb{K}$ above 2 and $v := v_{\mathfrak{P}}$ with $v(2) = 1$. In the notation of Section 4 (rescaling, anchored basis, $G_{ij} = \tilde{\Lambda}_X(a_i b_j)$), for $i, j < h$:*

(L) *if $\pi = i + j - K \geq 1$, then $v(G_{ij}) \geq 1 - \pi$; if $\pi = 0$, then $G_{ij} = 0$;*
(U) *if $N_0 = K - 1 - i - j \geq 1$, then $v(G_{ij}) \geq 2 - N_0 + s_2(N_0)$; if $N_0 = 0$, then $v(G_{ij}) = 2$.*

Consequently $n_2 \leq h^2/2 + h$ for every even h, so the prime 2 costs at most $c_2(0) = \frac{1}{2}\log 2$ per h^2.

Proof. The rescaling $\Delta_h = 2^{-3h}\tilde{\Delta}_h$ and the anchored basis do not involve z. Since 2 is inert, v is integer-valued and restricts to v_2 on $\mathbb{Q}$, so $n_2 = -v_2(\Delta_h) = 3h - v(\tilde{\Delta}_h)$ for the rational polynomial Δ_h. In the proof of Lemma 4.1 the quantity ν enters only through $v(y^b/(1-y)^{b+1}) = -\nu(b+1)$ in (L) and $v(\eta) = \nu$ for $\eta = y-1$ in (U). The other inputs are independent of z and of ν: the Newton coefficients of $s \mapsto (2s+1)^c$ have valuation $\geq 2b - s_2(b)$; $S(\zeta)$ has odd denominators and powers of the unit z, so $v(S(\zeta)) \geq 0$; the Newton coefficients γ_r of g (called γ_a there) satisfy $v_2(\gamma_r) \geq 2r - s_2(r)$; and $v_2(N_0!) = N_0 - s_2(N_0)$. Here y and $1-y = az$ are 2-adic units, so the proof applies with $\nu = 0$ (it never used $\nu \geq 1$), and the bounds of Lemma 4.1 at $\nu = 0$ read: in (L), $1 - \pi + \min_{b<\pi}(2b - s_2(b)) = 1 - \pi$ (at $b = 0$); in (U), $\min\{2-N_0+s_2(N_0),\ 2-N_0+s_2(N_0)+\min_{r<N_0}(2r-s_2(r))\} = 2-N_0+s_2(N_0)$. The cases $\pi = 0$ and $N_0 = 0$ are as there.

The content. Put $g(s) := \min(s - K + 3,\ K + 1 - s)$, a concave function. Then $v(G_{ij}) \geq g(i+j)$ for all $i, j < h$: for $s = i + j \leq K - 2$, (U) gives at least $2 - N_0 + 1 = s - K + 4 > g(s)$ (as $s_2(N_0) \geq 1$); for $s = K - 1$ the valuation is $2 = g(s)$; for $s = K$ the entry is 0; for $s \geq K + 1$, (L) gives $K + 1 - s = g(s)$. For a permutation σ of $\{0, \ldots, h-1\}$ the vector $(i + \sigma(i))_i$ is majorised by $(2i)_i$ (proof of Proposition 4.2), so Karamata's inequality gives $\sum_i g(i + \sigma(i)) \geq \sum_i g(2i)$. With $K = h + 1$, $g(2i) = 2i - h + 2$ for $2i \leq h$ and $g(2i) = h + 2 - 2i$ for $2i \geq h + 2$, so

$$\sum_{i<h} g(2i) = \left(\tfrac{h}{2} + 1\right)\left(2 - \tfrac{h}{2}\right) + \left(\tfrac{h}{2} - 1\right)\left(2 - \tfrac{h}{2}\right) = 2h - \tfrac{h^2}{2}.$$

By the Leibniz expansion $v(\tilde{\Delta}_h) \geq 2h - h^2/2$, hence $n_2 \leq h^2/2 + h$. □

The minorant is a tent around $s = K$, with slopes $+1$ and -1, instead of the monotone $\hat{g}$ of Proposition 4.2; diagonal rescalings cannot improve a Leibniz bound of this kind. With $c_2(0) = \frac{1}{2}\log 2$ the formula $c_2(\nu) = |\nu - 2|\frac{\log 2}{4}$ of Proposition 4.2 holds for every $\nu \geq 0$.

Checks. The bounds (L), (U) were checked on the exact Gram matrices at $a = 1, 3, 5, 11$ for $h = 8, 16, 24, 32$, and by an independent implementation at $a \in \{1, 3, 5, 11, 59, 135, 165\}$ for $h \in \{8, 16, \ldots, 48\}$, with no violation and with equality attained in both. The exact n_2 depends only on ν: for every odd a tested, $n_2 = 104, 352, 737, 1248$ at $h = 16, 32, 48, 64$, against the bound $144, 544, 1200, 2112$. The share $n_2 \log 2/h^2$ falls from 0.282 to 0.211 over this range, and to 0.195 at $h = 128$ (a proved lower bound for n_2); a three-term fit suggests the limit $\frac{1}{4}\log 2$, half the proved constant. We have not proved this. Lemma 3.8 and Lemma 6.1 were checked on exact data at odd a (for example $a = 3, 9, 15, 21$ for Lemma 3.8, and Lemma 6.1 at $a = 1, 3, 5, 7, 9, 11, 29, 59, 135$, including ramified ℓ and the index prime $\ell = 13$ at $a = 29$), with no violation; at $a = 7$, $\ell = 7$ (so $\ell \mid a$) Lemma 6.1 fails, as it must. For $p \geq K$ dividing a (for example $p = 59$ at $a = 59$) the exact values are $n_p = (h-2)^2/4$, so the term $e(h-2)^2/4$ of Lemma 3.8 is attained.

The margins. The proof of Section 7, with $\mathcal{H} := \{3\ell-1 : \ell \geq 37 \text{ prime},\ \ell \nmid a,\ \ell^2 \nmid b\}$, Lemma 8.1 at $p = 2$ and $\Phi(a)$ an admissible analytic bound at $z = \alpha_a$, gives

$$\limsup_{h\in\mathcal{H}} \frac{1}{h^2} \log|F_h(T)| \leq M_{\mathrm{odd}}(a) := \Phi(a) + \tfrac{4}{3} + \tfrac{1}{2}\log 2 + \sum_{p^e \| a} c_p(e), \tag{8.1}$$

where $\Phi(a)$ is either $\Phi_U - \frac{1}{2}L(a)$ (shift lemma) or the value of a direct certificate at α_a.

Proposition 8.2. *For odd $a \geq 5$, with the shift lemma and the ten certificates of Table 5, $M_{\mathrm{odd}}(a) < 0$ if and only if $a \notin \mathcal{E}_{\mathrm{odd}}$.*

Proof. Since $a+\sqrt{a^2+4} > 2a$, $-\frac{1}{2}L(a) < -\frac{1}{2}\log 2 - \frac{1}{2}\log a + \frac{1}{2}\log(4+\sqrt{20})$, and $c_2(0) = \frac{1}{2}\log 2$ cancels the $-\frac{1}{2}\log 2$. For odd p, $\frac{1}{4(p-1)} \leq \frac{1}{8} \leq \frac{e}{8}$, so $\sum_{p^e\|a} c_p(e) \leq \frac{3}{8}\log a$. Hence

$$M_{\mathrm{odd}}(a) < C_0 - \tfrac{1}{8}\log a, \quad C_0 := \Phi_U + \tfrac{4}{3} + \tfrac{1}{2}\log(4+\sqrt{20}) = 1.0425084777\ldots,$$

which is negative for $a > e^{8C_0} = 4188.3737\ldots$. For the odd a with $5 \leq a \leq 5001$ we evaluate $M_{\mathrm{odd}}(a)$ in Arb at 200 bits. With the shift lemma alone there are 49 values with $M_{\mathrm{odd}}(a) \geq 0$, the largest 195. Ten of them have negative margins with the certificates of Table 5; the remaining 39 are $\mathcal{E}_{\mathrm{odd}}$, where the margins are positive, also with the further certificates we computed there (for example $+0.0088$ at $a = 165$ and $+0.4580$ at $a = 11$, where $\alpha_{11} = \varphi^{-5}$, both with a direct certificate). □

a	$M_{\mathrm{odd}}(a)$, shift lemma	certified $\Phi(a) \leq$	$M_{\mathrm{odd}}(a) \leq$	$c_{\mathrm{cell}} \leq$
59	$+0.0406$	-2.730260	-0.013393	8.21
61	$+0.0318$	-2.746986	-0.022232	8.24
67	$+0.0072$	-2.794044	-0.047037	8.34
91	$+0.0492$	-2.947508	-0.005370	8.65
95	$+0.0455$	-2.969059	-0.009198	8.69
111	$+0.0275$	-3.047021	-0.027329	8.85
117	$+0.0427$	-3.073383	-0.012169	8.90
135	$+0.0541$	-3.145029	-0.000887	9.05
147	$+0.0133$	-3.187657	-0.041735	9.13
195	$+0.0156$	-3.329068	-0.039558	9.42

TABLE 5. The ten certificates on which Theorem 1.3 depends ($N = 1600$, $r = 32$; upper endpoints of Arb balls, rounded up). The second column is the margin with the shift lemma alone, rounded; it is positive, so these a need their certificate. The last column is the cell constant of Theorem 5.1(ii) for $h \geq 100$.

The certificates. They are produced by the program of Section 5.3 with only z changed to α_a; at $a = 4$ it reproduces the value Φ_U of Theorem 5.3

bit for bit. For this paper all ten were re-run, and they reproduced the stored upper endpoints bit for bit; the AI referee of Theorem 1.3 had re-run four of them ($a = 59, 91, 135, 195$) with the same result, and an independent floating-point primal computation lies below every certified value, by about $2.7 \cdot 10^{-4}$. The side conditions of Theorem 5.1 hold for each of the ten: at $x = 4$, $F_\mu - \lambda \le -8.87$ (required: -1.13), the slope of the tail bound is at most -3.57 (required: -0.94), and on $[-1/h, 0]$, $h \ge 100$, $F_\mu - \lambda \le 2\log(2 + 1/h) + 1 + |\log z|/h - \lambda \le c_{\mathrm{cell}}$ with c_{cell} as in Table 5; c_{cell} enters only the $O(h)$ term $E_{\mathrm{an}}(h)$. The other covered a need only the shift lemma, for instance $M_{\mathrm{odd}}(101) \le -0.099782$. The margins at $a = 135$ and $a = 91$ are rigorous but thin, so the effective h_0 there is large; $a = 123$ has $M_{\mathrm{odd}}(123) \le -0.000034$ with the shift lemma alone. (The formal proof uses a weaker constant in place of Φ_U and therefore needs a direct certificate at $a = 123$ too; Section 11.)

Proof of Theorem 1.3. Let $a \ge 5$ be odd with $a \notin \mathcal{E}_{\mathrm{odd}}$, and suppose $T = \chi_2(\alpha_a) = a'/b$. For $h \in \mathcal{H}$, Lemma 6.1 with $\ell \nmid a$ gives $F_h(a'/b) \ne 0$. Along $\mathcal{H}$ the odd primes $p \nmid a$ contribute at most $\frac{4}{3}$ (Theorem 3.7), the odd $p \mid a$ at most $c_p(v_p(a))$ (Lemma 3.8), the prime 2 at most $\frac{1}{2}\log 2$ (Lemma 8.1), and the size at most $\Phi(a)$ (Theorem 5.1 with the shift lemma, or with the certificate of Table 5). So the lim sup is at most $M_{\mathrm{odd}}(a) < 0$ (Proposition 8.2), contradicting Proposition 2.4. □

Remark 8.3. The cases $a = 1$ (φ^{-1}) and $a = 3$ are not covered: there $\alpha_a > \sqrt{5} - 2$, so the shift lemma does not apply, and numerically the margins are positive. Two improvements, neither proved, are suggested by the data: the 2-adic constant $c_2(0) = \frac{1}{4}\log 2$, and the removal of the term $\frac{1}{4(p-1)}\log p$ from $c_p(e)$ (cf. Remark 7.2); for the latter the bound $n_p \le 2h\,\mathrm{lev}^*(p) + v_p(h!) + e(h-2)^2/4$ at special primes held in every one of 39 exact data points. Each would shrink $\mathcal{E}_{\mathrm{odd}}$; $a = 11$ (φ^{-5}) seems out of reach of this machine even with both.

8.2. **The inverse tangent integral: Theorem V$_i$.** *Data.* Let $n \ge 3$, $\mathbb{K} := \mathbb{Q}(\sqrt{n^2-4})$, $\beta := \beta_n = (n - \sqrt{n^2-4})/2$ and $u_n := 1/\beta_n = (n + \sqrt{n^2-4})/2$. Then $\beta^2 - n\beta + 1 = 0$ and $\sigma(\beta) = 1/\beta$, so β is a unit of norm $+1$; $\beta_{2m} = m - \sqrt{m^2-1}$. Let χ_4 be the nontrivial character modulo 4, so that $\mathrm{Ti}_2(x) = \sum_{t\ \mathrm{odd}\ge 1} \chi_4(t) x^t/t^2$, and define the χ_4-twisted machine with the poles and atoms of Section 2.1:

$$\mathcal{M}^\natural(f) := \sum_{t\ \mathrm{odd}\ge 1} \chi_4(t)\beta^t f'(t), \qquad \Lambda^\natural(g) := \mathcal{M}^\natural(g/D), \qquad T^\natural := \mathrm{Ti}_2(\beta).$$

Put $w := i\beta$, $\mathcal{F} := \mathbb{K}(i)$, and let $\mathcal{M}_w$, Λ_w, A_w, B_w, P_w, $\Delta_{h,w}$ be the objects of Section 2 with z replaced by w (the series converge absolutely since $|w| = \beta < 1$, and Lemma 2.1 holds at w with the same proof, with T replaced by $\chi_2(w)$). Then

$$y := w^2 = -\beta^2, \qquad 1 - y = 1 + \beta^2 = n\beta.$$

This is the only change of data: in Theorem 1.1, $1 - y = 2mz$.

Lemma 8.4 (the identity)**.** *For every rational function f in the span of Section 2, $\mathcal{M}^\natural(f) = -i\,\mathcal{M}_w(f)$. Hence $\Lambda^\natural = A^\natural + T^\natural B^\natural$ with $A^\natural := -iA_w$ and $B^\natural := B_w$, both with values in $\mathbb{K}$, and*

$$\Delta_h^\natural(X) := \det\bigl(A^\natural(t^{i+j}) + XB^\natural(t^{i+j})\bigr)_{i,j<h} = (-i)^h\,\Delta_{h,w}(iX) \in \mathbb{Q}[X].$$

Proof. For odd t, $\chi_4(t) = i^{t-1} = -i \cdot i^t$, so $\chi_4(t)\beta^t = -i\,w^t$, which gives the first claim. Also $\chi_2(w) = \sum_{t\text{ odd}} i^t\beta^t/t^2 = i\ \mathrm{Ti}_2(\beta)$, so $-i\Lambda_w = -iA_w + \mathrm{Ti}_2(\beta)B_w$. For even x, $w^x = (-1)^{x/2}\beta^x \in \mathbb{K}$; so B_w takes values in $\mathbb{K}$, and so does $A^\natural$: by the computation of Lemma 2.1 applied to $\mathcal{M}^\natural$ directly (using $\chi_4(n+x) = i^x\chi_4(n)$ for even x), its pole part consists of $(-1)^{x/2}\beta^x$ times partial sums $\sum\chi_4(n)\beta^n/n^2$, and its polynomial part $\sum_{t\text{ odd}}\chi_4(t)\beta^t R'(t)$ is a rational function of β with rational coefficients. The determinant identity follows from $A^\natural + XB^\natural = -i(A_w + iXB_w)$.

Reflection. Extend σ to $\mathcal{F}$ by $\sigma(i) = i$. Then $\sigma(w) = i/\beta = -1/w$. The proofs of Lemma 2.2 and Corollary 2.3 use only $\sigma(z) = -1/z$ and that the coefficients are rational functions of z with rational coefficients (they are algebraic identities, so $|\sigma(w)| > 1$ is irrelevant). Hence $\Delta_{h,w} \in \mathcal{F}^\sigma[X] = \mathbb{Q}(i)[X]$, so $\Delta_h^\natural \in \mathbb{Q}(i)[X]$. Its coefficients also lie in $\mathbb{K}$, and $\mathbb{K}\cap\mathbb{Q}(i) = \mathbb{Q}$. □

Remark 8.5 (the sign rule). More generally, for weight k, atoms carrying a periodic character χ with parity $s_\chi := \chi(-t)/\chi(t)$, and a unit z of norm $N(z)$, the analogue of Lemma 2.2 reads $\sigma(a(f)) = \eta\,a(\tilde f)$, $\sigma(b(f)) = -b(\tilde f)$ with $\eta := N(z)\,s_\chi(-1)^k$, and the Hankel determinant polynomial lies in $\mathbb{Q}[X]$ when $\eta = -1$; when $\eta = +1$ it has the form $G(\sqrt{d}\,X)$ with $G \in \mathbb{Q}[Y]$ and the method proves only $T \notin \sqrt{d}\,\mathbb{Q}$. (Sketch: for the polynomial part, $S_r(q) := \sum_{t\ge1}\chi(t)q^t t^r$ is rational and $S_r(1/q) = -s_\chi(-1)^r S_r(q)$; for the pole part substitute $n \mapsto -n$ in the partial sums.) Theorem 1.1 has $(N, s_\chi, k) = (-1, +1, 2)$ and Theorem 1.5 has $(+1, -1, 2)$, both with $\eta = -1$; for the latter Lemma 8.4 gives a complete proof, and only that case is used. The general rule was checked exactly in 64 cases ($k \le 4$, two characters, eight units, $h = 6$), and in 8 more for a variant with even atoms.

Valuations. The coefficients of $\Delta_h^\natural$ and of $\Delta_{h,w}$ differ by powers of i, so at every prime of $\mathcal{F}$ their Gauss valuations agree, and since $\Delta_h^\natural \in \mathbb{Q}[X]$ the common value is the rational one. After the factor $-i$, every formula of Sections 2–6 involves w only through $y = w^2$, even powers $w^{2\xi} = (-1)^\xi\beta^{2\xi}$, and $-iw = \beta$; so each lemma may be read at w over $\mathcal{F}$, or directly in $\mathbb{K}$ with $y = -\beta^2$. We now list what transfers.

- *Odd $p \nmid n$.* β, y and $1 - y = n\beta$ are p-units, also for $p \mid n^2 - 4$ (as $\gcd(n, n^2-4) \mid 4$). Lemmas 3.1–3.5 and Theorem 3.7 hold verbatim: at most $\frac{4}{3}$ per h^2.
- *Odd $p \mid n$.* Lemma 3.8 with $e := v_p(n)$, since $v_p(1 - y) = v_p(n)$.

- *The prime 2.* $\nu := v(1-y) = v_2(n)$. For even $n = 2m$, $\nu = 1 + v_2(m) \geq 1$, and Lemma 4.1 and Proposition 4.2 hold verbatim (at any prime of $\mathbb{K}$ above 2, normalised by $v(2) = 1$). For odd n, $n^2 - 4 \equiv 5 \bmod 8$, so 2 is inert in $\mathbb{K}$ and $\nu = 0$; Lemma 8.1 holds verbatim, read in $\mathbb{K}$ with $y = -\beta^2$: its proof uses only that y and $1-y$ are 2-adic units, and z-free facts, and the signs $\chi_4(t) = \pm 1$ in the partial sums $S(\zeta)$ do not affect their 2-integrality. In all cases the cost is at most $c_2(v_2(n))$.
- *Size.* In the proof of Theorem 5.1 the weights enter only through step 2, $|\det \mathbb{M}_{S,S'}| \leq \prod_t (|\chi_4(t)\beta^t|\,\omega_t)^{n_t}$, so the signs never enter, and Theorem 5.1 holds for $T^\natural$ at every $0 < \beta \leq \sqrt{5} - 2$, i.e. for $n \geq 5$ ($\beta_5 = 0.2087\ldots$). The shift lemma gives $\Phi \leq \Phi_U - \frac{1}{2}L_i(n)$ with $L_i(n) := \log(u_n/(2+\sqrt{5})) = \log((\sqrt{5}-2)/\beta_n) > 0$.
- *Nonvanishing.* See Lemma 8.6.

Lemma 8.6 (Lemma P'_n)**.** *Let $\ell \geq 7$ be prime with $\ell \nmid n$, $h = 3\ell - 1$, and $c_j := [X^j]\Delta_h^\natural$. Then $v_\ell(c_0) = -2K$ and $v_\ell(c_j) \geq -2K + 2j$ for all j; hence $\Delta_h^\natural(a'/b) \neq 0$ whenever $v_\ell(b) \leq 1$.*

Proof. By Lemma 8.4, $v_\ell(c_j)$ is the valuation of the j-th coefficient of $\Delta_{h,w}$ at a prime $\mathfrak{l}$ of $\mathcal{F}$ above ℓ (normalised by $v_\mathfrak{l}(\ell) = 1$). We run the proof of Lemma 6.1 at w, with $\zeta := w^\ell \bmod \mathfrak{l}$. Since the conjugates of w are $\pm w$ and $\pm 1/w$, $\zeta \in \{w, -1/w, -w, 1/w\}$ (according to the splitting of ℓ in $\mathbb{K}$ and $\ell \bmod 4$; for $\ell \mid n^2-4$, $\beta \equiv 1/\beta$ and $\zeta = \pm w$), and in all cases $\zeta^2 \in \{w^2, w^{-2}\}$. So $1-\zeta^{\pm 2}$ and $1-w^{\pm 2\ell}$ reduce to $1-w^{\pm 2}$, and $1-w^2 = n\beta$, $1-w^{-2} = n/\beta$ are $\mathfrak{l}$-units exactly when $\ell \nmid n$; with w, y, $1-y$ also units, steps 1–3 of Lemma 6.1 go through. In steps 4–5 every quantity is a unit for each choice of ζ: $c = \zeta^3/(1-\zeta^2)$, $\det M = -64c\zeta^4$, $\mathbf{1}^\mathsf{T}\operatorname{adj}(M)\mathbf{1} = -64\zeta^4$, $U_\gamma \equiv -1$ (independent of the point), and $\sum_{\gamma<\ell} w^{-2\gamma} = (1-w^{-2\ell})/(1-w^{-2}) \equiv 1$ or $-w^2$. Step 6 is unchanged. Index primes are handled in $\mathcal{O}_\mathcal{F}$ as in Section 6. □

The conclusion of Lemma 8.6 was checked on the exact $\Delta_h^\natural$ for $\ell = 7, 11, 13$ at $n = 4, 8, 12$, and, independently, for one to four primes $\ell \leq 17$ each at $n = 5, 9, 10, 15, 21, 22, 197, 199$, covering the four Frobenius classes and ramified ℓ; at $\ell \mid n$ ($n = 21$, $\ell = 7$; $n = 22$, $\ell = 11$; $n = 7$, $\ell = 7$) it fails, as it must. The identity of Lemma 8.4 was checked exactly ($A^\natural = -iA_w$ and $B^\natural = \pm B_w$, the sign depending only on the normalisation of T) at five pairs (n, h) by an implementation independent of the one used for the profiles; Lemma 8.1 was checked entrywise on the χ_4 machine at $n = 3, 5, 9, 15, 197$, with no violation and equality attained, and with $n_2 = 104, 352, 737, 1248$ at $h = 16, 32, 48, 64$ for every odd n tested, the same values as for odd a.

The margins. The proof of Section 7, with $\mathcal{H} := \{3\ell - 1 : \ell \geq 37 \text{ prime}, \ \ell \nmid n, \ \ell^2 \nmid b\}$ and Lemma 8.6, gives the bound

$$M_i(n) := \Phi_U - \tfrac{1}{2}L_i(n) + \tfrac{4}{3} + c_2(v_2(n)) + \sum_{p^e \| n,\ p \text{ odd}} c_p(e) \qquad (n \geq 5). \tag{8.2}$$

For even $n = 2m$ the arithmetic is that of Theorem 1.1 term by term ($\nu = 1 + v_2(m)$, and the odd primes of n are those of m, with the same e), so

$$M_i(2m) = M(m) + \tfrac{1}{2}\log\frac{m+\sqrt{m^2+1}}{m+\sqrt{m^2-1}},$$

with $M(m)$ as in (1.2). The correction is positive and at most $\frac{1}{2}(\sqrt{m^2+1} - \sqrt{m^2-1})/(m+\sqrt{m^2-1}) \le \frac{1}{4(m^2-1)}$; its largest value for $m \ge 3$ is 0.027850 (at $m = 3$). For odd n there is no counterpart in Theorem 1.1: $\nu = 0$, every odd prime factor of n is special, and the shift is from $\sqrt{5}-2$ to β_n directly.

Proposition 8.7. *Let $n \ge 5$. Then $M_i(n) < 0$ if and only if n is not one of $6, 10, 14, 18, 30$, not an odd number ≤ 69, and not in $\mathcal{E}_i$.*

Proof. Uniform tail. For $n \ge 5$ we have $u_n \ge n-1$, $\sum_{p^e \| n,\, p \text{ odd}} \frac{e}{4}\log p = \frac{1}{4}\log n - \frac{\nu}{4}\log 2$, $c_2(\nu) - \frac{\nu}{4}\log 2 = (|\nu-2| - \nu)\frac{\log 2}{4} \le \frac{\log 2}{2}$, and, for any $P \ge 3$, the primes $p > P$ dividing n contribute at most $\frac{\log n}{4P}$ to the terms $\frac{\log p}{4(p-1)}$. Hence

$$M_i(n) \le A + \tfrac{1}{2}\log 2 + \tfrac{1}{4}S_P + \tfrac{1}{2}\log\frac{n}{n-1} - \tfrac{1}{4}\big(1-\tfrac{1}{P}\big)\log n,$$

with $A := \Phi_U + \frac{1}{2}\log(2+\sqrt{5}) + \frac{4}{3} = 0.6959348874\ldots$ and $S_P := \sum_{3 \le p \le P} \frac{\log p}{p-1}$. The right side decreases in n. For $P = 13$, $S_{13} = 1.7295192876\ldots$, and at $n = 599$ the right side is $-0.000105877\ldots < 0$ (Arb). So $M_i(n) < 0$ for every $n \ge 599$.

Table. For $5 \le n \le 20\,000$ we evaluate $M_i(n)$ in Arb at 200 bits; no ball contains 0. The even n with $M_i(n) > 0$ are exactly $6, 10, 14, 18, 30$, and the odd ones are the odd $n \le 69$ and the 17 elements of $\mathcal{E}_i$, the largest 195 (with $M_i(195) \ge +0.0156$). □

n	β_n	$\sin 2\theta$	ν	$M_i(n) \le$	n	$M_i(n) \le$
8	$4-\sqrt{15}$	1/4	3	−0.162496	115	−0.007466
12	$6-\sqrt{35}$	1/6	2	−0.131029	71	−0.007838
16	$8-\sqrt{63}$	1/8	4	−0.341821	129	−0.012699
22	$11-\sqrt{120}$	1/11	1	−0.015842	73	−0.015115
197	$(197-\sqrt{38805})/2$	2/197	0	−0.271540	42	−0.019792
6	$3-2\sqrt{2}$	1/3	1	$\ge +0.3998$	119	−0.026888
195		2/195	0	$\ge +0.0156$		

TABLE 6. Margins (8.2) for Theorem 1.5 (Arb; upper bounds rounded up, and lower bounds for the two exceptions below the line). The right half and $n = 22$ are all covered n with $M_i(n) > -0.03$.

All values in Table 6 rest only on proved constants and on the certified endpoint Φ_U. The tightest cases $n = 71$ and $n = 115$ use the constant $c_2(0) = \frac{1}{2}\log 2$ of Lemma 8.1 exactly.

Proof of Theorem 1.5. Let n be as in the theorem ($n = 2m$ in part (a)), so $n \geq 8$ and, by Proposition 8.7, $M_i(n) < 0$. Suppose $\mathrm{Ti}_2(\beta_n) = a'/b$. For $h \in \mathcal{H}$, Lemma 8.6 gives $F_h(a'/b) \neq 0$, where F_h is the primitive part of $\Delta_h^\natural$. Along $\mathcal{H}$ the arithmetic is bounded by the items above and the size by $\Phi_U - \frac{1}{2}L_i(n)$, so $\limsup_{h\in\mathcal{H}} h^{-2}\log|F_h(\mathrm{Ti}_2(\beta_n))| \leq M_i(n) < 0$, contradicting Proposition 2.4. □

Evidence and limits. At $n = 8$ the exact finite-h scores (Arb analytic part, exact contents) are -0.528, -0.379, -0.336, -0.308 at $h = 16, 32, 48, 64$, and $F_h(T^\natural) \neq 0$ in every computed case. At $n = 2m$ the exact per-prime profiles agree with those of Theorem 1.1 at m up to a few units. The case $n = 4$, i.e. $\mathrm{Ti}_2(2-\sqrt{3})$, is out of reach: $\beta_4 > \sqrt{5}-2$, so the shift lemma points the wrong way, and a numerical solution of the equilibrium problem suggests that the method misses by about 0.04 per h^2. Direct certificates at odd β_n, which reduced the exceptions for χ_2 at odd a from 49 to 39, have not been computed, and neither improvement mentioned after Theorem 1.3 is proved; any of them would shrink $\mathcal{E}_i$.

8.3. **Status of the extensions.** Theorems 1.3 and 1.5 are formalised in Lean (Section 11). Each was checked by one AI referee at a single gate, with code independent of the code used to find the results (Lemma 8.1, the transferred lemmas on exact data, the exception sets by independent interval arithmetic, and for Theorem 1.5 the identity of Lemma 8.4); both referees asked for minor fixes, which were applied. For Theorem 1.5 the referee reviewed the first write-up, and the detailed delta proof written afterwards to apply the fixes has not been re-read by a referee. No human has read either proof.

9. Closed forms and known results

9.1. **Closed forms.** Put $w_m := (1-z_m)/(1+z_m)$, so that Landen's identity reads

$$\chi_2(z_m) + \chi_2(w_m) = \frac{\pi^2}{8} - \frac{1}{2}\log z_m \log w_m. \tag{9.1}$$

For $m = 1$, $z_1 = w_1 = \sqrt{2}-1$, and (9.1) gives $\chi_2(\sqrt{2}-1) = \pi^2/16 - \frac{1}{4}\log^2(1+\sqrt{2})$. For $m = 2$, $w_2 = \varphi^{-1}$, and (9.1) with the value of $\chi_2(\varphi^{-1})$ gives (1.1).

Proposition 9.1. *Let $m \geq 1$. Then $w_m = (\sqrt{m^2+1}-1)/m$, $\sigma(w_m) = -1/w_m$ and $mw_m^2 + 2w_m - m = 0$; so w_m is an algebraic integer if and only if $m \mid 2$. For $m \geq 3$, z_m and w_m are multiplicatively independent, and $z_m \wedge w_m \neq 0$ in $\Lambda^2(\mathbb{K}^\times)\otimes\mathbb{Q}$.*

Proof. The formula for w_m and its quadratic equation are direct computations; the product of the roots is -1, so $\sigma(w_m) = -1/w_m$, and the monic equation $X^2 + (2/m)X - 1$ has integer coefficients iff $m \mid 2$. For $m \geq 3$, w_m is not integral, so $v_{\mathfrak{p}}(w_m) < 0$ at some prime $\mathfrak{p}$, while z_m is a unit. If $z_m^{\alpha} w_m^{\beta} = \pm 1$, then $\beta v_{\mathfrak{p}}(w_m) = 0$, so $\beta = 0$, and then $\alpha = 0$ since $0 < z_m < 1$. Hence z_m, w_m are linearly independent in $\mathbb{K}^{\times} \otimes \mathbb{Q}$, and $z_m \wedge w_m \neq 0$. $\square$

From $\delta \operatorname{Li}_2(x) = -(1-x) \wedge x$ one gets $\delta\chi_2(z) = \pm\frac{1}{2}\, w \wedge z$ modulo torsion, with $w = (1-z)/(1+z)$. The symbol of π^2 and of a product of two logarithms is 0 in Λ^2. So for $m \geq 3$ no dilogarithm identity that preserves the symbol expresses $\chi_2(z_m)$ through π^2 and products of logarithms; for $m = 1, 2$ the symbol vanishes, as Landen's evaluations require. This is a statement about symbols only: an unconditional statement that $\chi_2(z_m)$ is not of this form would need algebraic independence results that are not available.

By (9.1), Theorem 1.1 can equally be stated as $\chi_2(w_m) + \frac{1}{2} \log z_m \log w_m \notin \frac{\pi^2}{8} + \mathbb{Q}$. At $m = 4$, Hakimoglu-Brown [8, Cor. 14] (unrefereed) gives two identities for Rogers' dilogarithm L over $\mathbb{Q}(\sqrt{17})$, in the bases $a = (\sqrt{17} - 3)/4$ and $b = \sqrt{17} - 4$. Their sum isolates $4\operatorname{Li}_2(b) - \operatorname{Li}_2(b^2) = 4\chi_2(b)$, and gives

$$12\chi_2(\sqrt{17} - 4) = 18L(a) - 6L(a^2) + 18L(2a) - 3L(4a^2) - 2\pi^2 \\ - 6\log b \log(1-b) + 3\log b \log(1-b^2),$$

which we checked numerically to 10^{-120}. It is a relation among non-elementary values (a and $1-a$ are not units), not a closed form; Theorem 1.1 at $m = 4$ is equivalent to the irrationality of its right side.

9.2. **Known criteria.** Table 7 compares Theorem 1.1 with the general criteria of Section 1.1, evaluated at z_m at the real place where z_m is small. As there, $\chi_2(z) = z + \frac{z}{4}\Phi_2(\frac{1}{2}, z^2)$, and the height of z_m^{-2} at that place is $\log(1/z_m)$. The evaluations are ours; that of [4, Thm. 1] through $\Phi_2(\frac{1}{2}, z^2)$ was re-derived by the AI referee of Theorem 1.1 from the statement in the arXiv version (arXiv:2010.09167) of [4], and that of [3, Thm. 2.1] by a separate reference check from the statement in arXiv:1912.03811v2. The absolute values of [3] are not normalised by the degree; there, with $r = 2$, the point 1, $\beta = z_m^{-2}$, $x = \frac{1}{2}$ and $\mathbb{K}$ real quadratic, the quantity of Theorem 2.1 is $2\log(1/z_m) - 16 - \log 2\,851\,875 = 2\log(1/z_m) - 30.863487\ldots$, and the table lists half of it. The thresholds in the table and in Section 9.3 are certified in interval arithmetic from these formulas. None of the evaluations has been checked by the authors of the criteria.

At $m = 4$ the first quantity is -13.88 and the third -13.34. The criterion of [4] gives the linear independence of 1, $\Phi_1(\frac{1}{2}, z^2)$, $\Phi_2(\frac{1}{2}, z^2)$ over $\mathbb{K}$, hence $\chi_2(z_m) \notin \mathbb{K}$, a stronger conclusion than ours in its range. So the infinite tail $m \geq 4\,346\,237$ of Theorem 1.1 is due to [4]. Theorem 2.1 of [3], as stated, has the same conclusion and would give it already for $m \geq 2\,517\,046$.

criterion	key quantity at z_m	covers
[4, Thm. 1], via $\Phi_2(\frac{1}{2}, z^2)$	$\log(1/z_m) - 15.977968\ldots$	$m \geq 4\,346\,237$, with $\chi_2(z_m) \notin \mathbb{K}$
[4, Thm. 1], via $\mathrm{Li}_2(\pm z)$	$\frac{1}{2}\log(1/z_m) - 15.78$	$m \gtrsim e^{31.5}/2$
[3, Thm. 2.1] (proof only outlined), via $\Phi_2(\frac{1}{2}, z^2)$	$\log(1/z_m) - 15.431744\ldots$	$m \geq 2\,517\,046$, with $\chi_2(z_m) \notin \mathbb{K}$
[13, Thm. 2.2] (unrefereed)	$\frac{1}{2}\log(1/z_m) - 26.84$	$m \gtrsim e^{53.7}/2$
Bombieri–André via [22] (unrefereed)	height $\frac{1}{2}\log(1/z_m)$ against about 1450	$m \gtrsim e^{2900}$
results at rational points	—	none

TABLE 7. General criteria at z_m (our evaluations).

Its proof, however, is given only in outline, with the details deferred to forthcoming articles; [4, Rem. 1(iii)] refers to it as the result announced without proof, says that the result of [4] is better in general but not always, and its comparison [4, Prop. 4] does not cover the case $r = 2$ with one point used here. What Theorem 1.1 adds is therefore the range $2 \leq m \leq 2\,517\,045$ with $m \notin \mathcal{E}$, by one argument for all m, and, if the announced result of [3] is set aside, the range $2 \leq m \leq 4\,346\,236$. As far as our searches show, no result in the literature covers any m with $2 \leq m \leq 2\,517\,045$; for $m \geq 3$ the searches were limited (Section 1.1), and a MathSciNet search and an expert query are still owed.

9.3. **The extensions.** *Odd a.* Proposition 9.1 extends to every a.

Proposition 9.2. *Let $a \geq 1$, $\alpha = \alpha_a$ and $w := (1-\alpha)/(1+\alpha)$. Then $aw^2 + 4w - a = 0$, so w is an algebraic integer if and only if $a \mid 4$. For $a \notin \{1, 2, 4\}$, in particular for every odd $a \geq 3$, $\alpha \wedge w \neq 0$ in $\Lambda^2(\mathbb{K}^\times) \otimes \mathbb{Q}$.*

Proof. From $\alpha^2 + a\alpha - 1 = 0$, $(1+\alpha)^2(aw^2+4w-a) = a(1-\alpha)^2 + 4(1-\alpha^2) - a(1+\alpha)^2 = -4(\alpha^2 + a\alpha - 1) = 0$. The monic polynomial $X^2 + (4/a)X - 1$ has integer coefficients iff $a \mid 4$, and its roots have product -1. If w is not integral, $v_{\mathfrak{p}}(w) < 0$ at some prime $\mathfrak{p}$, while α is a unit; the rest is the proof of Proposition 9.1. □

For $a = 2m$ this is Proposition 9.1. So among the units α_a only $a = 1$ (φ^{-1}), $a = 2$ ($\sqrt{2} - 1$) and $a = 4$ ($\sqrt{5} - 2$) admit a Landen-type closed form; for odd $a \geq 3$, $\chi_2(\alpha_a)$ has no expression through π^2 and products of logarithms by symbol-preserving dilogarithm identities.

The inverse tangent integral. With $\theta := \arctan \beta_n$ we have $\sin 2\theta = 2\beta/(1+\beta^2) = 2/n$, and

$$\mathrm{Ti}_2(\tan\theta) = \theta \log\tan\theta + \tfrac{1}{2}\big(\mathrm{Cl}_2(2\theta) + \mathrm{Cl}_2(\pi - 2\theta)\big), \qquad \mathrm{Cl}_2(x) = \sum_{k \geq 1} \frac{\sin kx}{k^2},$$

a classical formula (see [18]), which we checked numerically to 80 digits at $n = 5, 8, 12, 22, 71$. For $n \geq 5$ the value $\sin 2\theta = 2/n$ is rational and not in $\{0, \pm\frac{1}{2}, \pm 1\}$, so 2θ is not a rational multiple of π [21, Cor. 3.12]. Only $n = 4$ gives a rational angle ($2\theta = \pi/6$), and there $\mathrm{Ti}_2(2-\sqrt{3}) = \frac{2}{3}G - \frac{\pi}{12}\log(2+\sqrt{3})$.

Proposition 9.3. *Let $n \geq 5$, $\zeta := e^{2i\theta} = (\sqrt{n^2-4}+2i)/n$ and $\mathcal{F} = \mathbb{Q}(i, \sqrt{n^2-4})$. Then ζ is not an algebraic integer, and $\bar{\zeta} \wedge \beta_n \neq 0$ in $\Lambda^2(\mathcal{F}^\times) \otimes \mathbb{Q}$. Modulo torsion, $\bar{\zeta} \wedge \beta_n$ is the symbol of $\mathrm{Ti}_2(\beta_n)$.*

Proof. ζ and $\bar{\zeta} = \zeta^{-1}$ are the roots of $X^2 - bX + 1$ with $b = 2\sqrt{n^2-4}/n$, and $b^2 = 4 - 16/n^2 \notin \mathbb{Z}$ for $n \geq 5$; so b, and hence ζ, is not integral, $v_{\mathfrak{p}}(\zeta) \neq 0$ at some prime $\mathfrak{p}$ of $\mathcal{F}$, and ζ is multiplicatively independent of the unit β_n. For the symbol, $\mathrm{Ti}_2(x) = (\mathrm{Li}_2(ix) - \mathrm{Li}_2(-ix))/(2i)$ and $\delta\,\mathrm{Li}_2(u) = -(1-u)\wedge u$ give, modulo torsion, $\pm\frac{1-i\beta}{1+i\beta}\wedge\beta$, and $(1-i\tan\theta)/(1+i\tan\theta) = e^{-2i\theta} = \bar{\zeta}$. $\square$

So for $n \geq 5$ no symbol-preserving identity expresses $\mathrm{Ti}_2(\beta_n)$ through π^2, G and products of logarithms. The symbol $\bar{\zeta} \wedge \beta_n$ lies in the part $S^1 \wedge \mathbb{R}_{>0}$ of $\Lambda^2(\mathbb{C}^\times) \otimes \mathbb{Q}$, while the symbols of the real values $\chi_2(\alpha_a)$ of Theorems 1.1 and 1.3 do not; this suggests, but does not prove, that the numbers of Theorem 1.5 are not rational combinations of those of Theorems 1.1 and 1.3. As a numerical check, PSLQ at 320 digits with coefficients up to 10^{12} was run on the 20 numbers $\mathrm{Ti}_2(\beta_n)$, 1, π^2, G, $\pi\log\beta_n$, $\log^2\beta_n$, $\pi\log 2$, $\log 2\log\beta_n$, $\log^2 2$, $\chi_2(\beta_n)$, $\mathrm{Li}_2(\beta_n)$, $\sqrt{|D|}\,L(2,\chi_D)$ (with D the discriminant of $\mathbb{Q}(\sqrt{-(n^2-4)})$), $\theta\log\beta_n$, $\theta\pi$, θ^2, $\theta\log n$, and four values $\chi_2(z_{m'})$ with m' near $n/2$ and $m' = n$. It found no relation at $n = 5, 8, 9, 10, 12, 22$. At $n = 6$ it found only a relation without Ti_2 among $\chi_2(\beta_6)$, $\mathrm{Li}_2(\beta_6)$, π^2, $\log^2\beta_6$, as expected since $\beta_6 = (\sqrt{2}-1)^2$, and none after removing Li_2; as a control, at $n = 4$ it found $3\,\mathrm{Ti}_2(\beta_4) = 2G + 3\theta\log\beta_4$.

Known criteria. The evaluations of [4, Thm. 1] and [3, Thm. 2.1] in Section 9.2 use only the real place and the height $\log(1/z)$ of z^{-2} (the point z^{-2} is an algebraic integer, so no denominator enters), so they do not depend on the parity of a: since $\log(1/\alpha_a) = \mathrm{arsinh}(a/2)$, the key quantity $\mathrm{arsinh}(a/2) - 15.977968\ldots$ of [4] changes sign between $a = 8\,692\,473$ and $a = 8\,692\,474$, and that of [3], $\mathrm{arsinh}(a/2) - 15.431744\ldots$, between $a = 5\,034\,091$ and $a = 5\,034\,092$. So [4] gives the stronger $\chi_2(\alpha_a) \notin \mathbb{K}$ for $a \geq 8\,692\,474$, and [3], as stated, for $a \geq 5\,034\,092$. For the inverse tangent integral, $\mathrm{Ti}_2(z) = z + \frac{z}{4}\Phi_2(\frac{1}{2}, -z^2)$: the point 1 is replaced by -1. Both criteria depend on the points only through their absolute values $|\cdot|_v$, and $|-1|_v = 1$ at every place, so every constant is unchanged. With $\log(1/\beta_n) = \mathrm{arcosh}(n/2)$ this gives $\mathrm{Ti}_2(\beta_n) \notin \mathbb{K}$ for $n \geq 8\,692\,474$ by [4], and for $n \geq 5\,034\,092$ by [3] as stated (for even $n = 2m$: $m \geq 4\,346\,237$ and $m \geq 2\,517\,046$). These are our evaluations, not checked by the authors of the criteria. What Theorems 1.3 and 1.5 add is therefore the odd a with $59 \leq a \leq 5\,034\,091$, $a \notin \mathcal{E}_{\mathrm{odd}}$; the odd n with $71 \leq n \leq 5\,034\,091$, $n \notin \mathcal{E}_i$; and the m with $4 \leq m \leq 2\,517\,045$, $m \notin \{5, 7, 9, 15\}$. If the result announced in

[3] is set aside (see Section 9.2), the upper limits become 8 692 473, 8 692 473 and 4 346 236.

Literature. For odd a we searched our own notes and the web (two queries); we found closed forms at φ^{-1} and φ^{-3}, results at rational points and identities, but no irrationality result for χ_2 or Li_2 at the units α_a with odd a. For Ti_2 we searched our own notes, which include an earlier sweep of arXiv (2007–2026) for irrationality results on related values, and the web (two queries); we found only identities for Ti_2 and Cl_2, and no irrationality result for Ti_2 or Cl_2 at points of this kind. These searches were carried out by AI systems; a MathSciNet search and an expert query are still owed, so we make no claim beyond the statement that we know of no earlier result in these ranges.

10. Two points

10.1. **The statement.** Let $n \geq 6$ be even and $P_n(X) := X^4 + nX^3 + 2X^2 - nX + 1$. It is irreducible with four real roots: two in $(0,1)$, which we call $z > w$, and $-1/z$, $-1/w$; moreover $w = (1-z)/(1+z)$, and $z \approx 1 - 2/n$, $w \approx 1/n$. Let $\mathcal{K} := \mathbb{Q}(z)$, a totally real biquadratic field with $\mathrm{Gal}(\mathcal{K}/\mathbb{Q}) = \{1, s, l, sl\}$, $s(z) = -1/z$, $l(z) = w$. Put $u := z - 1/z$; then $u^2 + nu + 4 = 0$, and $\mathcal{F} := \mathcal{K}^s = \mathbb{Q}(u) = \mathbb{Q}(\sqrt{n^2-16})$, on which l acts as the nontrivial automorphism, with $l(u) = 4/u$. With $\varepsilon_k := (k + \sqrt{k^2-4})/2$ we have $zw = \varepsilon_{n+2}^{-1}$ and $z/w = \varepsilon_{n-2}$. Put $T_z := \chi_2(z)$, $T_w := \chi_2(w)$; Landen's identity gives

$$T_n := T_z + T_w = \frac{\pi^2}{8} - \frac{1}{2}\log z \log w = \frac{\pi^2}{8} - \frac{1}{8}\log^2 \varepsilon_{n+2} + \frac{1}{8}\log^2 \varepsilon_{n-2}.$$

Definition 10.1. M1(n) is the statement: $T_n \notin \mathbb{Q}$, or $T_z - T_w \notin \sqrt{n^2-16}\,\mathbb{Q}$.

Equivalently, there is no $\alpha \in \mathcal{F}$ with $T_z = \alpha$ and $T_w = l(\alpha)$: such an α would have $\alpha + l(\alpha) \in \mathbb{Q}$ and $\alpha - l(\alpha) \in \sqrt{n^2-16}\,\mathbb{Q}$, and conversely. (Exchanging z and w only changes the sign of $T_z - T_w$.)

Theorem 10.2 (Theorem M1)**.** M1(n) *holds for every* $n \in \mathcal{N} := \{730, 740, 750, 760, 800, 1000\}$.

The proof is computer-assisted: for each $n \in \mathcal{N}$ separately it uses two dual certificates checked in interval arithmetic. It is a disjunction at six specific values of n, not a statement about a range of n, and it is not formalised.

The case $n = 730$ (Theorem 1.6). Here $z = 0.99726400650629232574\ldots$, $w = 0.0013698707255499998352\ldots$, $\varepsilon_{732} = 366 + \sqrt{133955} = 1/(zw)$, $\varepsilon_{728} = 364 + 11\sqrt{1095} = z/w$ and $\sqrt{730^2-16} = 22\sqrt{1101}$. Numerically $T_{730} = 1.2246689334990568135\ldots$, $T_z = 1.2232990624878816026\ldots$ and $T_w = 0.0013698710111752109\ldots$. The fields are $\mathcal{K} = \mathbb{Q}(\sqrt{133955}, \sqrt{1095})$ and $\mathcal{F} = \mathbb{Q}(\sqrt{1101})$, and the primes ramified in $\mathcal{K}$ are $2, 3, 5, 73, 367$. The other instances are in Table 8; in each case $\mathcal{K} = \mathbb{Q}(\sqrt{d_+}, \sqrt{d_-})$ with $d_\pm$ the squarefree parts of $(n \pm 2)^2 - 4$.

n	$\varepsilon_{n+2} = 1/(zw)$	$\varepsilon_{n-2} = z/w$	$\sqrt{n^2-16}$	ramified in $\mathcal{K}$
730	$366+\sqrt{133955}$	$364+11\sqrt{1095}$	$22\sqrt{1101}$	$2,3,5,73,367$
740	$371+2\sqrt{34410}$	$369+4\sqrt{8510}$	$16\sqrt{2139}$	$2,3,5,23,31,37$
750	$376+5\sqrt{5655}$	$374+5\sqrt{5595}$	$2\sqrt{140621}$	$2,3,5,13,29,373$
760	$381+2\sqrt{36290}$	$379+6\sqrt{3990}$	$12\sqrt{4011}$	$2,3,5,7,19,191$
800	$401+20\sqrt{402}$	$399+20\sqrt{398}$	$4\sqrt{39999}$	$2,3,67,199$
1000	$501+10\sqrt{2510}$	$499+10\sqrt{2490}$	$4\sqrt{62499}$	$2,3,5,83,251$

TABLE 8. The six instances of Theorem 10.2.

10.2. **What changes at two points.** The machine is that of Section 2 with z the root above: the same poles, atoms and functional, now with $A, B\colon \mathbb{Q}[t] \to \mathcal{K}$ and $\Delta_h \in \mathcal{K}[X]$. We write $y = z^2$; then $1 - y = -uz$. Since u is an algebraic integer with $u\,l(u) = 4$, u divides 4, and z, w are units of $\mathcal{K}$ (roots of a monic polynomial with constant term 1).

Reflection. Lemma 2.2 holds with $\sigma = s$, the generator of $\mathrm{Gal}(\mathcal{K}/\mathcal{F})$: its proof used only $\sigma(z) = -1/z$ and that the coefficients are rational functions of z with rational coefficients. So the odd-index entries lie in $e\,\mathcal{F}[X]$ with $e := z + 1/z$, $s(e) = -e$, $e^2 = u^2 + 4 = -nu \in \mathcal{F}$, and the rescaled matrix $E_{i+j}e^{[i\text{ odd}]-[j\text{ odd}]}$ shows $\Delta_h \in \mathcal{F}[X]$. The reflection removes the conjugates $-1/z$ and $-1/w$, where the series diverge.

The second place. The remaining conjugate, l, is not removed but convergent: the coefficients of A and B are rational functions of z, so $l(\Delta_h)(T_w) = \det(\Lambda_w(t^{i+j}))$, the Hankel determinant of the same machine at the weight w (the series converge since $0 < w < 1$). The two embeddings of $\mathcal{F}$ thus see the z-machine and the w-machine, and the endgame uses the norm $\mathcal{F} \to \mathbb{Q}$.

Odd primes. At every prime of $\mathcal{K}$ above an odd p, z, y and $1 - y = -uz$ are units. These are the only properties of z that Lemmas 3.1–3.5 and Theorem 3.7 use, so they hold verbatim, with the valuation normalised by $v(p) = 1$; there are no special odd primes, including the ramified ones and those dividing $n^2 - 16$. The bound is the same at every prime $\mathfrak{p}$ of $\mathcal{F}$ above p.

The prime 2. Here Lemma 4.3 fails: at $n = 400$, $h = 48$, the exact 2-adic content is 388.5, above $h\lfloor\log_2 h\rfloor + 3h = 384$. It is replaced by the following weaker bound, which is the analogue over $\mathcal{K}$ of the case $\nu = 1$ of Proposition 4.2.

Lemma 10.3 (Lemma 10″)**.** *Let n be even and $\mathfrak{P}$ a prime of $\mathcal{K}$ above* 2*, with $v = v_{\mathfrak{P}}$ normalised by $v(2) = 1$. Then $-v(\Delta_h) \le (h-2)^2/4 + 3h$.*

Proof. Since n is even, $u/2$ is a root of $X^2 + (n/2)X + 1$, hence a unit; so $\nu := v(1-y) = v(u) = 1$, instead of 2. Follow the proof of Lemma 4.1, which up to the valuations does not involve z. In (L), $y^b/(1-y)^{b+1}$ now has valuation $-(b+1)$, so $v(G_{ij}) \ge 1 - \pi + \min_{b<\pi}(v_2(b!) - 1) = -\pi$. In (U), with $v(\eta) = 1$, the first part has valuation $2 + s_2(N_0)$ and the second part valuation

$\geq 1 + s_2(N_0)$, so $v(G_{ij}) \geq 2$ for $N_0 \geq 0$; the bound no longer grows with N_0. Hence $v(G_{ij}) \geq -f(i+j)$ with $f(s) := (s-K)_+$, a convex function, and $v(\tilde{\Delta}_h) \geq -\max_\sigma \sum_i f(i+\sigma(i))$. If $i < i'$ and $\sigma(i) > \sigma(i')$, put $c = i + \sigma(i)$, $d = i' + \sigma(i')$, $a = i + \sigma(i')$, $b = i' + \sigma(i)$; then $a + b = c + d$ and $a < c, d < b$, so convexity gives $f(c) + f(d) \leq f(a) + f(b)$, and removing inversions does not decrease the sum. So the identity maximises, and $\sum_{i<h} f(2i) = 1 + 3 + \cdots + (h-3) = (h-2)^2/4$. Finally $-v(\Delta_h) = 3h - v(\tilde{\Delta}_h)$. □

So the prime 2 costs $\frac{1}{4}\log 2 = 0.1733\ldots$ per h^2 instead of 0. The mechanism that made it free in Lemma 4.3 is not available anywhere in the family:

Proposition 10.4. *For every n there is a prime $\mathfrak{P}$ of $\mathcal{K}$ above 2 with $v_\mathfrak{P}(1-y) \leq 1$ (normalised by $v_\mathfrak{P}(2) = 1$). In particular $v_2(1-y) = 2$, which holds at $\sqrt{5} - 2$ and makes the prime 2 free there, cannot hold at every prime above 2.*

Proof. z is a unit, so $v_\mathfrak{P}(1-y) = v_\mathfrak{P}(u)$. From $u\,l(u) = 4$ we get $v_\mathfrak{P}(u) + v_\mathfrak{P}(l(u)) = 2$, and $v_\mathfrak{P}(l(u)) = v_{l^{-1}(\mathfrak{P})}(u)$. If $v_\mathfrak{P}(u) \geq 2$ for every $\mathfrak{P} \mid 2$, then $v_\mathfrak{P}(l(u)) \leq 0$ for every $\mathfrak{P}$, so $v_{l^{-1}(\mathfrak{P})}(u) \leq 0$, a contradiction. □

For even n, $\nu = 1$ at every prime above 2; for odd n, $\nu \in \{0, 2\}$, and the prime with $\nu = 0$ is worse, so odd n are not used. Proposition 10.4 does not show that the loss $\frac{1}{4}\log 2$ is necessary: exact profiles suggest that the true 2-adic cost is about $\frac{1}{8}\log 2$ per h^2, but we have not proved this.

The analytic part. Theorem 5.1 holds at a general weight $q \in \{z, w\}$, with $L := \log(1/q)$, the field $v_q(x) := -Lx - \Psi(x)$, and three changes. (a) v_q is strictly decreasing on $(-\tanh L, \infty)$, which contains $[-1/h, \infty)$ if and only if $h > \coth L$; this is the range of h ($\coth L_z = 364.9986\ldots$ at $n = 730$, and $\coth L_w < 1.0001$). (b) The dual density is only assumed ≥ 0, of mass 1, with $\operatorname{supp}\mu \subset [0, X_e]$ for some $X_e \geq 4$ (at the z-place the support is long, $X_e \approx 1300$), and $\Phi_q(\mu, \lambda) := -E(\mu) + \lambda + \int_0^{X_e} (F_\mu - \lambda)_+\,dx$ with $F_\mu = 2U^\mu + v_q$. (c) The tail condition becomes: with $B(x) := 2\log x - \log(x-1) - Lx$, both $\beta := \lambda - B(X_e)$ and $\gamma := L - (X_e - 2)/(X_e(X_e-1))$ are positive. With $M_0 := 2\log(X_e+1) + 1 + L - \lambda$ and $\kappa := 2/(1 - e^{-2\gamma})$, the proof of Theorem 5.1 gives, for every even $h > \coth L$,

$$\log\bigl|\det(\Lambda_q(t^{i+j}))_{i,j<h}\bigr| \leq h^2\Phi_q(\mu,\lambda) + E_q(h),$$

$$E_q(h) := 2h\log(4h) + h\log 4 + (hX_e/2 + 2)\log 3 + h\,c_1(h) + 2.362h - h\log h + h\max(M_0, 0) + \kappa.$$

The pattern count below $X_e h$ replaces $(2h+1)\log 3$, the cell $[-1/h, 0]$ costs $h\max(M_0, 0)$ (here $U^\mu \leq \log(X_e+1)$ and $v_q \leq L+1$ there), and the far nodes cost κ. Since $E_q(h) = O(h\log h)$, the lim sup is again Φ_q.

Nonvanishing. Lemma 6.1 holds over $\mathcal{F}$ for $h = 3\ell - 1$ with $\ell \geq 7$ prime and unramified in $\mathcal{K}$. Now $\zeta := z^\ell \bmod \mathfrak{l}$ is $\mathrm{Frob}_\mathfrak{l}(z)$, one of four values:

$\mathrm{Frob}_{\mathfrak{l}}$	1	s	l	sl
ζ	z	$-1/z$	w	$-1/w$
$1-\zeta^2$	$-uz$	u/z	$-(4/u)w$	$(4/u)/w$
$1-\zeta^{-2}$	u/z	$-uz$	$(4/u)/w$	$-(4/u)w$

Every entry is an $\mathfrak{l}$-unit because $u \mid 4$ and ℓ is odd, and steps 1–5 of Lemma 6.1 use only these units and $1-z^{-2}=u/z$. Since ℓ is unramified, $v_{\mathfrak{l}}$ restricts to $v_{\mathfrak{L}}$ on $\mathcal{F}$ for $\mathfrak{L}=\mathfrak{l}\cap\mathcal{F}$, and step 6 gives $\Delta_h(\alpha)\neq 0$ for every $\alpha\in\mathcal{F}$ with $v_{\mathfrak{L}}(\alpha)\geq -1$, whether ℓ is split or inert in $\mathcal{F}$.

Lemma 10.5 (Lemma E′, the norm endgame)**.** *Let $n\geq 6$ be even, and suppose that for each $q\in\{z,w\}$ there is a pair (μ_q,λ_q) as in (b) above ($\mu_q\geq 0$, mass 1, $\operatorname{supp}\mu_q\subset[0,X_e]$ with $X_e\geq 4$; the cap $\mu_q\leq 1$ is not required) satisfying the tail condition (c), with*

$$M_1(n):=\tfrac{1}{2}\big(\Phi_z(\mu_z,\lambda_z)+\Phi_w(\mu_w,\lambda_w)\big)+\tfrac{4}{3}+\tfrac{1}{4}\log 2<0.$$

Then M1(n) *holds.*

Proof. Suppose $\alpha\in\mathcal{F}$ has $T_z=\alpha$ and $T_w=l(\alpha)$, and fix $b\geq 1$ with $b\alpha\in\mathcal{O}_{\mathcal{F}}$. Let $\mathcal{H}:=\{3\ell-1:\ell\geq 7 \text{ prime, unramified in } \mathcal{K},\ \ell\nmid b,\ 3\ell-1>\coth L_z\}$, an infinite set; note $\coth L_w<\coth L_z$. For $h\in\mathcal{H}$, Lemma P′ over $\mathcal{F}$ gives $\Delta_h(\alpha)\neq 0$. For a prime $\mathfrak{p}$ of $\mathcal{F}$ put $n_{\mathfrak{p}}:=-v_{\mathfrak{p}}(\Delta_h)$, which may be a half-integer at ramified primes, and $n_p:=\max(0,\max_{\mathfrak{p}\mid p}\lceil n_{\mathfrak{p}}\rceil)$, $c_h:=\prod_p p^{n_p}$. Then $c_h\Delta_h\in\mathcal{O}_{\mathcal{F}}[X]$, and by the odd-prime lemmas and Lemma 10.3,

$$\log c_h\leq\sum_{3\leq p<K}B_p\log p+\big((h-2)^2/4+3h\big)\log 2+\theta(K)+\log 2,$$

where B_p are the bounds used in Theorem 3.7; the term $\theta(K)=O(h)$ pays for all ceilings. So $\limsup_{h\in\mathcal{H}}h^{-2}\log c_h\leq\frac{4}{3}+\frac{1}{4}\log 2$. The algebraic integer $\beta:=b^h c_h\Delta_h(\alpha)$ is nonzero. Its two real embeddings are $b^h c_h\Delta_h(T_z)=b^h c_h\det(\Lambda_z(t^{i+j}))$ and $b^h c_h\, l(\Delta_h)(T_w)=b^h c_h\det(\Lambda_w(t^{i+j}))$, so $N_{\mathcal{F}/\mathbb{Q}}(\beta)=b^{2h}c_h^2\det(\Lambda_z(t^{i+j}))\det(\Lambda_w(t^{i+j}))$ is a nonzero integer. Taking logarithms, dividing by $2h^2$, and applying the analytic bound at both places, $0\leq\frac{4}{3}+\frac{1}{4}\log 2+\frac{1}{2}(\Phi_z+\Phi_w)=M_1(n)<0$, a contradiction. □

10.3. **The certificates.** For each $n\in\mathcal{N}$ and $q\in\{z,w\}$ the certificate is a piecewise constant $\mu_q\geq 0$ on a grid of $[0,X_e]$ with mass exactly 1, a rational λ_q, a rigorous Arb upper bound for $\Phi_q(\mu_q,\lambda_q)$, and an Arb check of the tail condition with $X_e\geq 4$; the logarithm $L=\log(1/q)$ is an Arb ball computed from the quartic, so every bound encloses the value at the true L. Two independent evaluators were used, and where both certified the same (n,q) the smaller bound is taken. Table 9 lists the results; $\frac{4}{3}+\frac{1}{4}\log 2=1.50662012847331966\ldots$, and every $M_1(n)$ is computed in Arb from the printed balls and lies entirely below 0.

n	$\Phi_z \le$	$\Phi_w \le$	$M_1(n) \le$
730	0.97551590832772	−3.98960878950	−0.000426312
740	0.98154355971108	−3.99641267680	−0.000814430
750	0.98568546476333	−4.00312517276	−0.002099725
760	0.98610919549783	−4.00974867868	−0.005199613
800	1.00091072617	−4.03494034059	−0.010394678
1000	1.05765433525	−4.14651125316	−0.037808330

TABLE 9. Certified bounds for Theorem 10.2 (upper endpoints of Arb balls, rounded up). At the z-place $X_e = 1300$ for $n \le 760$, and $X_e \approx 1202$ and 1503 for $n = 800$, 1000; at the w-place $X_e \approx 4$.

The constant Φ_z is positive because $z \approx 1-2/n$ gives the z-machine almost no decay; the w-place, with $w \approx 1/n$, pays for it. The side constants are comfortable: at $n = 730$, for example, $\beta = 0.5719$, $\gamma = 0.001971$, $M_0 \le 11.17$, $\kappa \le 508.4$ at the z-place, and $\beta = 16.41$, $\gamma = 6.426$ at the w-place. A third evaluator, written independently of both, confirms $M_1(n) < 0$ for all six n and gives $M_1(730) \le -0.000452$. As a validation, the three codes applied at the weight $\sqrt{5}-2$ of Corollary 1.2 give -1.35926599, -1.35939764 and -1.35939884, consistent with Theorem 5.3 and the primal value -1.3594016 of Remark 5.4.

10.4. **Scope.** We state the limits of Theorem 10.2 explicitly.

- It is a disjunction. We do not know which of the two statements holds, and neither is known by other means: $T_n \notin \mathbb{Q}$ is a statement of strong-four-exponentials type, and we know of no result on $T_z - T_w$.
- For very large n general criteria apply. Take the shift $\frac{1}{2}$ at w^2 (the point 1, $\beta = w^{-2}$), at the real place where w is small; the conjugates of w^{-2} are w^{-2}, z^{-2}, w^2, z^2. By our evaluation, the condition of [4, Thm. 1], $\frac{1}{2}\log(1/w) - \log(1/z) - 15.977968\ldots > 0$, holds for every even $n \ge 75\,559\,095\,378\,072 \approx 7.56 \cdot 10^{13}$, and that of [3, Thm. 2.1] (as stated, with its proof only outlined; see Section 9.2), $2\log(1/w) - 32 - \log(5^4 \cdot 3^5 \cdot 13^4) - 4\log(1/z) - 2\log(4+9z^{-2}) > 0$, for every even $n \ge 7\,608\,252\,420\,528 \approx 7.61 \cdot 10^{12}$. Either gives $\chi_2(w) \notin \mathcal{K}$, and hence M1($n$), for those n. The thresholds are certified in interval arithmetic from these formulas; our reading of the two statements has not been checked by a referee or by their authors. For $730 \le n \le 1000$ the two quantities (the second divided by $[\mathcal{K} : \mathbb{Q}] = 4$) lie between -12.7 and -12.5, and between -11.6 and -11.3, so Theorem 10.2 is not implied.
- It is proved for the six listed values of n only, each with its own certificates. Even n between them are not covered, and odd n are

not covered at all: Lemma 10.3 needs n even, and for odd n one prime above 2 has $v(1-y)=0$.

- For all even $n \geq 730$ the statement is supported only numerically. A proof would need interval certificates in n (the dual bound is monotone in L, but this needs to be written) and an argument as $n \to \infty$; neither exists.
- The method cannot reach $n = 720$: there the primal value of the equilibrium problem gives $M_1(720) \geq +0.00098$ (computed numerically, with a density admissible up to rounding of order 10^{-12}), so no dual certificate can succeed. Numerically the threshold of the method lies in $(720, 730)$.
- The theorem is not formalised in Lean. Its proof was checked by two AI referees and has not been read by a human.

11. The formalisation

11.1. **What is formalised.** The Lean project [16] proves Theorem 1.1, Theorem 1.3, Theorem 1.5, Corollary 1.2 and the identity (1.1) in Lean 4 [6] (version v4.35.0) with Mathlib [30] (version v4.35.0 [31]). The machine is formalised once, with the trace a of the unit $\alpha_a = (\sqrt{a^2+4}-a)/2$ as a parameter, so that Corollary 1.2 is the case $a = 4$, Theorem 1.1 the case $a = 2m$ and Theorem 1.3 the case of odd a. Theorem U was proved first; Theorem V adds the ingredients that depend on m: Lemma 4.1, Lemma 3.8, the shift lemma, the margins, and Lemma 6.1 for every m; Theorem 1.3 adds Lemma 8.1, eleven direct certificates and the margins at odd a. Theorem 1.5 uses a copy of the machine with the sign fixed, in the folder `Ti/` (namespace `Chi2U.Ti`, 35 modules), with the trace n of β_n as parameter; no file of the other theorems is changed. The project has 130 modules under `Chi2U/`, with about 30,600 lines, of which about 7,800 are ported verbatim from the formalisation of [15] [17]: the prime number theorem, the interval arithmetic, the generic analytic files, the criterion and the definition of χ_2. All declarations are in the namespace `Chi2U`. The project is available at `https://github.com/telaviv/chi2-quadratic-units-irrationality` [16], together with a copy of this paper and the scripts of Section 12. Theorem 10.2 is not formalised. Corollary 1.4 and the statement on the units of norm $+1$ after Theorem 1.5 are not stated in Lean; the first follows from Theorems 1.1 and 1.3 as explained after its statement, the second from Theorem 1.5, since every unit of norm $+1$ in $(0,1)$ is some β_n. The Clausen form of Theorem 1.5, through the classical formula (see [18]), is not formalised either.

11.2. **The statement.** The definitions that the reader has to check are in `Defs.lean`, `Statement.lean`, `V/Defs.lean`, `VOdd/Defs.lean`, `Ti/Defs.lean` and `Golden/Corollary.lean`:

```
noncomputable def chi2Term (z : ℝ) (n : ℕ) : ℝ :=
    z ^ (2 * n + 1) / (2 * (n : ℝ) + 1) ^ 2
noncomputable def chi2 (z : ℝ) : ℝ := ∑' n : ℕ, chi2Term
```

```
z n
def V.TheoremV : Prop := ∀ m :  ℕ, 2 ≤ m → m ∉ ({3, 5, 7,
9, 15} :  Finset ℕ) →
    Irrational (chi2 (Real.sqrt ((m :  ℝ) ^ 2 + 1) - m))
theorem V.theoremV : V.TheoremV
def VOdd.EOdd :  Finset ℕ := {5, 7, 9, 11, 13, 15, 17, 19,
21, 23, 25, 27, 29, 31, 33, 35,
    37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 63, 65, 69,
75, 77, 81, 85, 87, 93,
    99, 105, 165}
def VOdd.TheoremVOdd :  Prop := ∀ a :  ℕ, Odd a → 5 ≤ a →
a ∉ EOdd →
    Irrational (chi2 ((Real.sqrt ((a :  ℝ) ^ 2 + 4) - a) /
2))
theorem VOdd.theoremVOdd :  VOdd.TheoremVOdd
noncomputable def Ti.ti2Term (z :  ℝ) (k :  ℕ) :  ℝ :=
    (-1) ^ k * z ^ (2 * k + 1) / (2 * (k :  ℝ) + 1) ^ 2
noncomputable def Ti.Ti2 (z :  ℝ) :  ℝ :=
    ∑' k :  ℕ, ti2Term z k
def Ti.EOddI : Finset ℕ := {75, 77, 81, 85, 87, 91, 93,
    95, 99, 105, 111, 117, 123, 135, 147, 165, 195}
def Ti.TheoremVi :  Prop :=
    (∀ m :  ℕ, 4 ≤ m → m ∉ ({5, 7, 9, 15} :  Finset ℕ) →
      Irrational (Ti2
        (m - Real.sqrt ((m :  ℝ) ^ 2 - 1)))) ∧
    (∀ n :  ℕ, Odd n → 71 ≤ n → n ∉ EOddI →
      Irrational (Ti2
        ((n - Real.sqrt ((n :  ℝ) ^ 2 - 4)) / 2)))
theorem Ti.theoremVi :  Ti.TheoremVi
def TheoremU : Prop := Irrational (chi2 (Real.sqrt 5 - 2))
theorem chi2_sqrt5_sub_two_irrational :  TheoremU
theorem chi2_sqrt5_sub_two_eq :
    chi2 (Real.sqrt 5 - 2) = Real.pi ^ 2 / 24 - 3 / 4 *
Real.log goldenRatio ^ 2
theorem golden_irrational' :
    Irrational (Real.pi ^ 2 - 18 * Real.log goldenRatio ^ 2)
```

`V.TheoremV` and `V.theoremV` are in `V/Defs.lean` and `V/Interface.lean`, and `VOdd.EOdd`, `VOdd.TheoremVOdd` and `VOdd.theoremVOdd` in `VOdd/Defs.lean` and `VOdd/Interface.lean`; `EOdd` is the set $\mathcal{E}_{\mathrm{odd}}$ written out. `Ti.Ti2`, `Ti.EOddI`, `Ti.TheoremVi` and `Ti.theoremVi` are in `Ti/Defs.lean` and `Ti/Interface.lean`; `Ti2` is the inverse tangent integral, it does not depend on `chi2`, and `EOddI` is the set $\mathcal{E}_i$ written out. Of the other three theorems, the first is in `Main.lean`, the second in `Golden/ClosedForm.lean`, the third in `Golden/Corollary.lean`. The definition of χ_2 is identical to that of [17], and `Defs.lean` proves summability for $|z| \le 1$, so `chi2 (Real.sqrt 5 - 2)` and, since $0 < \sqrt{m^2+1} - m < 1$ and $0 < (\sqrt{a^2+4} - a)/2 < 1$, the values of `chi2` in `V.TheoremV` and `VOdd.TheoremVOdd` are the genuine

sums of the series. Likewise `Ti/PoleExpansion.lean` proves summability of the series of `Ti2` for $|z| \le 1$, and $0 < \beta_n < 1$, so the values of `Ti2` in `Ti.TheoremVi` are the genuine sums. The internal point `zT n` $= n/2 - \sqrt{(n/2)^2 - 1}$ is rewritten into the forms of the statement by proved lemmas. `goldenRatio` is Mathlib's $(1 + \sqrt{5})/2$ and `Irrational` is Mathlib's predicate $x \notin \mathbb{Q}$. The identity (1.1) is proved from Landen's identity for χ_2, Euler's reflection formula and Landen's identity for Li_2 (`Golden/Landen.lean`, `Golden/Dilog.lean`), so the golden-ratio form is not a separate assumption.

The command `lake env lean Axioms.lean` prints, among its other lines,

```
'Chi2U.V.theoremV' depends on axioms:
    [propext, Classical.choice, Quot.sound]
'Chi2U.VOdd.theoremVOdd' depends on axioms:
    [propext, Classical.choice, Quot.sound]
'Chi2U.Ti.theoremVi' depends on axioms:
    [propext, Classical.choice, Quot.sound]
'Chi2U.chi2_sqrt5_sub_two_irrational' depends on axioms:
    [propext, Classical.choice, Quot.sound]
'Chi2U.golden_irrational'' depends on axioms:
    [propext, Classical.choice, Quot.sound]
```

and the same list for `chi2_sqrt5_sub_two_eq` and for each of the main ingredients listed in `Axioms.lean`. These are Lean's three standard axioms. The development contains no `sorry`, no added axiom, no `native_decide`, no `implemented_by` and no `@[extern]`, so no step relies on compiled code.

11.3. **Correspondence with the paper.** Table 10 lists the main formal counterparts. Paths are relative to `Chi2U/`. The machine files take the trace a as a parameter, called `az` there (`Qd az` = `QuadraticAlgebra` $\mathbb{Q}$ $((a/2)^2+1)$ 0, `zK az`, `Delta az h`); Theorem 1.1 uses $a = 2m$, and the entries marked $m = 2$ are used only for Corollary 1.2. The files of Theorem 1.5 take the trace n of β_n as a parameter (`Qt n` = `QuadraticAlgebra` $\mathbb{Q}$ $((n/2)^2 - 1)$ 0, `zK n`, `Delta n h`, in the namespace `Chi2U.Ti`); most of their declarations have the names of their χ_2 counterparts.

11.4. **Differences from the written proof.** The formal proof establishes what the endgame needs, and in several places it takes a route that was shorter in Lean.

- *The machine.* Δ_h is *defined* as the determinant of the rational coordinates of the matrix $\tilde{E}$ of Corollary 2.3, so $\Delta_h \in \mathbb{Q}[X]$ holds by definition; Lemma 2.2 and Corollary 2.3 then identify it with the Hankel determinant over $\mathbb{K}$ and its value at T with $\det(\Lambda(t^{i+j}))$.
- *Theorem B'.* The formal proof does not follow Section 5. It follows the lattice-energy route of the formal proof of Theorem B in [15]: the Andreief identity for the node expansion, the bound $h^h(1+2/h)^{h^2}$ for determinants with derivative rows via averages over roots of unity

(instead of step 3), and a discrete Pólya-type lattice energy inequality comparing the patterns, which live on the odd nodes, with a reference sequence on all of $\mathbb{N}$ (instead of steps 5–6). The ingredient not present in [17] is the two-sided Jensen bound of step 4 (`log_abs_Dpoly_ge`), which gives the node weight (5.3). This route evaluates the dual at the nodes pointwise, so it never meets the cell $[-1/h, 0]$ and does not use the monotonicity (5.1) (which is nevertheless proved, as `vU_strictAntiOn`, on $[-\frac{1}{2}, \infty)$). It is stated asymptotically, like Theorem B of [15]: the explicit $E_{\mathrm{an}}(h)$ is not formalised.

- *The certificate.* A formal dual certificate (`DualCertU`) consists of a step density on cells of $[0, 2]$, a grid of width $1/L$ on $[0, X]$ with $X = 4$, rational cell bounds S_k for $2U^\mu + v$, and the tail conditions; its exponent is $-E(\mu) + \lambda + \frac{1}{L}\sum_k (S_k - \lambda)_+$. The certificates are written in a linear-majorant form (`LinCertU`, `Cert/Lin.lean`): on the k-th grid cell, $2U^\mu + v$ is bounded by an affine function ℓ_k with endpoint values $e_{0,k}$, $e_{1,k}$, obtained from chords and tangents of each term, as in [15], which keeps the cancellation between neighbouring cells without the near/far split of Section 5.3. The cell is charged $\frac{1}{2L}\big((e_{0,k} - \lambda)_+ + (e_{1,k} - \lambda)_+\big)$, the trapezoid rule, which by convexity is at least the integral of $(\ell_k - \lambda)_+$ over the cell. Splitting each cell into r subcells and taking on each the larger endpoint value of ℓ_k gives a certificate of the first kind whose exponent exceeds the linear-majorant one by at most $B/(2Lr)$, $B = \sum_k |e_{1,k} - e_{0,k}|$ (`LinCertU.exists_dual`), so the analytic theorem is applied unchanged. For Corollary 1.2 the step density has 10 cells of width $\frac{1}{5}$ and the grid has $L = 40$ (160 cells); the certified value is $\Phi_{\mathbb{Q}} = -27083980489111/20000000000000 \approx -1.3541990$, coarser than Theorem 5.3 but enough. The formal statements are $\Phi < \Phi_L := -1.3538792$ (`exists_certU'`; the rounded constant Φ_L is the one used by the shift lemma below) and $\Phi < -27/20$ (`exists_certU`); the assembly (`boundsU`) uses $\beta = -27/20$, so the formal margin is $\frac{4}{3} - \frac{27}{20} = -\frac{1}{60}$.
- *Lemma 8′.* For the generic primes only the single-level range $K/3 \le p$ used by Theorem 3.7 is formalised, by a crude row bound on the Lagrange Gram matrix and Lemma 3.4 instead of the atom expansion. The multi-level statement needed by Lemma 3.8 is proved directly (`V/LemmaS` and its helper files) instead of through the expansion of Lemma 7′: a Schur-complement form of the determinant, a multilinear expansion of its bottom rows, and a Newton (divided-difference) factorisation that extracts the squared Vandermonde factor. The polynomial factor enters through integer row and column exponents $(i - J)_+$, $(j - J - 1)_+$ with the same total $(J - 1)^2$. This proof does not use $p \mid m$.

- *Lemma* 10_ν. The Karamata step is replaced by a weighted Leibniz bound: by concavity $\hat g(2i)+\hat g(2j) \le 2\hat g(i+j)$, so integer row weights $\lfloor \hat g(2i)/2 \rfloor$ and column weights $\lceil \hat g(2j)/2 \rceil$ give $v_2(\det G) \ge \sum_i \hat g(2i)$. Some entry bounds are coarser than in Lemma 4.1, where the slack of its final bound allows it. For odd m no half-integral valuations occur: every bound is a rational power of 2 times an element of the order.
- *The shift lemma* is applied to the formal certificate of Theorem U, in its discrete format: each grid cell is split into r subcells, and the left-endpoint sum loses $L(m)/(2rL)$, which is absorbed by taking r large. So the formal analytic constant at z_m is $\Phi_L - \frac{1}{2}L(m) + \delta$ for any $\delta > 0$, with $\Phi_L = -1.3538792$ in place of $\Phi(z_2)$ of Theorem 5.3.
- *The margins* are proved with this coarser constant, and the exception set is still exactly $\mathcal{E}$. Even $m \ge 4$ and odd $m \ge 69$ are handled by a closed form written with $\log m$ (the tightest case is $m = 69$, with slack about 10^{-3}), $m = 2$ exactly, and the 28 odd $m \in [11, 67] \setminus \mathcal{E}$ one by one with rational bounds on the logarithms (for instance $\log(2+\sqrt{5}) \le \frac{25}{12}\log 2$).
- *Lemma* P'. Congruences are taken in a residue field of the order at a prime above ℓ, with $\zeta := z^\ell$ itself; no identification of ζ by Frobenius is needed, only $1 - \bar z^{2\ell} = (2m\bar z)^\ell$. The formal Lemma P′ (`lemmaP`) needs only $\ell \nmid 2m$, for every m; the instance used for Theorem 1.1 (`lemmaP_m`) assumes $\ell \nmid 2m(m^2+1)$, which still leaves infinitely many ℓ. The progression sums are handled by an exact shift identity instead of the Newton form.
- *The endgame* uses only even h, through $P_h := \Delta_{2\lfloor h/2 \rfloor}$ in the criterion of [17].
- *Lemma* 10_0. As for Lemma 10_ν, the Karamata step is replaced by the weighted Leibniz bound: by concavity $g(2i) + g(2j) \le 2g(i + j)$, and since h is even every $g(2i)$ is even, so the integer row and column weights $\lfloor g(2i)/2 \rfloor$, $\lceil g(2j)/2 \rceil$ lose nothing and give $v(\tilde\Delta_h) \ge \sum_i g(2i) = 2h - h^2/2$. The bound (U) is used in the weaker form $v(G_{ij}) \ge 2 - N_0$, which is all the tent minorant needs.
- *The analytic constant at odd* a is $\Phi_L - \frac{1}{2}L(a)$ (shift lemma), with $\Phi_L = -1.3538792$, which is 0.0053 weaker than Φ_U. So $a = 123$, where the shift lemma alone gives $M_{\mathrm{odd}}(123) \le -0.000034$ in Proposition 8.2, has a positive formal shift margin and needs a direct certificate too: the formal proof uses eleven certificates, at $a = 59, 61, 67, 91, 95, 111, 117, 123, 135, 147, 195$, against the ten of Table 5. They are Lean's own certificates in the linear-majorant form, on grids with L between 20 and 100 (against $N = 1600$ in Table 5), and their values lie 0.0006 to 0.015 above those of Table 5 (for example -2.718256 against -2.730260 at $a = 59$). The formal margins

are therefore thinner: the thinnest is at $a = 135$, about -0.00025 (against -0.000887), followed by $a = 59$, about -0.0014.

- *The margins at odd a.* Instead of the bound $C_0 - \frac{1}{8}\log a$ and the Arb table for $a \le 5001$, the formal proof splits off the primes 3, 5, 7 in $\sum_{p^e \| a} c_p(e)$, which gives a closed form negative for every odd $a \ge 275$, and checks the 96 odd $a < 275$ outside $\mathcal{E}_{\mathrm{odd}}$ one by one, with rational enclosures of the logarithms. Only the "if" part of Proposition 8.2 is formalised; the "only if" part is not needed for Theorem 1.3.
- *Nonvanishing at odd a* uses `lemmaP_m` with $\ell \ge 7$, $\ell \nmid a(a^2+4)$, along $h = 3\ell - 1$, instead of the set $\mathcal{H}$ of Section 8.1 with $\ell \ge 37$; infinitely many ℓ remain.
- *Theorem 1.5: a copy of the machine.* Lemma 8.4 is not used. The χ_4-twisted machine is defined directly over $\mathbb{K} = \mathbb{Q}(\sqrt{(n/2)^2 - 1})$ with the weights $\chi_4(t)\beta^t$, i.e. with $y = -\beta^2$, $1 - y = n\beta$ and $\sigma(\beta) = 1/\beta$, by a copy of the files of the χ_2 machine with the sign fixed; each copied file names its original. Lemma 2.2, Corollary 2.3, the valuation lemmas and Lemma 8.6 are proved again for this machine, with the proofs of the χ_2 files and the departures listed above (Lemma 8′, Lemmas 10_ν and 10_0, Lemma P′); the signs $\chi_4(t) = \pm 1$ are units and do not change any bound. A sign parameter in the shared files would have touched every proof that unfolds a weight.
- *The prime 2 at even n.* The valuations are taken in the order $\mathbb{Z}_{(2)}[\sqrt{d}]$, $d = (n/2)^2 - 1$, which is not the maximal order when $n \equiv 2 \bmod 4$. Every bound is obtained from rational powers of 2 times elements of this order, so it is a valid lower bound for the valuation at the prime above 2.
- *Lemma P'_n* is stated for primes $\ell \ge 7$ with $\ell \nmid n(n^2-4)$, as `lemmaP_m` for odd a, and used along $h = 3\ell - 1$ instead of the set $\mathcal{H}$ of Section 8.2; the proof uses only $\ell \nmid n$, and infinitely many ℓ remain. It is read in $\mathbb{K}$, in a residue field at a prime above ℓ, so no case split by the Frobenius class of ℓ is needed.
- *The analytic constant for Theorem 1.5.* The formal Theorem B′ above holds at β_n with the signed weights, since only $|\chi_4(t)\beta^t| = \beta^t$ enters (`DualCertT.analytic`). A copy of the shift lemma moves the formal certificate of Corollary 1.2 to β_n, so the formal analytic constant is $\Phi_L - \frac{1}{2}L_i(n) + \delta$ for any $\delta > 0$. No further certificate is needed.
- *The margins of Theorem 1.5* are proved with Φ_L, which is 0.0053 weaker than Φ_U; the exception set is still exactly that of Proposition 8.7. The thinnest formal margins are at $n = 115$, about -0.00213 (against -0.007466), and $n = 71$, about -0.0025 (against -0.007838). Instead of the uniform tail and the Arb table for $n \le 20\,000$, the formal proof uses the sharper bound $-\log \beta_n \ge \log n -$

$1/(n-1)^2$, three closed forms, negative for $n \equiv 0 \bmod 4$ with $n \geq 8$, for $n \equiv 2 \bmod 4$ with $n \geq 138$ and for odd $n \geq 275$, and checks the remaining 113 values of n (28 with $n \equiv 2 \bmod 4$ and 85 odd) one by one, with rational enclosures of the logarithms. Only the "if" part of Proposition 8.7 is formalised.

11.5. **How the certificates are checked.** The certificate data were produced by the scripts `scripts/gen_cert_lin.py` (at any trace) and `scripts/gen_cert_u.py`, which runs it at $a = 4$ and writes `Cert/DataU.lean` and `Cert/LogTab.lean`; the eleven certificates of Theorem 1.3 are in `VOdd/CertData/A`a`.lean`, each with its generating command in its header. The scripts are not trusted: every check is a decidable statement about the rational data, proved by `decide +kernel`. For Corollary 1.2 these are the 160 grid cells in 4 chunks of 40, the energy bound, the tail and the enclosure of $\log z$ (`Cert/DataU.lean`), together with a table of 1004 logarithm enclosures (`Cert/LogTab.lean`) shared with the certificates at odd a. The dyadic enclosure of $\log 2$ used by the table, and the logarithm enclosures in the finite cases of the margins at odd a (`VOdd/Margin.lean`) and of Theorem 1.5 (`Ti/Margin.lean`, generated by `scripts/ti/margin_ti_cases.py`), are also checked by `decide +kernel`. Theorem 1.5 uses no certificate of its own. A wrong value would make the build fail but could not make a false statement provable.

11.6. **Cost.** The kernel check of the certificate of Corollary 1.2, with the logarithm table, takes about 71 seconds. A build of a fresh clone of the repository, with Mathlib's prebuilt cache and at most four parallel Lean processes, took 9 minutes 35 seconds (9110 jobs, no errors) on a Windows 11 desktop with 32 logical processors and 31.7 GB of RAM, before the files for general m, for odd a and for Ti_2 were added; the complete project has 9178 jobs, and `Axioms.lean` reports only the three standard axioms.

11.7. **What has to be trusted.** Beyond the Lean kernel and the three axioms, the reader has to trust that the definitions displayed above express Theorem 1.1, Theorem 1.3, Theorem 1.5 and Corollary 1.2, and the Mathlib definitions they use (real numbers, `tsum`, `Real.sqrt`, `Real.log`, `Real.pi`, `goldenRatio`, `Irrational`, `Odd`, `Finset`). Nothing else, in particular no numerical data and no generating script, needs to be trusted. To avoid trusting the elaborator's handling of the environment, every declaration of all 130 project modules was replayed through the kernel with `leanchecker` (shipped with the Lean toolchain), each module against the compiled environment of its imports, on 9 November 2026; all modules passed, in about 56 minutes in total. Mathlib itself was not replayed from scratch (`--fresh`). Four AI referees, for Corollary 1.2, Theorem 1.1, Theorem 1.3 and Theorem 1.5, checked the fidelity of the statements (the definitions above, traced to Mathlib, and the absence of hypotheses, custom axioms and compiler trust) and found them faithful; the second also confirmed the exception set

$\mathcal{E}$ independently in interval arithmetic for $m \le 3000$, the third compared `EOdd` with $\mathcal{E}_{\mathrm{odd}}$ element by element and recomputed the exception set for the odd $a \le 6001$, with the formal constants and with those of Section 8.1, and the fourth compared `EOddI` with $\mathcal{E}_i$ element by element and recomputed the exception set for $5 \le n \le 5000$, with the formal constants and with those of Section 8.2. Apart from the check by the Lean kernel, all verification of the formalisation so far was carried out by AI systems; no human expert has yet reviewed the Lean code.

12. Computations and reproducibility

12.1. **Software.** The computations use python-flint 0.9.0 [29]: exact arithmetic over $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{d_m})$ for every statement about valuations, and Arb ball arithmetic [12] for every real number that feeds a claim. Hankel determinants are evaluated at T in ball arithmetic with working precision about $3.6h^2 + 8000$ bits at $m = 2$, and at least $2 \cdot$ (bits of the coefficients) $+ 4000$ bits in the independent check for general m. The formalisation uses Lean v4.35.0 and Mathlib v4.35.0.

12.2. **What each computation is used for.**

- *Proof inputs.* The certificate of Theorem 5.3 (`cert_U.py`, run with arguments `1600 32 2 4`, 128 bits, about one minute); the margin table for the odd $m \le 65$ in the proof of Proposition 7.1 (`margins_V.py`, Arb at 200 bits, which also lists every $m < 200$); the certificates of Theorem 10.2 (`dual_ref.py`, and `dualcert_arb2.py` with its library `dualcert.py`), the margins and the explicit numbers of Section 10 (`facts_M1.py`, Arb at 3000 bits). These are rigorous: floating point is used only to choose the density and λ, and any choice gives a valid bound.
- *Not used by the proof of Theorem 1.1.* The direct certificates of Table 3 (`cert_V.py`, the program `cert_U.py` with only z changed); the effective thresholds (`h0_U.py`, `h0_scan.py` at $m = 2$; `h0_V.py`, `h0_scan_V.py` at $m = 4$): exact sums, using the constants of Theorem 5.1; the identity at $m = 4$ in Section 9 (`id17_check.py`).
- *Checks of the lemmas* (not used by the proof). At $m = 2$: the machine and the matrix $\tilde{E}$ (`umach_core.py`); the bordered identity at $h = 2, \dots, 12$ and $X \in \{0, 1, -3/7\}$ (`bordered_check.py`); the row claim (`rowclaim_check.py`); Lemmas 3.2, 3.5 on exact profiles up to $h = 256$ (`law_check.py`, `u_profile.py` with `fastcontent.py`); Lemma 3.3 by brute force for every odd $d < K + 40$ and 194 values of K (`law_check.py`); Lemma 4.3 entrywise for $h \le 64$ (`lemma10_check.py`); Lemma 6.1 (`blocks_check.py`, `lemmaP_check.py`). An AI referee reimplemented the machine without sharing code and reproduced the numbers it checked; the certificate reproduced bit for bit at $N = 100, 200, 1600$. For general m: Lemma 4.1 entrywise (`lemma10nu_check.py`), Lemmas 3.2,

3.5, 3.8 and 6.1 on exact data (`oddp_check.py`), and the scores of Table 12 (`vprofile.py`). The AI referee of Theorem 1.1 wrote an independent implementation of the machine for general m, which checked these lemmas in 47 pairs (m, h) and reproduced the scores it recomputed; the referee also re-ran `cert_V.py` at $m = 4$ (same result bit for bit), compared it with an independent primal computation, and recomputed the margins with its own code.

- *The extensions* (Section 8). Proof inputs: `cert_za.py`, the program `cert_V.py` with $z = \alpha_a$, for the ten certificates of Table 5 (run with arguments `1600 32 2 4` and the value of a in the environment variable `CERT_A`; about ten seconds each); `margins_odd.py` (Arb at 200 bits) for the margin table of Proposition 8.2 (odd $a \le 5001$, with and without the certificates); `margins_ti2.py` (Arb at 200 bits) for the table $5 \le n \le 20\,000$ and the uniform tail of Proposition 8.7. Checks, not used by the proofs: `vodd.py` (Lemma 8.1 entrywise, Lemma 3.8 and Lemma 6.1 on exact data at odd a); `sharpS_check.py` (the sharper, unproved special-prime bound in the remark after Theorem 1.3); `wlib.py` (exact library for the weighted machines, including the χ_4 machine) with `wprofile.py` (exact contents and Arb scores, e.g. at $n = 8$); `margins_ti.py` (the first margin table for Theorem 1.5, up to $n \le 2000$, and a check of the Clausen form; superseded by `margins_ti2.py`); `lemmaP_ti.py` (Lemma 8.6 at $n = 4, 8, 12$); `step1_parity.py` (the exact check of the sign rule of Remark 8.5); `energy.py` (a floating-point model of the equilibrium problem, used only for the statement about $n = 4$). The AI referees of the two extensions wrote independent implementations (the machine at odd a, the identity of Lemma 8.4 with its own arithmetic in $\mathbb{Q}(i, \sqrt{d})$, entrywise and Newton-polygon checks, interval margins with mpmath, the PSLQ runs); for this revision the ten certificates were re-run, bit for bit, and every number of Section 8 and Section 9.3 was recomputed with separate code.

The scripts are in the folder `scripts/paper` of the repository [16] (those for Section 10 in `scripts/paper/m1`), with a README that gives the command line for each of them and the small result files quoted here. Not included are the referees' own check scripts (apart from `dual_ref.py`) and the earlier script that produced the exact profiles for $h \le 128$ in Table 11; those profiles are included as data, and `u_profile.py` computes the same quantities (checked at $h = 16$). The scripts for general m named above are in `scripts/paper/v`. The scripts of the extensions named above are in `scripts/paper/ext/vodd` and `scripts/paper/ext/ti2`.

12.3. **Numerical evidence.** Table 11 shows exact scores at $m = 2$, and Table 12 at other m. The arithmetic column is $h^{-2} \sum_p n_p \log p$ for the exact contents, so the score is the sum of the two previous columns. The values approach the limits slowly, as $O(\log h / h)$.

12.4. **Controls.** No false instance of the method is available, since no unit with rational χ_2 is known. As substitutes: the proved finite-h bound of Remark 7.3 lies above the exact score at every tested h, and every per-prime inequality holds on the exact profiles, often with equality; for a unit of norm $+1$ the reflection breaks (for the untwisted machine; with χ_4 on the atoms it holds again, Remark 8.5); with poles on one side only the conjugate place costs $+0.50$ per h^2 at $h = 16$; and at the points $\sqrt{2}-1$ and φ^{-1} the corresponding scores are positive (primal lower bounds $+0.3286$ and $+0.6572$ for the limiting exponent), so the method does not conclude there. At $\sqrt{2}-1 = z_1$ the exact scores are positive and rising (Section 7.2), and Lemma 6.1 fails at $\ell \mid m$ (Section 6), as it must. For Theorem 10.2, the method does not conclude at $n = 6$ ($M_1(6) \approx +0.477$), $n = 400$ ($M_1(400) \geq +0.070$) or $n = 720$ (numerical values).

Acknowledgements and disclosure

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paper	Lean declaration (file)
$\mathbb{K}$, z, σ	`Qd`, `zK`, `sig`, `emb` (`Field`); $m = 2$: `Q5 = Qd 4`
Lemma 2.1, (2.1)	`lemma1`, `aeval_chi2_Delta` (`PoleExpansion`); `Lmom_eq_mahler` (`Mahler`)
Lemma 2.2, Cor. 2.3	`eval₂_Delta`, `aeval_Delta`, `Delta_map` (`LemmaR`)
Prop. 2.4	`irrational_of_poly_approx` (`Criterion`); `MachineBoundsU`, `irrational_of_boundsU` (`MainCond`)
Lemma 3.3	`AK_eq` (`Arith/ClassCount`)
Lemma 3.4	`lemmaB` (`Arith/LemmaB`)
Lemmas 3.2, 3.5	`lemma9` (`Arith/OddPrimes`, `Arith/Lemma9Levels`); `Arith/Lemma8One`; `padicNorm_Delta_odd` (`Arith/OddExp`)
Thm. 3.7	`theoremA` (`Arith/PrimeSums`); `denAll_int`, `log_denAll_le` (`Arith/Assembly`)
Lemma 4.3 ($m = 2$)	`lemma10_L`, `lemma10_U`, `padicNorm_Delta_two` (`Arith/TwoAdic`)
Lemma 4.1	`lemma10nu_proof` (`V/Lemma10nu`, `V/TwoAdicNu`); `lemma10nu_size` (`V/Defs`)
Lemma 3.8	`lemmaS_proof` (`V/LemmaS`, `V/S*`); `AK_levels_size_proof` (`V/AKSize`)
Thm. 5.1 (other route)	`log_abs_Dpoly_ge`, `DualCertU`, `DualCertU.analytic` (`Analytic/TheoremBU`)
Thm. 5.3 (coarser)	`exists_certU`, `exists_certU'` (`Cert/CertU`); data `Cert/DataU`, `Cert/LogTab`; format `LinCertU`, `LinCertU.exists_dual` (`Cert/Lin`), `CDataLin`, `exists_dual_le` (`Cert/LinInstance`)
Lemma 5.5	`shift_lemma_proof` (`V/Shift`)
margins, Section 7.2	`margin_V_proof` (`V/Margin`)
Lemma 6.1	`lemmaP`, `nonvan_frequently` (`NonVan/LemmaP`); `lemmaP_m_proof` (`V/NonVan`)
Section 7	`denV0_int`, `log_denV_le`, `exists_boundsV`, `theoremV` (`V/Interface`); $m = 2$: `boundsU`, `chi2_sqrt5_sub_two_irrational` (`Main`)
(1.1)	`chi2_landen`, `Li2_reflection`, `Li2_landen` (`Golden/Landen`); `chi2_sqrt5_sub_two_eq` (`Golden/ClosedForm`)
Lemma 8.1	`lemma10_0_proof` (`VOdd/TenZero`); `lemma10_0_size` (`VOdd/Defs`)
Table 5 (own values)	`exists_certOdd_proof` (`VOdd/CertOdd`); data `VOdd/CertData/A`a for 11 values of a
Prop. 8.2 ("if" part)	`margin_VOdd_proof` (`VOdd/Margin`)
proof of Thm. 1.3	`denO0_int`, `log_denO_le`, `exists_boundsO`, `theoremVOdd` (`VOdd/Interface`)
Ti_2, $\mathbb{K}$, β_n, σ	`Ti2` (`Ti/Defs`); `Qt`, `zK`, `sig`, `emb`, `zT` (`Ti/Field`)
Lemma 8.4 (not used)	the χ_4 machine directly: `Ti/Machine`, `Ti/LemmaR`; `aeval_Ti2_Delta` (`Ti/PoleExpansion`)
valuations, Section 8.2	`padicNorm_Delta_odd_proof` (`Ti/OddExp`); `lemmaS_proof` (`Ti/LemmaS`); `lemma10nu_proof` (`Ti/Lemma10nu`); `lemma10_0_proof` (`Ti/TenZero`)
size, Section 8.2	`DualCertT`, `DualCertT.analytic` (`Ti/Analytic/TheoremBT`); `shift_lemma_proof` (`Ti/Analytic/ShiftT`); `analytic_T_proof` (`Ti/Analytic`)
Lemma 8.6	`lemmaP_n_proof` (`Ti/NonVan`, `Ti/NonVan/*`); `nonvan_frequently_T` (`Ti/Interface`)

h	score $h^{-2}\log\lvert F_h(T)\rvert$	analytic	arithmetic	n_2
16	−0.2930	−1.4379	1.1449	54
32	−0.1734	−1.4195	1.2462	142
64	−0.1147	−1.4015	1.2869	350
96	−0.0951	−1.3930	1.2980	576
128	−0.0764	−1.3876	1.3112	830
192	−0.0659	−1.3815	1.3156	1344
256	−0.0596	−1.3781	1.3185	1918

TABLE 11. Exact scores for Corollary 1.2 ($m = 2$). At $h = 192$ and 256 the contents are proved lower bounds for each n_p (fast content computation), which can only make the score look worse.

m	h	score	analytic	arithmetic	n_2
4	16	−0.5479	−1.7948	1.2469	94
4	32	−0.3829	−1.7739	1.3910	349
4	48	−0.3502	−1.7637	1.4135	737
4	64	−0.3142	−1.7572	1.4429	1277
4	96	−0.2903	−1.7491	1.4587	2721
4	128	−0.2662	−1.7443	1.4781	4733
6	32	−0.5176	−1.9798	1.4622	142
6	64	−0.4434	−1.9639	1.5205	350
8	32	−0.5976	−2.1254	1.5278	558
8	64	−0.5096	−2.1097	1.6001	2205
11	32	−0.5330	−2.2865	1.7534	196
11	48	−0.4407	−2.2769	1.8362	401
3	32	−0.1326	−1.6267	1.4941	196
3	48	−0.0734	−1.6165	1.5431	401

TABLE 12. Exact scores at other m. The row $m = 3$ is in $\mathcal{E}$ and is not evidence for it (its score rises to −0.0340 at $h = 64$). At $m = 4$, $h = 96, 128$, the first computation gave proved lower bounds for the contents; an independent exact computation gave the same values.