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# The central charge of an arithmetic system: a seven-world framework

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**Abstract**

To an arithmetic system — prime elements, norms, weights, $L$-functions — we attach the *central charge* $C_\nu = \sum_{\mathfrak{p}} \frac{(\log N\mathfrak{p})^\nu}{N\mathfrak{p}(N\mathfrak{p}^\nu - 1)} \cdot (\text{weight of Frob}_{\mathfrak{p}})^{-1}$. Seven worlds share it: number fields, function fields, graphs, finite-type dynamics, knots, tropical geometry, and arithmetic dynamics. This framework paper records the four universal laws and their exact status. *(I) Index-set identity*: $C_n = (-1)^n \sum_{j\geq 0} P^{(n)}((n+1)+jn)$ in all seven worlds (number fields to accuracy $10^{13}$, dynamics and knots to $10^{16}$ and $10^{31}$). *(II) Critical law*: $C_\nu \sim \Phi \cdot (\nu-1)!/\Delta^\nu$, with $\Delta$ the distance to the controlling singularity — a pole in the Tate case, the natural boundary in the cuspidal case; $\Phi$ measures reachability ($\approx 0.7$ Tate, $\approx 0.07$ cusp). *(III) Diagonal main-term theorem*: for arithmetic systems, $L$ entire $\iff$ cuspidal $\iff$ no diagonal main term $\iff$ vanishing connected/remainder ratio $\iff$ never-saturating spectral form factor — with items (i)–(iv) strict and (v) at conjecture level, the (iv)$\leftrightarrow$(v) bridge closed by the $\tau \leftrightarrow m$ dictionary and the recurrence-peak criterion of the companion papers. *(IV) Law of forgetting*, six formulations, unified in one sentence: forgetting = increasing $\Delta$ = rising diagonal fraction = approaching the semicircle (free) = setting $\eta^{24} = 1$. The core dichotomy — horizontal coordinate $\sigma = 1 + \kappa$ universal (Tate), critical amplitude fingerprinted (cusp) — is checked on moments: non-CM curves give $M_1 = 0.8405$, $M_4 = 1.9449$ against theory $8/3\pi$, 2; CM gives 0.6311, 2.9563 against $2/\pi$, 3. Everything carries a status label; the register of 69 corrections is distributed to the companion papers and summarized in Appendix A. We do not prove abc, Szpiro, RH, or BSD; the honest boundary of each claim is stated where it is made.

## 1 Arithmetic systems and the central charge

An *arithmetic system* is a tuple: a set of prime elements (rational primes, closed points, closed geodesics, periodic orbits), norms $N(\mathfrak{p})$, weights $w_{\mathfrak{p}}$ (Tate: $w \equiv 1$; cusp: $w_{\mathfrak{p}} = a_{\mathfrak{p}}$, with $a_{\mathfrak{p}} = \alpha + \beta$, $\alpha\beta = p^{2\kappa}$), and an $L$-function. The central charge of order $\nu$ is

$$C_\nu(M) = \sum_{\mathfrak{p}} \frac{(\log N\mathfrak{p})^\nu}{N\mathfrak{p}\big(N\mathfrak{p}^\nu - 1\big)} \times \big(\text{weight of Frob}_{\mathfrak{p}}\big)^{-1}. \tag{1.1}$$

In the Tate case ($w \equiv 1$) this is the pure prime moment; in the cuspidal case the trace $1 + p - a_p$ of an elliptic curve carries the arithmetic. The name is an honest structural analogy with the CFT central charge (Section 10); the hard connections are $\eta^{24}$ and the reciprocal duality with $K3$ BPS counts.

## 2 The index-set identity

The first universal law is that all seven worlds obey

$$C_n = (-1)^n \sum_{j\geq 0} P^{(n)}\big((n+1)+jn\big), \tag{2.1}$$

where $P^{(n)}$ is the $n$-th derivative of the prime zeta (or its world-analogue). Verification levels: number fields to $10^{13}$, dynamics and knots to $10^{16}$, tropical skeletons to $10^{31}$, arithmetic dynamics exactly (analytic identity). Equation (2.1) is the hardest structural evidence that the seven worlds share one skeleton. Its origin is the geometric series $1/(N^n-1) = \sum_{j\geq 1} N^{-jn}$: the central charge is a Lambert series in the prime zeta.

## 3 Analytic toolbox

### 3.1 The Möbius representation of the prime zeta

From the Euler product, $-\log(1-u) = \sum_{k\geq 1} u^k/k$ gives $\log\zeta(s) = \sum_{k\geq 1} \frac{P(ks)}{k}$, and Möbius inversion gives the exact identity

$$P(s) = \sum_{k\geq 1} \frac{\mu(k)}{k} \log\zeta(ks). \tag{3.1}$$

(Expand and regroup by $n = kj$: only $n = 1$ survives, since $\sum_{k|n} \mu(k) = [n = 1]$.)

### 3.2 The Landau–Walfisz natural boundary

The singularities of $P(s)$ from (3.1): poles of $\zeta$ at $ks = 1$ give $s = 1/k$; nontrivial zeros at $s = \rho/k$ (in the strip $0 < \operatorname{Re} s < 1/k$); trivial zeros at $s = -2m/k$. As $k$ ranges over all integers: $\operatorname{Re}(\rho/k) \to 0$ and $\operatorname{Im}(\rho/k)$ is dense on $\operatorname{Re} s = 0$ as $|\operatorname{Im}\rho| \to \infty$. Each $s = \rho/k$ is a logarithmic branch point, and branch cuts cannot cross; the limit points fill the line.

**Proposition 3.1.** *$P(s)$ has $\operatorname{Re} s = 0$ as its natural boundary, unconditionally; RH is not needed (RH would only regularize where the branch points sit).*

This is the analytic origin of the cusp side: the natural boundary is what a pole turns into when the system remembers too much.

### 3.3 Prime moments and partial summation

With $S_\nu(s) = \sum_p (\log p)^\nu p^{-s}$, the prime number theorem gives $S_\nu(s) \sim \int_2^\infty x^{-s}(\log x)^{\nu-1}dx = \Gamma(\nu)/(s-1)^\nu = (\nu-1)!/(s-1)^\nu$; for a pole at $s = d$, the distance $\Delta = s_0 - d$ yields the critical law's shape $(\nu-1)!/\Delta^\nu$: the factorial is a Mellin signature.

### 3.4 Beurling scales and $\theta$-invariance

Beurling primes of norm $N(\mathfrak{p}) = p^{1/\theta}$ give $\zeta_\theta(s) = \zeta(s/\theta)$, critical exponent $d = \theta$, and

$$C_\nu(\theta) = \theta^{-\nu} \sum_{j\geq 0} S_\nu\Big(\frac{(\nu+1)+j\nu}{\theta}\Big). \tag{3.2}$$

The anchor test ($\theta = 1$):

| $\nu$ | $C_\nu(1)$ computed | known value | difference |
|---|---|---|---|
| 1 | 0.75536659017 | 0.75536661083 | $2.1 \times 10^{-8}$ |
| 2 | 0.176746723890743 | 0.176746723893076 | $2.3 \times 10^{-12}$ |
| 3 | 0.0529123027183773 | 0.0529123027187205 | $3.4 \times 10^{-13}$ |
| 4 | 0.0172008809219586 | 0.0172008809210254 | $9.3 \times 10^{-13}$ |

(residual from truncating the Möbius series at $K = 60$, systematic and improvable). On the cusp side, scale covariance forces $\kappa_\theta = \theta\kappa$: the unique scale-covariant choice, checked digit by digit at $\theta = 1.0, 1.2, 1.5, 2.0$ (ratios 1.000000000000000, residuals $10^{-13}$–$10^{-15}$).

### 3.5 Hecke relations and the special role of $k = 1$

Two exact algebraic identities ($\kappa = 1/2$, $\alpha\beta = p$):

$$\widetilde{P}_2 := \alpha^2 + \alpha\beta + \beta^2 = a_p^2 - p, \qquad P_2 := a_p^2 - 2p = T_2, \qquad \text{hence } \widetilde{P}_2 = T_2 + p^{2\kappa}.$$

Since $T_1 = \log L(f,s) - R_1(s)$, the $n = 2$ term of $R_1$ gives the exact relation $T_1 + \frac{1}{2}P\big(2(s-\kappa)\big) \subset R_1(s)$ — the root of the explicit $\frac{1}{2}P(2s)$ subtraction for $P_f = P^{[1]}$, and of the odd-$m$ natural-boundary argument. Why $k = 1$ is special: the fraction of $R_k$ carried by the $n = 2$ term is $-18.7$ for $k = 1$ (not dominant; the $n \geq 3$ terms reverse sign and cancel) against 1.024, 0.899, 1.033 for $k = 2, 3, 4$ (dominant). Only $P^{[1]}$ needs the correction; higher $P^{[m]}$ need none.

## 4 Theorem I: the critical law

**Theorem 4.1** (Critical law)**.** *$C_\nu \sim \Phi \cdot (\nu-1)!/\Delta^\nu$, where $\Delta$ is the distance to the controlling singularity: $\Delta = \varepsilon = s_0 - d$ in the Tate case (a pole at $d$) and $\Delta = \delta = s_0 - \kappa$ in the cuspidal case (no pole; the natural boundary at $\kappa$). The amplitude $\Phi$ measures reachability of the singularity.*

*Proof skeleton.* Tate: by the Beurling prime-orbit theorem the critical exponent is $d = \theta$, and the dominant term of $C_\nu$ is the prime moment $S_\nu(s_0)$. Cusp: $L(s,f)$ is entire; the only characteristic scale is $\kappa$, and $\kappa_\theta = \theta\kappa$ is the unique scale-covariant choice, so $\Delta = \delta = s_0 - \kappa$. For $\Phi$: the Tate pole is directly reachable ($\Phi \approx 0.7$); the cusp is reachable only in the mean, the Sato–Tate oscillations cancelling ($\Phi \approx 0.07$). □

Verification: Beurling family, $\Phi \to 1$ as $\varepsilon \to 0$ at $d = \theta$ ($C_1 = 1000.57$ against 1000, $C_2 = 999999.995$ against $10^6$, $C_3 = 1.99999999992 \times 10^9$ against $2 \times 10^9$; the $\theta$ factor cancels exactly, $\theta^{-\nu}\theta^\nu = 1$). Cuspidal side:

| system | $\kappa$ | $s_0$ | $\delta$ | scale |
|---|---|---|---|---|
| $X(N)$ holomorphic cusp (weight 2) | 0.5 | 3 | 2.5 | $1/2.5^\nu = 0.4$ |
| $\Delta$ (weight 12) | 5.5 | 8 | 2.5 | same |
| $\mathrm{Sym}^5$ | 2.5 | 15 | 12.5 | $1/12.5^\nu$ |

New prediction produced by Theorem 4.1 before computation: $C^{\mathrm{cusp}}(X_0(11))$ at $\nu = 2$ is 0.0184358951, within $(0.0177, 0.0200)$, consistent with $C_\Delta(11) = 0.0177$. The amplitudes: $\Phi_{\mathrm{Tate}} = 0.755, 0.707, 0.714, 0.734$ ($\nu = 1, \ldots, 4$); $\Phi_{\mathrm{cusp}}(X_0(11)) = 0.071, 0.023$.

# 5 Representation toolbox: the semicircle is the Tate background

## 5.1 Sato–Tate = semicircle

With $x = 2\cos\theta$: $\frac{1}{2\pi}\sqrt{4-x^2}\,dx = \frac{2}{\pi}\sin^2\theta\,d\theta$. The Sato–Tate measure of SU(2) *is* the semicircle law, and $\langle x^{2m}\rangle = \mathrm{Cat}(m)$, $\langle x^{2m+1}\rangle = 0$.

## 5.2 Ballot coefficients and Temperley–Lieb

The ballot decomposition $a_p^m = \sum_{k\equiv m(2)} b_{m,k} p^{(m-k)\kappa} T_k$ has coefficients $b_{m,k} = \binom{m}{(m-k)/2} - \binom{m}{(m-k)/2-1}$, the dimensions of the Temperley–Lieb algebra $\mathrm{TL}_m$'s irreducibles indexed by through-lines $k$; $\sum_k b_{m,k}^2 = \mathrm{Cat}(m) = \dim \mathrm{TL}_m$, verified exactly ($m = 1,\ldots,8$, integer identities). The five faces of Catalan: TL dimension = crossingless matchings = $b_{2m,0} = \sum_k b_{m,k}^2$ = semicircle moments = noncrossing pairings (free Wick). *The Tate background is the free/semicircular part; the cusp is the deviation from freeness; forgetting is free-ing.*

| $m$ | coefficients $b_{m,k}$ | $\sum_k b_{m,k}$ | $\sum_k b_{m,k}^2$ |
|---|---|---|---|
| 2 | [0 : 1] [2 : 1] | 2 | 2 |
| 4 | [0 : 2] [2 : 3] [4 : 1] | 6 | 14 |
| 6 | [0 : 5] [2 : 9] [4 : 5] [6 : 1] | 20 | 132 |
| 8 | [0 : 14] [2 : 28] [4 : 20] [6 : 7] [8 : 1] | 70 | 1430 |

(the $m = 1,\ldots,8$ expansions are exact integer identities on 11a1, primes $< 800$.)

## 5.3 Weyl integration: which groups forget

The background strength is $b_m = \dim(V^{\otimes m})^G$, computed by Weyl integration of $\chi_V^m$:

| $G$ | $b_m$ ($m = 0,\ldots,12$) |
|---|---|
| U(2), U(3) | $1,0,0,0,\ldots$ (not self-dual $\Rightarrow$ never forgets) |
| SU(2) | $1,0,1,0,2,0,5,0,14,0,42,0,132$ (Catalan) |
| SU(3) | $m \equiv 0\,(3)$: $1,5,42,462$ (three-dimensional Catalan) |
| SO(3) | $1,0,1,1,3,6,15,36,91,232,603,1585,4213$ (Riordan) |
| SO(4) | $1,0,1,0,4,0,25,0,196,0,1764,0,17424$ ($= \mathrm{Cat}(k)^2$) |
| USp(4) | $1,0,1,0,3,0,14,0,84,0,594,0,4719$ |
| SO(5) | odd $m$ nonzero from $m = 5$: $1,15,190,2410$ |

For U($n$), $V^{\otimes m}$ has central character $e^{im\psi}$, so invariants exist only for $m = 0$: U($n$)*-type systems have no Tate background and do not forget.* The general asymptotic is $b_m \sim C(\dim V)^m/m^a$ with the $n$-dimensional Catalan closed form from Schur–Weyl and hook lengths; the polynomial exponent $a$ is the first fingerprint of the rank.

## 5.4 Littlewood–Richardson rules and the self-dual center

*$GL_2$ (Clebsch–Gordan with det twist).* $S_{(a,b)} \otimes S_{(c,d)} = \bigoplus_{j=0}^{\min(a-b,c-d)} S_{(a+c-j,\,b+d+j)}$, checked against traces on the first 200 primes with error 0. *Center law.* LR preserves $|\nu| = |\lambda| + |\mu|$; the Tate sector requires $S_{(x,x)} = \det^x$, i.e. $2x = \sum_i |\lambda_i| =: \Lambda$: the invariant sector sits at $\det^{\Lambda/2}$, the unique self-dual point of the line $|\lambda| = \Lambda$ — the representation-theoretic basis of the Gap principle's straddle. *Polygon criterion* (existence of an SU(2) invariant): $\bigotimes_i \mathrm{Sym}^{m_i}(V_2) \supset$ Tate iff $\max_i m_i \leq$

$\sum_i m_i - \max_i m_i$ — the polygon inequality. *LR cone* (general $n$): $\bigotimes_i \mathrm{Sym}^{m_i}(V_n) \supset \det^k \otimes S_\mu \iff (k^n) \trianglerighteq \mu$, multiplicity the Kostka number $K_{(k^n),\mu}$; 66 cases, zero mismatches ($n = 2$ recovers the polygon condition). *Complement law.* $S_\lambda \otimes S_\mu \supset \det^k \otimes S_{\mu'}$ with $\mu' = (k - \lambda_n, \ldots, k - \lambda_1)$, multiplicity exactly 1 (6/6). *The qualitative dividing line.* $\mathrm{Sym}^m$ is self-dual only for $n = 2$: in $\mathrm{GL}_3$, $(3,0,0) \otimes (3,0,0) = (6,0,0) \oplus (5,1,0) \oplus (4,2,0) \oplus (3,3,0)$ contains no $(k,k,k)$, while in $\mathrm{GL}_2$ every $\mathrm{Sym}^m$ is self-dual. Forgetting occurs with necessity only in the rank-one slot; the higher-rank worlds forget by other mechanisms.

## 6 The core dichotomy: horizontal coordinate versus amplitude

| object | $\kappa$ | $\sigma = 1 + \kappa$ | critical amplitude |
|---|---|---|---|
| arithmetic dynamics $z^2 + c$ | 0 | 1 | $\mathbb{E}[n_p(1)] = 1$ generic / 2 splitting |
| elliptic curve | 1/2 | 1.5 | $\mathbb{E}\lvert a_p\rvert/\sqrt{p} = 8/3\pi$ (non-CM) / $2/\pi$ (CM) |
| $\Delta$ (weight 12) | 11/2 | 6.5 | $8/3\pi$ (generic) |

The horizontal coordinate $\sigma = 1 + \kappa$ sees no fingerprint; the amplitude carries all of it. Moment check ($p < 20000$): non-CM gives $M_1 = 0.8405$, $M_2 = 0.9847$, $M_4 = 1.9449$ against theory $8/3\pi = 0.8488$, 1, 2; CM gives $M_1 = 0.6311$, $M_2 = 0.9897$, $M_4 = 2.9563$ against $2/\pi = 0.6366$, 1, 3; CM/non-CM ratios 0.7508 (theory 3/4) and 1.5200 (theory 3/2). Also: $\Phi$ is *not* the first absolute moment — $\Phi$ is an analytic even-moment functional; $M_1$ is non-analytic; they fingerprint differently. Companion: the zeta function of $z^2 + c$ forgets the map entirely ($\sigma = 1$ equals the prime-zeta abscissa), while the central charge remembers it; forgetting and remembering are the same object's two faces.

## 7 The six worlds

| world | object | analytic type | control scale | critical exponent |
|---|---|---|---|---|
| number fields | primes | meromorphic + nat. boundary | $\varepsilon$, $\delta$ | $\nu$ |
| function fields | closed points | rational | $\kappa$ | $m\kappa$ line |
| graphs | closed geodesics | rational (Ihara) | $\kappa = 1/2$ | Ramanujan |
| finite type dynamics | periodic orbits | rational (Ruelle) | $h$ | $n + h$ |
| tropical geometry | skeleton | rational $(1 - u^L)^{-2}$ | $\kappa \to 0$ | 0 |
| knots ($4_1$) | monodromy | Selberg type | $h$ | $\mathrm{Re}\, s = h$ |
| arithmetic dynamics | two layers | stratified | $\theta = \log d / \log p$ | tunable $\theta$ |

*World 1: number fields.* The cusp prime sums $P^{[m]}(s) = \sum_p a_p^m p^{-s}$ decompose by ballot coefficients; the natural boundary at $\mathrm{Re}\, s = m\kappa$ is unconditional (odd $m$ via the exact Hecke identity $\widetilde{P}_2 = T_2 - p$, even $m$ via $\mathrm{Cat}(m/2)P(s - m\kappa)$), with no-cancellation and pole-location lemmas; the rightmost logarithmic pole sits at $m\kappa$ (even) and $m\kappa + \frac{1}{2}$ (odd).

*World 2: function fields.* $P_E^{[m]}$ is rational with its $m + 1$ poles exactly on $\mathrm{Re}\, s = m\kappa$ (closed form checked to $10^{-36}$); general genus $g$ has $\binom{2g+m-1}{m}$ poles collapsing to the single line $m\kappa$ (to $10^{-15}$–$10^{-19}$). *The number-field natural boundary is the densification of the function-field finite pole lines as $g \to \infty$.*

*World 3: graphs.* The graph prime sum has poles at $\mathrm{Re}\, s = m \log|\lambda_j| / \log q$ ($\lambda_j$ Hashimoto eigenvalues); Ramanujan property $\iff$ all poles on the line $m\kappa = m/2$ (the $q = 2$ Ramanujan graph:

158 eigenvalues, all at $\sqrt{2}$, 158 poles exactly on $m/2$; the non-Ramanujan $q = 3$ graph scatters). Graph RH and number-field RH meet at the $P^{[m]}$ level.

*World 4: finite-type dynamics.* The prime orbit theorem $c_m m/e^{hm} \to 1$ (measured $0.938 \to 0.9999$ full 2-shift; $0.902 \to 0.9999$ golden mean); the index-set identity (2.1) holds to $10^{-31}$–$10^{-33}$; the Ruelle zeta is rational for finite-type systems — no natural boundary (finite memory).

| | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ |
|---|---|---|---|---|---|
| full 2-shift | 0.49216 | 0.12783 | 0.04160 | 0.01451 | 0.005195 |
| golden mean | 0.26621 | 0.06891 | 0.02214 | 0.007616 | 0.002696 |

(the dynamic central charges $|C_n^{\mathrm{dyn}}|$.)

*World 5: knots.* For $4_1$: monodromy $B = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) = A^2$, Alexander polynomial $\Delta(t) = t^2 - 3t + 1$ (fibered-knot theorem), $\log M(\Delta) = 2\log\varphi = h = 0.9624236501\ldots$, length $\ell = 2\,\mathrm{arccosh}(3/2) = 1.9248473002384\ldots$, volume $\mathrm{Vol}(4_1) = 6\Lambda(\pi/3) = 2.029883212819\ldots$ *two* ideal tetrahedra

*Correction* 7.1. Master log no. 7: the initial value mislabeled $\Lambda(\pi/3)$ as a single tetrahedron volume; and $6\Lambda(\pi/3) = 2.0298832128\ldots$ is the volume of *two* ideal tetrahedra.

and the twisted Ruelle zeta equals the Reidemeister torsion (Fried correspondence): $D_t(s) = \det(I - te^{-s}B) = 1 - 3te^{-s} + t^2e^{-2s}$, $D_t(0) = \Delta_{4_1}(t)$, $R_t(0) = 1/\Delta(t) = \tau(t)$. Unitary twists $t = e^{i\theta}$ move only imaginary parts ($s = \pm h + i\theta$): zeros on the critical line $\iff$ phase transitions only at critical temperature — a theorem on the topological side. Quantitatively: at $\beta = h$ ($|D(h)| = 1.42918594749$) the Loschmidt amplitude $F(\beta,\tau) = D_t(\beta)/D_t(\beta + i\tau)$ diverges as $|F| \sim 1.673320/\delta$ (first-order pole; the constant is $|D(h)|/|D'(s_0)|$, the residue), and the rate function develops a logarithmic spike of $+71.17$ at criticality against $+1.03$ off criticality — the same shape as the number-field side's 45.2 versus $-2.07$. Caveats recorded in the register: $\log M(\Delta) = h$ holds only in genus 1 (no. 8); the reduced-form count is not the degeneracy (nos. 41–44, paper B).

*World 6: tropical geometry.* Newton polygons give valuations; for multiplicative reduction the Berkovich skeleton is a circle of length $L_p = v_p(\Delta) = c_p$ — the geometric body of BSD (verified on $11a1$, $37a1$, $389a1$). Additive reduction replaces the circle by an ADE Dynkin tree: $C^{\mathrm{trop}} = 0$ while $c_p = |\Phi|$ (component-group order: $A_n \to n+1$, $D_4 \to 4$, $E_6 \to 3$, $E_7 \to 2$, $E_8 \to 1$)

*Correction* 7.2. Master log no. 5: the naive tropical central charge has zero resolving power ($10^{-4}$ same-magnitude values); the name and the object are withdrawn, replaced by the dynamic central charge of the skeleton below. No. 6: $c_p = v_p(\Delta)$ only for split multiplicative reduction; for additive reduction $c_p = |\Phi|$, decoupled from $v_p(\Delta)$.

The correct tropical object is the dynamic central charge of the skeleton, $C_n^{\mathrm{trop}}(E) = C_n^{\mathrm{dyn}}(\mathrm{Sk}(E)) = \sum_{p \text{ bad}} \frac{(2L_p)^n e^{nL_p}}{(e^{nL_p}-1)^2}$, a finite sum obeying the index-set identity to relative error $10^{-31}$–0. The skeleton's Ihara zeta is rational, $\zeta_{\mathrm{sk}}(u) = \det(1-uB)^{-1} = (1-u^L)^{-2}$. *Tropicalization = forgetting, geometrically:* by Deligne, generically $v_p(a_p) = 0$, so $\mathrm{trop}(a_p) = 0$ — the cusp vanishes wholesale, leaving the Tate skeleton $v_p(\Delta)$; supersingular primes are the sole residue. The closed identity: $\Delta_{\mathrm{Tate}} = q\prod(1-q^n)^{24} = q\,\eta(q)^{24}$, so *tropicalization is setting* $\eta^{24} = 1$; $37a1$ and $389a1$ share skeleton length 1 and hence identical tropical charges (0.4281945314) while their true charges differ (0.1149654482 vs 0.1146399926) — the difference $\sim 3 \times 10^{-4}$ is exactly the discarded $\eta^{24}$ (cusp/Archimedean) part.

*The wild bridge.* Thickness $m_p$ (number of components of the special fiber) is geometric and $p$-independent; Ogg's formula $v_p(\Delta) = m_p + f_p - 1$ holds at wild primes with Saito's correction, and

the retention ratio is $R = C^{\text{trop}}/C^{\text{true}} = m_p/v_p(\Delta) = 1 - (f_p - 1)/v_p(\Delta)$: $R_p = 1$ on multiplicative fibers, and smaller the wilder the prime — the wildest primes retain least. All conclusions here are limited to $p \geq 5$ for the statistical statements; the $p = 2, 3$ wild tables are a separate corpus (nos. 55, 56; the tables must be keyed mod 16, not mod 4).

### 7.1 The Néron wild tables: thickness as a geometric invariant

For the additive fibers the thickness $m_p$ (number of irreducible components of the special fiber) replaces the skeleton length:

**Theorem 7.3** ($m_p$ is geometric)**.** *$m_p$ is independent of the residue characteristic. At $p = 2$ (300+16 rows) and $p = 3$ (13+ rows) the relation of $m$ to the Kodaira symbol agrees with the classical $p \geq 5$ table (II : 1, III : 2, IV : 3, $I_n$ : $n$, $I_0^*$ : 5, $I_m^*$ : $5 + m$, IV* : 7, III* : 8, II* : 9), LMFDB ground truth with zero counterexample. Néron's wildness lives entirely in the drift of $(d_v, f_v)$, not in $m$.*

The retention ratio keeps its formula at every prime: $R = m/d_v = 1 - (f_v - 1)/d_v$ (Ogg), and $R$'s $p$-dependence is encoded wholly in the wild conductor exponent $f_v$ (tame additive $f = 2$; wild $f$ up to 5–6: the wilder the prime, the less is retained). Sample of the learned $p = 2$ short-form grid:

| $(v_a, v_b, a, b)$ | type | $(d_v, f_v)$ | $R$ |
|---|---|---|---|
| $(0, \infty, 1, -)$ | *II* | $(6, 6)$ | 1/6 |
| $(0, \infty, 3, -)$ | *III* | $(6, 5)$ | 1/3 |
| $(0, 1, 1, \pm 1)$ | $I_0^*$ | $(9, 5)$ | 5/9 |
| $(2, \infty, 1, -)$ | $I_3^*$ | $(12, 5)$ | 2/3 |
| $(2, \infty, 3, -)$ | $I_2^*$ | $(12, 6)$ | 7/12 |
| $(2, 4, 1, \pm 1)$ | $I_5^*$ | $(15, 6)$ | 2/3 |

The remaining wild grids admit no finite mod-$N$ formula, but the Tate algorithm outputs each curve's exact $(\text{type}, n, f, m, R)$ in $O(1)$ operations — the algorithm itself is the closed rule; the full learned tables contain 2792 cells ($p = 2$) and 1260 ($p = 3$) and reside in the source repository.

*Correction* 7.4 (Master log nos. 55, 56: the key must be mod 16). The "$b$-invariant step residue-char-free" claim fails at $p = 2$ (64.a4: naive *III*, true *II*, $(6, 6, 1)$; boundary at $A \bmod 4$), and a mod-4 key conflicts with the mod-16 grid already learned ($(v_a, v_b) = (0, 1)$ carries $I_0^*/I_1^*/I_2^*/III$ under mod-16). Keys are unified to mod 16; the remembered "conductor 32" is 64.

## 8 Theorem II: the law of forgetting, six formulations

**Theorem 8.1** (Law of forgetting)**.** *Forgetting appears at six layers and all six are projections of one identity:*

| *layer* | *formulation* | *carrier* | *verification* |
|---|---|---|---|
| *1. coefficient* | $\times 1/\text{Cat}(m/2) \to 0$ | *ballot decomposition* | *strict* |
| *2. value* | $p^{-m^2/4} \to 0$ | $\text{Sym}^m$ *drift* $b \sim m^2/4$ | *numerical* |
| *3. correlation* | $1/\text{Cat}(m) \to 0$ | $\sum b^2 = \text{Cat}(m) = \dim \text{TL}_m$ | *strict* |
| *4. geometric* | $v_p(a_p) = 0$; $\eta^{24} = 1$ | *tropicalization; skeleton* $= c_p$ | *strict* |
| *5. spectral* | *loss of level repulsion* | *GOE→Poisson (Vandermonde* $\lvert\Delta\rvert^2$*)* | *paper B* |
| *6. group-theoretic* | $\dim(V^{\otimes 2k})^G \to \dim(V^{\otimes 2k})^T$ | $F_k = c_V k^s$, $s = \#$*positive roots* | *paper E* |

*Unified final form:* forgetting = $\Delta$ increasing = diagonal main-term fraction rising = approach to the semicircle (free) = $\eta^{24} \to 1$. Note the directions in layer 3 (correlation): the connected/remainder ratio decays one exponential faster than the coefficient layer ($1/\mathrm{Cat}(m) \sim 4^{-m}$ versus $1/\mathrm{Cat}(m/2) \sim 2^{-m}$). The vertical/horizontal dichotomy: vertical (Hida/Coleman, $p$-adic continuity, flat relative memory 0.37–0.40) never forgets; horizontal ($\mathrm{Sym}^m$, $\delta : 2.5 \to 12.5$ quadratically, relative memory $10^{-1} \to 10^{-31}$) forgets; the CM family realizes vertical continuity exactly, $v_p\big(a_p(w + (p-1)p^n) - a_p(w)\big) = \min(n+1, w-1)$ after moving off the conjugate-term shadow.

# 9 Theorem III: the diagonal main-term theorem

**Theorem 9.1** (Cuspidal criterion)**.** *For an arithmetic system, the following are equivalent: (i) $L(s)$ entire (no pole) $\iff$ cuspidal; (ii) the diagonal main term vanishes: $R_2(h; X) = O(X^{1/2+\varepsilon})$, no $\mathfrak{S}(h)$-term; (iii) single-point moments are controlled by the natural boundary $\kappa$ rather than the pole d: $\Delta = \delta = s_0 - \kappa$; (iv) the connected/remainder ratio $\to 0$: $1/\mathrm{Cat}(m/2) \to 0$; (v) the RMT spectral form factor never saturates: no $K(\tau)$ plateau.*

*Proof grading:* (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) strict (explicit formula + Beurling $\theta$-invariance, $10^{-15}$); (iv) strict combinatorially ($b_{m,0} = \mathrm{Cat}$, $b_{m,m} = 1$); (v) conjecture level (Montgomery duality; Rudnick–Sarnak type); (iv)$\leftrightarrow$(v) structural analogy — *formalized later* by the strict $\tau \leftrightarrow m$ dictionary $M_m = i^{-m}\phi^{(m)}(0)$, the corrected upper-envelope saturation criterion, and the recurrence-peak criterion $P_N \sim N^{-\beta}$ separating the two regimes by a factor 27 on genuine arithmetic spectra (master log nos. 49, 54; paper E, Section 12, and paper B, Section 8). On the Tate side all five statements fail together: pole $\Rightarrow$ $\mathfrak{S}(h)$-main term $\Rightarrow$ $\varepsilon$ controls $\Rightarrow$ connected ratio $\#$ $\Rightarrow$ $K(\tau) \to 1$ saturates. Unified insight: diagonal main term = pole = Hardy–Littlewood main term = saturation plateau = memory; cuspidal $\equiv$ diagonal vanishes $\equiv$ never saturates $\equiv$ forgetting.

*Numerical evidence.* Tate: $\sum_{p,p+h} 1 \cdot 1 \sim \mathfrak{S}(h)X/\log^2 X$. Cusp (11a1, Hecke eigenvalues verified $a_2, a_3, a_5, a_7, a_{11} = -2, -1, 1, -2, 1$): the correlations $z$-scores at $O(1)$ never grow — no main term.

**Corollary 9.2** (Parity theorem)**.** *For odd $h$, $p$ and $p+h$ have opposite parity, so $C_\nu^{(2)}(h) = \frac{(\log 2)^\nu (\log(2+h))^\nu}{(2(2+h))^{\nu+1}}$ if $2+h$ is prime, 0 otherwise. This single deterministic term explains the entire 149–757$\times$ scatter observed for odd $h$ (the Hardy–Littlewood singular series vanishes for odd $h$, and division by it diverges): the scatter is not an anomaly, it is parity.*

*Region splitting:* in windows above $\sim 10^5$ the two-point central charge's $h$-dependence enters exactly through $\mathfrak{S}(h)$ ($\nu = 1, 2, 3$ ratios to the pairing law 0.9799–1.0107 at $5 \times 10^5$; twin-prime constant measured 0.6601649 vs true 0.6601618); below, the arithmetic region lives by its own law. The moment kernel $(\delta^2 + \gamma^2)^{-\nu/2}$ concentrates in $|\gamma| \lesssim \delta$, giving the time–energy duality $\tau_H \sim 1/\delta$: the divergence of $C_\nu$ as $\delta \to 0$ is the Weyl (smooth) part, not information; the central charge is a first-order single-point invariant, RMT fluctuations the second-order correlation layer. *Pairing critical law* (statistical window): the two-point central charge obeys

$$C_\nu^{(2)}(h) \sim \mathfrak{S}(h) \cdot \frac{(2\nu-2)!}{(2\nu+1)^{2\nu-1}},$$

with $h$-dependence entering exactly through the singular series; at $5 \times 10^5$ the ratios to the pairing law for $h = 2, 4, 6, 10, 12, 30$ and $\nu = 1, 2, 3$ lie in 0.9769–1.0107 (twin-prime constant measured 0.6601649 against the true 0.6601618).

# 10 The 24

$\Delta = \eta^{24}$ exactly: $\Delta(q) = (E_4^3 - E_6^2)/1728 = q\prod(1-q^m)^{24}$ coefficient-by-coefficient, with $\tau(n)$: $1, -24, 252, -1472, 4830, -6048, \ldots$ The 24-network:

| object | value | relation to 24 |
|---|---|---|
| $\tau(2)$ | $-24$ | $= -c$ (Monster CFT central charge) |
| Leech lattice dimension | 24 | unique 24-dimensional even unimodular rootless lattice |
| $\Delta = \eta^{24}$ exponent | 24 | $= c$ |
| Leech kissing number | 196560 | $= 24 \times (2^{13} - 2) = 24 \times 8190$ |
| Monster minimal faithful representation | 196883 | $47 \cdot 59 \cdot 71$ |
| $J = j - 744$ coefficient | 196884 | $= 196883 + 1 = 196560 + 182$ |

And 24 sits in the denominator of $C_\Delta$: the $p = 2$ term dominates (87.37%) with denominator $2^5 + 2^6 - \tau(2) = 32 + 64 + 24 = 120$ — the Tate skeleton 96 plus $c = 24$: the leading term is suppressed by exactly 4/5. $C_\Delta$ (primes $< 60$) $= 0.002291220813543$, matching the high-precision value $0.002291220813775$ to $2 \times 10^{-13}$.

**Proposition 10.1** (Falsifiable, confirmed)**.** *Among all systems computed, only* $\Delta$ *(the unique level-one weight-*12 *cusp form* $= \eta^{24}$*) has* 24 *entering the denominator at the prime* 2*: the corrected term* $a_2(\mathrm{Sym}^m)$ *is never* $-24$ *elsewhere (checked on* 12 *systems: CM weights* $3, 5, 7, 9, 11$*;* $11a1$ *and its* $\mathrm{Sym}^2, \mathrm{Sym}^3, \mathrm{Sym}^4$*).*

*Mirror side:* $L(11a1, 1)/\Omega = 1/5 \in \mathbb{Q}$, with $\Omega = \int_\gamma \omega$ the CY one-fold period, the D-brane central charge $Z(\gamma)$; the difference between $L(11a1, 1)$ and $\Omega/5$ is $5.55 \times 10^{-17}$ (an exact rational match). $q/\Delta(q) = 1/\eta^{24} = 1 + 24q + 324q^2 + 3200q^3 + 25650q^4 + \ldots$ reproduces the Yau–Zaslow counts $N_1 = 24, N_2 = 324, N_3 = 3200, N_4 = 25650$ digit by digit: the cusp correction $= \eta^{24}$ is reciprocal to the $K3$ BPS partition function, and $N_1 = 24$ is the same 24 as $c = 24$, $-\tau(2) = 24$, and Leech. The Möbius weights $n_d = \sum_{k|d} \frac{\mu(k)}{k^3} N_{d/k}$ (CY3: $D = 3$) match exactly for $d = 1, \ldots, 8$: multi-cover counting with weight $k^{-D}$, where $D$ is the dimension — primes side $D = 1$, CY threefold side $D = 3$. The mirror-quintic basic period $\varpi_0(z) = \sum_n \frac{(5n)!}{(n!)^5}(z/5^5)^n$ is finite at the conifold ($\varpi_0(z_C) = 1.0707258684$): the basic period stays finite on the discriminant locus; what diverges is the special geometry metric.

## 10.1 The Gap straddle criteria

For the straddle $[a, b]$ with $a = \kappa - d/2$, $b = \kappa + d/2$, the straddling ratio

$$\rho(d) = \frac{p^a + p^b}{2p^\kappa} = \cosh\Big(\frac{d}{2}\ln p\Big)$$

depends only on the width $d$, not on $\kappa$; with the dominant prime $p^\star = 2$ in all ten systems, $\rho$ is a universal ruler: $d = 1 : 1.061$, $d = 2 : 1.250$, $d = 3 : 1.591$, $d = 4 : 2.125$, $d = 5 : 2.917$, $d = 6 : 4.063$. Two independent criteria select the nearest strict straddle uniquely: $C2'$: $\rho \le 1.25 = \cosh(\ln 2)$ (the threshold calibrated from the integral-$\kappa$ side, where positivity forces $d = 2$, so the calibration is not circular); $C3'$: $|S| \ge 5\%$ ($|S|$ in place of the signed $S$, master log no. 59). On all 10/10 systems the intersection is the single width $d_{\min}$ (odd $m$: $d = 1$; even $m$: $d = 2$); on the integral side the criteria independently recover the known answer ($\mathrm{Sym}^4$ excludes $d = 4$ at $\rho = 2.125$; $\mathrm{Sym}^6$ excludes $d = 4, 6$). The Gap principle remains an inductive hypothesis, but its "tightness" is criterion-selected, not chosen.

## 11 Companion papers and the cross-validation

The material of this framework splits into four companion papers: the group-theoretic core (master identity, forgetting factor $F_k = k{+}1$ and its complete asymptotics, Howe duality = Rankin–Selberg); the seventh world (arithmetic dynamics: dynamic Lehmer constant $0.2036772613\ldots$ proved minimal over $\mathbb{Q}$, the three-axis fingerprint theorem, the PCF criterion over number fields); the spectral axis (length-spectrum degeneracy and the Maass level-one verdict); and the abc interface (the gap formula, Ogg's $m_p - 1$ form, the exact discrete RG equation for the Szpiro ratio). *Cross-validation:* the constant $C_1 = 0.75536661\ldots$ (the $\nu = 1$ central charge, the Euler product $\sum_p \frac{\log p}{p(p-1)}$) has now been computed independently in three separate lines of this program — the present framework, the entropy-collapse analysis of the Collatz-type recursions, and the power-kernel moments — each from unrelated definitions; the three computations agree, and this agreement, recorded here, is the strongest single cross-check the program possesses.

## 12 Honest boundaries

- This framework does not prove abc, Szpiro, RH, BSD, or any case of the Birch–Swinnerton-Dyer conjecture; Mochizuki's IUT remains disputed (Scholze–Stix).
- Our central charge is not the CFT central charge; the name is structural analogy, and the hard connections are $\eta^{24}$ and the $K3$ BPS reciprocal duality.
- The Gap principle is an inductive hypothesis: its representation-theoretic basis (invariants at the self-dual center) is a theorem, but tightness of the straddle is a criterion-selected choice (master log no. 59: the straddling ratio $\rho = \cosh(\frac{d}{2}\log p^\star)$ depends only on $d$, and $\rho \le 1.25$ with $|S| \ge 5\%$ selects the nearest strict straddle uniquely on 10/10 systems).
- Item (v) of Theorem 9.1 is at conjecture level (Montgomery–Rudnick–Sarnak); the (iv)↔(v) bridge is closed only at the level of the finite-$N$ dictionary and the recurrence-peak criterion.
- The CM family is the computable model of a Hida family; the cap $\min(n+1, w-1)$ is CM-specific, and general non-CM Hida families are expected to have no cap (unverified).
- Every numerical value above carries its provenance in the master compendium; corrections applied are distributed to the companion papers and summarized in Appendix A.

# A Corrections touching this framework

| # | error as made | correction |
|---|---|---|
| 2 | Maass $\kappa$ written $1/2$ | $\kappa = 0$; critical line $\kappa + \frac{1}{2} = \frac{1}{2}$ (paper B) |
| 3 | degeneracy scripts: two bugs | fixed; $g(D) = h^+(D)$ (paper B, nos. 41–44) |
| 5 | naive $C^{\mathrm{trop}}$ claimed resolving power | zero resolving power; name withdrawn; replaced by the skeleton dynamic charge |
| 6 | additive reduction $c_p = v_p(\Delta)$ | $c_p = \lvert\Phi\rvert$ (ADE component group order) |
| 7 | $\mathrm{Vol}(4_1)$ initial value | $6\Lambda(\pi/3)$, two ideal tetrahedra |
| 8 | $M(\Delta) = h$ universal | only genus 1; in general $\log M \geq h$ |
| 9 | $p$-adic valuation cap universal | cap is CM-specific |
| 34 | $\Phi_1 = 0.755362$ miscopied | $\Phi_1 = C_1 = 0.75536661$; Möbius truncation $K = 12$ insufficient at $\nu = 1$ ($j = 13$ term $\approx \log 2 \cdot 2^{14} \approx 4 \times 10^5$), closed at $K = 60$ |
| 36 | "$\Phi_n \to 0.734$?" | $\Phi_n$ does not converge to a constant: limit 0 (base $r = e \log 3/3 = 0.9954459403$; peak $0.847045989$ at $n^\star = 103$; valley $0.707$ at $n = 2$) |
| 45 | Gap straddle degenerates at integral $\kappa$ | strict straddle: integral $\kappa \Rightarrow d = 2$, half-integral $\kappa \Rightarrow d = 1$; selected uniquely by $\rho \leq 1.25$ (no. 59) |
| 47 | dangling cross-reference to §6.5 | renumbered; all cross-references carry the part name |
| 55 | "$b$-invariant step residue-char-free" | false at $p = 2$ (64.a4 vs 32.a3); boundary at $A$ mod 4; conductor is 64 |
| 56 | wild-table key precision conflict | keys must be unified to mod-16 |
| 59 | straddle criteria defects | $\lvert S\rvert$ in place of signed $S$; threshold $\rho \leq 1.25 = \cosh(\log 2)$ calibrated from the integral side |

# B Supplementary derivations

*B.1. Generating function of the symmetric-power L-functions.* For $|t|$ small,

$$\sum_{k\geq 0} \log L(s, \mathrm{Sym}^k f)\, t^k = \sum_p \sum_{n\geq 1} \frac{p^{-ns}}{n\,(1-\alpha^n t)(1-\beta^n t)},$$

since $\sum_{k\geq 0} \mathrm{tr}(\mathrm{Sym}^k A_p) t^k = 1/\det(1 - tA_p)$; verified at 11a1, $s = 10$, $t = 0.1$ to $1.1 \times 10^{-10}$ ($P/K/N = 400/10/14$). *B.2. Two closed forms for $R_k$.* With $P_n = \alpha^n + \beta^n$,

$$h_k^{(n)} = \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \binom{k-j}{j} p^{2\kappa n j} P_n^{k-2j} = p^{n\kappa k} U_k(\cos n\theta_p),$$

the second by replacing $\theta$ with $n\theta$ in $T_k = p^{k\kappa} U_k(\cos\theta)$; agreement to $\leq 10^{-35}$. *B.3. Tate content at general $n$.* $\langle U_k(\cos n\theta)\rangle$: $k$ odd: $(0, -1, 0, 0)$ at $n = 1, 2, 3, 4$; $k$ even: $(0, +1, +1, +1)$. Mechanism: the Sato–Tate measure gives $\langle \cos 2\theta \rangle = -1/2$ as the only nonzero Fourier moment, and $2j - k = \pm 1$ only for $k$ odd. Measured (400 primes, $k = 1..6$): $n = 3$: $0.02, 1.09, 0.01, 1.13, -0.07, 1.09$; $n = 4$: $0.02, 0.97, 0.06, 0.92, 0.29, 0.93$. *B.4. General fibered knots: beyond genus 1.* For fibered $\Sigma_g$ with monodromy $B$, $\log M(\Delta) \geq h$, equality iff the unstable roots share one quadratic

field: among 16 exact genus-2 examples, 9 give $\log M = h$ (e.g. $[1,-4,5,-4,1]$, $\lambda = \varphi^2$, the same field as $4_1$) and 7 give $\log M > h$ ($[1,-6,11,-6,1] = \Delta_2 \cdot \Delta_{4_1}$ gives $2h$; $[1,-7,13,-7,1]$, $[1,-10,19,-10,1]$ give $+0.609$, $+0.411$). The critical amplitude $A$ is predicted exactly: single maximum $A = 1$ (1.00291/1.00031/1.0000086/0.99950 at $\delta = 0.2/0.1/0.05/0.02$), $\Delta_2 \cdot 4_1$: $A = 2$ (2.01109/2.00340/2.00094/1.99917), alternating $\pm\lambda$ pairs return to $A = 1$ by parity cancellation. *B.5. Catalan asymptotics.* $\mathrm{Cat}(m/2) \sim 2^{m+3/2}/(\sqrt{\pi}\, m^{3/2})$ (leading term within 7% at $m = 30$; with the Stirling correction the match improves to 0.9995): the boundary position $m\kappa$ grows linearly while the Tate weight grows exponentially — the quantitative form of layer-1 forgetting. *B.6. Two-prime central charge.* For 11a1 with 467 twin-prime pairs the disconnected background is $\langle x^m \rangle^2 = \mathrm{Cat}(m/2)^2$: measured versus background, $m = 2$: 1.0113 vs 0.9862; $m = 4$: 3.8007 vs 3.8939 (background value 1, 4) — the correlation layer forgets one exponential faster than the coefficient layer. *B.7. Khovanov lift (categorification).* $4_1 = \mathrm{closure}((\sigma_1\sigma_2^{-1})^2) \in B_3$, $\rho(\sigma_i) = A + A^{-1}e_i$, $\delta = -A^2 - A^{-2}$; $V_{4_1}(t) = t^2 - t + 1 - t^{-1} + t^{-2}$. At the Sato–Tate phase $A = e^{i\theta/2}$ the loop parameter is $\delta_p = -2\cos\theta = -a_p/p^\kappa$, and Deligne's bound is the Jones-index bound $\delta^2 \le 4$: each prime carries a Temperley–Lieb algebra. The Khovanov lift promotes $\delta$ to $\dim V = 2$: chain groups $C(s) = V^{\otimes|s|}$, $2^4 = 16$ resolutions, basis of crossingless matchings ($\mathrm{Cat}(4) = 14$), total chain dimension $\sum_s 2^{|s|} = 66$, Euler characteristic $\sum_s (-1)^{|s|}2^{|s|} = 2 = 2V_{4_1}(1)$ — categorification lifts $\delta$ to $\dim V$, reproducing the Jones value exactly.

## References


[1] A. Walfisz, *Weylsche Exponentialsummen in der neueren Zahlentheorie*, Mathematische Forschungsberichte XVI, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

[2] A. Beurling, Analyse de la loi asymptotique de la distribution des nombres premiers généralisés, *Acta Math.* **68** (1937), 255–291.

[3] P. Deligne, La conjecture de Weil. II, *Publ. Math. IHÉS* **52** (1980), 137–252.

[4] J. Tate, Algorithm for determining the type of a singular fiber in an elliptic pencil, in *Modular Functions of One Variable IV*, Lecture Notes in Math. **476**, Springer, 1975, 33–52.

[5] A. P. Ogg, Abelian curves of 2-power conductor, *Proc. Cambridge Philos. Soc.* **62** (1966), 143–148.

[6] T. Saito, Conductor, discriminant, and the Noether formula of arithmetic surfaces, *Duke Math. J.* **57** (1988), 151–173.

[7] Y. Ihara, On discrete subgroups of the two by two projective linear group over $\mathfrak{p}$-adic fields, *J. Math. Soc. Japan* **18** (1966), 219–235.

[8] R. Bowen, Periodic points and measures for Axiom A diffeomorphisms, *Trans. Amer. Math. Soc.* **154** (1971), 377–397.

[9] W. Parry, M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics, *Astérisque* **187–188** (1990).

[10] D. Fried, Analytic torsion and closed geodesics on hyperbolic manifolds, *Invent. Math.* **84** (1986), 523–540.

[11] J. H. Conway, N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Grundlehren Math. Wiss. **290**, Springer, 1988.

[12] S.-T. Yau, E. Zaslow, BPS states, string duality, and nodal curves on K3, *Nuclear Phys. B* **471** (1996), 503–512.

[13] G. Shimura, The special values of the zeta functions associated with cusp forms, *Comm. Pure Appl. Math.* **29** (1976), 783–804.

[14] B. Gross, D. Zagier, Heegner points and derivatives of $L$-series, *Invent. Math.* **84** (1986), 225–320.

[15] P. Diaconis, M. Shahshahani, On the eigenvalues of random matrices, *J. Appl. Probab.* **31A** (1994), 49–62.

[16] E. Rains, High powers of random elements of compact Lie groups, *Probab. Theory Related Fields* **107** (1997), 219–241.

[17] R. Howe, Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond, *Israel Math. Conf. Proc.* **8** (1995), 1–182.

[18] W. Fulton, J. Harris, *Representation Theory: A First Course*, Grad. Texts in Math. **129**, Springer, 1991.

[19] J. H. Silverman, *The Arithmetic of Elliptic Curves*, 2nd ed., Grad. Texts in Math. **106**, Springer, 2009.

[20] H. L. Montgomery, The pair correlation of zeros of the zeta function, *Proc. Sympos. Pure Math.* **24** (1973), 181–193.

[21] Z. Rudnick, P. Sarnak, Zeros of principal $L$-functions and random matrix theory, *Duke Math. J.* **81** (1996), 269–322.

[22] J. E. Cremona, *Algorithms for Modular Elliptic Curves*, 2nd ed., Cambridge University Press, 1997.