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SMALE'S MEAN VALUE CONSTANT (1981) EXCEEDS 1.1559

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Abstract. We prove that Smale's mean value constant exceeds 1.1559837, improving the bound 1.1018 of Jatar and Ng. Smale proved in 1981 that for every complex polynomial P of degree at least 2 and every point z with $P'(z) \neq 0$, some critical point ζ of P satisfies $|P(z)-P(\zeta)| \leq K\,|z-\zeta|\,|P'(z)|$ with $K = 4$, and he conjectured that $K = 1$ suffices. The best constant K^* is called Smale's mean value constant. Smale's conjecture is false: a proof that $K^* > 1$ appeared in 2026. Jatar and Ng showed that every pair (W, B) of a normalized univalent map W of the unit disk and a finite Blaschke product B with $B(0) > 0$, such that $(WB)'$ has no zero in the disk, gives $K^* \geq 1/B(0)$. We construct such a pair explicitly, with B of degree two, from a polynomial of degree 18 with Gaussian rational coefficients, and we verify the required properties by a computation in exact arithmetic. Since the best constant $K(d)$ for degree d is nondecreasing in d (Crane), $K(d)$ also exceeds 1.1559837 for all large d. All theorems, lemmas and corollaries of this paper, and the results of Jatar and Ng that it uses, are also proved in the Lean 4 proof assistant and checked with the Lean comparator.

AI Disclosure. This work was done overwhelmingly by AI models and supervised by the author. The models were Anthropic's Claude and OpenAI's ChatGPT. The author takes full responsibility for the paper.

1. Introduction

We prove that Smale's mean value constant K^*, defined below, exceeds 1.1559837. The best lower bound proved before was 1.1018, due to Jatar and Ng [10].

Smale's mean value problem. For $d \geq 2$ let $\mathcal{P}_d$ be the set of complex polynomials p of degree d with $p(0) = 0$ and $p'(0) = 1$. Every critical point β of such a p is nonzero, as $p'(0) = 1$, and we put

$$m(p) = \min_{p'(\beta)=0} \frac{|p(\beta)|}{|\beta|}, \qquad K(d) = \sup_{p\in\mathcal{P}_d} m(p), \qquad K^* = \sup_{d\geq 2} K(d).$$

If P is a polynomial of degree d and $P'(z) \neq 0$, then $p(u) = (P(z+u) - P(z))/P'(z)$ lies in $\mathcal{P}_d$, and its critical points are the points $\beta = \zeta - z$ with ζ a critical point of P. So $K(d)$ is the best constant K in the inequality

$$\left|\frac{P(z) - P(\zeta)}{z - \zeta}\right| \leq K\,|P'(z)| \qquad \text{for some critical point } \zeta \text{ of } P, \tag{1}$$

over all polynomials P of degree d and all z with $P'(z) \neq 0$, and K^* is the best constant over all degrees: *Smale's mean value constant*. In words, (1) says that the mean rate of change of P between z and some critical point ζ is at most K times its rate of change at z; hence the name.

Smale proved (1) with $K = 4$ [13], so $K^* \leq 4$. The polynomial $p(z) = z - z^d/d$ gives $K(d) \geq (d-1)/d$ (its critical points β are the $(d-1)$-st roots of unity, and $p(\beta)/\beta = 1 - \beta^{d-1}/d = (d-1)/d$ at each). Smale conjectured that $K^* = 1$, and even that $K(d) = (d-1)/d$ for every d; this is *Smale's mean value conjecture*. The mean value problem is the first of the additional problems

Date: October 28, 2026. Revised: November 9, 2026.

2020 *Mathematics Subject Classification.* Primary 30C10; Secondary 30C15, 30C45, 30J10.

Key words and phrases. Smale's mean value conjecture, univalent functions, Blaschke products.

The Lean 4 proof and the programs and data of Section 5 are at `https://github.com/david44hello/papers_supporting_materials/tree/smale-mean-value-constant-v2.90/smale-mean-value-constant`.

in Smale's list of problems for the next century [14]. The equality $K(d) = (d-1)/d$ holds for $d \le 4$ (Tischler [15]; see also [12, Sect. 7.2]), and Crane announced a computer-assisted proof for $d = 5$ [6, Sect. 1]. Every later improvement of the constant 4 is a bound for $K(d)$ that depends on d. For example, Beardon, Minda and Ng proved $K(d) \le 4^{(d-2)/(d-1)}$ using an estimate for the hyperbolic metric [3]; Conte, Fujikawa and Lakic proved $K(d) \le 4(d-1)/(d+1)$ using Bieberbach's bound for the second coefficient of univalent functions [4]; Fujikawa and Sugawa combined the two methods [8]; and Crane proved $K(d) < 4 - 2.263/\sqrt{d}$ for $d \ge 8$ [6]. All these bounds tend to 4 as $d \to \infty$, so 4 is still the best known upper bound for K^*. Crane also showed that $K(d)$ is nondecreasing in d [5].

Smale's conjecture is false: a proof that $K^* > 1$ was made public in September 2026 [1]; it shows that some p has $m(p) > 1$, but does not make the degree of p explicit. Jatar and Ng [10] then recast the problem in terms of pairs of a univalent map of the unit disk and a finite Blaschke product (Section 2) and proved

$$K^* \ge \sqrt{8081/6656} = 1.1018586\ldots;$$

this bound comes from an explicit pair. (In [10], $m(p)$, $K(d)$ and K^* are written $S(P)$, $\mathcal{M}_d$ and $\mathcal{M}$.) Adamczewski and Colbrook [2] explain the mechanism behind the proof in [1] and give an explicit polynomial p of degree 29 984 with $m(p) > 1.0000004$, verified by interval arithmetic; it gives $K(d) > 1$ for every $d \ge 29\,984$.

Main result. The table compares the best earlier bounds with ours.

Best result before this paper	**New result**
$K^* > 1.1018$ [10, Thm. 1.3]; $K^* > 1.1030$, stated without proof in [10, Rem. 6.7]	$K^* > 1.1559837$ (Theorem 5.1)
$K(d) > 1.1018$ for all large d [10, Thm. 1.3]	$K(d) > 1.1559837$ for all large d (Corollary 5.4)

Theorem 1.1. *Smale's mean value constant satisfies $K^* \ge \sqrt{\Lambda}$, where Λ is the rational number*

$$\Lambda = \frac{267952521936226337027481600174795544467587176908574201599 7}{200518481134769781691297951341495479358548025660497985536 0}$$

and $\sqrt{\Lambda} = 1.15598372919\ldots$; in particular $K^* > 1.1559837$, and $K(d) > 1.1559837$ for all sufficiently large d.

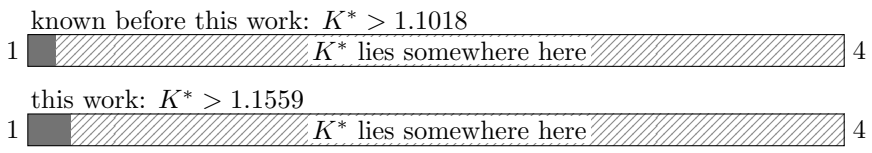


FIGURE 1. What is known about K^*, to scale. Each bar runs from 1, the value Smale conjectured, to 4, his upper bound of 1981. The dark part is excluded by the best lower bound; the hatched part is left open.

The bound comes from a single explicit admissible pair (W, B) in the sense of Jatar and Ng (Definition 2.1), with B of degree two. We build it from a polynomial Q of degree 18, whose derivative has a zero of order 5 at $z = 1$, by a construction of Jatar and Ng [10, Lemma 6.2] that we extend from one zero of B to several (Lemma 3.1). The coefficients of Q and the zeros of Q used in the construction are Gaussian rationals. That the pair has the required properties reduces, by Dieudonné's univalence criterion and the Schur–Cohn test, to four finite checks (Lemma 4.3); the main check says that a certain polynomial of degree 578 in one real variable has no zero in $[-1, 1]$. All four checks are made in exact arithmetic, on the exact data printed in Section 5. An expanded version of this paper, with further results, is posted on Zenodo [9].

2. Admissible pairs

Let $\mathbb{D} = \{|w| < 1\}$ be the unit disk and $\mathcal{S}$ the class of univalent (holomorphic and injective) maps W of $\mathbb{D}$ with $W(0) = 0$ and $W'(0) = 1$. For $a \in \mathbb{D}$ let $B_a(w) = (w - a)/(1 - \bar{a}w)$. A *finite Blaschke product of degree* $k \geq 0$ is a function $B = \lambda B_{a_1} \cdots B_{a_k}$ with $|\lambda| = 1$ and $a_1, \ldots, a_k \in \mathbb{D}$.

Definition 2.1 ([10, Def. 1.1])**.** An *admissible pair* is a pair (W, B) of a map $W \in \mathcal{S}$ and a finite Blaschke product B with $B(0) > 0$ such that the derivative $(WB)'$ has no zero in $\mathbb{D}$. We write σ_k for the supremum of $1/B(0)$ over the admissible pairs with $\deg B \leq k$.

Theorem 2.2 (Jatar and Ng)**.** *For every admissible pair* (W, B) *and every* $A < 1/B(0)$ *there is a polynomial* $p \in \mathcal{P}_d$*, for some* $d \geq 2$*, with* $m(p) > A$ [10, Prop. 4.2]*. Hence* $K^* \geq 1/B(0)$ *for every admissible pair.*

The condition in Definition 2.1 concerns the open disk only: if WB extends holomorphically across the unit circle, $(WB)'$ may vanish there. The pair of Section 5 has such a zero, of order 5. It is still admissible, and [10, Prop. 4.2] is stated for every admissible pair. Its proof first reduces to pairs with no zero of $(WB)'$ on the closed disk [10, Lemma 2.1], as follows. Let B be nonconstant, and let $t < 1$ satisfy $t > B(0)$ and $t > |B(z)|$ at every critical point z of B in $\mathbb{D}$. Then the set $\Omega_t = \{z \in \mathbb{D} : |B(z)| < t\}$ is a Jordan domain containing 0, with real-analytic boundary and closure in $\mathbb{D}$. Let ψ_t map $\mathbb{D}$ conformally onto Ω_t with $\psi_t(0) = 0$ and $\psi_t'(0) > 0$. Then ψ_t extends conformally across the circle, and $(W^{(t)}, B^{(t)}) = \big(W \circ \psi_t/\psi_t'(0),\, B \circ \psi_t/t\big)$ is an admissible pair. Moreover $(W^{(t)}B^{(t)})' = (WB)'(\psi_t)\,\psi_t'/(t\psi_t'(0))$ has no zero even on the closed disk, and $1/B^{(t)}(0) = t/B(0)$ tends to $1/B(0)$ as $t \to 1$.

Remark 2.3 (From polynomials to pairs). The link between pairs and polynomials is easiest to see by going from polynomials to pairs [10, Prop. 3.1]. Let $p \in \mathcal{P}_d$ with $m(p) > 1$, let $1 < R < m(p)$, put $q(w) = p(w)/w$, and let Ω be the component of $\{|q| < R\}$ that contains 0 (note $q(0) = 1$); it is bounded, as q is a nonconstant polynomial ($d \geq 2$), and simply connected, by the maximum principle. Let ψ map $\mathbb{D}$ conformally onto Ω with $\psi(0) = 0$ and $\psi'(0) = \kappa > 0$, and put $W = \psi/\kappa$ and $B = (q \circ \psi)/R$. Then $W \in \mathcal{S}$ and $B(0) = 1/R$. Moreover B is a finite Blaschke product: since $|q| < R$ in Ω and $|q| = R$ on $\partial\Omega$, it maps $\mathbb{D}$ into $\mathbb{D}$ with $|B(z)| \to 1$ as $|z| \to 1$, and every such holomorphic map is a finite Blaschke product. Finally $(WB)' = p'(\psi)\,\psi'/(\kappa R)$ has no zero in $\mathbb{D}$, since every critical point β of p has $|q(\beta)| \geq m(p) > R$ and so lies outside Ω. So (W, B) is admissible with $1/B(0) = R$; letting $R \to m(p)$ gives $m(p) \leq \sup_k \sigma_k$. This also holds when $m(p) \leq 1$, since the admissible pair $W(w) = w$, $B = 1$ gives $\sigma_k \geq 1$ for every k. Hence $K^* \leq \sup_k \sigma_k$. Theorem 2.2 goes the other way, so $K^* = \sup_k \sigma_k$ [10, Thm. 1.2]. From a given pair, Jatar and Ng build polynomials p with $m(p)$ close to $1/B(0)$, using Runge's theorem and Hilbert's lemniscate theorem.

3. A construction of admissible pairs

For $a, b \in \mathbb{D}$ with $a \neq b$ put

$$J(a, b) = 1 + \frac{(1 - |a|^2)(1 - |b|^2)}{|a - b|^2}.$$

Since $|1 - \bar{a}b|^2 - |a - b|^2 = (1 - |a|^2)(1 - |b|^2)$, we have $1/|B_a(b)|^2 = J(a, b)$. For $b \in \mathbb{D}$ let $\varphi(w) = (w + b)/(1 + \bar{b}w)$; it maps $\mathbb{D}$ onto $\mathbb{D}$, with $\varphi(0) = b$ and $\varphi'(0) = 1 - |b|^2$.

The idea of the next lemma is to start from a polynomial Q without critical points in $\mathbb{D}$ and to divide out the Blaschke factors B_{a_j} of some of its zeros $a_1, \ldots, a_k$ in $\mathbb{D}$. If the quotient H is univalent, then $Q = H \prod_j B_{a_j}$ is a univalent function times a Blaschke product, and the derivative of this product is Q', which has no zero in $\mathbb{D}$ by the choice of Q; composition with the automorphism φ moves a further zero b of Q to 0. The bound $1/B(0) = \prod_j 1/|B_{a_j}(b)|$ is large when the a_j are close to b in the pseudo-hyperbolic distance $|B_{a_j}(b)|$; what limits it is that Q must keep its critical points out of $\mathbb{D}$ while H stays univalent. For $k = 1$ the lemma is [10, Lemma 6.2].

Lemma 3.1. *Let $k \geq 1$ and let Q be a polynomial such that Q' has no zero in $\mathbb{D}$ and Q has $k+1$ distinct zeros $b, a_1, \dots, a_k$ in $\mathbb{D}$. Suppose that the polynomial*

$$H(z) = Q(z) \prod_{j=1}^{k} \frac{1 - \bar{a}_j z}{z - a_j}$$

is univalent in $\mathbb{D}$. Put $N = (1-|b|^2)H'(b)$, $\eta = \prod_{j=1}^{k} B_{a_j}(b)$,

$$W = \frac{H \circ \varphi}{N}, \qquad B = \frac{|\eta|}{\eta} \prod_{j=1}^{k} B_{a_j} \circ \varphi.$$

Then (W, B) is an admissible pair, $\deg B = k$, and

$$\frac{1}{B(0)^2} = \prod_{j=1}^{k} J(a_j, b).$$

Proof. Since the a_j are distinct zeros of Q, H is a polynomial. As $Q(b) = 0$ and $b \neq a_j$ for every j, $H(b) = 0$. As H is univalent, $H'(b) \neq 0$, so $N \neq 0$. Since φ is a conformal self-map of $\mathbb{D}$, W is univalent in $\mathbb{D}$, with $W(0) = H(b)/N = 0$ and $W'(0) = H'(b)\varphi'(0)/N = 1$; so $W \in \mathcal{S}$. Each $B_{a_j} \circ \varphi$ is a conformal automorphism of $\mathbb{D}$, hence of the form λB_γ with $|\lambda| = 1$ and $\gamma \in \mathbb{D}$; so B is a finite Blaschke product of degree k. Moreover $\eta \neq 0$, since $b \neq a_j$, and $B(0) = (|\eta|/\eta)\,\eta = |\eta| > 0$. Since $H \prod_j B_{a_j} = Q$, we have $WB = (|\eta|/(\eta N))\, Q \circ \varphi$, so

$$(WB)' = \frac{|\eta|}{\eta N}\, Q'(\varphi)\, \varphi',$$

which has no zero in $\mathbb{D}$, because $\varphi(\mathbb{D}) = \mathbb{D}$, Q' has no zero in $\mathbb{D}$ and φ' has no zero at all. Finally $1/B(0)^2 = 1/|\eta|^2 = \prod_j 1/|B_{a_j}(b)|^2 = \prod_j J(a_j, b)$. □

4. Reduction to finite checks

Lemma 3.1 needs two facts that are not finite checks as they stand: that H is univalent in $\mathbb{D}$, and that Q' has no zero in $\mathbb{D}$. This section reduces both to finitely many exact computations.

Univalence. Let $H(z) = \sum_{\ell=0}^{n} h_\ell z^\ell$ be a polynomial. The *Dieudonné polynomials* of H are

$$P_c(w) = \sum_{\ell=1}^{n} h_\ell\, U_{\ell-1}(c)\, w^{\ell-1} \qquad (c \in [-1, 1]),$$

where $U_{\ell-1}$ is the Chebyshev polynomial of the second kind, $U_{\ell-1}(\cos\theta) = \sin \ell\theta / \sin\theta$. For $c = \cos\theta$ they satisfy $H(ze^{i\theta}) - H(ze^{-i\theta}) = 2i \sin\theta\, z P_c(z)$; moreover $P_1 = H'$, since $U_{\ell-1}(1) = \ell$, and $P_c(0) = h_1$. Since $U_{\ell-1}(-c) = (-1)^{\ell-1} U_{\ell-1}(c)$, also $P_{-c}(w) = P_c(-w)$; in particular $P_{-1}(w) = H'(-w)$. So if $-1 < c < 1$ and $P_c(z) = 0$ with $0 < |z| < 1$, then H takes the same value at the two distinct points $ze^{\pm i\theta}$ of $\mathbb{D}$, and H is not univalent in $\mathbb{D}$. Dieudonné's criterion [7] (see also [10, Lemma 6.3]) is a converse: if no P_c with $c \in [-1, 1]$ has a zero in $\mathbb{D}$, then H is univalent in $\mathbb{D}$. It is usually stated for $H(0) = 0$ and $H'(0) = 1$, but it applies to every polynomial H: the hypothesis gives $h_1 = P_c(0) \neq 0$, and neither adding a constant to H (the P_c do not involve h_0) nor dividing H by h_1 changes univalence or the zeros of the P_c.

That no P_c has a zero in $\mathbb{D}$, for all c in an interval, is not a finite check. The next lemma reduces it to the absence of zeros on the unit circle, which Lemma 4.3 below tests with a resultant.

Lemma 4.1. *Suppose that H' has no zero in the closed disk $\overline{\mathbb{D}}$ and that no P_c with $c \in [-1, 1]$ has a zero on the unit circle. Then H is univalent in $\mathbb{D}$.*

Proof. Since $P_c(0) = h_1 = H'(0) \neq 0$, no P_c vanishes identically. Since no P_c has a zero on the circle, the number $\frac{1}{2\pi i}\oint_{|w|=1} P_c'(w)/P_c(w)\,dw$ of zeros of P_c in $\mathbb{D}$ depends continuously on $c \in [-1,1]$; being an integer, it is constant. At $c = 1$ it is 0, since $P_1 = H'$. So no P_c has a zero in $\mathbb{D}$, and Dieudonné's criterion applies. □

Zeros in the closed disk. For a polynomial $f(w) = \sum_{\ell=0}^{n} e_\ell w^\ell$ of *formal degree* n (that is, e_n may vanish) put $f^*(w) = w^n\overline{f(1/\bar{w})} = \sum_{\ell=0}^{n} \bar{e}_{n-\ell} w^\ell$; on the circle, $|f^*| = |f|$.

Lemma 4.2 (Schur–Cohn; see [11])**.** *Let $f(w) = \sum_{\ell=0}^{n} e_\ell w^\ell$ with $n \geq 1$, and put $Tf = \bar{e}_0 f - e_n f^*$, of formal degree $n-1$, with constant term $|e_0|^2 - |e_n|^2$. If $|e_0| > |e_n|$, then f and Tf have the same zeros on the circle, and if f has none there, the same number of zeros in $\mathbb{D}$. Consequently, if $f_0 = f$, $f_j = Tf_{j-1}$ (with f_{j-1} at formal degree $n-j+1$), and the constant terms δ_j of f_j $(1 \leq j \leq n)$ are all positive, then f has no zero in $\overline{\mathbb{D}}$.*

Proof. On the circle $|f^*| = |f|$. If $f(w) = 0$ there, then $f^*(w) = 0$ and $Tf(w) = 0$; if $Tf(w) = 0$ there, then $|e_0|\,|f(w)| = |e_n|\,|f(w)|$, so $f(w) = 0$. If f has no zero on the circle, then $|e_n f^*| < |\bar{e}_0 f|$ there, and Rouché's theorem gives the same number of zeros in $\mathbb{D}$ for $\bar{e}_0 f$ and Tf. For the last claim, let $1 \leq j \leq n$. The constant term δ_j of $f_j = Tf_{j-1}$ is $|u_0|^2 - |u_{n-j+1}|^2$, where u_0 and u_{n-j+1} are the constant and the top coefficient of f_{j-1}, of formal degree $n-j+1$. So $\delta_j > 0$ is the hypothesis of the first claim for f_{j-1}. Now $f_n = \delta_n$ is a nonzero constant, so it has no zeros; going back from $j = n$ to $j = 1$, each f_{j-1} has no zero on the circle and none in $\mathbb{D}$. □

The four checks.

Lemma 4.3. *Let $k \geq 1$, let Q be a polynomial of degree $n \geq 2$, let $b, a_1, \ldots, a_k \in \mathbb{C}$, and let $H(z) = Q(z)\prod_{j=1}^{k}(1 - \bar{a}_j z)/(z - a_j)$. Suppose:*

- (D) *$|b|, |a_1|, \ldots, |a_k| < 1$, the points $b, a_1, \ldots, a_k$ are pairwise distinct, and $Q(b) = Q(a_1) = \cdots = Q(a_k) = 0$.*

By (D), H is a polynomial of degree at most n; let P_c be its Dieudonné polynomials. Suppose further:

- (C) *$Q' = ZV$, where V is a polynomial without zeros in $\overline{\mathbb{D}}$ and Z is a nonzero constant times $\prod_\nu (z - \zeta_\nu)^{\mu_\nu}$ (possibly an empty product), with all $|\zeta_\nu| = 1$ and all μ_ν positive integers;*
- (R) *$\mathrm{Res}_w(P_c, P_c^*) \neq 0$ for every $c \in [-1,1]$, where $P_c^*(w) = \sum_{\ell=1}^{n} \bar{h}_\ell\, U_{\ell-1}(c)\, w^{n-\ell}$ (for real c this is $(P_c)^*$, with P_c taken at formal degree $n-1$), and $\mathrm{Res}_w(P_c, P_c^*)$ is the determinant of the $(2n-2) \times (2n-2)$ Sylvester matrix of P_c and P_c^*, both taken at formal degree $n-1$;*
- (S) *H' has no zero in $\overline{\mathbb{D}}$.*

Then the pair (W, B) of Lemma 3.1 is admissible, $\deg B = k$, and $1/B(0)^2 = \prod_{j=1}^{k} J(a_j, b)$. Hence $K^* \geq \prod_{j=1}^{k} J(a_j, b)^{1/2}$.

Proof. Each P_c has formal degree $n-1$. Let $c \in [-1,1]$ and suppose that $P_c(w_0) = 0$ for some w_0 with $|w_0| = 1$. Since $1/\bar{w}_0 = w_0$, we get $P_c^*(w_0) = w_0^{n-1}\overline{P_c(w_0)} = 0$, so w_0 is a common zero of P_c and P_c^*, and the resultant vanishes at c (a common zero makes the Sylvester matrix singular, at any formal degrees). This contradicts (R). So no P_c with $c \in [-1,1]$ has a zero on the circle, and by (S) and Lemma 4.1, H is univalent in $\mathbb{D}$. By (C), Q' has no zero in $\mathbb{D}$. So Lemma 3.1 applies, and Theorem 2.2 gives $K^* \geq 1/B(0)$. □

The labels (D), (C), (R), (S) stand for the data, the critical points of Q, the resultant, and the Schur–Cohn test for H'. Since the h_ℓ are fixed and each $U_{\ell-1}$ is a polynomial, the formulas for P_c and P_c^* make sense for every c, and $\mathrm{Res}_w(P_c, P_c^*)$ is a polynomial in c, so (R) is a statement about one polynomial in one real variable. In (C) and (S) the absence of zeros in $\overline{\mathbb{D}}$ is checked with Lemma 4.2, applied to V and to H'. Lemma 4.2 can only show that a polynomial has no zero in

the closed disk, while the Q' of Section 5 has a zero on the circle, at $z = 1$; this is why (C) splits off the factor Z.

5. Proof of Theorem 1.1

The polynomial Q of our pair has degree 18, and we describe it through its critical points. Fifteen of its seventeen critical points are prescribed: $z = 1$, with multiplicity 5, and ten Gaussian rationals ρ_j; the other two, the zeros of a quadratic G, are then fixed by requiring that Q vanish at a_1 and a_2 as well as at b. The numbers b, a_1, a_2, $\rho_1, \dots, \rho_{10}$ below are Gaussian rationals whose denominators are powers of 2; G, Q and H are determined by them exactly. These thirteen numbers were found by a numerical search that made $1/B(0)$ large (see Remark 5.2), with the degree 18 and the critical point of order 5 at $z = 1$ held fixed, and were then rounded; the proof uses only the printed numbers. Let

$$b = -\tfrac{3238273}{16777216} + \tfrac{6869081}{8388608}\,i, \qquad a_1 = -\tfrac{4425427}{16777216} - \tfrac{256269}{2097152}\,i, \qquad a_2 = \tfrac{2126359}{8388608} - \tfrac{6838355}{8388608}\,i,$$

and let $\rho_1, \dots, \rho_{10}$ be the Gaussian rationals

$$\begin{aligned}
\rho_1 &= \tfrac{524767}{524288} + \tfrac{3303}{1048576}\,i, & \rho_2 &= \tfrac{262319}{262144} + \tfrac{3785}{1048576}\,i, & \rho_3 &= \tfrac{510667}{1048576} + \tfrac{596603}{524288}\,i,\\
\rho_4 &= \tfrac{140827}{1048576} + \tfrac{1177373}{1048576}\,i, & \rho_5 &= -\tfrac{20639}{32768} + \tfrac{439225}{524288}\,i, & \rho_6 &= -\tfrac{105435}{131072} - \tfrac{652651}{1048576}\,i,\\
\rho_7 &= -\tfrac{291711}{524288} - \tfrac{113639}{131072}\,i, & \rho_8 &= -\tfrac{96673}{524288} - \tfrac{1084035}{1048576}\,i, & \rho_9 &= \tfrac{524719}{524288} - \tfrac{3195}{262144}\,i,\\
\rho_{10} &= \tfrac{262337}{262144} - \tfrac{4087}{1048576}\,i.
\end{aligned}$$

Put $L(t) = (t-1)^5 \prod_{j=1}^{10}(t - \rho_j)$. There is a unique pair (g_0, g_1) of Gaussian rationals such that the quadratic $G(t) = t^2 + g_1 t + g_0$ satisfies

$$\int_b^{a_1} G(t)L(t)\,dt = \int_b^{a_2} G(t)L(t)\,dt = 0.$$

This is a linear system in g_0, g_1 with nonzero determinant; the real and imaginary parts of g_0 and g_1 have denominators of 451 to 458 digits. Define

$$Q(z) = 18\int_b^z G(t)L(t)\,dt, \qquad \text{so that} \qquad Q'(z) = 18\,(z-1)^5\,G(z)\prod_{j=1}^{10}(z-\rho_j).$$

Since GL is monic of degree 17, Q is a monic polynomial of degree 18; $Q(b) = 0$ by definition, and the two conditions on G say exactly that $Q(a_1) = Q(a_2) = 0$. We put $V(z) = 18\,G(z)\prod_j(z-\rho_j)$, of degree 12, and $H(z) = Q(z)(1-\bar a_1 z)(1-\bar a_2 z)/((z-a_1)(z-a_2))$, a polynomial of degree 18. Numerically the zeros of G are $1.00000688\ldots - 0.00152261\ldots\,i$ and $-0.41009053\ldots + 1.04265219\ldots\,i$. The exact coefficients of Q, Q', V and H are in the repository named on the first page. Figure 2 shows the configuration: ten of the seventeen critical points of Q, counted with multiplicity, lie within 0.0123 of $z = 1$ (the zero of order 5, the first zero of G, and ρ_1, ρ_2, ρ_9, ρ_{10}), and the other seven have moduli between 1.017 and 1.238. The two factors of $\Lambda = J(a_1, b)\,J(a_2, b)$ are $J(a_1, b) = 1.30038\ldots$ and $J(a_2, b) = 1.02761\ldots$.

Theorem 5.1. *For these data, (D), (C) with $Z = (z-1)^5$, (R) and (S) of Lemma 4.3 hold, with $k = 2$ and $n = 18$. Hence the pair (W, B) of Lemma 3.1 is admissible,* $\deg B = 2$, *and* $1/B(0)^2 = J(a_1, b)\,J(a_2, b) = \Lambda$, *the rational number of Theorem 1.1; so $K^* \geq \sqrt{\Lambda} > 1.1559837$.*

Proof. Each of the four facts reduces to finitely many exact computations over $\mathbb{Q}(i)$, which we carry out by computer, from the numbers printed above, as follows. We first solve the linear system for g_0, g_1 and expand Q, Q' and H into exact lists of coefficients; all the checks are made on these lists. The programs and their output are in the repository named on the first page; the whole computation takes about eight minutes on one processor core.

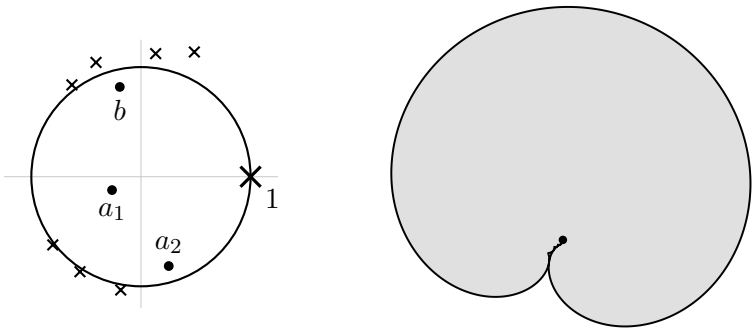


FIGURE 2. Left: the data of this section. The dots are the zeros b, a_1, a_2 of Q in $\mathbb{D}$, and the crosses are the 17 critical points of Q, counted with multiplicity; none lies in $\mathbb{D}$. The large cross at $z = 1$ stands for the ten of them within 0.0123 of 1, named in the text; the small crosses are the other seven. Right: the image $H(\mathbb{D}) = N\,W(\mathbb{D})$, computed numerically and shown without its scale. The dot is $H(b) = 0$, next to the tip of a narrow inlet whose two sides almost touch (Remark 5.2).

(D) We compute $|b|^2$, $|a_1|^2$, $|a_2|^2$ exactly, compare them with 1, and check that the three points are distinct and that Q vanishes at each of them. We also check that the coefficient list of Q' is that of the derivative of Q, and that $H(z)(z-a_1)(z-a_2) = Q(z)(1-\bar{a}_1 z)(1-\bar{a}_2 z)$.

(C) We divide Q' by $(z-1)^5$ exactly, with remainder 0, and apply Lemma 4.2 to the quotient V: the 12 numbers δ_j are positive.

(R) After multiplying H by a positive integer, so that its coefficients are Gaussian integers (this multiplies the resultant by a positive constant), we compute $F(c) = \mathrm{Res}_w(P_c, P_c^*)$ as an exact polynomial in c. The resultant is the determinant of a 34×34 Sylvester matrix whose entries are polynomials in c of degree at most 17, so F has degree at most 578, and its values at 579 points determine it. We compute the determinant at the 581 integer points $c = 0, 1, \ldots, 580$, where the entries are Gaussian integers since each $U_{\ell-1}$ has integer coefficients. We work modulo the 1128 largest primes $r \equiv 1 \pmod 4$ below 2^{62}, mapping i to each of the two square roots of -1 modulo r (the two images give the real and imaginary parts modulo r), and recover the values exactly by the Chinese remainder theorem: Hadamard's inequality bounds the real and imaginary parts at every point by 2^E with $E = 69921$, and the product of the primes exceeds 2^{E+2}. The imaginary parts of all 581 values are 0; since F has degree at most 578, F has real coefficients. We interpolate F from its values at 579 of the points, check the result at the other two, and find that F has degree 578. Finally $F(\pm 1) \neq 0$, and F has no zero in $(-1, 1)$: we split $[-1, 1]$ by repeated bisection, checking that F does not vanish at any midpoint, until on each piece $[c_0, c_1]$ the coefficients of $(1+x)^{578} F\big((c_0+c_1 x)/(1+x)\big)$ have no sign change; since $x \mapsto (c_0+c_1 x)/(1+x)$ maps $(0, \infty)$ onto (c_0, c_1), Descartes' rule of signs shows that F has no zero in (c_0, c_1). For our F this takes 47 applications of Descartes' rule, 24 of them on final subintervals.

(S) We apply Lemma 4.2 to H', of degree 17: the 17 numbers δ_j are positive.

We also check that $J(a_1, b)\, J(a_2, b)$ equals the rational number Λ of Theorem 1.1. In exact arithmetic, $\Lambda - 1.1559837^2 = 6.75\ldots \cdot 10^{-8} > 0$, so $\sqrt{\Lambda} > 1.1559837$. Lemma 4.3 gives the rest. □

Remark 5.2 (The margins). The zeros of V, H' and the P_c lie outside $\overline{\mathbb{D}}$. We call the distance from the unit circle to the nearest zero of such a polynomial its *margin*. Some margins are very small, which is why every step is done in exact arithmetic. They are small because the data were chosen to make $1/B(0)$ large, and, as explained before Lemma 3.1, what limits $1/B(0)$ is that these zeros

must stay out of $\mathbb{D}$. The zero of V nearest to the circle, a zero of G, has modulus $1.00000804\ldots$. The zero of H' nearest to the circle has modulus $1.00001141\ldots$; since $P_{\pm1}(w) = H'(\pm w)$, this gives the margin at $c = \pm1$. Numerically the zeros of the P_c come closest to the circle near $c = \pm0.9231$, within about $7.03 \cdot 10^{-6}$; see Figure 3. A small margin has a geometric meaning. For $-1 < c < 1$ and $c = \cos\theta$, a zero w of P_c gives $H(we^{i\theta}) = H(we^{-i\theta})$ with $|w| > 1$; so when $|w|$ is close to 1, two points of the boundary curve of $H(\mathbb{D})$ nearly coincide. Numerically, at the dips of Figure 3 this happens along the narrow inlet of Figure 2, whose two sides almost touch.

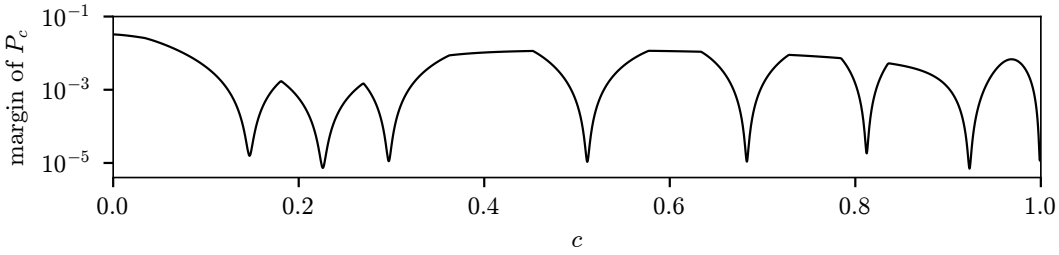


FIGURE 3. The margin of P_c, that is, the distance from the unit circle to the nearest zero of P_c (Remark 5.2), computed numerically for $0 \le c \le 1$ and drawn on a logarithmic scale. Since $P_{-c}(w) = P_c(-w)$, the curve for $-1 \le c \le 0$ is the mirror image of this one. Check (R) proves that the curve never reaches 0. Its deep dips are the places where zeros of P_c are pushed against the circle.

The degree statement of Theorem 1.1 rests on the monotonicity of $K(d)$, which is due to Crane [5] (see also [10, proof of Cor. 4.6]); we include the short proof.

Lemma 5.3 (Crane)**.** *$K(d+1) \ge K(d)$ for every $d \ge 2$.*

Proof. Let $p \in \mathcal{P}_d$, and for $\varepsilon \ne 0$ put $p_\varepsilon(z) = p(z) + \varepsilon z^{d+1}$, which lies in $\mathcal{P}_{d+1}$. As $\varepsilon \to 0$, $d-1$ of the d critical points of p_ε, counted with multiplicity, tend to those of p, and the remaining one, β_ε, tends to ∞. The first claim follows from Hurwitz's theorem, as $p'_\varepsilon \to p'$ locally uniformly; the second because the product of the d critical points of p_ε has modulus $1/((d+1)|\varepsilon|) \to \infty$, as $p'_\varepsilon(0) = 1$, while the other $d-1$ stay bounded. Since $p'_\varepsilon(\beta_\varepsilon) = 0$ gives $\varepsilon\beta_\varepsilon^d = -p'(\beta_\varepsilon)/(d+1)$, we have

$$\frac{p_\varepsilon(\beta_\varepsilon)}{\beta_\varepsilon} = \frac{p(\beta_\varepsilon)}{\beta_\varepsilon} - \frac{p'(\beta_\varepsilon)}{d+1},$$

and if α is the leading coefficient of p, the right side is $\alpha\beta_\varepsilon^{d-1}/(d+1) + O(\beta_\varepsilon^{d-2})$, which tends to ∞ in modulus, since $d \ge 2$. As the critical points of p are nonzero, it follows that $m(p_\varepsilon) \to m(p)$. Hence $K(d+1) \ge m(p)$ for every $p \in \mathcal{P}_d$, and $K(d+1) \ge K(d)$. □

Corollary 5.4. *$K(d) > 1.1559837$ for all sufficiently large d.*

Proof. Let $A = 1.1559837$. By Theorem 5.1, $A < K^* = \sup_d K(d)$, so $K(d_0) > A$ for some d_0. By Lemma 5.3, $K(d) > A$ for every $d \ge d_0$. □

The proof gives no explicit value of d_0, since Theorem 2.2 does not bound the degree of the polynomial it gives. Theorem 5.1 and Corollary 5.4 prove Theorem 1.1.

ACKNOWLEDGMENTS

The author is immensely grateful to Edward Crane for mentorship, and for feedback and suggestions that greatly improved this work and its exposition.

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