SPECTRAL SYNTHESIS FOR THE BÁEZ–DUARTE STEP SPACE

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Abstract. We prove that the closed Báez–Duarte subspace of the Mellin step space is exactly the common kernel of the derivative evaluations prescribed by the zeros of the Riemann zeta function in $\operatorname{Re} s > 1/2$, with their multiplicities. No finiteness or sparsity assumption on this zero set is required. The proof combines a uniform fractional regularity estimate for Mellin transforms of step functions with a comparison obtained by reflecting zeros in a shifted vertical line. These estimates produce a holomorphic Hardy-space family whose membership in the Báez–Duarte subspace continues from large positive shifts to every sufficiently small positive shift. A uniformly bounded weak limit then gives the synthesis theorem.

hexagon:2611.00257v1

1. The step space and the synthesis theorem

For $a \geq 0$, set

$$\Omega_a = \{s \in \mathbb{C} : \operatorname{Re} s > \tfrac{1}{2} + a\}, \qquad \Omega = \Omega_0.$$

We use the normalization

$$\|h\|^2_{H^2(\Omega_a)} = \sup_{\sigma > 1/2+a} \frac{1}{2\pi} \int_{\mathbb{R}} |h(\sigma + it)|^2 \, dt. \tag{1.1}$$

The Mellin transform

$$(\mathcal{M}h)(s) = \int_0^1 h(u) u^{s-1} \, du$$

is an isometry from $L^2(0,1)$ onto $H^2(\Omega)$. Let S be the Mellin image of the closed subspace of functions constant almost everywhere on each interval

$$C_m = \left(\frac{1}{m+1}, \frac{1}{m} \right], \qquad m \in \mathbb{N}.$$

For $f \in S$, write f_m for its value on C_m before taking the Mellin transform. Thus

$$\|f\|_S^2 = \sum_{m \geq 1} \frac{|f_m|^2}{m(m+1)}, \qquad e_m(s) = \frac{m^{-s} - (m+1)^{-s}}{s} \tag{1.2}$$

are, respectively, the norm and an orthogonal basis of S. This is the Mellin formulation of the Báez–Duarte system [1, 2].

For each integer $l \geq 2$, define

$$G_l(s) = (l^{-s} - l^{-1}) \frac{\zeta(s)}{s}, \qquad V = \overline{\operatorname{span}\{G_l : l \geq 2\}}^S. \tag{1.3}$$

The singularity at $s = 1$ is removable. Indeed, G_l is the Mellin transform of the negative of the step function

$$q_l(u) = \left\{ \frac{1}{lu} \right\} - \frac{1}{l} \left\{ \frac{1}{u} \right\}. \tag{1.4}$$

Here $\{\cdot\}$ denotes fractional part. On C_m, $q_l = m/l - \lfloor m/l \rfloor$, so $G_l \in S$.

Date: November 5, 2026.

2020 *Mathematics Subject Classification.* Primary 11M26; Secondary 30H10.

Key words and phrases. Riemann zeta function, Báez–Duarte criterion, spectral synthesis, Hardy space, Blaschke product.

Let Z^+ be the set of zeros of ζ in Ω, and let $m(\rho)$ be the multiplicity of ρ. Define

$$J = \{f \in S : f^{(j)}(\rho) = 0 \text{ for all } \rho \in Z^+ \text{ and } 0 \le j < m(\rho)\}. \tag{1.5}$$

Interior derivative evaluations are continuous on $H^2(\Omega)$, so J is closed. The formula for G_l gives $V \subseteq J$.

Theorem 1.1 (Full spectral synthesis)**.** *The spaces in* (1.3) *and* (1.5) *satisfy*

$$V = J.$$

The conclusion holds whether Z^+ is empty, finite, or infinite, and requires no additional condition on the distribution or multiplicities of its zeros.

The theorem identifies the closed subspace determined by the off-line zero jets. It does not assert that Z^+ is empty. The finite-zero case and a conditional infinite-zero case are treated in [5]; the proof below does not use either result.

Throughout, implied constants may depend on explicitly fixed compact parameter ranges. We use the standard inner–outer factorization and Smirnov boundary theorem for half-plane Hardy spaces [4]. In particular, a function in the Smirnov class with square-integrable boundary values belongs to H^2, and division by a Blaschke product whose zeros are canceled preserves the H^2 norm.

2. Uniform fractional regularity of Mellin steps

Lemma 2.1. *Let $A > 0$ and $0 < \alpha_0 < 1/2$. There is a constant $C = C(A, \alpha_0)$ such that, for every $f \in S$,*

$$\frac{1}{2\pi}\int_{\mathbb{R}} (1+|t|)^{2\alpha} |f(\tfrac{1}{2} + a + it)|^2 \, dt \le C\|f\|_S^2 \tag{2.1}$$

whenever $0 \le a \le A$ and $0 \le \alpha \le \min(a, \alpha_0)$. At $a = 0$ the boundary values are understood in the Hardy-space sense. Moreover,

$$(s+1)^\alpha f(s) \in H^2(\Omega_a), \qquad \|(s+1)^\alpha f(s)\|_{H^2(\Omega_a)} \le C'\|f\|_S. \tag{2.2}$$

The power uses the holomorphic logarithm on $\operatorname{Re}(s+1) > 0$.

Proof. Put $u = e^{-y}$ and extend all functions of y by zero to $y < 0$. The function $t \mapsto f(\frac{1}{2}+a+it)$ is the Fourier transform of

$$h_a(y) = \sum_{m \ge 1} h_{a,m}(y), \qquad h_{a,m}(y) = f_m e^{-(1/2+a)y} \mathbf{1}_{I_m}(y),$$

where

$$I_m = [\log m, \log(m+1)), \qquad \ell_m = |I_m| \asymp m^{-1}.$$

If $M_m = \|h_{a,m}\|_2^2$, then

$$M_m \le m^{-2a} \frac{|f_m|^2}{m(m+1)}. \tag{2.3}$$

For $0 < \alpha < 1/2$, the Fourier characterization of the fractional seminorm is

$$\frac{1}{2\pi}\int_{\mathbb{R}} |t|^{2\alpha} |\widehat{h}(t)|^2 \, dt = d_\alpha \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|h(y) - h(v)|^2}{|y-v|^{1+2\alpha}} \, dy \, dv, \tag{2.4}$$

where

$$d_\alpha^{-1} = \int_{\mathbb{R}} \frac{|e^{ir} - 1|^2}{|r|^{1+2\alpha}} \, dr, \qquad d_\alpha \asymp \alpha \quad (0 < \alpha \le \alpha_0).$$

The identity follows from Plancherel and Tonelli; see also [3]. The comparison near $\alpha = 0$ follows by splitting the last integral at $|r| = 1$: the tail is of order $1/\alpha$ and the integral near zero stays bounded on the indicated range.

Consider one cell $I = [L, R)$ of length $\ell = \ell_m$. The maximum and minimum of $e^{-(1/2+a)y}$ on I have uniformly bounded ratio. The part of the double integral in (2.4) with one variable outside I is therefore bounded by

$$\frac{CM_m\ell^{-2\alpha}}{\alpha(1-2\alpha)}. \tag{2.5}$$

Indeed, integration in the outside variable produces the factors $(y-L)^{-2\alpha}/(2\alpha)$ and $(R-y)^{-2\alpha}/(2\alpha)$, while $\|h_{a,m}\|_\infty^2 \le CM_m/\ell$. For $y, v \in I$, the mean value theorem gives

$$|h_{a,m}(y) - h_{a,m}(v)|^2 \le C(M_m/\ell)|y-v|^2.$$

Consequently the contribution from $I \times I$ is at most $CM_m\ell^{2-2\alpha} \le CM_m\ell^{-2\alpha}$. Multiplying these estimates by $d_\alpha \ll \alpha$ yields

$$\|h_{a,m}\|_2^2 + \frac{1}{2\pi}\int_{\mathbb{R}} |t|^{2\alpha}|\widehat{h}_{a,m}(t)|^2\,dt \le C\ell_m^{-2\alpha}M_m, \tag{2.6}$$

uniformly for $0 < \alpha \le \alpha_0$.

At any pair (y, v), at most two of the differences $h_{a,m}(y) - h_{a,m}(v)$ are nonzero. Hence, for finite sums,

$$|h_a(y) - h_a(v)|^2 \le 2\sum_m |h_{a,m}(y) - h_{a,m}(v)|^2.$$

Using (2.3) and (2.6), the total cost is bounded by

$$C\sum_{m\ge1} m^{2\alpha-2a}\frac{|f_m|^2}{m(m+1)} \le C\|f\|_S^2.$$

Finite truncations converge in the corresponding fractional norm, so the same estimate holds for every $f \in S$. Since $(1+|t|)^{2\alpha} \le C(1+|t|^{2\alpha})$ uniformly in α, this proves (2.1) for $\alpha > 0$. For $\alpha = 0$ it is Plancherel's theorem.

Finally, $f|_{\Omega_a} \in H^2(\Omega_a)$ and $(s+1)^\alpha$ is the reciprocal of a bounded outer function there. Thus their product is in the Smirnov class. The weights $|\frac{3}{2} + a + it|^\alpha$ and $(1+|t|)^\alpha$ are uniformly comparable in the prescribed range. The Smirnov boundary theorem and (2.1) now imply (2.2). □

Remark 2.2. Suppose that B is a Blaschke product on Ω_a and that $f \in S$ vanishes at all its zeros with at least their multiplicities. Under the hypotheses of Lemma 2.1, inner division gives

$$\left\|\frac{(s+1)^\alpha f(s)}{B(s)}\right\|_{H^2(\Omega_a)} = \|(s+1)^\alpha f(s)\|_{H^2(\Omega_a)} \le C'\|f\|_S. \tag{2.7}$$

In particular, the constant does not depend on the number or separation of the zeros of B.

3. Reflection of the completed zeta function

Write

$$\xi(s) = A(s)\zeta(s), \qquad A(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2), \qquad X(z) = \xi(\tfrac{1}{2}+z). \tag{3.1}$$

The function X is even, entire of order one, and satisfies $X(\overline{z}) = \overline{X(z)}$. Its zeros lie in $|\operatorname{Re} z| < 1/2$ and none are real. We use the usual canonical product and zero-counting formula for ξ [6].

Let $\nu(\beta,\gamma)$ be the multiplicity of X at $\beta + i\gamma$. For $0 \le a < 1/2$, define

$$B_a(z) = \prod_{\substack{\beta+i\gamma\in Z(X)\\ \beta>a,\ \gamma>0}} \left(\frac{(z-\beta)^2+\gamma^2}{(z+\beta-2a)^2+\gamma^2}\right)^{\nu(\beta,\gamma)}. \tag{3.2}$$

Each zero with $\beta > a$ and positive ordinate is indexed; the displayed factor includes its conjugate. Each factor equals one at $z = a$. The product is a Blaschke product on $\operatorname{Re} z > a$ and

extends meromorphically to the plane. In fact, the paired factors differ from one by $O_K(\gamma^{-2})$ on any fixed compact set away from the poles, and the multiplicity-weighted inverse-square sum over these zeros converges. Set

$$D_a(z) = X(z)/B_a(z). \tag{3.3}$$

The quotient extends to an entire function. It replaces the zeros $\beta \pm i\gamma$, with $\beta > a$, by $2a - \beta \pm i\gamma$; every other zero stays in place. In particular, D_a has no zeros in $\operatorname{Re} z > a$.

For clarity, the canonical factor of X associated with an off-line quadruple $\{\pm\beta \pm i\gamma\}$, $\beta > 0$, is

$$\frac{\big((z-\beta)^2+\gamma^2\big)\big((z+\beta)^2+\gamma^2\big)}{(\beta^2+\gamma^2)^2}. \tag{3.4}$$

If $\beta > a$, its counterpart in D_a replaces the first numerator factor by $(z+\beta-2a)^2+\gamma^2$ and leaves the positive denominator unchanged. A line-zero pair contributes $(z^2+\gamma^2)/\gamma^2$. These products converge locally uniformly and have the same prefactor $X(0)$, so no additional exponential factor is introduced.

Lemma 3.1 (Real-shift comparison)**.** *If $x > 0$, $0 \le a < 1/2$, and $a \le x/\sqrt{2}$, then*

$$|X(it)| \le |D_a(x+it)| \qquad (t \in \mathbb{R}). \tag{3.5}$$

Proof. We compare the canonical factors at each ordinate; the normalizing denominators are identical. For a zero on the line, the required inequality is $|u| \le |x+iu|$, with $u = t-\gamma$, and the same inequality holds at $-\gamma$.

For an off-line pair at ordinate γ, the magnitude contributed at it is β^2+u^2. If $0 < \beta \le a$, the pair is unchanged, and

$$\begin{aligned}\big((x+\beta)^2+u^2\big)\big((x-\beta)^2+u^2\big) - (\beta^2+u^2)^2 \\ = x^2(2u^2+x^2-2\beta^2) \ge 0.\end{aligned}$$

If $\beta > a$, put $r_1 = x+\beta$ and $r_2 = x+\beta-2a$. Both are positive, and

$$r_1 r_2 - \beta^2 = 2(x-a)\beta + x(x-2a) \ge x^2 - 2a^2 \ge 0.$$

It follows that $r_1^2 + r_2^2 \ge 2r_1r_2 \ge 2\beta^2$, whence

$$(r_1^2+u^2)(r_2^2+u^2) \ge (\beta^2+u^2)^2.$$

The same comparison applies at $-\gamma$. Multiplying the comparisons over finite products and passing to their locally uniform limits proves (3.5). □

Lemma 3.2 (Vertical variation)**.** *Fix $0 \le a < 1/2$ and a compact interval $I \subset (a,\infty)$. There is $C = C(a,I)$ such that*

$$\left|\frac{\partial}{\partial t}\log|D_a(x+it)|\right| \le C\log(2+|t|), \qquad x \in I,\ t \in \mathbb{R}. \tag{3.6}$$

Consequently, if also $a \le x/\sqrt{2}$ for $x \in I$, then, uniformly for $x \in I$ and $|v| \le 1$,

$$\frac{|X(it)|}{|D_a(x+i(t+v))|} \le C_1(1+|t|)^{C_2|v|}. \tag{3.7}$$

Proof. Let $b \pm i\gamma$ run over the conjugate pairs of zeros of D_a, including multiplicity. All b lie in $[-1/2, a]$. Differentiation of the paired canonical product gives terms

$$K_r(t-\gamma) + K_r(t+\gamma), \qquad K_r(u) = \frac{u}{r^2+u^2}, \qquad r = x-b.$$

For $x \in I$, the distances r belong to a fixed compact subinterval of $(0,\infty)$. Thus

$$|K_r(u) - K_1(u)| \le C(1+|u|)^{-3}.$$

The unit-interval zero bound $N(T+1)-N(T)=O(\log(2+T))$ shows that replacing all the kernels by K_1 incurs an error $O(\log(2+|t|))$.

Here $N(T)$ counts the positive ordinates of all nontrivial zeta zeros, with multiplicity; reflection preserves this counting function. Write

$$N(T)=M(T)+E(T),\qquad M(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi},\qquad E(T)=O(\log(2+T)). \tag{3.8}$$

By conjugation it suffices to consider $t\geq 10$. For $\gamma>2t$,

$$K_1(t-\gamma)+K_1(t+\gamma)=O(t/\gamma^2),$$

and summing by (3.8) gives $O(\log t)$. The ranges $\gamma<t/2$ and $3t/2<\gamma\leq 2t$ also contribute $O(\log t)$, since each kernel is $O(1/t)$ there. On the remaining range $[t/2,3t/2]$, the terms $K_1(t+\gamma)$ satisfy the same bound.

The only remaining sum is

$$\int_{t/2}^{3t/2}K_1(t-v)\,dN(v).$$

The contribution of dE is $O(\log t)$ by integration by parts, because K_1' is integrable on $\mathbb{R}$ and the endpoint terms are bounded by $O(\log t/t)$. For the main term, the constant $M'(t)$ integrates to zero by oddness on this symmetric interval, and

$$|M'(v)-M'(t)|\leq C|v-t|/t.$$

Its remaining integral is therefore $O(1)$. The estimates on a fixed bounded range of t follow from local convergence and the positive lower bound on r. This proves (3.6).

Integrating (3.6) between t and $t+v$ and applying Lemma 3.1 proves (3.7). □

We record the form of these comparisons that will be used below. For $s=1/2+it$ and $w=x+iv$, (3.1) gives

$$\frac{\zeta(s)}{\zeta(s+w)/B_a(s+w-1/2)}=\frac{X(it)}{D_a(x+i(t+v))}\frac{A(s+w)}{A(s)}. \tag{3.9}$$

Stirling's formula, uniformly for w in a fixed compact subset of $\operatorname{Re} w>0$, gives

$$\left|\frac{A(s+w)}{A(s)}\right|\leq C(1+|t|)^{x/2}. \tag{3.10}$$

The denominator has no zero on $\operatorname{Re} s=1/2$; bounded t is handled by continuity. Thus the hypotheses of Lemma 3.2 give

$$\left|\frac{\zeta(s)}{\zeta(s+w)/B_a(s+w-1/2)}\right|\leq C(1+|t|)^{x/2+C_2|v|}. \tag{3.11}$$

For real shifts, there is a useful uniform version: for each fixed $x_*>0$,

$$\left|\frac{\zeta(s)}{\zeta(s+x)/B_a(s+x-1/2)}\right|\leq C_{x_*}(1+|t|)^{x/2} \tag{3.12}$$

when $0<x\leq x_*$, $0\leq a<1/2$, and $a\leq x/\sqrt{2}$. Indeed, the comparison in Lemma 3.1 has constant one, while Stirling's formula is uniform for $0\leq x\leq x_*$. All ratios are understood by removing their removable singularities.

4. A holomorphic family in the Hardy space

We first justify the boundary-to-Hardy step needed for the family.

Lemma 4.1. *The bounded holomorphic function*

$$Z(s) = \frac{(s-1)\zeta(s)}{(s+1)^2}$$

on Ω has a factorization

$$Z(s) = B_0(s-1/2)O(s), \tag{4.1}$$

where O is bounded and outer, with a unimodular constant absorbed into O. Every interior translate $O(s+w)$, $\operatorname{Re} w > 0$, is also outer on Ω.

Proof. The classical convexity estimate in a bounded vertical strip and absolute convergence farther to the right imply boundedness of Z. The pole at 1 is canceled, and its zeros in Ω are precisely those of ζ, with their multiplicities.

The function Z extends analytically across every finite point of the boundary line. Away from its isolated boundary zeros this excludes singular inner measure; at a zero of finite order its logarithmic boundary behavior excludes a singular atom as well. Thus its inner factor has no singular measure on the finite boundary. A nonconstant singular inner factor at infinity would contain $e^{-c(s-1/2)}$ for some $c > 0$, which is impossible since $Z(\sigma) \sim 1/\sigma$ as $\sigma \to +\infty$. The remaining inner factor is exactly $B_0(s-1/2)$, proving (4.1).

After scaling O to have modulus at most one, its log modulus is the Poisson extension of its boundary log modulus, with no linear term at infinity. The Poisson semigroup identity shows the same assertion after a positive horizontal translate. A vertical translate merely translates the boundary data. This proves the last assertion. □

Fix $f \in J$. Define

$$g(s) = \frac{sf(s)}{\zeta(s)}. \tag{4.2}$$

The prescribed zero jets remove every possible pole in Ω, and $g(1) = 0$. For $\operatorname{Re} w > 0$ put

$$F_w(s) = \frac{\zeta(s)}{s}\big(g(s+w) - g(1+w)\big). \tag{4.3}$$

This is holomorphic in $s \in \Omega$: the numerator in parentheses vanishes at $s = 1$, canceling the only pole of $\zeta(s)/s$.

Proposition 4.2. *There is an open connected set $U \subset \{\operatorname{Re} w > 0\}$ containing $(0, 3/4)$ such that $w \mapsto F_w$ is an $H^2(\Omega)$-valued holomorphic map on U. Its Hardy norm is locally bounded on U.*

Proof. We first place F_w in the Smirnov class. Since $f \in J$, inner division gives $f(s) = B_0(s-1/2)f_1(s)$ with $f_1 \in H^2(\Omega)$. Lemma 4.1 yields

$$g(s) = \frac{s(s-1)}{(s+1)^2}\frac{f_1(s)}{O(s)}.$$

For each $\operatorname{Re} w > 0$, the translate $f_1(s+w)$ belongs to $H^2(\Omega)$ and $O(s+w)$ is bounded and outer. Thus $g(s+w)$ and $h_w(s) = g(s+w) - g(1+w)$ are in the Smirnov class on Ω. The function

$$b_1(s) = \frac{s-1}{s}$$

is the Blaschke factor of Ω at 1. Since $h_w(1) = 0$, h_w/b_1 remains in the Smirnov class, and

$$F_w(s) = Z(s)\frac{(s+1)^2}{s^2}\frac{h_w(s)}{b_1(s)} \tag{4.4}$$

does also.

Fix $x_0 \in (0, 3/4)$. Choose

$$\frac{x_0}{2} < \alpha < a < \min\left(\frac{x_0}{\sqrt{2}}, \frac{1}{2}\right). \tag{4.5}$$

Choose a small compact interval I about x_0 with $a < x/\sqrt{2}$ for $x \in I$. By making a complex neighborhood U_{x_0} smaller if necessary, Lemma 3.2 gives, for $w = x + iv \in U_{x_0}$,

$$x \in I, \qquad x/2 + C_2|v| < \alpha. \tag{4.6}$$

On $\operatorname{Re} s = 1/2$, factor the first term of (4.3) as

$$\frac{\zeta(s)}{s} g(s+w) = \frac{s+w}{s} \frac{\zeta(s)}{\zeta(s+w)/B_a(s+w-1/2)} \frac{f(s+w)}{B_a(s+w-1/2)}. \tag{4.7}$$

The rational factor is uniformly bounded. By (3.11) and (4.6), the middle factor is bounded by $C(1+|t|)^{\alpha}$. Choose a fixed $\alpha_0 \in (\alpha, 1/2)$. The zeros of $B_a(s-1/2)$ are among the zeros canceled by f, so Lemma 2.1 and (2.7) give

$$H_{a,\alpha}(z) = \frac{(z+1)^{\alpha} f(z)}{B_a(z-1/2)} \in H^2(\Omega_a), \qquad \|H_{a,\alpha}\|_{H^2(\Omega_a)} \le C\|f\|_S.$$

For $z = s + w$ we have $\operatorname{Re} z = 1/2 + x > 1/2 + a$ and $|z+1|^{\alpha} \asymp (1+|t|)^{\alpha}$, uniformly in U_{x_0}. Evaluating the Hardy norm on this interior vertical line therefore shows that the boundary function in (4.7) is locally bounded in $L^2(\mathbb{R})$.

The remaining boundary term is $g(1+w)\zeta(s)/s$. Its L^2 norm is locally bounded because $g(1+w)$ is holomorphic and

$$\int_{\mathbb{R}} \left| \frac{\zeta(1/2+it)}{1/2+it} \right|^2 dt < \infty. \tag{4.8}$$

For example, the convexity estimate $\zeta(1/2+it) \ll (1+|t|)^{1/3}$ suffices for this integrability. The two boundary terms are used here only as L^2 functions; their poles at $s = 1$ cancel in the complete expression (4.3). Combining the boundary bound with (4.4), the Smirnov boundary theorem gives $F_w \in H^2(\Omega)$ with locally bounded norm.

For each fixed $s \in \Omega$, including the removable value at 1, $F_w(s)$ is holomorphic in w. Testing against the reproducing kernels of $H^2(\Omega)$ gives weak holomorphy on their linear span. This span is dense; local norm boundedness extends holomorphy to every bounded linear functional. The standard weak-to-strong holomorphy principle now gives Hardy-space-valued holomorphy on U_{x_0}. The union of these neighborhoods over $x_0 \in (0, 3/4)$ is open and connected and proves the proposition. □

5. Membership, continuation, and the zero-shift limit

Lemma 5.1 (Membership at large shifts)**.** *For every $f \in S$ and real $x \in (1/2, 3/4)$, the expression (4.3), formed using $g(z) = zf(z)/\zeta(z)$ at the translated arguments, belongs to V.*

Proof. First suppose f has finite step support. Then

$$P(s) = sf(s)$$

is a Dirichlet polynomial. On $\operatorname{Re} s > 1$,

$$\frac{P(s)}{\zeta(s)} = \sum_{l \ge 1} c_l l^{-s}, \qquad \sup_l |c_l| < \infty,$$

because $1/\zeta(s) = \sum_{n \ge 1} \mu(n) n^{-s}$ converges absolutely there and P has finitely many coefficients. For $x > 1/2$ this gives

$$F_x = \sum_{l \ge 2} c_l l^{-x} G_l. \tag{5.1}$$

To justify norm convergence, use (1.4). On $(0, 1/l]$ one has $|q_l| \leq 1 + 1/l$, while on $[1/l, 1]$,

$$q_l(u) = \frac{1}{l} \left\lfloor \frac{1}{u} \right\rfloor.$$

Hence

$$\|G_l\|^2_{H^2(\Omega)} = \|q_l\|^2_{L^2(0,1)} \leq \frac{C}{l} + \frac{1}{l^2} \int_{1/l}^{1} u^{-2}\, du \leq \frac{C'}{l}.$$

The series in (5.1) is therefore absolutely convergent in $H^2(\Omega)$, since $\sum_l l^{-x-1/2} < \infty$. It agrees pointwise with (4.3) in an absolutely convergent half-plane and hence throughout Ω. Thus $F_x \in V$.

Now let $f \in S$ be arbitrary and let $f^{(N)}$ be its finite step truncations. For fixed $x > 1/2$, absolute convergence gives a uniform bound for $1/\zeta(s+x)$ on $\operatorname{Re} s = 1/2$. The convexity bound used in (4.8) and Lemma 2.1, with $a = x$ and $\alpha = 1/3$, imply

$$\left\| \frac{s+x}{s} \frac{\zeta(s)}{\zeta(s+x)} \big(f^{(N)}(s+x) - f(s+x)\big) \right\|_{L^2(\operatorname{Re} s = 1/2)} \leq C_x \left\| f^{(N)} - f \right\|_S.$$

Evaluation at $1+x$ is also continuous in the step norm. It follows that the boundary expressions for $F_x^{(N)}$ converge in L^2. Since $F_x^{(N)} \in V \subset H^2(\Omega)$, they converge in Hardy norm to an element of V. Interior point evaluation identifies this limit with F_x as defined in (4.3). No zero-jet condition on the truncations has been used. □

Proof of Theorem 1.1. It remains to prove $J \subseteq V$. Fix $f \in J$ and form F_w as in Proposition 4.2. Let $L \in (H^2(\Omega))^*$ be any continuous complex-linear functional with $L|_V = 0$. The scalar function $w \mapsto L(F_w)$ is holomorphic on the connected set U of Proposition 4.2. Lemma 5.1 shows that it vanishes for every $x \in (1/2, 3/4)$. By the identity theorem it vanishes throughout U. The space S is closed in $H^2(\Omega)$, and V is closed in S, hence V is also closed in $H^2(\Omega)$. The Hahn–Banach separation theorem therefore gives

$$F_x \in V \qquad (0 < x < 3/4). \tag{5.2}$$

We next show that

$$\sup_{0<x<1/4} \|F_x\|_{H^2(\Omega)} < \infty. \tag{5.3}$$

Set

$$a = \frac{2x}{3}, \qquad \alpha = \frac{x}{2}.$$

Then $\alpha < a < x/\sqrt{2}$. On the boundary $s = 1/2 + it$, (4.7) and the uniform real-shift estimate (3.12) imply

$$\left| \frac{\zeta(s)}{s} g(s+x) \right| \leq C(1+|t|)^{x/2} \left| \frac{f(s+x)}{B_{2x/3}(s+x-1/2)} \right|.$$

Lemma 2.1 and inner division, with fixed upper bounds for a and α, make the right-hand side uniformly bounded in $L^2(\mathbb{R})$. Explicitly,

$$\left\| \frac{(z+1)^{x/2} f(z)}{B_{2x/3}(z-1/2)} \right\|_{H^2(\Omega_{2x/3})} \leq C\|f\|_S,$$

and the vertical line $z = s + x$ lies inside $\Omega_{2x/3}$. The comparison of $|z+1|^{x/2}$ with $(1+|t|)^{x/2}$ is uniform for $0 < x < 1/4$.

For the subtraction term, the simple pole of ζ at 1 and boundedness of evaluation near 1 give

$$|g(1+x)| = \left| \frac{(1+x) f(1+x)}{\zeta(1+x)} \right| \leq Cx\|f\|_S.$$

Its boundary norm therefore tends to zero by (4.8). Since each F_x already belongs to $H^2(\Omega)$, the boundary estimates prove (5.3).

Finally, g is holomorphic on Ω and $g(1) = 0$. Thus, at every $s \in \Omega \setminus \{1\}$,

$$\lim_{x \downarrow 0} F_x(s) = \frac{\zeta(s)}{s}\big(g(s) - g(1)\big) = f(s),$$

including the canceled zeta zeros by holomorphic continuation. At $s = 1$ the removable value is $F_x(1) = g'(1 + x)$, which tends to $g'(1) = f(1)$. Hence $F_x(s) \to f(s)$ at every point of Ω. The uniform Hardy bound and density of the linear span of the reproducing kernels imply

$$F_x \rightharpoonup f \quad \text{in } H^2(\Omega) \qquad (x \downarrow 0).$$

By (5.2) and weak closedness of the closed linear subspace V, we obtain $f \in V$. Together with $V \subseteq J$, this proves the theorem. □

Acknowledgment of AI assistance

The authors acknowledge assistance from OpenAI's AI tools in refining the argument, checking intermediate steps, and preparing the mathematical exposition.

References

[1] L. Báez-Duarte, *A strengthening of the Nyman–Beurling criterion for the Riemann hypothesis*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **14** (2003), no. 1, 5–11. arXiv:math/0202141.

[2] B. Bagchi, *On Nyman, Beurling and Baez-Duarte's Hilbert space reformulation of the Riemann hypothesis*, Proc. Indian Acad. Sci. Math. Sci. **116** (2006), no. 2, 137–146. doi:10.1007/BF02829783.

[3] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573. doi:10.1016/j.bulsci.2011.12.004.

[4] P. L. Duren, *Theory of H^p Spaces*, Pure and Applied Mathematics, vol. 38, Academic Press, New York–London, 1970.

[5] `span_of_dilates`, *The Báez-Duarte subspace has codimension equal to the number of zeta zeros off the critical line, with distance and rate bounds not assuming RH*, rStation, Paper 1, version 12, October 27, 2026. `https://rstation.ai/problems/riemann_hypothesis/papers/1`.

[6] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Clarendon Press, Oxford, 1986.

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