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PERIOD RELATIONS AS PERIOD TERMS: LINEAR INDEPENDENCE AT CM AND BETA PERIODS

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Abstract. We prove linear independence results for periods of CM abelian varieties and Beta values by using a known relation between periods as the period term in the algebraic subgroup theorem. For example, 1, $B(\frac{2}{7},\frac{2}{7})$, $B(\frac{3}{7},\frac{3}{7})$ and $B(\frac{3}{7},\frac{3}{7})/B(\frac{1}{7},\frac{1}{7})$ are linearly independent over $\overline{\mathbb{Q}}$, and families of such relations attached to the CM types of $\mathbb{Q}(\zeta_N)$ give the linear independence of 1, a common ratio and $\kappa \geq 2$ periods. The same method proves Waldschmidt's Conjecture 42 at six classes of CM ratios (for instance, $\wp(\log 2/\varpi)$ and $\wp(\log 3/\varpi)$ are not both algebraic on the lemniscatic curve) and several four exponentials statements at CM periods. All results are explicit consequences of theorems of Waldschmidt and of Roy and Waldschmidt.

1. Introduction

Waldschmidt's algebraic subgroup theorem with period term [28, Thm. 4.1] bounds the rank of a group of logarithms of algebraic points lying on a subspace V, with a saving κ for the periods that lie on V; the theorem of Roy and Waldschmidt [23] puts it in a form with number-field coefficients. A period on V comes from a relation among periods. In the companion note [16] the period data come from the choice of the slope (for instance the period $(2\pi i, 2ic\Omega)$ on the line of slope $c\Omega/\pi$) and from Legendre's relation and its analogue $P_\sigma P_{\bar\sigma} \in \pi\overline{\mathbb{Q}}^\times$ for CM abelian varieties. Here we use quadratic relations among first-kind periods of four different eigen-slots, which exist on CM abelian varieties of dimension ≥ 3 by Deligne's theorem in the form of Koblitz and Ogus, and, in the other results, the quotients of periods of two different eigen-slots. Each relation, placed on the line, is the period term of the count.

Notation. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, $\mathcal{L} = \{z \in \mathbb{C} : e^z \in \overline{\mathbb{Q}}^\times\}$ and $\widetilde{\mathcal{L}} = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}$. For an elliptic curve $E\colon y^2 = 4x^3 - g_2x - g_3$ with $g_2, g_3 \in \overline{\mathbb{Q}}$ we write $\wp, \zeta$ for its Weierstrass functions, Λ_E for its period lattice, $\eta(\lambda) = \zeta(z+\lambda) - \zeta(z)$ for the quasi-periods, and $\mathcal{L}_E = \{u \in \mathbb{C} : u \in \Lambda_E \text{ or } \wp(u) \in \overline{\mathbb{Q}}\}$. We use two curves throughout:

$$E_i\colon y^2 = 4x^3 - 4x, \qquad \Lambda = \varpi\,\mathbb{Z}[i], \qquad \varpi = \tfrac{\Gamma(1/4)^2}{2\sqrt{2\pi}}, \qquad \eta(\varpi) = \tfrac{\pi}{\varpi};$$
$$E_\rho\colon y^2 = 4x^3 - 4, \qquad \Lambda = \Omega_\rho\mathbb{Z}[\rho], \qquad \Omega_\rho = \tfrac{\Gamma(1/3)^3}{2^{4/3}\pi}, \qquad \eta(\Omega_\rho) = \tfrac{2\pi}{\sqrt{3}\,\Omega_\rho}.$$

(The quasi-periods follow from the Legendre relation and $\zeta(iz) = -i\zeta(z)$, $\zeta(\rho z) = \bar\rho\zeta(z)$. The real periods are Beta integrals, $\varpi = \int_1^\infty (x^3-x)^{-1/2}dx = \frac{1}{2}B(\frac{1}{4},\frac{1}{2})$ and $\Omega_\rho = \int_1^\infty (x^3-1)^{-1/2}dx =$

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$\frac{1}{3}B(\frac{1}{6},\frac{1}{2})$, and the Γ-forms follow from the reflection and duplication formulas; these are instances of the Chowla–Selberg formula [10].)

Let $A/\overline{\mathbb{Q}}$ be a simple abelian variety with complex multiplication by a CM field F of degree $2g$, with CM type Φ. For an embedding σ of F, an *eigen-differential* is a class $\omega_\sigma \in H^1_{\mathrm{dR}}(A/\overline{\mathbb{Q}})$ with $[x]^*\omega_\sigma = \sigma(x)\omega_\sigma$; it is of the first kind (holomorphic) iff $\sigma \in \Phi$. An *eigen-period* is a nonzero number $P_\sigma = \int_\gamma \omega_\sigma$, $\gamma \in H_1(A,\mathbb{Z})$. By Riemann's bilinear relations $P_\sigma P_{\bar\sigma} \in \pi\overline{\mathbb{Q}}^\times$. For a CM abelian variety that is not simple, every nonzero eigen-period is an algebraic multiple of an eigen-period of a simple factor, of the same kind (§3.1). For $N \geq 3$ and $0 < a, b < N$ with $a + b \neq N$, the Beta value $B(a/N, b/N) = \Gamma(a/N)\Gamma(b/N)/\Gamma((a+b)/N)$ is, up to $\overline{\mathbb{Q}}^\times$, an eigen-period of a CM abelian variety (a factor of the Jacobian of the Fermat curve of degree N), of the first kind iff $a + b < N$ (§3.1).

The level-seven example. Products of CM abelian varieties involving a simple factor of dimension ≥ 3 can have homogeneous quadratic relations among first-kind periods of four different eigen-slots (Remark 4.10). On the Jacobian A of $C\colon y^7 = x(1-x)$ (genus 3, CM by $\mathbb{Q}(\zeta_7)$), with E an elliptic curve with CM by $\mathbb{Q}(\sqrt{-7})$, the Beta cocycle and one reflection give

$$B(\tfrac{1}{7},\tfrac{1}{7})\,B(\tfrac{2}{7},\tfrac{2}{7}) = \frac{\sin(3\pi/7)}{\sin(\pi/7)}\,B(\tfrac{3}{7},\tfrac{3}{7})\,B(\tfrac{1}{7},\tfrac{2}{7}),$$

where $B(j/7, j/7)$ $(j = 1, 2, 3)$ are the periods of the holomorphic eigen-differentials dx/y^{7-j} on C and $B(1/7, 2/7) \in \overline{\mathbb{Q}}^\times\Omega_E$ (Chowla–Selberg [10]). Theorem 4.2 (§4) uses such a relation as the period term. On $y^7 = x(1-x)$ it gives:

Corollary 1.1 (Corollary X7). (a) *For $O, Q \in C(\overline{\mathbb{Q}})$ with $Q - O$ of infinite order in A,*

$$\int_O^Q \frac{dx}{y^5} \Big/ \int_O^Q \frac{dx}{y^6} \;\notin\; \overline{\mathbb{Q}} \cdot \frac{B(3/7,3/7)}{B(1/7,1/7)}.$$

(b) *The numbers 1, $B(\frac{2}{7},\frac{2}{7})$, $B(\frac{3}{7},\frac{3}{7})$, $B(\frac{3}{7},\frac{3}{7})/B(\frac{1}{7},\frac{1}{7})$ are linearly independent over $\overline{\mathbb{Q}}$; equivalently, so are π, $\Gamma(\frac{2}{7})^2\Gamma(\frac{3}{7})$, $\Gamma(\frac{1}{7})\Gamma(\frac{3}{7})^2$, $\Gamma(\frac{2}{7})\Gamma(\frac{3}{7})^2/\Gamma(\frac{1}{7})$.*

Corollary 1.1 is Corollary 4.7(a), (c). The relation is the period term in the following sense. Let z_1, z_2, z_3 be the eigen-coordinates on $\mathrm{Lie}(A)$ dual to $dx/y^6, dx/y^5, dx/y^4$, with periods $P_j \in \overline{\mathbb{Q}}^\times B(\frac{j}{7},\frac{j}{7})$. On $G = A \times E \times \mathbb{G}_{\mathrm{a}}^2$ read the map $\Psi = (z_1 + z_E + q_1,\ z_3 + z_2 + q_2)\colon \mathrm{Lie}(G) \to \mathbb{C}^2$ and the line $V = \mathbb{C}(P_1, P_3)$. The relation puts two elements of the K-span of $\Psi(\Omega(G))$ on V, linearly independent over a number field K, and the theorem of Roy and Waldschmidt then allows no further logarithm of an algebraic point on V (Theorem 4.6); Corollary 1.1 follows by specialising the logarithms. The general statement, for four distinct first-kind eigen-slots with $P_aP_b \in \overline{\mathbb{Q}}^\times P_cP_d$, is Theorem 4.2.

Several parallel pairs. At level N, let $\tau \in (\mathbb{Z}/N)^\times \setminus \{\pm 1\}$ and let $T(\tau)$ be the set of CM types Φ of $\mathbb{Q}(\zeta_N)$ with $1, \tau \in \Phi$; it has $\kappa = 2^{g-2}$ elements, where $2g = \varphi(N)$. Replacing τ by $-\tau$ in each Φ gives a partner type Φ', and by Deligne's theorem the ratios $P_{\Phi'}/P_\Phi$ of the corresponding first-kind periods all lie in one class $\overline{\mathbb{Q}}^\times r$. Theorem 4.14 places all κ pairs on one line and gives: *1, r and the κ periods $P_{\Phi'}$ are linearly independent over $\overline{\mathbb{Q}}$*, and the same for every subfamily. For $\kappa = 1$ this follows from Wüstholz's analytic subgroup theorem (multiply by $\pi/P_{\Phi'}$); the point is $\kappa \geq 2$, i.e. $g \geq 3$. For instance, at $N = 15$, $\tau = 2$ $(\kappa = 4)$, the numbers

$$1, \quad \frac{B(\frac{4}{15},\frac{4}{15})}{B(\frac{1}{15},\frac{2}{15})}, \quad B(\tfrac{4}{15},\tfrac{4}{15}), \quad B(\tfrac{1}{15},\tfrac{7}{15}), \quad B(\tfrac{2}{15},\tfrac{6}{15}), \quad B(\tfrac{1}{3},\tfrac{1}{3})$$

are linearly independent over $\overline{\mathbb{Q}}$. Levels 9 and 24 give analogues of Corollary 1.1 with elementary proofs of the relations (Theorem 4.12): for instance 1, ϖ, $B(\frac{1}{24},\frac{1}{12})$, $B(\frac{1}{24},\frac{1}{12})/B(\frac{1}{24},\frac{1}{24})$ are linearly independent over $\overline{\mathbb{Q}}$.

Products of periods are in general not 1-periods, so these statements are outside the period theorem of Huber and Wüstholz [11], which describes the $\overline{\mathbb{Q}}$-linear relations among 1-periods. They would reduce to it if a single Gamma monomial multiplied every number of the list into $\overline{\mathbb{Q}}^\times$ times a 1-period. In the Koblitz–Ogus dictionary there is no such multiplier for the lists of Corollary 1.1, Corollary 4.13 and the example above (Remark 4.9, [Computed]).

Other results. The same method, with the quotient of the periods of two different eigen-slots as the slope of the line, gives the following (§§5–8).

- *Waldschmidt's Conjecture 42 at CM ratios* (Theorem 7.1). For instance $\wp_i(\log 2/\varpi)$ and $\wp_i(\log 3/\varpi)$ are not both algebraic, nor are $\wp_i(\Omega_\rho \log 2/\pi)$ and $\wp_i(\Omega_\rho \log 3/\pi)$.
- *Strong elliptic four exponentials* at $t \in \overline{\mathbb{Q}}^\times(\Omega'/\Omega)^{\pm1} \cup \overline{\mathbb{Q}}^\times(\pi/\Omega\Omega')^{\pm1}$ for two non-isogenous CM curves (Corollary 5.2), and on eigen-coordinates of CM abelian varieties at $t \in \overline{\mathbb{Q}}^\times(P'/P_\sigma)^{\pm1}$ (Theorem 5.5). For instance $\wp_i(\Omega_\rho/\varpi)$ and $\wp_i(\Omega_\rho u/\varpi)$ are not both algebraic for every $u \in \mathcal{L}_{E_i} \setminus \{0\}$.
- *Four exponentials for quasi-logarithms* (Theorem 6.2). For instance, if $u_1, u_2 \in \mathcal{L}_{E_i}$ and ϖ are linearly independent over $\mathbb{Q}(i)$, then $(\varpi/\Omega_\rho)\zeta_i(u_1)$ and $(\varpi/\Omega_\rho)\zeta_i(u_2)$ are not both $\overline{\mathbb{Q}}$-linear combinations of 1 and quasi-logarithms $\zeta_i(u)$.
- *Collinearity of period points* of three CM curves with distinct CM fields (Theorem 8.1), a codimension-two application.
- *Imaginary quadratic multiplication* of signature (r,s) (Theorem 8.4): on a simple Picard Jacobian, the abelian integrals $\int^P dx/y^2$, $\int^P x\,dx/y^2$ are proportional, with factor $\log\alpha/2\pi i$, to the two periods along one cycle only if P is torsion and α a root of unity.

Position. All of this is an explicit application of published theorems: Waldschmidt's subgroup theorem with period term [28, Thm. 4.1], its hyperplane case [28, §5d], and the theorem of Roy and Waldschmidt [23, Théorème], used with the weight of the quotient group that appears inside its proof (§2). We claim no new method. Several results are direct instances of the theorem of Roy and Waldschmidt as printed: Theorem 5.1(1), Theorem 5.5 for elliptic curves, Theorems 4.6, 4.12(1), the families of Theorem 4.14 (and through them Theorem 4.12(2)), and rows (iii), (iv) of the line form of Theorem 7.1 (Remark 2.4). What we contribute is the choice of the period relation placed on the line, the explicit statements and examples, and the proofs written out at these periods: the group, the map, the line, the period datum, the quotient, and either the exclusion of every algebraic subgroup or the computation of the weight. Theorems 8.1 and 8.4 use Waldschmidt's theorem on subspaces of codimension ≥ 2, subgroup by subgroup.

As far as we found, none of these statements is in print, with the following exceptions and partial overlaps (§9.1). For Theorem 4.14 with $\kappa = 1$ the independence statement is Wüstholz's theorem in substance. Theorem 6.2 in the class $\overline{\mathbb{Q}}^\times\Omega^{\pm1}$ is of degree one by Laurent's identity, and we expect it to follow from the analytic subgroup theorem [Heuristic]. The cases $t \in \overline{\mathbb{Q}}$ of (U1), (U3) follow easily from Wüstholz's theorem. Corollary 7.2(b) follows from known results and complex conjugation when $\log\alpha, \log\beta$ and their complex conjugates span a $\mathbb{Q}$-space of dimension ≥ 3; what is new is the conjugation-stable case, such as $(\log 2, \log 3)$. Likewise the conclusion of Theorem 5.5, and that of Corollary 5.2, follow from the elliptic and abelian strong six exponentials theorems of Roy and Waldschmidt and complex conjugation when $\overline{\mathbb{Q}}y_1 + \overline{\mathbb{Q}}y_2$ is not stable under complex conjugation (for Theorem 5.5 we checked this only at the level of a proof sketch). Our examples lie in the conjugation-stable cases.

Conventions. Unless labelled otherwise, every statement is proved here given the published theorems cited; we use the labels [Lit], [Proof sketch], [Heuristic] (depending on the period conjecture or the Rohrlich–Lang conjecture), [Numerical] and [Computed] (exact combinatorics by a script). Every result relies on Waldschmidt's Theorem 4.1 as printed, including the one step of its proof that [28] indicates by reference to other papers (§2.1). The results were found with substantial use of AI tools, and all review so far is by AI referees; no human expert has checked the proofs. §9 says which statements are checked in Lean.

Organisation. §2 states the published theorems and the shape of the arguments, §3 the linear algebra of CM abelian varieties and of universal vector extensions. §4 proves Theorem 4.2 and its instances at levels 7, 9, 24 and Theorem 4.14; §5 the strong elliptic and eigen-slot statements, §6 four exponentials for quasi-logarithms, §7 Conjecture 42 at CM ratios and §8 the codimension-two applications. §9 is on formal verification and the literature.

2. The published theorems

2.1. **Waldschmidt's subgroup theorem.**

Theorem 2.1 ([28, Thm. 4.1, pp. 385–386])**.** *Let $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times G_2$ be a connected commutative algebraic group over $\overline{\mathbb{Q}}$ of dimension $d = d_0 + d_1 + d_2 \geq 1$, with projections π_0, π_1 onto the first two factors (d_0, d_1 need not be maximal). Let $V \subset T_G(\mathbb{C})$ be a subspace of dimension $N < d$, $W \subset V$ a subspace defined over $\overline{\mathbb{Q}}$, and $Y \subset V$ a finitely generated subgroup with $\Gamma = \exp_G Y \subset G(\overline{\mathbb{Q}})$; put $\kappa = \mathrm{rk}_{\mathbb{Z}}(Y \cap \ker \exp_G)$. Then there is a connected algebraic subgroup $G' \neq G$ defined over $\overline{\mathbb{Q}}$ such that, with $\delta = \dim G/G'$, $\delta_0 = \dim \mathbb{G}_{\mathrm{a}}^{d_0}/\pi_0(G')$, $\delta_1 = \dim \mathbb{G}_{\mathrm{m}}^{d_1}/\pi_1(G')$, $\delta_2 = \delta - \delta_0 - \delta_1$, $\lambda = \mathrm{rk}_{\mathbb{Z}}\,\Gamma/(\Gamma \cap G')$ and $\tau = \dim W/(W \cap T_{G'})$, one has $\delta > \tau$ and*

$$(\lambda + \delta_1 + 2\delta_2)(d - N) \leq (\delta - \tau)(d_1 + 2d_2 - \kappa). \tag{1}$$

($G' = 0$ is allowed.) The proof [28, §§6–7] combines an auxiliary function [28, Prop. 6.1] with Philippon's zero estimate [20]. Prop. 6.1 asks for a polynomial that vanishes with multiplicity $\geq T$ along W at the points of $\exp_G(Y(S))$, with the saving κ in the parameters. Waldschmidt says [28, p. 396] that this is proved in [27, Prop. 2.4] when $W = 0$ and the exponent b is 1, and that the estimates for the derivatives are provided by Bertrand's Lemma 7 [26, App. I], and he says to compare with Bertrand [3] and Philippon–Waldschmidt [21]. So for $W \neq 0$ the printed proof only indicates this step. All our direct applications of Theorem 2.1 have $W \neq 0$ and $\kappa > 0$, and so do the instances of it behind Theorem 2.3 below. We use Theorem 2.1 as printed. Put

$$2N \geq d_1 + 2d_0 + \kappa. \tag{2}$$

This is not a hypothesis of Theorem 2.1; it is the range in which this step has been checked outside print. For $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times E_1 \times \cdots \times E_m$, with E_j elliptic curves, Theorem 2.1 under (2) is proved in Lean [14, 15]. For other groups, in particular for CM abelian varieties of dimension ≥ 2 and for universal vector extensions, we rely on the printed theorem; no human expert has checked this step. Every direct application of Theorem 2.1 below satisfies (2), and each such proof says so. Theorem 2.1 is stated for any d_0 (p. 385); Waldschmidt himself applies its consequence [28, Thm. 1.1] with $d_0 = 2$, $W \neq 0$, $\kappa = 0$ in the proof of [28, Cor. 2.1]. When $d_0 \geq 1$ the printed proof uses an exponent $b > 1$ in the case $\delta = \tau$ (p. 397), which belongs to the same step. Several of our applications (Theorem 5.1(1), Proposition 4.1, Theorem 4.14) have $d_0 = 2$.

2.2. **The hyperplane case.**

Theorem 2.2 ([28, Thm. 1.1, pp. 375–376, under (5.4) of §5d, p. 394])**.** *Let $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times G_2$ be a connected commutative algebraic group over $\overline{\mathbb{Q}}$ of dimension $d \geq 1$, with d_0, d_1 maximal*

and $d_2 = d - d_0 - d_1$. Let V be a hyperplane of $T_G(\mathbb{C})$, $W \subset V$ a subspace of dimension $t \geq 0$ defined over $\overline{\mathbb{Q}}$, and $Y = \mathbb{Z}y_1 + \cdots + \mathbb{Z}y_m \subset V$ a subgroup of rank m with $\exp_G Y \subset G(\overline{\mathbb{Q}})$. Put $\kappa = \mathrm{rk}_{\mathbb{Z}}(Y \cap \ker \exp_G)$. If

$$m > (d_1 + 2d_2 - \kappa)(d - 1 - t), \tag{5.4}$$

then V contains a nonzero algebraic Lie subalgebra of $T_G(\mathbb{C})$ defined over $\overline{\mathbb{Q}}$, that is, $T_{G'} \subset V$ for some connected algebraic subgroup $G' \neq 0$ of G defined over $\overline{\mathbb{Q}}$.

The printed Theorem 1.1 has the hypothesis $m > (d_1 + 2d_2)(d - 1 - t)$, without κ; [28, §5d] proves it under the weaker hypothesis (5.4) (p. 394 prints $\kappa = \ell - m$, a misprint for $m - \ell$). That proof applies Theorem 2.1 to G and then again to an auxiliary subgroup [28, Lemma 5.1]. Since $\alpha(G) = d_1 + 2d_2$ for a maximal decomposition (§2.3), Theorem 2.2 is, in contrapositive form, the case $K = \mathbb{Q}$, $\Psi = \mathrm{id}$ of Theorem 2.3 below for hyperplanes V (stated for a finitely generated Y, with κ the rank of its periods). Theorems 2.1 and 2.2 are stated for literal products. Every connected commutative G over $\overline{\mathbb{Q}}$ is isogenous to $\mathbb{G}_{\mathrm{a}}^{d_0(G)} \times \mathbb{G}_{\mathrm{m}}^{d_1(G)} \times G_2$ (a surjection onto $\mathbb{G}_{\mathrm{a}}$, resp. $\mathbb{G}_{\mathrm{m}}$, restricts onto a $\mathbb{G}_{\mathrm{a}}$, resp. a subtorus, of the linear part, which splits off up to isogeny), and both statements are invariant under isogeny (as [28, p. 389] notes), so we apply them to groups that are only isogenous to such a product.

2.3. **The theorem of Roy and Waldschmidt.** Let G be a connected commutative algebraic group over $\overline{\mathbb{Q}}$, T_G its tangent space at the origin with its $\overline{\mathbb{Q}}$-structure, $\Omega(G) = \ker \exp_G$ its periods and $\Lambda(G) = \exp_G^{-1}(G(\overline{\mathbb{Q}}))$ the logarithms of its algebraic points. For a number field K of degree n, $\Omega_K(G)$ and $\Lambda_K(G)$ are the K-spans of $\Omega(G)$ and $\Lambda(G)$ in $T_G(\mathbb{C})$. Fix a basis $\theta = (\theta_1, \ldots, \theta_n)$ of $K/\mathbb{Q}$ and let $p_\theta \colon T_{G^n} = T_G^n \to T_G$, $(z_j) \mapsto \sum_j \theta_j z_j$. Let $d_0(G)$ (resp. $d_1(G)$) be the largest d with a surjection $G \to \mathbb{G}_{\mathrm{a}}^d$ (resp. $G \to \mathbb{G}_{\mathrm{m}}^d$) over $\overline{\mathbb{Q}}$, and $\alpha(G) = 2\dim G - 2d_0(G) - d_1(G)$. Let $\mathcal{E}$ be a finite-dimensional complex vector space with a $\overline{\mathbb{Q}}$-structure, $\Psi \colon T_G(\mathbb{C}) \to \mathcal{E}$ a $\overline{\mathbb{Q}}$-rational linear map, and $\Psi_\theta = \Psi \circ p_\theta$. Let H_0 (resp. H_Ψ) be the largest connected algebraic subgroup of G^n with $p_\theta(T_{H_0}) = 0$ (resp. $T_{H_\Psi} \subset \ker \Psi_\theta$), and put

$$\alpha_K(G) = \alpha(G^n/H_0)/n \ \text{ [23, p. 360]}, \qquad \alpha_{K,\Psi}(G) = \alpha(G^n/H_\Psi)/n \ \text{ (the \textit{inner weight})};$$

neither depends on θ (a change of basis induces an isogeny of G^n), and $\alpha_{K,\Psi}(G) \leq \alpha_K(G)$ [23, p. 364]. Finally $S_K(G)$ is the set of subspaces $p_\theta(T_{G'})$ of T_G, for G' an algebraic subgroup of G^n.

Theorem 2.3 ([23, Théorème, (i)⇒(iii), p. 363, with the inequality of step (b), p. 364])**.** *Let $V \subset \mathcal{E}$ be a complex subspace containing no nonzero $\Psi(S)$, $S \in S_K(G)$, and $W \subset V$ a subspace defined over $\overline{\mathbb{Q}}$. Put $\kappa = \dim_K\big(V \cap \Psi(\Omega_K(G))\big)$. Then $V \cap \Psi(\Lambda_K(G))$ is a finite-dimensional K-vector space, and*

$$\dim_K\big(V \cap \Psi(\Lambda_K(G))\big) \leq \big(\alpha_{K,\Psi}(G) - \kappa\big)\dim_{\mathbb{C}}(V/W) \leq \big(\alpha_K(G) - \kappa\big)\dim_{\mathbb{C}}(V/W).$$

Roy and Waldschmidt state the second inequality, for every connected commutative G over $\overline{\mathbb{Q}}$, every number field K and every V. Their proof has two steps. Step (b) (p. 364) reduces to $K = \mathbb{Q}$, $\Psi = \mathrm{id}$ on G^n/H_Ψ; it is written in full, and the inequality obtained before its last line ("On peut aussi majorer $\alpha(G')$ par $n\alpha_K(G)$") is the first inequality above. (Step (b) also applies (a) to $\ker \Psi'$, to get equalities in its (4); the inequality needs only the surjections there, and that instance is in any case Wüstholz's theorem.) Step (a) (p. 363) is the case $K = \mathbb{Q}$, $\Psi = \mathrm{id}$. It follows formally from [28, Thm. 4.1] by Roy's categorical method [22, §6]: Roy shows that the statements of his Proposition 2.1 hold with the period-carrying function $a'(X) = \alpha(G) - \mathrm{rk}(Y \cap \Omega(G))$ [22, p. 266], and step (a) outlines how the bound follows from them. In the proofs, Theorem 2.3 is used only with $\mathcal{E} = \mathbb{C}^2$, Ψ onto, V a line and $W = 0$ (the

remarks of §8 also use the case $K = \mathbb{Q}$, $\Psi = \mathrm{id}$, i.e. Theorem 2.2 and [28, Thm. 4.1] directly); then the subspace of T_{G^n/H_Ψ} to which step (b) reduces is a hyperplane, and for hyperplanes [23, Remarque 2] cites step (a) from [28, Thm. 1.1], i.e. Theorem 2.2. By [23, Remarque 1], the condition on V holds whenever V has no nonzero $\overline{\mathbb{Q}}$-rational point (and then $W = 0$); this is the only case we use. We call $\alpha_{K,\Psi}(G)$ the *inner weight*.

Upper bounds for the inner weight. For a connected algebraic subgroup H of a connected commutative group G one has $\alpha(G/H) \le \alpha(G)$, since $\alpha(H) + \alpha(G/H) \le \alpha(G)$ [22, Lemme 4.3] and $\alpha(H) \ge 0$. If $U \subset G^n$ is a connected algebraic subgroup with $T_U \subset \ker \Psi_\theta$, then $U \subset H_\Psi$, so G^n/H_Ψ is a quotient of G^n/U and

$$\alpha_{K,\Psi}(G) \le \alpha(G^n/U)/n.$$

Moreover $\alpha(G'') \le 2 \dim G'' - 2a - b$ whenever G'' admits surjections onto $\mathbb{G}_{\mathrm{a}}^a$ and onto $\mathbb{G}_{\mathrm{m}}^b$. So an explicit U and explicit surjections bound the inner weight from above, and an upper bound is all that the applications need.

2.4. **The shape of the arguments.** Every result of §§4–7 has the following shape. There are a group G (a product of copies of $\mathbb{G}_{\mathrm{a}}$, $\mathbb{G}_{\mathrm{m}}$, CM elliptic curves, simple CM abelian varieties and their universal vector extensions), a $\overline{\mathbb{Q}}$-rational map $\Psi\colon T_G \to \mathbb{C}^2$ (onto), a line $V = \mathbb{C}(1, s)$ (or $\mathbb{C}w$) whose slope is transcendental, and $W = 0$. So the condition of Theorem 2.3 holds by [23, Remarque 1], and $\dim V/W = 1$. A period relation gives a period datum on V, so $\kappa \ge 1$ (for Theorem 4.2 two periods on V give $\kappa \ge 2$, and κ pairs give κ for Theorem 4.14), and a counterexample to the statement gives further vectors on V, K-independent of the period data. Their total number is bounded by $\alpha_{K,\Psi}(G) - \kappa \le \alpha(G'')/n - \kappa$ for an explicit quotient $G'' = G^n/U$. The table lists the numbers (for $n = [K : \mathbb{Q}]$ large enough; "data" counts the period data together with the vectors a counterexample would give).

result	G''	$\alpha(G'')/n$	κ	data	bound
Prop. 4.1 (Thms. 4.2, 4.6, 4.12)	$\mathbb{G}_{\mathrm{a}}^2 \times \prod_j A_j''$	4	2	3	2
Thm. 4.14 (κ pairs)	$\mathbb{G}_{\mathrm{a}}^2 \times \prod_j A_j''$	2κ	κ	$\kappa + 1$	κ
Thm. 5.1(1), Thm. 5.5 (EA)	$\mathbb{G}_{\mathrm{a}}^2 \times A_1'' \times A_2'' \times A'''$	3	1	3	2
Thm. 5.1(2), Thm. 5.5 (EA$^\natural$)	$\mathbb{G}_{\mathrm{a}} \times A_1'' \times A_2'' \times Q''$	$3 + 2/n$	1	3	$2 + 2/n$
Thm. 6.2 ($X = \mathbb{G}_{\mathrm{m}}$)	$Q_1 \times \mathbb{G}_{\mathrm{m}}^n \times Q_3$	$3 + 4/n$	1	3	$2 + 4/n$
Prop. 7.3, rows (iii), (iv)	$\mathbb{G}_{\mathrm{m}}^n \times A'' \times A'''$	3	1	3	2
Prop. 7.3, rows (i), (ii), (v), (vi)	$\mathbb{G}_{\mathrm{m}}^n \times A'' \times Q''$	$3 + 2/n$	1	3	$2 + 2/n$

Here the A'' are abelian varieties of dimension $n/2$ *for every* g: reading one eigen-coordinate kills all but $n/2$ dimensions of A^n (Lemma 3.1); and Q'', Q_1, Q_3 are extensions of such an abelian variety by $\mathbb{G}_{\mathrm{a}}$, of dimension $n/2 + 1$. In each row the bound is below the number of data for n large enough ($n \ge 3$, resp. $n \ge 5$ for Theorem 6.2), which is the contradiction.

We give two kinds of proofs. In the *direct* proofs (Proposition 4.1, Theorems 4.14, 5.1(1), 6.2, and row (iv) of Proposition 7.3 in Remark 7.4) we apply Theorem 2.1 once, to $G'' = G^n/H_\Psi$, and exclude every connected algebraic subgroup by an explicit case check, with (2) checked at G''. In these applications $V'' = \Psi''^{-1}(V)$ is a hyperplane, $N = d - 1$, and $W'' = \ker \Psi''$, so that $\tau = \delta + r' - 2$ where $r' = \dim \Psi''(T_{G'})$; thus only G' with $r' \le 1$ can satisfy $\delta > \tau$, and then $\delta - \tau = 2 - r'$. In the other proofs (Theorem 5.5, hence Theorem 5.1(2), and Proposition 7.3) we apply Theorem 2.3 with the upper bound for the inner weight of §2.3, which needs only an explicit U, the dimension of G^n/U and κ. The direct proofs show which subgroups are tight and by how much; the others are shorter. Theorems 8.1 and 8.4 (§8) are not of this shape: there V'' has codimension ≥ 2.

Remark 2.4 (The printed weight)**.** For several results the printed weight α_K of Theorem 2.3 already suffices, on the group G itself, with the weights of §3.3: (i) Theorem 5.1(1), and Theorem 5.5 for elliptic curves: $\alpha_K(\mathbb{G}_a\times E\times\mathbb{G}_a\times E\times E')=3$, $\kappa\geq 1$; (ii) Proposition 4.1 when the four slots are all the first-kind slots of the varieties involved (levels 7 and 9): $\alpha_K(G)=\sum_A \dim A=4$, $\kappa=2$ (Remark 4.5); (iii) Theorem 4.14 for a whole family $T(\tau)$: $\alpha_K=2\kappa$; through it, and the slot-splitting step of §4.3, Theorem 4.12(2) and every subfamily of Theorem 4.14 (the paragraph after the proof of Theorem 4.12); (iv) rows (iii), (iv) of Proposition 7.3: $\alpha_K(\mathbb{G}_a\times\mathbb{G}_m\times E\times E')=3$, $\kappa\geq 1$. With Wüstholz's and Masser's theorems for the final deductions, these results are direct instances of [23, Théorème (iii)] as printed. For Theorem 5.5 with $\max(g,g')\geq 2$, Theorem 5.1(2), the fibre rows of Proposition 7.3 and Theorem 6.2, the printed weight on the groups used here gives no conclusion (for instance $\alpha_K(\mathbb{G}_a\times A\times\mathbb{G}_a\times A\times A')=2g+g'$, and $\alpha_K(\mathbb{G}_m\times E^\natural\times E)=4+2/n$ by Lemma 3.3), and the inner weight is used.

3. CM abelian varieties and universal vector extensions

This section gives the linear algebra of CM abelian varieties of any dimension g and of their universal vector extensions (§§3.1–3.2), and the weights (§3.3). Most of it is also in [16, §3].

3.1. **CM abelian varieties.** Let $A/\overline{\mathbb{Q}}$ be a simple abelian variety of dimension g with CM by the CM field F, $[F:\mathbb{Q}]=2g$, with (primitive) CM type Φ. For $\sigma\in\Phi$ let $z_\sigma\colon \mathrm{Lie}(A)\to\mathbb{C}$ be the $\overline{\mathbb{Q}}$-rational eigen-coordinate dual to a holomorphic eigen-differential ω_σ, so that $z_\sigma([x]v)=\sigma(x)z_\sigma(v)$ for $x\in\mathcal{O}_F$. Since $H_1(A,\mathbb{Q})=F\gamma_0$ for any $0\neq\gamma_0\in\Omega(A)$, we have $z_\sigma(x\gamma_0)=\sigma(x)P_\sigma$ with $P_\sigma=z_\sigma(\gamma_0)$, so $z_\sigma(\Omega(A))\subset\sigma(F)P_\sigma$.

Reduction to the simple case. A CM abelian variety over $\overline{\mathbb{Q}}$ is isogenous to $\prod_i B_i^{m_i}$ with B_i simple CM abelian varieties of primitive type, with CM fields F_i. By its *CM algebra* we mean any commutative semisimple $\mathbb{Q}$-subalgebra $\mathcal{E}\subset\mathrm{End}^0(A)$ of degree $2\dim A$ (so $\mathcal{E}=\prod_i\mathcal{E}_i$ with $\mathcal{E}_i\supseteq F_i$ maximal commutative in $M_{m_i}(F_i)$), and eigen-differentials of A are taken with respect to $\mathcal{E}$; the argument below does not depend on this choice. Being maximal commutative, $\mathcal{E}$ contains the centre $\prod_i F_i$ of $\mathrm{End}^0(A)$ and its central idempotents. An eigen-character $\mathcal{E}\to\overline{\mathbb{Q}}$ is nonzero on exactly one of these, so the eigen-differential lives on one isotypic part $B_i^{m_i}$, and the character restricts to an embedding σ_0 of F_i, which acts on $B_i^{m_i}$ diagonally. An eigen-differential is therefore, on that part, a σ_0-eigen class for F_i, i.e. a $\overline{\mathbb{Q}}$-combination of pull-backs $\mathrm{pr}_k^*\omega_{\sigma_0}$ of one σ_0-eigen differential of B_i, of the same kind. So every nonzero eigen-period lies in $\overline{\mathbb{Q}}\,\sigma_0(F_i)P_{\sigma_0}(B_i)\setminus\{0\}=\overline{\mathbb{Q}}^\times P_{\sigma_0}(B_i)$, as stated in the introduction.

Lemma 3.1 (Lemma K′)**.** *Let $F\subset K$, where K is a number field containing the Galois closure of F, $n=[K:\mathbb{Q}]$, and let $\theta_1,\dots,\theta_n$ be a basis of $K/\mathbb{Q}$. For an embedding $\sigma\colon F\to K$ put $\psi_\sigma(m)=\sum_j\theta_j\sigma(m_j)$ on F^n. Then $(\psi_\sigma)_\sigma\colon F^n\to K^{2g}$ is a $\mathbb{Q}$-linear bijection, each ψ_σ is σ-semilinear and onto K, and for every set Σ of embeddings the common kernel of $(\psi_\sigma)_{\sigma\in\Sigma}$ is an F-subspace of F-dimension $n-|\Sigma|\,n/(2g)$.*

Proof. $F^n\cong F\otimes_{\mathbb{Q}}K$ via $m\mapsto\sum m_j\otimes\theta_j$, and $F\otimes_{\mathbb{Q}}K\cong K^{2g}$ via $x\otimes y\mapsto(\sigma(x)y)_\sigma$, since K contains the Galois closure; the composite is $(\psi_\sigma)_\sigma$. Semilinearity is clear. The common kernel is the preimage of $\{0\}^\Sigma\times K^{2g-|\Sigma|}$, of $\mathbb{Q}$-dimension $(2g-|\Sigma|)n$, i.e. of F-dimension $(2g-|\Sigma|)n/(2g)$. □

For $g=1$, $F=k$ is an imaginary quadratic field, ψ_{id} and ψ_- are the maps ψ, ϕ below, and Lemma 3.1 becomes the following lemma, which we use throughout for CM elliptic curves (we keep its direct proof).

Lemma 3.2. *Let $k\subset K$ be an imaginary quadratic field inside a number field of degree n, $\theta_1,\dots,\theta_n$ a basis of $K/\mathbb{Q}$, and on k^n put $\psi(m)=\sum\theta_j m_j$, $\phi(m)=\sum\theta_j\bar m_j$. Then $(\psi,\phi)\colon k^n\to$*

$K \times K$ is a $\mathbb{Q}$-linear bijection; $\ker\psi$ *and* $\ker\phi$ *are k-subspaces of dimension $n/2$, and* $\ker\psi \cap \ker\phi = 0$.

Proof. $k^n \cong K \otimes_{\mathbb{Q}} k$ via $m \mapsto \sum \theta_j \otimes m_j$, and $K \otimes_{\mathbb{Q}} k \cong K \times K$ via $x \otimes y \mapsto (xy, x\bar{y})$; the composite is (ψ, ϕ). ψ is k-linear and ϕ is k-semilinear, and both are onto K, which has k-dimension $n/2$. □

An *eigen-slot* is a pair (A, σ) with $A/\overline{\mathbb{Q}}$ a simple CM abelian variety and σ an embedding of its CM field; its periods are the eigen-periods P_σ of ω_σ (on $A^\natural$ if $\sigma \notin \Phi$). Two eigen-slots are equal iff the abelian varieties are isogenous and the embeddings correspond under the isogeny. An eigen-slot is of the first kind iff $\sigma \in \Phi$, and of the second kind otherwise. We use the following standard facts.

- (S1) [Lit] (standard; see Shimura [24]). $\mathrm{End}^0(A^n) = M_n(F)$, and Hom between non-isogenous simple abelian varieties is 0. Abelian subvarieties of A^n are the B_S, $S \subset F^n$ an F-subspace, with $\dim B_S = g \dim_F S$ and $T(B_S) = \{(\sigma(M)u_\sigma)_{\sigma\in\Phi} : u_\sigma \in \mathbb{C}^{\dim S}\}$ in the eigen-coordinates (M a matrix whose columns span S). On $T(B_S)$ the form $\theta^{\mathsf{T}} z_\sigma$ vanishes iff $\psi_\sigma(S) = 0$, and different $\sigma \in \Phi$ carry independent u_σ.
- (P) [Proved] Every connected algebraic subgroup of $\mathbb{G}_{\mathrm{a}}^a \times \mathbb{G}_{\mathrm{m}}^b \times B$, B an abelian variety, is a product $(H' \cap \mathbb{G}_{\mathrm{a}}^a) \times (H' \cap \mathbb{G}_{\mathrm{m}}^b) \times B'$; subgroups of $\mathbb{G}_{\mathrm{a}}^a$ are $\overline{\mathbb{Q}}$-subspaces and of $\mathbb{G}_{\mathrm{m}}^b$ rational subtori. (The image of H' in B is an abelian subvariety B', and $H'/(H' \cap M)$, M the linear part, maps to the linear group $M/(H' \cap M)$ trivially; so $H' = (H' \cap M)^0 \times B'$ by dimension, and $H' \cap M$ splits since $\mathrm{Hom}(\mathbb{G}_{\mathrm{a}}, \mathbb{G}_{\mathrm{m}}) = \mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{a}}) = 0$.)
- (Ch) By Dedekind's independence of characters, the vectors $(\sigma(x))_{\sigma\in\Phi}$, $x \in F$, span K^g over K. Hence $\Lambda_K(A)$, read in the eigen-coordinates, is $\prod_{\sigma\in\Phi} K\, z_\sigma(\Lambda(A))$, and likewise for $\Omega_K(A)$: the slot coordinates of K-combinations of logarithms (or periods) can be prescribed independently.
- (W) [Lit] (Wüstholz [37]). The numbers 1, π and nonzero periods of pairwise distinct eigen-slots (first or second kind, on one or several simple CM abelian varieties) are linearly independent over $\overline{\mathbb{Q}}$. In particular $P_\sigma \notin \overline{\mathbb{Q}}$ and $P_\sigma/\pi \notin \overline{\mathbb{Q}}$. A nonzero eigen-period of A (resp. a second-kind eigen-period of $A^\natural$) is not in $\overline{\mathbb{Q}}\mathcal{L}$.

All of these follow from Wüstholz's theorem on $\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}} \times \prod_i A_i^\natural$ (resp. with $\mathbb{G}_{\mathrm{m}}^r$ added), applied to the point with coordinates 1, $2\pi i$ (resp. the logarithms) and the cycles, as in the following example. For example, $P/\pi \notin \overline{\mathbb{Q}}$: otherwise the point $(\gamma_0, 2\pi i)$ of $\mathrm{Lie}(A \times \mathbb{G}_{\mathrm{m}})$ lies in a $\overline{\mathbb{Q}}$-hyperplane $z_\sigma = a\, x$; the smallest algebraic subgroup whose Lie algebra contains it is $A \times \mathbb{G}_{\mathrm{m}}$ (A is simple and the subgroup is a product by (P)), whose Lie algebra is not in the hyperplane.

Universal vector extension. For $\sigma \notin \Phi$ the eigen-coordinate w_σ lives on $\mathrm{Lie}(A^\natural)$. We use the following facts. The input from the literature is [Lit] (Mazur–Messing [19]): $\mathrm{Lie}(A^\natural) = H^1_{\mathrm{dR}}(A)^\vee$ functorially, and $A^\natural$ is universal among extensions of A by vector groups (every such extension of B by U is the push-out of $B^\natural$ along a unique linear map $W_B \to U$, where W_B is the fibre of $B^\natural$). The rest is [Proved] by the one-line arguments given.

- (F1) $\mathrm{Lie}(A^\natural) = H^1_{\mathrm{dR}}(A)^\vee$ with functorial $\mathcal{O}_F$-action. In the dual eigenbasis (e_τ) the fibre Lie algebra is $\bigoplus_{\tau\notin\Phi} \mathbb{C}e_\tau$, and w_σ ($\sigma \notin \Phi$) is nonzero exactly on e_σ, which lies in the fibre. *Proof:* the Lie algebra sequence of $0 \to V \to A^\natural \to A \to 0$ is the dual of $0 \to H^0(A, \Omega^1) \to H^1_{\mathrm{dR}}(A) \to H^1(A, \mathcal{O}) \to 0$, so the fibre is the annihilator of $H^0(A, \Omega^1) = \bigoplus_{\tau\in\Phi} \overline{\mathbb{Q}}\omega_\tau$.
- (F2) For $B \subset A^n$, $B^\natural \to (A^n)^\natural$ is injective on fibres and $\pi^{-1}(B) = B^\natural + V^n$ (V the fibre). *Proof:* on fibres the map is dual to the restriction $H^1(A^n, \mathcal{O}) \to H^1(B, \mathcal{O})$, which is onto

because B is a direct factor of A^n up to isogeny (Poincaré reducibility); and $B^\natural$ maps onto B.

(F3) $\mathrm{Ext}(B,\mathbb{G}_\mathrm{a}) \cong H^1(B,\mathcal{O})$ and $\mathrm{Hom}(B,\mathbb{G}_\mathrm{a}) = 0$: a connected H over B with given intersection W' with the fibre exists iff $W' \supseteq W_B$, and then $H = B^\natural + W'$ is unique. *Proof:* such an H is a splitting over B of $\pi^{-1}(B)/W'$, which by (F2) and universality is the push-out of $B^\natural$ along $W_B \to V^n/W'$; it splits iff this map is 0, i.e. $W_B \subseteq W'$, and the splitting is unique since $\mathrm{Hom}(B, V^n/W') = 0$.

(F4) $T(B^\natural_S) = \bigoplus_\tau \tau(S)\otimes\mathbb{C}$ and $W_{B_S} = \bigoplus_{\tau\notin\Phi}\tau(S)\otimes\mathbb{C}$. *Proof:* under the comparison isomorphism $H^1_\mathrm{dR}(B_S)^\vee \cong H_1(B_S,\mathbb{Q})\otimes\mathbb{C}$, with $H_1(B_S,\mathbb{Q}) = S\gamma_0 \subset F^n\gamma_0$, the τ-eigen component is $\tau(S)\otimes\mathbb{C}$; the fibre part is $\tau\notin\Phi$ by (F1).

The periods of $A^\natural$ satisfy $\Omega(A^\natural)\otimes\mathbb{Q} = F\tilde\gamma_0$ with $w_\sigma(x\tilde\gamma_0) = \sigma(x)P_\sigma$, and the values $w_\sigma(\Lambda(A^\natural))$ ("σ-fibre values") are second-kind abelian integrals $\int\omega_\sigma + \overline{\mathbb{Q}}$.

Beta values [Lit] (Weil [35]; Rohrlich's appendix to [10]; Koblitz–Ogus [12]). For $N\ge 3$ and $0<a,b<N$ with $a+b\not\equiv 0 \bmod N$, $B(a/N,b/N)$ is, up to $\overline{\mathbb{Q}}^\times$, the period of an eigen-differential on the Fermat curve of degree N, with Koblitz–Ogus label $\Phi_{a,b} = \{u\in(\mathbb{Z}/N)^\times : \langle ua/N\rangle + \langle ub/N\rangle + \langle u(N-a-b)/N\rangle = 1\}$ (after reducing $a/N, b/N$ to lowest common denominator); the corresponding isotypic factor of the Jacobian is a power of a simple CM abelian variety, and the differential is of the first kind iff $a+b<N$. Deligne's theorem (stated by Koblitz–Ogus [12]; it rests on Deligne's theorem that Hodge cycles on abelian varieties are absolutely Hodge [6]) gives: if $\mathbf{1}_{\Phi_a}+\mathbf{1}_{\Phi_b} = \mathbf{1}_{\Phi_c}+\mathbf{1}_{\Phi_d}$ on $(\mathbb{Z}/N)^\times$, then $P_aP_b \in \overline{\mathbb{Q}}^\times P_cP_d$. Together with the rule that π corresponds to the constant function 1, and $\mathbf{1}_\Phi + \mathbf{1}_{-\Phi} \equiv 1$, the same dictionary contains $P_\sigma P_{\bar\sigma} \in \pi\overline{\mathbb{Q}}^\times$. That these are all multiplicative relations is the Rohrlich–Lang conjecture, used below only in remarks labelled [Heuristic].

3.2. **CM elliptic curves and their universal vector extensions.** For a CM elliptic curve E let $E^\natural$ be its universal vector extension, with base coordinate z and conjugate-eigen fibre coordinate w: $\mathcal{O} = \mathrm{End}(E)$ acts on $T(E^\natural) = H^1_\mathrm{dR}(E)^\vee$ by its identity character on z and by the conjugate character on w (the eigenline of H^1_dR for $\bar a$ is spanned by $x\,dx/y + c\,dx/y$ with $c\in\overline{\mathbb{Q}}$, so these coordinates are rational; this is (F1) for $g=1$). In these coordinates the periods are $(\lambda,\eta^*(\lambda))$, $\lambda\in\Lambda_E$, where $\eta^*(\lambda) = \eta(\lambda) - \kappa_1\lambda$ satisfies $\eta^*(a\lambda) = \bar a\,\eta^*(\lambda)$; the logarithms of points of $E^\natural(\overline{\mathbb{Q}})$ are $(u,\zeta^*(u)+t)$ with $u\in\mathcal{L}_E$, $t\in\overline{\mathbb{Q}}$ and $\zeta^* = \zeta - \kappa_1\,\mathrm{id}$. On E_i and E_ρ, $\kappa_1 = 0$. Since the characters $a\mapsto a, \bar a$ of k are K-independent for $K\supseteq k$,

$$\Omega_K(E^\natural) = K\lambda_0 \times K\eta^*(\lambda_0) \qquad (\Lambda_E\otimes\mathbb{Q} = k\lambda_0). \tag{3}$$

We call *fibre values* the w-coordinates of $\Lambda(E^\natural)$, that is, $\zeta^*(u)+\overline{\mathbb{Q}}$ for $u\in\mathcal{L}_E\setminus\Lambda_E$ and $\eta^*(\lambda)+\overline{\mathbb{Q}}$ at lattice points, and write Fib for the set of fibre values. By Legendre and $\eta^*(a\lambda) = \bar a\eta^*(\lambda)$, $\Omega\eta^*(\Omega)\in\overline{\mathbb{Q}}^\times\pi$ for every CM period Ω.

3.3. **Weights.** $\alpha_K(\mathbb{G}_\mathrm{a}) = 0$, $\alpha_K(\mathbb{G}_\mathrm{m}) = 1$, and for an elliptic curve $\alpha_K(E) = 1$ if E has CM by k and $K\supseteq k$, else 2 [23, Lemme, p. 361]; more generally $\alpha_K(A) = g$ for a simple CM abelian variety A of dimension g when K contains the Galois closure of its CM field ([23, Lemme, p. 361]; it is also Lemma 3.1 with $\Sigma=\Phi$); α_K is additive on products [23, (2)]. For the universal vector extension $E^\natural$ of a CM curve and $K\supseteq k$ (used only to say when the printed weight gives no conclusion) we have:

Lemma 3.3. $\alpha_K(E^\natural) = 2 + 2/n$.

Proof. In the coordinates (z,w) of §3.2, let $H\subset(E^\natural)^n$ be connected with $p_\theta(T_H) = 0$, with abelian image $B\subset E^n$. Then $B = B_S$ for a k-subspace $S\subset k^n$ on which both ψ (base) and ϕ (fibre) of Lemma 3.2 vanish; since $\ker\psi\cap\ker\phi = 0$, $B = 0$. Hence H lies in the fibre $\mathbb{G}_\mathrm{a}^n$, and

the largest such H is the rational hyperplane $\sum \theta_j t_j = 0$, of dimension $n-1$. The quotient has dimension $n+1$ and admits no surjection to $\mathbb{G}_{\mathrm{a}}$ or $\mathbb{G}_{\mathrm{m}}$ (it is a non-split extension of E^n by $\mathbb{G}_{\mathrm{a}}$), so $\alpha = 2(n+1)$ and $\alpha_K = 2 + 2/n$. □

Laurent's identity. We use it only in remarks, to say which statements are linear among periods. Let $\Lambda_E = \Omega\,\mathbb{Z}[\alpha]$ (a sublattice of finite index suffices), and write Masser's relation [18] as $\eta(\alpha\Omega) = \bar\alpha\eta(\Omega) + c\,\Omega$ with $c \in \overline{\mathbb{Q}}$. Legendre's relation gives $(\alpha - \bar\alpha)\eta(\Omega) - c\Omega = 2\pi i/\Omega$, and $f(z) = \zeta(\alpha z) - \bar\alpha\zeta(z) - cz$ is an elliptic function defined over $\overline{\mathbb{Q}}$. With the third-kind period $\lambda(u,\Omega) = \Omega\zeta(u) - \eta(\Omega)u$ one obtains, for $u \in \mathcal{L}_E$ not a pole of f,

$$\frac{2\pi i\, u}{\Omega} = \bar\alpha\,\lambda(u,\Omega) - \lambda(\alpha u,\Omega) + \Omega f(u), \qquad f(u) \in \overline{\mathbb{Q}}. \tag{4}$$

On E_i this reads $2\eta(\varpi)u = i\lambda(iu,\varpi) - \lambda(u,\varpi)$ (cf. [17, §9, p. 160], where the general CM form is derived; the display above is its case $\mu = i$). So $\Omega\zeta^*(u)$ and $\eta^*(\Omega)u$ are, up to $\overline{\mathbb{Q}}$-linear combinations, periods of 1-motives; the identity concerns a period and a logarithm of the same curve.

4. First-kind period relations as period terms

4.1. **Set-up.** A *first-kind slot* is an eigen-slot of the first kind (§3.1), written $j = (A_j, \sigma_j)$ with $A_j/\overline{\mathbb{Q}}$ a simple CM abelian variety (CM field F_j, primitive type Φ_j) and $\sigma_j \in \Phi_j$; z_j is the corresponding eigen-coordinate on $\mathrm{Lie}(A_j)$ and $P_j = z_j(\gamma_j) \neq 0$ for a fixed $\gamma_j \in \Omega(A_j)$, so $z_j(\Omega(A_j)) \subset \sigma_j(F_j)P_j$. As in §3.1, two slots are equal iff the abelian varieties are isogenous and the embeddings correspond; distinct slots of one A have distinct embeddings. By (W), periods of distinct slots are $\overline{\mathbb{Q}}$-linearly independent. Let a, b, c, d be four pairwise distinct first-kind slots (distinct in the sense just defined, i.e. up to isogeny) with a *first-kind relation*

$$P_a P_b = \mu\, P_c P_d, \qquad \mu \in \overline{\mathbb{Q}}^{\times}. \tag{5}$$

Let $G = \prod_A A \times \mathbb{G}_{\mathrm{a}}^2$, the product over the isogeny classes among A_a, A_b, A_c, A_d, each taken once (so the factors A are pairwise non-isogenous; a slot z_j on an A_j isogenous to the chosen representative A is read on A through an isogeny, which rescales P_j and $z_j(\Lambda(A_j))$ by $\overline{\mathbb{Q}}^{\times}$), let $\beta \in \overline{\mathbb{Q}}^{\times}$ and

$$\Psi = \big(z_a + z_d + q_1,\ z_c + z_b + q_2\big)\colon T_G \to \mathbb{C}^2, \qquad V = \mathbb{C}w, \quad w = (P_a, \beta P_c),$$

where q_1, q_2 are the $\mathbb{G}_{\mathrm{a}}$-coordinates. Then $w' := (P_d, \beta P_b/\mu) = (P_d/P_a)\,w$ by (5). Let K be a number field containing the Galois closures of all F_j, μ and β, and $n = [K : \mathbb{Q}]$. By (Ch) and $z_j(\Omega(A_j)) \subset \sigma_j(F_j)P_j$,

$$\Psi(\Omega_K(G)) = \mathcal{P} := (KP_a + KP_d) \times (KP_c + KP_b),$$

so $w, w' \in \Psi(\Omega_K(G))$, and they are K-independent since $P_d/P_a \notin \overline{\mathbb{Q}}$.

Proposition 4.1. *$V \cap \Psi(\Lambda_K(G)) = Kw + Kw'$.*

Theorem 4.2 (Theorem X)**.** *In the set-up above, let $v_j \in \overline{\mathbb{Q}}\Lambda(A_j)$ for $j = a, b, c, d$ and $q_1, q_2 \in \overline{\mathbb{Q}}$ satisfy*

$$\big(z_a(v_a) + z_d(v_d) + q_1\big) \cdot \beta P_c = \big(z_c(v_c) + z_b(v_b) + q_2\big) \cdot P_a. \tag{6}$$

Then $q_1 = q_2 = 0$ and $z_j(v_j) \in \overline{\mathbb{Q}}P_j$ for each j. If $v_j \in \Lambda(A_j)$, then $\exp_{A_j}(v_j)$ is a torsion point.

When several of the slots lie on the same A (as in §4.4), the v_j may be logarithms of different points: by (Ch), a single element of $\Lambda_K(A)$ has slot coordinates $z_j(v_j)$ for any prescribed v_j.

4.2. **Proof of Proposition 4.1.** Suppose $v \in V \cap \Psi(\Lambda_K(G))$, $v \notin Kw + Kw'$. Let θ, p_θ and $\Psi_\theta = \Psi \circ p_\theta$ be as before.

The group. For each isogeny class A let Σ_A be the set of slots on A used by Ψ. By (P) and (S1), the largest connected subgroup with tangent space in $\ker \Psi_\theta$ is $U = \prod_A B_{\ker(\psi_\sigma)_{\sigma\in\Sigma_A}} \times \{\theta^{\mathsf{T}} q_1 = \theta^{\mathsf{T}} q_2 = 0\}$, and $G'' = G^n/U = \mathbb{G}_{\mathrm{a}}^2 \times A''$ with $\dim A'' = \sum_A |\Sigma_A|\, n/2 = 2n$ by Lemma 3.1. So $d = 2n+2$, $d_0 = 2$, $d_1 = 0$, $d_2 = 2n$. The induced $\Psi''\colon T_{G''} \to \mathbb{C}^2$ is onto, $V'' = \Psi''^{-1}(V)$ is a hyperplane ($N = 2n+1$), and $W'' = \ker \Psi'' \subset V''$ is $\overline{\mathbb{Q}}$-rational of dimension $2n$.

The data. Let Y'' be generated by the images of lifts of $D\theta_h w$, $D\theta_h w'$ (in $\Omega(G^n)$) and $\theta_h v$, $h = 1, \dots, n$. Since U is connected, $\Omega(G'')$ is the image of $\Omega(G^n)$, so $\Psi''(\Omega(G'')) \subset \mathcal{P}$, and $V \cap \mathcal{P} = Kw + Kw'$ (Lemma 4.4 with $L = 0$). Hence $\operatorname{rk} Y'' = 3n$, $\kappa = 2n$ exactly, and $\Gamma = \exp Y''$ has rank n. Since $d_1 + 2d_2 - \kappa = 2n$ and $d - N = 1$, (1) reads

$$\lambda + 2\delta_2 \le (2 - r') \cdot 2n. \tag{7}$$

Lemma 4.3 (Lemma S′)**.** *Every connected algebraic subgroup of G'' is $L \times B$ with L a $\overline{\mathbb{Q}}$-subspace of $\mathbb{G}_{\mathrm{a}}^2$ and B an abelian subvariety of A''. $\Psi''(T_B)$ is spanned by the target directions of the slots on which B is not killed: it is 0 iff $B = 0$; it is $\subset \mathbb{C}(1,0)$ iff the slots c, b are killed, and then $\dim B \le n$; similarly for $\mathbb{C}(0,1)$ (slots a, d killed); otherwise it is $\mathbb{C}^2$.*

Proof. The product decomposition is (P). By (S1) the slot forms $\theta^{\mathsf{T}} z_j$ on $T(B_S)$ involve independent vectors u_σ, and each slot feeds exactly one target coordinate, so $\Psi''(T_B)$ is a coordinate subspace; it is 0 only for $B = 0$ by maximality of U. If only the slots of one target survive, $S/\ker$ embeds into K^2 via the two surviving ψ_σ, so $\dim B \le 2 \cdot n/2 = n$ (Lemma 3.1). □

Lemma 4.4 (Lemma L′)**.** *For $L \in \{0,\ \mathbb{C}(1,0),\ \mathbb{C}(0,1)\}$ one has $V \cap (L + \mathcal{P}) = Kw + Kw'$.*

Proof. If $L \ne \mathbb{C}(1,0)$ and $\xi w \in L + \mathcal{P}$, then $\xi P_a \in KP_a + KP_d$, so $\xi = x + yP_d/P_a$ $(x, y \in K)$ and $\xi w = xw + yw'$. If $L = \mathbb{C}(1,0)$, then $\xi\beta P_c \in KP_c + KP_b$, so $\xi = x + yP_b/P_c = x + y\mu P_d/P_a$ and $\xi w = xw + y\mu w'$. □

The case check. Let $G' = L \times B$ with $r' = \dim \Psi''(T_{G'}) \le 1$, $e = \dim B$. Then $\delta_2 = 2n - e$. If $\Psi''(T_{G'}) \in \{0, \mathbb{C}(1,0), \mathbb{C}(0,1)\}$, then $\lambda = n$: if $\exp(\sum c_h \tilde{y}_h) \in G'$ for the lifts $\tilde{y}_h$ of $\theta_h v$, then $xv \in \Psi''(T_{G'}) + \mathcal{P}$ with $x = \sum c_h \theta_h \in K$, so $x = 0$ by Lemma 4.4.

G'	r'	λ	(7) reads	holds?
0	0	n	$5n \le 4n$	never
$B \ne 0$, image an axis ($e \le n$), $L \in \{0, \text{same axis}\}$	1	n	$5n - 2e \le 2n$	never ($e \le n$)
L an axis, $B = 0$	1	n	$5n \le 2n$	never
L a non-axis line, $B = 0$	1	≥ 0	$\lambda + 4n \le 2n$	never
$r' = 2$			$\delta > \tau$ fails	

So no connected $G' \ne G''$ satisfies the conclusion of Theorem 2.1, for any n; the minimum excess (left side minus right side) is n, attained at $G' = 0$ and at axis subgroups with $e = n$. Condition (2) reads $4n+2 \ge 2n+4$; $d_0 = 2$ is allowed (§2.1). This contradiction proves Proposition 4.1. □

Remark 4.5 (Second derivation)**.** Waldschmidt's [28, §5d] (κ-refined Theorem 1.1, hyperplane case) applies to (G'', V'', W'', Y'') directly: condition (5.4) there reads $m = 3n > (d_1 + 2d_2 - \kappa)(d - 1 - t) = 2n$, and V'' contains no nonzero Lie algebra of an algebraic subgroup, by Lemma 4.3 and $P_c/P_a \notin \overline{\mathbb{Q}}$. Here $m = \operatorname{rk} Y''$; p. 394 of [28] prints "$\kappa = \ell - m$", which must be read $m - \ell$ ($\ell = \operatorname{rk}\Gamma$, cf. pp. 392–393), as noted after Theorem 2.2.

Relation to [23, Théorème (iii)]. With the weights of §3.3, $\alpha_K(G) = \sum_A \dim A \ge 4$, with equality exactly when the four slots are all the first-kind slots of the varieties involved. In that case (levels 7 and 9 below) Theorem 2.3 with $\kappa = 2$ gives $\dim_K(V \cap \Psi(\Lambda_K(G))) \le (4-2)\cdot 1 = 2$, so Proposition 4.1 already follows from it (Remark 2.4); otherwise (level 24 below) the printed

weight on G itself is larger, and Theorem 2.3 with the inner weight $\alpha_{K,\Psi} = 4$ is used (§2.4). But the level-24 quadruple consists of two of the four pairs of a full family of Theorem 4.14, so Theorem 4.2 at level 24 follows from that family by the slot-splitting step of §4.3 (the paragraph after the proof of Theorem 4.12).

4.3. **Proof of Theorem 4.2 from Proposition 4.1.** Enlarge K to contain q_1, q_2 and the coefficients of the v_j, so that $v_j \in K\Lambda(A_j)$; the proposition holds for every such K. When several slots lie on one A, (Ch) gives a single element of $\Lambda_K(A)$ with the prescribed slot coordinates $z_j(v_j)$. So hypothesis (6) says that the point $\Psi(v_a, \dots, v_d, q_1, q_2)$ lies in $V \cap \Psi(\Lambda_K(G))$, and Proposition 4.1 gives $x, y \in K$ with

$$z_a(v_a - x\gamma_a) + z_d(v_d - y\gamma_d) + q_1 = 0,$$

and similarly in the second coordinate. *Slot splitting.* Write $v_a - x\gamma_a$ and $v_d - y\gamma_d$ as K-combinations of finitely many $u_1, \dots, u_m \in \Lambda(A)$ (for each isogeny class A involved). The point $\xi = (1, (u_i)_i) \in \mathrm{Lie}(\mathbb{G}_\mathrm{a} \times \prod_A A^m)$ has $\exp \xi$ algebraic, and the displayed relation is a $\overline{\mathbb{Q}}$-linear form ℓ vanishing at ξ. By Wüstholz's theorem [37], the smallest algebraic subgroup H whose Lie algebra contains ξ has $T_H \subset \ker \ell$. By (P) and (S1), $H = L \times \prod_A B_{S_A}$, and $L = \mathbb{G}_\mathrm{a}$ because the $\mathbb{G}_\mathrm{a}$-coordinate of ξ is 1; so $(1, 0) \in T_H$ and $q_1 = 0$. On $T(B_S)$ the slot forms involve independent vectors u_σ (S1), and the slots a, d are distinct; so each slot form vanishes on T_H separately, and $z_a(v_a) = xP_a$, $z_d(v_d) = yP_d$. Likewise $q_2 = 0$, $z_c(v_c) \in KP_c$ and $z_b(v_b) \in KP_b$. *Torsion.* Let $v_j \in \Lambda(A_j)$ with $z_j(v_j) = x_jP_j$. If $x_j = 0$, Wüstholz on the simple A_j gives $v_j = 0$. Otherwise the smallest algebraic subgroup of A_j^2 whose Lie algebra contains (v_j, γ_j) is a proper abelian subvariety B_S (its tangent space lies in $\ker(z_j \oplus -x_jz_j)$) that projects onto the second factor. So $\dim_F S = 1$ and B_S is the graph of some $\alpha \in F_j = \mathrm{End}^0(A_j)$; hence $v_j \in F_j\gamma_j \subset \mathbb{Q}\,\Omega(A_j)$ and $\exp(v_j)$ is torsion. □

4.4. **The level-seven instance.** Let $C\colon y^7 = x(1-x)$. It has genus 3 (branch points $0, 1, \infty$), and dx/y^k is holomorphic exactly for $k = 4, 5, 6$ (orders $6-k$ at 0 and 1, $2k-8$ at ∞). The automorphism $y \mapsto \zeta y$ ($\zeta = e^{2\pi i/7}$) multiplies $dx/y^6, dx/y^5, dx/y^4$ by ζ, ζ^2, ζ^3, so $A = \mathrm{Jac}(C)$ has CM by $\mathbb{Q}(\zeta_7)$ with type $\{1, 2, 3\}$, which has trivial stabiliser in $(\mathbb{Z}/7)^\times$; so A is simple. Let z_1, z_2, z_3 be the eigen-coordinates dual to $dx/y^6, dx/y^5, dx/y^4$. On the closed cycle $\gamma =$ (lift of $[0, 1]$ to the sheet $y > 0$) minus (its image under $y \mapsto \zeta y$),

$$\int_\gamma \frac{dx}{y^{7-j}} = (1 - \zeta^j)\, B(\tfrac{j}{7}, \tfrac{j}{7}) =: P_j \qquad (j = 1, 2, 3).$$

Let $E = E_{-7}$ be an elliptic curve over $\overline{\mathbb{Q}}$ with CM by $\mathbb{Q}(\sqrt{-7})$; by Chowla–Selberg [Lit] [10] it has a period $\Omega \in \overline{\mathbb{Q}}^\times B(\frac{1}{7}, \frac{2}{7})$. The Beta cocycle $B(x, y)B(x + y, z) = B(y, z)B(x, y + z)$ at $(\frac{1}{7}, \frac{1}{7}, \frac{2}{7})$ and one reflection give, exactly,

$$B(\tfrac{1}{7}, \tfrac{1}{7})B(\tfrac{2}{7}, \tfrac{2}{7}) = B(\tfrac{1}{7}, \tfrac{2}{7})B(\tfrac{1}{7}, \tfrac{3}{7}), \qquad B(\tfrac{1}{7}, \tfrac{3}{7}) = \mu\, B(\tfrac{3}{7}, \tfrac{3}{7}),$$

$$\mu = \frac{\Gamma(\frac{1}{7})\Gamma(\frac{6}{7})}{\Gamma(\frac{3}{7})\Gamma(\frac{4}{7})} = \frac{\sin(3\pi/7)}{\sin(\pi/7)} = 1 + 2\cos\tfrac{2\pi}{7},$$

so $P_1P_2 \in \overline{\mathbb{Q}}^\times P_3\Omega$, a relation (5) with $a = z_1$, $b = z_2$, $c = z_3$, $d = E$. (In Koblitz–Ogus labels: $\{1, 2, 3\} + \{1, 4, 5\} = \{1, 3, 5\} + \{1, 2, 4\}$; the label $\{1, 2, 4\}$ is induced from $\mathbb{Q}(\sqrt{-7})$.) No use of Deligne's theorem is needed at this level.

Theorem 4.6 (Theorem X7)**.** *Let $v_1, v_2, v_3 \in \Lambda(A)$, $u \in \mathcal{L}_E$, $q_1, q_2 \in \overline{\mathbb{Q}}$ and $\beta \in \overline{\mathbb{Q}}^\times$. If*

$$\big(z_1(v_1) + u + q_1\big) \cdot \beta P_3 = \big(z_3(v_3) + z_2(v_2) + q_2\big) \cdot P_1,$$

then $q_1 = q_2 = 0$ and $\exp(v_1), \exp(v_2), \exp(v_3)$, $\exp_E(u)$ are torsion points.

This is Theorem 4.2 (with Ω normalised so that $P_1P_2 = \mu' P_3\Omega$ and $\mu' \in K$); here n is a multiple of 6 and the margin is n. Since z_1, z_2, z_3 are all the first-kind slots of A, Proposition 4.1 for these slots already follows from Theorem 2.3 ($\alpha_K(A \times E \times \mathbb{G}_{\mathrm{a}}^2) = 3 + 1$, $\kappa = 2$; Remark 4.5, Remark 2.4).

Corollary 4.7. (a) *For every non-torsion point* $\exp(v) \in A(\overline{\mathbb{Q}})$*:* $z_2(v)/z_1(v) \notin \overline{\mathbb{Q}} \cdot P_3/P_1 = \overline{\mathbb{Q}} \cdot B(\frac{3}{7}, \frac{3}{7})/B(\frac{1}{7}, \frac{1}{7})$*. In particular Corollary 1.1(a) holds.*

(b) *For every non-torsion* $u \in \mathcal{L}_E$ *and every* $v \in \Lambda(A)$*:* $z_3(v)/u \notin \overline{\mathbb{Q}}^{\times} \cdot P_3/P_1$*.*

(c) P_1P_2*,* P_1P_3*,* P_1*,* P_3 *are* $\overline{\mathbb{Q}}$*-linearly independent; equivalently, Corollary 1.1(b).*

That (c) is not a consequence of the analytic subgroup theorem is [Heuristic] (Remark 4.8).

Proof. (a) Data $v_1 = v_2 = v$, $v_3 = 0$, $u = 0$, $q_1 = q_2 = 0$, $\beta = \gamma$: if $z_2(v) = \gamma\, z_1(v) P_3/P_1$ then the hypothesis of Theorem 4.6 holds, so $\exp(v)$ is torsion. The values 0 and ∞ of the ratio do not occur for $v \neq 0$: $z_j(v) = 0$ forces $v = 0$ by Wüstholz on the simple A. For $O, Q \in C(\overline{\mathbb{Q}})$, z_j of the Abel–Jacobi logarithm of $Q - O$ is $\int_O^Q$ of the dual differential. (b) Data $v_3 = v$, u, $\beta = \gamma \in \overline{\mathbb{Q}}^{\times}$, all others 0. (c) Substituting (5) into $\alpha_1 P_1P_2 + \alpha_2 P_1P_3 + \alpha_3 P_1 + \alpha_4 P_3 = 0$ gives $(\alpha_2 P_1 + \alpha_1\mu'\Omega + \alpha_4)\cdot P_3 = (-\alpha_3)\cdot P_1$. So, with $\beta = 1$ and $K \ni \alpha_i$, $v = (\alpha_2 P_1 + \alpha_1\mu'\Omega + \alpha_4,\ -\alpha_3) \in V \cap \Psi(\Lambda_K(G))$, and Proposition 4.1 gives $v = (xP_1 + y\Omega,\ xP_3 + yP_2/\mu')$ with $x, y \in K$. The $\overline{\mathbb{Q}}$-independence of $1, P_2, P_3$ (W) gives $x = y = \alpha_3 = 0$, and then that of $1, P_1, \Omega$ gives $\alpha_1 = \alpha_2 = \alpha_4 = 0$. Dividing the four numbers by P_1 gives $P_2, P_3, 1, P_3/P_1$, which are $B(\frac{2}{7}, \frac{2}{7}), B(\frac{3}{7}, \frac{3}{7}), 1, B(\frac{3}{7}, \frac{3}{7})/B(\frac{1}{7}, \frac{1}{7})$ up to factors in $\mathbb{Q}(\zeta_7)^{\times}$. The Gamma form follows from $B(\frac{1}{7}, \frac{1}{7}) = \Gamma(\frac{1}{7})^2/\Gamma(\frac{2}{7})$ (the definition) and, by the reflection formula, $B(\frac{2}{7}, \frac{2}{7}) = \Gamma(\frac{2}{7})^2\Gamma(\frac{3}{7})\sin(\frac{3\pi}{7})/\pi$ and $B(\frac{3}{7}, \frac{3}{7}) = \Gamma(\frac{3}{7})^2\Gamma(\frac{1}{7})\sin(\frac{\pi}{7})/\pi$, after multiplying by π. □

No hidden symmetry. C is hyperelliptic: $(2x-1)^2 = 1 - 4y^7$, and $x \mapsto 1 - x$ is the hyperelliptic involution, acting as $[-1]$ on A and multiplying z_1, z_2, z_3 by -1 together. $\mathrm{Aut}(C) \cong \mathbb{Z}/14$ and $\mathrm{End}^0(A) = \mathbb{Q}(\zeta_7)$ acts diagonally, $z_j([x]v) = \sigma_j(x) z_j(v)$; so no symmetry links z_1, z_2, z_3 of one point in a way that could produce P_3/P_1 in (a).

4.5. **Remarks: linearisation and other relations.**

Remark 4.8 (Not linearisations)**.** [Heuristic] By an abelian analogue of Laurent's identity (4), which we do not reproduce, each $\ell_j := \pi z_j(v)/P_j$ is a $\overline{\mathbb{Q}}$-combination of 1-periods; it linearises only products $P_{\bar\sigma} z_\sigma$ of a period with the logarithm of the conjugate slot on the same variety. Dividing (6) by P_aP_c and using (5) gives

$$\ell_c - \beta\ell_a + \frac{P_d}{P_a}\,(\mu\ell_b - \beta\ell_d) + \pi\Big(\frac{q_2}{P_c} - \frac{\beta q_1}{P_a}\Big) = 0,$$

where the ratio P_d/P_a of periods of two different slots multiplies a combination of 1-periods. Unless $\mu\ell_b = \beta\ell_d$ this is not a $\overline{\mathbb{Q}}$-linear relation among 1-periods, so the analytic subgroup theorem does not apply as it does to (4). For (c): $P_3/P_1 \in \overline{\mathbb{Q}}^{\times} P_3\overline{P_1}/\pi$ has Koblitz–Ogus function $\mathbf{1}_{\{1,3,5\}} - \mathbf{1}_{\{1,2,3\}}$, which takes three values, while every slot period and π have functions with at most two values; under the Rohrlich–Lang (or period) conjecture, P_3/P_1 is not in the $\overline{\mathbb{Q}}$-span of 1-periods, and Wüstholz's theorem does not decide (c). Chudnovsky's theorem gives the algebraic independence of π and $\Omega_{-7} \sim \Gamma(\frac{1}{7})\Gamma(\frac{2}{7})/\Gamma(\frac{3}{7})$, which does not suffice, since (a)–(c) involve P_1, P_2, P_3. By the period conjecture (the Mumford–Tate group of A is a 4-dimensional torus and Ω adds no transcendence degree), Theorem 4.6 is a consequence of the period conjecture.

Remark 4.9 (No Gamma-monomial multiplier)**.** [Computed] In the dictionary of Koblitz and Ogus (§3.1), a product of values $\Gamma(a/N)$ and of a power of π corresponds to a $\mathbb{Q}$-valued function

on $(\mathbb{Z}/N)^\times$, with $\Gamma(a/N) \mapsto (t \mapsto 1 - \langle ta/N\rangle)$ and $\pi \mapsto 1$, products going to sums; by Deligne's theorem, products with the same function have an algebraic ratio. The numbers 1, π and the eigen-periods, of either kind, attached to the CM types of $\mathbb{Q}(\zeta_N)$ correspond (up to $\overline{\mathbb{Q}}^\times$) to the functions 0, 1 and the indicators of the half-sets $\Phi \subset (\mathbb{Z}/N)^\times$ with $\Phi \sqcup (-\Phi) = (\mathbb{Z}/N)^\times$. An exact computation (a Python script of about fifty lines) checks, for each of the following lists, that there is no function g on $(\mathbb{Z}/N)^\times$ such that $g+f$ is one of these functions for every member f of the list:

- 1, $B(\frac{2}{7},\frac{2}{7})$, $B(\frac{3}{7},\frac{3}{7})$, $B(\frac{3}{7},\frac{3}{7})/B(\frac{1}{7},\frac{1}{7})$ (Corollary 4.7(c));
- 1, $B(\frac{4}{9},\frac{4}{9})$, $B(\frac{2}{9},\frac{2}{9})$, $B(\frac{2}{9},\frac{2}{9})/B(\frac{1}{9},\frac{1}{9})$ (Corollary 4.13(2));
- 1, ϖ, $B(\frac{1}{24},\frac{1}{12})$, $B(\frac{1}{24},\frac{1}{12})/B(\frac{1}{24},\frac{1}{24})$ (Corollary 4.13(4), with $\varpi = \frac{1}{2}B(\frac{1}{4},\frac{1}{2})$);
- the six numbers of the example after Theorem 4.14 ($N = 15$, $\kappa = 4$).

Since each list contains 1, g itself must be one of these functions, so the search is finite. As controls, the list 1, r, $P_{\Phi'}$ of a single pair at $N = 15$ and the list 1, P_3/P_1 at $N = 7$ do admit such a g. The classical Gamma identities (reflection, multiplication, the Beta cocycle) preserve these functions, so no Gamma monomial can be shown by them to turn all the numbers of one of these lists into $\overline{\mathbb{Q}}^\times$-multiples of 1, π or CM eigen-periods at once; under the Rohrlich–Lang conjecture no Gamma monomial does [Heuristic]. So no reduction of this kind to Wüstholz's theorem or to the period theorem for 1-periods of Huber and Wüstholz [11] can be obtained from the classical Gamma identities; under the Rohrlich–Lang conjecture none exists [Heuristic]. This is more than Remark 4.8, which concerns P_3/P_1 alone.

Remark 4.10 (Where first-kind relations occur)**.** *Elliptic slots.* Among CM elliptic slots (any number of non-isogenous curves) the relations of Deligne's theorem are only the Legendre relations [Proved]: the indicators are $(1\pm\chi_k)/2$ with χ_k odd quadratic characters, which are linearly independent. (That there are no other relations is the Rohrlich–Lang conjecture, [Heuristic].) So for products of CM elliptic curves, the two-curve statements of [16, §6] (two Legendre relations on a line) are the only constructions of this kind.

Labels. [Proved] (elementary combinatorics). For CM types of $\mathbb{Q}(\zeta_N)$ containing 1, a first-kind partner of (Φ_a, Φ_b) exists iff Φ_a, Φ_b differ on at least two $\pm$-pairs: with $\Delta = \Phi_a \triangle \Phi_b$, the partners are $(\Phi_a \cap \Phi_b) \cup X$ and $(\Phi_a \cap \Phi_b) \cup (\Delta \setminus X)$, X one element from each $\pm$-pair in Δ, $X \notin \{\Phi_a \setminus \Phi_b, \Phi_b \setminus \Phi_a\}$. Hence first-kind relations among labels exist at every level N with $\varphi(N)/2 \geq 3$. The passage from labels to periods is Deligne's theorem [Lit] (§3.1 for Beta values; in general [6]).

Examples. The smallest levels are $N = 7$ and $N = 9$, with one relation each: at $N = 7$ the instance above; at $N = 9$ one with slots of a simple CM 3-fold with CM by $\mathbb{Q}(\zeta_9)$ and of E_ρ, $B(\frac{1}{9},\frac{1}{9})B(\frac{1}{9},\frac{4}{9}) = 3^{1/3}\frac{\sin(2\pi/9)}{\sin(\pi/9)}B(\frac{1}{9},\frac{2}{9})B(\frac{1}{3},\frac{1}{3})$ ([Proved], Lemma 4.11, from Gauss's multiplication formula and the reflection formula). At $N = 24$ there is one whose slots are two CM elliptic curves (CM by $\mathbb{Q}(\sqrt{-6})$ and $\mathbb{Q}(i)$) and two slots of a simple CM 4-fold: $B(\frac{1}{24},\frac{1}{24})B(\frac{1}{24},\frac{1}{4}) \in \overline{\mathbb{Q}}^\times B(\frac{1}{24},\frac{1}{12})B(\frac{1}{24},\frac{1}{8})$ ([Proved], Lemma 4.11). Both give instances of Theorem 4.2 (§4.6).

4.6. **Levels 9 and 24, and several pairs.**

Lemma 4.11. (1) $B(\frac{1}{9},\frac{1}{9})\,B(\frac{1}{9},\frac{4}{9}) = 3^{1/3}\,\frac{\sin(2\pi/9)}{\sin(\pi/9)}\,B(\frac{1}{9},\frac{2}{9})\,B(\frac{1}{3},\frac{1}{3})$, *and* $B(\frac{1}{9},\frac{1}{9})\,B(\frac{4}{9},\frac{4}{9}) = 16\sin(\frac{\pi}{9})\sin^3(\frac{2\pi}{9})\,B(\frac{2}{9},\frac{2}{9})\,B(\frac{1}{3},\frac{1}{3})$.

(2) $B(\frac{1}{24},\frac{1}{24})\,B(\frac{1}{24},\frac{1}{4}) = \mu_{24}\,B(\frac{1}{24},\frac{1}{12})\,B(\frac{1}{24},\frac{1}{8})$, *where*

$$\mu_{24} = 2^{-1/12}\,3^{1/8}\,\frac{\sin(7\pi/24)\sin(9\pi/24)}{\sin(10\pi/24)\sin(6\pi/24)}.$$

Both sides of each identity also agree numerically to 60 digits [Numerical].

Proof. We use Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$ and Gauss's multiplication formulas $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$ and $\Gamma(z)\Gamma(z+\frac{1}{3})\Gamma(z+\frac{2}{3}) = 2\pi\,3^{1/2-3z}\Gamma(3z)$.

(1) With $g_k = \Gamma(k/9)$ and $s_k = \sin(k\pi/9)$, the case $m=3$ at $z = 1/9, 2/9$ gives $g_1g_4g_7 = 2\pi\,3^{1/6}g_3$ and $g_2g_5g_8 = 2\pi\,3^{-1/6}g_6$, and reflection gives $g_1g_8 = \pi/s_1$, $g_2g_7 = \pi/s_2$. Hence $g_4 = 2\cdot 3^{1/6}s_2\,g_2g_3/g_1$ and $g_5 = 2\cdot 3^{-1/6}s_1\,g_1g_6/g_2$, and the first ratio equals $g_1^2g_4g_6/(g_2^2g_3g_5) = 3^{1/3}s_2/s_1$. The second identity follows from the first combined with $B(\frac{1}{9},\frac{2}{9})/B(\frac{2}{9},\frac{2}{9}) = 2\cdot 3^{1/6}s_2$ and $B(\frac{1}{9},\frac{4}{9})/B(\frac{4}{9},\frac{4}{9}) = s_4/s_1$ (consequences of the same formulas) and $8s_1s_2s_4 = \sqrt{3}$.

(2) With $g_k = \Gamma(k/24)$ the left side divided by $B(\frac{1}{24},\frac{1}{12})B(\frac{1}{24},\frac{1}{8})$ is $g_1g_4g_6/(g_2^2g_7)$. Write R_k for reflection at $z = k/24$, D_z for duplication at z and T_z for triplication at z. The combination $R_6 - R_7 - R_9 + R_{10} - D_{2/24} + D_{3/24} + T_{1/24} - T_{2/24}$ (exponents of the g_k added) has exponent vector $e_1 = e_4 = e_6 = 1$, $e_2 = -2$, $e_7 = -1$, all others 0; its constants multiply to μ_{24} (the powers of π cancel; the powers of 2 give $2^{-5/6+3/4} = 2^{-1/12}$, those of 3 give $3^{3/8-1/4} = 3^{1/8}$). □

The slots [Lit] (Koblitz–Ogus labels [12]; §3.1; Chowla–Selberg [10]). At level 9, $B(\frac{1}{9},\frac{1}{9})$, $B(\frac{2}{9},\frac{2}{9})$, $B(\frac{4}{9},\frac{4}{9})$ are, up to factors in $\mathbb{Q}(\zeta_9)^\times$, periods of three distinct first-kind eigen-differentials dx/y^8, dx/y^7, dx/y^5 of the curve $C_9\colon y^9 = x(1-x)$ (genus 4), whose Jacobian is isogenous to $A_9 \times E_\rho$ with A_9 a simple CM threefold (CM by $\mathbb{Q}(\zeta_9)$, type $\{1,2,4\}$), and $B(\frac{1}{3},\frac{1}{3}) \in \overline{\mathbb{Q}}^\times\Omega_\rho$ is the period of dx/y^6, which comes from E_ρ. At level 24, $B(\frac{1}{24},\frac{1}{24}) \in \overline{\mathbb{Q}}^\times\Omega_{-6}$ and $B(\frac{1}{24},\frac{1}{4}) \in \overline{\mathbb{Q}}^\times\varpi$ are periods of CM curves E_{-6}, E_i with CM by $\mathbb{Q}(\sqrt{-6})$ and $\mathbb{Q}(i)$, and $B(\frac{1}{24},\frac{1}{12})$, $B(\frac{1}{24},\frac{1}{8})$ are, up to $\overline{\mathbb{Q}}^\times$, periods P_c, P_d of two distinct first-kind eigen-coordinates z_c, z_d of a simple CM fourfold A_{24} (CM by $\mathbb{Q}(\zeta_{24})$). By Lemma 4.11, $P_1P_4 \in \overline{\mathbb{Q}}^\times P_2\Omega_\rho$ and $\Omega_{-6}\varpi \in \overline{\mathbb{Q}}^\times P_cP_d$, so Theorem 4.2 applies to both quadruples.

Theorem 4.12 (Theorems X9 and X24)**.** (1) (Level 9.) *On A_9 let z_1, z_2, z_4 be the coordinates dual to the eigen-differentials dx/y^8, dx/y^7, dx/y^5, with periods $P_j \in \overline{\mathbb{Q}}^\times B(\frac{j}{9},\frac{j}{9})$. If $v_1, v_2, v_4 \in \Lambda(A_9)$, $u \in \mathcal{L}_{E_\rho}$, $q_1, q_2 \in \overline{\mathbb{Q}}$, $\beta \in \overline{\mathbb{Q}}^\times$ and $(z_1(v_1) + u + q_1)\beta P_2 = (z_2(v_2) + z_4(v_4) + q_2)P_1$, then $q_1 = q_2 = 0$ and all of $\exp(v_j)$, $\exp_{E_\rho}(u)$ are torsion.*

(2) (Level 24.) *If $u \in \mathcal{L}_{E_{-6}}$, $u' \in \mathcal{L}_{E_i}$, $v_c, v_d \in \Lambda(A_{24})$, $q_1, q_2 \in \overline{\mathbb{Q}}$, $\beta \in \overline{\mathbb{Q}}^\times$ and $(u + z_d(v_d) + q_1)\beta P_c = (z_c(v_c) + u' + q_2)\Omega_{-6}$, then $q_1 = q_2 = 0$ and all exponentials are torsion.*

Corollary 4.13. (1) *For every non-torsion point $\exp(v) \in A_9(\overline{\mathbb{Q}})$, the ratio $z_4(v)/z_1(v)$ is not in $\overline{\mathbb{Q}}\,B(\frac{2}{9},\frac{2}{9})/B(\frac{1}{9},\frac{1}{9})$.*

(2) *The numbers 1, $B(\frac{4}{9},\frac{4}{9})$, $B(\frac{2}{9},\frac{2}{9})$, $B(\frac{2}{9},\frac{2}{9})/B(\frac{1}{9},\frac{1}{9})$ are linearly independent over $\overline{\mathbb{Q}}$.*

(3) *For non-torsion $u \in \mathcal{L}_{E_{-6}}$ and every $u' \in \mathcal{L}_{E_i}$: $u'/u \notin \overline{\mathbb{Q}}^\times B(\frac{1}{24},\frac{1}{12})/B(\frac{1}{24},\frac{1}{24})$.*

(4) *The numbers 1, ϖ, $B(\frac{1}{24},\frac{1}{12})$, $B(\frac{1}{24},\frac{1}{12})/B(\frac{1}{24},\frac{1}{24})$ are linearly independent over $\overline{\mathbb{Q}}$.*

That (2) and (4) are not consequences of the analytic subgroup theorem is [Heuristic], as for Corollary 4.7(c) (Remark 4.8).

Proof of Theorem 4.12 and Corollary 4.13. As for Theorem 4.6 and Corollary 4.7, taking for (a,b,c,d) the slots (z_1, z_4, z_2, E_ρ), resp. (E_{-6}, E_i, z_c, z_d) in the relation (5) and in (6). The case analysis of Proposition 4.1 uses only that the four slots are distinct and of the first kind, the relation (5), and $\dim A'' = 2n$, which holds here ($3n/2+n/2$, resp. $n+n/2+n/2$, by Lemma 3.1). In (3), Theorem 4.12(2) makes u torsion, so no condition on u' is needed. □

At level 9 the slots z_1, z_2, z_4 are all the first-kind slots of A_9, so, as at level 7, Proposition 4.1 for them already follows from Theorem 2.3 ($\alpha_K = 3+1$, $\kappa = 2$; Remark 4.5). At level 24 only two of the four first-kind slots of A_{24} are used ($\alpha_K = 4+1+1$), so the printed weight on its

own group gives the bound $(6-2)\cdot 1 = 4$ and no conclusion; Theorem 2.3 with the inner weight applies (§2.4). Moreover its two pairs are two of the four pairs of the full family $T(11)$ at $N = 24$ of Theorem 4.14 below: in Koblitz–Ogus labels [12], $B(\frac{1}{24},\frac{1}{24}) \leftrightarrow \{1,5,7,11\} \mapsto B(\frac{1}{24},\frac{1}{12}) \leftrightarrow \{1,5,7,13\}$ ($E_{-6} \to A_{24}$) and $B(\frac{1}{24},\frac{1}{8}) \leftrightarrow \{1,5,11,17\} \mapsto B(\frac{1}{24},\frac{1}{4}) \leftrightarrow \{1,5,13,17\}$ ($A_{24} \to E_i$); the other two pairs are $\{1,7,11,19\} \mapsto \{1,7,13,19\}$ (A_{24} to a curve with CM by $\mathbb{Q}(\sqrt{-3})$) and $\{1,11,17,19\} \mapsto \{1,13,17,19\}$ (a curve with CM by $\mathbb{Q}(\sqrt{-2})$ to A_{24}). So it follows from item (iii) of Remark 2.4 by the slot-splitting step of §4.3: (Ch) moves a point of Λ_K of the level-24 group into Λ_K of the group of $T(11)$ with the same Ψ-image (after rescaling the second target by a constant in $\overline{\mathbb{Q}}^\times$, which keeps it an instance), Theorem 4.14 applies, and Wüstholz's theorem splits the slots, which kills the coefficients of the two unused pairs. This is the same kind of input as the Wüstholz and Masser steps used for the other direct instances. Hence Theorem 4.12(2) and Corollary 4.13(3) also follow from the printed form [23, Théorème (iii)] (with Wüstholz's theorem, and Deligne's theorem for the common slope of $T(11)$ as in the proof of Theorem 4.14), and Corollary 4.13(4) is part of the independence statement of Theorem 4.14 for $T(11)$ (there $r \in \overline{\mathbb{Q}}^\times B(\frac{1}{24},\frac{1}{12})/B(\frac{1}{24},\frac{1}{24})$, and the $P_{\Phi'}$ include $B(\frac{1}{24},\frac{1}{12})$ and $B(\frac{1}{24},\frac{1}{4}) \in \overline{\mathbb{Q}}^\times\varpi$). The same applies to every proper subfamily of Theorem 4.14. What is ours at level 24 is the elementary proof of the relation (Lemma 4.11(2)), which avoids Deligne's theorem.

Theorem 4.14 (Theorem X$^\kappa$: several pairs)**.** *Let $N \geq 3$, $G_N = (\mathbb{Z}/N)^\times$ of order $2g$, $\tau \in G_N \setminus \{\pm 1\}$, and let $T(\tau)$ be the set of CM types $\Phi \subset G_N$ with $1, \tau \in \Phi$ (so $\kappa := |T(\tau)| = 2^{g-2}$). For $\Phi \in T(\tau)$ put $\Phi' = (\Phi \setminus \{\tau\}) \cup \{-\tau\}$, and let P_Φ, $P_{\Phi'}$ be periods of the corresponding first-kind slots (the eigen-slot at the embedding 1 of a CM abelian variety with CM by $\mathbb{Q}(\zeta_N)$ of type Φ, resp. Φ'). Then the 2κ slots are distinct, there are $\beta_\Phi \in \overline{\mathbb{Q}}^\times$ and r with $\beta_\Phi P_{\Phi'} = rP_\Phi$ for all Φ, and with $G = \mathbb{G}_{\mathrm{a}}^2 \times \prod A$ (each isogeny class once), $\Psi = (\sum_\Phi z_\Phi + q_1,\ \sum_\Phi \beta_\Phi z_{\Phi'} + q_2)$, $V = \mathbb{C}(1,r)$ and every number field $K \supseteq \mathbb{Q}(\zeta_N)$ containing the β_Φ:*

$$V \cap \Psi(\Lambda_K(G)) = \bigoplus_{\Phi \in T(\tau)} K\, P_\Phi(1,r).$$

Consequently 1, r and the κ numbers $P_{\Phi'}$ are linearly independent over $\overline{\mathbb{Q}}$. The same holds for every subfamily of $T(\tau)$.

Proof. Distinctness. $\Phi \ni \tau$, $\Phi' \not\ni \tau$, and $\Phi \mapsto \Phi'$ is injective, so the 2κ labels are distinct; distinct labels give distinct slots, since $A_{\Phi_2} \sim A_\Phi$ forces $\Phi_2 = \Phi u$ and then the 1-eigenline of A_{Φ_2} is the $u^{\mp 1}$-eigenline of A_Φ, another embedding unless $u \in \operatorname{Stab}\Phi$. For an induced type the slot lives on the simple factor B with CM field $F_0 = \mathbb{Q}(\zeta_N)^{\operatorname{Stab}\Phi}$ and primitive type Φ_0 (the factors A of G are these simple factors, each isogeny class once). Isogenous simple factors have equal abelian CM fields $F_0 \subset \mathbb{Q}(\zeta_N)$, so $B_{\Phi_0'} \sim B_{\Phi_0}$ forces $\Phi_0' = \Phi_0 u_0$, and the embedding $1|_{F_0}$ of $B_{\Phi_0'}$ is $u_0^{\mp 1}|_{F_0}$ on B_{Φ_0}, which is $1|_{F_0}$ only if $u_0 \in \operatorname{Stab}\Phi_0 = 1$; so distinct types give distinct slots in this case too. *Common slope.* $\mathbf{1}_{\Phi'} - \mathbf{1}_\Phi = \delta_{-\tau} - \delta_\tau$ for all Φ, so $P_{\Phi_1'}P_{\Phi_2} \in \overline{\mathbb{Q}}^\times P_{\Phi_1}P_{\Phi_2'}$: for Fermat labels by Deligne's theorem [6] in the form of [12] (§3.1), and in general by the linearity in Φ of Shimura's period symbol, proved by Blasius [5] (via Deligne's theorem on absolute Hodge cycles) and, for abelian CM fields, by Anderson [1]. *The count.* The proof of Proposition 4.1 goes through with κ pairs: $G'' = \mathbb{G}_{\mathrm{a}}^2 \times A''$ with $\dim A'' = \kappa n$ (Lemma 3.1); the periods contribute κn to $\operatorname{rk}(Y'' \cap \ker\exp)$; one extra datum gives $m = (\kappa+1)n$; every slot occurs in one target only, and distinct slots (several on one A are distinct embeddings) carry independent coordinates on T_B (S1), so $\Psi''(T_B) \in \{0, \mathbb{C}(1,0), \mathbb{C}(0,1), \mathbb{C}^2\}$; and for an axis image ℓ, $V \cap (\ell + \Psi(\Omega_K(G))) = \bigoplus_\Phi KP_\Phi(1,r)$ (as in Lemma 4.4), whence $\lambda = n$. Condition (1) then reads $n + 2\kappa n \leq 2\kappa n$ for $G' = 0$, $n + 2(\kappa n - e) \leq \kappa n$ with $e \leq \kappa n/2$ for axis images, and $\lambda + 2\kappa n \leq \kappa n$ for a non-axis line in $\mathbb{G}_{\mathrm{a}}^2$: never (margin n). Condition (2): $2\kappa n + 2 \geq \kappa n + 4$.

Independence. A relation $a_0 + a_1 r + \sum \gamma_\Phi P_{\Phi'} = 0$ (K containing the coefficients) gives the point $(a_1, -a_0 - \sum \gamma_\Phi P_{\Phi'}) = a_1(1, r)$ of $V \cap \Psi(\Lambda_K(G))$, whose first coordinate must lie in $\sum K P_\Phi$; so $a_1 = 0$ by (W), and then $\gamma = 0$, $a_0 = 0$ by (W). □

Remark 4.15. For one pair ($\kappa = 1$, i.e. $g = 2$, or a one-pair subfamily) the independence statement follows from Wüstholz's theorem alone (multiply by $\pi/P_{\Phi'}$, which makes every member a 1-period); for $\kappa \geq 2$ we know no common Gamma-monomial multiplier that does this (Remark 4.9 excludes one for the example below), and we do not know a linear proof. (That there is none is [Heuristic], as in Remark 4.8.) For the whole family $T(\tau)$ the theorem already follows from the printed form [23, Théorème (iii)] [Proved]: every CM type contains exactly one of $\tau, -\tau$, so the 2κ types Φ, Φ' are all the CM types of $\mathbb{Q}(\zeta_N)$ containing 1, and the 2κ slots are all the first-kind slots of the simple varieties involved; hence $\alpha_K(G) = \sum_A \dim A = 2\kappa$, and Theorem 2.3 gives $\dim_K \leq (2\kappa - \kappa) \cdot 1 = \kappa$ (Remark 2.4). This covers the example below. For a proper subfamily the printed weight on its own group can be larger, but the subfamily follows from its full family $T(\tau)$ by the slot-splitting step of §4.3, and its independence statement is part of that of the full family (as at level 24, after the proof of Theorem 4.12).

Example 4.16. $N = 15$, $\tau = 2$: the four pairs $B(\frac{1}{15}, \frac{2}{15}) \mapsto B(\frac{4}{15}, \frac{4}{15})$, $B(\frac{1}{15}, \frac{3}{15}) \mapsto B(\frac{1}{15}, \frac{7}{15})$, $B(\frac{2}{15}, \frac{2}{15}) \mapsto B(\frac{2}{15}, \frac{6}{15})$, $B(\frac{1}{15}, \frac{1}{15}) \mapsto B(\frac{1}{3}, \frac{1}{3})$ form the whole family $T(2)$ ($\kappa = 4$; their common slope class was also checked exactly), so 1, $B(\frac{4}{15}, \frac{4}{15})/B(\frac{1}{15}, \frac{2}{15})$, $B(\frac{4}{15}, \frac{4}{15})$, $B(\frac{1}{15}, \frac{7}{15})$, $B(\frac{2}{15}, \frac{6}{15})$, $B(\frac{1}{3}, \frac{1}{3})$ are linearly independent over $\overline{\mathbb{Q}}$.

5. Strong elliptic four exponentials on eigen-slots

In this section the strong four exponentials statement is read on a CM curve or on one eigen-coordinate of a CM abelian variety instead of on $\mathbb{G}_\mathrm{m}$. Let E, E' be elliptic curves over $\overline{\mathbb{Q}}$ with complex multiplication by k, k', Ω, Ω' nonzero periods, $\eta = \eta^*(\Omega)$ and $\eta' = \eta^*(\Omega')$ the conjugate-eigen quasi-periods ($\Omega\eta, \Omega'\eta' \in \overline{\mathbb{Q}}^\times \pi$). Write $\widetilde{\mathcal{L}}_E = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}_E$, and Fib_E, $\mathrm{Fib}_{E'}$ for the sets of fibre values (§3.2; Fib_E contains $\overline{\mathbb{Q}}$, the values $\zeta^*(u)$, $u \in \mathcal{L}_E$, and η). Two CM curves are isogenous iff $k = k'$.

Theorem 5.1 (Theorems SE, SE$^\natural$)**.** *Let $c \in \overline{\mathbb{Q}}^\times$ and $y_1, y_2 \in \widetilde{\mathcal{L}}_E$ with Ω, y_1, y_2 linearly independent over $\overline{\mathbb{Q}}$.*

1. (SE) *If $k \neq k'$ and $t = c\Omega'/\Omega$, then ty_1, ty_2 are not both in $\overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}_E + \overline{\mathbb{Q}}\mathcal{L}_{E'}$.*
2. (SE$^\natural$) *If $k \neq k'$ and $t = c\eta'/\Omega$, then ty_1, ty_2 are not both in $\overline{\mathbb{Q}}\mathcal{L}_E + \overline{\mathbb{Q}}\mathrm{Fib}_{E'}$.*

The analogue at $t = c\eta/\Omega \in \overline{\mathbb{Q}}^\times \pi/\Omega^2$, with ty_1, ty_2 not both in $\overline{\mathbb{Q}}\mathcal{L}_E + \overline{\mathbb{Q}}\mathrm{Fib}_E$, is Theorem SE$_0$ of [16, Thm. 5.7], which also gives Corollary 5.2 below in the classes $\overline{\mathbb{Q}}^\times(\pi/\Omega^2)^{\pm 1}$ [16, Cor. 5.8].

Corollary 5.2 (Strong elliptic four exponentials)**.** *Let $k' \neq k$, and let $t \in \overline{\mathbb{Q}}^\times(\Omega'/\Omega)^{\pm 1}$ or $t \in \overline{\mathbb{Q}}^\times(\pi/\Omega\Omega')^{\pm 1}$. If $y_1, y_2 \in \widetilde{\mathcal{L}}_E$ are linearly independent over $\overline{\mathbb{Q}}$, then ty_1 and ty_2 are not both in $\widetilde{\mathcal{L}}_E$. In particular, if $u_1, u_2 \in \mathcal{L}_E$ are linearly independent over k, then $\wp_E(tu_1)$ and $\wp_E(tu_2)$ are not both algebraic.*

Lemma 5.3. *$\Omega' \notin \widetilde{\mathcal{L}}_E$ and $\eta' \notin \widetilde{\mathcal{L}}_E$ if $k' \neq k$; in particular $t \notin \overline{\mathbb{Q}}$ in both cases of Theorem 5.1.*

Proof. [Lit: Wüstholz [37]] If $\Omega' = \beta_0 + \sum \beta_i \ell_i$ with $\ell_i \in \mathcal{L}_E$, the vector $(1, \ell, \Omega')$ of $\mathrm{Lie}(\mathbb{G}_\mathrm{a} \times E^r \times E')$ lies in a $\overline{\mathbb{Q}}$-hyperplane; the smallest algebraic subgroup through it is $H \times B \times E'$ ($\mathrm{Hom}(E, E') = 0$), whose tangent space is not in the hyperplane. For η' use $\mathbb{G}_\mathrm{a} \times E^r \times E'^\natural$: the smallest subgroup contains $0 \times 0 \times E'^\natural$ (its image in $\mathbb{G}_\mathrm{a} \times E'^\natural$ maps onto E' and has $\mathbb{G}_\mathrm{a}$-coordinate 1, so it is $\mathbb{G}_\mathrm{a} \times E'^\natural$; the kernel part cannot mix since $\mathrm{Hom}(E', E) = 0$ and $\mathrm{Hom}(\mathbb{G}_\mathrm{a}, E^r) = 0$), and the fibre direction is not in the hyperplane. □

Proof of Corollary 5.2. The statement is invariant under $t \mapsto \gamma t$ ($\gamma \in \overline{\mathbb{Q}}^\times$) and under $t \mapsto 1/t$ (replace y_j by ty_j), and $\overline{\mathbb{Q}}^\times \pi/\Omega\Omega' = \overline{\mathbb{Q}}^\times \eta'/\Omega$. If $\Omega \in \overline{\mathbb{Q}}y_1 + \overline{\mathbb{Q}}y_2$ then $t\Omega \in \widetilde{\mathcal{L}}_E$, i.e. Ω' or η' lies in $\widetilde{\mathcal{L}}_E$; this is excluded by Lemma 5.3. So Theorem 5.1 applies, since $\widetilde{\mathcal{L}}_E$ lies in each of its targets ($\overline{\mathbb{Q}} \subset \mathrm{Fib}_{E'}$). For the last sentence: $\wp(z) \in \overline{\mathbb{Q}} \cup \{\infty\}$ means $z \in \mathcal{L}_E$, and k-independent elliptic logarithms of a CM curve are $\overline{\mathbb{Q}}$-independent (Masser [18]). □

Proof of Theorem 5.1. Let $K \supseteq kk'(c)$ contain all coefficients; then $n = [K : \mathbb{Q}] \geq 4$. Let θ be a $\mathbb{Q}$-basis of K, and $\psi(m) = \sum \theta_j m_j$, $\phi(m) = \sum \theta_j \bar{m}_j$ on k^n, with kernels of k-dimension $n/2$ (Lemma 3.2). By Lemma 5.3, $t \notin K$.

(1) $G = \mathbb{G}_\mathrm{a} \times E \times \mathbb{G}_\mathrm{a} \times E \times E'$, $\Psi = (a_1 + z_1,\, a_2 + z_2 + z')$, $V = \mathbb{C}(1, t)$. Then $G'' = \mathbb{G}_\mathrm{a}^2 \times A_1 \times A_2 \times A'$ with each abelian factor of dimension $n/2$ ($d_0 = 2$, $d_1 = 0$, $d_2 = 3n/2$); the data are $w = (\Omega, c\Omega')$ and $v_j = (y_j, ty_j)$, so $m = 3n$ and $\kappa = n$ (a period in Y'' has first coordinate in $K\Omega$). Connected subgroups are $H \times B_{12} \times B'$ ($\mathrm{Hom}(E, E') = 0$); for $B_{12} \subset A_1 \times A_2$ the image $\Psi''(T_{B_{12}})$ is K-rational, and $\dim B_{12} \leq n/2$ when it is a line. For $r' = \dim \Psi''(T_{G'}) = 1$, (1) reads $\mathrm{rk}\, Y_{G'} + 2 \dim B \geq 4n$ (B the abelian part of G'), and for $G' = 0$ it reads $6n - \kappa \leq 2(3n - \kappa)$, i.e. $n \leq 0$ since $\kappa = n$: excluded, with margin n. Image $\mathbb{C}(0, 1)$ gives $\mathrm{rk}\, Y_{G'} \leq n$ and $\dim B \leq n$; a non-K-rational line from H gives $\mathrm{rk}\, Y_{G'} \leq 3n$, $B = 0$; a K-rational line $\ell \neq \mathbb{C}(0, 1)$ gives $\dim B \leq n/2$ and $\mathrm{rk}\, Y_{G'} \leq 2n$, because $v_j \in \ell + \Psi(\Omega_K(G))$ for $j = 1, 2$ would give $y_j \in K\Omega + K\Omega^2/\Omega'$ if $\ell = \mathbb{C}(1, 0)$, and $y_j(1 - pt) \in K\Omega + K\Omega' = \Omega(K + Kt)$ if $\ell = \mathbb{C}(p, 1)$ (the period datum is $\Omega' = t\Omega/c$, and $1 - pt \neq 0$ as $t \notin K$); either way Ω, y_1, y_2 would be K-dependent. No subgroup satisfies (1); the minimum margin is n. Condition (2): $3n + 2 \geq 4 + n$.

(2) This is the case $g = g' = 1$ of Theorem 5.5 (EA$^\natural$) below: take $A = E$, $\sigma = \mathrm{id}$, $A' = E'$, and for $\tilde{\sigma}$ the conjugate embedding of k', read on $E'^\natural$ through its conjugate-eigen fibre coordinate w' (§3.2). Then $P' = \eta'$, $z_\sigma(\Lambda(E)) = \mathcal{L}_E$, $w_{\tilde{\sigma}}(\Lambda(E'^\natural)) = \mathrm{Fib}_{E'}$ and $\widetilde{\mathcal{L}}_\sigma = \widetilde{\mathcal{L}}_E$. (Likewise (1) is the case $g = g' = 1$ of Theorem 5.5 (EA), so the proof of Theorem 5.5 gives a second proof of (1).) □

Example 5.4. With $E = E_i$ (period ϖ) and $E' = E_\rho$ (period Ω_ρ): for every $u \in \mathcal{L}_{E_i} \setminus \{0\}$, e.g. the logarithm of $(2, \sqrt{24})$, the numbers $\wp_i(\Omega_\rho/\varpi)$ and $\wp_i(\Omega_\rho u/\varpi)$ are not both algebraic. This is Corollary 5.2 in the class Ω'/Ω, with $(y_1, y_2) = (1, u)$, which are $\overline{\mathbb{Q}}$-independent since $u \neq 0$ is transcendental; it is a consequence of Theorem 5.1(1).

Provenance. Theorem 5.1(1) is a direct instance of [23, Théorème (iii)] as printed (Remark 2.4); the proof above applies Theorem 2.1 directly and shows that the tight subgroups have margin n. For (2) the printed weight gives no conclusion on the group used here, and the proof uses the inner weight. As far as we found, neither (1) nor (2) is stated in print; for spans $\overline{\mathbb{Q}}y_1 + \overline{\mathbb{Q}}y_2$ that are not stable under complex conjugation, Corollary 5.2 also follows from the elliptic strong six exponentials theorem [23, Cor. 1] on $\mathbb{G}_\mathrm{a} \times E$ and complex conjugation, since $\widetilde{\mathcal{L}}_E$ is conjugation-stable for CM E (§9.1).

Eigen-slots. Let $A/\overline{\mathbb{Q}}$ be simple with CM by F and primitive type Φ, $\sigma \in \Phi$, $P_\sigma \neq 0$ a period of z_σ, and $\widetilde{\mathcal{L}}_\sigma = \overline{\mathbb{Q}} + \overline{\mathbb{Q}} z_\sigma(\Lambda(A))$.

Theorem 5.5 (Theorems EA, EA$^\natural$)**.** *Let (A', σ') be another first-kind slot ($A' = A$, $\sigma' \neq \sigma$ allowed), resp. $(A', \tilde{\sigma})$ a second-kind slot ($\tilde{\sigma} \notin \Phi'$, read on $A'^\natural$ through $w_{\tilde{\sigma}}$; $A' = A$ allowed), where an A' isogenous to A is replaced by A (this rescales P' and the target by $\overline{\mathbb{Q}}^\times$); then either $A' = A$ or $\mathrm{Hom}(A, A') = 0$, and we require $\sigma' \neq \sigma$ under the induced identification of the F-actions), with period P', and $t = cP'/P_\sigma$, $c \in \overline{\mathbb{Q}}^\times$. If $y_1, y_2 \in \widetilde{\mathcal{L}}_\sigma$ and P_σ, y_1, y_2 are linearly independent over $\overline{\mathbb{Q}}$, then ty_1, ty_2 are not both in $\widetilde{\mathcal{L}}_\sigma + \overline{\mathbb{Q}} z_{\sigma'}(\Lambda(A'))$, resp. in $\overline{\mathbb{Q}} z_\sigma(\Lambda(A)) + \overline{\mathbb{Q}} w_{\tilde{\sigma}}(\Lambda(A'^\natural))$. Consequently, for $t \in \overline{\mathbb{Q}}^\times (P'/P_\sigma)^{\pm 1}$ and $\overline{\mathbb{Q}}$-independent $y_1, y_2 \in \widetilde{\mathcal{L}}_\sigma$, ty_1, ty_2 are not both in $\widetilde{\mathcal{L}}_\sigma$.*

The slot convention matters: if A' is isogenous to A but not equal to it, then without the convention t can be algebraic and the statement fails.

Proof. We apply Theorem 2.3 with the upper bound for the inner weight of §2.3. Let K be a number field containing the Galois closures of F and F', the number c and all coefficients needed below, with $n = [K : \mathbb{Q}] \geq 4$, and let θ be a basis of $K/\mathbb{Q}$. For EA let $G = \mathbb{G}_{\mathrm{a}} \times A \times \mathbb{G}_{\mathrm{a}} \times A \times A'$ and $\Psi = (a_1 + z_\sigma,\, a_2 + z_\sigma + z_{\sigma'})$, where the first z_σ is read on the first factor A, the second on the second factor A, and $z_{\sigma'}$ on A' (if $A' = A$, the last factor is a third copy of A). For EA$^\natural$ let $G = \mathbb{G}_{\mathrm{a}} \times A \times A \times A'^\natural$ and $\Psi = (a + z_\sigma,\, z_\sigma + w_{\tilde\sigma})$, read in the same way. Let $V = \mathbb{C}(1, t)$ and $W = 0$.

The condition on V. P' and P_σ are periods of distinct eigen-slots (by the slot convention), so $t \notin \overline{\mathbb{Q}}$ by (W). Hence V has no nonzero $\overline{\mathbb{Q}}$-rational point, and Theorem 2.3 applies.

The data. $\Lambda_K(G)$ is the product of the K-spans of the logarithms of the factors, and $\Lambda_K(\mathbb{G}_{\mathrm{a}}) = \overline{\mathbb{Q}}$. So $\Psi(\Lambda_K(G))$ is $(\overline{\mathbb{Q}} + Kz_\sigma(\Lambda(A))) \times (\overline{\mathbb{Q}} + Kz_\sigma(\Lambda(A)) + Kz_{\sigma'}(\Lambda(A')))$ for EA and $(\overline{\mathbb{Q}} + Kz_\sigma(\Lambda(A))) \times (Kz_\sigma(\Lambda(A)) + Kw_{\tilde\sigma}(\Lambda(A'^\natural)))$ for EA$^\natural$. Suppose that y_1, y_2 are as in the statement and that ty_1, ty_2 lie in the target. Since K contains the coefficients, $v_j = (y_j, ty_j)$ lies in $V \cap \Psi(\Lambda_K(G))$ for $j = 1, 2$. The period datum $w = (P_\sigma, cP')$ lies in $V \cap \Psi(\Omega_K(G))$, since $P_\sigma = z_\sigma(\gamma_0)$ for a period γ_0 of A, $P' = z_{\sigma'}(\gamma_0')$ for a period γ_0' of A' (resp. $P' = w_{\tilde\sigma}(\tilde\gamma_0')$ for a period $\tilde\gamma_0'$ of $A'^\natural$, §3.1), and $c \in K$. So $\kappa \geq 1$. The first coordinates P_σ, y_1, y_2 of w, v_1, v_2 are linearly independent over $\overline{\mathbb{Q}}$, hence over K, so $\dim_K(V \cap \Psi(\Lambda_K(G))) \geq 3$.

The inner weight. By Lemma 3.1, $\ker \psi_\sigma \subset F^n$ has F-dimension $n - n/(2g)$, so $B_{\ker \psi_\sigma} \subset A^n$ has dimension $gn - n/2$, and by (S1) the form $\theta^{\mathsf{T}} z_\sigma$ vanishes on its tangent space; likewise for σ' on A'^n. For EA put

$$U = \{\theta \cdot a_1 = 0\} \times B_{\ker \psi_\sigma} \times \{\theta \cdot a_2 = 0\} \times B_{\ker \psi_\sigma} \times B_{\ker \psi_{\sigma'}}.$$

Then U is connected, $T_U \subset \ker \Psi_\theta$, and $G^n/U = \mathbb{G}_{\mathrm{a}} \times A_1'' \times \mathbb{G}_{\mathrm{a}} \times A_2'' \times A'''$ with three abelian varieties of dimension $n/2$; it has dimension $3n/2 + 2$ and surjects onto $\mathbb{G}_{\mathrm{a}}^2$, so $\alpha(G^n/U) \leq 3n + 4 - 4 = 3n$. For EA$^\natural$, let $t_{\tilde\sigma}$ be the $\tilde\sigma$-coordinate of the fibre of $(A'^\natural)^n$ and put

$$U = \{\theta \cdot a = 0\} \times B_{\ker \psi_\sigma} \times B_{\ker \psi_\sigma} \times \big(B'^\natural_{\ker \psi_{\tilde\sigma}} + \{\theta^{\mathsf{T}} t_{\tilde\sigma} = 0\}\big).$$

By (F1) and (F4), $\theta^{\mathsf{T}} w_{\tilde\sigma}$ vanishes on the tangent space of the last factor, and the quotient Q'' of $(A'^\natural)^n$ by it is an extension of $A'^n/B'_{\ker \psi_{\tilde\sigma}}$ (dimension $n/2$) by $\mathbb{G}_{\mathrm{a}}$, of dimension $n/2 + 1$ (the fibre part $W_{B'}$ of $B'^\natural_{\ker \psi_{\tilde\sigma}}$, of dimension $g'n - n/2$, lies in $\{\theta^{\mathsf{T}} t_{\tilde\sigma} = 0\}$; compare [16, proof of Thm. 4.6]). So $G^n/U = \mathbb{G}_{\mathrm{a}} \times A_1'' \times A_2'' \times Q''$ has dimension $3n/2 + 2$ and surjects onto $\mathbb{G}_{\mathrm{a}}$, and $\alpha(G^n/U) \leq 3n + 4 - 2 = 3n + 2$.

Hence $\alpha_{K,\Psi}(G) \leq 3$, resp. $\leq 3 + 2/n$, and Theorem 2.3 gives $\dim_K(V \cap \Psi(\Lambda_K(G))) \leq 2$, resp. $\leq 2 + 2/n < 3$, a contradiction. For the consequence: the statement is symmetric under $t \mapsto 1/t$ (replace y_j by ty_j); if $P_\sigma \in \overline{\mathbb{Q}} y_1 + \overline{\mathbb{Q}} y_2$, then $cP' = tP_\sigma \in \widetilde{\mathcal{L}}_\sigma$, which Wüstholz's theorem excludes (as in the proof of Lemma 5.3, on $\mathbb{G}_{\mathrm{a}} \times A^r \times A'$, resp. $\mathbb{G}_{\mathrm{a}} \times A^r \times A'^\natural$, using (S1) and, for $A' = A$, that the slot forms of distinct embeddings involve independent parameters); otherwise the first part applies, since $\widetilde{\mathcal{L}}_\sigma$ lies in both targets. □

Example 5.6. On $A = \mathrm{Jac}(y^7 = x(1-x))$ with z_j dual to dx/y^{7-j} and $P_j \in \overline{\mathbb{Q}}^\times B(\frac{j}{7}, \frac{j}{7})$ (§4.4), put $t = B(\frac{2}{7}, \frac{2}{7})/B(\frac{1}{7}, \frac{1}{7})$. For every non-torsion $\exp(v) \in A(\overline{\mathbb{Q}})$ (then P_1, 1, $z_1(v)$ are $\overline{\mathbb{Q}}$-linearly independent, by Wüstholz's theorem on $\mathbb{G}_{\mathrm{a}} \times A$ as in §4.3), the numbers t and $tz_1(v)$ are not both in $\overline{\mathbb{Q}} + \overline{\mathbb{Q}} z_1(\Lambda(A)) + \overline{\mathbb{Q}} z_2(\Lambda(A))$.

For $g = g' = 1$, EA with non-isogenous curves is Theorem 5.1(1), EA$^\natural$ with $A' = E'$ non-isogenous to E is Theorem 5.1(2), and EA$^\natural$ with $A' = E$, $\tilde\sigma = \bar\sigma$ is Theorem SE$_0$ of [16].

For $g \geq 2$ or $g' \geq 2$ the proof uses the inner weight. As far as we found, Theorem 5.5 for $\max(g, g') \geq 2$ is not in print.

6. Four exponentials for quasi-logarithms

Let E have CM by k, $\Omega \neq 0$ a period, $\eta = \eta^*(\Omega)$, and Fib_E the set of fibre values of E (§3.2: constants, quasi-logarithms $\zeta^*(u) + \overline{\mathbb{Q}}$, $u \in \mathcal{L}_E$, and $\eta^*(\Lambda_E)$). The following lemma describes the mixed subgroups of two fibre factors.

Lemma 6.1 (Fibre images are rational)**.** *Let $K \supseteq k$ have degree n, θ a $\mathbb{Q}$-basis of K, $\phi(m) = \sum \theta_j \bar{m}_j$ on k^n, and $Q = (E^\natural)^n/(B^\natural_{\ker\phi} + \{\theta^{\mathsf{T}} w = 0\})$, with fibre coordinate $\varsigma = \theta^{\mathsf{T}} w$ (the quotient Q'' of the proof of* [16, Thm. 4.3]*). Put $Q_1 = Q_3 = Q$. If $H \subset Q_1 \times Q_3$ is a connected algebraic subgroup which is not a vector group and $(\varsigma_1, \varsigma_3)(T_H)$ is a line ℓ, then ℓ is spanned by a vector of K^2, and the abelian image of H has dimension at most $n/2$.*

Proof. Let $\tilde{H} \subset (E^\natural)^{2n}$ be the connected preimage, with abelian image B_S, $S \subset k^{2n}$, and let M be a matrix over $\mathcal{O}_k$ whose columns span S. Since $\mathcal{O}_k$ acts on $\mathrm{Lie}(E^\natural)$ by $\mathrm{diag}(a, \bar{a})$, $\mathrm{Lie}(B^\natural_S) = \{(Mu, \bar{M}u')\}$, and $\tilde{H} \supseteq B^\natural_S$ by the universal property of vector extensions ((F3) for $g = 1$). The fibre vectors $(0, \bar{M}u')$ have fibre coordinates $(\phi(M_1)u', \phi(M_3)u')$, which span the K-rational space $R_S = \mathrm{span}\{(\phi(s_1), \phi(s_3)) : s \in S\}$; $R_S = 0$ exactly when $S \subset \ker\phi \oplus \ker\phi$, i.e. when H is a vector group. So $0 \neq R_S \subseteq \ell$ and $\ell = R_S$. The map $S/(\ker\phi)^2 \to K^2$, $s \mapsto (\phi(s_1), \phi(s_3))$, is injective with image in the K-line $\ell \cap K^2$, whence the dimension bound. □

Theorem 6.2 (Theorem QL)**.** *Let p_2 be one of: $2\pi i$; a period Ω' of a CM curve E' ($E' = E$ allowed); the quasi-period $\eta' = \eta^*(\Omega')$ of a CM curve E' with CM field $k' \neq k$; a first-kind eigen-period P of a simple CM abelian variety A. Let X_2 be the corresponding set of values $\mathcal{L}$, $\mathcal{L}_{E'}$, $\mathrm{Fib}_{E'}$, $z_\sigma(\Lambda(A))$. Let $c \in \overline{\mathbb{Q}}^\times$ and $t = cp_2/\eta$. If $f_1, f_2 \in \overline{\mathbb{Q}}\mathrm{Fib}_E$ and η, f_1, f_2 are linearly independent over $\overline{\mathbb{Q}}$, then tf_1 and tf_2 are not both in $\overline{\mathbb{Q}}\mathrm{Fib}_E + \overline{\mathbb{Q}}X_2$.*

Corollary 6.3. *Let $t \in \overline{\mathbb{Q}}^\times\{\Omega,\ \Omega\Omega'/\pi,\ \Omega/\Omega',\ \Omega P/\pi\}^{\pm 1}$, with Ω', P as in Theorem 6.2 (in the class Ω/Ω' the curve E' has CM field $k' \neq k$). If $f_1, f_2 \in \overline{\mathbb{Q}}\mathrm{Fib}_E$ are linearly independent over $\overline{\mathbb{Q}}$, then tf_1, tf_2 are not both in $\overline{\mathbb{Q}}\mathrm{Fib}_E$. For example (class Ω/Ω' with $E = E_i$, $E' = E_\rho$), if $u_1, u_2 \in \mathcal{L}_{E_i}$ and ϖ are linearly independent over $\mathbb{Q}(i)$, then $(\varpi/\Omega_\rho)\zeta_i(u_1)$ and $(\varpi/\Omega_\rho)\zeta_i(u_2)$ are not both $\overline{\mathbb{Q}}$-linear combinations of 1 and quasi-logarithms $\zeta_i(u)$, $u \in \mathcal{L}_{E_i} \setminus \varpi\mathbb{Z}[i]$.*

In the class $\overline{\mathbb{Q}}^\times\Omega^{\pm 1}$ ($p_2 = 2\pi i$) the statement is of degree one: by Laurent's identity (4), $\Omega\zeta^*(u) = \lambda(u, \Omega) + \eta u$ with $\eta u \in \overline{\mathbb{Q}}^\times(\pi/\Omega)u$ a $\overline{\mathbb{Q}}$-combination of third-kind periods, so the hypothesis $tf_1, tf_2 \in \overline{\mathbb{Q}}\mathrm{Fib}_E$ is a pair of $\overline{\mathbb{Q}}$-linear relations among periods of 1-motives, and that case is within reach of the analytic subgroup theorem (we keep it in Theorem 6.2 as a calibration case). The bilinear content is in the other three classes, where a period of another slot multiplies a fibre value, as in the example.

Proof of Theorem 6.2. Let K be a number field of degree $n \geq 6$ containing k, c, the CM field of the second factor (the Galois closure of F if X_2 comes from A) and all coefficients needed to write $f_j \in \overline{\mathbb{Q}}\mathrm{Fib}_E$ and $tf_j \in \overline{\mathbb{Q}}\mathrm{Fib}_E + \overline{\mathbb{Q}}X_2$ ($j = 1, 2$), so that $v_j \in \Psi(\Lambda_K(G))$. Take $G = E^\natural \times X \times E^\natural$ with $X = \mathbb{G}_\mathrm{m}, E', E'^\natural, A$ respectively, $\Psi = (w_1,\ x_2 + w_3)$ (w_1, w_3 the fibre coordinates, x_2 the coordinate of X read by X_2) and $V = \mathbb{C}(1, t)$. If $X = E'^\natural$, the two fibre coordinates read in the second target are first merged (the vector subgroup $\{\theta^{\mathsf{T}} w' + \theta^{\mathsf{T}} w_3 = 0\}$ lies in the kernel). Then $G'' = Q_1 \times X'' \times Q_3$ with Q_1, Q_3 as in Lemma 6.1 (for $X = E'^\natural$: $Q_1 \times (X'' \times Q_3)/D$, $D = \{\varsigma' + \varsigma_3 = 0\}$, with one fibre $\mathbb{G}_\mathrm{a}$ over the abelian part of $X'' \times Q_3$). The data are $w = (\eta, cp_2)$ and $v_j = (f_j, tf_j)$; $m = 3n$, $\kappa = n$ (a period in Y'' has first coordinate

in $K\eta$), and $\Psi(\Omega_K(G)) = K\eta \times (K\eta + Kp_2)$. Here $t \notin K$: $t \in \overline{\mathbb{Q}}$ would put p_2 in $\overline{\mathbb{Q}}\eta \subset \overline{\mathbb{Q}}\mathrm{Fib}_E$, which Wüstholz's theorem [37] excludes. For $r' = 1$, (1) reads $\operatorname{rk} Y_{G'} + l + 2\dim(G'/T') \geq 4n$, where T' is the torus part of G' and $l = \dim T'$. Subgroups with image $\mathbb{C}(0,1)$ have $\operatorname{rk} Y_{G'} \leq n$ and $l + 2\dim(G'/T') \leq 2n+2$. Subgroups of Q_1 have image $\mathbb{C}(1,0)$, dimension $\leq n/2+1$ and $\operatorname{rk} Y_{G'} \leq 2n$. Mixed subgroups of $Q_1 \times Q_3$ that are not vector groups have, by Lemma 6.1, a K-rational image line, dimension $\leq n/2+1$, and $\operatorname{rk} Y_{G'} \leq 2n$: if $v_1, v_2 \in \ell + \Psi(\Omega_K(G))$ with $\ell = \mathbb{C}(p,1)$, $p \in K$, then $f_j(1-pt) \in K\eta + Kp_2 = \eta(K + Kt)$ (period datum $p_2 = t\eta/c$), and if $\ell = \mathbb{C}(1,0)$ then $tf_j \in \eta(K+Kt)$; either way η, f_1, f_2 lie in a 2-dimensional K-space, against their K-independence. Vector lines have dimension 1 and $\operatorname{rk} Y_{G'} \leq 3n$. These cases are complete. If X'' has an abelian part isogenous to the base of E ($E' = E$, or A isogenous to a power of E), abelian subvarieties may mix Q_1, X'' and Q_3 (and, for $X = E'^{\natural}$, Q_1 and the merged factor); if the abelian image of G' projects nontrivially to X'' (resp. to the E'-part of the merged factor), its image contains $(0,1)$, so $r' = 1$ forces image $\mathbb{C}(0,1)$, hence $\phi(S_1) = 0$ and $G' \subset X'' \times Q_3$, which is the case of image $\mathbb{C}(0,1)$ above; otherwise Lemma 6.1 applies verbatim (to $Q_1 \times Q_3$, resp. to Q_1 and the merged factor). So every subgroup with $r' = 1$ has total at most $3n+2 < 4n$. For $G' = 0$ the inequality fails with margin $n - 4 > 0$, since $n \geq 6$ (it obstructs exactly for $n \leq 4$); this is the minimum margin. No subgroup satisfies (1). Condition (2) holds ($d_0 = 0$, $2N = 2d - 2 \geq 3n + 2 \geq d_1 + \kappa$). □

Proof of Corollary 6.3. The statement is invariant under $t \mapsto \gamma t$ ($\gamma \in \overline{\mathbb{Q}}^\times$) and under $t \mapsto 1/t$ (replace f_j by tf_j). By $\Omega\eta \in \overline{\mathbb{Q}}^\times\pi$ and $\Omega'\eta' \in \overline{\mathbb{Q}}^\times\pi$, the classes $\overline{\mathbb{Q}}^\times p_2/\eta$ of Theorem 6.2 are $\overline{\mathbb{Q}}^\times\Omega$, $\overline{\mathbb{Q}}^\times\Omega\Omega'/\pi$, $\overline{\mathbb{Q}}^\times\Omega/\Omega'$ and $\overline{\mathbb{Q}}^\times\Omega P/\pi$. If $\eta \in \overline{\mathbb{Q}}f_1 + \overline{\mathbb{Q}}f_2$, then $tf_1, tf_2 \in \overline{\mathbb{Q}}\mathrm{Fib}_E$ gives $cp_2 = t\eta \in \overline{\mathbb{Q}}\mathrm{Fib}_E$, which Wüstholz's theorem [37] excludes; otherwise Theorem 6.2 applies. For the example: $t = \varpi/\Omega_\rho \in \overline{\mathbb{Q}}^\times\eta_\rho/\eta_i$ is the class $\overline{\mathbb{Q}}^\times p_2/\eta$ with $p_2 = \eta' = \eta(\Omega_\rho)$ of $E' = E_\rho$, whose CM field $\mathbb{Q}(\rho)$ differs from $\mathbb{Q}(i)$; and $f_j = \zeta_i(u_j) \in \mathrm{Fib}_{E_i}$ ($\kappa_1 = 0$ on E_i). By Wüstholz's theorem on $(E_i^\natural)^3$, whose smallest algebraic subgroup through $((\varpi,\eta_i),(u_1,\zeta_i(u_1)),(u_2,\zeta_i(u_2)))$ is $(E_i^\natural)^3$ when ϖ, u_1, u_2 are $\mathbb{Q}(i)$-independent, the numbers $\eta_i, \zeta_i(u_1), \zeta_i(u_2)$ are $\overline{\mathbb{Q}}$-independent, so in particular f_1, f_2 are. Finally $\overline{\mathbb{Q}}\mathrm{Fib}_{E_i}$ is the $\overline{\mathbb{Q}}$-span of 1 and the $\zeta_i(u)$, $u \in \mathcal{L}_{E_i}$ off the lattice, since $\eta_i = 2\zeta_i(\varpi/2)$ and $\eta(\Lambda) \subset \mathbb{Q}(i)\eta_i$. □

Remark 6.4. Without Lemma 6.1 a mixed line $\mathbb{C}(\mu,1)$ with $\mu \notin K$ would only give $\operatorname{rk} Y_{G'} \leq 3n$, which is an obstruction. The same lemma, with $\psi_{\tilde\sigma}$ in place of ϕ and $K \supseteq \tilde\sigma(F)$, gives, by (F1)–(F4), the analogue for second-kind eigen-coordinates of CM abelian varieties (not written out).

7. Conjecture 42 at CM ratios

In this section $E/\overline{\mathbb{Q}}$ has CM by k, $\Omega \neq 0$ is a period, η^* the eigen quasi-period (§3.2), so $\eta^*(\Omega) \in \overline{\mathbb{Q}}^\times\pi/\Omega$; Fib_E is the set of fibre values; $\widetilde{\mathcal{L}} = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}$. A CM curve E' "not isogenous to E" is defined over $\overline{\mathbb{Q}}$ with CM field $k' \neq k$, and Ω' is a nonzero period of E'.

Waldschmidt's Conjecture 42 [31, p. 173] asks that a rank-one matrix $\left(\begin{smallmatrix} u_1 & \log\alpha_1 \\ u_2 & \log\alpha_2 \end{smallmatrix}\right)$ with $u_j \in \mathcal{L}_E$, $\log\alpha_j \in \mathcal{L}$ have $\mathbb{Q}$-proportional rows; equivalently, the middle inequality $\delta(e^z, \wp(tz)) \leq 1$ of Ramachandra's δ-conjecture in the form of [33, §2(a)]. Theorem M of [16, Thm. 6.1] is its CM case with a period row.

Theorem 7.1 (Theorem C42)**.** *Let $\rho \in R_E := \overline{\mathbb{Q}}^\times \cdot \{\Omega,\ \Omega',\ \pi/\Omega',\ \Omega'/\Omega,\ \pi/\Omega^2,\ \pi/(\Omega\Omega')\}$. If $u_1, u_2 \in \mathcal{L}_E \setminus \{0\}$ satisfy $\rho u_1, \rho u_2 \in \widetilde{\mathcal{L}}$, then $u_2/u_1 \in k$. If moreover $\rho u_j \in \mathcal{L}$ ($j = 1,2$), then $u_2/u_1 \in \mathbb{Q}$.*

Corollary 7.2. (a) *Conjecture 42 of* [31] *holds for every rank-one matrix whose column ratio $\log\alpha_j/u_j$ lies in R_E.*

(b) *If* $\log\alpha, \log\beta$ *are linearly independent over* $\mathbb{Q}$ *and*

$$t \in \overline{\mathbb{Q}}^{\times} \cdot \{1/\Omega,\ 1/\Omega',\ \Omega'/\pi,\ \Omega/\Omega',\ \Omega^2/\pi,\ \Omega\Omega'/\pi\},$$

then $\wp_E(t\log\alpha)$ *and* $\wp_E(t\log\beta)$ *are not both algebraic (a pole counts as algebraic). For instance* $\wp_i(\log 2/\varpi)$ *and* $\wp_i(\log 3/\varpi)$ *are not both algebraic, nor are* $\wp_i(\Omega_\rho \log 2/\pi)$ *and* $\wp_i(\Omega_\rho \log 3/\pi)$.

(c) *For* $\rho \in R_E$ *and* k*-independent* $u_1, u_2 \in \mathcal{L}_E$, ρu_1 *and* ρu_2 *are not both in* $\widetilde{\mathcal{L}}$*; e.g.* $e^{\varpi u_1}$ *and* $e^{\varpi u_2}$ *are not both algebraic for* $\mathbb{Q}(i)$*-independent* $u_1, u_2 \in \mathcal{L}_{E_i}$*, nor are* $e^{\pi u_j/\varpi^2}$*, nor* $e^{\Omega_\rho u_j/\varpi}$.

Proof of Corollary 7.2. (a) and (c) are restatements. For (b) put $\rho = 1/t$ and $u_j = t\log\alpha_j$; if both values are algebraic then $u_j \in \mathcal{L}_E$ and $\rho u_j \in \mathcal{L}$, so $\log\beta/\log\alpha = u_2/u_1 \in \mathbb{Q}$. □

The line form. Each class of ρ is a cell $\sigma_1 \,|\, \sigma_2 + \sigma_3$: a factor σ_1 read into the first coordinate of $\mathbb{C}^2$, two factors σ_2, σ_3 read into the second, a slope $r = c\,p_2/p_1$ with p_i the period of σ_i, and the set

$$\mathcal{S}_K = \{\lambda \in \Lambda_K(\sigma_1) :\ r\lambda \in \Lambda_K(\sigma_2) + \Lambda_K(\sigma_3)\},$$

where $\Lambda_K(\mathbb{G}_{\mathrm{m}}) = K\mathcal{L}$, $\Lambda_K(E) = K\mathcal{L}_E$ and $\Lambda_K(\text{fibre}) = K \cdot \text{Fib}$.

row	σ_1	σ_2	σ_3	slope r	$p_1 \mapsto rp_1$
(i)	$\mathbb{G}_{\mathrm{m}}$	fibre of E	E	c/Ω	$2\pi i \mapsto \overline{\mathbb{Q}}^{\times}\eta^*(\Omega)$
(ii)	$\mathbb{G}_{\mathrm{m}}$	fibre of E'	E	c/Ω'	$2\pi i \mapsto \overline{\mathbb{Q}}^{\times}\eta^{*\prime}(\Omega')$
(iii)	$\mathbb{G}_{\mathrm{m}}$	E'	E	$c\Omega'/\pi$	$2\pi i \mapsto 2ic\Omega'$
(iv)	E	E'	$\mathbb{G}_{\mathrm{m}}$	$c\Omega'/\Omega$	$\Omega \mapsto c\Omega'$
(v)	E	fibre of E	$\mathbb{G}_{\mathrm{m}}$	$c\pi/\Omega^2$	$\Omega \mapsto \overline{\mathbb{Q}}^{\times}\eta^*(\Omega)$
(vi)	E	fibre of E'	$\mathbb{G}_{\mathrm{m}}$	$c\pi/(\Omega\Omega')$	$\Omega \mapsto \overline{\mathbb{Q}}^{\times}\eta^{*\prime}(\Omega')$

with $\rho = 1/r$ in rows (i)–(iii) and $\rho = r$ in rows (iv)–(vi).

Proposition 7.3 (Line form)**.** *In each row, for every number field* $K \supseteq kk'(i,c)$ *with* $[K:\mathbb{Q}] \geq 4$, $\dim_K \mathcal{S}_K \leq 2$*, and* $p_1 \in \mathcal{S}_K$*. The same holds with a factor* $\mathbb{G}_{\mathrm{a}}$ *added to the coordinate that carries the ordinary logarithms (target* 1 *in rows (i)–(iii), target* 2 *in rows (iv)–(vi)), i.e. with* $K\mathcal{L}$ *replaced by* $K + K\mathcal{L}$.

Proof. We apply Theorem 2.3 with the upper bound for the inner weight of §2.3. Fix a row, let $n = [K:\mathbb{Q}]$ and let θ be a basis of $K/\mathbb{Q}$. Let G be the product of three groups carrying the three factors: $\mathbb{G}_{\mathrm{m}}$; E or E', read through the base coordinate z, resp. z'; $E^\natural$ or $E'^\natural$, read through the conjugate-eigen fibre coordinate w, resp. w', of §3.2. (In rows (i) and (v) the curve E occurs twice, as E and as $E^\natural$; these are two different factors of G.) Let $\Psi\colon T_G \to \mathbb{C}^2$ read σ_1 into the first coordinate and the sum of σ_2 and σ_3 into the second, $V = \mathbb{C}(1,r)$ and $W = 0$. Since $\Lambda_K(G)$ is the product of the K-spans of the logarithms of the factors, $\Psi(\Lambda_K(G)) = \Lambda_K(\sigma_1) \times (\Lambda_K(\sigma_2) + \Lambda_K(\sigma_3))$, and so

$$V \cap \Psi(\Lambda_K(G)) = \{(\lambda, r\lambda) :\ \lambda \in \mathcal{S}_K\}, \qquad \dim_K \mathcal{S}_K = \dim_K\big(V \cap \Psi(\Lambda_K(G))\big).$$

The condition on V. Up to $\overline{\mathbb{Q}}^{\times}$, r is $1/\Omega$, $1/\Omega'$, Ω'/π, Ω'/Ω, $\eta^*(\Omega)/\Omega$ or $\eta^{*\prime}(\Omega')/\Omega$ (rows (i)–(vi); recall $\eta^*(\Omega) \in \overline{\mathbb{Q}}^{\times}\pi/\Omega$). None of these is algebraic, by (W): Ω, Ω', $\eta^*(\Omega)$, $\eta^{*\prime}(\Omega')$ are periods of four distinct eigen-slots, and periods of distinct slots are linearly independent over $\overline{\mathbb{Q}}$ together with 1 and π. So V has no nonzero $\overline{\mathbb{Q}}$-rational point, and Theorem 2.3 applies.

The period datum. By Legendre's relation and $\eta^*(a\lambda) = \bar{a}\,\eta^*(\lambda)$, $\Omega\,\eta^*(\Omega) \in \pi i\, k^{\times}$: if $\lambda_1, \lambda_2 = \tau\lambda_1$ is a basis of Λ_E with $\operatorname{Im}\tau > 0$, then $\eta^*(\lambda_1)\lambda_2 - \eta^*(\lambda_2)\lambda_1 = 2\pi i$ gives $(\tau - \bar{\tau})\lambda_1\eta^*(\lambda_1) = 2\pi i$, and $\Omega\eta^*(\Omega) = a\bar{a}\,\lambda_1\eta^*(\lambda_1)$ for $\Omega = a\lambda_1$, $a \in k^{\times}$. Likewise for E'. Since $K \supseteq kk'(i,c)$, in each row rp_1 lies in the K-span of the periods of σ_2: $K\eta^*(\Omega)$, $K\eta^{*\prime}(\Omega')$, $K\Omega'$, $K\Omega'$, $K\eta^*(\Omega)$,

$K\eta^{*\prime}(\Omega')$ (here $\eta^*(\Omega)$ is the fibre coordinate of the period of $E^\natural$ above Ω, §3.2). And $p_1 \in \Omega(\sigma_1)$. So $(p_1, rp_1) \in V \cap \Psi(\Omega_K(G))$, $p_1 \in \mathcal{S}_K$ (periods are logarithms), and $\kappa \geq 1$.

The inner weight. Let ψ, ϕ (resp. ψ', ϕ') be the maps of Lemma 3.2 for k (resp. k'). In E^n the abelian subvariety $B_{\ker\psi}$ has dimension $n/2$ and $\theta^{\mathsf{T}} z$ vanishes on its tangent space. In $(E^\natural)^n$, put $U^\natural = B^\natural_{\ker\phi} + \{\theta^{\mathsf{T}} w = 0\}$: the tangent space of $B^\natural_S$ is $\{(Mu, \bar{M}u')\}$ for a matrix M whose columns span S ((F4) for $g = 1$), so $\theta^{\mathsf{T}} w = \phi(M)u'$ vanishes on it for $S = \ker\phi$, and the quotient $(E^\natural)^n/U^\natural$ is an extension of $E^n/B_{\ker\phi}$ (dimension $n/2$) by $\mathbb{G}_{\mathrm{a}}$, of dimension $n/2+1$ (as in [16, proof of Thm. 4.3]). The same holds for E' with ψ', ϕ'. Let $U \subset G^n$ be the product of these subgroups in the factors E, E', $E^\natural$, $E'^\natural$, and of 0 in the factor $\mathbb{G}_{\mathrm{m}}^n$. Then U is connected, $T_U \subset \ker\Psi_\theta$, and

$$G^n/U = \mathbb{G}_{\mathrm{m}}^n \times A'' \times A''' \quad \text{(rows (iii), (iv))}, \qquad G^n/U = \mathbb{G}_{\mathrm{m}}^n \times A'' \times Q'' \quad \text{(rows (i), (ii), (v), (vi))},$$

with $\dim A'' = \dim A''' = n/2$ and $\dim Q'' = n/2 + 1$. It surjects onto $\mathbb{G}_{\mathrm{m}}^n$, so $\alpha(G^n/U) \leq 2 \cdot 2n - n = 3n$, resp. $\leq 2(2n+1) - n = 3n + 2$.

Hence $\dim_K \mathcal{S}_K \leq \alpha_{K,\Psi}(G) - \kappa \leq 3 - 1 = 2$ in rows (iii), (iv), and $\leq 3 + 2/n - 1 < 3$ in the other rows, since $n \geq 4$; so $\dim_K \mathcal{S}_K \leq 2$.

Affine variant. Add a factor $\mathbb{G}_{\mathrm{a}}$ to G, read into the coordinate that carries $\mathbb{G}_{\mathrm{m}}$, and add $\{\theta \cdot a = 0\}$ to U. Then $\Lambda_K(\mathbb{G}_{\mathrm{a}}) = \overline{\mathbb{Q}} \supseteq K$, the quotient G^n/U acquires a factor $\mathbb{G}_{\mathrm{a}}$, which raises $2\dim$ by 2 and $2d_0$ by 2, so the bound for $\alpha(G^n/U)$ is unchanged; the condition on V and the period datum are as before. So $\{\lambda \in \overline{\mathbb{Q}} + K\mathcal{L} : \ r\lambda \in \Lambda_K(\sigma_2) + \Lambda_K(\sigma_3)\}$ in rows (i)–(iii), and $\mathcal{S}_K$ with $\Lambda_K(\mathbb{G}_{\mathrm{m}})$ replaced by $\overline{\mathbb{Q}} + K\mathcal{L}$ in rows (iv)–(vi), have K-dimension ≤ 2, which contains the statement. □

Remark 7.4 (A direct proof of row (iv))**.** Let $G = E \times E' \times \mathbb{G}_{\mathrm{m}}$ with coordinates (z, z', x), $\Psi(z, z', x) = (z, z' + x)$, $V = \mathbb{C}(1, r)$, $n = [K : \mathbb{Q}]$, θ, p_θ, Ψ_θ as in §2.3. Connected algebraic subgroups of $E^n \times E'^n \times \mathbb{G}_{\mathrm{m}}^n$ are products $B_S \times B'_{S'} \times T_L$ ($\mathrm{Hom}(E, E') = 0$), and such a subgroup has tangent space in $\ker\Psi_\theta$ iff $\psi(S) = 0$, $\psi'(S') = 0$ and $L = 0$. So $U = B_{\ker\psi} \times B'_{\ker\psi'}$, and $G'' = G^n/U = A'' \times A''' \times \mathbb{G}_{\mathrm{m}}^n$ with $\dim A'' = \dim A''' = n/2$: $d_0 = 0$, $d_1 = n$, $d_2 = n$, $d = 2n$, weight $3n$. Ψ'' is onto $\mathbb{C}^2$, V'' is a hyperplane and $W'' = \ker\Psi''$. The period datum is $w = (\Omega, c\Omega') \in \Psi(\Omega_K(G))$. Suppose $\Omega, u_1, u_2 \in \mathcal{S}_K$ are K-independent, put $v_j = (u_j, ru_j)$, and let Y'' be generated by lifts of $D\theta_h w$, $\theta_h v_1$, $\theta_h v_2$: then $m = 3n$, and $\kappa = n$ (a period of G'' has first coordinate in $K\Omega$, which kills the v-parts). With $d - N = 1$ and $\delta - \tau = 2 - r'$, $r' = \dim\Psi''(T_{G'})$, an obstruction needs $(m - \mathrm{rk}\, Y_{G'}) + (3n - \mathrm{wt}\, G') \leq (2 - r') \cdot 2n$, where $Y_{G'} = Y'' \cap (T_{G'} + \Omega(G''))$. Since U is maximal, a nonzero B'' has image $\mathbb{C}(1, 0)$ and a nonzero $B''' \times T_L$ has image containing $\mathbb{C}(0, 1)$; so $r' \leq 1$ leaves: $G' = 0$ ($5n \leq 4n$ fails, margin n); $G' = B''' \times T_L \neq 0$ with image $\mathbb{C}(0, 1)$ ($\mathrm{rk}\, Y_{G'} \leq n$, so the left side is $\geq 3n > 2n$); $G' = B'' \neq 0$ with image $\mathbb{C}(1, 0)$: an element of $Y_{G'}$ has Ψ''-image (X, rX) with $rX \in K\Omega' + K \cdot 2\pi i$, i.e. $X \in K\Omega + K \cdot 2\pi i\Omega/(c\Omega')$, a 2-dimensional K-space, so $\mathrm{rk}\, Y_{G'} \leq 2n$ and the left side is $\geq n + 2n > 2n$. No G' satisfies (1). (2) reads $2(2n - 1) \geq 2n$. The minimum margin is n, for every even n.

Proof of Theorem 7.1. Take K large (containing the coefficients needed below). *Rows (i)–(iii).* $x_j := \rho u_j \in \widetilde{\mathcal{L}} \subset K + K\mathcal{L}$ and $rx_j = u_j \in K\mathcal{L}_E$, so $2\pi i, x_1, x_2 \in \mathcal{S}_K$ (affine variant) and $a \cdot 2\pi i + bx_1 + c'x_2 = 0$ with $(a, b, c') \neq 0$. If $a \neq 0$, then $bu_1 + c'u_2 = -2\pi i a/\rho$ is a nonzero element of $\overline{\mathbb{Q}}^\times \eta^*(\Omega)$, $\overline{\mathbb{Q}}^\times \eta^{*\prime}(\Omega')$ or $\overline{\mathbb{Q}}^\times \Omega'$ lying in $\overline{\mathbb{Q}}\mathcal{L}_E$. This is impossible by Wüstholz's theorem [37]: $\overline{\mathbb{Q}}\mathcal{L}_E \cap \overline{\mathbb{Q}}\eta^*(\Omega) = 0$ (apply it to $(u_1, \ldots, u_r, (\Omega, \eta^*(\Omega)))$ in $E^r \times E^\natural$: the smallest algebraic subgroup whose tangent space contains this point contains the conjugate-eigen part of $E^\natural$ (F3), which carries the fibre coordinate w independently of the coordinates $z(u_i)$, so a form $\sum_i \beta_i z(u_i) - \gamma w$ with $\gamma \neq 0$ does not vanish on it), $\overline{\mathbb{Q}}\mathcal{L}_E \cap \overline{\mathbb{Q}}\eta^{*\prime} = 0$ (on $E \times E'^\natural$) and $\overline{\mathbb{Q}}\mathcal{L}_E \cap \overline{\mathbb{Q}}\Omega' = 0$ (on $E \times E'$). So $a = 0$. *Rows (iv)–(vi).* $\Omega, u_1, u_2 \in \mathcal{S}_K$ (affine variant in target 2), so $a\Omega + bu_1 + c'u_2 = 0$;

if $a \neq 0$ then $\rho(bu_1 + c'u_2) = -a\rho\Omega \in \widetilde{\mathcal{L}}$, but $\rho\Omega \in \overline{\mathbb{Q}}^\times\{\Omega', \eta^*(\Omega), \eta^{*\prime}(\Omega')\}$, none of which lies in $\widetilde{\mathcal{L}}$ (Wüstholz on $\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}^N \times E'$, $\times E^\natural$, $\times E'^\natural$). So $a = 0$. In all rows u_1, u_2 are $\overline{\mathbb{Q}}$-dependent, so $u_2/u_1 \in k$ by Masser's theorem [18]. If $\rho u_j = \log \alpha_j$ and $u_2 = \gamma u_1$ with $\gamma \in k$, then $\log \alpha_2 = \gamma \log \alpha_1 \neq 0$, and the Gel'fond–Schneider theorem forces $\gamma \in \mathbb{Q}$. □

Provenance. Rows (iii), (iv) of Proposition 7.3 are direct instances of [23, Théorème (iii)] as printed (Remark 2.4); for the fibre rows the printed weight gives no conclusion on the groups used here, and the inner weight is used. The linearisation filter: on a CM curve only $\eta^* \cdot u$ and $\Omega \cdot \zeta^*(u)$ are periods of 1-motives (Laurent's identity (4)), so $\rho \in \overline{\mathbb{Q}}^\times \pi/\Omega$ (Theorem M of [16, Thm. 6.1]) is the unique linearisable CM ratio, and we do not expect Theorem 7.1 to follow from the analytic subgroup theorem [Heuristic]. On what is known: all six classes satisfy $\bar\rho/\rho \in \overline{\mathbb{Q}}^\times$ and $\bar E$ is isogenous to E, so a datum u gives a second datum $\bar u$. Write $V(\alpha, \beta)$ for the $\mathbb{Q}$-span of $\log \alpha, \log \beta, \overline{\log \alpha}, \overline{\log \beta}$. If $\dim V(\alpha, \beta) = 4$, Corollary 7.2(b) already follows [Proof sketch] from the Lang–Ramachandra mixed six exponentials theorem [31, Thm. 24] plus conjugation (Ramachandra; Diaz [7]); if $\dim V(\alpha, \beta) = 3$, it is a one-line consequence of Theorem 2.3 on $E \times \mathbb{G}_{\mathrm{m}}$ ($\alpha_K = 2$, $\kappa = 0$) plus conjugation [Proof sketch]. As far as we found, the conjugation-stable case $\dim V(\alpha, \beta) = 2$ (for instance α, β real), which contains all our examples, is not in print. Corollary 7.2(b) is the inequality $\delta(e^z, \wp(tz)) \leq 1$ of Ramachandra's δ-conjecture in the form of [33, §2(a)], at six CM classes of t; the best bounds listed there are $\delta \leq 3$ in general and $\delta \leq 2$ at the slope of Theorem M, with no case of $\delta \leq 1$.

8. Applications in codimension ≥ 2

The two results of this section live on subspaces V'' of codimension ≥ 2, where Theorem 2.1 applied subgroup by subgroup is stronger than the uniform bound of Theorem 2.3: at $G' = 0$, (1) divides the threshold by $d - N$ [28, (4.2)]. In the configurations we use, Theorem 2.3 gives no conclusion or a weaker one (bound 3 against 2 data for Theorem 8.1, $k \leq g - 2$ for (U1), and its condition on V fails when K contains the imaginary quadratic field of Theorem 8.4). We apply Theorem 2.1 as printed, subgroup by subgroup; both applications satisfy (2). The other application of this kind we know is Theorem P of [16, §7]. Notation: $\widetilde{\mathcal{L}} = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}$; a CM curve is an elliptic curve over $\overline{\mathbb{Q}}$ with complex multiplication, and η_j^* is the conjugate-eigen quasi-period of a period Ω_j (§3.2), so $\eta_j^* \in \overline{\mathbb{Q}}^\times \pi/\Omega_j$.

8.1. **Collinearity of period points.**

Theorem 8.1 (Theorem CL4)**.** *Let E_1, E_2, E_3 be CM elliptic curves over $\overline{\mathbb{Q}}$ with pairwise distinct CM fields k_1, k_2, k_3, and Ω_j a nonzero period of E_j. If $s, t \in \mathbb{C}$ satisfy*

$$s + 2\pi i\, t \in \overline{\mathbb{Q}}\mathcal{L} \quad \text{and} \quad s + \Omega_j t \in \overline{\mathbb{Q}}\mathcal{L}_{E_j}\ (j = 1, 2, 3),$$

then $s = 0$. Equivalently, if the four points $(2\pi i, x)$, (Ω_j, y_j) of $\mathbb{C}^2$ with $x \in \overline{\mathbb{Q}}\mathcal{L}$, $y_j \in \overline{\mathbb{Q}}\mathcal{L}_{E_j}$ lie on one line, the line passes through the origin. The same holds with $\mathbb{G}_{\mathrm{m}}$ replaced by a fourth CM curve E_0 with a CM field distinct from the others.

Corollary 8.2. *Put $X = \Omega_1(2\pi i - \Omega_2)/(\Omega_1 - \Omega_2)$ and $Y = \Omega_1(\Omega_3 - \Omega_2)/(\Omega_1 - \Omega_2)$. Then e^X and $\wp_{E_3}(Y)$ are not both algebraic.*

Proof. Take $y_1 = \Omega_1$, $y_2 = 0$: then $t = \Omega_1/(\Omega_1 - \Omega_2)$ and $s = -\Omega_2 t \neq 0$, and the theorem forbids $x = X \in \mathcal{L}$ together with $y_3 = Y \in \mathcal{L}_{E_3}$. (Some such proviso is needed: $y_j = \Omega_j/2$ gives $s = 0$, $x = \pi i$.) □

Proof of Theorem 8.1. Let $K \supseteq k_1k_2k_3(i)$, $n = [K:\mathbb{Q}] \geq 8$, $G = \mathbb{G}_{\mathrm{m}} \times E_1 \times E_2 \times E_3$, $\Psi = \mathrm{id}$ (one slot per coordinate). Subgroups of G^n are products, $U = 0 \times B_{\ker \psi_1} \times B_{\ker \psi_2} \times B_{\ker \psi_3}$, and $G'' = \mathbb{G}_{\mathrm{m}}^n \times A_1'' \times A_2'' \times A_3''$ with $\dim A_j'' = n/2$: $d = 5n/2$, $d_0 = 0$, $d_1 = n$, $d_2 = 3n/2$, weight $4n$. Put $w = (1,1,1,1)$, $p = (2\pi i, \Omega_1, \Omega_2, \Omega_3)$, $V = \mathbb{C}w \oplus \mathbb{C}p$, $W = \mathbb{C}w$ ($\overline{\mathbb{Q}}$-rational). So $V'' = \Psi''^{-1}(V)$ has codimension 2 and $W'' = \Psi''^{-1}(W)$ is $\overline{\mathbb{Q}}$-rational of dimension $d-3$. A datum $v = sw + tp$ with $s \neq 0$ lies in $\Psi(\Lambda_K(G))$; Y'' is generated by lifts of $D\theta_h p$ and $\theta_h v$, so $m = 2n$.

$\kappa = n$. A K-period vector on V is $s'w + t'p$ with $s' + 2\pi i t' \in K \cdot 2\pi i$ and $s' + \Omega_j t' \in K\Omega_j$. Writing $s' = 2\pi i(a - t') = \Omega_j(b_j - t')$ and eliminating t' between $j = 1, 2$ gives $(b_1 - b_2)/(2\pi i) - (b_1 - a)/\Omega_2 + (b_2 - a)/\Omega_1 = 0$; as $1/\Omega_j \in \overline{\mathbb{Q}}^{\times}\eta_j^*/\pi$, this is a $\overline{\mathbb{Q}}$-linear relation among $1, \eta_1^*, \eta_2^*$, trivial by Wüstholz's theorem on $\mathbb{G}_{\mathrm{a}} \times E_1^\natural \times E_2^\natural$. So $b_1 = b_2 = a$, $s' = 0$, and $V \cap \Psi(\Omega_K(G)) = Kp$.

Lemma S: if three of the four coordinates of a datum $v = sw + tp$ are K-periods, then $s = 0$. If they are the E_j-coordinates, $s = \Omega_j(b_j - t)$ for $j = 1, 2, 3$; eliminating t pairwise gives $(b_2 - b_3)/\Omega_1 + (b_1 - b_2)/\Omega_3 + (b_3 - b_1)/\Omega_2 = 0$, a $\overline{\mathbb{Q}}$-relation among $\eta_1^*, \eta_2^*, \eta_3^*$ (after multiplying by π), trivial by Wüstholz on $E_1^\natural \times E_2^\natural \times E_3^\natural$; so $b_1 = b_2 = b_3 = t$ and $s = 0$. If they include the $\mathbb{G}_{\mathrm{m}}$-coordinate, the computation of κ applies. (The eliminations use $2\pi i \notin \overline{\mathbb{Q}}\Omega_j$ and $\Omega_1/\Omega_2 \notin \overline{\mathbb{Q}}$.)

Obstructions. Connected subgroups are $G' = T_L \times B_1 \times B_2 \times B_3$; every nonzero factor maps onto its coordinate line (U is maximal), so $\Psi''(T_{G'}) = \mathbb{C}^S$ for the set S of nonzero factors, and $\delta - \tau = 3 - |S| + [S = \{0,1,2,3\}]$. (1) reads $(\lambda + 4n - \operatorname{wt} G') \cdot 2 \leq (\delta - \tau) \cdot 3n$: for $S = \emptyset$, $\lambda = n$ and $10n \leq 9n$ fails; for one slot, $\mathrm{wt} \leq n$ and $\lambda = n$ (by Lemma S, the datum's other three coordinates are not all periods), so $8n \leq 6n$ fails (without Lemma S, $\lambda = 0$ would give $6n \leq 6n$); for two slots the left side is $\geq 4n > 3n$; for three or four slots $\delta > \tau$ fails. Condition (2): $2(5n/2 - 2) \geq 2n$. So no datum with $s \neq 0$ exists. The four-curve version has the same table (weight $4n$, Lemma S with any three of the four η^*'s, (2) $4n - 4 \geq n$). □

Controls. With $W = 0$, or without κ, one datum is allowed; a non-CM E_3 gives $G' = 0$ at equality. Theorem 2.3 on this plane ($\alpha_K = 4$, $\kappa = 1$, $\dim V/W = 1$) gives the bound 3 against 2 vectors, hence nothing.

8.2. **A unitary source of the period term.** Let $\mathbf{K} \subset \mathbb{C}$ be imaginary quadratic, ι the inclusion, and $A/\overline{\mathbb{Q}}$ an abelian variety of dimension g, simple over $\overline{\mathbb{Q}}$, with $\mathcal{O}_{\mathbf{K}} \subset \mathrm{End}(A)$. Then $\operatorname{Lie} A = L_\iota \oplus L_{\bar\iota}$, where $[a]$ acts by $\iota(a)$ on L_ι and by $\bar\iota(a)$ on $L_{\bar\iota}$; both are $\overline{\mathbb{Q}}$-rational. The signature is $(r, s) = (\dim L_\iota, \dim L_{\bar\iota})$; for A simple and $r \geq 2$ we have $s \geq 1$. Examples: Picard curves $y^3 = f_4(x)$ (signature $(2,1)$; dx/y^2, $x\,dx/y^2$ span one eigenspace) and $y^3 = f_6(x)$ (signature $(3,1)$), whose Jacobians are simple for generic f (simplicity must be checked for each curve). These are generically not of CM type. For $x \in \operatorname{Lie} A$ write $x = x_\iota + x_{\bar\iota}$.

Lemma 8.3. *Assume $r \geq 2$. For a nonzero period ω of A, $\omega_\iota \neq 0$ and $\omega_{\bar\iota} \neq 0$. If $r > s$ (for instance if $s = 1$) and ω, ω' are periods with $\omega_\iota' \in \mathbb{C}\omega_\iota$, then $\omega' \in \mathbf{K}\omega$ in $\Omega(A) \otimes \mathbb{Q}$.*

Proof. If $\omega \in L_{\bar\iota}$, then $(\mathbf{K} \otimes \mathbb{R})\omega$ is a complex line spanned by the rank-2 lattice $\mathcal{O}_{\mathbf{K}}\omega$, an elliptic curve in A, impossible for A simple of dimension ≥ 2. For the second claim with $s = 1$: if ω, ω' are $\mathbf{K}$-independent with $\omega_\iota' = t\omega_\iota$, take $\theta_t \in \mathbf{K} \otimes \mathbb{R}$ with $\iota(\theta_t) = t$; then $0 \neq \omega' - \theta_t\omega \in U_{\mathbb{R}} \cap L_{\bar\iota}$ for $U = \mathbf{K}\omega + \mathbf{K}\omega'$, and $U_{\mathbb{R}} \cap L_{\bar\iota}$ is a $\mathbf{K} \otimes \mathbb{R}$-submodule, on which $\mathbf{K} \otimes \mathbb{R}$ acts through $\bar\iota$ by all complex scalars; so it is a nonzero complex subspace of the line $L_{\bar\iota}$, and $L_{\bar\iota} \subset U_{\mathbb{R}}$. As $g \geq 3$, U lies in the kernel of a nonzero $\mathbf{K}$-linear $\varphi\colon \Omega(A)_{\mathbb{Q}} \to \mathbf{K}$, whose extension kills $L_{\bar\iota}$ and is complex-linear on L_ι, hence is a holomorphic 1-form with periods in $\iota(\mathbf{K})$; the $\mathbf{K}$-span of $\mathrm{Tr} \circ \varphi$ is a 2-dimensional rational sub-Hodge structure of H^1, i.e. an elliptic quotient of A, a contradiction. For $s \geq 2$, $r > s$: Theorem 2.1 with $Y = \mathcal{O}_{\mathbf{K}}\omega + \mathcal{O}_{\mathbf{K}}\omega'$ as in the proof below ($\kappa = 4$, $\lambda = 0$, and

$d - N = r - 1 \geq 1$ since $r \geq 2$) gives $2g(r-1) \leq r(2g-4)$, i.e. $r \leq s$, a contradiction; here (2) holds strictly. □

The first claim needs only $g \geq 2$. In the second, $r \geq 2$ is what makes "$s = 1$" imply $r > s$: the formulation "if $s = 1$ or $r > s$" would fail at $(r, s) = (1, 1)$, $g = 2$, where L_ι is a line, so $\omega'_\iota \in \mathbb{C}\omega_\iota$ holds for all periods ω, ω', while $\Omega(A) \otimes \mathbb{Q}$ has $\mathbf{K}$-dimension 2. (The proof uses $g \geq 3$ for $s = 1$, and $d - N \geq 1$ for $s \geq 2$.) Theorem 8.4 assumes $r \geq 2$.

Theorem 8.4 (Theorem U)**.** *Let A be as above, of signature (r, s) with $r \geq 2$, and ω a nonzero period.*

- (U1) *If $r \geq s + 2$ and $u \in \Lambda(A)$ has $u_\iota \in \mathbb{C}\omega_\iota$, then $\exp_A(u)$ is torsion.*
- (U2) *The $u \in \Lambda(A)$ with $u_\iota \in \mathbb{C}\omega_\iota$ form a group whose image in $A(\overline{\mathbb{Q}})/\mathrm{tors}$ is an $\mathcal{O}_\mathbf{K}$-module of rank $\leq \lfloor s/(r-1) \rfloor$ (≤ 1 for Picard curves).*
- (U3) *If $r \geq s + 1$ and $u \in \Lambda(A)$ has $u_\iota = t\,\omega_\iota$ with $2\pi i t \in \mathcal{L} + \iota(\tau)\mathcal{L}$ ($\mathcal{O}_\mathbf{K} = \mathbb{Z}[\tau]$), then $t \in \mathbf{K}$ and $\exp_A(u)$ is torsion; in particular $u_\iota = (\log\alpha/2\pi i)\,\omega_\iota$ with $\alpha \in \overline{\mathbb{Q}}^\times$ forces α to be a root of unity.*
- (U4) *If $r \geq s + 1$, E is a CM curve with CM by $\mathcal{O}_\mathbf{K}$ (acting on $\operatorname{Lie} E$ through ι) with period Ω, and $u \in \Lambda(A)$, $v \in \mathcal{L}_E$ satisfy $u_\iota = (v/\Omega)\,\omega_\iota$, then $\exp_E(v)$ and $\exp_A(u)$ are torsion.*

Concretely, on a simple Picard Jacobian,

$$\Big(\int^P \frac{dx}{y^2}, \int^P \frac{x\,dx}{y^2}\Big) = \frac{\log\alpha}{2\pi i}\Big(\int_\gamma \frac{dx}{y^2}, \int_\gamma \frac{x\,dx}{y^2}\Big)$$

only for P torsion and α a root of unity: the unitary analogue of $\wp(\varpi \log\alpha/\pi) \notin \overline{\mathbb{Q}}$ ([16, Cor. 6.2, (M2)]).

Proof. Let $G = A \times E^a \times T^b$ ($a, b \in \{0, 1\}$; $T = \mathbb{G}_\mathrm{m} \otimes \mathcal{O}_\mathbf{K}$, a split torus of dimension 2), so $d = g + a + 2b$, $d_0 = 0$, $d_1 = 2b$, $d_2 = g + a$, weight $2g + 2a + 2b$. $L_\iota(G) = L_\iota(A) \oplus \operatorname{Lie} E^a \oplus \mathbb{C}_\iota^b$ has dimension $R = r + a + b$, and $L_{\bar\iota}(G) = L_{\bar\iota}(A) \oplus \mathbb{C}_{\bar\iota}^b$ has dimension $S = s + b$; both are $\overline{\mathbb{Q}}$-rational. Let $w = (\omega, \Omega^a, (2\pi i \otimes 1)^b)$ be the period vector, and

$$V = \mathbb{C}w_\iota \oplus L_{\bar\iota}(G), \quad W = L_{\bar\iota}(G), \quad N = 1 + S, \quad t = S, \quad d - N = R - 1.$$

Since $[a]$ acts on $L_\iota(G)$ by the scalar $\iota(a)$, $\mathcal{O}_\mathbf{K} w \subset V$, and every datum below lies in V.

Data. A datum is $y \in \Lambda(G)$ with $y_\iota = t' w_\iota$; put $Y = \mathcal{O}_\mathbf{K} w + \sum_j \mathcal{O}_\mathbf{K} y_j$. If a component of y is torsion, the datum is trivial: e.g. if its torus component is torsion then $t' \in \iota(\mathbf{K})$ and $y - [c]w \in W$ for some $c \in \mathbf{K}$, a $\overline{\mathbb{Q}}$-rational subspace containing a logarithm of an algebraic point, so by Wüstholz's theorem it lies in the Lie algebra of an algebraic subgroup inside $L_{\bar\iota}(G)$, which is 0; if the A-component is torsion and $r > s$, Lemma 8.3 gives $t' \in \mathbf{K}$. So for nontrivial data each orbit $\mathcal{O}_\mathbf{K} y_j$ has rank 2 modulo $\mathcal{O}_\mathbf{K} w$, $\kappa = 2$, and for $G' \subset G$, $\lambda(G') \geq 2$ per independent datum unless G' contains all factors of the datum (a rational line $l \subset T$ meets an orbit $\mathcal{O}_\mathbf{K} z$ of a non-torsion z in rank ≤ 1).

Subgroups. A and E are simple, $\mathrm{Hom}(A, E) = 0$, and there are no homomorphisms between abelian varieties and tori; so $G' = A' \times E' \times T_L$ with $A' \in \{0, A\}$, $E' \in \{0, E\}$, $T_L \in \{0, T, \text{a rational line } l\}$, and $\tau = S - \dim(W \cap T_{G'})$ with $W \cap T_{G'} = L_{\bar\iota}(A') + (\mathbb{C}_{\bar\iota}$ if $T_L = T)$.

The inequality. G' obstructs iff $\delta > \tau$ and $(\lambda + \delta_1 + 2\delta_2)(R - 1) \leq (\delta - \tau)(2b + 2g + 2a - 2)$. For $G = A$ ($a = b = 0$), $G' = 0$ and k data: $\lambda = 2k$, $\delta = g$, $\tau = s$, and the inequality is $(2k + 2g)(r - 1) \leq r(2g - 2)$, i.e. $k(r - 1) \leq s$. So k data are excluded iff $k > s/(r - 1)$; with $G' = A$ one has $\delta = 0$. This gives (U1) ($k = 1$, $r \geq s + 2$) and (U2). For (U3), (U4) take $G = A \times T$, resp. $A \times E$, and one nontrivial datum y. Then $d - N = R - 1 = r$ and $d_1 + 2d_2 - \kappa = 2g$ in both cases, and $\lambda = 2$ for every G' in the table below except $A \times l$, by the

remark on orbits above (for $G' = l$ a point of $\Gamma \cap G'$ has torsion A-component). The tangent line of a one-dimensional subtorus $l \subset T$ is not an eigenline of $\mathcal{O}_{\mathbf{K}}$ (it comes from a rank-one subgroup of the cocharacters, which is not an $\mathcal{O}_{\mathbf{K}}$-module), so $W \cap T_l = 0$. The proper connected subgroups are:

G	G'	δ	δ_1	δ_2	τ	$(\lambda+\delta_1+2\delta_2)\,r$	$(\delta-\tau)\cdot 2g$
$A\times T$	0	$g+2$	2	g	$s+1$	$(2g+4)r$	$2g(r+1)$
	A	2	2	0	1	$4r$	$2g$
	T	g	0	g	s	$(2g+2)r$	$2gr$
	l	$g+1$	1	g	$s+1$	$(2g+3)r$	$2gr$
	$A\times l$	1	1	0	1	$\delta=\tau$	
$A\times E$	0	$g+1$	0	$g+1$	s	$(2g+4)r$	$2g(r+1)$
	A	1	0	1	0	$4r$	$2g$
	E	g	0	g	s	$(2g+2)r$	$2gr$

A subgroup obstructs iff the left column is at most the right one. Since $g = r + s$, both $(2g+4)r \le 2g(r+1)$ and $4r \le 2g$ are equivalent to $r \le s$, and the rows T, l, E never obstruct. So for $r \ge s+1$ no subgroup satisfies (1). For Picard $(2,1)$ ($g=3$) the rows read $20 > 18$, $8 > 6$, $16 > 12$, $18 > 12$ for $A \times T$ and $20 > 18$, $8 > 6$, $16 > 12$ for $A \times E$. Condition (2): $2 + 2S \ge 2b + 2$, strict as $s \ge 1$. For (U3), $t \in \mathbf{K}$ and $\log\alpha = 2\pi i(c_1 + c_2\sqrt{-D})$ give $|\alpha| = e^{-2\pi c_2\sqrt{D}}$, and Gel'fond–Schneider forces $c_2 = 0$. □

Controls. With the period term deleted, or with $W = 0$, the count obstructs; for Picard $(2,1)$ one datum is allowed ($8 \le 8$ at $G' = 0$); for a non-simple $A = B \times E_\rho$ of signature $(3,1)$, (U1) is false and the count finds the obstruction $G' = B$. *Provenance.* (U2) for $r = 2$ (Picard curves included) is also the case $K = \mathbb{Q}$, $\Psi = \mathrm{id}$, $G = A$, V a hyperplane, of Theorem 2.3, i.e. of Theorem 2.2 ($\alpha = 2g$, $\kappa = 2$, so $2+2k \le 2g-2$, i.e. $k \le s$). For $r \ge 3$, and for (U1), (U3), (U4), V has codimension $r - 1 \ge 2$ in T_G (resp. $R-1$), and the subgroup-by-subgroup count gives more than Theorem 2.3; for $K \supseteq \mathbf{K}$ the condition on V in Theorem 2.3 fails, since $L_{\bar\iota} \in S_K(A)$. The relation $u_\iota \parallel \omega_\iota$ is bilinear in a logarithm and a period of the same abelian variety; it is $\overline{\mathbb{Q}}$-linear only for algebraic t [Heuristic: not linearisable].

9. Formal verification and literature

Formal verification. Two statements of this paper have been checked in Lean 4 with Mathlib, both relative to explicit hypotheses; everything else is not formalised.

- Theorem 4.6, with its torsion clause, and Corollary 4.7(a), (b) (for Abel–Jacobi integrals) and (c) are checked in the certificate [13] (declarations `CM4E.AbVar.thmX7`, `thmX7_torsion`, `x7a_AJ`, `x7b_AJ`, `x7c`) *conditionally* on a named instance of Waldschmidt's theorem for the genus-three group (`Thm11KappaX7`), which neither [13] nor [14] proves, and on named inputs (Chowla–Selberg at level seven, and forms of Wüstholz's theorem; for the torsion clause and the Abel–Jacobi forms also a named torsion step). Lean has no Jacobians: the logarithms of A enter as a parameter set, and Abel–Jacobi integrals with endpoints at the branch points are not covered. The generic Theorem 4.2, without the torsion clause, is checked conditionally in the same way (`AbVar.XSetup.thmXA`). In the terminology of [15, §12] these are of status W (verified conditionally on a named form of [28] that is not proved).
- Theorem 5.1(1) for $(E, E') = (E_i, E_\rho)$, at $t = c\,\Omega_\rho/\varpi$ ($c \in \overline{\mathbb{Q}}^\times$), is checked (`CM4E.Unconditional.thmSE_varpi_omegaRho`) with no form of [28] assumed: the instance of Waldschmidt's theorem it uses is proved in the Lean project [14] described in [15], and

the one hypothesis left is that g_3 of the lattice $\Omega_\rho\mathbb{Z}[\rho]$ is algebraic ($g_3 = 4$ is classical but not formalised). This is status I of [15, §12].

In both cases the Lean proofs go through the hyperplane theorem (Theorem 2.2), not through our case checks, and the group models are hand-written and reviewed, not derived in Lean. Theorems 4.12, 4.14, 5.1(2), 5.5, 6.2, 7.1, 8.1, 8.4 and their corollaries are not formalised.

9.1. **What we read.** *Quadratic relations.* Gao, Ullmo and Yafaev [9] and Gao and Ullmo [8] describe algebraic relations among holomorphic CM periods beyond the linear ones as largely open, apart from cases reducible to the linear case such as $\theta\theta'/\pi \notin \overline{\mathbb{Q}}$ [9, Cor. 3.1], and ask whether all quadratic relations are elementary [9, Question 1.3]; the relations used in Theorems 4.2 and 4.14 are elementary in their sense. Wolfart and Wüstholz [36] determine the $\overline{\mathbb{Q}}$-linear relations among Beta values; Corollary 1.1(b) involves the quotient $B(\frac{3}{7},\frac{3}{7})/B(\frac{1}{7},\frac{1}{7})$, whose Koblitz–Ogus function takes three values, so it is not covered. Huber and Wüstholz [11] prove the period conjecture for 1-periods; they exclude products of periods explicitly ([11, Remark 13.2], and the Prologue), and their Beta material is linear. By Remark 4.9, no Gamma monomial reduces our lists to theirs within the Koblitz–Ogus dictionary.

The other results. For Theorem 7.1: Waldschmidt's survey [31] (Conjecture 42 and Thm. 24), the account of Ramachandra's δ-conjecture in [33], and §3.5 of [25] (one-parameter subgroups; mixed $E \times \mathbb{G}_\mathrm{m}$ is not treated there, and there is no CM refinement). For Theorems 5.1, 5.5: [23, Cor. 1] (elliptic and abelian strong six exponentials) and the conjugation arguments of Diaz [7] and Waldschmidt [32, Cor. 2.7, Thm. 4.1], which give the non-conjugation-stable cases as said in the introduction; [30, Thms. 4.2, 4.3], which need many more data and involve no period slope. For Theorem 6.2: Laurent [17], whose results are of degree one. For Theorem 8.1: the count [28, (4.2), Cor. 4.3] and [30, Thm. 4.2] (mixed $\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m} \times E$, seven columns, no CM); the numbers in Corollary 8.2 are artificial, and the content is the mechanism. For Theorem 8.4: [28, 27, 23] (for signature (r, s) with $r, s \geq 1$ the weight of [23] is $\alpha_K(A) = 2g$, with no saving) and [11], which gives only the case $t \in \overline{\mathbb{Q}}$. Roy's Luminy paper [22] (read in full) has only general bounds, not statements at particular periods. We also looked in Waldschmidt's survey [34] (read in full) and in [30, 32], and searched arXiv for 2024–26 papers stating four exponentials or a subgroup-theorem count at CM, Beta or abelian periods, without finding any.

Not consulted. We could not consult Bertrand [4] (Schneider–Lang theorems on abelian varieties with endomorphisms; the most likely earlier source for Theorem 8.4), Waldschmidt [29], Baker–Wüstholz [2, §6.2], Ramachandra's papers in Acta Arith. 14 (1968), or the MathSciNet citation lists of [23] and [28] (the zbMATH lists we saw are nearly empty). Any of these might contain some of our statements. Every result here is a specialisation of the same published theorems, so this residual risk is shared by all of them.

Review. All review so far is by AI referees; no human expert has reviewed the definitions or the proofs. The results and proofs were found and written with substantial use of AI tools. Recent literature in this area includes unreviewed, partly AI-generated preprints; we cite none of them.

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