Critical edges in labelled complete graphs

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Abstract

In a complete graph whose edges have labels $1, \dots, \ell$, an edge is critical for a matching if its label is smaller than that of a matching edge sharing an endpoint. The parameter $\gamma(\ell)$ is the asymptotic worst-case, over labelings, of the smallest critical-edge count obtainable by choosing the matching, normalized by the number of pairs of covered vertices; $\gamma'(\ell)$ is its perfect-matching version. These parameters arise in bounds for finding cliques in random graphs by edge queries in a prescribed number of rounds. We prove

$$0.419772820486\ldots \le \gamma'(4) \le \gamma(4) \le 0.4334035.$$

For every $\ell \ge 2$ we give an explicit construction proving $\gamma'(\ell) \ge 1/2 - 1/[4(\ell-1)]$. We also improve the state-of-the-art upper bounds for general ℓ, obtaining

$$\gamma(\ell) \le \frac{1}{2} - \frac{4}{5 \cdot 2^{\ell}} + O(2^{-3\ell/2}) \qquad (\ell \to \infty),$$

together with explicit finite bounds. The proofs of upper bounds compare matchings obtained by local exchanges. The proofs of lower bounds use labelings that are constant on pairs of vertex parts; the stronger four-label construction has six parts and an algebraic value of degree six. Exact-arithmetic verification programs and their data are included for the computer-assisted parts of the proofs.

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1 Introduction

A matching in an edge-labelled complete graph assigns a label to each vertex it covers: the label of its matching edge. We ask how to choose a matching so that few graph edges have a smaller label than at least one of the matching edges incident to them, that is, smaller than at least one of its endpoints. This extremal problem was introduced by Csóka and Pongrácz [1] in their study of algorithms that find cliques and dense subgraphs by querying edges of a random graph. We improve both upper bounds for arbitrary labelings and lower bounds obtained from explicit constructions.

1.1 The extremal parameters

Let K_N be a complete graph on N vertices. Let $\lambda : E(K_N) \to \{1, \dots, \ell\}$ be an edge labeling and let $\mathcal{M}$ be a matching of size M. For a covered vertex v, define $b_{\mathcal{M}}(v)$ to be the label of the unique matching edge incident to v, and put $b_{\mathcal{M}}(v) = 0$ for an uncovered vertex. We call this number the *birth label* of v. An edge uv is *critical* for $\mathcal{M}$ when

$$\lambda(uv) < \max\{b_{\mathcal{M}}(u), b_{\mathcal{M}}(v)\}.$$

For example, if u is matched by an edge of label 3, then every edge of label 1 or 2 incident to u is critical. A matching edge itself is never critical. An edge from a covered vertex to an uncovered vertex can be critical; such an edge is called *outward critical*.

Write $C_\lambda(\mathcal{M})$ for the total number of critical edges, including outward critical edges. Following [1], define

$$\gamma(\ell) = \limsup_{M\to\infty} \sup_{N\geq 2M} \max_{\lambda} \min_{|\mathcal{M}|=M} \frac{C_\lambda(\mathcal{M})}{\binom{2M}{2}}. \tag{1}$$

Thus the labeling is chosen first and the matching second. The normalizing denominator counts pairs among the $2M$ covered vertices, even though the numerator also counts outward critical edges. The perfect-matching version is

$$\gamma'(\ell) = \limsup_{M\to\infty} \max_{\lambda : E(K_{2M}) \to [\ell]} \min_{\mathcal{M} \text{ perfect}} \frac{C_\lambda(\mathcal{M})}{\binom{2M}{2}}, \qquad [\ell] = \{1, \dots, \ell\}.$$

In particular $\gamma'(\ell) \le \gamma(\ell)$. An upper bound for $\gamma(\ell)$ must work for arbitrarily many uncovered vertices; a lower construction on $2M$ vertices already bounds both parameters from below.

Csóka and Pongrácz [1] proved $\gamma'(2) = \gamma(2) = 1/4$ and $\gamma'(3) = \gamma(3) = 3/8$. For four labels their estimates were

$$\frac{5}{12} \le \gamma'(4) \le \gamma(4) \le \frac{7}{16}.$$

For every $\ell \ge 4$ they established

$$\gamma(\ell) \le \frac{1}{2} - \frac{1}{3 \cdot 2^{\ell-1} - 4\ell + 8}. \tag{2}$$

They also showed that the corresponding parameter with arbitrarily many labels equals $1/2$.

1.2 Results

Theorem 1.1. *Let α_L be the algebraic number specified in Section 17. Then*

$$0.41977282048624007772\ldots = \alpha_L \le \gamma'(4) \le \gamma(4) \le \frac{866807}{2000000} = 0.4334035.$$

The lower bound is realized by a labeling on six vertex parts whose proportions belong to a number field of degree six.

The lower bound extends the search for explicit extremal labelings, whereas the upper bound asserts that every four-label instance has an appropriate matching. The two estimates leave a gap; no equality between $\gamma'(4)$ and $\gamma(4)$ is assumed.

Theorem 1.2. *For every integer $\ell \ge 2$,*

$$\gamma'(\ell) \ge \frac{1}{2} - \frac{1}{4(\ell-1)}.$$

For $\ell \ge 5$, put $k = 2^{\ell-2}$ and $S = 3k - 2\ell + 4$. Then

$$\gamma(\ell) \le \frac{1}{2} - \frac{1}{2S} - \frac{1}{648k}.$$

In addition,

$$\gamma(\ell) \le \frac{1}{2} - \frac{1}{5k} + \frac{2\sqrt{2}}{25k^{3/2}} + O\left(\frac{\log k}{k^2}\right) \qquad (\ell \to \infty).$$

The general lower construction has ℓ vertex parts. Its exact finite matching minimum and the optimality of its part proportions are proved in Section 2. The optimality is within the specified labeling family, with one part of mass at least $1/2$. For $\ell = 4$ it recovers $5/12$; the six-part construction of Theorem 1.1 gives the stronger value above.

The finite general upper bound improves (2) by $1/(162 \cdot 2^{\ell})$. The asymptotic upper bound increases the leading gap below $1/2$ from $1/(6k)$ to $1/(5k)$, a factor of $6/5$. The error term in that statement is necessary. For finite ℓ, Theorem 16.1 gives a rational two-variable quadratic formula; it is this finite formula, and the bound in Section 15, that produce the following certified estimates.

ℓ	Lower bound	Upper bound
4	0.419772820486	0.4334035
5	7/16	0.471876610486
6	9/20	0.487382111461
7	11/24	0.494116218018
8	13/28	0.497175259512
9	15/32	0.498622167790
10	17/36	0.499299501525
11	19/40	0.499630626700
12	21/44	0.499810572146

The lower bounds apply to γ' and hence to γ. The upper bounds apply to γ and hence to γ'. Displayed decimal lower bounds are rounded downward, and upper bounds upward. The general-label upper entries are verified rational choices of the parameters in the finite formulas; they are not claims of optimality over all possible matching arguments.

1.3 Proof strategy and organization

The general lower construction in Section 2 provides an elementary example of the problem. Its proof counts critical edges for an arbitrary perfect matching, without a computer calculation.

For the upper bounds, we start with a matching chosen by the order of [1]. This order prevents several small exchanges that would reduce its weight. Those restrictions bound the number of critical edges inside and between the sets of vertices covered by matching edges of each label. The original estimate is sharp only if several of these counts are simultaneously large. We show that such a configuration permits other matchings with smaller critical-edge counts.

The four-label argument begins in Section 3. We describe the exchanges explicitly and keep track of the edges that prevent each exchange. This leads to a finite family of quadratic inequalities. Section 13 combines these inequalities to prove the numerical bound. Each combination has rational coefficients, and its validity is checked by an exact matrix factorization. The combinatorial derivations are given in the text; the supplied programs check the resulting identities, signs, and coverage of the parameter ranges.

For general ℓ, alternating paths provide a systematic family of exchanges. Sections 14–16 derive the corresponding finite and asymptotic upper bounds. The estimates account for outward critical edges, so the number of uncovered vertices does not enter the error terms.

Section 17 proves the six-part lower bound. For a labeling constant on pairs of vertex parts, a matching is described by the amount it matches between each pair of parts. These amounts satisfy linear constraints, and the noncritical-edge count is a quadratic function of them. We bound this function over the entire feasible set using exact algebraic arithmetic. The algebraic number and the part proportions are specified explicitly. Appendix A examine small perturbations of this construction. Section 18 gives the verification commands and identifies the data needed for each computational claim.

Two normalizations occur in the proofs. Upper-bound counts are divided by M^2, so their conversion to γ includes a factor of $1/2$, since $\binom{2M}{2} = 2M^2 - M$. Lower constructions use vertex masses summing to one and the density of ordered critical pairs; this is already on the γ' scale. The convention is stated again at the start of each argument.

1.4 Application to clique queries

Consider an Erdős–Rényi graph $G(n, 1/2)$ whose edges are initially unknown. An algorithm may query at most n^δ pairs in ℓ rounds, choosing all queries of a round before receiving the answers to that round. Let $\alpha_*(\delta, \ell)$ be the supremum of the constants α for which such an algorithm finds a clique of size $(\alpha + o(1)) \log_2 n$ with probability tending to one. Computational time is unrestricted.

Feige, Gamarnik, Neeman, Rácz, and Tetali [2] obtained upper bounds in this query model, and Feige and Ferster [3] sharpened the analysis for two and three rounds. The clique-query theorem of Csóka and Pongrácz [1] gives, for $1 \le \delta \le 2$ and $\ell \ge 3$,

$$\alpha_*(\delta, \ell) \le 1 + \sqrt{1 - \frac{(2-\delta)^2}{4\gamma(\ell)}}.$$

Consequently any upper bound G_ℓ for $\gamma(\ell)$ proved here can replace $\gamma(\ell)$ on the right. In particular one may use $G_4 = 0.4334035$. Our lower constructions bound the combinatorial parameters; they do not assert the existence of clique-query algorithms attaining the corresponding expression.

2 A lower bound for every number of labels

The following construction extends the four-part labeling of Csóka and Pongrácz [1], which gives the lower bound 5/12 for four labels. Its advantage is that the minimum over all perfect matchings can be calculated exactly for every number of labels.

Theorem 2.1. *For every integer $\ell \geq 2$,*

$$\gamma(\ell) \geq \gamma'(\ell) \geq \frac{1}{2} - \frac{1}{4(\ell-1)}.$$

In particular these lower bounds tend to $1/2$ as $\ell \to \infty$.

Put $m = \ell - 1$. The construction has parts $A_1, \dots, A_m, R$ with vertex proportions

$$a_1 = \frac{1}{4m}, \qquad a_i = \frac{1}{2m} \quad (2 \leq i \leq m), \qquad r = \frac{2m+1}{4m}. \tag{3}$$

The part R provides a reservoir of possible matching partners for the other parts, which we call satellites. Labels are constant on each pair of parts and are given by

$$\lambda(A_i, A_j) = \max\{1, \min(i,j) - 1\}, \qquad \lambda(A_i, R) = i, \qquad \lambda(R,R) = m+1. \tag{4}$$

The first rule includes $i = j$ and hence specifies all internal labels. When $\ell = 4$, the proportions are $(1,2,2,7)/12$, and the label matrix is exactly the one in [1].

Proof of Theorem 2.1. We prove an exact finite statement. For any positive integer q, take $N = 4mq$ vertices and parts of sizes

$$|A_1| = q, \qquad |A_i| = 2q \ (2 \leq i \leq m), \qquad |R| = (2m+1)q.$$

Write $A = \bigcup_{i=1}^{m} A_i$, $s = |A| = (2m-1)q$, and $r_N = |R|$. We claim that the minimum number of critical edges over all perfect matchings is

$$(4m^2 - 2m)q^2 - (m-1)q. \tag{5}$$

Fix a perfect matching. Let x_i count its edges between A_i and R, and put

$$X = \sum_{i=2}^{m} x_i, \qquad p_i = \sum_{j<i} |A_j|.$$

Thus $p_1 = 0$ and $p_i = (2i-3)q$ for $i \geq 2$. Every vertex in A_i has birth at most i, because every edge incident to A_i has label at most i. An edge between A_i and R, whose label is i, is therefore critical if and only if its endpoint in R has birth greater than i. Exactly $r_N - \sum_{j \leq i} x_j$ vertices of R have this property. Consequently the number of critical edges between A and R is

$$\sum_i |A_i| \left(r_N - \sum_{j \leq i} x_j \right) = s r_N - \sum_i x_i (s - p_i). \tag{6}$$

Let U consist of those vertices in A_i, $i \geq 2$, that are matched to R. Every edge within A incident to U is critical: the vertex in $A_i \cap U$ has birth i, whereas its edge to any satellite has label smaller than i. Since $|U| = X$, these edges number

$$\binom{s}{2} - \binom{s-X}{2} = sX - \frac{X^2 + X}{2}.$$

Adding this to (6) yields

$$C_\lambda(\mathcal{M}) \geq s(r_N - x_1) + \sum_{i=2}^{m} p_i x_i - \frac{X^2 + X}{2}. \tag{7}$$

All $0 \leq x_i \leq 2q$ for $i \geq 2$. The identity

$$2\sum_{i=2}^{m} p_i x_i - X^2 = \sum_{i=2}^{m} x_i(2q - x_i) + 2 \sum_{2 \leq i < j \leq m} x_j(2q - x_i) \geq 0$$

therefore applies. Moreover $x_1 \leq q$ and $X \leq 2(m-1)q$. Equation (7) now gives

$$C_\lambda(\mathcal{M}) \geq s(r_N - q) - (m-1)q = (4m^2 - 2m)q^2 - (m-1)q.$$

Empty sums are interpreted as zero, so the argument includes $m = 1$.

Equality is attained by matching every vertex of A to a distinct vertex of R, and matching the remaining $2q$ vertices of R internally. Indeed $x_1 = q$, $x_i = 2q$ for $i \geq 2$, and the only critical edges within A are precisely those incident to U. No edge inside R is critical. This proves (5). Dividing by $\binom{4mq}{2}$ and letting $q \to \infty$ gives

$$\lim_{q \to \infty} \frac{(4m^2 - 2m)q^2 - (m-1)q}{\binom{4mq}{2}} = \frac{2m-1}{4m},$$

which proves the theorem by the definition of $\gamma'(\ell)$. □

The theorem gives $1/4, 3/8, 5/12, 7/16, 9/20$ for $\ell = 2, 3, 4, 5, 6$, respectively. For four labels the six-part construction of Theorem 17.1 gives a stronger bound. The present family supplies a simple bound for every ℓ.

Proposition 2.2. *Keep the labels in (4), and allow arbitrary nonnegative satellite proportions $a_1, \ldots, a_m$ and reservoir proportion $r = 1 - \sum_i a_i \geq 1/2$. Among these choices, the largest possible minimum asymptotic critical density is $(2m-1)/(4m)$. It is attained only at the proportions (3).*

Proof. Put $S = \sum_i a_i \leq 1/2$. There is a matching sending every satellite vertex into the reservoir, with the remaining reservoir vertices matched internally. Counting ordered pairs, its critical density is

$$G(a) = 2S - 2S^2 - \left(2a_1^2 + \sum_{i=2}^{m} a_i^2 \right).$$

For clarity, its satellite contribution is $S^2 - a_1^2$ and its satellite–reservoir contribution is $2\sum_i a_i(r - \sum_{j \leq i} a_j)$; their sum is the displayed expression. The minimum critical density over all matchings cannot exceed $G(a)$.

Weighted Cauchy–Schwarz gives

$$2a_1^2 + \sum_{i=2}^{m} a_i^2 \geq \frac{2S^2}{2m-1}.$$

It follows that

$$G(a) \leq 2S - \frac{4m}{2m-1} S^2 = \frac{2m-1}{4m} - \frac{4m}{2m-1} \left(S - \frac{2m-1}{4m} \right)^2.$$

Equality requires $S = (2m-1)/(4m)$ and $a_2 = \cdots = a_m = 2a_1$, giving exactly (3). For $m = 1$ the same conclusion follows directly from $S = a_1$. The finite construction already proved attains this value. Approximation by rational proportions justifies the count for arbitrary real masses. □

This optimality statement concerns the specified labels and the condition $r \geq 1/2$. It makes no assertion about other labelings or about the same labeling with a reservoir of mass less than $1/2$.

3 The initial matching and its structural properties

In the four-label problem, fix a labeling of K_N and an integer $M \le N/2$. Among all M-edge matchings choose $\mathcal{M}_0$ minimizing lexicographically

$$(n_2 + 2n_3 + 4n_4,\ n_3 + 2n_4,\ n_4),$$

where n_i is the number of its edges of label i. This is the matching order of Csóka and Pongrácz [1]. Formal weights

$$w_1 = 0, \quad w_2 = 1, \quad w_3 = 2 + \varepsilon, \quad w_4 = 4 + 2\varepsilon + \varepsilon^2$$

give an equivalent notation for this order. We compare polynomials at their first differing coefficient. Alternatively one may choose a sufficiently small positive ε for the given finite instance. The formal order avoids any need for a uniform choice of ε as M grows.

The coefficients of $1, \varepsilon, \varepsilon^2$ are called the primary, secondary, and tertiary weights, respectively. A replacement that decreases the first coefficient where they differ contradicts the choice of $\mathcal{M}_0$.

Let X_i be the endpoints of the initial label-i matching edges. An *outward* edge joins a covered vertex to an uncovered vertex. An outward critical edge could replace its incident matching edge and strictly decrease the weight, so none exists. Between two differently labeled matching edges, the four cross edges form two complementary pairs. Both edges of a complementary pair cannot be critical: if the larger original label is j, their labels would both be below j, and $2w_{j-1} < w_j$ shows that the replacement has smaller formal weight. Thus each such rectangle has at most two critical edges. Its deficit from two is nonnegative, even before summing over rectangles. There are no internal critical edges in X_1. There are none in X_4 either, since an edge of label at most three and its complementary edge would replace two label-4 matching edges at smaller primary weight.

3.1 Decomposing the label-1 graph inside X_2

Color the original X_2 matching red and its label-1 edges blue. An alternating path or cycle has edges alternating between red and blue; the paths used here begin and end with a red edge. There is no alternating cycle. There is also no alternating path with four blue edges: completing such a path by the edge between its ends would replace five red edges of primary cost five by four blue edges and an edge of primary cost at most four.

The following special case of the peeling argument in [1] gives the decomposition we need. We include the proof.

Lemma 3.1. *Let a graph consist of a red perfect matching and a disjoint set of blue edges. Suppose there is no alternating cycle and no alternating path with four blue edges. Remove every red pair having an endpoint of blue degree at most three. The remaining graph has no alternating path with two blue edges, and each of its red pairs has an endpoint of residual blue degree at most one.*

Proof. For a longest red-ended alternating path with p blue edges, each endpoint has blue degree at most p. Indeed, a blue neighbor outside the path and its red mate would extend the path. Among the p other red pairs on the path, at most one endpoint of each pair can be a blue neighbor: the other choice closes an alternating cycle.

Let G' be the remaining graph. If it is empty there is nothing to prove. Otherwise let P be a longest red-ended alternating path in G', with endpoints u, v and p blue edges. We have $p \le 3$, while every vertex of G' has blue degree at least four in the original graph. If $p = 3$, u has a blue neighbor in a deleted pair. Adding that blue edge and its neighbor's red edge extends P to a path with four blue edges, a contradiction. If $p = 2$, both u and v have at least two blue neighbors among the deleted vertices. Choose distinct such neighbors u', v'. They cannot be red mates: the two blue edges, the path P, and the red edge $u'v'$ would form an alternating cycle.

Thus their red pairs are distinct and outside P. Extending at both ends by the two blue edges and these red pairs again gives four blue edges. Hence $p \le 1$.

Finally, if both ends of a residual red edge had blue degree at least two, choose distinct blue neighbors of its endpoints. The two neighbors cannot be red mates, by the alternating-cycle condition, so adding their red mates gives a red-ended alternating path with two blue edges. This contradiction proves the last claim. □

Apply the lemma by removing every red pair having an endpoint of blue degree at most three in the original blue graph. Call the low-degree endpoints B and their mates A. The remaining graph has no alternating path with two blue edges. In each remaining red pair select an endpoint of residual blue degree at most one, call it D, and call its mate C. Deleting at most $O(M)$ blue incidences therefore leaves blue edges only in

$$AA, \quad AC, \quad AD, \quad CC.$$

The AB and CD pairs have masses d and q after division by M. Here $d + q = x = |X_2|/(2M)$. The number of possible blue template pairs is

$$\frac{d^2}{2} + 2dq + \frac{q^2}{2} = \frac{3x^2}{4} - \frac{(d-q)^2}{4}$$

in units of M^2, up to $O(1/M)$. Write $E_{AA}, E_{AC}, E_{AD}, E_{CC}$ for their missing-blue densities. We call these four permitted regions the *template*. A missing blue pair in a region is a vertex pair whose edge has label at least two. At most a constant times M blue edges have been discarded, independently of the number of uncovered vertices.

Inside X_3, the graph of labels at most two has no alternating cycle or alternating path with two low-label edges. In the path case, three old label-3 edges have primary cost six; the two low-label edges and a closing label-4 edge have primary cost at most six, and in the tie case secondary cost at most two instead of three. The last paragraph of the preceding proof shows that each original label-3 pair has an endpoint of degree at most one in the low-label graph. Discard all edges incident to one such endpoint from each pair; this removes at most zM edges, where $z = |X_3|/(2M)$. All remaining low-label edges lie on the other zM endpoints, giving the bound $z^2M^2/2 + O(M)$.

3.2 Matching facts used below

We use three classical results with their finite forms. Kőnig's theorem [4] identifies the maximum matching size with the minimum vertex-cover size in a bipartite graph. In a bipartite graph with sides of sizes $m \le D$ and a missing edges, it implies a matching of size at least $m - a/D$: a cover of size t covers at most tD edges.

If an n-vertex graph has matching number k, the Erdős–Gallai bound [5] is

$$e(G) \le \max\left\{ \binom{2k+1}{2}, \; kn - \binom{k+1}{2} \right\}.$$

In particular $e(G) \le kn + O(n)$. Finally the Tutte–Berge formula [6, 7, 8] gives

$$2\nu(G) = \min_{S \subseteq V(G)} \big(|V(G)| + |S| - o(G - S) \big),$$

where $o(G - S)$ counts odd components. We explicitly derive the additional stability estimate needed here, rather than assume a stability theorem from the literature.

We will also use the following dense-graph consequence of the Erdős–Gallai inequality. Suppose a graph has tM vertices and at most $cM^2 + O(M)$ missing edges, where $c \le 9t^2/50$. If a maximum matching covers RM vertices, then

$$R^2 \ge t^2 - 2c - o(1), \qquad R \ge 4t/5 - o(1). \tag{8}$$

Indeed the two asymptotic branches of the edge bound are $R^2/2$ and $Rt/2 - R^2/8$. For $R \le 4t/5$, both are at most $8t^2/25$, whereas the edge density is at least $t^2/2 - c \ge 8t^2/25$. For $R \ge 4t/5$ the first branch is the larger one, which gives (8). Equality at the branch boundary and the $O(M)$ terms are accommodated by the stated $o(1)$ errors.

We repeatedly augment a matching while preserving all vertices it already covers. This is possible because an augmenting path changes only the matching status of its two previously uncovered endpoints; the internal vertices stay covered [7]. A related elementary count will be useful. If U is the unmatched set of a maximum matching, then the endpoints of any matched edge have at most $|U| + 1$ neighbors in U in total. Otherwise there are two distinct unmatched neighbors, one at each end, giving an augmenting path of three edges. The possible extra one contributes only $O(M)$ in the applications below.

4 A matching replacement on alternating paths

We construct several M-edge matchings and bound the smallest of their critical-edge counts. Throughout the four-label upper-bound proof, vertex-set sizes are divided by M and edge counts by M^2. In particular, write $C_0 = C_\lambda(\mathcal{M}_0)/M^2$, and use analogous subscripts for the other matchings. A bound $C \le 2G + o(1)$ therefore implies $\gamma(4) \le G$. An expression such as $UV = 3$ means $\lambda(UV) = 3$, and $UV \le 3$ means $\lambda(UV) \le 3$. All $o(1)$ terms tend to zero uniformly in $N \ge 2M$; only $O(M)$ covered vertices enter the auxiliary graphs.

Write y, x, z, w for the normalized numbers of edges of labels $1, 2, 3, 4$ in $\mathcal{M}_0$, respectively; their sum is one. Thus $|X_1|/M = 2y$, and similarly for the other three classes. We call vertices of X_4 *top vertices*. The decomposition in Section 3 gives label-2 pairs AB and CD, of masses d, q, where $d + q = x$. Set

$$b = |d - q|/2, \quad m = \min(d, q) = x/2 - b, \quad B = 1 - y^2 - x^2/4 - z^2/2 - w^2, \qquad B_0 = B - b^2.$$

We write $h = |d - q| = 2b$ when specifying the imbalance between the two families of label-2 pairs. Let $E_{AA}, E_{AC}, E_{AD}, E_{CC}$ count missing blue pairs in the four permitted blue regions, and let E be their sum. Let C_{ij} be the normalized original critical count between X_i and X_j when $i \ne j$, and within X_i when $i = j$. For $i < j$, define the cross-class deficit e_{ij} as the difference between the two-critical-edges-per-matching-pair bound and the actual critical count between X_i and X_j. For example, if this count is $C_{12}M^2$, then $e_{12} = 2xy - C_{12} \ge 0$. We also use the abbreviations

$$a = E_{AD}, \qquad c = E_{CC}, \qquad j = E_{AC}.$$

The original count satisfies

$$C_0 \le B - b^2 - E - \sum_{i<j} e_{ij} + o(1). \tag{9}$$

Here the X_2 count is at most $3x^2/4 - b^2 - E + o(1)$, the X_3 count is at most $z^2/2 + o(1)$, and the other internal counts vanish. Summing these and the cross-class bounds proves (9).

4.1 The retained alternating paths

Choose a blue matching from some A vertices to some D vertices, and a blue perfect matching on the original C mates of those D vertices. These two matchings, together with the original red edges, form vertex-disjoint alternating paths of the form

$$B_1A_1D_1C_1C_2D_2A_2B_2.$$

The edges $B_1A_1, D_1C_1, C_2D_2, A_2B_2$ are original label-2 edges, and A_1D_1, C_1C_2, D_2A_2 are blue. Replace the four original edges by those three blue edges and B_1B_2. Minimality forces B_1B_2 to

have label 4, so the primary weight remains four. We call a union of these paths a *neutral core*. The four sets of vertices retained in it each have the same mass r, where $0 \le r \le \min(d, q)$.

Fix a neutral core with maximum possible r among all such joint choices of blue AD and CC matchings. Its replacement matching uses AD and CC edges at label 1 and BB edges at label 4, and agrees with $\mathcal{M}_0$ elsewhere. Subsequent references to retained or omitted vertices are relative to this choice, unless a different core is explicitly constructed. Its normalized critical count is C_r.

Let H_3M^2 count the label-3 edges from retained B to X_3, and put $h_3 = rz - H_3$. We call an original label-3 pair *good* if it can be oriented PQ with P having at least four label-3 neighbors in retained B; the remaining pairs are *exceptional*. The switching argument below shows that $h_3 \ge -o(1)$ and that the exceptional pair mass is at most $h_3/r + o(1)$ when r is bounded away from zero. This condition, and other parameter conditions used below, are established in Sections 8 and 13 before the corresponding inequalities are used in the final optimization. For brevity write $f_R = r(x + z - 3y) - 3r^2$.

For $z, w > 0$, the core comparison derived in Section 4.2 gives

$$\begin{aligned} C_r \le B - b^2 + r(x + z - 3y) - 3r^2 - (1 - y/z)h_3 - e_{13} - e_{34} \\ + (r/(2w) - 1)e_{14} + (y/z - 1)e_{23} + (y/w - 1)e_{24} + o(1). \end{aligned} \tag{10}$$

4.2 Derivation of the core comparison

We derive (10) by counting the changes on disjoint edge classes. The retained sets A, B, C, D each have mass r. The new matching uses blue AD and CC edges and label-4 BB edges, whereas the original matching used $AB(2)$ and $CD(2)$. Every alternating four-red, three-blue path has a label-4 edge between its two B ends: an edge of label at most three would replace primary cost four by cost at most two. Thus the rematching is primary-minimal. Its entire new label-4 class is a label-4 clique, and its B vertices may be paired arbitrarily.

For each original label-3 edge PQ, its two endpoints cannot have distinct label-3 neighbors B, B' in the retained set: re-pairing the B vertices would permit $BB'(4), PQ(3) \to BP(3), B'Q(3)$, which decreases the primary weight. Thus its total label-3 degree into retained B is at most $rM + O(1)$. If the pair is exceptional, that degree is at most six. Summing proves $H_3 \le rz + o(1)$ and the exceptional-mass bound $h_3/r + o(1)$ used in Section 4.

For each retained D, write A for its new mate and B for the old mate of A. For every other retained B', the switch

$$BB'(4), AD(1) \longrightarrow B'D(\le 3), AB(2)$$

would reduce primary weight. Hence there are at most rM exceptional BD contacts. The analogous switch including an original top edge shows $DT = 4$ for every old top vertex T.

Let $PQ(3)$ be good, with P having at least four retained B neighbors of label three. Switching PQ with a suitably re-paired BB' shows $QB = 4$. Choosing a neighbor B' distinct from the old mate B of the A paired with a given D, the switch

$$BB'(4), AD(1), PQ(3) \longrightarrow PB'(3), QD(\le 3), AB(2)$$

shows $QD = 4$. If CC' is a core edge and D is the old mate of C', adding CC' to this switch and replacing it by $QC(\le 2), C'D(2)$ proves $QC \ge 3$. These arguments use only strict primary improvements. The secondary order will be used separately for the internal label-3 deficit.

Put $h_1 = 2ry - H_1$, where H_1 counts contacts of labels at most three between retained B and old X_1. The changes from the old matching to the core satisfy the following count table:

Pairs affected	Upper bound for the change/M^2
inside the four retained sets	$-r^2 + E + o(1)$
retained versus omitted X_2	$r(x - 2r) + o(1)$
retained versus X_1	$-2ry - h_1 + e_{12} + o(1)$
retained versus X_3	$rz - h_3$
retained versus X_4	0

For the first row, at least $3r^2 - E - o(1)$ old blue contacts were critical; afterwards only BA, BC and $O(M)$ exceptional BD contacts can be critical, giving at most $2r^2 + o(1)$. For the second row, only B promotions can increase the count. Pairwise minimality between a new $BB(4)$ edge and an omitted label-2 matching edge bounds its critical cross contacts by two.

For the third row, the old selected AB, CD pairs have at least $4ry - e_{12}$ blue contacts with X_1, since the cross-rectangle deficits are individually nonnegative. The change is their removal plus the H_1 contacts made critical by B. Against X_3, only the newly critical label-3 contacts from B contribute; all other birth changes are below the old birth three. Against X_4 no maximum endpoint birth changes. Both matchings have no outward critical edges. Adding these changes to (9) gives

$$C_r \le B - b^2 + r(x + z - 2y) - 3r^2 - h_1 - h_3 - \sum_{(i,j)\neq(1,2)} e_{ij} + o(1).$$

It remains to relate h_1 and h_3. For a good PQ let g_P be its retained-B label-3 degree, and let b_Q, d_P count, respectively, the retained A, D contacts that fail to have label at most two. The original half bounds and the exclusions $QB = QD = 4, QC \ge 3$ imply $b_Q + d_P \le e_P + r - g_P$, where e_P is its share of e_{23}. For $V \in X_1$ and $T \in X_4$, let a_V, c_T count the failures of $VB \le 3$ and $TC \le 3$ in the aligned core slots. Then $\sum_V a_V = h_1$ and $\sum_T c_T \le e_{24}$.

For original matching edges $UV(1), TT'(4), AB(2), CD(2), PQ(3)$, the replacement

$$UT(\le 3),\ T'C(\le 3),\ DP(\le 2),\ QA(\le 2),\ BV(\le 3)$$

has weight at most $8 + 3\varepsilon$, whereas the old weight is $8 + 3\varepsilon + \varepsilon^2$. Thus whenever UT is critical, the four failures cover the aligned core indices, yielding $a_V + c_{T'} \ge g_P - e_P$. Sum first over good pairs. A nongood pair has only $O(1)$ retained B contacts of label three, so the omitted contribution is $o(1)$ after normalization. Sum next over the $2yw - e_{14}$ critical X_1–X_4 edges. The endpoint degree bounds $2w$ and $2y$ give

$$(2yw - e_{14})(rz - h_3 - e_{23}) \le 2z(wh_1 + ye_{24}) + o(1).$$

On the localized domain $z, w > 0$, expanding and discarding favorable products proves

$$h_1 \ge ry - \frac{y}{z}h_3 - \frac{r}{2w}e_{14} - \frac{y}{z}e_{23} - \frac{y}{w}e_{24} - o(1).$$

Substituting in the count table proves (10). The same proof retains $-I_3$ if the old internal-X_3 count is written as $z^2/2 - I_3$.

In particular, on $2y \le z$, $y \le w$, $r \le 2w$,

$$C_r \le B - b^2 + r(x + z - 3y) - 3r^2 - h_3/2 - e_{13} - e_{34} + o(1). \tag{11}$$

If $y \le 3z/8$, the coefficient $1/2$ of h_3 in (11) can be replaced by $5/8$, directly from $1 - y/z \ge 5/8$ in (10).

5 Bounds on the size of the neutral core

Let $a = E_{AD}$ and $c = E_{CC}$ be the missing blue-edge densities in the indicated rectangles, let $m = \min(d, q)$, and let $D = \max(d, q)$. All densities are normalized by M^2, while vertex-set sizes are normalized by M. The inequalities below have uniform $o(1)$ errors in the finite problem. We suppress these errors in the displayed asymptotic formulas. The formulas involving $1/m$ or $1/D$ concern $m > 0$; this is guaranteed on the parameter domain retained in Section 8.

First find a maximum blue AD matching, of size tM with $t \ge m - a/D$, by the bipartite matching bound in Section 3. Then find a maximum blue CC matching on the corresponding C

endpoints. Its endpoints and their assigned AD edges form a neutral core. In the dense regime $E \le 9m^2/50$, (8) gives

$$r^2 \ge t^2 - 2c \ge (m - a/D)^2 - 2c.$$

Indeed $c \le 9m^2/50 - a \le (9/50)(m - a/D)^2$, so the density hypothesis of that corollary holds. Consequently

$$\frac{m}{D}a + c \ge \frac{m^2 - r^2}{2} + \frac{a^2}{2D^2}. \tag{12}$$

Missing AA and AC edges have no role in this packing obstruction. Keeping a and c separate will allow the later count bounds to use the constraints on each edge class.

In the larger-CD branch, write $d = m$ and $q = d + h$. If $h \ge 3d/10$ and $E \le 9d^2/50$, first find a maximum blue matching in the *entire* C graph. Its covered vertex mass is at least

$$\sqrt{q^2 - 2c} \ge \sqrt{(13d/10)^2 - 9d^2/25} = \sqrt{133/100}\, d > d.$$

Here (8) is applicable since $c \le 9d^2/50 \le 9q^2/50$. Select $dM + O(1)$ endpoints that are paired by this blue matching. Between their old D mates and A, a maximum blue matching misses at most $(a/d)M + O(1)$ vertices, by the bipartite vertex-cover bound. Delete every selected CC matching edge that meets one of these unmatched D mates. At most twice as many endpoints are deleted. The surviving AD and CC edges give disjoint alternating paths, and hence an actual neutral core satisfying

$$r \ge d - \frac{2a}{d}, \qquad a \ge \frac{d(d - r)}{2}. \tag{13}$$

Choose a core of maximum size among all joint blue AD and CC packings in the peeled template. Such a core is at least as large as each of the preceding constructions. Increasing its size preserves the lower bounds in both (12) and (13). All switching and counting statements used subsequently apply to this actual core.

In particular, if $r < d$ in this branch, (13) forces a positive number of missing AD edges, regardless of the number of missing CC edges.

5.1 Averaging the reverse construction

We can also average the choice of the dM endpoints in this construction. Suppose a blue CC matching covers LM endpoints, where $L \ge \sqrt{q^2 - 2c} - o(1)$. Choose $dM/2$ of its matching edges uniformly. The expected number of missing AD edges on the chosen D mates is at most $(d/L)aM^2$. Some choice attains this bound. Its balanced AD matching therefore misses at most $(a/L)M + O(1)$ vertices. Deleting whole CC edges again costs at most twice this number. Choosing the largest actual core among all constructions gives

$$2a \ge \sqrt{q^2 - 2c}\,(d - r) - o(1). \tag{14}$$

For a ratio interval $v_0 \le h/d \le v_1$ with $v_0 \ge 3/10$, set

$$\alpha = \frac{1}{1 + v_0}, \qquad \beta = \frac{1}{9(1 + v_0)}, \qquad K = \frac{d^2 - r^2}{2}, \quad J = \frac{q(d - r)}{2}.$$

The elementary inequality $\sqrt{1 - 2t} \ge 1 - 10t/9$ for $0 \le t \le 9/50$ follows by squaring. Since $(d - r)/q \le 1/[5(1 + v_0)]$, the two packing inequalities imply

$$\alpha a + c \ge K, \qquad a + \beta c \ge J. \tag{15}$$

These inequalities apply to the fixed maximum core of Section 4. A core constructed later on a restricted set of vertices will have its own size parameter.

6 Restrictions on labels near the retained vertices

The count bounds will depend on missing edges both inside and outside the retained sets. This section relates those quantities and proves label restrictions that permit later matching replacements without creating critical edges to uncovered vertices.

6.1 Edges incident to the Q endpoint of a good pair

The core replacements imply

$$Qv \neq 1 \qquad \text{for every good } PQ \text{ and every old } v \in X_2. \tag{16}$$

Indeed the neutral core is a union of disjoint alternating paths, each consisting of four original red label-2 edges and three blue links, with two B endpoints. Four retained B neighbors of P belong to at least two such paths. Given an old red edge uv, choose one of these paths disjoint from uv, and name its endpoints B, B' so that $PB = 3$. If $Qu = 1$, replace the four red edges of the path, $uv(2)$, and $PQ(3)$ by its three blue links and $PB(3), Qu(1), vB'(\leq 4)$. The primary weight decreases from seven to at most six. This proves (16) using the original matching, with no assumption that u belongs to the retained core.

A good Q also has label four to every originally uncovered vertex. Re-pair a core edge $BB'(4)$ with $PB = 3$; then $BB'(4), PQ(3) \to PB(3), Qu(\leq 3)$ would reduce primary weight, leaving B' uncovered instead of u. Including an original top edge gives the analogous conclusion $QT = 4$. These label restrictions justify the later increases in the matching labels of good Q vertices, including when arbitrarily many vertices remain uncovered.

6.2 A Tutte–Berge packing stability inequality

Let c_* be the missing-blue density inside the actual retained C set, and put $K = (q^2 - r^2)/2$. The following relation holds when $d \geq q$:

$$\left(5 - \frac{4q}{r}\right) c_* \leq c - K + \frac{2r}{d}a + \frac{2a^2}{d^2} + o(1) \leq c - K + \frac{59}{25}a + o(1). \tag{17}$$

Its left coefficient is nonnegative on $r \geq 4q/5$.

Here is the argument. Extend the retained blue CC matching to a maximum matching covering mass R, without uncovering its initial vertices. Then $R \geq r$. Match the original D mates of its covered vertices to A using blue edges. The bipartite matching bound leaves at most $aM/d + O(1)$ of these D vertices unmatched. Deleting every CC edge meeting such a vertex's mate removes at most $2aM/d + O(1)$ endpoints and produces a neutral core. Maximality of the fixed core therefore gives $R \leq r + 2a/d + o(1)$. Let c_R count missing pairs within the covered set, and let $\delta = q - R$. A Tutte–Berge equality set S, of mass s, is fully matched to distinct odd components of the graph outside S; each remaining odd component has one unmatched vertex. Thus unmatched-to-covered blue edges have density at most $\delta s + o(1)$: edges to S give this term, while contacts within components total only $O(M)$. The covered vertices outside S occupy at least sM distinct components. Concentrating all remaining vertices in one component minimizes their missing pairs, giving

$$c_R \geq s(R - 3s/2) - o(1) \geq sR/4 - o(1), \qquad s \leq R/2.$$

Since the unmatched set is independent,

$$c \geq \frac{q^2 - R^2}{2} + c_R - \delta s - o(1) \geq \frac{q^2 - R^2}{2} + \left(5 - \frac{4q}{R}\right) c_R - o(1).$$

Use $c_* \le c_R$, the bounds on R, and $a \le 9d^2/50$ to obtain (17). The only standard input is the Tutte–Berge formula [6, 7, 8]; the displayed stability estimate follows from the component count above.

In particular, at $a = 0$, $c = K$, and $r > 4q/5$, this forces $c_* = o(1)$. The same missing-CC allowance cannot both reduce the core size and make its retained graph substantially incomplete.

6.3 A stronger bound when the retained graph is dense

In the proof of (17), the barrier set S has normalized size $s \le R/2$. A density hypothesis forces s to be smaller and therefore strengthens both the internal missing-edge estimate and the bound on blue edges from retained to omitted vertices.

Lemma 6.1. *Suppose $r \ge 7q/8$ and $c \le q^2/8$. Then*

$$\frac{7 - 4q/r}{3}\, c_* \le c - K + \frac{2r}{d}a + \frac{2a^2}{d^2} + o(1), \tag{18}$$

$$U \le \frac{2r}{d}a + \frac{4(q-r)}{7r-4q}\left(c - K + \frac{2r}{d}a + \frac{2a^2}{d^2}\right) + o(1). \tag{19}$$

Here c_* is the missing-blue density inside the retained C set, $K = (q^2 - r^2)/2$, and U is its cross-blue density to omitted C.

Proof. Extend the retained blue matching to a maximum one covering mass R, where $r \le R \le r + 2a/d + o(1)$, and let c_R be the missing density inside its covered set. The barrier calculation proving (17) gives

$$c \ge \frac{q^2 - R^2}{2} + \left(5 - \frac{4q}{R}\right) c_R - o(1).$$

If $c_R \ge R^2/8$, its right side is at least

$$\frac{q^2}{8} + \frac{(q-R)(3q-R)}{8} - o(1).$$

Consequently, in the limiting region $R < q$ and $c \le q^2/8$, we have $c_R < R^2/8$. The concave function $s(R - 3s/2)$ is at least $R^2/8$ throughout $s \in [R/6, R/2]$. Hence the barrier mass satisfies $s < R/6$, and therefore

$$c_R \ge \frac{3R}{4}s - o(1).$$

Its unmatched-to-covered blue density is at most $(q-R)s + o(1) \le 4(q-R)c_R/(3R) + o(1)$. Thus

$$c \ge \frac{q^2 - R^2}{2} + \frac{7 - 4q/R}{3}c_R - o(1).$$

Use $c_* \le c_R$ and $(R^2 - r^2)/2 \le 2ra/d + 2a^2/d^2 + o(1)$ to obtain (18).

To obtain the cross-blue bound, contacts with the newly covered vertices cost at most $r(R - r) \le 2ra/d + o(1)$. The remaining contacts are at most $4(q - R)c_R/(3R)$; substitute the last inequality and use the monotonicity of $4(q - R)/(7R - 4q)$ in R. This proves (19).

For completeness the boundary and finite errors do not require a strict hypothesis $c < q^2/8$. Given any sequence of finite instances, pass to a subsequence on which the bounded normalized parameters converge. If $R < q$ in the limit, the preceding argument applies. If $R = q$, the unmatched mass of the full maximum matching tends to zero, so its unmatched-to-covered blue density is $o(1)$ (the separate newly-covered contribution $r(R-r)$ to U remains), and the coefficient in the internal inequality tends to one. These alternatives prove the displayed inequalities with uniform $o(1)$ on the compact localized domain. The original finite errors are $O(M)$ and involve only the $O(M)$ covered vertices, so this uniformity is independent of N. □

The complementary region $c \geq q^2/8$ is easy to exclude separately. There the original and core bounds give

$$C_0 \leq B_0 - q^2/8 + o(1), \qquad C_r \leq B_0 + f_R + o(1).$$

A rational positive-definite certificate on $7q/8 \leq r \leq q$ proves $\gamma(4) \leq 0.43321$ in this region. Its checker reconstructs the weighted quadratic form and subtracts only nonnegative multiples of $(x+y+z+w)(r-7q/8)$ and $(r-7q/8)(q-r)$. Thus the stronger resource estimates apply throughout the remaining region. On a ratio box $h/q \in [v_0, v_1]$, $r/q \in [\rho_0, \rho_1]$, we may take

$$L = \frac{4(1-\rho_0)}{7\rho_0 - 4}, \qquad \beta = \frac{2\rho_1}{1+v_0} + \frac{9/25}{(1+v_0)^2}.$$

6.4 Using maximality of the joint choice of matchings

The core was chosen to have maximum size among all feasible joint blue AD and blue CC matchings. Its maximality gives a second relation between missing edges and blue edges leaving the retained set. Write its retained mass as r, let a be the missing-blue AD density, and let c be the missing-blue CC density. Assume $d \geq q$ and $r > 4q/5$. Define

$$u_0 = a/d, \qquad K = (q^2 - r^2)/2, \qquad E_{\mathrm{pack}} = c - K + qu_0 - u_0^2/2.$$

Lemma 6.2. *With uniform vanishing finite errors,*

$$E_{\mathrm{pack}} \geq 0, \qquad c_* \leq \frac{E_{\mathrm{pack}}}{5 - 4q/r} + o(1), \qquad U \leq ru_0 + \frac{4(q-r)}{5r-4q} E_{\mathrm{pack}} + o(1). \tag{20}$$

Here c_* is the missing-blue density in the retained C set and U its blue contact density to all omitted C.

Proof. Augment the core's blue AD matching to a maximum blue AD matching. Its covered D set contains the initial retained set and has mass $t \geq q - a/d - o(1)$. Let T be its original C mates. The initial retained CC matching is maximum in $G[T]$: a larger blue matching in T would, together with the distinct AD mates already assigned to every member of T, give a larger feasible core. Let c_T count missing blue pairs in T, and let s be a Tutte–Berge barrier mass for this graph. The usual component argument applied inside T gives

$$0 \leq s \leq r/2 + o(1), \quad c_* \geq s(r - 3s/2) - o(1) \geq rs/4 - o(1),$$

and the blue density from its unmatched set to its covered set is at most $(t-r)s + o(1)$. Consequently

$$c_T \geq \frac{t^2 - r^2}{2} + c_* - (t-r)s - o(1) \geq \frac{t^2 - r^2}{2} + (5 - 4t/r)c_* - o(1).$$

Since $c_T \leq c$, $t \leq q$, and $t \geq q - u_0 - o(1)$, this proves the claimed internal resource. The omitted C outside T have mass at most $u_0 + o(1)$ and give at most $ru_0 + o(1)$ blue contacts. The remaining omitted contacts are at most $4(t-r)c_*/r + o(1)$. Substitution proves the external resource.

The maximum-core choice preserves all label restrictions and count inequalities proved for arbitrary neutral cores, as well as the lower bounds on r in Section 5. All auxiliary graphs have $O(M)$ vertices, so their finite errors are uniform in the total number of original vertices. □

The same argument also gives inequalities retaining the barrier size. For this purpose retain an actual $u = q - t \in [0, u_0]$ and barrier mass $s \in [0, r/2]$ satisfying

$$c_* \geq s(r - 3s/2), \qquad c_* \leq c - K + qu - u^2/2 + (q - u - r)s,$$

$$U \le ru + (q - u - r)s.$$

Thus

$$c - K + qu - u^2/2 \ge s(2r - q + u - 3s/2).$$

In particular, at equality in the packing constraint, $E_{\text{pack}} = 0$, the retained graph is blue complete and $U \le ra/d$. Thus equality in this constraint restricts the blue edges leaving the retained set as well as the edges inside it.

6.5 The internal count on the label-3 class

Write Z for the mass of good original pairs P_iQ_i and $\beta = z - Z$ for the remaining label-3 pair mass. Set $I_3 = z^2/2 - C_{33} \ge 0$, and let E_{PP} count good-P pairs with label at least 3. Then

$$E_{PP} \le I_3 + o(1). \tag{21}$$

First, a good Q_i has no label-at-most-2 edge to any endpoint R of another original pair $RS(3)$. In the neutral core, the replacement

$$BB'(4), P_iQ_i(3), RS(3) \longrightarrow P_iB(3), Q_iR(\le 2), B'S(\le 4)$$

would reduce primary cost from 8 to at most 7. Second, if $RP_i \le 2$ and $SP_j \le 2$ for distinct good indices, replace the three *original* label-3 matching edges by

$$RP_i(\le 2), \quad SP_j(\le 2), \quad Q_iQ_j(\le 4).$$

Primary cost is at most 6; if it equals 6, the secondary cost decreases from 3 to 2. This contradicts the choice of $\mathcal{M}_0$. Each remaining pair therefore has at most $ZM + O(1)$ critical contacts to the good pairs. Summing the good, mixed, and remaining internal counts gives

$$C_{33} \le Z^2/2 - E_{PP} + Z\beta + \beta^2/2 + o(1) = z^2/2 - E_{PP} + o(1),$$

which proves (21).

7 Rematching through the label-4 class

We next construct a matching by pairing retained C vertices with vertices of X_4. Their original D mates are then available for a separate matching using labels at most three. We call this the *reservoir comparison*; the auxiliary set of D vertices can later be enlarged.

Every retained D has label 4 to every originally uncovered vertex. Indeed, if its new mate is A, whose old mate is B, then

$$BB'(4), AD(1) \longrightarrow Du(\le 3), AB(2)$$

would reduce primary weight from 4 to at most 3. The matching size is unchanged; B' becomes uncovered in place of u. Similarly, using C' as the new mate of a retained C and D as the old mate of C', the switch

$$BB'(4), AD(1), CC'(1) \longrightarrow Cu(\le 2), C'D(2), AB(2)$$

proves that retained C has outward labels at least 3. The corresponding switches with an original top edge $TT'(4)$ give $DT = 4$ and $CT \ge 3$. Thus at most $e_{24}M^2$ of the retained-C/top contacts fail to have label 3.

Choose a label-3 matching from retained C to original top vertices, using tM vertices on each side, with tM even after discarding at most one edge. Thus we match mass t of retained C to top vertices at label 3. The top matching supplies $t \ge [r - e_{24}/(2w)]_+ - o(1)$ when $r \le 2w$. Choose

this matching first. On its selected old D mates, take a maximum matching of labels at most 3; suppose its covered mass is s. The remaining $u_D = t - s$ vertices form a label-4 clique and can be paired internally. Keep all original AB edges and all original CD edges outside the selected target, and pair the remaining top vertices internally. The latter is possible because X_4 is a label-4 clique and an even number of its vertices remain. All original label-1 and label-3 matching edges are unchanged. This specifies an M-edge matching on exactly the original covered set; write its normalized critical count as C_T.

A matched edge has at most $u_D + o(1)$ label-at-most-3 contacts from its two endpoints to these unmatched vertices: two contacts with distinct unmatched endpoints would give a three-edge augmenting path. Only the unmatched D have birth 4. Hence new critical label-3 DD contacts number at most

$$u_D s/2 + u_D(q-t) = u_D(q - t/2) - u_D^2/2 \le (q - t/2)^2/2.$$

Every new label-2 DD contact is charged to a nonblue mate CC edge, since blue CC together with $DD(2)$ would improve the two old $CD(2)$ edges. Label-1 DD contacts were already critical. Thus the DD contribution is at most $(q - t/2)^2/2 + E_{CC}$.

The remaining direct contact count, with $a = E_{AD}$, $c = E_{CC}$, $j = E_{AC}$, is

$$\begin{aligned} C_T - C_0 \le t(\tfrac{3}{2}q - r - z + 2y + 3h_3/r) - \tfrac{7}{8}t^2 + q^2/2 \\ + 2c + j + (2 + t/r)a + e_{12} + e_{23} + e_{24} + e_{34} + o(1). \end{aligned}$$

Indeed CD gains cost at most $tq + t(q-t)$; demoted top vertices lose at least $tr - e_{24}$ contacts to retained C and $t(z - h_3/r) - e_{34}$ to good P. Bad old label-3 pairs cost at most $2th_3/r$. The original half bounds supply the X_1 term and the cross-deficit charges. For omitted B contacts, two insertions into the initial core force label 4 unless the omitted A row misses almost all initial retained D; those rows cost at most $(t/r)a$. The separate c charges for C promotion and $DD(2)$ gains are retained, since both may occur on the same mate pair.

The derivative of the displayed t polynomial is at least $-(5r/4 + z)$ for $0 \le t \le r \le q$. Substituting the top-matching deficiency and (9) gives

$$\begin{aligned} C_T \le B - b^2 + \tfrac{3}{2}qr - \tfrac{15}{8}r^2 - rz + 2ry + q^2/2 + 3h_3 \\ - E_{AA} + 2a + c - e_{13} - e_{14} + \frac{5r/4 + z}{2w} e_{24} + o(1). \end{aligned} \tag{22}$$

All selected C, D retain their exact outward caps, including when some D edges have label less than 4.

7.1 Missing edges wholly outside the retained set

Here $q = m \le d$. The maximum-core choice includes the following construction: first take a maximum blue matching on all C, then match its old D mates to A, deleting whole CC edges when an endpoint cannot be matched. The rectangular vertex-cover bound loses at most a/d D vertices, and deletion costs at most twice this many endpoints. If the full CC matching covers r_C, maximality of the chosen core therefore gives $r_C \le r + 2a/d$.

Outside the actual core's C vertices, a blue matching consequently has at most $aM/d + o(M)$ edges. A graph with matching number k has at most $k|V| + O(|V|)$ edges, by the matching extremal bound. Its missing CC density outside the core is thus at least

$$c_{\text{out}} \ge (q - r)^2/2 - a(q - r)/d - o(1).$$

Both positive c charges in the reservoir comparison must touch a selected C index, so both exclude c_{out}. Subtracting $2c_{\text{out}}$ from (22) is valid. Since $q - r \le q/5 \le d/5$, the resulting count has missing-edge terms at most $12a/5 + c - E_{AA}$, and main polynomial

$$f_T = \tfrac{7}{2}qr - \tfrac{23}{8}r^2 - rz + 2ry - q^2/2. \tag{23}$$

On $r + z \le 6w$ and $r \le 2w$, the positive e_{24} coefficient in (22) is at most $13/4$.

7.2 Averaging the choice of top vertices

Select uniformly a subset T_0 of even mass r from the original top class of mass $2w$. Match retained C to T_0 at label 3 as far as possible; let the used mass be $t = r - \delta$, after harmless parity rounding. The expected number of missing label-3 pairs in this balanced bipartite graph is at most $re_{24}/(2w)$. A vertex cover for a matching of size tM forces at least $r\delta M^2$ missing pairs. Therefore

$$\mathbb{E}\delta \le e_{24}/(2w) + o(1).$$

Now choose the maximum low-label matching on the corresponding D vertices, as in the reservoir construction. Its count bounds hold for every choice of T_0 and the subsequent CT matching.

Every old X_3–top edge has label at least 3: a label at most 2 would, together with an arbitrary complementary edge, replace an old label-3 and label-4 matching edge at strictly smaller primary cost. Consequently the total label-3 density between these classes is $2zw - e_{34}$. Uniform sampling, followed by deletion of at most δM unused top vertices, gives expected losses

$$L_{TC} \ge r^2 - r\mathbb{E}\delta - \frac{r}{2w}e_{24} - o(1), \qquad L_{T3} \ge rz - 2z\mathbb{E}\delta - \frac{r}{2w}e_{34} - o(1).$$

This counts contacts with *all* original label-3 pairs, so there is no exceptional-pair allowance for these top-vertex losses. The remaining D promotion allowance is at most $2th_3/r \le 2h_3$.

After these losses are separated, the t polynomial is $F(t) = t(3q/2 + 2y) - 7t^2/8$. Since $F'(t) \ge -r/4$ for $0 \le t \le r \le q$, replacing t by r costs at most $r\mathbb{E}\delta/4$. Combining all terms with the old count gives the same main reservoir polynomial, but replaces its $3h_3$ by $2h_3$. The e_{24}, e_{34} coefficients become respectively

$$\frac{9r/4 + 2z}{2w} - 1 \le \frac{21}{4}, \qquad \frac{r}{2w} - 1 \le 0.$$

The first bound uses $r + z \le 6w$ and $r \le 2w$. Including the outside-CC correction, some actual matching therefore satisfies

$$C_T \le B - b^2 + f_T + 2h_3 + \frac{12}{5}a + c - E_{AA} - e_{13} - e_{14} + \frac{21}{4}e_{24} + o(1). \tag{24}$$

The averaging selects among valid finite matchings. Its error terms are uniform in the number of originally uncovered vertices.

7.3 Separating the two labels in the DD count

We can bound together two types of newly critical DD edges: those of label 2, and those of label 3 whose other endpoint is outside the target. When either is charged to its missing CC mate, the same DD edge cannot contribute to both types. We therefore count their shared set of missing CC mates only once. Let U_{out} count blue pairs between the initial retained C set and its omitted complement. For an actual target I of mass $t \le r$, let c_I count nonblue CC pairs touching I, and let u be the unmatched target-D mass in the low-label matching.

Every newly critical label-2 DD pair has a nonblue CC mate. Charge these pairs together with outside-target label-3 DD pairs whose CC mates are nonblue. Their combined density is at most c_I, because a DD pair cannot have both labels 2 and 3. Unmatched-to-matched target contacts contribute at most $u(t-u)/2$. The remaining outside-target contacts have blue CC mates and contribute at most $u(r-t) + U_{\text{out}}$. Hence the whole DD gain is

$$c_I + u(r - t/2) - u^2/2 + U_{\text{out}} \le c_I + (r - t/2)^2/2 + U_{\text{out}} + o(1).$$

The corresponding bound used in (24) was $c_I + (q - t/2)^2/2$. For every $t \le r$,

$$\frac{(r - t/2)^2 - (q - t/2)^2}{2} = -\frac{(q-r)(q+r-t)}{2} \le -\frac{q(q-r)}{2}.$$

This inequality holds before averaging, so it incurs no additional e_{24} or sampling error. The separate C-promotion charge is unchanged, as is the correction for pairs entirely outside the core.

The blue resource U_{out} is itself controlled. Extend the retained CC matching to a maximum one covering mass R, with $r \le R \le r + 2a/d + o(1)$, and let c_R count missing pairs inside its covered set. The Tutte–Berge calculation in (17) gives

$$\text{blue}(\text{unmatched}, \text{covered}) \le \frac{4(q-R)}{R} c_R + o(1), \qquad \left(5 - \frac{4q}{R}\right) c_R \le c - \frac{q^2 - R^2}{2} + o(1).$$

Contacts with the newly covered vertices cost at most $r(R-r)$. Consequently, for $r > 4q/5$, put

$$K = \frac{q^2 - r^2}{2}, \quad L = \frac{4(q-r)}{5r - 4q}, \quad \beta = \frac{2r}{d} + \frac{2a}{d^2}.$$

Then

$$U_{\text{out}} \le \frac{2r}{d} a + L(c - K + \beta a) + o(1). \tag{25}$$

Here $K(r) - K(R) \le 2ra/d + 2a^2/d^2 \le \beta a$; all expressions being multiplied by $L \ge 0$ are nonnegative up to $o(1)$. The same conclusion holds at $r = q$ with $L = 0$.

Before rounding its coefficient to $12/5$, the random-top comparison has a coefficient $2 + 2(q - r)/d$. Combining it with (25) therefore gives

$$\begin{aligned} C_T \le B - b^2 + f_T - \frac{q(q-r)}{2} - LK + 2h_3 \\ + \left(2 + \frac{2q}{d} + L\beta\right) a + (1+L)c - E_{AA} - e_{13} - e_{14} + \frac{21}{4} e_{24} + o(1). \end{aligned} \tag{26}$$

On a ratio box $h/q \ge v_0$, $r/q \ge \rho_0 > 4/5$, one may replace L by $4(1-\rho_0)/(5\rho_0 - 4)$ and β by $2/(1+v_0) + (9/25)/(1+v_0)^2$. This uses $a \le 9q^2/50$. Replacing L by an upper bound is valid for its entire nonnegative resource $c - K + \beta a$.

7.4 Including the exceptional label-3 pairs

Write b_3 for the mass of old label-3 pairs outside the good subset; thus $b_3 \le h_3/r + o(1)$. For a selected CT target of mass t, form the graph of labels at most 3 on the target D vertices together with all $2b_3$ endpoints of these exceptional pairs. Start with their old label-3 matching edges and augment to a maximum matching. Every exceptional endpoint stays covered, so the unmatched set lies entirely in D. Write u for its mass and pair it internally at label 4. Exceptional vertices now have birth at most 3; all their other count changes are nonpositive. The existing outward caps therefore remain valid.

The low-label matching covers mass $t + 2b_3 - u$. The three-edge augmentation obstruction bounds its contacts with unmatched D by $u(t + 2b_3 - u)/2 + o(1)$. In particular this jointly bounds the new internal $DD(3)$ contacts and the new label-3 contacts with exceptional vertices. Adding the same outside-target and disjoint missing-CC charges as before gives

$$c_I + u(r - t/2 + b_3) - u^2/2 + U_{\text{out}} \le c_I + (r - t/2 + b_3)^2/2 + U_{\text{out}} + o(1).$$

There is no separate $2tb_3$ allowance.

Before top-demotion losses, the target-dependent polynomial is

$$g(t) = t(2q + 2y) - t^2 + (r - t/2 + b_3)^2/2, \qquad g'(t) \ge -r/4 - b_3/2.$$

The same random sampling therefore adds at most $b_3 e_{24}/(4w)$ to the former deficiency error. At $t = r$, the square is $r^2/8 + rb_3/2 + b_3^2/2$. Consequently (26) strengthens to

$$\begin{aligned} C_T \le B - b^2 + f_T - \frac{q(q-r)}{2} - LK + \frac{h_3}{2} + \frac{h_3^2}{2r^2} \\ + \left(2 + \frac{2q}{d} + L\beta\right) a + (1+L)c - E_{AA} - e_{13} - e_{14} + \frac{13}{2} e_{24} + o(1). \end{aligned} \tag{27}$$

The last coefficient follows from $b_3 \le z$ and $r+z \le 6w$: $(9r/4+2z+b_3/2)/(2w)-1 \le 13/2$. This inequality holds throughout its stated parameter domain. The exact certificates in Section 13 use its complete defect terms.

On the additional low-c domain, (19) improves (27) by replacing L with $4(q-r)/(7r-4q)$. On a box with $h/q \ge v_0$ and $r/q \in [\rho_0, \rho_1]$, we use

$$L \le \frac{4(1-\rho_0)}{7\rho_0 - 4}, \qquad \beta \le \frac{2\rho_1}{1+v_0} + \frac{9/25}{(1+v_0)^2}.$$

As before, the coefficient L multiplies the complete nonnegative resource $c - K + \beta a$; its negative $-LK$ term is retained.

For the final bound we use maximality of the joint AD and CC choice in the fixed core. Lemma 6.2, applied before substituting for U_{out}, gives the sharper joint-reservoir inequality

$$\begin{aligned} C_T \le{}& B - b^2 + f_T - \frac{q(q-r)}{2} - LK + \frac{h_3}{2} + \frac{h_3^2}{2r^2} \\ &+ \left(2 + \frac{2q-r}{d} + L\frac{q}{d}\right) a + (1+L)c - E_{AA} - e_{13} - e_{14} + \frac{13}{2}e_{24} - I_3 + o(1), \qquad (28) \\ L ={}& \frac{4(q-r)}{5r-4q}, \qquad K = \frac{q^2-r^2}{2}. \end{aligned}$$

The favorable quadratic packing term $-a^2/(2d^2)$ was discarded. The same substitution is used for the deterministic profile reservoir. It is valid throughout $r > 4q/5$ and requires no additional dense-C case distinction.

8 A preliminary bound and restriction of the parameter domain

We first combine only the original and neutral-core counts. Their quadratic bounds give both the preliminary estimate 23/53 and a quantitative restriction on the parameters of any instance that could exceed the final bound. All parameter restrictions in this section therefore use inequalities valid before the additional matching constructions are invoked. Write

$$f = y^2 + x^2/4 + z^2/2 + w^2, \quad B_0 = 1 - f - b^2, \quad m = x/2 - b,$$

where $x + y + z + w = 1$ and all coordinates are nonnegative.

8.1 Entry into the dense region

If $E \ge 9m^2/50$, completing the square gives

$$\frac{x^2}{4} + b^2 + \frac{9}{50}(x/2-b)^2 = \frac{17x^2}{59} + \frac{59}{50}(b - 9x/118)^2.$$

Weighted Cauchy–Schwarz consequently bounds the original count by

$$C_0 \le 1 - y^2 - \frac{17x^2}{59} - \frac{z^2}{2} - w^2 + o(1) \le \frac{110}{127} + o(1).$$

Indeed the reciprocals of the four coefficients sum to 127/17. This handles the entire nondense region.

Otherwise the packing construction gives an actual core with

$$r^2 \ge m^2 - 2E - o(1) \ge \frac{16}{25}m^2 - o(1), \qquad E \ge \frac{m^2 - r^2}{2} - o(1).$$

The maximum core fixed in Section 4 is at least as large as the core used to establish these inequalities, so both remain valid for its size r. For a positive working count threshold $2G$, the original inequality also gives a fixed positive lower bound for m whenever $C_0 > 2G$. In fact

$$1 - f = \frac{7}{8} - Q, \quad Q = (y - 1/8)^2 + (x - 1/2)^2/4 + (z - 1/4)^2/2 + (w - 1/8)^2,$$

and

$$\frac{(m - 1/4)^2}{2} \le Q + b^2 < 7/8 - 2G + o(1).$$

Thus

$$m \ge \frac{1}{4} - \sqrt{2(7/8 - 2G)} - o(1). \tag{29}$$

For every final threshold used below the right side exceeds $1/9$. In particular $r \ge 4m/5 - o(1)$, and $m/2 \le r \le m$ for all sufficiently large M. A fixed positive bound such as $1/9$ suffices; no stronger lower bound for m is assumed.

The other initial absorption conditions follow just as directly. Minimizing f on the indicated halfspaces gives

$$y \ge 4w \implies f \ge 17/127, \qquad x \ge 16w \implies f \ge 65/484.$$

For the first calculation the active boundary is $y = 4w$, and the remaining x, z contribution is at least $(1 - 5w)^2/6$; minimizing $17w^2 + (1 - 5w)^2/6$ gives $17/127$. For the second boundary $x = 16w$, minimize $65w^2 + (1 - 17w)^2/3$, obtaining $65/484$. Hence $y < 4w$ and $x < 16w$ whenever the original count exceeds a threshold satisfying

$$2G > \max\{110/127,\ 1 - 17/127,\ 1 - 65/484\}. \tag{30}$$

These inequalities are checked exactly for the final value of G. They also hold for $G = 23/53$.

8.2 The two comparison quadratics

Let $S = e_{14} + e_{23} + e_{24}$. The five-edge replacement at the end of Section 4.2, retaining the terms h_1, h_3, gives

$$h_1 + h_3 \ge \frac{[(2yw - e_{14})(rz - e_{23}) - 2zye_{24}]_+}{\max(2zw, 2yw - e_{14})} - o(1).$$

Put $D_* = \max(2zw, 2yw)$. Increasing the denominator and discarding the favorable product $e_{14}e_{23}$ shows that this is at least

$$r \min(y, z) - \frac{rz}{D_*} e_{14} - \frac{2yw}{D_*} e_{23} - \frac{2zy}{D_*} e_{24} - o(1).$$

The inequality remains valid if its untruncated right side is negative. The entry conditions give

$$\frac{rz}{D_*} \le \frac{r}{2w} \le \frac{x}{4w} < 4, \qquad \frac{2yw}{D_*} \le 1, \qquad \frac{2zy}{D_*} \le \frac{y}{w} < 4.$$

Zero-denominator cases are either excluded by the original-count threshold or handled by assigning penalty zero. Consequently $h_1 + h_3 \ge r \min(y, z) - 4S - o(1)$.

Define

$$A = B_0 - \frac{m^2 - r^2}{2}, \qquad B_* = B_0 + r(x + z - 2y) - 3r^2 - r \min(y, z).$$

The original and core counts obey

$$C_0 \le A - S + o(1), \qquad C_r \le B_* + 3S + o(1).$$

Here unused nonpositive deficits have simply been discarded. Taking weights $4/5, 1/5$ leaves the nonpositive term $-S/5$.

For $V = (x, y, z, w, b, r)^T$ and $\ell = (1, 1, 1, 1, 0, 0)^T$, define symmetric matrices by their diagonal and upper-triangular nonzero off-diagonal entries; all unspecified entries are zero:

	diagonal	off-diagonal entries
H_0	$(3/8, 1, 1/2, 1, 3/2, -1/2)$	$(H_0)_{xb} = -1/4$
H_y	$(1/4, 1, 1/2, 1, 1, 3)$	$(H_y)_{xr} = -1/2,\ (H_y)_{yr} = 3/2,\ (H_y)_{zr} = -1/2$
H_z	$(1/4, 1, 1/2, 1, 1, 3)$	$(H_z)_{xr} = -1/2,\ (H_z)_{yr} = 1$

Then $A = 1 - V^T H_0 V$, while $B_* = 1 - V^T H_y V$ on $y \le z$ and $B_* = 1 - V^T H_z V$ on $z \le y$. Let

$$c_0 = (1/2, 0, 0, 0, -1, -1)^T, \qquad s_0 = (-1/2, 0, 0, 0, 1, 2)^T, \qquad N_0 = (c_0 s_0^T + s_0 c_0^T)/2.$$

Its quadratic form is $(m - r)(2r - m) \ge 0$. The exact identities needed for the bound are

$$\mathcal{A} = \tfrac{4}{5} H_0 + \tfrac{1}{5} H_y - \tfrac{7}{53} \ell\ell^T - \tfrac{1}{5} N_0 \succeq 0, \tag{31}$$

$$\mathcal{Z} = \tfrac{4}{5} H_0 + \tfrac{1}{5} H_z - \tfrac{7}{53} \ell\ell^T \succ 0. \tag{32}$$

Their exact rational LDL^T pivots are respectively

$$\left(\frac{71}{265}, \frac{57}{71}, \frac{29}{114}, \frac{15}{29}, \frac{17}{20}, 0\right), \qquad \left(\frac{231}{1060}, \frac{26}{33}, \frac{3}{13}, \frac{5}{12}, \frac{29}{35}, \frac{74}{725}\right).$$

The zero pivot has zero remaining Schur-complement column. Thus no numerical eigenvalue test is involved. These identities prove

$$\min(C_0, C_r) \le 46/53 + o(1), \qquad \gamma(4) \le 23/53.$$

All matrix entries and decompositions are reconstructed by `previous_certificate.py`.

8.3 Coercivity and exact halfspace losses

The matrix $\mathcal{A}$ has rank five and kernel spanned by

$$V_* = (26, 4, 16, 7, 3, 10)^T/53, \qquad \ell^T V_* = 1.$$

The kernel claim follows either by multiplication or from the displayed pivots and $\mathcal{A} V_* = 0$. On the mass hyperplane we therefore have

$$\min(C_0, C_r) \le \frac{46}{53} - (V - V_*)^T \mathcal{A} (V - V_*) - \frac{1}{5}(m - r)(2r - m) + o(1). \tag{33}$$

The quadratic form is strictly positive away from V_* on that hyperplane. An explicit coordinate matrix makes every localization check reproducible. Let W have columns $e_x - e_w, e_y - e_w, e_z - e_w, e_b, e_r$. Then

$$K_0 = W^T \mathcal{A} W = \begin{pmatrix} 7/5 & 1 & 1 & -3/10 & -1/4 \\ 1 & 2 & 1 & 0 & 3/10 \\ 1 & 1 & 3/2 & 0 & -1/10 \\ -3/10 & 0 & 0 & 8/5 & 3/10 \\ -1/4 & 3/10 & -1/10 & 3/10 & 3/5 \end{pmatrix} \succ 0.$$

For a retained linear condition $f(V) = f(V_*) + a_f^T \eta \ge 0$, where $V - V_* = W\eta$ and $f(V_*) > 0$, Cauchy–Schwarz in the K_0 inner product gives

$$f(V) \le 0 \quad \Longrightarrow \quad \eta^T K_0 \eta \ge \frac{f(V_*)^2}{a_f^T K_0^{-1} a_f}.$$

This is the exact count loss used for each halfspace exclusion. The checker `localize.py` evaluates it with rational arithmetic.

The other branch has a convenient uniform loss: even $\mathcal{Z} - (1/1000)\ell\ell^T$ is positive definite. Its exact pivots are

$$\left(\frac{11497}{53000}, \frac{180569}{229940}, \frac{81827}{361138}, \frac{32456}{81827}, \frac{3201}{4057}, \frac{62923}{640200}\right).$$

Thus $z \le y$ gives count at most $46/53 - 1/1000 + o(1)$.

Finally, when using (33) below at a target $G < 23/53$, repeat the original-count entry tests (29)–(30) with that G. They justify density, $r \ge m/2$, and all absorption coefficients without using the additional matching comparisons. The exact checker then verifies that every excluded halfspace loss exceeds $2(23/53 - G)$, or replaces it by an explicitly stronger certificate. This is the complete logical order of the localization.

9 Rematching when the CD family is larger

Suppose $q \ge d$, and put $h = q - d$, $a = E_{AD}$, $c = E_{CC}$, and $j = E_{AC}$. We use the surplus h of CD pairs to construct a matching different from the reservoir matching of Section 7. Throughout this section we assume the following normalized parameter conditions:

$$3d/10 \le h \le 5d/4, \quad 4d/5 \le r \le d, \quad E \le 9d^2/50,$$

$$y \le 13d/20, \quad y \le 3z/8, \quad z \le 3d, \quad y \le w, \quad d \le 2w.$$

Section 13 shows that all instances in this orientation outside these conditions already satisfy the asserted upper bound. No additional condition $h \le r$ or $h \le d$ is used.

9.1 Which omitted pairs can replace a retained pair

Index the initial retained AD links as A_iD_i, write C_i for the original mate of D_i, and let σ be the fixed-point-free involution defined by the retained blue edges $C_iC_{\sigma(i)}$. An omitted C_0D_0 inserts into the initial core at index i if A_iD_0 and $C_0C_{\sigma(i)}$ are both blue: replace A_iD_i and $C_iC_{\sigma(i)}$ by those two edges, and exchange the retained original pair C_iD_i for C_0D_0. This gives another neutral core of the same size. Let U be the actual set of noninsertable pairs, of mass u. Let a_U count missing AD edges from initial core A to D_U, and c_U missing CC edges between C_U and initial core C. Failure at every index gives

$$ru \le a_U + c_U, \qquad u \le \frac{a + c}{r} \le 9d/40 < h.$$

The other CD indices are *eligible*; call their set V, of mass $q - u$. Its internal missing-edge counts a_V (all missing AD with eligible D) and c_V (missing CC with both indices eligible) satisfy

$$a_U + a_V \le a, \qquad c_U + c_V \le c.$$

Inserting any eligible pair and applying the outward switches proves that its D has outward label 4 and its C has outward labels at least 3. The initial retained B set does not change, so its good PQ orientation still gives $QD = 4$ and $QC \ge 3$ for eligible pairs.

9.2 Transporting the initial core deficit

Let I, J be the initial and rebuilt retained B sets, of masses r, s. For a subset W of either actual core, let $H(W)$ count its label-3 contacts to old X_3 and set $h(W) = |W|z - H(W)$. We have $h(W) \ge -o(1)$. Indeed each core B set is a label-4 clique and can be re-paired

arbitrarily. Distinct label-3 neighbors B, B' of the two endpoints of an old $PQ(3)$ would allow $BB'(4), PQ(3) \to BP(3), B'Q(3)$, reducing primary weight. Thus all such contacts use one endpoint of PQ, or there are at most two contacts. This proves the claim for every subset of size at least two; a singleton costs only $O(M)$ unnormalized. Additivity therefore gives

$$h(J) = h(I) - h(I \setminus J) + h(J \setminus I) \geq h_3 - z(d-s) - o(1).$$

If $\overline{R}(s)$ is (10) without its h_3 term, and $\kappa = 1 - y/z$, then

$$C_s \leq \overline{R}(s) - \kappa h_3 + (z-y)(d-s) + o(1). \tag{34}$$

The two cores may have different matchings and different good-pair orientations; their union need not be a core.

9.3 Completing the target without a repair loss

Put $T = h - u > 0$, so there are $d + T$ eligible indices in normalized units. On these indices define a graph H by

$$ij \in E(H) \quad \Longleftrightarrow \quad C_iC_j \text{ is blue and } D_iD_j \text{ has label 3.}$$

A blue CC pair cannot have its DD mate at label at most 2, because that would reduce the primary weight of two old $CD(2)$ edges. Let a maximum matching of H cover mass k. If $k \geq T$, use T of its endpoints as the target, pairing C blue and D at 3 along that matching.

If $k < T$, retain all its k endpoints. Among its unmatched indices, whose mass is $d + (T-k)$, a blue CC matching covers at least $T - k$. Otherwise the unmatched vertices of a maximum blue matching would have mass at least d, forming a blue-independent set and requiring at least $d^2/2 - o(1)$ missing pairs, contrary to $c \leq 9d^2/50$. Choose $T - k$ such endpoints and add them to the target. Their DD mate edges have label 4, since unmatched vertices of H are independent in H. Thus the entire target has a blue CC perfect matching, with D birth 3 on the first part and 4 on the rest. No random pairing or target repair is necessary.

In the second case only the last $T - k$ target D vertices can create new label-3 DD contacts. Each matched edge of H has at most $T - k + o(1)$ total H contacts to this unmatched target: two contacts at distinct unmatched endpoints would augment the maximum matching by one edge. Hence the new H contacts number at most

$$\frac{k(T-k)}{2} \leq \frac{T^2}{8}.$$

Any other eligible label-3 DD contact is charged to a missing CC mate edge. Label-2 DD contacts also have missing CC mates, and these two label categories are disjoint. Their combined allowance is therefore at most c_V, not twice c_V. Contacts to noneligible D are bounded separately by uT.

9.4 Counting the new matching and allocating missing edges

Rebuild a separate neutral core on the eligible complement of the target, which has mass exactly d. Let its size be s. Use its neutral replacement, the CC and DD edges just constructed on the target, and all unchanged original matching edges elsewhere. The chosen sets are disjoint, so this is an M-edge matching; call its normalized critical count C_D. Comparing the target matching to this core gives

$$\begin{aligned} C_D \leq \overline{R}(s) - \kappa h_3 + (z-y)(d-s) + T(q-2s) + uT - \tfrac{3}{8}T^2 \\ + j + (1 + T/r)a_V + 2c_V + e_{12} + e_{23} + (2T/r)h_3 + o(1). \end{aligned} \tag{35}$$

Here target D–C contacts cost at most Tq, demoted C removes $2Ts + T^2/2$ blue contacts, and the preceding matching argument returns at most $T^2/8$ label-3 DD contacts. The X_1 and good PQ half bounds give e_{12}, e_{23}; bad pairs cost $2Th_3/r$. The outside-B allowance $(T/r)a_V$ uses missing AD into the initial retained D, all of which are eligible.

The missing-edge terms can be allocated more precisely. Write a_J for missing AD into complement D, and c_J for missing CC entirely inside the complement. The raw $+a_V$ in (35) is actually supported on target D columns, disjoint from a_J. Both raw c charges are supported on pairs incident to target indices, disjoint from c_J. The two c charges may overlap each other, so retain coefficient 2 on their common resource.

Let $v = h/d$ and set

$$P_0 = \frac{10}{9}(53/10 - v), \qquad P(T) = P_0 + \frac{20T}{9d}.$$

The packing derivative, including (34), is bounded by

$$(6d - x - z + 3y)_+ + z - y + 2T = \max(4d - h + 2y, z - y) + 2T \le \frac{9}{10} dP(T).$$

The last inequality uses $y \le 13d/20$, $z \le 3d$, and $v \le 5/4$. The balanced case of (12) gives $s^2 \ge d^2 - 2(a_J + c_J) - o(1)$ and $s \ge 4d/5 - o(1)$. Dividing $d^2 - s^2$ by $d + s$ therefore gives $d - s \le 10(a_J + c_J)/(9d) + o(1)$. Since $P(T) \ge 9/2 > 2$, disjoint allocation proves

$$P(T)a_J + a_{\text{target}} \le P(T)a_V, \qquad P(T)c_J + 2c_{\text{incident}} \le P(T)c_V.$$

Thus after replacing s by d, the two eligible defect coefficients are only $P(T) + T/r$ and $P(T)$.

9.5 Accounting for the noneligible pairs

Normalize lengths by d and defects by d^2. Write $v = h/d$, $\rho = r/d$, $\sigma = u/d$, and $l = T/d = v - \sigma$. The main target polynomial is

$$f(l) = l(2v - 1) - \frac{11}{8} l^2, \qquad \frac{df}{d\sigma} = 1 - 2v + \frac{11}{4} l \le 31/16.$$

The eligible coefficients $P_0 + 20l/9 + l/\rho$ and $P_0 + 20l/9$ both decrease with σ. Their minimum is at least $P_0 \ge 9/2$. Increasing σ removes at least $\rho\, d\sigma$ eligible missing-edge resource; its count cost is at least $18/5$ per unit. All target h_3 coefficients also decrease. The resource envelope therefore has derivative at most

$$31/16 - 18/5 = -133/80 < 0.$$

This holds at resource-allocation kinks by comparing optimal allocations at two nearby values of σ. Consequently the actual count is bounded by the formula evaluated at $u = 0$, $T = h$. This is a comparison of count bounds; the actual construction still uses $T = h - u$.

Put $P_h = \frac{10}{9}(53/10 + h/d)$. After discarding favorable cross terms on the stated domain, the resulting inequality is

$$\begin{aligned} C_D \le B - b^2 + d(x + z - 3y) - 3d^2 - dh + \tfrac{5}{8}h^2 \\ + (P_h + h/r)a + P_h c + j + e_{12} + \tfrac{3}{8}e_{23} + (2h/r - \tfrac{5}{8})h_3 + o(1). \end{aligned} \tag{36}$$

The certificate in Section 13 uses these coefficients. The allocation above is essential: the target and complementary packing use disjoint parts of the same missing-edge count.

9.6 Using C vertices with several retained blue neighbors

To bound contacts with omitted B vertices, call an original CD index rich if its C vertex has at least four blue neighbors in the initial retained C set, and write g for the rich mass. If an omitted A_0B_0 has two blue A_0D links to rich indices, then $B_0D_* = 4$ for every eligible D_*. To see this, insert C_*D_* into the initial core. Choose a distinct rich neighbor of A_0 and insert A_0B_0 through it, retaining the target slot. Four blue neighbors suffice to avoid the removed vertex and the target's mate. Re-pair the resulting B clique through B_0 and the old mate B_i of the target's A_i. The switch $B_0B_i(4), A_iD_*(1) \to B_0D_*(\le 3), A_iB_i(2)$ would reduce primary weight, a contradiction. A row with only one rich neighbor has at most one exceptional target contact. Charging zero- and one-neighbor rows to their missing rich AD links bounds the outside-B term by $(T/g)a_{\mathrm{out}} + o(1)$.

If the nonrich mass is b_C, with b_0 of it initially retained, then $c \ge rb_C - b_0^2/2 - o(1)$. The bound $c \le 9d^2/50$ first implies $b_C < r + o(1)$, and then

$$b_C \le r - \sqrt{r^2 - 2c} + o(1) \le \frac{5c}{4r} + o(1), \qquad g \ge q - 9d/32 - o(1).$$

The last estimate uses $r \ge 4d/5$. The resource a_{out} uses A rows outside the initial core, whereas a_U uses initial rows; these resources are disjoint. Thus the preceding noneligible-pair argument remains valid, with outside coefficient T/g. More precisely, before substituting $r \ge 4d/5$, the same argument gives $g \ge q - 9d^2/(40r) - o(1)$. On a box with $v = h/d \in [v_0, v_1]$ and $r/d \ge \rho_0$, the a coefficient in (36) can consequently be replaced by

$$A_D = \frac{10}{9}(53/10 + v_1) + \frac{v_1}{1 + v_1 - 9/(40\rho_0)}, \qquad \rho_0 \le r/d.$$

Also retain the exact deficit term $-(1 - y/z)h_3 + 2hh_3/r$, rather than its constant envelope. These coefficients and the unrounded h_3 term are the ones used for this orientation in Section 13.

10 Auxiliary matchings through good label-3 pairs

This section concerns the orientation $d \ge q$. We construct a bipartite matching that uses both blue AD edges and certain edges from A to the P endpoints of good label-3 pairs. Either this matching has a prescribed size, giving a direct comparison, or its minimum vertex cover yields another counting inequality. Put $\kappa = 1 - y/z$ and $I_3 = z^2/2 - C_{33} \ge 0$. The original, core, and reservoir counts all retain an additional $-I_3$. The original and core matchings leave the original label-3 edges unchanged; the joint reservoir may rematch exceptional pairs, but their births only decrease, so the original internal-count deficit is still available. Let $Z = z - h_3/r + o(1)$ be a chosen mass of good pairs, discarding surplus good pairs when necessary. We use the global internal estimate $E_{PP} \le I_3 + o(1)$ proved in (21), and the retained-C resource (17).

Call an original A strong if it has at least two blue neighbors among the initial retained D. Their mass is at least $d - a/r - o(1)$. Any two strong rows can be placed in distinct AD slots; keeping the original CC links gives an actual neutral core containing both of their B endpoints. Build a bipartite graph from all A to the retained D and good P. Allow every blue AD edge; allow AP precisely when A is strong, $AP = 1$, and its corresponding $AQ = 2$. Start with the actual initial AD matching and augment. Every initially covered A and every retained D stays covered. A matching with $r + t$ edges consists of r new AD links and t compatible AP links.

Keep the original blue CC matching. Pair the new core's B vertices at label 4, and match the selected B to the corresponding good Q at label 4. The latter endpoints have exact outward label 4. Every new core A is insertable into the initial core, so its B satisfies $PB \ge 3$ and $QB = 4$ against the original good orientation. No core vertices have been deleted.

Any two vertices in the union of new-core and selected B can coexist in a neutral core. Within either group this is established as before. For a cross pair, insert the strong selected A while

avoiding the other vertex's D slot. Consequently every subset of this union has nonnegative label-3 deficit, up to $o(1)$. The union contains every initial B. If h_{new} and h_{sel} are the new-core and selected deficits, then

$$h_{\text{new}} + h_{\text{sel}} \geq h_3 - o(1).$$

The selected B gain is exactly $tz - h_{\text{sel}}$. Thus $-\kappa h_{\text{new}} - h_{\text{sel}} \leq -\kappa h_3 + o(1)$.

Write $R(r)$ for the right side of (10) without its $-\kappa h_3$, and with $-I_3$ retained. The contact comparison gives

$$\begin{aligned} C_{AP} \leq R(r) + t(2z + d + 2q - 6r) - 2t^2 + E_{AA} + j + a + e_{12} + e_{13} + e_{23} + I_3 \\ + \frac{3t}{r}a + (t/r - \kappa)h_3 + o(1). \end{aligned} \tag{37}$$

Here compatibility $AP = 1, AQ = 2$ is useful in the omitted CD pairs: $PD = 1$ forces $BC = 4$, and $PC = 1$ forces $BD = 4$, by replacing the three original AB, CD, PQ edges. Hence their total contribution is at most $2t(q - r) + e_{23}$, with no missing-CC charge. Selected PP losses use $E_{PP} \leq I_3$. The X_1 half bounds supply $e_{12} + e_{13}$; the remaining bad-pair allowance is th_3/r. These give (37) with all contacts outside the matching controlled by the exact outward caps.

10.1 The saturated alternative

Set $H = d - r$ and $\tau = (H - a/r)_+$. If the joint matching reaches τ, stop an augmentation sequence there. Concavity of $f(t) = t(2z + d + 2q - 6r) - 2t^2$ gives

$$f(\tau) \leq f(H) + \frac{a}{r}(3d - 2q + 2r - 2z)_+.$$

This includes $\tau = 0$, when the unchanged core is used. Thus (37) yields the global cap inequality

$$\begin{aligned} C_{AP} &\leq R(r) + f_A + E_{AA} + j + e_{12} + e_{13} + e_{23} + I_3 + A_{AP}a + (H/r - \kappa)h_3 + o(1), \\ f_A &= (d - r)(2z - d + 2q - 4r), \\ A_{AP} &= 1 + 3H/r + \frac{1}{r}(3d - 2q + 2r - 2z)_+. \end{aligned} \tag{38}$$

There is no loss from rebuilding a smaller core.

10.2 The deficient alternative

Otherwise take a minimum vertex cover A_0, D_0, P_0 of the joint bipartite graph, of total mass $r + t < r + \tau$. Put $k = r - |D_0|$. Write $\eta = a/r$ and note $0 \leq k \leq r$. If $k > \eta$, the empty AD rectangle and $t < H - \eta$ would give

$$a \geq (d - |A_0|)k \geq (d - t - k)k > (r + \eta - k)k = r\eta + (k - \eta)(r - k) \geq a,$$

a contradiction. Therefore $k \leq \eta$. Thus $A_0 \cup P_0$ covers the ordinary compatible AP graph with size $t + k < H$. If W is the label-2 density between all A and good P, its nonedge rectangle implies

$$W + e_{23} + \frac{Z}{r}a \geq rZ - H(d - Z)_+. \tag{39}$$

The last allowance accounts for the nonstrong rows. The two-neighbor lemma gives exactly the Za/r allowance used here.

Let v be the additional P mass covered by a maximum blue matching on retained C and good P, starting from the original CC matching. Write U_{PC}, U_{PP} for its ambient blue densities, and c_* for missing blue pairs within retained C. On $v < r$, $Z \geq r + v$,

$$(r - v)\{U_{PC} + 2(1 - r/Z)U_{PP} - vZ\} \leq 4Zc_* + o(1). \tag{40}$$

To prove this, take a maximum PP matching covering mass $p \le v$, then a maximum PC matching of size k after deleting those P. The remaining C graph and the matching bound $e(G) \le \nu(G)|V(G)|$ give $p + k \le v + 4c_*/(r - v) + o(1)$. Also $U_{PC} \le rp + k(Z - p)$ and $U_{PP} \le pZ/2 + o(1)$. Substitution gives (40). All matchings used in this argument are disjoint on their prescribed vertices.

For a second actual comparison, assign r distinct good PQ pairs uniformly to the actual AD links, and perform $AD(1), PQ(3) \to AQ(2), DP(2)$ whenever compatible. Its expected failed mass is at most e_{23}/Z: $PD = 1$ would contradict primary minimality when $AQ \le 2$, and the other failures are old cross deficits. Reverting a failed group raises only its P, Q births from 2 to 3. The half bounds charge at most $2x + 2y + r$ label-2 contacts per failed pair. The resulting actual matching therefore satisfies

$$C_{AD} - C_r \le r(4y - d) - \frac{r}{Z}W + \frac{r}{Z}U_{PC} + \left(\frac{r}{Z}\right)^2 (U_{PP} + I_3) + a + \frac{2x + 2y + 3r}{Z} e_{23} + o(1). \tag{41}$$

In the formal count, core blue gains are $5r^2/2$, the retained PC, PD losses total $2r^2$, and selected PP losses give $r^2/2$. The QA loss against all A is rd; promotions against X_1 cost at most $4ry$. This explains the main term and the three blue-density corrections in (41).

For the exact certificate, further retained domains are $Z \ge 3r/2$, $d \le 23Z/20$, and $r \ge 7q/8$; their exceptional regions are excluded in Section 13. Choose $0 < \alpha \le 1/2$ for each ratio box. If $v \ge \alpha r$, stop the blue augmentation at αr and use the CP comparison (42), whose actual matching and edge count are given in Section 10.3. If $v \le \alpha r$, combine (39), (40), (41), and Lemma 6.2. Since $r/Z \le 2/3$, the blue-density contribution is at most $\alpha r^2 + 4c_*/(1 - \alpha)$. The cover error obeys

$$\frac{r}{Z} H(d - Z)_+ \le \frac{3}{20} r(d - r),$$

since $d/Z \le 23/20$. The jointly maximal core gives, on a box with $r/q \in [\rho_0, \rho_1]$ and $h/q \in [v_0, v_1]$, the coefficients

$$K_C = \frac{4}{(1 - \alpha)(5 - 4/\rho_0)}, \qquad \beta_0 = \frac{1}{1 + v_0}.$$

The small-v count has main polynomial

$$f_R + r(4y - d - r) + \frac{3}{20} r(d - r) + \alpha r^2 - K_C K, \qquad K = (q^2 - r^2)/2,$$

and coefficients at most $2 + \beta_0 K_C$ on a, K_C on c, $57/8$ on e_{23}, and $-5/8$ on h_3. To see the cross-deficit constant explicitly, (39) contributes r/Z and (41) contributes $(2x + 2y + 3r)/Z$; adding the core coefficient $y/z - 1$ gives at most $57/8$ on the retained domains. All other unused terms are nonpositive. These are the quadratic inequalities used in Section 13.

10.3 Edge counts for the two auxiliary comparisons

We now construct and count the two matchings used in Section 10. The first extends a blue matching from retained C to good P vertices; the second replaces an AD edge together with a good PQ edge. Both use the label restriction (16).

For the CP construction, retain all actual AD and BB core edges. Start with its blue CC matching in the blue graph on retained C and all good P, then augment to a maximum matching. Every retained C remains covered. If its additionally covered P mass is v, the blue matching covers $r + v$ vertices. Replace the old CC edges and these vM original PQ edges by this blue matching and an internal label-4 matching on their Q mates. The finite integer vM is even, since rM and $rM + vM$ are even. All unselected PQ edges remain. Edge counts agree: $(r + v)M/2 + vM/2 = rM/2 + vM$. Primary weight also agrees, since the new QQ edges have

total weight $2vM$, the same as the removed label-3 edges. Thus this is an actual primary-neutral matching with the same covered vertices; in particular it has no outward critical edges.

In the CP construction only selected P births decrease from 3 to 1 and their Q mates increase from 3 to 4. Retained A, C, D give net change at most $-2rv$ plus their portion of e_{23}: the three P losses total $3rv$ up to this deficit, and only the Q–C label-3 gain, at most rv, can offset them. An insertable omitted AB pair has $QB = 4$ and $PB \geq 3$ and costs only its cross deficit. Noninsertable rows have mass at most a/r, so they cost at most $2va/r$. An eligible omitted CD pair costs at most v; noneligible pairs have mass at most $(a + c)/r$ and cost at most one further contact each. These terms total $v(q - r) + v(a + c)/r$.

Inside X_3, selected Q can gain at most vZ contacts with good P and $2v(z-Z)$ with exceptional endpoints. Selected P loses at least $v^2/2 - I_3$ internal contacts, using $E_{PP} \leq I_3$. Thus this part is at most $vz - v^2/2 + I_3 + v(z - Z)$. The X_1 contribution is at most e_{13} by comparing its old half bound with the at most two contacts from each selected Q. All uses of e_{23} here are on disjoint old pair rectangles. Their sum gives

$$C_{CP} - C_r \leq v(z + q - 3r) - v^2/2 + e_{13} + e_{23} + I_3 + (v/r)h_3 + (3v/r)a + (v/r)c + o(1). \quad (42)$$

For AD/PQ, let $\pi = r/Z$ and first perform the formal birth change on every assigned group, before repairing incompatible assignments. Only retained A, D promote from 1 to 2; selected P, Q demote from 3 to 2. Promotions cost at most $4ry$ against X_1 and $5r^2/2$ inside the retained core: the latter consists of AA, AC, AD. The expected $PA(2)$ loss is πW. The $PC(2), PD(2)$ losses are at least $2r^2 - \pi U_{PC}$, except for their cross deficits and blue PD edges. A blue PD edge forces its current AD partner A to have $AQ > 2$, since otherwise $AD(1), PQ(3)$ could be replaced at smaller primary cost. These failures are charged to the same AB–PQ cross deficit used below.

The $QA(2)$ loss is at least $rd - a$ up to its cross deficit: the preliminary exclusion rules out blue QA, every insertable row has $QB = 4, PB \geq 3$, and noninsertable rows have mass at most a/r. The combined two uses of the AB cross deficit and the one use of the CD cross deficit cost at most $2\pi e_{23}$. Selected $PP(2)$ loses at least $r^2/2 - \pi^2(U_{PP} + I_3) + o(1)$ in expectation. All other contact changes are nonpositive. Summation gives the formal bound

$$r(4y - d) - \pi W + \pi U_{PC} + \pi^2(U_{PP} + I_3) + a + 2\pi e_{23} + o(1).$$

The expected failed assignment mass is at most e_{23}/Z. Restoring a failed group decreases its A, D births and raises only its P, Q births from 2 to 3. The latter creates only label-2 contacts. The old per-pair half bounds give at most $2x + 2y$ contacts to $X_1 \cup X_2$, and good-Q low-label exclusion leaves at most r further contacts to selected P endpoints. Thus repair costs at most $(2x + 2y + r)e_{23}/Z$, proving the stated coefficient $(2x + 2y + 3r)/Z$. Every repaired assignment is an actual primary-neutral matching. The formal assignment is also outward-safe: A, D never exceed their original birth 2, and P, Q only demote. Consequently no error depends on the number of uncovered vertices.

11 Enlarging the retained sets by re-pairing

We work in the case $d \geq q$. The purpose of this section is to enlarge the sets of C and D vertices available for the reservoir matching. The enlargement must preserve the label restrictions that make changes of matching safe at uncovered vertices.

Index the original CD matching edges as C_iD_i. An index is *retained* if its pair belongs to the neutral core, and *omitted* otherwise. For a retained index i, write A_i for the vertex joined to D_i by the core's blue AD matching. The retained and omitted index sets have masses r and $\delta = q - r$, respectively. Put $H = h_3$, and define

$$B_0 = B - b^2, \qquad F(R) = 4qR - \frac{23}{8}R^2 - Rz + 2Ry - q^2.$$

Here $b = |d - q|/2$ is the imbalance parameter; b_3 below denotes the mass of exceptional label-3 pairs. The comparisons in this section are conditional on

$$r \geq 7q/8, \quad q \leq 2w, \quad z \leq 3q, \quad r + z \leq 6w, \quad Z = z - H/r \geq 3r/2. \tag{43}$$

Section 13 proves that the complementary cases already satisfy the asserted upper bound, using comparisons that do not assume these conditions. All statements here concern finite matchings. Harmless parity adjustments and the bounded-degree discarded blue edges cost $O(M)$, with constants uniform in $N \geq 2M$.

Let C_{ret} and C_{out} denote the retained and omitted C sets. Write

$$U = \mathrm{blue}(C_{\mathrm{ret}}, C_{\mathrm{out}}),$$

where $\mathrm{blue}(U_1, U_2)$ denotes the number of blue edges between the indicated disjoint sets, divided by M^2. Joint maximality of the core gives

$$U \leq \frac{r}{d}a + L(c - K + \beta a) + o(1), \qquad L = \frac{4(q-r)}{5r-4q}, \quad K = \frac{q^2 - r^2}{2}, \quad \beta = \frac{q}{d}. \tag{44}$$

This follows from Lemma 6.2 by dropping its favorable $-a^2/(2d^2)$ term. The complete bracket is nonnegative. The following arguments keep U explicit until the final optimization.

11.1 A comparison with specified unmatched mass

Choose a maximum label-3 matching from the initial retained C to the original label-4 vertices X_4. Its even target mass t satisfies

$$t \geq r - e_{24}/(2w) - o(1).$$

On the corresponding target D vertices together with all endpoints of the exceptional old label-3 pairs, start with the exceptional matching edges and augment to a maximum matching of labels at most 3. Every exceptional endpoint remains covered. Let s be the mass of target D vertices covered by this low-label matching; its unmatched set has mass $t - s$ and is contained in D. Pair that set at label 4. Put $\sigma = s/r \in [0, 1]$.

The reservoir counts bound possible gains by counting eligible pairs of vertices. Some of those pairs cannot actually produce a gain. For a covered target index i and an omitted index o, record the following two failures:

1. C_iC_o is nonblue but does not have label 2;
2. C_iC_o is nonblue and D_iD_o does not have label 2.

Let ξ be the sum of their counts divided by M^2. A pair satisfying both conditions contributes twice, because it removes one possible gain from each of two separate edge counts. Also let

$$\zeta = \frac{1}{M^2}\#\{(i,o) : i \in S,\ C_iD_o \geq 3\} + \frac{1}{M^2}\#\{(i,o) : i \in S,\ D_iC_o \geq 3\}, \qquad V = \xi + \zeta,$$

where S is the set of covered target-D indices, and an expression such as $C_iD_o \geq 3$ abbreviates $\lambda(C_iD_o) \geq 3$. These failures remove possible direct CD gains: C_i has birth three, and D_i has birth at most three, while omitted vertices retain birth two. The corrections ξ and ζ apply to disjoint types of edges, namely CC, DD and CD, DC, respectively. The discarded blue CD incidences contribute only $O(M)$.

Lemma 11.1. *Some actual matching satisfies*

$$\begin{aligned} C_I \leq B_0 + F(r) - \frac{1}{2}(s - r/2)^2 + (2 - \sigma)H - V \\ + \left(2 + \frac{2\delta}{d}\right)a + c - E_{AA} + U + 4e_{24} - e_{13} - e_{14} - I_3 + o(1). \end{aligned} \tag{45}$$

Proof. Let $b_3 \le H/r + o(1)$ be the exceptional pair mass. The low-label matching covers $s + 2b_3$ vertices. The three-edge augmentation obstruction therefore bounds the combined internal $DD(3)$ and D–exceptional label-3 gains by $(t-s)(s/2+b_3)+o(1)$. The remaining outside-target label-3 gains with blue CC mates cost at most $(t-s)(r-t)+U$. Nonblue-mate gains share the single DD missing-edge allowance, as in the disjoint reservoir argument.

The direct CD gain is at most $2tq - t^2$. The top losses to retained C and good P are at least $tr - e_{24}$ and $t(z-b_3) - e_{34}$. The X_1 comparison costs at most $2ty + e_{12}$; good-P failures cost at most e_{23}. After separating the missing-edge terms, the target-dependent polynomial is

$$G(t) = t(2q - r - z + b_3 + 2y) - t^2 + (t-s)(r - t + s/2 + b_3).$$

Keeping s, b_3 fixed, its derivative is

$$G'(t) = 2q - z + 2b_3 + 2y - 4t + 3s/2 \ge -2r - z.$$

Replacing t by r consequently costs at most $(2r+z)e_{24}/(2w) \le 4e_{24}$. At $t = r$, the last term together with the good-P loss is

$$\frac{r^2}{8} - \frac{1}{2}(s - r/2)^2 + (2r - s)b_3.$$

Use $(2r-s)b_3 \le (2-\sigma)H + o(1)$.

The direct A, B allowances are $2a + j + (t/r)a$. Both positive CC charges exclude pairs entirely outside the initial retained C, whose missing density is at least $\delta^2/2 - a\delta/d - o(1)$. Add the initial matching count and subtract the two corrections defining V. The e_{12}, e_{23}, e_{34} terms cancel, the a coefficient is at most $2 + 2\delta/d$, and the main polynomial becomes $F(r)$. This proves the claim. □

11.2 Two ways to re-pair a retained and an omitted pair

On the covered target indices and omitted indices permit either of the links

	required contacts		replacement label-two pairs
straight	$C_iC_o = D_iD_o = 2$,	$A_iC_o = 1$	$C_iC_o,\ D_iD_o$
crossed	$C_iD_o = D_iC_o = 2$,	$A_iD_o = 1$	$C_iD_o,\ D_iC_o.$

(46)

A pair lacking a straight link is charged to ξ, U, or a missing AC contact. A pair lacking a crossed link is charged to ζ, a missing AD contact, or a discarded blue CD incidence. A pair lacking both must be charged in both lists. Thus the density of missing links in their union is at most $(V + U + a + j)/2 + o(1)$. The rectangular vertex-cover bound gives a matching of $kM + O(1)$ links with

$$k \ge \min(s, \delta) - \frac{V + U + a + j}{2\max(s, \delta)} - o(1). \tag{47}$$

If both sides have zero mass, take $k = 0$. When $s \ge \delta > 0$, this reduces to $k \ge \delta - (V + U + a + j)/(2s) - o(1)$. The same inequality also permits enlargement when $s < \delta$. For each link use one available orientation. The disjoint label-two replacements preserve the entire formal weight vector, every original vertex birth, the old critical count, and all cross-class deficits. For a straight link add D_o to the eligible C pool and C_o to the eligible D pool; for a crossed link add C_o to the C pool and D_o to the D pool. Denote the resulting pools by C^* and D^*. Both contain their original retained sets and have mass $R = r + k$. The replacement label-2 pairs match the two pools.

Lemma 11.2. *Every vertex of D^* has exact outward and top label 4, and label 4 to every good Q. Every vertex of C^* has outward and top labels at least 3, and labels at least 3 to every good Q.*

Proof. The assertions for initially retained vertices were already proved. For a straight selected pair, the actual core replacement

$$A_iD_i(1), C_oD_o(2) \longrightarrow A_iC_o(1), D_iD_o(2)$$

is primary-neutral and preserves all initial CC and BB links. It places C_o in an AD role. The usual switches with a $BB(4)$ edge prove its outward, top, and good-Q label-4 caps.

For D_o, the link $C_oC_i = 2$ proves the outward cap by mate release. If $D_ou \leq 2$ for an originally uncovered u, replacing $C_oD_o(2)$ by $D_ou(2)$ would preserve primary weight and leave C_o uncovered, contradicting the retained C_i cap. A label-1 edge is already a primary improvement. For the top cap, let $C_iC_j(1)$ be its retained CC link, let $A_jD_j(1)$ be the corresponding AD link, and pair the old mate B_j of A_j with another initial B' at label 4. A putative $D_oT \leq 2$ permits

$$\begin{aligned} &B_jB'(4), C_iC_j(1), A_jD_j(1), C_oD_o(2), TT'(4) \\ &\quad\longrightarrow D_oT(\leq 2), C_oC_i(2), C_jD_j(2), A_jB_j(2), B'T'(\leq 4), \end{aligned}$$

reducing primary cost from 9 to at most 8. For a good pair PQ, replace $TT'(4)$ by $PQ(3)$ and choose B' with $PB' = 3$. The same switch would reduce cost from 7 to at most 6 if $D_oQ \leq 2$. The fixed threshold of four neighbors makes all choices disjoint. □

Explicit caps for a crossed link. Assume $C_iD_o = D_iC_o = 2$ and $A_iD_o = 1$, where $A_iD_i = 1$ belongs to the old neutral core and $A_iB_i = 2$ was an original red pair. The primary-neutral replacement

$$A_iD_i(1), C_oD_o(2) \longrightarrow A_iD_o(1), D_iC_o(2)$$

keeps all old CC and BB core edges. Thus D_o becomes an AD endpoint in an actual primary-minimal matching, and C_o has remaining birth two.

The following switches prove that D_o has label four to an originally uncovered vertex u, an old top vertex T, and every good Q, respectively. Choose the auxiliary B' distinct from B_i; in the last line choose it among the retained label-three neighbors of P:

$$\begin{aligned} B_iB'(4), A_iD_o(1) &\longrightarrow A_iB_i(2), D_ou(\leq 3), \\ B_iB'(4), A_iD_o(1), TT'(4) &\longrightarrow A_iB_i(2), D_oT(\leq 3), B'T'(\leq 4), \\ B_iB'(4), A_iD_o(1), PQ(3) &\longrightarrow A_iB_i(2), D_oQ(\leq 3), PB'(3). \end{aligned}$$

Their primary costs decrease from $4, 8, 6$ to at most $3, 7, 5$. The first switch leaves B' uncovered instead of u.

For C_o, use the original matching and $A_iD_i = 1$. A putative outward or top label at most two gives

$$\begin{aligned} A_iB_i(2), C_iD_i(2), C_oD_o(2) &\longrightarrow A_iD_i(1), C_iD_o(2), C_ou(\leq 2), \\ A_iB_i(2), C_iD_i(2), C_oD_o(2), TT'(4) &\longrightarrow A_iD_i(1), C_iD_o(2), C_oT(\leq 2), B_iT'(\leq 4). \end{aligned}$$

The costs decrease from $3, 7$ to at most $2, 6$. For the good-Q cap, let $C_iC_j = 1$ be the old core edge and let $A_jD_j = 1$ be the core link at its other endpoint. Choose a core B' with $PB' = 3$ and re-pair B_jB' at label four. Then

$$\begin{aligned} &B_jB'(4), C_iC_j(1), A_jD_j(1), C_oD_o(2), PQ(3) \\ &\quad\longrightarrow C_oQ(\leq 2), D_oC_i(2), C_jD_j(2), A_jB_j(2), B'P(3) \end{aligned}$$

decreases primary cost from 7 to at most 6. The fixed good-pair threshold supplies disjoint auxiliary choices. These switches prove all the claimed label restrictions for the crossed orientation. Each restriction follows from the existence of the stated edges and the minimality of the initial matching.

An original B vertex whose mate A_0 has two blue neighbors among the initial retained D sees every D^* vertex at label 4, except for $O(M)$ own-slot incidences. For a selected new D endpoint, first make its single core exchange in the preceding proof, then insert A_0 at a different original D slot. The usual $BB(4), AD(1)$ switch applies. These are individual witnesses; the inserted rows need not all coexist in one core. The remaining weak rows have mass at most $a/r + o(1)$.

11.3 The enlarged reservoir count

Lemma 11.3. *For every $R = r + k$ supplied by (47), there is an actual matching satisfying*

$$C_E \le B_0 + F(R) + c - E_{AA} + \left(1 + \frac{R}{r} + \frac{2\delta}{d}\right) a + j + 2U + \frac{R}{2r} H + \frac{H^2}{2r^2} + \frac{35}{4} e_{24} - e_{13} - e_{14} - I_3 + o(1). \tag{48}$$

Proof. Uniformly sample RM old top vertices and match C^* to them at label 3. By Lemma 11.2 and the unchanged original cross half bounds, the total missing C^*–top label-3 density is at most e_{24}. The actual target mass t' therefore satisfies $\mathbb{E}(R - t') \le e_{24}/(2w) + o(1)$. Use the joint low-label matching on its D^* mates and all exceptional old label-3 endpoints. All exceptional births only decrease; every promoted vertex has the exact outward cap proved above.

Orient all CD pairs after the swaps as $C_{\text{new}}, D_{\text{new}}$. Their missing A-edge densities obey $a_{\text{new}} + j_{\text{new}} = a + j$. The direct gains involving A, B cost at most $2a_{\text{new}} + j_{\text{new}} + (t'/r)a$. Indeed a new C–B label-2 edge forces its complementary AD link to be missing; new D–A and C–A gains require missing AD and AC links, respectively. The last term is the weak-row allowance just proved. After subtracting the old $a + j$ this is at most $(1 + R/r)a + j$.

Partition the originally omitted indices into straight-selected O_s, crossed-selected O_c, and unselected O_0. Only O_s exchanges its old C, D memberships. Let e_{out} count all old blue pairs inside the omitted C set, and let subscripts indicate their support. Up to discarded $O(M)$ incidences,

$$c_{\text{new}} = c + U_{O_s} + e_{\text{out,incident } O_s}.$$

Both positive missing-CC charges exclude pairs entirely outside the expanded C pool, whose density is $(q - R)^2/2 - e_{\text{out},O_0O_0} + o(1)$. The remaining disjoint DD blue-mate allowance is at most $U_{O_0} + e_{\text{out},O_cO_0} + o(1)$. Hence, including the old $-c$, their sum is bounded by

$$\begin{aligned} c - (q - R)^2 + 2U_{O_s} + U_{O_0} + 2e_{\text{out,incident } O_s} + 2e_{\text{out},O_0O_0} + e_{\text{out},O_cO_0} \\ \le c - (q - R)^2 + 2U + 2e_{\text{out}} \\ \le c - (q - R)^2 + 2U + 2a\delta/d + o(1). \end{aligned}$$

The three outside-edge supports on the first line are disjoint. This proves that mixing the two orientations introduces no additional resource charge. In particular, this estimate is valid when links of both orientations are selected in the same matching.

The direct CD gain is $2t'q - t'^2$. The joint internal-DD and exceptional-pair allowance is $(R - t'/2 + b_3)^2/2$, in addition to the missing-edge charges just handled. Random top demotion loses at least $R^2 - R\mathbb{E}(R - t') - Re_{24}/(2w)$ contacts to C^* and $Rz - 2z\mathbb{E}(R - t') - Re_{34}/(2w)$ to old X_3. The target polynomial before these losses has derivative at least $-R/4 - b_3/2$. At $t' = R$ its exceptional contribution is $Rb_3/2 + b_3^2/2 \le RH/(2r) + H^2/(2r^2) + o(1)$. The complete e_{24} coefficient is

$$\frac{9R/4 + 2z + b_3/2}{2w} - 1 \le \frac{35}{4},$$

using $R \le q \le 2w$, $z \le 3q$, and $b_3 \le z$. The e_{34} coefficient is $R/(2w) - 1 \le 0$. The X_1 half bound costs $2Ry + e_{12}$, and the good-P failure allowance is e_{23}; both deficits cancel their old terms. Collecting the main terms gives $F(R)$ and proves the claim. □

11.4 An optimized tangent envelope

For $0 \le \varepsilon \le 1$, concavity of F gives the tangent bound

$$F(R) \le F(q) + \frac{23}{8}\varepsilon^2 q^2 + \left[\left(\frac{7}{4} - \frac{23}{4}\varepsilon\right) q + z - 2y\right](q - R).$$

It is the tangent at $R = (1-\varepsilon)q$. Replacing the bracket by its positive part remains valid because $R \le q$. If $\sigma = s/r \ge 1/7$, then $s \ge \delta$, so put

$$J_\varepsilon = \frac{[(7/4 - 23\varepsilon/4)q + z - 2y]_+}{2\sigma r}.$$

The doubled denominator records the availability of both orientations. Substituting (47) in the extended count yields

$$\begin{aligned} C_E \le B_0 + F(q) + \tfrac{23}{8}\varepsilon^2 q^2 + c - E_{AA} \\ + \left(1 + \frac{q}{r} + \frac{2\delta}{d} + J_\varepsilon\right) a + (1 + J_\varepsilon) j + (2 + J_\varepsilon) U + J_\varepsilon V \\ + \frac{q}{2r} H + \frac{H^2}{2r^2} + \frac{35}{4} e_{24} - e_{13} - e_{14} - I_3 + o(1). \end{aligned} \tag{49}$$

The comparison (45) has coefficient -1 on V. Thus a convex combination eliminates the positive V term in (49) whenever the weight on (45) is at least J_ε times the weight on (49). Equation (44) and the split packing inequality then control the other missing-edge terms. On parameter intervals where this enlargement is not used, (45) remains available with its negative square term.

The choice $\varepsilon = 0$ gives the endpoint tangent. Choosing ε closer to the actual fractional enlargement retains more of the concave polynomial. Every value of ε gives a valid bound throughout the stated domain; the choice only changes its strength.

11.5 The two reservoir matchings share a cardinality constraint

The initial and enlarged reservoir constructions use related auxiliary graphs. Comparing their maximum matching sizes yields an additional constraint. Write $R = r + k$ for the enlarged eligible-pool mass, $t \le r$ for the initial target mass, and $t' \le R$ for the enlarged target mass. Let b_3 be the exceptional label-three pair mass. Both auxiliary graphs include all $2b_3$ exceptional endpoints and start from their old matching. The old maximum low-label matching covers s target D vertices, and the enlarged one covers s' target D^* vertices.

Put $\delta_0 = r - t$ and $\delta_1 = R - t'$. The corresponding target sets I, I' satisfy

$$|I' \setminus I|/M \le k + \delta_0 + o(1).$$

Deleting the edges incident to these added vertices from the new maximum matching leaves a matching in the old auxiliary graph. Each deleted vertex destroys at most one matching edge. Because both maximum matchings preserve every exceptional endpoint, this yields

$$\frac{s' + 2b_3}{2} \le \frac{s + 2b_3}{2} + k + \delta_0 + o(1), \qquad s' \le s + 2k + 2\delta_0 + o(1). \tag{50}$$

This argument compares ordinary maximum matching cardinalities; it does not assume that the new matching contains the old matching's edges. Starting with the exceptional pairs and augmenting guarantees that a maximum matching preserving their covered vertices exists in each graph.

The unmatched new target mass is $u' = t' - s'$. Before maximizing over u', the contribution of the low-label matching to the enlarged critical-edge count is

$$u'\big(R - t'/2 + b_3\big) - \frac{(u')^2}{2}.$$

The unconstrained upper bound is $(R - t'/2 + b_3)^2/2$. Completing the square and using (50) gives an additional loss at least

$$\frac{1}{2}\left(\frac{r}{2} - s - \frac{3k}{2} - b_3 - 2\delta_0 - \frac{3}{2}\delta_1\right)_+^2. \tag{51}$$

Indeed $u' \geq t' - s - 2k - 2\delta_0$, and the unconstrained maximizing value is $R - t'/2 + b_3$; their difference is the expression inside the positive part. All other terms in the enlarged count bound are unchanged.

Condition on the old target and the selected swap links, so that r, s, k, δ_0 are fixed during the new top sampling. We have

$$\delta_0 \leq \frac{e_{24}}{2w} + o(1), \qquad \mathbb{E}\delta_1 \leq \frac{e_{24}}{2w} + o(1), \qquad b_3 \leq H/r + o(1).$$

The function $x \mapsto (x_+)^2/2$ is convex. Jensen's inequality applied to (51) therefore proves that one may subtract

$$\frac{1}{2}\left(\frac{r}{2} - s - \frac{3k}{2} - \frac{H}{r} - \frac{7e_{24}}{4w}\right)_+^2 \tag{52}$$

from the right side of the enlarged-reservoir bound (48). The coefficient $7/(4w)$ is the sum of the old-target contribution $2/(2w)$ and the new-target contribution $(3/2)/(2w)$. The bounds on the positive terms due to unused top vertices remain valid, so they require no additional error term in (52). All errors are uniform in $N \geq 2M$, as both auxiliary graphs have only $O(M)$ vertices.

Quadratic forms for exact certificates. Let $g = r/2 - s - 3k/2 = 2r - s - 3R/2$. The inequality

$$-\tfrac{1}{2}(x_+)^2 \leq \tfrac{1}{2}\tau^2 - \tau x \qquad (\tau \geq 0)$$

provides useful rational quadratic bounds. Choosing $\tau = \eta r$ for any fixed rational $\eta \geq 0$ adds to the right side of (48)

$$\eta r(s - r/2 + 3k/2) + \frac{\eta^2 r^2}{2} + \eta H + \frac{7\eta r}{4w} e_{24}. \tag{53}$$

The last coefficient is at most $7\eta/2$ on $r \leq 2w$. Writing the enlarged polynomial as $F(R)$, its modified part is $F(R) + (3\eta/2)rR$. First taking the tangent at $R = q$, thus using $\varepsilon = 0$ in the preceding subsection, the two-orientation swap envelope uses

$$J_\eta = \frac{(7q/4 + z - 2y - 3\eta r/2)_+}{2\sigma r}, \qquad \sigma = s/r,$$

and its main polynomial is

$$F(q) + \eta r\big(s - r/2 + 3(q - r)/2\big) + \eta^2 r^2/2.$$

Its H coefficient increases by η, and its e_{24} coefficient increases by at most $7\eta/2$; the usual two-orientation defect coefficients use J_η in place of J.

Alternatively choose $\tau = \lambda q$ with fixed rational $\lambda \geq 0$. Using $q/r \leq 8/7$ and $q \leq 2w$, the new defect costs are at most

$$\frac{8\lambda}{7} H + \frac{7\lambda}{2} e_{24}.$$

In the envelope, the main polynomial changes by

$$\left(\frac{\lambda^2}{2} + \frac{3\lambda}{2}\right) q^2 - \lambda(2 - \sigma) qr,$$

and

$$J_\lambda = \frac{((7/4 - 3\lambda/2)q + z - 2y)_+}{2\sigma r}.$$

These formulas introduce no new optimization coordinate: a rational η or λ may be chosen separately on each ratio box.

A coarser consequence, useful when fixed coefficients on the deficits are preferred, is the additional term

$$-\frac{1}{2}g_+^2 + \frac{H}{2} + \frac{7}{4}e_{24}.$$

To verify it, note that $g_+ \le r/2$ and $\frac{1}{2}g_+^2 - \frac{1}{2}(g-D)_+^2 \le (r/2)D$ for $D \ge 0$. Take $D = b_3 + 2\delta_0 + 3\delta_1/2$, average, and use $r \le 2w$. The certificate below uses the tangent form (53).

The envelope when the covered side is smaller. The full rectangular bound (47) gives

$$q - R \le (\delta - s)_+ + \frac{V + U + a + j}{2\max(s,\delta)} + o(1).$$

After the rank tangent with parameter η and the reservoir tangent at $R = (1-\varepsilon)q$, the coefficient of $q - R$ is bounded by

$$\left[(7/4 - 23\varepsilon/4)q + z - 2y - 3\eta r/2\right]_+ .$$

To make this bound uniform on a parameter box, suppose

$$\sigma_0 \le s/r \le \sigma_1, \qquad \rho_0 \le r/q \le \rho_1, \qquad \zeta_0 \le z/q \le \zeta_1, \qquad \upsilon_0 \le y/q \le \upsilon_1.$$

Put

$$D_{\mathrm{box}} = \max\{\sigma_0\rho_0, 1-\rho_1\}, \qquad A_{\mathrm{box}} = [7/4 - 23\varepsilon/4 + \zeta_1 - 2\upsilon_0 - 3\eta\rho_0/2]_+.$$

If $D_{\mathrm{box}} > 0$, use

$$J = \frac{A_{\mathrm{box}}}{2D_{\mathrm{box}}}, \qquad E_{\mathrm{box}} = A_{\mathrm{box}}[1 - (1+\sigma_0)\rho_0]_+. \tag{54}$$

The resulting enlarged column has the same resource coefficients as before, with this J, and the additional main term $E_{\mathrm{box}}q^2$. Indeed the slope is at most $A_{\mathrm{box}}q$, $\max(s,\delta) \ge D_{\mathrm{box}}q$, and $(\delta - s)_+ \le [1-(1+\sigma_0)\rho_0]_+ q$. If $D_{\mathrm{box}} = 0$, give this comparison zero weight. On a box satisfying $\sigma_0\rho_0 \ge 1 - \rho_0$, the additional term vanishes and the formula agrees with the envelope for complete enlargement. Otherwise it accounts explicitly for the part of the omitted set that the matching cannot absorb.

12 The supply of label-4 vertices and the imbalance of the two families

The reservoir construction matches retained C vertices to the original label-4 vertex set, whose mass is $2w$. When that set is smaller than the retained set, a comparison that simply replaces the matched mass by r is not applicable. We first give a bound that treats both possible relative sizes. It will also exclude small values of $|d - q|$ before any lower bound on that imbalance is assumed.

Write $H = h_3$ and $B_0 = B - b^2$. The following comparison applies in either orientation and uses only $r \le q$, which every neutral core satisfies. It assumes neither $q \le 2w$ nor a lower bound on the mass of good P vertices. Put

$$h_0 = \min(r, 2w), \qquad D_0 = \max(r, 2w),$$

and define the quadratic polynomial, separately on its two capacity regions,

$$g(h_0) = h_0(3q/2 - r - z + 2y) - 7h_0^2/8 + q^2/2.$$

A maximum label-3 matching between retained C and the original label-4 vertices covers target mass

$$t \ge h_0 - e_{24}/D_0 - o(1).$$

This is the rectangular vertex-cover bound and applies regardless of which side is larger. The exact outward caps of retained C, D were proved independently of top capacity.

The direct reservoir count, with the matched mass t left unspecified, gives

$$\begin{aligned} C_T - C_0 \le t(3q/2 - r - z + 2y + 3H/r) - 7t^2/8 + q^2/2 \\ + 2c + j + (2 + t/r)a + e_{12} + e_{23} + e_{24} + e_{34} + o(1). \end{aligned}$$

For $0 \le t \le r \le q$, the derivative of its part independent of H is at least $-5r/4 - z$. Also $3tH/r \le 3H$ and $t/r \le 1$. Adding the initial count and using this bound on the number of unmatched target vertices proves

$$\begin{aligned} C_T \le B_0 + g(h_0) + 3H + 2a + c - E_{AA} - e_{13} - e_{14} \\ + \frac{5r/4 + z}{D_0} e_{24} + o(1). \end{aligned} \tag{55}$$

No $Z \ge 3r/2$ hypothesis enters this comparison.

On the domain $r, w \ge 1/10$ established from the initial quadratic certificate, the coefficient of e_{24} is at most $33/8$, and in particular at most 5. Indeed $x \ge 2r$ and $x + y + z + w = 1$ imply $5r/4 + z \le 1 - 3r/4 - w$. If $r \le 2w$, the ratio is at most $33/8$ by $r, w \ge 1/10$; if $r \ge 2w$, it is at most $15/4$. The core estimate (10) is also valid here. On $y \le 3z/8$ and $y \le w$ it is enough to retain its $-5H/8$ and to bound its possibly positive e_{14} coefficient by

$$(r/(2w) - 1)_+ \le 3/2,$$

using $r \le \min(d, q) \le 1/2$ and $w \ge 1/10$.

Let p, b_R, b_T be nonnegative weights, summing to one, for the initial count, the core count, and (55), respectively. It suffices to impose

$$\frac{5}{8} b_R \ge 3b_T, \qquad p \ge 5b_T, \qquad p \ge 2b_T, \qquad p + b_T \ge \frac{3}{2} b_R.$$

These absorb respectively H, e_{24}, the positive a, c terms, and e_{14}; all remaining cross-defect terms are nonpositive. Writing $m = \min(d, q)$ and $K = (m^2 - r^2)/2$, the packing estimate $E \ge K - o(1)$ then yields

$$C \le B_0 + b_R f_R + b_T g(h_0) - (p - 2b_T)K + o(1).$$

This is a rational quadratic expression on each of the regions $r \le 2w$ and $r \ge 2w$.

Small imbalance. Apply the same comparison to $0 \le h \le 3m/10$, where $h = |d - q|$. Keep the actual CD mass q in $g(h_0)$, but use $m = \min(d, q)$ in the packing term K. There are four cases: the two orientations, each with $r \le 2w$ or $r \ge 2w$. In the former case use $h_0 = r$; in the latter use $h_0 = 2w$. Retain the same conservative $3e_{14}/2$ core coefficient and the same defect budgets. Four rational positive-definite matrices certify

$$\gamma(4) \le 433403/1000000 = 0.433403 \qquad (0 \le h \le 3m/10).$$

The checker explicitly requires all four orientation/capacity cases. It assumes neither $m \le 2w$ nor $h \ge 3m/10$. We therefore first dispose of the small-imbalance region. The comparison for $q \ge d \ge 2w$ below can then use $h \ge 3m/10$. This order is essential because that comparison depends on a positive lower bound for h/m.

In the case $d \ge q$, the two matrices in `ab-boundary-certificate.json` prove

$$\gamma(4) \le 433/1000 \quad \text{when } d \ge q \ge 2w$$

on the shared localized domain. Its checker verifies the defect budgets, positive definiteness by rational LDL^T, and the presence of both capacity regions. This includes both relative sizes of r and $2w$ whenever $q \ge 2w$.

The larger-CD orientation. The full surplus-target construction uses no top vertices. Its assumption $d \le 2w$ is needed only when discarding the rebuilt-core term $(s/(2w)-1)e_{14}$ and the analogous initial-core term. Since $s, r \le d \le 1/2$ and $w \ge 1/10$, retain at most $3e_{14}/2$ in each core or full-target comparison. Bound this term before the packing replacement $s \to d$; then every derivative and resource-allocation argument for that replacement is unchanged. All eligibility, matching-intersection, unsafe-vertex, and rich-row arguments are independent of top capacity. Consequently the full-target comparison is valid on $d \ge 2w$ provided that its convex-combination weights obey the additional inequality $p \ge 3(b_R + b_D)/2$, where p, b_R, b_D are the weights of the initial, core, and full-target counts. The rational eight-variable certificate in `cd-boundary-certificate.json` gives $\gamma(4) \le 0.43301$ on this region. Together with the larger-AB bound 0.433, this excludes $m \ge 2w$ in both orientations; since $r \le m$, it also excludes $r \ge 2w$.

13 Combining the matching comparisons

We now prove that the preceding inequalities cover every labelled complete graph and imply $\gamma(4) \le 0.4334035$. Throughout this section, $h = |d - q| = 2b$ and $m = \min(d, q)$. Numerical bounds on γ include the factor $1/2$ that converts counts divided by M^2 to counts divided by $\binom{2M}{2}$.

13.1 How the quadratic certificates prove an upper bound

For a fixed labelled graph, let $C_0, C_1, \dots$ denote the counts, divided by M^2, of the actual matchings constructed above. For nonnegative weights θ_j summing to one,

$$\min_j C_j \le \sum_j \theta_j C_j.$$

We choose the weights so that the coefficients of all uncontrolled nonnegative missing-edge counts on the right are nonpositive. Packing inequalities remove the remaining such quantities. The result is an upper bound $1 - \Phi(V) + o(1)$, where Φ is a homogeneous quadratic form in the remaining parameters V.

Write $\Lambda = x + y + z + w$, which equals one for the normalized parameters. Suppose the parameter domain gives homogeneous quadratic inequalities $g_i(V) \ge 0$. A certificate consists of nonnegative rational numbers μ_i such that

$$\Phi(V) - (1 - 2G)\Lambda^2 - \sum_i \mu_i g_i(V) = V^T A V$$

for a positive semidefinite rational matrix A. This identity implies $\Phi(V) \ge 1 - 2G$ on the domain, hence some matching has count at most $2G + o(1)$, and the resulting bound is $\gamma(4) \le G$. Equality constraints may be multiplied by arbitrary rational numbers. The checkers reconstruct the identity from the displayed comparisons and the certificate data. They verify positive definiteness by an exact LDL^T factorization, so no floating-point eigenvalue calculation is used.

The case division proceeds as follows. The initial and core comparisons first restrict the parameter domain. The bounds of Section 12 then dispose of small imbalance and insufficient supply of label-4 vertices. We divide the remaining cases into $d \ge q$ and $q \ge d$. In the first case the alternatives in the AP and CP matching comparisons cover the domain; in the second case the full-target comparison applies. Every restriction below is established using comparisons valid before that restriction is imposed.

13.2 Common parameter restrictions

The starting certificate and its entry conditions are proved in Section 8.

The retained conditions are established in two ways. The starting quadratic residual gives the desired bound whenever any of the following inequalities fails:

$$r, w \geq 1/10, \quad z \geq 1/5, \quad y \geq 187m/1000, \quad y \leq w, \quad 2y \leq z,$$
$$r + z \leq 6w, \quad h \leq 5m/4, \quad y \leq m, \quad z \leq 3m.$$

These conditions depend only on $m = \min(d, q)$ and apply in both orientations. The exact checker verifies that each residual count gap exceeds $2(23/53 - 0.4334035)$.

The remaining boundary regions are optimized directly, using the initial and core inequalities with weights $1 - \beta, \beta$, where $0 \leq \beta \leq 1/4$. The initial deficit coefficient -1 and core coefficient at most 3 are then absorbed. Nonnegative multiples of

$$(m - r)(2r - m), \quad \Lambda(m - r), \quad \Lambda(r - 4m/5)$$

and the relevant branch and violating-halfspace constraints are subtracted from the quadratic form. Exact rational positive-definite residuals give the following *bounds on* γ, already divided by two:

Excluded region	Certified bound
$y \geq 3z/8$, with $y \leq z$	0.43324206
$y \geq 13m/20$	0.43335152
$y \geq z$	0.42503599

Their rational weights and multipliers are in `exclusion-certificate.json`. The checker reconstructs each matrix independently. At this point the retained common conditions include $y \leq 3z/8$, $y \leq 13m/20$, $z \leq 3m$, and $h \leq 5m/4$, but no top-capacity condition or positive lower bound on h/m.

Section 12 supplies the following additional exceptions, in this order:

Region	Matrices	Bound on γ
$0 \leq h \leq 3m/10$, both orientations and capacity cases	4	0.433403
$d \geq q \geq 2w$, both capacity cases	2	0.433
$q \geq d \geq 2w$, with $h \geq 3d/10$	1	0.43301

The first uses the raw reservoir with target $\min(r, 2w)$ and assumes no capacity bound. We may then retain $h \geq 3m/10$. The next two exclude $m \geq 2w$ in the two orientations, and $r \leq m$ also gives $r \leq 2w$. This ordering is necessary: the surplus-target capacity argument uses the positive imbalance established in the first step. The exact checkers are `verify_small_imbalance.py`, `verify_ab_boundary.py`, and `verify_cd_boundary.py`. Their domain forms, weights, and coverage checks are reconstructed from the accompanying rational data. Here $h = 2b$ in both orientations; no condition $h \leq r$ is used.

13.3 Parameter restrictions when $d \geq q$

In this orientation $q = m$, $d = q + h$. The original random-top comparison and the starting packing inequality give the following excluded-region bound:

region	bound on γ
$r \leq 7q/8$	0.43336741

This bound is verified by `verify_ab_good_domains.py`; it uses neither a bound on the missing-edge density within the retained C set nor either alternative of the AP comparison. Here and below $Z = z - h_3/r$.

On $r \ge 7q/8$, the joint-reservoir inequality gives

$$Z \le 3r/2 \quad \Longrightarrow \quad \gamma \le 0.433392.$$

Five rational seven-variable certificates use the violating form $zu - rz + 3r^2/2 \ge 0$, where $u = h_3/z$. Their ratio domain is $h/q \in [3/10, 5/4]$, $r/q \in [7/8, 1]$, $z/q \in [2/5, 3]$; the last lower endpoint follows from $z \ge 1/5$ and $q \le 1/2$. They use the weaker Tutte–Berge resource and impose no good-P lower bound. The checker `verify_z_domain.py` verifies the defect budgets, matrices, and full domain coverage from `z-domain-certificate.json`.

After excluding $r < 7q/8$, split according to $c \ge q^2/8$. In that case the initial and core bounds have the majorants $B_0 - q^2/8$ and $B_0 + f_R$. Put $F = x^2/4 + y^2 + z^2/2 + w^2 + b^2$, so $B_0 = 1 - F$. Take core weight $135283/10^6$ and initial weight its complement. The corresponding loss form is

$$F + \frac{864717}{10^6}\frac{q^2}{8} - \frac{135283}{10^6} f_R - \frac{69682}{10^6}\Lambda(r - 7q/8) - \frac{21247}{10^6}(r - 7q/8)(q - r).$$

Here $\Lambda = x + y + z + w$. Subtracting $(1 - 2 \cdot 0.43321)\Lambda^2$ gives a positive-definite rational six-variable form, checked by `verify_high_c.py`. Thus this case satisfies $\gamma \le 0.43321$. On its complement Lemma 6.1 applies.

After also excluding $Z < 3r/2$, a further two-matrix certificate gives

$$d \ge 23Z/20 \quad \Longrightarrow \quad \gamma \le 0.433378.$$

It combines the initial, core, and joint-reservoir bounds and subtracts a nonnegative multiple of

$$h_3 - rz + 20rd/23 = zu - rz + 20rd/23 \ge 0, \qquad u = h_3/z.$$

The data are in `good-domain-new-certificate.json`; its checker verifies the resource budgets, both ratio boxes, and rational positive definiteness. It uses the Tutte–Berge inequality (17), without the additional dense-case estimate.

Both orientations also retain $y \ge 187m/1000$. The exact count loss for its violating halfspace is $23138190/20583940031$, larger than $2(23/53 - 0.4334035)$. This localization uses only the starting certificate. The surviving larger-AB domain is consequently

$$3/10 \le h/q \le 5/4, \quad 7/8 \le r/q \le 1, \quad 21/16 \le z/q \le 3, \quad 187/1000 \le y/q \le 13/20,$$
$$Z \ge 3r/2, \quad d \le 23Z/20, \quad c \le q^2/8,$$

together with the common conditions already stated.

13.4 Three exhaustive alternatives on the larger-AB domain

All three alternatives now use seven coordinates $V = (x, y, z, w, b, r, u)$, where $u = h_3/z$. In particular, the exact core loss $(1 - y/z)h_3$ is $(z - y)u$, a quadratic form. The ratio coordinates are

$$(v, \rho, \zeta, \sigma, \upsilon) = (h/q, r/q, z/q, s/r, y/q)$$

and their common root box is

$$[3/10, 5/4] \times [7/8, 1] \times [21/16, 3] \times [0, 1] \times [187/1000, 13/20]. \tag{56}$$

Here s is the covered target-D mass in Section 11. It is not an independent coordinate of the quadratic form: each box encloses its actual ratio s/r. The AP lemma gives either its saturated comparison or its deficient comparison. In the latter case use the two CP alternatives with the single fixed threshold $\alpha = 179/1000$. Each of these three cases is certified on a separate complete partition of (56).

On a box with endpoints $v_i, \rho_i, \zeta_i, \sigma_i, \upsilon_i$, $i = 0, 1$, put

$$L = \frac{4(1-\rho_0)}{5\rho_0 - 4}, \qquad \beta = \alpha_0 = \frac{1}{1+v_0}, \qquad K = \frac{q^2 - r^2}{2},$$

$$A_T = 2 + \frac{2-\rho_0}{1+v_0} + L\beta, \quad C_T = 1 + L, \qquad T_0 = f_T - \frac{q(q-r)}{2} - LK.$$

These coefficients use the globally maximal joint-core resource (20), dropping its favorable quadratic term in a. This substitution preserves the negative term $-LK$, which is needed in the quadratic comparison. For the two-orientation swap comparison choose the tangent parameters $\epsilon_* = 5/64$ and $\eta_{\mathrm{rk}} = 1/10$ in all three alternatives. The second parameter uses the cardinality constraint of Section 11.5. Put

$$g = \tfrac{1}{2}\operatorname{dist}([\sigma_0, \sigma_1], 1/2)^2, \qquad A_{\mathrm{box}} = [7/4 - 23\epsilon_*/4 + \zeta_1 - 2\upsilon_0 - 3\eta_{\mathrm{rk}}\rho_0/2]_+.$$

For saturated AP, use the full rectangular swap bound, setting

$$D_{\mathrm{box}} = \max\{\sigma_0\rho_0, 1-\rho_1\}, \qquad J = \frac{A_{\mathrm{box}}}{2D_{\mathrm{box}}}, \qquad E_{\mathrm{box}} = A_{\mathrm{box}}[1 - (1+\sigma_0)\rho_0]_+$$

when $D_{\mathrm{box}} > 0$. If $D_{\mathrm{box}} = 0$, assign the enlarged comparison zero weight and set $J = E_{\mathrm{box}} = 0$. For the two deficient alternatives, use the enlarged comparison only on boxes satisfying $\sigma_0\rho_0 \geq 1 - \rho_0$, and set $J = A_{\mathrm{box}}/(2\sigma_0\rho_0)$ and $E_{\mathrm{box}} = 0$ there. On their other boxes its weight is zero and $J = E_{\mathrm{box}} = 0$. These formulas are proved in (54) and include boxes where only part of the omitted mass can be absorbed. All conditions are checked exactly, with no division by zero.

After removing the common $B_0 = 1 - F$, the core, joint-reservoir, profiled initial-reservoir, and enlarged-reservoir columns are

$$\begin{aligned}
P_R &= f_R - (z-y)u,\\
P_T &= T_0 + zu/2 + (\zeta_1/\rho_0)^2u^2/2,\\
P_D &= T_0 - gr^2 + (2-\sigma_0)zu - V_{\mathrm{fail}},\\
P_X &= q^2/8 - qz + 2qy + (\tfrac{23}{8}\epsilon_*^2 + E_{\mathrm{box}})q^2 - (2+J)LK\\
&\qquad + zu/(2\rho_0) + (\zeta_1/\rho_0)^2u^2/2 + JV_{\mathrm{fail}}\\
&\qquad + \eta_{\mathrm{rk}}(\sigma_1 - 2 + \eta_{\mathrm{rk}}/2)r^2 + \tfrac{3}{2}\eta_{\mathrm{rk}}qr + \eta_{\mathrm{rk}}zu.
\end{aligned}$$

The additional terms in P_X follow by applying the tangent (53) and then taking the tangent of $F(R) + (3\eta_{\mathrm{rk}}/2)rR$ at $R = (1-\epsilon_*)q$. The $s = \sigma r$ term uses the upper endpoint σ_1. Here $V_{\mathrm{fail}} = \xi + \zeta$ is the unused-charge resource from Section 11; the letter ζ in that resource is unrelated to the ratio coordinate z/q. The enlarged column has coefficients

$$A_X = 1 + \frac{1}{\rho_0} + \frac{2(1-\rho_0)}{1+v_0} + J + (2+J)\left(\frac{\rho_1}{1+v_0} + L\beta\right), \qquad C_X = 1 + (2+J)L$$

on a, c, and coefficient $1 + J$ on $j = E_{AC}$.

The remaining auxiliary column depends on the alternative. For saturated AP, put

$$\eta = \frac{1+v_1}{\rho_0} - 1, \qquad A_{AP} = \max\left\{-2 + \frac{3(1+v_1)}{\rho_0}, \frac{4 + 6v_1 - 2\zeta_0}{\rho_0}\right\}.$$

Then

$$P_A = f_R + f_A - (z-y)u + \eta zu, \qquad (A_A, C_A, j_A, e_A) = (A_{AP}, 0, 1, 3/8).$$

The last four entries bound its positive coefficients on (a, c, j, e_{23}). For the deficient AP case, define

$$K_C = \frac{4}{(1-\alpha)(5 - 4/\rho_0)}, \qquad \alpha = 179/1000.$$

The large-CP alternative has

$$P_A = f_R + \alpha r(z + q - 3r) - \alpha^2 r^2/2 + (\alpha - 5/8)zu,$$
$$(A_A, C_A, j_A, e_A) = (3\alpha, \alpha, 0, 3/8).$$

The small-CP alternative has

$$P_A = f_R + r(4y - d - r) + \tfrac{3}{20} r(d - r) + \alpha r^2 - K_C K - 5zu/8,$$
$$(A_A, C_A, j_A, e_A) = (2 + \beta K_C, K_C, 0, 57/8).$$

Here the $3/20$ coefficient follows from $d \le 23Z/20$, and K_C follows from the retained-core part of the joint-packing lemma.

Let $p, b_R, b_T, b_A, b_D, b_X$ be the initial and five comparison weights, nonnegative and summing to one. A nonnegative split-packing multiplier μ uses $\alpha_0 a + c \ge K$. The exact resource budgets are

$$\begin{aligned}
&p \ge A_T(b_T + b_D) + A_A b_A + A_X b_X + \alpha_0 \mu,\\
&p \ge C_T(b_T + b_D) + C_A b_A + C_X b_X + \mu,\\
&p \ge j_A b_A + (1 + J) b_X,\\
&p \ge 13 b_T/2 + 4 b_D + (35/4 + 7\eta_{\mathrm{rk}}/2) b_X,\\
&p \ge e_A b_A, \qquad b_D \ge J b_X.
\end{aligned}$$

The last inequality absorbs V_{fail}. For saturated AP, the e_{23} inequality follows already from the j inequality. The remaining AP allowances on e_{12} and E_{AA} are also absorbed; all other omitted resource coefficients are nonpositive. Thus no unused-charge resource is treated as a free parameter after combination.

13.5 Rational matrices and exhaustive coverage

For clarity we specify the domain multipliers. Let $\mathcal{L}$ contain the six box faces for v, ρ, ζ, the two faces for υ, and

$$\begin{gathered}
3z/8 - y, \quad 13q/20 - y, \quad 2w - q, \quad 6w - r - z,\\
z - (\zeta_0/\rho_1)u - 3r/2, \qquad 23[z - (\zeta_0/\rho_1)u]/20 - d,\\
x, y, z, w, q, h, d - r, w - y.
\end{gathered}$$

The good-P forms are valid because $Z = z - (z/r)u \le z - (\zeta_0/\rho_1)u$; one must not substitute $z - u$ for Z. With $\Lambda = x + y + z + w$, subtract the certificate's nonnegative multiples of $\Lambda L, uL$ for $L \in \mathcal{L}$, of $u(r - u)$ and Λu, of products of opposite box faces, and of q times those faces. After adding the split-packing loss $+\mu K$ and subtracting $(1 - 20.4334035)\Lambda^2$, all reconstructed rational residual matrices have strictly positive LDL^T pivots.

The complete active partitions are

Alternative	Matrices	Checker
Saturated AP	1629	`verify_swap_cap.py`
Deficient AP, large CP	36	`verify_cp_large.py`
Deficient AP, small CP	38	`verify_cp_small.py`

Each checker reads its corresponding JSON certificate, reconstructs the forms from the displayed scalars, checks every resource budget, and verifies root-box containment, disjoint interiors, and exact total five-dimensional volume. The partitions for the two CP alternatives need not agree: their shared threshold α is fixed globally, and each partition covers the entire root domain. All three alternatives therefore give $\gamma \le 0.4334035$.

13.6 The case $q \geq d$

Use eight coordinates (x, y, z, w, b, r, u, t) with $u = h_3/z$, $t = h_3/r$, so $zu = rt$. Set

$$d = x/2 - b, \quad q = x/2 + b, \quad h = 2b, \qquad K = (d^2 - r^2)/2, \quad J_C = q(d - r)/2,$$

$$f_D = d(x + z - 3y) - 3d^2 - dh + 5h^2/8.$$

The 31 boxes in `cd-domain-certificate.json` partition $(h/d, r/d) \in [3/10, 5/4] \times [4/5, 1]$. On a box put

$$P = \frac{10}{9}(53/10 + v_1), \qquad A_D = P + \frac{v_1}{1 + v_1 - 9/(40\rho_0)}, \qquad \alpha_0 = \frac{1}{1 + v_0}, \quad \beta_0 = \frac{1}{9(1 + v_0)}.$$

The initial, core, and full-target weights p, b_1, c_1 sum to one and obey

$$p \geq c_1, \qquad \alpha_0\mu + \nu \leq p - A_D c_1, \qquad \mu + \beta_0\nu \leq p - Pc_1.$$

These absorb defects using $K \leq \alpha_0 a + c$ and $J_C \leq a + \beta_0 c$. The resulting loss form is

$$F - b_1 f_R - c_1 f_D + \mu K + \nu J_C + (b_1 + c_1)(z - y)u - 2c_1 ht.$$

The valid linear forms are the four ratio-box faces and

$$3z/8 - y, \quad 13d/20 - y, \quad 3d - z, \quad 2w - d, \quad 6w - r - z,$$
$$w - y, \quad y - 187d/1000, \quad r - u, \quad z - t, \quad u, t, x, y, z, w, d, h.$$

Subtract nonnegative multiples of Λ, u, t times every listed form, of opposite box-face products, and of d times the box faces. Add the certificate's unrestricted multiple of $zu - rt = 0$. In particular, the terms $\Lambda(r - u)$ and $\Lambda(z - t)$ use the bound $h_3 \leq rz$, expressed as the linear inequalities $u \leq r$ and $t \leq z$. The exact checker reconstructs the matrices and all budgets and coverage tests. After subtracting $(1 - 20.4334035)\Lambda^2$, all rational LDL^T pivots are positive. Thus this orientation gives $\gamma \leq 0.4334035$ without any of the larger-AB assumptions.

Taking the maximum over both orientations and all exceptional cases proves

$$\boxed{\gamma(4) \leq \frac{866807}{2000000} = 0.4334035.}$$

14 Upper bounds from alternating paths for general ℓ

In this section and the next two sections, $\ell \geq 5$. The argument uses a matching that minimizes the polynomial weight order of Csóka and Pongrácz [1], followed by a second matching obtained by exchanging edges along disjoint alternating paths. The formal edge weights are

$$w_1(\varepsilon) = 0, \qquad w_t(\varepsilon) = \sum_{j=0}^{t-2} 2^{t-2-j}\varepsilon^j \quad (2 \leq t \leq \ell).$$

Coefficients are compared starting at the constant term. The *primary weight* is the constant coefficient, so its values are $0, 1, 2, 4, \ldots, 2^{\ell-2}$. Fix a matching $\mathcal{M}_0$ of size M that minimizes this order. Let X_t be the set of endpoints of its label-t edges, and write $x_t = |X_t|/(2M)$, so $\sum_t x_t = 1$. Put

$$k = 2^{\ell-2}, \qquad S = 3k - 2\ell + 4, \qquad c_1 = c_\ell = 1, \quad c_2 = k, \quad c_t = 2^{\ell-t+1} - 2 \ (3 \leq t < \ell).$$

Here and below, an endpoint mass is divided by M, whereas an edge count is divided by M^2. Thus x_t is also the number of label-t matching edges divided by M. Write $C_0 = C_\lambda(\mathcal{M}_0)/M^2$.

No critical edge joins a covered vertex to an uncovered vertex: replacing the incident matching edge by such an edge would decrease the primary weight. Between distinct classes X_i, X_j, the complementary-pair argument of Section 3 gives at most $2x_ix_jM^2$ critical edges. Denote the difference between this upper bound and the actual count, divided by M^2, by $e_{ij} \geq 0$. Put $x = x_2$ and

$$Q = \sum_{t=1}^{\ell} \frac{(x_t - c_t/S)^2}{c_t}.$$

Color the matching edges inside X_2 red and its label-1 edges blue. An alternating path alternates red and blue edges; it is *red-ended* if its first and last edges are red. There is no alternating cycle, and no red-ended alternating path with k blue edges, since either would give a matching of smaller primary weight. We use the following consequence of the alternating-path lemma of [1]: a graph on v disjoint red pairs with no alternating cycle and no red-ended alternating path containing b blue edges has at most $(1 - 1/b)v^2 + O_b(v)$ blue edges. When the forbidden number k is even, the peeling proof also partitions the red pairs into $p = k/2$ layers. After deleting $O_\ell(M)$ blue edges, each pair in layer i can be oriented as L_iR_i so that all remaining blue edges belong to left–left pairs or to L_iR_j with $i < j$. The mass a_i of layer i is its number of red pairs divided by M; hence $\sum_i a_i = x$. This set of permitted pairs is the *blue template*. Let E be the number of its missing blue edges divided by M^2, and put

$$R = \frac{1}{2}\sum_{i=1}^{p}(a_i - 2x/k)^2.$$

The cross-class bounds just described give $2x_ix_j$ for each $i < j$. The internal critical count in class t is at most $(1 - 1/c_t)x_t^2 + o_\ell(1)$, by the alternating-path bound of [1]; for $t = 1, \ell$ it is zero. The actual X_2 template has density

$$x^2 - \frac{1}{2}\sum_i a_i^2 = x^2 - x^2/k - R,$$

so its missing blue density gives the further loss E. Since $\sum_t c_t = S$ and $\sum_t x_t = 1$, completing squares gives $\sum_t x_t^2/c_t = 1/S + Q$. Combining these estimates gives a nonnegative deficit D such that

$$C_0 \leq 1 - \frac{1}{S} - D + o_\ell(1), \qquad Q + R + E + \sum_{i<j} e_{ij} \leq D + o_\ell(1). \tag{57}$$

One can make D literally nonnegative at finite M by adding a fixed $O_\ell(1/M)$ allowance to its definition. Every error below involves only the original $2M$ covered vertices and is independent of N.

14.1 A constraint from a maximal collection of disjoint paths

Choose an integer b with $k/2 \leq b < k$, and put $L = b + 1$. Pack vertex-disjoint red-ended alternating paths in X_2, each with b blue edges and L original red edges, until no further path can be added. Let μ be the mass of original red edges in this packing. There are $\mu M/L$ paths, up to rounding.

The graph on the remaining $(x - \mu)M$ red pairs has no alternating cycle and no alternating path with b blue edges. Lemma 2.6 of Csóka–Pongrácz therefore bounds its blue density by $(1 - 1/b)(x - \mu)^2 + o_\ell(1)$. If a_i' are the remaining layer masses, the template gives the lower bound

$$(x - \mu)^2 - \frac{1}{2}\sum_i (a_i')^2 - E - o_\ell(1).$$

Since $0 \le a_i' \le a_i$ and $\frac{1}{2}\sum_i a_i^2 = x^2/k + R$, we obtain

$$R + E \ge \frac{(x-\mu)^2}{b} - \frac{x^2}{k} - o_\ell(1). \tag{58}$$

The mass constraint in Q implies

$$Q \ge \frac{S}{k(S-k)}(x - k/S)^2.$$

Minimizing the sum of this quadratic and the right side of (58) over $x \in \mathbb{R}$ proves

$$D \ge d_b(\mu) - o_\ell(1), \qquad d_b(\mu) = \frac{(k-b)/S - 2\mu + \mu^2}{S-k+b} = \frac{(1-\mu)^2}{S-k+b} - \frac{1}{S}. \tag{59}$$

The unconstrained minimizer is $x = [b + (S-k)\mu]/(S-k+b)$; in particular this minimization is just completion of a square.

14.2 Constructing the comparison matching

The closing edge of each packed path has label ℓ. Indeed, if its label were smaller, replacing the $L > k/2$ red edges by the b blue edges and the closing edge would lower primary cost from L to at most $k/2$. Exactly the same argument proves that *each endpoint has label ℓ to every originally uncovered vertex*: close the path to that vertex instead, leaving its other endpoint uncovered. Matching cardinality is unchanged in this switch.

Replace each path by its blue edges and its label-ℓ closing edge. All path-internal vertices now have matching label 1; its two endpoints have label ℓ. We call these changes a demotion and a promotion, respectively. Every promoted endpoint has only label-ℓ edges to originally uncovered vertices, as just proved. No new outward critical edge is therefore created. This conclusion holds for arbitrary $N \ge 2M$ and does not require the comparison matching to minimize primary weight.

Every original red pair has one original left endpoint. Among the μM left endpoints of packed pairs, at most $2\mu M/L$ are path endpoints. Thus at least $(1 - 2/L)\mu M$ original left vertices are demoted to label 1. Their blue pairs become noncritical, giving a loss of at least

$$\frac{1}{2}(1-2/L)^2\mu^2 M^2 - EM^2 - O_\ell(M).$$

Only the $2\mu M/L$ promoted endpoints can create new critical pairs. There are at most $4\mu M^2/L$ such pairs with original covered vertices, and none with original uncovered vertices. Hence

$$C_{\text{path}} - C_0 \le E + \frac{4\mu}{L} - \frac{1}{2}(1-2/L)^2\mu^2 + o_\ell(1).$$

The E allowance is already present in D, so

$$C_{\text{path}} \le 1 - \frac{1}{S} - h_b(\mu) + o_\ell(1), \qquad h_b(\mu) = \frac{1}{2}(1-2/L)^2\mu^2 - \frac{4\mu}{L}. \tag{60}$$

We may stop the packing at any smaller prescribed mass, with an $O_\ell(1/M)$ rounding error.

Theorem 14.1. *For every integer b with $k/2 \le b < k$ and every $0 < u < 1$,*

$$\gamma(\ell) \le \frac{1}{2} - \frac{1}{2S} - \frac{1}{2}\min\{d_b(u), h_b(u)\}.$$

In particular one may maximize the last minimum over u and over the allowed integers b, and combine this with any other valid bound.

Proof. If $D \ge d_b(u) - o_\ell(1)$, use the original matching. Otherwise, (59) and the strict decrease of d_b on $[0, 1)$ force the maximal path packing to cover at least mass $u - o_\ell(1)$. Stop at that mass and use (60). Dividing the count bound by 2 proves the assertion. Strict inequalities can first be used with an arbitrarily small fixed margin, followed by passage to the limit, so boundary cases are included. □

14.3 Exact optimization and asymptotics

Set $b = k/2$, $L = k/2 + 1$, $T = 2S - k$, and

$$A = \frac{1}{2}(1 - 2/L)^2 - \frac{2}{T}, \quad B = \frac{4}{T} - \frac{4}{L}, \quad d_0 = \frac{k}{ST}.$$

Whenever $A > 0$, the positive solution of $h_b(u) = d_b(u)$ is

$$u_* = \frac{-B + \sqrt{B^2 + 4Ad_0}}{2A}, \qquad \delta_k = \frac{k/S - 4u_* + 2u_*^2}{T}. \tag{61}$$

If $0 < u_* < 1$ and $\delta_k > 0$, this is the optimum of the one-variable comparison: d_b is decreasing, while h_b is increasing on its positive branch. Therefore

$$\gamma(\ell) \le \frac{1}{2} - \frac{1}{2S} - \frac{\delta_k}{2}. \tag{62}$$

The hypotheses hold for every $\ell \ge 10$. Indeed $k \ge 256$ and $L \ge 129$, so the positive zero $8L/(L-2)^2$ of h_b is less than $1/12$, whereas $d_b(1/12) > 0$ follows from $k/S > 1/3$. Also $A > 0$, since $A \ge \frac{1}{2}(127/129)^2 - 1/384 > 0$. Therefore the decreasing curve d_b meets the positive increasing branch of h_b before the positive zero of d_b.

This choice of path length is optimal within the whole family of Theorem 14.1 when $\ell \ge 10$, not merely within the μ optimization. To prove this, temporarily allow real $b \in [k/2, k)$ and put $T_b = S - k + b$, $L = b + 1$, and $a = 1 - 2/L$. At any positive intersection $h_b(u) = d_b(u) > 0$, we have $h_u = a^2 u - 4/L > a^2 u/2$ and $d_u = -2(1-u)/T_b < 0$. The intersection value is strictly decreasing with b if

$$(1 - u)h_u > (4au^2 + 8u)T_b/L^2.$$

This follows by implicit differentiation, using $\partial_b d_b = -(1-u)^2/T_b^2$ and $\partial_b h_b = (2au^2 + 4u)/L^2$. For $\ell \ge 10$, $S \ge 14k/5$; positivity of $d_b(u)$ then gives $(1-u)^2 > 23/28 > 81/100$, so $u < 1/10$. Since $a \ge 127/129$ and $T_b/L^2 \le 12/k$, the left side is greater than

$$\frac{9}{20}(127/129)^2 u,$$

while the right side is at most $(42/5)(12/256)u = (63/160)u$. The former constant is strictly larger. This proves monotonicity on every positive branch. A nonpositive comparison cannot improve a positive one, so the best allowed integer path length is $b = k/2$. The rational comparisons and root brackets are also verified by `general-certificate/verify_general.py`.

As $\ell \to \infty$, we have $S = 3k + O(\log k)$, $A = 1/2 + O(k^{-1})$, $B = O(k^{-1})$, and $d_0 = 1/(15k) + O((\log k)/k^2)$. Thus

$$u_* = \sqrt{\frac{2}{15k}} + O(k^{-1}), \qquad \delta_k = \frac{k}{S(2S-k)} + O(k^{-3/2}).$$

Combining the two principal terms in (62) gives

$$\boxed{\gamma(\ell) \le \frac{1}{2} - \frac{1}{2S-k} + O(k^{-3/2}) = \frac{1}{2} - \frac{1}{5k} + O(k^{-3/2}).} \tag{63}$$

The leading gap in the upper bound of [1] is $1/(6k)$, whereas the leading gap in (63) is $1/(5k)$, larger by a factor $6/5$. The treatment of uncovered vertices follows from the label restriction proved for path endpoints, and is therefore uniform in the number of such vertices.

15 An explicit upper bound for every $\ell \geq 5$

We continue with the notation and deficit decomposition of Section 14, and put $y = x_1$. Rather than selecting paths one at a time, we now choose equally many red pairs from several layers and rematch their endpoints simultaneously. Choosing fewer than all $p = k/2$ layers permits larger errors in the layer sizes and in the number of missing blue edges. The resulting criterion yields an explicit improvement for each $\ell \geq 5$, including the values for which the asymptotic path estimate is weak.

15.1 Selecting layers with few missing edges

Fix an integer j with $k/4 < j \leq p$, and put $n = p - j + 1$. The missing template degree of a vertex is the number of its permitted blue neighbors in the template that are not actual blue neighbors. Delete an original red pair if one of its endpoints has missing template degree greater than hM, where $h = \theta/S$ and $\theta > 0$. At most $2EM/h + o(M)$ pairs are deleted. Among the surviving layers, the jth largest mass is at least

$$\frac{2x}{k} - \sqrt{\frac{2(j-1)}{pn}R} - \frac{2E}{nh} - o_\ell(1). \tag{64}$$

Indeed for any n original layers, the sum of their masses is at least

$$\frac{nx}{p} - \sqrt{2R\,\frac{n(p-n)}{p}}.$$

This is Cauchy–Schwarz applied to the centered indicator of those layers and the vector $(a_i - x/p)_i$. Apply it to the n smallest surviving layers and subtract the total deletion allowance.

The mass constraint improves the x part of Q to

$$|2x/k - 2/S| \leq 2\sqrt{\frac{S-k}{kS}Q}.$$

Consequently, with

$$\chi = \frac{4(S-k)}{kS} + \frac{2(j-1)}{pn} = \frac{2}{n} - \frac{4}{S}, \qquad B_c = \frac{2S}{n\theta},$$

the jth largest surviving mass is at least

$$\frac{2}{S} - \sqrt{\chi(Q+R)} - B_c E - o_\ell(1).$$

For $D \leq \delta$, the joint budget $Q + R + E \leq D + o(1)$ and a completed square give the uniform lower bound

$$r_0 = \frac{2}{S} - \frac{\chi}{4B_c} - B_c\delta. \tag{65}$$

Explicitly, $\sqrt{\chi(\delta - E)} + B_c E \leq B_c\delta + \chi/(4B_c)$. Choose the same even number rM of surviving red pairs from each of these j layers, with $r = r_0 - o(1)$. If

$$r_0 > 2\theta/S, \tag{66}$$

every selected endpoint has fewer than $rM/2 - O(1)$ missing template neighbors. The selected layers inherit their original order.

15.2 Rematching the selected vertices

Number the selected layers $1, \dots, j$ in their inherited order. They support an alternating path with layer sequence

$$1, 2, \dots, j, j, j-1, \dots, t$$

and $2j - t + 1$ red edges, with any prescribed distinct endpoints in R_1 and R_t. To construct it, traverse the red pairs from layer 1 up to layer j, use a left–left blue edge between two pairs of layer j, and return through the layers down to t. At the join between the two traversals, an intermediate pair must satisfy at most two prescribed blue adjacencies; all other choices require at most one. The missing-degree bound excludes at most $2hM + O_\ell(1) < rM$ choices, including pairs already used. Thus all vertices can be chosen distinctly. Every selected R_1–R_t pair must consequently have label ℓ when $2j - t + 1 > k/2$. In particular, R_1 is a label-ℓ clique, and its vertices have exact outward label ℓ: use a path with $2j$ red edges and replace its closing edge by an edge to an originally uncovered vertex. A smaller label would decrease original primary weight from $2j > k/2$ to at most $k/2$.

Make blue perfect matchings between L_i and R_{i+1}, a blue perfect matching inside L_j, and any perfect matching inside R_1. The bipartite graphs in the first step have minimum degree greater than half the part size, so Hall's theorem gives perfect matchings. The graph on L_j also has minimum degree at least half its even order and hence has a perfect matching: otherwise two uncovered vertices of a maximum matching would admit an augmenting path of three edges. Only R_1 is promoted to matching label ℓ; all other selected vertices are demoted to matching label 1. The label restriction to originally uncovered vertices proves that no outward critical edge is created.

We also have a conditional cross half bound. Let uv be an original matching edge outside the selected layers, of label $t < \ell$. If

$$2j + 2^{t-2} > k \qquad (t \geq 2),$$

two complementary critical contacts from a new R_1R_1 pair to uv are impossible. Replace the $2j$ red edges of an alternating path joining the R_1 endpoints and the old edge uv by the path's blue edges and these two contacts. The original primary cost is strictly greater than k; the new cost is at most k. Thus the new critical density to X_t is at most rx_t under this condition and at most $2rx_t$ otherwise. This argument uses the original matching; it does not assume primary minimality of the new comparison matching.

Set

$$\eta = \begin{cases} 1, & j = p, \\ 2, & j < p, \end{cases} \qquad \tau = 2j - p,$$

and for $3 \leq t < \ell$ set $\alpha_t = 1$ if $2^{t-2} > k - 2j$, and $\alpha_t = 2$ otherwise. The X_1 comparison is sharper than a trivial bound: original cross half bounds show that the selected pairs had at least $2jry - e_{12}$ critical contacts to X_1, while the new matching has at most $2ry$.

The selected template originally has $(j^2 - j/2)r^2 + O_\ell(1/M)$ blue density, with at most E missing. The critical count among selected vertices is at most $(2j - \tau)r^2$: a critical edge must touch R_1, and all its edges to $R_1, \dots, R_\tau$ have label ℓ. There are at most jr^2 possible critical pairs to the left sides and $(j - \tau)r^2$ to the remaining right sides. Contacts to omitted X_2 cost at most $\eta r(x - jr)$; contacts to X_ℓ cannot change. Summing these statements gives

$$C_j - C_0 \leq E + e_{12} + r\Lambda - Ar^2 + o_\ell(1), \tag{67}$$

where

$$A = j^2 + (\eta - 5/2)j + \tau, \qquad \Lambda = \eta x - 2(j-1)y + \sum_{t=3}^{\ell-1} \alpha_t x_t.$$

Write λ_t for these coefficients of x_t, including $\lambda_\ell = 0$, and put

$$\Lambda_0 = \frac{1}{S}\sum_t c_t\lambda_t, \qquad V = \sum_t c_t\lambda_t^2 - \frac{(\sum_t c_t\lambda_t)^2}{S}.$$

Weighted Cauchy–Schwarz on the mass hyperplane gives $\Lambda \le \Lambda_0 + \sqrt{VQ} \le \Lambda_0 + \sqrt{V\delta}$. Because $E + e_{12} \le D + o(1)$, these two terms are already included in the deficit of the initial matching.

Theorem 15.1. *Let $\ell \ge 5$. Choose j, θ, δ as above and define r_0 by (65). If*

$$r_0 > 2\theta/S, \qquad r_0\big(Ar_0 - \Lambda_0 - \sqrt{V\delta}\big) > \delta,$$

then

$$\gamma(\ell) \le \frac{1}{2} - \frac{1}{2S} - \frac{\delta}{2}.$$

Nonstrict limiting choices are valid by approximation whenever the inequalities hold for a sequence of positive smaller δ.

Proof. If $D \ge \delta - o(1)$, the original matching suffices. Otherwise construct the shortened layered matching. Equations (57) and (67) give $C_j \le 1 - 1/S + r\Lambda - Ar^2 + o(1)$. The hypotheses and $r = r_0 - o(1)$ give the claim. All comparisons concern actual finite matchings and the errors are uniform in N. □

15.3 An explicit bound for every finite number of labels

Corollary 15.2. *For every $\ell \ge 5$,*

$$\boxed{\gamma(\ell) \le \frac{1}{2} - \frac{1}{3\cdot 2^{\ell-1} - 4\ell + 8} - \frac{1}{162\cdot 2^{\ell}}.}$$

Proof. Take $j = 3k/8 + 1$ and $\delta = 1/(324k)$. For a convenient simple uniform calculation, replace the coefficient $4(S-k)/(kS)$ in the layer selection estimate by $4/k$; the total square-root coefficient is then $\chi = 16/k$. Use

$$\theta = \begin{cases} 1/2, & k = 8, \\ 2/3, & k = 16, \\ 4/9, & k \ge 32. \end{cases}$$

The resulting guaranteed layer masses are

$$r_0 = 11/144 \quad (k=8), \qquad r_0 = 37/1080 \quad (k=16), \qquad r_0 = \frac{17k^2 - S^2}{9Sk^2} \quad (k \ge 32).$$

They strictly exceed $2\theta/S$. In the last case their difference is $(9k^2 - S^2)/(9Sk^2) > 0$, since $S < 3k$.

For $k = 8$, use $A = 14$, $\Lambda_0 = 5/9$, and the conservative variance upper bound $V = 52$. For $k \ge 16$, suitable upper bounds are

$$A = \frac{9k^2}{64} + \frac{13k}{16} + \frac{5}{2}, \qquad \Lambda_0 = 2 - \frac{3k/4 + 12}{S}, \qquad V = \frac{9k^2}{16} + 4S - 32.$$

Here the variance bound drops the nonnegative subtracted term in the formula for V. To verify the three smallest cases directly, put $Z = Ar_0 - \Lambda_0 - \delta/r_0$. The following rational bounds give $Z > 0$ and $Z^2 > V\delta$, which are equivalent to the second inequality in Theorem 15.1:

k	r_0	lower bound for Z	upper bound for $V\delta$
8	11/144	1/2	1/40
16	37/1080	1/3	1/16
32	2503/198144	1/2	1/9

All inequalities in the last two columns are strict. They can be checked by substitution and are also verified in `general-certificate/verify_general.py`.

For $k \geq 64$, use $r_0 \geq 8/(27k)$, $\Lambda_0 \leq 2$, and

$$\sqrt{V\delta} \leq \sqrt{k/576 + 1/27} \leq k/144.$$

The last inequality is equivalent to $k^2 - 36k - 768 \geq 0$, valid for $k \geq 64$. Consequently

$$Ar_0 - \Lambda_0 - \sqrt{V\delta} \geq \frac{5k}{144} - \frac{95}{54} \geq \frac{25}{54}.$$

Since $2\delta/r_0 \leq 1/48 < 25/54$, the count criterion holds even with 2δ on its right side. This proves the corollary. □

For small fixed ℓ the full criterion is stronger than this simple corollary. Rational choices of (j, θ, δ) obtained by optimizing the criterion are saved in `general-certificate/cleanup-certificates.json`; their verification requires no numerical solver and no floating-point computation.

16 A weighted comparison of the path matchings

The path comparison in Section 14 bounds the initial matching and the matching obtained from disjoint paths separately. Both bounds depend on the same deviations Q, R of the matching-class and layer masses. Keeping those common terms while taking a weighted average gives a stronger estimate. Use the notation of that section, including a maximal path packing of mass μ, and define

$$P(x,\mu) = \frac{(x-\mu)^2}{b} - \frac{x^2}{k}, \qquad h(\mu) = c\mu^2 - d\mu, \quad c = \frac{1}{2}\left(1 - \frac{2}{b+1}\right)^2, \quad d = \frac{4}{b+1}.$$

The two actual matchings satisfy

$$C_0 \leq 1 - 1/S - Q - R - E + o(1), \qquad C_{\text{path}} \leq 1 - 1/S - Q - R - h(\mu) + o(1).$$

Moreover $R + E \geq P(x,\mu) - o(1)$ by (58). For any $0 \leq \lambda \leq 1$, give weight λ to the original matching and $1-\lambda$ to the path matching. Since $R \geq 0$,

$$Q + R + \lambda E + (1-\lambda)h \geq Q + \lambda P + (1-\lambda)h - o(1).$$

In particular, the terms Q and R are used only once in this comparison.

Put

$$q_0 = \frac{S}{k(S-k)}, \qquad x_0 = \frac{k}{S}, \qquad K_0 = \frac{k}{S(S-k)},$$

and define the symmetric matrix and vector

$$M_\lambda = \begin{pmatrix} q_0 + \lambda(1/b - 1/k) & -\lambda/b \\ -\lambda/b & \lambda/b + (1-\lambda)c \end{pmatrix}, \qquad v_\lambda = \begin{pmatrix} 1/(S-k) \\ (1-\lambda)d/2 \end{pmatrix}. \tag{68}$$

Then the loss is at least

$$q_0(x - x_0)^2 + \lambda P(x,\mu) + (1-\lambda)h(\mu) = \begin{pmatrix} x & \mu \end{pmatrix} M_\lambda \begin{pmatrix} x \\ \mu \end{pmatrix} - 2v_\lambda^T \begin{pmatrix} x \\ \mu \end{pmatrix} + K_0.$$

Both endpoint matrices are positive definite: $M_0 = \operatorname{diag}(q_0, c)$, and $\det M_1 = 1/[b(S-k)] > 0$. Thus $M_\lambda \succ 0$ throughout $[0,1]$. Completing the square, without imposing additional restrictions on x or μ, gives the following theorem.

Theorem 16.1. *For every integer $k/2 \le b < k$ and every $\lambda \in [0,1]$, set*

$$\Delta_b(\lambda) = K_0 - v_\lambda^T M_\lambda^{-1} v_\lambda.$$

Then

$$\boxed{\gamma(\ell) \le \frac{1}{2} - \frac{1}{2S} - \frac{1}{2}\Delta_b(\lambda).}$$

All entries are rational when λ is rational. The bound is uniform in $N \ge 2M$ and uses the original matching order.

For fixed ℓ and b, the best weight in this theorem can be found by solving a polynomial of degree at most four. Write

$$D(\lambda) = \det M_\lambda, \qquad N(\lambda) = (M_\lambda)_{22}(v_\lambda)_1^2 - 2(M_\lambda)_{12}(v_\lambda)_1(v_\lambda)_2 + (M_\lambda)_{11}(v_\lambda)_2^2.$$

Here D has degree at most two and N degree at most three, and

$$\Delta_b(\lambda) = K_0 - \frac{N(\lambda)}{D(\lambda)}.$$

Its stationary equation is the explicit quartic

$$N(\lambda)D'(\lambda) - N'(\lambda)D(\lambda) = 0. \tag{69}$$

The function Δ_b is concave: before minimizing in (x, μ), the displayed loss is affine in λ, and an infimum of affine functions is concave. Thus an interior stationary point is a global maximum. If the derivative has opposite signs at the endpoints, the maximum is the unique stationary point; uniqueness follows also from analyticity, since a nonconstant analytic concave function cannot have an interval of maxima. Otherwise the appropriate endpoint is used. This describes the optimum for each fixed b exactly.

The accompanying checker sets $b = k/2$, reconstructs the quartic with rational coefficients, isolates its maximizing root by exact rational bisection, and verifies rational weights arbitrarily close to it. The finite bounds in the introduction use these rational weights when they improve the estimate from Theorem 15.1. The optimization concerns only the weight of these two matching comparisons for the fixed path length b.

The weighted comparison also determines the next term in an asymptotic upper bound. Substitute $b = k/2$ and $\lambda = 1 - \sqrt{8/k}$ into the rational formula. Expansion of its two-by-two inverse gives

$$\Delta_{k/2}(\lambda) = \frac{1}{15k} - \frac{4\sqrt{2}}{25k^{3/2}} + O\left(\frac{\log k}{k^2}\right).$$

Thus

$$\gamma(\ell) \le \frac{1}{2} - \frac{1}{5k} + \frac{2\sqrt{2}}{25k^{3/2}} + O\left(\frac{\log k}{k^2}\right).$$

For example, this expansion can be checked after writing $\varepsilon = 1 - \lambda$: the first two losses below $1/(15k)$ are $\varepsilon/(25k)$ and $8/(25\varepsilon k^2)$, whose sum is minimized at $\varepsilon = \sqrt{8/k}$. This expansion includes its correction terms; it does not assert the finite inequality $\gamma(\ell) \le 1/2 - 1/(5k)$.

17 An algebraic six-block lower construction

For a symmetric matrix $L \in \{1,2,3,4\}^{s\times s}$ and a positive mass vector $a \in \mathbb{R}^s$ with $\sum_i a_i = 1$, partition the vertex set into blocks of asymptotic proportions a_i and give every pair between blocks i, j label L_{ij}. Diagonal entries specify labels inside a block. These are finite-part labelings in the graph-limit formulation of [1]; see [9] for general background on graph limits. In this section all vertices are matched, so our construction gives a lower bound on $\gamma'(4)$ and hence on $\gamma(4)$.

17.1 Reducing the matching problem to a quadratic optimization

For $i < j$, let x_{ij} be the number of matching edges between the blocks divided by the total number N of vertices. Let x_{ii} be the proportion of vertices in block i matched internally. Thus

$$x_{ii} + \sum_{j \neq i} x_{ij} = a_i, \qquad x_{ij} \geq 0. \tag{70}$$

The convention on x_{ii} is essential: an internal matching edge accounts for two vertices. Write $\mathcal{P}(a)$ for this polytope. For an edge type $e = ij$, let $V(e) = \{i, j\}$, with a singleton when $i = j$, and let $L_e = L_{ij}$. Define the symmetric matrix

$$H_{ef} = \sum_{u \in V(e)} \sum_{v \in V(f)} \mathbf{1}\{L_e \leq L_{uv},\ L_f \leq L_{uv}\}.$$

The asymptotic proportion of noncritical pairs is

$$Q(x) = x^T H x. \tag{71}$$

Indeed an endpoint born on an edge of label b is noncritical to an endpoint born at c precisely when its ambient edge label is at least both b and c. Counting ordered pairs gives the displayed formula; diagonals and matching edges contribute only $O(N)$ finite error. Consequently the construction's minimum critical density is $1 - \max_{x \in \mathcal{P}(a)} Q(x)$.

A useful reduction removes every cross type ij with $L_{ij} \geq \max(L_{ii}, L_{jj})$: replace its mass by the same amount of internal vertex mass in each endpoint block. This preserves (70) and weakly decreases the birth label at every affected endpoint. It therefore cannot decrease Q. It is enough to maximize over the reduced flow polytope.

The maximum of a quadratic need not occur at a polytope vertex. We will use a sufficient condition that controls the whole polytope. If $v_1, \ldots, v_m$ are all reduced flow vertices and

$$v_i^T H v_j \leq T \quad (1 \leq i, j \leq m), \tag{72}$$

then every convex combination $x = \sum_i \theta_i v_i$ satisfies $Q(x) = \sum_{i,j} \theta_i \theta_j v_i^T H v_j \leq T$. Thus it is enough to check the mixed bilinear values as well as the values at the vertices. We refer to (72) as the bilinear certificate.

17.2 The value and masses

Let τ be the unique root in $(0.359275945611134729, 0.359275945611134730)$ of

$$P(t) = 34006t^6 - 148176t^5 + 211627t^4 - 226450t^3 + 422009t^2 - 423516t + 105476.$$

Set $D(t) = 694t^3 - 1512t^2 + 469t + 285$ and

$$\alpha_L = \frac{-49\tau^4 + 3367\tau^3 - 7130\tau^2 + 2072\tau + 1484}{12D(\tau)}.$$

Then

$$\alpha_L = 0.419772820486240077721086607635089744919175\ldots.$$

Use blocks A, B, C, D, E, F with label matrix

$$L = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 4 & 4 & 1 \\ 1 & 3 & 1 & 2 & 2 & 1 \\ 1 & 4 & 2 & 3 & 3 & 1 \\ 1 & 4 & 2 & 3 & 4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and masses $a^* = n(\tau)/(6D(\tau))$, where

$$n(t) = \begin{pmatrix} -77t^3 + 312t^2 - 389t + 226 \\ 1469t^3 - 2981t^2 + 675t + 637 \\ (t-1)(1337t^2 - 1993t - 452) \\ 222t^3 - 440t^2 + 154t + 32 \\ 835t^3 - 1995t^2 + 886t + 250 \\ 378t^3 - 638t^2 - 53t + 113 \end{pmatrix}.$$

The masses are approximately

$$(.0705314504, .3229020494, .3659074690, .0234231357, .2005210862, .0167148093).$$

Theorem 17.1. *The minimum asymptotic critical density of this construction is exactly α_L. In particular $\gamma(4) \geq \gamma'(4) \geq \alpha_L$.*

Proof. We first exhibit three matching flows that attain the proposed value. Write the six masses as (p, b, z, d, e, f). The flows, numbered 1, 2, and 3, have the following coordinates; all coordinates omitted from the table are zero:

	1	2	3
AB	p	0	p
AE	0	p	0
BC	$b-p-f$	0	$z-d-e$
BF	f	f	f
CD	0	d	d
CE	$p-b+z+f$	$e-p$	e
DE	d	0	0
BB	0	$b-f$	$b-p-z+d+e-f$
CC	0	$p+z-d-e$	0
EE	$b-p-z-d+e-f$	0	0

The six nonzero coordinates in each flow are strictly positive at a^*. Let C_1, C_2, C_3 be their noncritical quadratic matrices, obtained by substituting this table in (71). We call these flows *active* because they attain the maximum noncritical density at the chosen masses. Put $K(t) = 111t^3 - 220t^2 + 77t + 16$ and

$$\lambda_1 = \frac{(\tau-1)(7\tau^3 - 6\tau^2 - 32\tau - 14)}{2K(\tau)}, \quad \lambda_2 = \tau, \quad \lambda_3 = 1 - \lambda_1 - \lambda_2.$$

These weights are positive. Exact arithmetic modulo P verifies

$$(a^*)^T C_k a^* = 1 - \alpha_L =: T \quad (k = 1, 2, 3), \qquad \left(\sum_k \lambda_k C_k\right) a^* = T\mathbf{1}.$$

Thus the active flows attain noncritical density T.

For the reverse inequality, the remaining cross types after the reduction above are

$$AB,\ AD,\ AE,\ BC,\ BF,\ CD,\ CE,\ DE,\ DF,\ EF,$$

together with the six internal types. Let B be their 6×16 endpoint-incidence matrix: $B_{i,e} = 1$ if type e has an endpoint in block i, and 0 otherwise, with a single 1 for an internal type. The reduced polytope is $Bx = a^*$, $x \geq 0$. It is bounded, and B has rank six. Its vertices can therefore be enumerated by choosing six columns J, solving $B_J x_J = a^*$ when B_J is nonsingular, and setting every other coordinate to zero.

Here are the exact checks used in that enumeration. Let $\bar{a}$ be the rational vector

$$(0.070531450396389,\ 0.322902049431553,\ 0.365907468971544,$$
$$0.023423135669300,\ 0.200521086237747,\ 0.016714809293466).$$

The exact algebraic vector a^* lies in the coordinate box $|a_i - \bar{a}_i| \le 10^{-13}$. Among the column choices, 2880 are nonsingular. Of these, exactly 240 have every coordinate of $B_J^{-1}a$ strictly positive throughout the box; each of the others has a coordinate strictly negative throughout the box. Thus this procedure gives every vertex, with the same feasible column choices throughout the box. Of the $240 \cdot 241/2 = 28920$ unordered bilinear pairs in (72), 28915 are strictly below T; the five remaining pairs are identically equal to one of the three active quadratic forms. The strict pairs have certified margin greater than 0.000548643283. These assertions are checked using rational interval arithmetic, not floating-point tolerances. It follows that all matching flows have noncritical density at most T.

The computational part of this proof consists of these finite rational sign and polynomial-identity checks. The accompanying `lower-certificate/verify_algebraic.py` checks normalization, positivity, the three active equalities, and the stationarity identity in $\mathbb{Q}[t]/(P)$, and isolates the specified root by Sturm sequences. It then invokes `verify_flow_bound.py`, which reconstructs B, H, every nonsingular column choice, and every bilinear pair from the displayed label matrix and flow maps. The interval box and the rational input data are included in `lower-certificate/stationarity-input.json`. No optimization routine is needed to verify the assertions.

Finally, approximate a^* by rational mass vectors and take finite blow-ups with even total order. Any sequence of their matching flows has a convergent subsequence, and its limit belongs to $\mathcal{P}(a^*)$. Continuity of Q therefore preserves the lower bound on critical density in the limit. Conversely, the displayed active flows can be approximated by finite matchings, with only $o(N)$ endpoints adjusted for integrality and parity. Their critical counts change by $o(N^2)$. This proves both the finite realization of the lower bound and the asserted asymptotic equality. □

The number α_L has degree six over $\mathbb{Q}$. Its primitive integer minimal polynomial is

$$\begin{aligned}&57541351029485327769 60X^6 - 13507216477071755171328X^5\\ &\quad + 13208173323262094551248X^4 - 6886878440990980289832X^3\\ &\quad + 2019436570431176007853X^2 - 315750351353916573206X + 20566231563858477973.\end{aligned}$$

Both this polynomial and P are irreducible modulo 41. The designated roots of P and the displayed minimal polynomial lie in the isolating interval stated for τ and, respectively,

$$(0.41977282048624007772, 0.41977282048624007773).$$

The polynomial relation and irreducibility checks are included in the symbolic certificate.

17.3 Deriving the algebraic parameters

The degree-six equation follows by eliminating the stationarity equations of the three active quadratic forms. Write $q = d + e$. Among the three active noncritical forms only the first depends on the split between d and e with q fixed, and its stationary split is

$$d = (b + q - p - z - f)/4.$$

After substituting this expression, let $\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3$ be the quadratic matrices in (p, b, z, q, f). For weights $u, v, 1-u-v$, put $H(u,v) = u\widetilde{C}_1 + v\widetilde{C}_2 + (1-u-v)\widetilde{C}_3$. Near the certified solution this matrix is positive definite, and minimizing its quadratic on the mass hyperplane gives

$$T(u,v) = \frac{1}{\mathbf{1}^T H(u,v)^{-1}\mathbf{1}} = \frac{N(u,v)}{4S(u,v)},$$

where

$$\begin{aligned} N &= u^2(-20v^3 + 420v^2 - 224v - 128) \\ &\quad + u(7v^4 + 559v^3 - 842v^2 + 164v + 112) \\ &\quad + 4v^4 + 156v^3 - 248v^2 + 52v + 36, \\ S &= u^2(156v^2 - 84v - 56) + u(234v^3 - 344v^2 + 61v + 49) \\ &\quad + v^4 + 65v^3 - 102v^2 + 20v + 16. \end{aligned}$$

At the stationary solution $T = 1 - \alpha_L$ and $T_u = T_v = 0$. The polynomial $F = N - 4(1 - \alpha_L)S$ is quadratic in u, and its discriminant factors as

$$\begin{aligned} \operatorname{disc}_u F = (v-1)&(104\alpha_L v^2 - 56\alpha_L v - 28\alpha_L + v^3 - 38v^2 + 20v + 12) \\ &\times \big(8328\alpha_L v^3 - 18144\alpha_L v^2 + 5628\alpha_L v + 3420\alpha_L \\ &\qquad + 49v^4 - 3367v^3 + 7130v^2 - 2072v - 1484\big). \end{aligned}$$

The first two factors do not vanish in the isolating box. The last factor and its v derivative vanish. Solving the last factor for α_L yields its stated rational formula with $v = \tau$, and

$$\frac{d}{dt}\left(\frac{-49t^4 + 3367t^3 - 7130t^2 + 2072t + 1484}{12D(t)}\right) = -\frac{P(t)}{12D(t)^2}.$$

The double-root formula for u gives λ_1, and the linear stationarity equations give the cubic numerators of the masses. Eliminating t gives the degree-six polynomial for α_L. The checker `lower-certificate/verify_elimination_independent.py` reconstructs $N/(4S)$ from determinants of the active quadratic forms and checks the entire displayed discriminant identity with exact polynomial arithmetic.

18 Exact verification and accompanying files

The computer-assisted claims are finite algebraic inequalities. Their inputs are the matching comparisons and constructions proved in the preceding sections. The verification programs reconstruct the relevant quadratic forms and check the signs, identities, and parameter ranges with integer and rational arithmetic. Numerical optimization is not needed to run them. This use of exact certificates after numerical exploration is related to the rational certification methods discussed by Peyrl and Parrilo [10].

18.1 Contents and commands

The source archive contains `main.tex`, the included section files, and an `anc/` directory containing all verification programs and data. The bibliography is in `references.tex`; there is no external bibliography database or downloaded data set. The root file `main.tex` compiles with PDFLaTeX. For a local compilation, run it until the log no longer requests another pass:

```
pdflatex -interaction=nonstopmode -halt-on-error main.tex
pdflatex -interaction=nonstopmode -halt-on-error main.tex
pdflatex -interaction=nonstopmode -halt-on-error main.tex
```

The source uses the standard packages `geometry`, `amsmath`, `amssymb`, `amsthm`, `mathtools`, `booktabs`, `microtype`, `xurl`, and `hyperref`.

For verification, use Python 3.9 or later with its standard library. No additional Python packages, numerical solvers, network access, or external research directories are required. From the root of the extracted source archive, run

```
python3 anc/check_package.py
python3 anc/verify_all.py
```

The first command checks the file inventory and SHA-256 hashes. The second runs every mathematical certificate. Successful completion has exit status zero and ends with

```
PASS: all exact certificates.
```

Run Python normally: its `-O` option disables the assertions used by the checkers and must not be used. The driver rejects such an invocation. To run an individual checker, use the paths listed below relative to `anc/`. The working directory does not matter because paths to data are resolved relative to each script.

The programs print progress as each group of certificates passes. They may take several minutes, particularly for the matching-flow and stability calculations. Three deterministic summaries are regenerated: `general-certificate/verified-general-table.json`, `general-certificate/path-budget-certificates.json`, and `lower-certificate/neutral-pair-exact-results.json`. These are included in the archive so that their contents can also be inspected without running Python.

18.2 What each computation verifies

1. `verify_global.py` proves the algebraic part of the four-label upper bound. It reconstructs the quadratic comparisons from Sections 4–13, checks the signs of their rational weights, verifies the matrix factorizations, and checks that the parameter subregions cover the entire required domain. The JSON files beside it contain the weights and subdivisions. The helper modules in the same directory are included in full.

2. `general-certificate/verify_general.py` checks the explicit general-label upper bounds, the rational choices giving the numerical table in the introduction, and the finite quadratic formula in Theorem 16.1. It also brackets the optimizing weight for the fixed path-length choice. The alternating-path counting arguments and the asymptotic expansion are given in the text.

3. `lower-certificate/verify_algebraic.py` checks the sextic polynomial, root isolation, mass normalization and positivity, and the identities for the three matching flows in Section 17. It then invokes `lower-certificate/verify_flow_bound.py`, which enumerates the feasible vertices of the reduced matching-flow polytope and checks all their bilinear pairings. The latter step bounds every feasible flow, not only the vertices. The separate script `lower-certificate/verify_elimination_independent.py` checks the elimination that produces the algebraic value.

4. `lower-certificate/verify_atom_stability.py`, `lower-certificate/check_neutral_pairs_exact.py`, and `lower-certificate/verify_stationarity.py` check the finite profile and matrix calculations used in Appendix A. The appendix states precisely which perturbations are covered.

The general lower bound of Theorem 2.1 and its weight-optimality statement have complete elementary proofs in Section 2; neither depends on a computation.

The verification programs check the finite algebra arising from the proofs. The mathematical justification that each inequality applies to actual matchings is provided in the preceding sections. Classical matching theorems and the alternating-path bound of [1] are stated where used and cited to the literature. In particular, computer verification is not being used to infer a universal combinatorial assertion from a finite sample of graphs.

A Local optimality under the addition of one or two blocks

We next ask whether the six-part construction can be improved by adjoining parts of small positive mass. Throughout this section the labels between its six existing parts remain fixed. The result is local: neither a change to those labels nor a configuration far from the specified mass vector is covered by the theorem.

Theorem A.1. *There is a neighborhood of $(a^*, 0, 0)$ such that adjoining one or two new blocks to the fixed matrix L, with arbitrary labels in $\{1, 2, 3, 4\}$, and choosing masses in that neighborhood cannot produce a minimum critical density greater than α_L. The six old masses may vary simultaneously, and all masses sum to one. Zero masses for one or both new blocks are allowed.*

We use the notation $T = 1 - \alpha_L$ and C_k, λ_k from Section 17. To rule out a larger minimum critical density, it suffices to exhibit a matching with noncritical density at least T. A weighted average of the noncritical densities of several feasible matchings is a lower bound for their maximum, so the same conclusion follows from such an average being at least T.

Write the three active flow maps from the table in the proof of Theorem 17.1 as $x = P_k a$, and put $H_0 = \sum_k \lambda_k C_k$. An interval LDL^T calculation verifies $H_0 \succ 0$. Together with $H_0 a^* = T\mathbf{1}$, this gives

$$a^T H_0 a \geq T \quad \text{whenever } \mathbf{1}^T a = 1. \tag{73}$$

Indeed, writing $a = a^* + h$ with $\mathbf{1}^T h = 0$ gives $a^T H_0 a = T + h^T H_0 h$. The flows defining this inequality remain feasible whenever their six strictly positive basic coordinates remain positive; this holds in an open neighborhood of a^*.

A.1 One added block

The *label profile* of a new block G is the tuple consisting of its six labels to A, B, C, D, E, F and its internal label L_{GG}. For a block G of mass s, an extension of flow k chooses a partner $j \in \{A, B, C, D, E, F, G\}$. If $j \neq G$, match the new mass to j and apply the old flow map to $a - se_j$. If $j = G$, match G internally and apply the old map to a. These are feasible for all sufficiently small s and old masses near a^*. Let $c_{k,j}$ be the derivative of their noncritical count with respect to s, holding the old masses fixed at a^*. Since the old gradients satisfy

$$\sum_k \lambda_k \nabla (a^T C_k a)|_{a=a^*} = 2T\mathbf{1},$$

a selected set of partners gives a strictly positive first-order increase in the weighted noncritical density, after accounting for the mass removed from the six existing blocks, whenever

$$\sum_k \lambda_k c_{k,j_k} - 2T > 0. \tag{74}$$

The subtraction of $2T$ accounts for the constraint that the total mass remains one. The identity for the old gradients shows that arbitrary simultaneous first-order variations of the six existing masses have the same contribution.

It suffices to consider *capped* profiles, meaning that $L_{Gi} \leq \max(L_{GG}, L_{ii})$. Indeed lowering an uncapped contact to that maximum weakly improves the construction: the affected cross-matching type is dominated by its two internal types both before and after the change, while the ambient edge label decreases. There are exactly

$$48 + 384 + 1296 + 4096 = 5824$$

capped profiles. The checker supplies explicit partners for each. For 5801 profiles, the left side of (74) is strictly greater than $19/1000$. The smallest certified lower endpoint exceeds

0.0190748046715198. For the remaining 23 profiles, the weighted first-order expression vanishes; call these profiles *neutral*. They fall into the following classes. The first column specifies a part with which the new mass will be combined in the quadratic identity below; profile entries are ordered A, B, C, D, E, F.

old role	$(L_{GA}, L_{GB}, L_{GC}, L_{GD}, L_{GE}, L_{GF})$	L_{GG}	number
A	$(1, 2, 1, 1, 1, 1)$	$1, 2, 3, 4$	4
B	$(2, 4, 3, 4, 4, 1)$	4	1
C	$(1, 3, 1, 2, 2, 1)$	$1, 2, 3, 4$	4
D	$(1, 4, 2, 3, 3, 1)$	$3, 4$	2
E	$(1, 4, 2, 3, 4, 1)$	4	1
F	$(1, 1, 1, 1, 1, k)$	$\delta, 1 \le k \le \delta \le 4$	10
D	$(1, 4, 2, 4, 3, 1)$	4	1

For each neutral profile there are explicit partners such that its extended quadratic in strategy k is exactly

$$Q_k^{\mathrm{ext}}(a, s) = Q_k(a + se_{\mathrm{role}}) + \kappa_k s^2, \qquad \kappa_k \ge 0. \tag{75}$$

These are identities of rational quadratic matrices. Thus every neutral one-block extension is excluded exactly by (73), not merely to first order. For a nonneutral profile, positive definiteness on the old mass coordinates and its positive linear s term imply $\sum_k \lambda_k Q_k^{\mathrm{ext}}(a, s) \ge T$ in a sufficiently small neighborhood. Cross terms are absorbed by the positive old quadratic and the linear s margin. This proves the one-block assertion.

A.2 Two added blocks

For two neutral profiles of masses s, t, extend the same three flows by their designated partners independently. Their mixed quadratic is

$$Q_0(a + se_i + te_j) + \kappa s^2 + \kappa' t^2 + 2Kst, \qquad Q_0(x) = x^T H_0 x, \quad \kappa, \kappa' \ge 0.$$

There are 979 unordered neutral-profile/contact-label cases after capping the new mutual contact. Exact checks find 453 cases with $K = 0$, 469 with $K > 0$, and 57 with $K < 0$. The first two classes are excluded by (73). In every negative case, at least one active flow admits a strictly positive first-order increase in noncritical density: match equal portions of the two new blocks to each other instead of their designated old partners. Its weighted first-order gain is greater than $19/500$ per unit of this matched mass, verified by rational intervals. At old masses near a^*, this remains a positive margin. Taking a portion of size $\min(s, t)$ gains a positive constant times $\min(s, t)$, while $|2Kst| \le 2|K|(s + t)\min(s, t)$ and all quadratic changes of the response have the same smaller order. This excludes an improvement for sufficiently small $s + t$.

If at least one new profile is nonneutral, its positive linear mass term similarly absorbs all mixed terms involving it. A neutral profile's remaining contribution is controlled by (75). Since the profile lists are finite, a single sufficiently small neighborhood works for all cases. This completes the proof of Theorem A.1.

The exact checks for this section are `verify_atom_stability.py` and `check_neutral_pairs_exact.py`, with data in `atom-certificate.json`, all in the `lower-certificate` directory. They reconstruct the selected flow extensions directly from L and P_k, check all quadratic identities, and use the exact algebraic root intervals for every strict sign. The theorem allows new labels and simultaneous old-mass changes, but keeps all old label cells fixed and is local in the added mass. It does not exclude improvements from larger changes in the masses or from changes to labels between the six existing parts.

A.3 Fixed directions with more added blocks and uncovered vertices

For any fixed finite number of added blocks, a related statement holds along each direction in which their masses tend to zero at positive relative rates. We also allow a small excess of vertices over those covered by the required matching. The threshold for the added mass may depend on the direction; thus this statement differs from the neighborhood valid for all one- and two-block extensions in Theorem A.1.

Theorem A.2. *Keep the old six-by-six label matrix fixed and add a fixed finite collection of blocks with fixed labels. Measure masses per $2M$, where the required matching has M edges. Suppose the new masses are $u\alpha_s + o(u)$, with every $\alpha_s > 0$ and $\sum_s \alpha_s = 1$, and the old masses are $a^* + uh + o(u)$. Assume the total mass is at least one and $\sum_i h_i \geq -1$. For each sufficiently small fixed positive u, the minimum critical-edge count divided by $\binom{2M}{2}$ is at most $\alpha_L + o_M(1)$ in the associated finite blow-ups.*

Proof. First consider total mass one. Use the active flow maps, weights, and $T = 1 - \alpha_L$ from Section 17. As in the preceding section, an increase in noncritical density provides a matching that prevents an improvement of the lower construction. The check allowing profiles without the capping restriction enumerates all $4^7 = 16384$ new-block profiles. Of these, 16354 have the strictly positive insertion-price margin $> 19/1000$. The remaining thirty profiles each have a *parent*, namely an existing part i whose incident labels are no larger than the corresponding labels of the new part, including its internal label. The two label profiles agree on every matching type used by each of the three active flows. These are finite comparisons of integer labels. The identities giving the zero weighted first-order price are checked as linear forms in the old masses.

If a profile with positive first-order noncritical gain has positive relative mass, that gain dominates all quadratic errors. Suppose therefore all profiles are neutral. If a pair of new blocks with parents i, j has mutual label $k < L_{ij}$, matching equal portions to each other instead of their designated old partners produces a strict first-order noncritical gain in at least one active strategy. The exact checker examines all 81 such profile-pair/label cases and proves an unweighted gain greater than $9/100$ in an active strategy. The strategy weight is strictly positive, as are both relative block masses. This again prevents an improvement for small u.

It remains to treat neutral profiles whose mutual labels are all at least the corresponding parent labels. Aggregate their masses with their parents, obtaining s with $\sum_i s_i = 1$. Let β_i be the fraction of aggregate class i lying in added parts. For small u, all $\beta_i < 1/2$ and the old active flows $P_k s$ remain feasible. Lift each such flow so each added part inherits its parent's matching-label distribution, using only new–old and old–old matching edges. For a cross flow t_{ij}, route $\beta_i t_{ij}$ from new i to old j, $\beta_j t_{ij}$ from old i to new j, and $(1 - \beta_i - \beta_j)t_{ij}$ between old endpoints. For an internal endpoint flow t_{ii}, match $\beta_i t_{ii}$ new endpoints to the same mass of old endpoints, and match the remaining $(1 - 2\beta_i)t_{ii}$ old endpoints internally. Divide every new quota among added parts with the same parent in proportion to their masses. All matching-edge labels are the same as in the parent flow. Ambient labels can only increase, so the lifted noncritical density is at least $Q_k(s)$. Thus

$$Q_{\max} \geq \sum_k \lambda_k Q_k(s) = s^T H_0 s \geq T.$$

This is exact, and resolves the remaining directions at every order.

Now allow total mass larger than one. The old B column is an entrywise maximum of each old row:

$$L_{iB} \geq L_{ij} \qquad \text{for every old } i, j.$$

Leave the entire surplus, of mass $u(1 + \sum_i h_i) + o(u)$, uncovered inside old block B. This is possible for small u because $a^*_B > 0$. The remaining covered set has total mass exactly one, and the preceding perfect-matching insertion argument applies to it.

For a strict first-order case, the old–old matching endpoints have no critical edges to the uncovered B set by the displayed maximum-column property. Only the $O(u)$ new endpoints and the $O(u)$ old endpoints now matched to them can contribute outward critical edges. The uncovered set also has mass $O(u)$; hence this extra contribution is $O(u^2)$ and cannot cancel the strict linear improvement in the covered-set count.

In the residual neutral case, the proportional lift has no outward critical edges at all. Every lifted old endpoint has a parent birth at most its label to B. A new part with parent i inherits the parent's birth distribution and has label to B at least L_{iB}. Thus the exact covered-set bound remains valid after reinstating the uncovered vertices. The normalization is unchanged because all masses were measured per $2M$. □

The certificate `lower-certificate/verify_stationarity.py` checks the 16384 profiles, parent domination, active-edge agreement, and all 81 strict pair responses. Its stationary root lies in the same rational box as the exact algebraic construction of Theorem 17.1. This result is directional: it does not assert one radius uniform over varying numbers of added blocks or relative masses tending to zero. It also keeps all labels between the six existing parts fixed and does not concern configurations a fixed positive distance away.

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