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HOLOGRAPHIC ENTROPY INEQUALITIES WITHOUT UNSCALED CONTRACTION PROOFS

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Abstract. We show that a valid holographic entropy inequality can lack a proof by contraction until all its coefficients are multiplied by a common integer, even when it is balanced, that is, every party has the same total coefficient on both sides, or superbalanced, that is, this remains true after any party is exchanged with the purifier. Every facet of the holographic entropy cone other than subadditivity is superbalanced. A proof by contraction is a map between Boolean cubes that respects the terms in which each party occurs and does not increase weighted Hamming distances. A balanced seven-party inequality and a superbalanced six-party inequality fail at a single point of the cube. Six of their right-hand side terms correspond to the edges of two disjoint triangles of parties, and at that point an unscaled map would have to choose three of these edges that together touch all six vertices, which is impossible. The seven-party one has a proof after doubling. Superbalanced eight- and seven-party inequalities have no unscaled proof even on their realizable domains, the subsets of the cube on which a map already proves validity, and they keep this property when instances of the monogamy of mutual information are subtracted in five successive steps. Splitting further into valid inequalities leads to a superbalanced facet F_7 of the seven-party cone that we did not find in the literature. It has unscaled maps on its realizable domain, all of which break its symmetry. For star inequalities, whose left-hand side contains only single-party entropies, the least scaling is the least k for which an explicit polytope contains a point with coordinates in $\frac{1}{k}\mathbb{Z}$. From the weighing matrices of Alon and Vũ we obtain valid N-party star inequalities whose least scaling exceeds $N^{(1/4-\varepsilon)N}$ for every $\varepsilon > 0$ and large N, while every finite least scaling of an N-party star inequality is below $N^{N/2}$. Non-existence rests on DRAT proofs, validity on exact rational maps or identities, and the facet property on 126 explicit graph models, except for the star inequalities of arbitrary size, whose results rest on written proofs.

1. Introduction

In holographic theories the entropy of a boundary region is computed by the area of an extremal surface in the bulk, by the formula of Ryu and Takayanagi and its covariant extension [1, 2]. Beyond subadditivity and strong subadditivity [3], these entropies obey further linear inequalities, the first found being the monogamy of mutual information (MMI) [4], which also holds for the covariant formula under the assumptions of [5]. For N parties $A, B, C, \ldots$ and a purifier O, the valid linear inequalities cut out the holographic entropy cone $\mathrm{HEC}_N \subset \mathbb{R}^{2^N-1}$. Bao, Nezami, Ooguri, Stoica, Sully and Walter showed that HEC_N is a rational polyhedral cone and that it is the set of entropy vectors of graph models, weighted graphs in which the entropy of a set of parties is the weight of a minimal cut [6]. The cone is known completely for $N \leq 5$ [6, 7], and two infinite families of its facets are known [8, 9].

The main method for proving that a candidate inequality $\sum_l \alpha_l S(I_l) \geq \sum_r \beta_r S(J_r)$ is valid is proof by contraction [6]. Each party has an occurrence word on each side, which records the terms that contain it. A contraction map sends the Boolean cube of the left-hand side to that of the right-hand side, maps occurrence words to occurrence words and does not increase weighted Hamming distances. Bao et al. showed that such a map proves the inequality, and proved the five-party facets in this way [6, 7]. The list of five-party facets was shown to be complete in [7]. Much work since

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has gone into understanding and constructing contraction maps [10, 11, 12, 13, 14, 15, 16, 17]; see also [18].

Whether a valid inequality has a contraction proof depends on how it is presented. A coefficient can be kept as a weight or split into unit copies of its term, on either side, and the whole inequality can be multiplied by a positive integer k before a map is sought [6, 19, 16]. Bao, Furuya and Naskar claim that every valid holographic entropy inequality with rational coefficients has a contraction map, and in the arXiv version of their paper of 14 July 2026 they specify that a map may be sought for any presentation obtained by merging or expanding terms and by uniformly rescaling all coefficients [20]. The claim also appears, without that specification, in the PhD thesis of Naskar of 15 June 2026 [21]. Several recent works assume that the inequalities they study have contraction proofs [22, 23]. Grimaldi, Headrick, Hubeny and Shteyner remark that, empirically, when the fully split presentation of an inequality has no map, multiplying the inequality by an integer does not create one; their discussion concerns primitive, that is facet-defining, inequalities [16]. Whether the scaling is ever needed, and for which inequalities, is the subject of this paper.

An inequality is balanced if every party has the same total coefficient on both sides [6], and superbalanced if, in addition, every inequality obtained from it by exchanging a party with the purifier is balanced, a notion introduced in [24] and put in this form in [25]. He, Hubeny and Rangamani proved that every facet of HEC_N other than the instances of subadditivity is superbalanced [25]. For the facets at small N we found no failure. For $N \leq 4$ every facet of HEC_N is an instance of subadditivity or MMI [6], and these have unscaled maps. Of the eight orbits of facets of HEC_5 under relabelings [7], our computer search found an unscaled map on the full cube for a representative of each. These maps are not among our certificates, and nothing below uses them. This raises three questions. Does every valid balanced, or superbalanced, inequality have an unscaled contraction proof? If not, how large must the scaling be? And what happens at the facets?

We answer the first question in the negative for both classes, give partial answers to the second, and follow one failure to a facet. A map for an inequality Q on a domain $\Omega \subseteq \{0,1\}^L$ is an integer count map in the sense of Definition 2.6, that is, a contraction map of Bao et al. for the presentation with merged left-hand side and split right-hand side, which on the full cube is the most permissive unscaled presentation (Section 2.3). The domain is either the full cube or the realizable domain $\mathcal{A}(Q)$ of Definition 2.11, and $k_{\min}(Q)$ and $k^{\mathcal{A}}_{\min}(Q)$ are the least k for which kQ has a map on these domains (Definition 2.10). In this paper we give a proof of the following statement.

Theorem 1.1.

(a) *The seven-party inequality B_7^* of* (3.1) *is balanced and valid. It has no map on the full cube and none on its realizable domain, and $2B_7^*$ has a map on the full cube, so $k_{\min}(B_7^*) = k^{\mathcal{A}}_{\min}(B_7^*) = 2$.*

(b) *The six-party inequality U_6 of* (4.1) *is superbalanced and valid. It has no map on the full cube and none on its realizable domain, and $2U_6$ has a map on its realizable domain, so $k^{\mathcal{A}}_{\min}(U_6) = 2$. The eight-party inequality Q_8 of* (4.7) *and the seven-party inequalities $E_0, \ldots, E_5$ of Section 4, where E_0 is obtained from $3Q_8$ by subtracting a positive multiple of a sum of instances of* MMI *and deleting a party, and each E_{s+1} from E_s by subtracting a positive multiple of a sum of instances of* MMI*, are superbalanced and valid. None of them has a map on the full cube or on its realizable domain.*

(c) *The seven-party superbalanced inequality F_7 of* (5.1) *defines a facet of* HEC_7*, and this facet contains the face of* HEC_7 *on which E_0 vanishes. The inequality F_7 has a map on its realizable domain, and no such map is equivariant under the stabilizer* $\mathrm{Stab}(F_7) \cong \mathfrak{S}_3$*, that is, commutes with the relabelings that fix F_7 (Appendix A.4).*

(d) *The inequality $3Q_8$ has no map on the full cube or on its realizable domain, and $2E_0$, $2E_1$, $2E_3$ and $2E_4$ have maps on the full cube, so $k_{\min}(E_s) = k^{\mathcal{A}}_{\min}(E_s) = 2$ for $s = 0, 1, 3, 4$. For*

every $j \geq 0$, the star inequality D_j of Section 6.3, on $N = 4j + 7$ parties, is valid and $k_{\min}(D_j) = 2^{j+2} = 2^{(N+1)/4}$. The seven-party star inequality Z_7 of (6.20) has $k_{\min}(Z_7) = 8$. For every $N \geq 3$ there is a valid N-party star inequality Q_N, implied by subadditivity and not balanced, with $k_{\min}(Q_N) < \infty$ and $\log_2 k_{\min}(Q_N) \geq \frac{N}{4} \log_2 N - (\frac{5}{4} + \varepsilon_N)N$ with $\varepsilon_N \to 0$ as $N \to \infty$, and every valid N-party star inequality Q with $k_{\min}(Q) < \infty$ has $k_{\min}(Q) \leq 2(N+1)^{(N+1)/2} 2^{-N} < N^{N/2}$.

Part (a) shows that balance does not guarantee an unscaled contraction proof. The failure is visible at a single point of the cube, where an integer map would need three edges of two disjoint triangles of parties that together touch all six vertices (Section 3). Part (b) shows that superbalance does not guarantee one either, already for six parties. The six-party inequality U_6 fails at a single point by the same argument (Section 4.1), and for the eight- and seven-party inequalities the failure persists when sums of instances of MMI, which have unscaled maps, are subtracted in five steps. For E_0, E_1, E_3 and E_4 it disappears after doubling, by part (d). Part (c) is the one place where we can follow a failure to a facet. There the facet has an unscaled map on its realizable domain, though only one that breaks its symmetry. Part (d) shows that the scaling needed for a valid inequality can grow faster than exponentially with N. So far we know this only for star inequalities, which are implied by subadditivity and are not balanced (Section 6.5), and for them the least scaling, when finite, is below $N^{N/2}$. Our results are consistent with the claim of Bao, Furuya and Naskar in the form of their arXiv version of 14 July 2026 [20], and they show that the uniform rescaling allowed there cannot be dropped.

Each statement has one of three statuses. It is *certified* if it rests on an explicit certificate, a DRAT proof, an exact map, a list of graph models or an exact identity, checked by a program independent of the one that produced it (Appendix A). A DRAT proof on a subdomain to which every map on a larger domain restricts, such as the subdomain $\mathcal{U}(Q)$ of Lemma 2.15, counts as a certificate on that larger domain. It is *derived* if it follows from certified statements by a short proof given in the text, and *argued* if it rests on a written proof that was checked only by other instances of the models that wrote it, as described below. In Theorem 1.1, the statements that a map exists or does not exist are certified in parts (a)–(c) and, in part (d), for $3Q_8$, $2E_0$, $2E_1$, $2E_3$, $2E_4$ and Z_7. The one exception is the statement that B_7^* has no map on its realizable domain, which rests on the written proof of Proposition 3.2(iii) and is argued. For U_6 the non-existence is certified, and Proposition 4.3(iii) gives in addition a written proof. For the family D_j they are argued, and for D_0 and D_1 they are also certified at the scalings listed in Table 3. All results on star inequalities for general N, namely Theorems 6.10, 6.14 and 6.17, Corollary 6.22 and Proposition 6.24, are argued, with proofs in Appendix B, and the lower bound of part (d) rests in addition on a theorem of Alon and Vũ [26]. Validity is certified for B_7^*, U_6 and Q_8, and for E_0, E_1, E_3 and E_4 by the maps of part (d) on the full cube. It is derived for E_2, E_5 and F_7 from certified rational maps on their realizable domains through Lemma 2.13. The facet statement and the containment of faces in part (c) are derived (Theorem 5.2, Proposition 5.4).

Use of large language models and status of the proofs. This paper was produced by large language models, instances of the Claude models of Anthropic running as autonomous agents, in October and November 2026. They found the inequalities, wrote the proofs and the programs of Appendix A, ran the computations and wrote the text. The arguments were checked only by other instances of the same models, acting as referees and recomputing the certificates with programs of their own. Such checks may share the blind spots of the models that wrote the proofs. The author has read the paper and checked that the references lead to the cited works, but the statements attributed to them were checked only by the models: each statement was matched to a verbatim passage of the cited work, and another instance of the model ruled on whether the passage supports the sentence. No human expert has checked the proofs. The steps we regard as least secure are

the SAT encodings and the argued results on star inequalities of Sections 6.2–6.4. A DRAT proof certifies only that a formula is unsatisfiable, while the step from the map problem to the formula rests on the encoder (a single one for E_5, $3Q_8$ and the equivariant statement for V_7 of Section 5). The validity of E_2, E_5 and F_7 rests on Lemma 2.13, which for our rational certificates reduces to a theorem of Avis and Hernández-Cuenca (Section 2.4).

Code availability. The programs that check the certificates of Appendix A, the certificates themselves, and a program that regenerates the Boolean formulas and the integer maps are available under the MIT license at `https://github.com/ahippelainen/unscaled-contraction-public`.

Conventions. Parties are $A, B, C, \ldots$, the purifier is O, and $[N] = \{1, \ldots, N\}$ is the set of parties. A term written ABC in an inequality stands for $S(ABC)$, and $2ABC$ for $2S(ABC)$. Terms that contain O are replaced by the entropy of the complementary set of parties, so every inequality is a vector $Q \in \mathbb{Z}^{2^N-1}$ and reads $Q \cdot S \geq 0$, and kQ is the inequality with all coefficients multiplied by k. We write MMI(X:Y:Z) for the coefficient vector of the instance $S(XY) + S(XZ) + S(YZ) \geq S(X) + S(Y) + S(Z) + S(XYZ)$ of MMI, with parties concatenated, so that MMI(BC:DE:FG) has $X = \{B, C\}$. We write $[P]$ for 1 if the statement P holds and 0 otherwise. In asymptotic statements $o(1)$ stands for a function of N that tends to 0 as $N \to \infty$. So $k \geq N^{(c-o(1))N}$ means that for every $\varepsilon > 0$ we have $k \geq N^{(c-\varepsilon)N}$ for all large enough N, and $k \leq N^{(c+o(1))N}$ means that $k \leq N^{(c+\varepsilon)N}$ for all large enough N.

The method. All our statements about contraction proofs concern integer maps $n = (n_r)_{r=1}^{R} \colon \Omega \to \prod_r \{0, 1, \ldots, \beta_r\}$ on a domain $\Omega \subseteq \{0,1\}^L$ with

(1.2)
$$n_r\big(x^{(i)}\big) = \beta_r[i \in J_r] \quad (i \in [N]) \ , \qquad n(0) = 0 \ , \qquad \sum_{r=1}^{R} \big|n_r(u) - n_r(v)\big| \leq d_\alpha(u, v) \quad (u, v \in \Omega) \ ,$$

where $x^{(i)} \in \{0,1\}^L$ records which LHS terms contain the party i and d_α is the Hamming distance weighted by the LHS coefficients. The value $n_r(u)$ counts how many of the β_r unit copies of $S(J_r)$ the point u is sent into, and real values define real maps. Non-existence is certified by DRAT proofs [27], machine-checkable proofs that a Boolean formula is unsatisfiable, for SAT encodings of the map problem (Appendix A.1) on the full cube or on a small subdomain $\mathcal{U}(Q)$ to which every map on the full cube or on $\mathcal{A}(Q)$ restricts (Lemma 2.15). Validity is certified by exact identities with instances of subadditivity and MMI or by exact rational real maps on the full cube or on $\mathcal{A}(Q)$, the latter through Lemma 2.13: a real map on $\mathcal{A}(Q)$ proves that Q is valid. Facets are certified by $2^N - 2$ explicit tight graph models with linearly independent entropy vectors. For star inequalities, whose left-hand side is a weighted sum of single-party entropies, the contraction conditions at one point of the cube cut out a polytope $\Pi(Q)$, and kQ has a map exactly when $\Pi(Q)$ contains a point of the grid $\frac{1}{k}\mathbb{Z}^m$ (Theorem 6.10). Balance conditions of coin weighing make $\Pi(Q)$ a single point whose least common denominator is the total weight of the coins (Theorem 6.14), and the ill-conditioned weighing matrices of Alon and Vũ [26] make this weight superexponentially large (Corollary 6.22). The upper bound comes from Cramer's rule and Hadamard's inequality at a vertex of $\Pi(Q)$ (Proposition 6.24).

Section 2 defines contraction maps in count form and the realizable domain and proves two lemmas that are used throughout the paper. Sections 3, 4, 5 and 6 prove parts (a), (b), (c) and (d). Appendix A describes the certificates, Appendix B contains the omitted proofs, and Appendix C lists the inequalities that the main text does not print. We conclude in Section 7.

What is new. As far as we could determine, the following are new. First, valid balanced and superbalanced inequalities without an unscaled contraction proof in the presentation with split right-hand side, both on the full cube and on the realizable domain (Theorem 1.1(a), (b)), the

smallest of them with six parties. The earlier failures that we know of are failures of the presentation with merged right-hand side [19, 16]. In particular, the answer to Problem 3 of Avis and Hernández-Cuenca [19], in the formulation stated in Section 2.4, is negative. For four of the seven-party superbalanced examples, E_0, E_1, E_3 and E_4, doubling suffices on the full cube, so that $k_{\min} = 2$ (Theorem 1.1(d)). Second, the description of the scaling spectrum of star inequalities by lattice points of a polytope, the construction of star inequalities from coin weighings, with the explicit family D_j, whose least scaling is $2^{(N+1)/4}$, and the bounds $N^{(1/4-o(1))N}$ and $N^{(1/2+o(1))N}$ on the largest finite least scaling of an N-party star inequality. The ill-conditioned matrices behind the lower bound, and the estimate of their kernel vectors, are due to Alon and Vũ [26], who use them for coin-weighing problems (see also [28]). In the literature we read we found no earlier statement that scaling beyond small factors can be necessary. Third, the superbalanced facet F_7 of HEC_7, which does not appear among the seven-party inequalities we found in the literature (Section 5), together with its unscaled map on the realizable domain, which necessarily breaks the symmetry of F_7. Proof by contraction is due to Bao et al. [6], the realizable domain to Avis and Hernández-Cuenca [19], and the superbalance of facets to He, Hubeny and Rangamani [25].

2. Preliminaries: contraction maps and the realizable domain

In this section we fix graph models and the holographic entropy cone, the Bell-pair conditions that define balanced and superbalanced inequalities, contraction maps in the count form used by our map and non-existence certificates, and the realizable domain. Two lemmas at the end are used throughout Sections 3–5: Lemma 2.13 proves validity from a real-valued map on the realizable domain, and Lemma 2.15 transfers non-existence from a small subdomain to the full cube and to the realizable domain.

2.1. **Graph models and the holographic entropy cone.** A *graph model* is a finite undirected graph (V, E) with edge weights $w_e \geq 0$ and a labeling $b\colon \partial V \to [N] \cup \{O\}$ of a set $\partial V \subseteq V$ of boundary vertices. For $I \subseteq [N]$, an *I-cut* is a set $W \subseteq V$ with $W \cap \partial V = b^{-1}(I)$, its weight $C(W)$ is the total weight of the edges with exactly one endpoint in W, and the entropy of I is $S(I) = \min\{C(W)\colon W \text{ an } I\text{-cut}\}$. A cut attaining the minimum is a *min cut* of I. Since $C(W) = C(V \setminus W)$, the entropy of a set containing O is that of its complement in $[N] \cup \{O\}$. Bao, Nezami, Ooguri, Stoica, Sully and Walter showed that the holographic entropy cone $\mathrm{HEC}_N \subset \mathbb{R}^{2^N-1}$, defined through the Ryu–Takayanagi formula, is the set of entropy vectors $S = (S(I))_{\emptyset \neq I \subseteq [N]}$ of graph models, and that it is a rational polyhedral cone [6]. It is full-dimensional [19].

An entropy inequality is written

$$Q: \qquad \sum_{l=1}^{L} \alpha_l S(I_l) \;\geq\; \sum_{r=1}^{R} \beta_r S(J_r)\ , \tag{2.1}$$

with L left-hand side (LHS) terms, R right-hand side (RHS) terms, positive integers α_l, β_r and nonempty sets $I_l, J_r \subseteq [N]$, the I_l distinct, the J_r distinct and no I_l equal to a J_r. The same letter denotes the coefficient vector $Q \in \mathbb{Z}^{2^N-1}$, so that (2.1) reads $Q \cdot S \geq 0$, and $|\alpha| = \sum_l \alpha_l$, $|\beta| = \sum_r \beta_r$. The inequality is *valid* if $Q \cdot S \geq 0$ for every graph model. A valid $Q \neq 0$ defines a *facet* of HEC_N if the face $\mathrm{HEC}_N \cap Q^\perp$ has dimension $2^N - 2$. Since HEC_N is full-dimensional, this holds exactly when there are $2^N - 2$ linearly independent entropy vectors of graph models on which Q vanishes, a standard fact of polyhedral theory (see e.g. [19]). We call such graph models *tight*. For pairwise disjoint nonempty $X, Y, Z \subseteq [N]$, subadditivity and MMI are the valid inequalities

$$S(X)+S(Y) \geq S(XY)\ , \qquad S(XY)+S(XZ)+S(YZ) \geq S(X)+S(Y)+S(Z)+S(XYZ)\ , \tag{2.2}$$

where $XY = X \cup Y$. Hayden, Headrick and Maloney proved MMI for the Ryu–Takayanagi formula [4]; see also [6, 29] for proofs by contraction and by bit threads. The symmetric group

$\mathfrak{S}_{N+1}$ acts on $[N]\cup\{O\}$ and, through purification, on $\mathbb{R}^{2^N-1}$, and HEC_N is invariant under it [6]. The *stabilizer* $\mathrm{Stab}(Q)\subseteq\mathfrak{S}_{N+1}$ is the group of relabelings that fix the coefficient vector of Q.

2.2. **Bell pairs, balance and superbalance.** For distinct $i,j\in[N]\cup\{O\}$, the *Bell-pair vector* r_{ij} is the entropy vector of the graph model with a single edge of weight 1 between a boundary vertex of i and one of j [6, 25]: $r_{ij}(I)=1$ if I contains exactly one of i,j, and 0 otherwise. The *occurrence words* $x^{(i)}\in\{0,1\}^L$ and $y^{(i)}\in\{0,1\}^R$ of a party i are

$$x^{(i)}_l=[i\in I_l]\ ,\qquad y^{(i)}_r=[i\in J_r]\ , \tag{2.3}$$

and $x^{(O)}=0$, $y^{(O)}=0$. With the weighted Hamming distance

$$d_\alpha(u,v)=\sum_{l=1}^{L}\alpha_l|u_l-v_l|\ ,\qquad u,v\in\{0,1\}^L\ , \tag{2.4}$$

and d_β on $\{0,1\}^R$ defined likewise, a term $S(I)$ equals 1 on r_{ij} exactly when I separates i from j, so the *Bell value* of Q on the pair i,j is

$$Q\cdot r_{ij}=d_\alpha\big(x^{(i)},x^{(j)}\big)-d_\beta\big(y^{(i)},y^{(j)}\big)\ . \tag{2.5}$$

A valid inequality has nonnegative Bell values. The inequality Q is *balanced* if $Q\cdot r_{iO}=0$ for every party i, that is, if every party has the same total coefficient on both sides [6], and *superbalanced* if $Q\cdot r_{ij}=0$ for all distinct $i,j\in[N]\cup\{O\}$. This pairwise form [16] is equivalent to the form of He, Hubeny and Rangamani [25], that Q and all its purifications be balanced, of the notion introduced in [24], because the relabeling that exchanges j and O carries r_{iO} to r_{ij}. They proved that every facet of HEC_N other than the instances of subadditivity is superbalanced [25]. Instances of MMI are superbalanced.

2.3. **Contraction maps in count form.** Bao et al. proved an inequality valid by a map between Boolean cubes that respects the occurrence words and does not increase weighted Hamming distances [6]. A coefficient β_r can be kept as a weight or split into β_r unit copies of $S(J_r)$, and likewise on the left [6, 30]. Whether a map exists depends on this choice [19, 16]. We merge the left-hand side, split the right-hand side, and record a map by how many of the β_r copies of each RHS term contain a given point.

Definition 2.6 (Count maps)**.** Let Q be as in (2.1), and let $\Omega\subseteq\{0,1\}^L$ contain the occurrence words $x^{(i)}$ of all parties $i\in[N]$ and the word 0. A *real map for Q on Ω* is a map $n=(n_r)_{r=1}^R\colon\Omega\to\prod_{r=1}^R[0,\beta_r]$ such that

$$n_r\big(x^{(i)}\big)=\beta_r[i\in J_r]\quad\text{for all } i\in[N]\text{ and all } r\ ,\qquad n(0)=0\ , \tag{2.7}$$

and

$$\sum_{r=1}^{R}\big|n_r(u)-n_r(v)\big|\le d_\alpha(u,v)\qquad\text{for all } u,v\in\Omega\ . \tag{2.8}$$

It is an *integer map*, or an *unscaled map*, if $n_r(u)\in\{0,1,\dots,\beta_r\}$ for all u and r. For an integer $k\ge1$, a map for kQ is a map for the inequality with coefficients $k\alpha_l$ and $k\beta_r$.

The conditions (2.7) are the *pins*, and the condition at 0 is the pin of the purifier. We call $\Omega=\{0,1\}^L$ the *full cube*, and say that Q has an *unscaled contraction proof* on a domain if it has an integer map there. Write $\overline{y}^{(i)}\in\{0,1\}^{|\beta|}$ for the occurrence word of i in the presentation in which the RHS lists β_r unit copies $(r,1),\dots,(r,\beta_r)$ of each $S(J_r)$, so that $\overline{y}^{(i)}_{r,s}=y^{(i)}_r$, and let d_1 be the unit Hamming distance on $\{0,1\}^{|\beta|}$.

Lemma 2.9 (Count maps are split-RHS contraction maps)**.** *Let Ω be as in Definition 2.6. The inequality Q has an integer map on Ω if and only if there is a map $f\colon \Omega \to \{0,1\}^{|\beta|}$ with $f(x^{(i)}) = \overline{y}^{(i)}$ for all $i \in [N]$, $f(0) = 0$, and $d_1(f(u), f(v)) \le d_\alpha(u, v)$ for all $u, v \in \Omega$.*

The proof, by sorting the copies of each term, is in Appendix B.1. On the full cube, an integer map is therefore a contraction map of Bao et al. for the presentation with split right-hand side, and their theorem [6] shows that Q is valid. This presentation is the most permissive one on the full cube. A *presentation* of Q divides each coefficient α_l and each β_r into positive integer parts and lists the term once for each part, with that part as its coefficient. A contraction map f of Bao et al. for any presentation gives an integer map for Q on $\{0,1\}^L$. On the right, repeating each output bit as many times as the coefficient of its term turns f into a map into $\{0,1\}^{|\beta|}$ with unit distance, without changing any distance [16]. On the left, the map δ that copies bit l of $u \in \{0,1\}^L$ into the positions of all parts of α_l preserves weighted distances and sends occurrence words to occurrence words and 0 to 0, so the composition with δ is defined on $\{0,1\}^L$ and satisfies the conditions of Lemma 2.9. So every statement in this paper that Q has no unscaled map on the full cube excludes a contraction proof of Bao et al. for Q in every presentation, in particular the fully split one and the one with merged right-hand side. The failures of unscaled maps in the literature that we know of are failures of the merged right-hand side that disappear after splitting [19, 16].

A real map for kQ is k times a real map for Q, so the existence of a real map does not depend on the scaling, while the existence of an integer map does.

Definition 2.10 (Least scaling)**.** For a valid inequality Q, $k_{\min}(Q)$ is the least integer $k \ge 1$ such that kQ has an integer map on the full cube $\{0,1\}^L$, and $k^{\mathcal{A}}_{\min}(Q)$ is the least k such that kQ has an integer map on the realizable domain $\mathcal{A}(Q)$ of Definition 2.11. Either is ∞ if no such k exists.

Since $\mathcal{A}(Q) \subseteq \{0,1\}^L$, $k^{\mathcal{A}}_{\min}(Q) \le k_{\min}(Q)$. If n is a map for $k_1 Q$ and n' a map for $k_2 Q$ on the same domain, then $n + n'$ is a map for $(k_1 + k_2)Q$, because pins, ranges and the contraction condition all add, so the set of k for which kQ has an integer map is closed under addition. A real map with rational values and common denominator M gives the integer map Mn for MQ.

2.4. **The realizable domain.** In the proof of Bao et al., every vertex of a graph model receives the word that records which of the chosen LHS min cuts contain it. If the min cuts are chosen minimal, these words satisfy two conditions, and the map is needed only on the words that satisfy them [19, 16]. On the role of nesting in contraction proofs see [15].

Definition 2.11 (Realizable domain)**.** The *realizable domain* $\mathcal{A}(Q) \subseteq \{0,1\}^L$ of Q is the set of words x such that

(R1) if $x_l = x_{l'} = 1$, then $I_l \cap I_{l'} \neq \emptyset$, and
(R2) if $x_l = 1$ and $I_l \subseteq I_{l'}$, then $x_{l'} = 1$.

This is the domain of Avis and Hernández-Cuenca [19], who describe its elements through the antichains of their minimal elements; it is called the realizable subset in [16]. It is defined through the terms $I_l \subseteq [N]$ of (2.1), that is, after every term containing O has been replaced by its complement, and the pin $n(0) = 0$ singles out O in the same way. Every statement in this paper about $\mathcal{A}(Q)$ refers to this presentation, the one in which Q is printed, with O as the purifier. The domain depends only on the LHS terms, so $\mathcal{A}(kQ) = \mathcal{A}(Q)$. It contains 0 and every occurrence word $x^{(i)}$. On a subset of the cube, the contraction condition cannot in general be reduced to pairs at Hamming distance 1, as it can on the full cube [6], but it can be reduced to the following pairs.

Lemma 2.12 (Irreducible pairs)**.** *Let $\Omega \subseteq \{0,1\}^L$, and call two distinct points $u, v \in \Omega$* reducible *in Ω if some $z \in \Omega \setminus \{u, v\}$ lies in the box between them, $z_l = u_l$ whenever $u_l = v_l$, and* irreducible *otherwise. If $n\colon \Omega \to \mathbb{R}^R$ satisfies* (2.8) *for every irreducible pair, it satisfies* (2.8) *for all $u, v \in \Omega$.*

The proof is a short induction (Appendix B.1). On the full cube the irreducible pairs are the pairs at Hamming distance 1. We can now state the criterion behind our validity maps.

Lemma 2.13 (Realizable-domain criterion)**.** *Let Q be as in* (2.1)*. Suppose there is a map $n\colon \mathcal{A}(Q) \to \prod_{r=1}^{R}[0,\beta_r]$ with $n_r(x^{(i)}) = \beta_r[i \in J_r]$ for every party $i \in [N]$ and every r, with $n(0) = 0$, and with $\sum_r |n_r(u) - n_r(v)| \le d_\alpha(u,v)$ for all pairs $u, v \in \mathcal{A}(Q)$, or equivalently, by Lemma 2.12, for all pairs irreducible in $\mathcal{A}(Q)$. Then Q is valid on* HEC_N*.*

The proof is in Appendix B.2. Each vertex of a graph model receives the word that records which minimal min cuts of the LHS terms contain it, as in [6, 19]; these words lie in $\mathcal{A}(Q)$ by the nesting and disjointness properties of minimal min cuts [19]; and each RHS term is bounded by integrating the weights of the threshold cuts $\{v\colon n_r(v) > t\}$ over t.

For maps of the form $n_r = \beta_r f_r$ with $f\colon \mathcal{A}(Q) \to \{0,1\}^R$, for which $\sum_r |n_r(u) - n_r(v)| = d_\beta(f(u), f(v))$, Lemma 2.13 is the theorem of Avis and Hernández-Cuenca on their domain [19]. Grimaldi, Headrick, Hubeny and Shteyner record that a contraction map on the realizable domain proves validity [16]. For maps with rational values, which are all our validity certificates, Lemma 2.13 also follows from the proof of the theorem of Avis and Hernández-Cuenca: if M is a common denominator of n, then Mn is an integer map for MQ, by Lemma 2.9 with $\Omega = \mathcal{A}(Q)$ it is a $\{0,1\}$-valued contraction map into the cube of the split right-hand side, and that proof applies to this presentation [19], although the theorem is stated for distinct terms. The proof in Appendix B.2 covers real values directly.

Avis and Hernández-Cuenca ask in their Problem 3 whether the existence of a contraction map on their domain becomes a necessary condition for validity once every RHS term $\beta_r S(J_r)$ with $\beta_r \ge 2$ is replaced by β_r unit terms [19]. By Lemma 2.9 with $\Omega = \mathcal{A}(Q)$, in the formulation with merged left-hand side and no scaling this asks whether $k^{\mathcal{A}}_{\min}(Q) = 1$ for every valid Q.

2.5. **Restriction to a subdomain.** To show that Q has no integer map we exhibit a small set of points on which already no map exists. In most of our examples a distinguished party, always labeled A, carries the obstruction, and the set is

$$\mathcal{U}(Q) = \{x^{(1)}, \dots, x^{(N)}, 0\} \cup [0, x^{(A)}]\ , \qquad [0, x^{(A)}] = \{x \in \{0,1\}^L\colon x_l \le x_l^{(A)} \text{ for all } l\}\ , \tag{2.14}$$

the occurrence words together with the 2^{L_A} words supported on the L_A LHS terms that contain A.

Lemma 2.15 (Restriction)**.** *Let $k \ge 1$. (a) If kQ has no integer map on $\mathcal{U}(Q)$, then kQ has no integer map on $\{0,1\}^L$. (b) The inclusion $\mathcal{U}(Q) \subseteq \mathcal{A}(Q)$ holds if and only if the LHS terms that contain A form an antichain, that is, no two of them are nested. In that case, if kQ has no integer map on $\mathcal{U}(Q)$, then kQ has no integer map on $\mathcal{A}(Q)$ either.*

Proof. (a) A map on $\{0,1\}^L$ restricts to any subset that contains the occurrence words and 0. (b) Let $x \in [0, x^{(A)}]$. Any two terms with $x_l = x_{l'} = 1$ contain A, so (R1) holds. If $x_l = 1$ and $I_l \subseteq I_{l'}$, then $A \in I_{l'}$, and if the terms containing A form an antichain, $l' = l$ and (R2) holds. Conversely, if $I_l \subsetneq I_{l'}$ are two terms containing A, the word with a single 1 in position l lies in $[0, x^{(A)}]$ and violates (R2). The last statement follows as in (a). □

By Lemma 2.12, the contraction condition on $\mathcal{U}(Q)$ needs to be imposed only on the pairs irreducible in $\mathcal{U}(Q)$. The superbalanced inequalities of Sections 4.3, 4.4 and 5 satisfy the antichain condition, so for them every statement that no unscaled map exists holds both on the full cube and on the realizable domain.

3. Balanced inequalities without unscaled proofs

Our example has seven parties, and its failure can be checked by hand at a single point of the cube, where the contraction conditions reduce to covering the vertices of two disjoint triangles by edges.

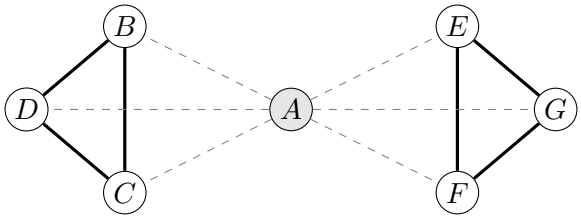


FIGURE 1. The structure of B_7^*. Each solid edge ϵ of the two triangles is an RHS term $S(A\epsilon)$, and the hub A (dashed) lies in all six. At the point z of the proof of Proposition 3.2, an integer map must choose three edges that touch all six vertices, which is impossible. Weight $\frac{1}{2}$ on every edge touches every vertex with total weight 1 and costs 3.

Let $N = 7$ with parties $A, \ldots, G$. Party A is a hub, and $\{B, C, D\}$ and $\{E, F, G\}$ are two triangles with the six edges $\mathcal{E} = \{BC, BD, CD, EF, EG, FG\}$. The inequality is

$$\begin{aligned} B_7^*: \quad & 2A + AO + 3\,(BO + CO + DO + EO + FO + GO) \\ & \geq \; ABC + ABD + ACD + AEF + AEG + AFG + 14\,O\ . \end{aligned} \tag{3.1}$$

By purification AO stands for $S(BCDEFG)$, iO for the entropy of the six parties other than i, and O for $S(ABCDEFG)$. So B_7^* has $L = 8$ LHS terms with $|\alpha| = 21$, and its $R = 7$ RHS terms are the six terms $S(A\epsilon)$, $\epsilon \in \mathcal{E}$, with $\beta = 1$ and $S(ABCDEFG)$ with $\beta = 14$, so $|\beta| = 20$. Every party has net coefficient 0, so B_7^* is balanced. The Bell values on pairs of parties are 4 on the six pairs inside a triangle and 2 on the other fifteen, so B_7^* is not superbalanced.

Proposition 3.2. *The inequality B_7^* has the following properties.*

(i) *It is a nonnegative combination, with coefficients in $\frac{1}{2}\mathbb{Z}$, of* 43 *instances of subadditivity and* MMI. *In particular it is valid.*
(ii) *$2B_7^*$ has an integer map on the full cube $\{0,1\}^8$.*
(iii) *B_7^* has no integer map on the full cube, and none on its realizable domain $\mathcal{A}(B_7^*)$.*

Hence $k_{\min}(B_7^*) = k_{\min}^{\mathcal{A}}(B_7^*) = 2$.

Part (i) is an identity between integer vectors with 26 instances of subadditivity and 17 of MMI, with coefficients in $\{\frac{1}{2}, 1, \frac{3}{2}, 2\}$, checked in exact rational arithmetic and given in the code repository. Part (ii) is certified by an explicit integer map whose pins, ranges and contraction condition on all 32,640 pairs of points of $\{0,1\}^8$ were checked in exact arithmetic. On the full cube, part (iii) is also certified by DRAT proofs (Appendix A). We now prove (iii) by hand on both domains. The last sentence then follows from (ii), since a map on the full cube restricts to $\mathcal{A}(B_7^*)$.

Proof of Proposition 3.2(iii). Order the LHS terms as $S(A)$, $S(AO)$ and the six terms $S(iO)$, $i \in \{B, \ldots, G\}$, with weights 2, 1 and 3. Party A occurs in $S(A)$ and in every $S(iO)$ (since iO stands for a set containing A), and not in $S(AO)$. A party $i \in \{B, \ldots, G\}$ occurs in $S(AO)$ and in $S(jO)$ for $j \neq i$, and not in $S(A)$ or $S(iO)$. Let $z \in \{0,1\}^8$ be the word that contains every LHS term except $S(A)$. Then

$$d_\alpha\big(z, x^{(A)}\big) = 2{+}1 = 3\,, \qquad d_\alpha\big(z, x^{(i)}\big) = 3 \quad (i \in \{B, \ldots, G\})\,, \qquad d_\alpha(z, 0) = 1{+}6{\cdot}3 = 19\,. \tag{3.3}$$

The point z lies in $\mathcal{A}(B_7^*)$, taken as in Section 2.4 for the purified terms. Indeed, the terms present in z are $S(AO)$ and the six $S(iO)$, whose purified sets $[N]\setminus\{A\}$ and $[N]\setminus\{i\}$ are six-element subsets of the seven parties, so any two of them intersect. The only containments among the LHS sets are $A \subseteq iO$, which impose nothing at z because z does not contain $S(A)$. The occurrence words and 0 lie in $\mathcal{A}(B_7^*)$ by Definition 2.11. It therefore suffices to show that no integer value $n(z)$ satisfies the contraction condition (2.8) between z and the eight pinned points $x^{(A)}, \ldots, x^{(G)}, 0$.

Write a candidate value as $n(z) = (\theta, 14 - t)$, with $\theta = (\theta_\epsilon)_{\epsilon \in \mathcal{E}} \in [0,1]^{\mathcal{E}}$ the values on the six terms $S(A\epsilon)$ and $14 - t \in [0, 14]$ the value on $S(ABCDEFG)$. By the pins (2.7), $n(x^{(A)})$ is 1 on every $S(A\epsilon)$, $n(x^{(i)})$ for $i \in \{B, \ldots, G\}$ is $[i \in \epsilon]$ on $S(A\epsilon)$, both are 14 on $S(ABCDEFG)$, and $n(0) = 0$. With (3.3) the contraction conditions at the eight pinned points read

$$(3.4)\ (A)\colon\ \sum_{\epsilon}(1-\theta_\epsilon)+t \le 3\ , \qquad (i)\colon\ \sum_{\epsilon}\big|\theta_\epsilon - [i\in\epsilon]\big|+t \le 3\ , \qquad (O)\colon\ \sum_{\epsilon}\theta_\epsilon + 14 - t \le 19\ .$$

For each vertex i, let $\mathcal{E}_i \subset \mathcal{E}$ be the two edges at i.

Suppose $n(z)$ is an integer, so that θ is the indicator of a set $\mathcal{C} \subseteq \mathcal{E}$ of edges and t is a nonnegative integer. Condition (i) becomes $|\mathcal{C}| + 2 - 2|\mathcal{C} \cap \mathcal{E}_i| + t \le 3$, and condition (A) becomes $6 - |\mathcal{C}| + t \le 3$, so $|\mathcal{C}| \ge 3$. If $|\mathcal{C}| = 3$, condition (A) forces $t = 0$, and condition (i) gives $|\mathcal{C} \cap \mathcal{E}_i| \ge 1$ for all six vertices: the three edges of $\mathcal{C}$ touch every vertex of both triangles. But an edge of $\mathcal{E}$ lies in one triangle and touches two of its three vertices, so each triangle needs two edges of $\mathcal{C}$, and $|\mathcal{C}| \ge 4$, a contradiction. If $|\mathcal{C}| \ge 4$, condition (i) gives $2|\mathcal{C} \cap \mathcal{E}_i| \ge |\mathcal{C}| - 1 \ge 3$, hence $|\mathcal{C} \cap \mathcal{E}_i| = 2$ for every vertex i. Then $\mathcal{C}$ contains both edges at every vertex, so $\mathcal{C} = \mathcal{E}$, and condition (i) reads $6 + 2 - 4 + t \le 3$, a contradiction. So no integer value is admissible at z, and no integer map exists on any domain that contains z, the occurrence words and 0, in particular on $\{0,1\}^8$ and on $\mathcal{A}(B_7^*)$. □

The real relaxation is feasible at z: the value $\theta_\epsilon = \frac{1}{2}$ for every edge and $t = 0$ satisfies (3.4) with $3 \le 3$ in (A), $\frac{1}{2} \cdot 6 = 3 \le 3$ in every (i), and $17 \le 19$ in (O). This is the integrality gap of edge covers (Figure 1): two disjoint triangles have fractional edge cover number 3 but need four edges in an integral cover. The point z lies coordinatewise between $x^{(i)}$ and $x^{(j)}$ for each of the 21 pairs of parties, and every one of these pairs has a positive Bell value.

Remark 3.5 (A second example). The balanced inequality

$$\begin{aligned} B_7:\quad & 3\,(AO + BO + CO + DO + EO + FO + GO) + DEFG \\ & \ge\ ABC + ADE + AFG + BDEF + BDEG + CDFG + CEFG + 15\,O \end{aligned}$$

is a nonnegative $\frac{1}{2}\mathbb{Z}$-combination of 46 instances of subadditivity and MMI and has $k_{\min}(B_7) = 2$ (Table 4), but its obstruction is not visible at one point: the contraction conditions against the pinned points force integer values at two points of $\{0,1\}^8$ at distance 1 whose values are at ℓ^1 distance 7.

Remark 3.6 (Scope). Both B_7^* and B_7 are positive combinations of instances of subadditivity and MMI, so they say nothing about facets of HEC_7. In every balanced valid inequality without an unscaled map whose least scaling we have determined, namely B_7^*, B_7, the inequality Q_8^A of Section 4 and the superbalanced inequalities E_0, E_1, E_3 and E_4 of that section, the least scaling is 2. The value 2 is not a general bound: in work to be reported separately we construct, for every odd prime p, valid balanced inequalities whose least scaling is finite and at least p, and valid superbalanced inequalities without a map for any scaling below p. The superbalanced, hence balanced, six-party inequality U_6 of Section 4.1 has no unscaled map either. We do not know whether a balanced valid inequality without an unscaled map exists with $N \le 5$.

4. Superbalanced inequalities without unscaled proofs

The obstruction of Section 3 sits at a point that lies between the occurrence words of every pair of parties, and it uses the positive Bell values of these pairs, which superbalance removes: if z lies between $x^{(i)}$ and $x^{(j)}$ and the Bell value of i, j is 0, both contraction conditions are tight, so $n(z)$ is forced onto an ℓ^1-geodesic between the pins. A superbalanced inequality can still fail at a single point, one that lies between no two of the pinned points. In Section 4.1 we give a six-party example that fails in this way, by the covering argument of Section 3. In Sections 4.2–4.4 we start from a balanced inequality Q_8^A whose terms all contain one party, complete it to a superbalanced

inequality Q_8 with eight parties, pass to seven parties, and split off instances of MMI in five steps. Each step is an identity $Q = Q' + cX$ of coefficient vectors, with $c > 0$ and X a sum of instances of MMI, and we call such a step a *splitting*. Every superbalanced inequality along the way is valid and has no unscaled map, on the full cube or on its realizable domain.

4.1. **Six parties.** Let $N = 6$ with parties $A, \dots, F$, and let $\{A, B, C\}$ and $\{D, E, F\}$ be two triangles with the six edges $\mathcal{E} = \{AB, AC, BC, DE, DF, EF\}$. The inequality

$$\begin{aligned} U_6 : \quad & AB + AC + AD + AE + AF + BC + BD + BE + BF + CD + CE + CF \\ & + DE + DF + EF + ABC + DEF + ABCDE + ABCDF + ABCEF \\ & + ABDEF + ACDEF + BCDEF \\ & \geq \; 4\,(A + B + C + D + E + F) + ABCD + ABCE + ABCF \\ & + ADEF + BDEF + CDEF + 3\,ABCDEF \end{aligned} \tag{4.1}$$

has on its left-hand side the 15 pairs of parties, the two triangles and the six sets of five parties, all with coefficient 1, so $L = |\alpha| = 23$, $R = 13$ and $|\beta| = 33$. By purification the six RHS terms with four parties are the terms $S(\epsilon O)$, $\epsilon \in \mathcal{E}$: $ABCD$ stands for $S(EFO)$, $ADEF$ for $S(BCO)$, and so on. They have the structure of the six terms $S(A\epsilon)$ of B_7^*, with the purifier in the role of the hub (Figure 1). All 21 Bell values of U_6 vanish, so U_6 is superbalanced, and in particular balanced. Its stabilizer, of order 72, fixes the purifier, permutes the vertices of each triangle and exchanges the two triangles.

For a vertex i let ϵ_i be the edge of its triangle that does not contain i, and for an edge ij let $\overline{ij}$ be the set of the other four parties. With x running over the three vertices of the triangle that does not contain i, let

$$\begin{aligned} \psi_1 &= \sum_{ij \in \mathcal{E}} \mathrm{MMI}(i{:}j{:}\overline{ij}) \;, \qquad \psi_2 = \mathrm{MMI}(A{:}B{:}C) + \mathrm{MMI}(D{:}E{:}F) \;, \\ \psi_3 &= \sum_{ij \in \mathcal{E}} \sum_{x} \mathrm{MMI}(i{:}j{:}x) \;, \qquad \psi_4 = \sum_{i} \sum_{x} \mathrm{MMI}(i{:}x{:}\epsilon_i) \;, \end{aligned} \tag{4.2}$$

sums of 6, 2, 18 and 18 instances of MMI, where i runs over the six vertices in ψ_4. Each ψ_k is invariant under the stabilizer of U_6, so the identity in Proposition 4.3(i) can be checked by hand on the 9 orbits of the stabilizer on the 63 terms.

Proposition 4.3. *The inequality U_6 has the following properties.*

(i) $6U_6 = 3\psi_1 + 3\psi_2 + \psi_3 + \psi_4$. *In particular U_6 is valid.*
(ii) $2U_6$ *has an integer map on its realizable domain $\mathcal{A}(U_6)$, which has* 580 *points.*
(iii) U_6 *has no integer map on the full cube $\{0,1\}^{23}$, and none on $\mathcal{A}(U_6)$.*

Hence $k_{\min}^{\mathcal{A}}(U_6) = 2$.

Part (i) is an identity between integer vectors, checked exactly by two programs. Part (ii) is certified by an explicit integer map whose pins, ranges and contraction condition on all 167,910 pairs of points of $\mathcal{A}(U_6)$ were checked in exact arithmetic. Part (iii) is also certified by DRAT proofs (Table 4). We now prove it by hand.

Proof of Proposition 4.3(iii). Let $z \in \{0,1\}^{23}$ be the word that contains the six LHS terms with five parties and no other term. Two sets of five of the six parties intersect, and no other LHS term contains a set of five parties, so z satisfies (R1) and (R2) and lies in $\mathcal{A}(U_6)$. The occurrence word $x^{(i)}$ contains the five pair terms at i, the triangle of i and the five sets of five parties that contain i, while z also contains the set of five parties without i, which $x^{(i)}$ does not, so

$$d_\alpha\big(z, x^{(i)}\big) = 5 + 1 + 1 = 7 \quad (i \in \{A, \dots, F\}) \;, \qquad d_\alpha(z, 0) = 6 \;. \tag{4.4}$$

It suffices to show that no integer value $n(z)$ satisfies the contraction condition (2.8) between z and the seven pinned points $x^{(A)}, \dots, x^{(F)}, 0$.

Write a candidate value as $n(z) = (a, b, c)$, with $a_i \in [0, 4]$ the value on $S(i)$, $b_\epsilon \in [0, 1]$ the value on $S(\epsilon O)$ for $\epsilon \in \mathcal{E}$ and $c \in [0, 3]$ the value on $S(ABCDEF)$, and let $|n| = \sum_i a_i + \sum_\epsilon b_\epsilon + c$. A party i lies in the set of parties that $S(\epsilon O)$ stands for exactly when $i \notin \epsilon$. So by the pins (2.7), $n(x^{(i)})$ is 4 on $S(i)$ and 0 on the other $S(j)$, 1 on the four terms $S(\epsilon O)$ with $i \notin \epsilon$ and 0 on the other two, and 3 on $S(ABCDEF)$, with total 11. Since every coordinate of $n(z)$ lies in its range $[0, \beta_r]$ and every pinned coordinate is 0 or β_r, $\sum_r |n_r(z) - n_r(x^{(i)})| = |n| + 11 - 2s_i$ with $s_i = a_i + \sum_{\epsilon \not\ni i} b_\epsilon + c$. With (4.4) the contraction conditions at the seven pinned points read

$$(O)\colon\ |n| \le 6\ , \qquad (i)\colon\ 2s_i \ge |n| + 4 \quad (i \in \{A, \dots, F\})\ . \tag{4.5}$$

Summing (i) over the six parties, in which each a_i occurs once, each b_ϵ four times and c six times, gives $\sum_\epsilon b_\epsilon + 3c \ge 2\sum_i a_i + 12$. By (O), $\sum_\epsilon b_\epsilon \le 6 - \sum_i a_i - c$, hence $2c \ge 3\sum_i a_i + 6$, and since $c \le 3$ this forces $a = 0$ and $c = 3$. Then $\sum_\epsilon b_\epsilon \ge 3$ by the sum of the conditions (i) and $\sum_\epsilon b_\epsilon \le 3$ by (O), so $\sum_\epsilon b_\epsilon = 3$ and $|n| = 6$. Condition (i) now reads $\sum_{\epsilon \not\ni i} b_\epsilon \ge 2$, that is $\sum_{\epsilon \ni i} b_\epsilon \le 1$, for every vertex i. These six sums add up to $2\sum_\epsilon b_\epsilon = 6$, so $\sum_{\epsilon \ni i} b_\epsilon = 1$ for every vertex: b is a fractional perfect matching of the two triangles. In a triangle with vertices i, j, k the equations $b_{ij} + b_{ik} = b_{ij} + b_{jk} = b_{ik} + b_{jk} = 1$ have the unique solution $b_{ij} = b_{ik} = b_{jk} = \frac{1}{2}$. So the only value that satisfies (4.5) is $a = 0$, $b_\epsilon = \frac{1}{2}$ for every edge and $c = 3$, which satisfies all seven conditions with equality and is not an integer. Hence no integer map exists on any domain that contains z, the occurrence words and 0, in particular on $\{0, 1\}^{23}$ and on $\mathcal{A}(U_6)$. □

As for B_7^*, an integer value at z would have to be three edges of two disjoint triangles that touch every vertex exactly once, which do not exist, while the value $\frac{1}{2}$ on every edge is admissible (Figure 1). The difference lies in the position of z. The point z of Proposition 3.2 lies between the occurrence words of every pair of parties, and these pairs have positive Bell values. The point z of Proposition 4.3 lies between no two of the pinned points: the words $x^{(i)}$ and $x^{(j)}$ of two parties both contain the pair term ij, which z does not contain, and $x^{(i)}$ and 0 both miss the set of five parties without i, which z contains. By Proposition 4.3(i), U_6 is a positive combination of instances of MMI that are not multiples of it, so it is valid but does not define a facet of HEC_6. The valid inequality without a map in the presentation with merged right-hand side given by Grimaldi, Headrick, Hubeny and Shteyner, a sum of two instances of MMI, is of the same kind [16]. We did not decide whether $2U_6$ has an integer map on the full cube, so on the full cube we know only $k_{\min}(U_6) \ge 2$.

4.2. **A balanced core.** Let $N = 8$ with parties $A, \dots, H$. The inequality

$$\begin{aligned} Q_8^A\colon\quad & ADE + AFG + ABCH + ABDFH + ABEGH + ACDGH + ACEFH \\ & \ge\ 2A + ABDGH + ABEFH + ACDFH + ACEGH + ABCDEFGH \end{aligned} \tag{4.6}$$

has every term containing A and unit coefficients except $2A$. It is balanced, and 16 of the 28 pairs of parties have Bell value 2 (the other 12 have 0), so it is not superbalanced. It is valid and $k_{\min}(Q_8^A) = 2$. The inequality $2Q_8^A$ has an integer map on the full cube $\{0, 1\}^7$, certified by an exact check on all pairs of points, which proves validity by the theorem of Bao et al. [6]. DRAT proofs certify that Q_8^A has none (Table 4).

4.3. **From eight to seven parties.** Let

(4.7)
$$\begin{aligned} Q_8: \quad & ADE + AFG + ABCH + ABDFH + ABEGH + ACDGH + ACEFH \\ & + 2\,(BCDE + BCFG + BDGH + BEFH + CDFH + CEGH + DEFG) \\ & \geq\ 2A + 4\,(B + C + D + E + F + G + H) + DE + FG + BCH \\ & + ABDGH + ABEFH + ACDFH + ACEGH + 3\,BCDEFGH + ABCDEFGH\ . \end{aligned}$$

Its terms that contain A are exactly those of Q_8^A. The other LHS terms are twice the complements in $\{B,\dots,H\}$ of the seven lines BCH, BDF, BEG, CDG, CEF, DEH, FGH of a Fano plane on $B,\dots,H$. The inequality Q_8 has $L = 14$, $R = 17$, $|\alpha| = 21$ and $|\beta| = 41$. All 36 Bell values vanish, so it is superbalanced, and $|\operatorname{Stab}(Q_8)| = 8$.

Q_8 is not a facet. Let R_8 be the inequality whose LHS is the sum of the seven complements of Fano lines above, with unit coefficients, and whose RHS is $2\,(B+C+D+E+F+G+H)+2\,BCDEFGH$. The vector $42R_8$ is a positive integer combination of 147 instances of MMI (with coefficients 1, 3 and 18 on three orbits), and $Q_8 = \frac{1}{3}Q_8' + \frac{2}{3}R_8$ with $Q_8' = 3Q_8 - 2R_8$. Since Q_8' is valid (Theorem 4.11), $Q_8 - tR_8$ is valid for $t \leq \frac{2}{3}$, and a graph model on which Q_8' vanishes and R_8 does not shows that $\frac{2}{3}$ is the largest such t (Table 4). The inequality $Q_8'' = Q_8' - 3\,\mathrm{MMI}(A{:}H{:}DE)$ is valid as well, and both are superbalanced.

Deleting a party preserves validity [6]: if Q is valid for $N+1$ parties, the inequality obtained by replacing every term $S(I)$ by $S(I\setminus\{H\})$ is valid for N parties, because adding an isolated boundary vertex labeled H to a graph model does not change the value of either inequality. Deleting H from Q_8', or equally from Q_8'', gives the seven-party inequality

(4.8)
$$\begin{aligned} E_0: \quad & 3\,(ABC + ADE + AFG) + 4\,(BDG + BEF + CDF + CEG) \\ & + 3\,(ABDF + ABEG + ACDG + ACEF) + 4\,(BCDE + BCFG + DEFG) \\ & \geq\ 6A + 8\,(B + C + D + E + F + G) + 3\,(BC + DE + FG) \\ & + 3\,(ABDG + ABEF + ACDF + ACEG) + 5\,BCDEFG + 3\,ABCDEFG\ , \end{aligned}$$

with $L = 14$, $R = 16$, $|\alpha| = 49$ and $|\beta| = 83$. Like every deletion of a superbalanced inequality, E_0 is superbalanced.

4.4. **The descent.** The seven parties of E_0 carry the structure of a Fano plane with lines ABC, ADE, AFG, BDG, BEF, CDF, CEG (Figure 2). Its stabilizer, of order 24, is the group of collineations of this plane that fix A (it fixes the purifier). We split E_0 along the instances and orbit sums of MMI

(4.9)
$$\begin{aligned} \mu_0 &= \mathrm{MMI}(BC{:}DE{:}FG)\ , \qquad \mu_1 = \mathrm{MMI}(B{:}D{:}F)\ , \\ \mu_2 &= \mathrm{MMI}(A{:}C{:}DF) + \mathrm{MMI}(A{:}E{:}BF) + \mathrm{MMI}(A{:}G{:}BD)\ , \\ \mu_3 &= \mathrm{MMI}(A{:}B{:}D) + \mathrm{MMI}(A{:}B{:}F) + \mathrm{MMI}(A{:}D{:}F)\ , \\ \mu_4 &= \mathrm{MMI}(B{:}E{:}G) + \mathrm{MMI}(C{:}D{:}G) + \mathrm{MMI}(C{:}E{:}F)\ , \end{aligned}$$

and define

(4.10)
$$E_1 = E_0 - 2\mu_0\ , \qquad E_2 = E_1 - 2\mu_1\ , \qquad E_3 = E_2 - \mu_2\ , \qquad E_4 = E_3 - \mu_3\ , \qquad E_5 = E_4 - 2\mu_4\ .$$

The first step gives E_1, the most symmetric member of the chain, with the same stabilizer as E_0. Its LHS consists of the seven lines, with weight 3 for the lines through A and 4 for the others, the complements of the four lines that miss A, with weight 3, and the complements of the three lines through A, with weight 2. The coefficient lists of $E_1,\dots,E_5$ are in the code repository (Appendix C).

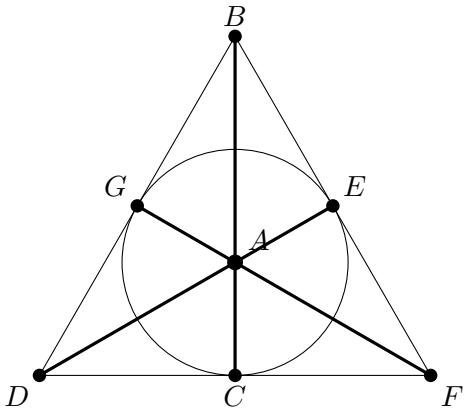


FIGURE 2. The Fano plane on the parties of $E_0, \ldots, E_5$ and F_7. Its seven lines are the three sides, the three medians and the circle. The thick lines ABC, ADE, AFG pass through A.

Theorem 4.11. *The inequalities Q_8, Q'_8, Q''_8 ($N = 8$) and $E_0, \ldots, E_5$ ($N = 7$) are superbalanced and valid. None of them has an integer map, either on the full cube or on its realizable domain.*

Proof. Superbalance is an exact integer computation of all Bell values. Validity of Q_8, Q'_8 and Q''_8 follows from exact rational real maps on the full cube, each checked in exact arithmetic on all edges of $\{0,1\}^{14}$ (which suffice by Lemma 2.12), through Lemma 2.13 or the theorem of Bao et al. applied to MQ_8, MQ'_8, MQ''_8, with M a common denominator of the map. Validity of E_0 follows by deletion from that of Q'_8. For $E_1, \ldots, E_5$ we have exact rational real maps on the realizable domains, checked in exact arithmetic, and Lemma 2.13 gives validity. For E_0, E_1, E_3 and E_4, validity follows also, without Lemma 2.13 and without the deletion, from the integer maps for $2E_0$, $2E_1$, $2E_3$ and $2E_4$ on the full cube of Proposition 6.1, by the theorem of Bao et al. (Section 2.3).

For the non-existence, in each of the nine inequalities the seven LHS terms that contain A form an antichain, so the subdomain $\mathcal{U}$ consisting of the occurrence words, the word 0 and the 2^7 points of the interval $[0, x^{(A)}]$ lies in the realizable domain, and by Lemma 2.15 it suffices to show that the integer map problem on $\mathcal{U}$ has no solution. Here $|\mathcal{U}| = 135$ for $N = 8$ and $|\mathcal{U}| = 134$ for $N = 7$. For every inequality this is certified by DRAT proofs checked with `drat-trim` (Table 4). □

By Section 2.3, Theorem 4.11 also excludes unscaled maps on the full cube in the fully split presentation and in the presentations with merged right-hand side.

Proposition 4.12 (The descent)**.** *$E_0 = E_5 + 2\mu_0 + 2\mu_1 + \mu_2 + \mu_3 + 2\mu_4$, and the steps along μ_0, μ_1 and μ_4 are maximal:* $\sup\{t\colon E_s - t\mu_s \text{ valid}\} = 2$ *for $s = 0, 1, 4$.*

The identities (4.10) are exact integer identities, the lower bounds 2 follow from the validity of the next inequality (Theorem 4.11), and the upper bounds from the graph models listed in Table 4. None of Q_8, Q'_8, $E_0, \ldots, E_5$ is a facet. Each of $Q_8, Q'_8, E_0, \ldots, E_4$ is a positive combination of the next inequality and instances of MMI. For the last one, $6E_5 = 2X_7 + Y_7 + 3\mu_5$, where μ_5 is the orbit sum (C.1) and X_7 the valid inequality (C.2) of Appendix C.1, and Y_7, defined by this identity, is valid by Proposition 5.4. Whether Q''_8 is a facet of HEC_8 we do not know. After each split the remaining inequality still has no unscaled map, while the subtracted instances of MMI have unscaled maps [6]. Section 5 follows the chain one step further, to a facet.

5. A superbalanced facet of HEC_7 and its unscaled map

In Section 4 the failure of unscaled maps survived five splittings of E_0, and the last inequality of the chain splits further as $6E_5 = 2X_7 + Y_7 + 3\mu_5$, where X_7 is printed in (C.2) and Y_7 is defined by

	N	L	R	$\lvert\alpha\rvert$	$\lvert\beta\rvert$	$\lvert\mathcal{A}\rvert$	validity	next step, $\sup t$
Q_8	8	14	17	21	41		map on $\{0,1\}^{14}$	$-tR_8$, $\frac{2}{3}$
Q'_8	8	14	17	49	91		map on $\{0,1\}^{14}$	$-t\,\mathrm{MMI}(A{:}H{:}DE)$, ≥ 3
Q''_8	8	14	18	49	88		map on $\{0,1\}^{14}$	delete H
E_0	7	14	16	49	83	2187	map for $2Q$ on $\{0,1\}^{14}$	$-t\mu_0$, 2
E_1	7	14	16	43	75	2187	map for $2Q$ on $\{0,1\}^{14}$	$-t\mu_1$, 2
E_2	7	15	19	45	75	2916	map on $\mathcal{A}$	$-t\mu_2$, ≥ 1
E_3	7	15	25	42	69	2916	map for $2Q$ on $\{0,1\}^{15}$	$-t\mu_3$, ≥ 1
E_4	7	15	24	42	66	2916	map for $2Q$ on $\{0,1\}^{15}$	$-t\mu_4$, 2
E_5	7	18	33	48	66	6912	map on $\mathcal{A}$	

TABLE 1. The superbalanced inequalities of Theorem 4.11. L and R count the distinct LHS and RHS terms, $|\mathcal{A}|$ is the size of the realizable domain (given for $N = 7$). "Validity" names the certificate: an exact rational map on the full cube, an integer map for $2Q$ on the full cube (Proposition 6.1), or an exact rational map on the realizable domain together with Lemma 2.13. The last column gives the next step of the chain and $\sup\{t\colon Q - tX \text{ valid}\}$ for the subtracted vector X. A lower bound $\geq c$ comes from the validity of the next inequality, and an exact value is attained by an explicit graph model on which the next inequality vanishes and X does not.

this identity. In this section we split Y_7 once more and reach a facet F_7 of HEC_7, compare it with the seven-party inequalities in the literature, and place it in an exact decomposition of E_0. We then show that F_7 has an unscaled contraction proof on its realizable domain and that every such proof breaks the symmetry of F_7.

5.1. **The inequality F_7.** Let $N = 7$ with parties $A, \dots, G$. The inequality

$$\begin{aligned} F_7:\quad & 2ABC + 2ADE + 2AFG + BDF + 4BDG + 4BEF + BEG + 4CDF \\ & + CDG + CEF + 2ABDF + 2ABEG + 2ACDG + 2ACEF + BCDE \\ & + BCFG + DEFG + 2BCDEG + 2BCEFG + 2CDEFG \\ & \geq\ 3B + 2C + 3D + 2E + 3F + 2G + 2AB + 2AD + 2AF + 3BE + 3BG \\ & + 3CD + 3CF + 3DG + 3EF + ABDG + ABEF + ACDF + 3ACEG \\ & + BCDEF + BCDFG + BDEFG + 4BCDEFG + 2ABCDEFG \end{aligned} \tag{5.1}$$

has $L = 20$, $R = 24$, $|\alpha| = 39$ and $|\beta| = 54$, and its coefficients, which go up to 4, have no common factor. All 28 of its Bell values vanish, so F_7 is superbalanced. Its seven LHS terms that contain A are those of E_0, the three lines through A of the Fano plane of Figure 2 and the complements of the four lines that miss A, now all with weight 2, and they form an antichain. The realizable domain $\mathcal{A}(F_7)$ has 11,046 of the 2^{20} points of the cube.

The stabilizer $\mathrm{Stab}(F_7)$ has order 6. It fixes A and the purifier and permutes the three ordered pairs (B, C), (D, E), (F, G) as blocks, so it is isomorphic to $\mathfrak{S}_3$, generated by $(BD)(CE)$ and $(BF)(CG)$. It is a subgroup of the stabilizer of E_0, which has order 24. The orbit of F_7 under the 8! relabelings of $A, \dots, G, O$ has 6,720 members. F_7 does not come from an inequality with fewer parties by letting one party stand for two (a zero-lift in the sense of [19]), since every term of such an inequality contains both or neither of the two merged points (for a pair i, O: i occurs in no term), and no pair of $A, \dots, G, O$ has this property for F_7. All statements of this subsection are exact integer computations from (5.1).

5.2. **The facet.**

Theorem 5.2. *The inequality F_7 is valid on* HEC_7 *and defines a facet of* HEC_7*.*

Proof. For validity, there is a map $n\colon \mathcal{A}(F_7) \to \prod_{r=1}^{24}[0,\beta_r]$ with rational values that satisfies the pins (2.7) and the contraction condition (2.8) on all 61,001,535 pairs of points of $\mathcal{A}(F_7)$, checked in exact arithmetic (Appendix A.2). By Lemma 2.13, F_7 is valid.

For the facet property, there are 126 explicit graph models with integer edge weights on which F_7 vanishes and whose entropy vectors are linearly independent. The entropy vectors are recomputed by exact max-flow computations and the rank is certified modulo primes (Appendix A.3). Since HEC_7 is full-dimensional in $\mathbb{R}^{127}$ (Section 2.1), the face $\mathrm{HEC}_7 \cap F_7^{\perp}$ has dimension $126 = 2^7 - 2$, so F_7 defines a facet. □

Lemma 2.13 keeps the validity certificate small, since the map needs values only on $\mathcal{A}(F_7)$ instead of all 2^{20} points of the cube.

Remark 5.3 (Comparison with the literature). The facet F_7 does not appear among the 43 seven-party inequalities of the works listed in Table 6 (Appendix C.3): none of them has the canonical form of F_7 under the 8! relabelings with purification, after division by the greatest common divisor of the coefficients (exact computation). Since a relabeling only permutes the coefficients, the coefficient 4 of F_7 against the maximum 3 in the table separates them also without canonical forms. Members of the three families with more than seven and at most 14 parties do not give a multiple of F_7 by merging or deleting parties either, and we found no published list of facets of HEC_7. Appendix C.3 gives the details and the scope of the search.

5.3. **Where F_7 comes from.** The facet arises from the chain of Section 4 by one more splitting, with a sum of other valid inequalities in the role of X. Let $g = (DF)(EG)$. It lies in the stabilizers of F_7, X_7 and Y_7, which coincide, but not in that of V_7. The inequalities X_7, V_7 and V_7' and the orbit sum μ_5 are written out in Appendix C.1, and $Y_7 = 6E_5 - 2X_7 - 3\mu_5$.

Proposition 5.4. *The identities*

$$4Y_7 = 3F_7 + 2V_7 + g(V_7) + 2V_7' \tag{5.5}$$

and

$$24E_0 = 3F_7 + 2V_7 + g(V_7) + 2V_7' + 8X_7 + 12\mu_5 + 48\mu_4 + 24\mu_3 + 24\mu_2 + 48\mu_1 + 48\mu_0 \tag{5.6}$$

hold exactly, and every inequality on their right-hand sides is valid. Consequently every graph model on which E_0 vanishes has $F_7 \cdot S = 0$, that is, the face $\mathrm{HEC}_7 \cap E_0^{\perp}$ *lies in the facet* $\mathrm{HEC}_7 \cap F_7^{\perp}$*. The same holds with E_0 replaced by any of $E_1, \dots, E_5$ or Y_7.*

Proof. The identity (5.5) is an identity of integer vectors in $\mathbb{Z}^{127}$, checked exactly from the coefficient vectors. With the definition $6E_5 = 2X_7 + Y_7 + 3\mu_5$ of Y_7 it gives $24E_5 = 8X_7 + 4Y_7 + 12\mu_5 = 3F_7 + 2V_7 + g(V_7) + 2V_7' + 8X_7 + 12\mu_5$, and with $E_0 = E_5 + 2\mu_0 + 2\mu_1 + \mu_2 + \mu_3 + 2\mu_4$ (Proposition 4.12) it gives (5.6). The inequality F_7 is valid by Theorem 5.2. The inequalities V_7, V_7' and X_7 are valid by Lemma 2.13, each with an exact rational map on its realizable domain, of 10,032, 10,032 and 6,912 points, checked like the map of Theorem 5.2 (Table 4). The relabeling $g(V_7)$ is valid because HEC_7 is invariant under relabelings. The μ_s are sums of instances of MMI [4], and Y_7 is valid by (5.5).

For a graph model with entropy vector S, every term on the right of (5.6), evaluated on S, is nonnegative. If $E_0 \cdot S = 0$, all of them vanish, and in particular $3F_7 \cdot S = 0$. For E_s with $1 \le s \le 5$ the same argument applies to the identity $24E_s = 24E_5 + (\text{a nonnegative combination of } \mu_0, \dots, \mu_4)$ given by (4.10), and for Y_7 to (5.5). □

Among the summands of (5.6), F_7 is the only one that we know to be a facet (Table 2).

5.4. **Unscaled maps of F_7.**

Theorem 5.7. (i) *F_7 has an integer map on its realizable domain $\mathcal{A}(F_7)$. Hence F_7 has an unscaled contraction proof on its realizable domain, and $k^{\mathcal{A}}_{\min}(F_7) = 1$.*
(ii) *No integer map for F_7 on $\mathcal{A}(F_7)$ is equivariant under* $\mathrm{Stab}(F_7)$.

Proof. (i) The certificate is a table of integers $n_r(u) \in \{0, \dots, \beta_r\}$ for the 11,046 points $u \in \mathcal{A}(F_7)$ and the 24 RHS terms. We rebuild $\mathcal{A}(F_7)$ from Definition 2.11, verify that it is the set of points of the certificate, and verify the pins, the ranges and the contraction condition on all pairs of points in exact arithmetic (Appendix A.2). An integer map is a real map, so Lemma 2.13 applies.

(ii) The equivariant encoding of Appendix A.4, in which the variables of (u, r) and $(g \cdot u, g(r))$ are identified for all $g \in \mathrm{Stab}(F_7)$, is unsatisfiable, and its DRAT proof is accepted by `drat-trim`. Every equivariant integer map would give a satisfying assignment. □

The inequality F_7 thus has $k^{\mathcal{A}}_{\min} = 1$, the property that Problem 3 of Avis and Hernández-Cuenca (Section 2.4) asks about, while the inequalities of Theorem 4.11 do not. The obstruction in (ii) is one of integrality. Averaging an integer map of (i) over the six elements of $\mathrm{Stab}(F_7)$, which fix the purifier, gives an equivariant real map with values in $\frac{1}{6}\mathbb{Z}$, and six times it is an equivariant integer map for $6F_7$. So the least k for which kF_7 has an equivariant integer map on $\mathcal{A}(F_7)$ lies between 2 and 6 (derived).

We did not attempt to construct an integer map for F_7 on the full cube $\{0,1\}^{20}$, so whether $k_{\min}(F_7) = 1$ is open. Lemma 2.15 cannot exclude it, because the map of (i) restricts to $\mathcal{U}(F_7) \subseteq \mathcal{A}(F_7)$.

Remark 5.8 (The other summands). The inequalities V_7 and V_7' also have integer maps on their realizable domains, each certified like the map of Theorem 5.7(i). As for F_7, no integer map of V_7 on $\mathcal{A}(V_7)$ is equivariant under its stabilizer, which has order 2 and is generated by $(BF)(CG)$, again certified by a DRAT proof. The stabilizer of V_7' is trivial. For each of V_7 and V_7' we have 125 explicit tight graph models with linearly independent entropy vectors, one fewer than a facet needs, so we do not know whether they are facets. For V_7 the 125 vectors also lie in $F_7^{\perp}$ (an exact computation), so they span $V_7^{\perp} \cap F_7^{\perp}$, and V_7 defines a facet if and only if some graph model has $V_7 \cdot S = 0$ and $F_7 \cdot S > 0$. For X_7 and Y_7 we know validity but not whether they have unscaled maps on their realizable domains. Hence (5.6) writes $24E_0$ as a positive combination of X_7 and of inequalities known to have unscaled contraction proofs on their realizable domains: F_7, V_7, $g(V_7)$, V_7', and the instances of MMI, which have them even on the full cube [6]. Remark 5.9 writes E_0 as a positive combination of inequalities known to have them, without X_7.

Remark 5.9 (A half-sum behind the failures). The seven-party superbalanced inequality P_7 of (C.5) in Appendix C.1 has an integer map on its realizable domain, of 6,912 points, certified like the map of Theorem 5.7(i), and is therefore valid by Lemma 2.13. With $g = (DF)(EG)$, the identities

$$2X_7 = V_7' + P_7 + 2g(P_7)\ , \qquad 2E_5 = V_7' + P_7 + 2\,\mathrm{MMI}(D{:}BG{:}CF) + 2\,\mathrm{MMI}(E{:}BF{:}CG) \tag{5.10}$$

hold exactly, and with (4.10) the second gives $2E_s = V_7' + P_7 + 2M_s$ for $s = 0, \dots, 5$, where M_s is $\mathrm{MMI}(D{:}BG{:}CF) + \mathrm{MMI}(E{:}BF{:}CG)$ plus a nonnegative integer combination of $\mu_s, \dots, \mu_4$. So X_7 and every E_s are positive combinations of valid inequalities with integer maps on their realizable domains, namely V_7', P_7, $g(P_7)$ and instances of MMI. The half-sum $\frac{1}{2}(V_7' + P_7)$ is an integer vector, valid and superbalanced. Its seven LHS terms that contain A are those of E_0 and form an antichain, and DRAT proofs on $\mathcal{U}$, of 134 points, show that it has no integer map, on the full cube or on its realizable domain (Lemma 2.15). So in these decompositions the coefficients $\frac{1}{2}$ fall on two inequalities that have unscaled maps on their realizable domains, and their half-sum has none. Whether V_7' and P_7 are facets, and whether they have unscaled maps on the full cube, are open questions.

	L	R	$\lvert\alpha\rvert$	$\lvert\beta\rvert$	$\lvert\,\mathrm{Stab}\,\rvert$	$\lvert\mathcal{A}\rvert$	integer map on $\mathcal{A}$	tight rank
F_7	20	24	39	54	6	11,046	yes, none equivariant	126, facet
V_7	20	26	35	48	2	10,032	yes, none equivariant	125, open
V_7'	20	23	32	44	1	10,032	yes	125, open
X_7	18	33	102	138	6	6,912	open	not a facet
Y_7	21	27	66	93	6	12,774	open	not a facet
P_7	18	35	60	80	8	6,912	yes	

TABLE 2. The inequalities in (5.5) and (5.6) other than the sums of MMI instances, and the inequality P_7 of Remark 5.9, all valid and superbalanced. "Integer map on $\mathcal{A}$" says whether an unscaled map on the realizable domain is known, and whether one equivariant under Stab exists. "Tight rank" is the number of linearly independent entropy vectors of explicit tight graph models, and 126 is the dimension of a facet of HEC_7. Empty entries were not determined.

In the one case where we followed a superbalanced inequality without unscaled maps down to a facet, the facet has an unscaled contraction proof on its realizable domain, and every such proof breaks its symmetry. The inequality X_7 is not a facet, since by (5.10) it is a positive combination of valid inequalities that are not multiples of it. If V_7, V_7' or P_7 is not a facet, the facets into which it decomposes are not known, and any of them could lack an unscaled proof.

6. How much scaling is needed

In this section we ask how large the least scaling $k_{\min}$ of Definition 2.10 can be. The claim of Bao, Furuya and Naskar [20] concerns every valid inequality and allows a uniform rescaling before a map is sought, so the question applies to every valid inequality, balanced or not. Star inequalities are the class in which we can answer it exactly; since they are implied by subadditivity and are not balanced (Section 6.5), they show how large the rescaling allowed by that claim may need to be for valid inequalities, and they say nothing about the scaling needed by the balanced and superbalanced inequalities of Sections 3 and 4. We first record what we know for the superbalanced inequalities of Section 4. We then show that for star inequalities $k_{\min}$ is the least k for which an explicit rational polytope contains a point of the grid $\frac{1}{k}\mathbb{Z}^m$, and with this criterion we turn families of coin weighings into star inequalities whose least scaling is the total weight of the coins. This gives an explicit family with $k_{\min} = 2^{(N+1)/4}$ exactly, and, from ill-conditioned weighing matrices, star inequalities whose least scaling grows faster than exponentially in N. We also bound the least scaling of star inequalities from above. Section 6.5 states the scope of the star results.

6.1. Scaled maps of the superbalanced failures.

For the inequalities of Theorem 4.11 we can decide the following scaled cases.

Proposition 6.1. (i) *$3Q_8$ has no integer map, either on the full cube or on its realizable domain.*

(ii) *$2E_0$, $2E_1$, $2E_3$ and $2E_4$ have integer maps on the full cube. Hence $k_{\min}(E_s) = k_{\min}^{\mathcal{A}}(E_s) = 2$ for $s = 0, 1, 3, 4$.*

Part (i) is certified like Theorem 4.11, by a DRAT proof for the problem for $3Q_8$ on $\mathcal{U}$. For part (ii), the maps are tables of integers on $\{0,1\}^{14}$ for E_0 and E_1 and on $\{0,1\}^{15}$ for E_3 and E_4. Each was checked in exact integer arithmetic by two independent checkers on all edges of the cube, which suffice by Lemma 2.12, and by at least one of them also on all 134,209,536 or 536,854,528 pairs of points of $\{0,1\}^{14}$ or $\{0,1\}^{15}$ (Appendix A.2). So $k_{\min}^{\mathcal{A}}(E_s) \le k_{\min}(E_s) \le 2$, and Theorem 4.11 excludes $k = 1$ on both domains. The map for $2E_0$ is the sum of the map for $2E_1$ and four times

an unscaled map of μ_0. This sum is a map for $2E_0 = 2E_1 + 4\mu_0$ because every term of μ_0 is a term of E_1 on the same side, so that pins, ranges and the contraction condition add. On $\mathcal{A}(E_4)$, which has 2,916 points, $2E_4$ also has an integer map equivariant under $\mathrm{Stab}(E_4)$, which has order 6 and fixes the purifier (Appendix A.4), checked in exact arithmetic on all 4,250,070 pairs of points. The least scaling of E_2, E_5, Q_8, Q_8' and Q_8'' is not known.

6.2. **Star inequalities and their scaling spectrum.**

Definition 6.2 (Star inequalities)**.** Let $m \geq 1$, let $J_1, \ldots, J_m \subseteq [N]$ be distinct sets with $2 \leq |J_K| \leq N-1$, let $\beta_1, \ldots, \beta_m$ be positive integers and let ρ_i, $i \in [N] \cup \{O\}$, be nonnegative integers, the *radii*. The *star inequality* with these data is

$$Q: \qquad \sum_{i\in[N]} \rho_i S(i) + \rho_O S([N]) \;\geq\; \sum_{K=1}^{m} \beta_K S(J_K)\ . \tag{6.3}$$

It has $R = m$ RHS terms in the notation of (2.1). The *word* of a party $i \in [N]$ is $\eta^{(i)} = (\beta_K[i \in J_K])_{K=1}^m$, and $\eta^{(O)} = 0$. The *star polytope* of Q is

$$\Pi(Q) = \Big\{ c \in \prod_{K=1}^{m} [0, \beta_K] \colon \|c - \eta^{(i)}\|_1 \leq \rho_i \text{ for all } i \in [N] \cup \{O\} \Big\}\ . \tag{6.4}$$

Since $S([N]) = S(O)$, the LHS is a weighted sum of the single-party entropies of all $N+1$ parties. The words are the pins of a map for Q, and $\Pi(Q)$ is the set of points of the box that lie in the ℓ_1-ball of radius ρ_i around $\eta^{(i)}$ for every i. Because the words are corners of the box, the distances to them are affine on the box.

Lemma 6.5 (Linearity on the box)**.** *Let $\eta \in \prod_K \{0, \beta_K\}$ and $c \in \prod_K [0, \beta_K]$. Then*

$$\|c - \eta\|_1 = \sum_{K\colon \eta_K = 0} c_K + \sum_{K\colon \eta_K = \beta_K} (\beta_K - c_K)\ , \tag{6.6}$$

and $\|c - \eta\|_1 + \|c - \bar\eta\|_1 = |\beta|$ for the complementary word $\bar\eta = \beta - \eta$. Hence $\Pi(Q)$ is cut out of the box by $N+1$ inequalities with coefficients ± 1 and integer right-hand sides, and if two words of Q are complementary and their radii add up to $|\beta|$, both of their constraints hold with equality on $\Pi(Q)$.

The proof is in Appendix B.8.

Validity of a star inequality follows from subadditivity once its polytope has a point. For each K let

$$\mathrm{SA}_K^1 : \sum_{i\in J_K} S(i) \;\geq\; S(J_K)\ , \qquad \mathrm{SA}_K^2 : \sum_{i\in[N]\setminus J_K} S(i) + S([N]) \;\geq\; S(J_K)\ , \tag{6.7}$$

where the second is subadditivity for the parties $([N] \setminus J_K) \cup \{O\}$, whose joint entropy is $S(J_K)$ by purification. Both are sums of instances of subadditivity, valid on HEC_N and for all quantum states [31]. We use the same symbols for their coefficient vectors and write P_I for that of $S(I) \geq 0$.

Lemma 6.8 (Validity from a real point)**.** *If $c \in \Pi(Q)$, then*

$$Q = \sum_{K=1}^{m} \big[(\beta_K - c_K)\,\mathrm{SA}_K^1 + c_K\,\mathrm{SA}_K^2 \big] + \sum_{i\in[N]} \big(\rho_i - \|c - \eta^{(i)}\|_1\big)\mathrm{P}_{\{i\}} + \big(\rho_O - \|c\|_1\big)\mathrm{P}_{[N]} \tag{6.9}$$

with all coefficients nonnegative. In particular Q is valid, and if $kc \in \mathbb{Z}^m$, then kQ is a nonnegative integer combination of instances of subadditivity and positivity.

The proof, a comparison of coefficients, is in Appendix B.8.

Theorem 6.10 (Scaling spectrum of star inequalities)**.** *Let Q be a star inequality and $k \geq 1$. Then kQ has an integer map on the full cube if and only if $\Pi(Q) \cap \frac{1}{k}\mathbb{Z}^m \neq \emptyset$. Consequently, if $\Pi(Q) \neq \emptyset$, then Q is valid and*

$$k_{\min}(Q) = \min\left\{k \geq 1 \colon \Pi(Q) \cap \tfrac{1}{k}\mathbb{Z}^m \neq \emptyset\right\} < \infty \ , \tag{6.11}$$

and if $\Pi(Q) = \emptyset$, then no kQ has an integer map on the full cube.

The proof is in Appendix B.4. The occurrence word of a party i has exactly the bits of $S(i)$ and $S([N])$ set, so the point z with only the bit of $S([N])$ set lies at weighted distance ρ_i from it and at distance ρ_O from 0, and for an integer map n of kQ the contraction condition at these pairs says exactly that $n(z)/k$ is a grid point of $\Pi(Q)$. Conversely, a grid point c gives a map that uses, on the term J_K, $k(\beta_K - c_K)$ copies of the obvious map of SA^1_K and kc_K copies of that of SA^2_K.

The smallest example is the three-party star inequality

$$Q' : \qquad A + B + C + ABC \ \geq \ AB + AC \ , \tag{6.12}$$

with all radii 1 and the words $(1,1)$, $(1,0)$, $(0,1)$ and 0 of A, B, C and O on (AB, AC). The pairs A, O and B, C have complementary words and radii adding up to $|\beta| = 2$, so by Lemma 6.5 every $c \in \Pi(Q')$ has $c_1 + c_2 = 1$ and $c_1 = c_2$, and $\Pi(Q') = \{(\frac{1}{2}, \frac{1}{2})\}$. Hence kQ' has an integer map exactly for even k, $k_{\min}(Q') = 2$, and $2Q'$ is a sum of four instances of subadditivity. In this case the theorem can be checked directly: the point $z = 0001$, with the LHS terms in the order A, B, C, ABC, differs from each of the four occurrence words in one bit of weight 1, so $n(z)$ lies within ℓ_1-distance k of all four corners of $[0,k]^2$, which forces $n(z) = (\frac{k}{2}, \frac{k}{2})$. We do not present Q' as new. It is the smallest case of the mechanism below.

6.3. **Coin weighing and single-point polytopes.** To make $k_{\min}$ large we need star polytopes that are single points with large denominators. Complementary pairs of words supply equalities (Lemma 6.5), and balance conditions from coin weighing make them determine the point. We use the language of coin weighing, and the superexponential lower bound of Section 6.4 rests on the ill-conditioned weighing matrices of Alon and Vũ [26].

We take $\beta = \mathbf{1}$, write $\mathbf{1}_X \in \{0,1\}^m$ for the indicator vector of $X \subseteq [m]$ and $\mathbf{1} = \mathbf{1}_{[m]}$, and call the coordinates $K \in [m]$ *coins*. A set $T \subseteq [m]$ stands for a weighing that puts the coins of T on one pan of a balance and the other coins on the other pan, and a vector $\xi \in \mathbb{R}^m$ of coin weights *balances* T if $\xi(T) = \xi(\bar{T})$, where $\bar{T} = [m] \setminus T$ and $\xi(X) = \sum_{K \in X} \xi_K$. This use of the word balance for weighings is unrelated to balanced inequalities in the sense of Section 2.2. For $\xi \in [0,1]^m$, (6.6) gives

$$\|\xi - \mathbf{1}_X\|_1 = \sum_{K \notin X} \xi_K + \sum_{K \in X} (1 - \xi_K) = \|\xi\|_1 + |X| - 2\xi(X) \ , \tag{6.13}$$

so a party with word $\mathbf{1}_T$ and radius $|T|$ requires the pan with T to be at least as heavy as the other, $\xi(T) \geq \xi(\bar{T})$.

Theorem 6.14 (Coin-weighing stars)**.** *Let $m \geq 2$ and $N \geq 3$, and let $\mathcal{T} = (T_1, \ldots, T_{N-1})$ be a list of subsets of $[m]$, in which a set may occur more than once, such that*

- (W1) *the vectors $\xi \in \mathbb{R}^m$ that balance every T_i form a line $\mathbb{R}\omega$, where $\omega \in \mathbb{Z}^m$ has positive entries without a common factor, and*
- (W2) *there are $\gamma_1, \ldots, \gamma_{N-1} > 0$ with $\sum_{i=1}^{N-1} \gamma_i(\mathbf{1} - 2 \cdot \mathbf{1}_{T_i}) = 0$.*

Put $q = \sum_K \omega_K$, and let $Q_{\mathcal{T}}$ be the inequality (6.3) with $\beta = \mathbf{1}$ in which the party $i \leq N-1$ has the word $\mathbf{1}_{T_i}$ and the radius $|T_i|$, the party N has the word $\mathbf{1}$ and the radius $m-1$, and the purifier has the radius 1, so that J_K is the set of parties whose word has a 1 at coin K. Then:

- *(a) $Q_{\mathcal{T}}$ is a star inequality on N parties with m RHS terms, and $\Pi(Q_{\mathcal{T}}) = \{\omega/q\}$;*
- *(b) $qQ_{\mathcal{T}} = \sum_{K=1}^{m} \left[(q - \omega_K)\,\mathrm{SA}^1_K + \omega_K\,\mathrm{SA}^2_K\right]$, so $Q_{\mathcal{T}}$ is valid, indeed for all quantum states;*

(c) $kQ_{\mathcal{T}}$ has an integer map on the full cube if and only if q divides k, and $k_{\min}(Q_{\mathcal{T}}) = q$.

The proof is in Appendix B.5. The party N and the purifier have complementary words and radii adding up to $m = |\beta|$, so every point ξ of $\Pi(Q_{\mathcal{T}})$ has $\|\xi\|_1 = 1$ (Lemma 6.5). The constraint of a party $i \le N-1$ says $\xi(T_i) \ge \xi(\bar{T}_i)$, and (W2) says that a combination of the differences $\xi(\bar{T}_i) - \xi(T_i)$ with positive coefficients vanishes for every ξ. So at every point of $\Pi(Q_{\mathcal{T}})$ every weighing balances, and (W1) fixes the point. In the language of weighing, (W2) says that the weighings can be given positive weights γ_i such that every coin lies on each pan with the same total weight. The conditions of Definition 6.2 on the sets J_K also follow from (W1) and (W2).

Two kinds of lists satisfy (W2) with all $\gamma_i = 1$. A *paired* list consists of sets $T_1, \dots, T_n$ and their complements $\bar{T}_1, \dots, \bar{T}_n$, in some order, and (W2) holds because $(\mathbf{1} - 2\cdot\mathbf{1}_T) + (\mathbf{1} - 2\cdot\mathbf{1}_{\bar{T}}) = 0$. In a *half-regular* list every coin lies in exactly half of the sets.

The family D_j is a paired case, built by a doubling recursion. Put $\omega^{(0)} = (1,1,1,1)$ and, recursively,

$$\omega^{(j+1)} = \left(\omega^{(j)}, \tfrac{q_j}{2}, \tfrac{q_j}{2}\right), \qquad q_j = \sum_K \omega_K^{(j)} . \tag{6.15}$$

Then $\omega^{(j)}$ has $m_j = 2j+4$ entries, $q_j = 2^{j+2}$, and $\omega_1^{(j)} = 1$. Level $j+1$ adds the coins $u_j = 2j+5$ and $v_j = 2j+6$, each weighing half the total of the old coins. The *halves* form the list $\mathcal{T}_0 = (\{1,2\},\{1,3\},\{1,4\})$ and, recursively, the list $\mathcal{T}_{j+1}$ of the sets

$$T \cup \{u_j\} \ \ (T \in \mathcal{T}_j, \text{ in order}), \qquad T_1^{(j)} \cup \{v_j\}, \qquad \{u_j, v_j\}, \tag{6.16}$$

where $T_1^{(j)}$ is the first member of $\mathcal{T}_j$. So $\mathcal{T}_j$ has $m_j - 1$ members, each of weight $q_j/2$ (Lemma B.6). The inequality D_j is $Q_{\mathcal{T}}$ for $m = m_j$ and the paired list $\mathcal{T}$ in which each half $T \in \mathcal{T}_j$, in order, is followed by its complement $\bar{T}$, so it has $N_j = 2(m_j - 1) + 1 = 4j+7$ parties. For each $T \in \mathcal{T}_j$ it has one party with word $\mathbf{1}_T$ and radius $|T|$ and one with word $\mathbf{1}_{\bar{T}}$ and radius $|\bar{T}|$, a last party has word $\mathbf{1}$ and radius $m_j - 1$, and the purifier has radius 1.

Theorem 6.17 (An explicit family)**.** *For every $j \ge 0$:*

- *(a) D_j is a star inequality on $N_j = 4j+7$ parties with $m_j = 2j+4$ RHS terms, and $\Pi(D_j) = \{c^{(j)}\}$ with $c^{(j)} = \omega^{(j)}/q_j$;*
- *(b) $q_j D_j = \sum_{K=1}^{m_j} \left[(q_j - \omega_K^{(j)})\,\mathrm{SA}_K^1 + \omega_K^{(j)}\,\mathrm{SA}_K^2 \right]$, so D_j is valid, indeed for all quantum states;*
- *(c) kD_j has an integer map on the full cube if and only if 2^{j+2} divides k, and $k_{\min}(D_j) = 2^{j+2} = 2^{(N_j+1)/4}$.*

The proof is in Appendix B.6. Condition (W2) holds because the list is paired, and a vector balances $\bar{T}$ exactly when it balances T. So the proof consists in showing (W1), that the balance conditions $\xi(T) = \xi(\bar{T})$ for $T \in \mathcal{T}_j$ force $\xi \in \mathbb{R}\omega^{(j)}$, which follows by induction on the levels, and then applies Theorem 6.14 with $q = q_j$. For $j = 0, 1, 2$ we have also verified (a) and (b) in exact rational arithmetic.

For $j = 0$, naming the parties with the words $\mathbf{1}_{12}$, $\mathbf{1}_{34}$, $\mathbf{1}_{13}$, $\mathbf{1}_{24}$, $\mathbf{1}_{14}$, $\mathbf{1}_{23}$, $\mathbf{1}$ (with $\mathbf{1}_{12} = \mathbf{1}_{\{1,2\}}$) as $A, \dots, G$ gives

$$D_0: \qquad 2A+2B+2C+2D+2E+2F+3G+ABCDEFG \ \ge\ ACEG+ADFG+BCFG+BDEG , \tag{6.18}$$

with the RHS terms in the order of the coins. Here $\Pi(D_0) = \{\frac{1}{4}(1,1,1,1)\}$, so every map of kD_0 takes the value $\frac{k}{4}(1,1,1,1)$ at z, and $4D_0 = \sum_{K=1}^{4}(3\ \mathrm{SA}_K^1 + \mathrm{SA}_K^2)$. Table 3 lists the certificates for D_0 and D_1 that do not use the theorem. For D_0 they confirm $k_{\min}(D_0) = 4$ directly. Single-point polytopes also arise without complementary pairs, with larger denominators at the same N.

Example 6.19 (A seven-party star with $k_{\min} = 8$)**.** The star inequality

$$\begin{aligned} Z_7: \quad & 4A + 3B + 4C + 4D + 4E + 3F + 4G + 2ABCDEFG \\ & \geq\ ABCG + ABDE + ACEF + BCEF + BEFG + CDEG + CDFG \end{aligned} \tag{6.20}$$

has no complementary pair of words. With the coordinates in the order of its RHS terms, the point $c = \frac{1}{8}(2,1,1,6,3,1,2)$ attains all eight radii, and the eight constraint rows of Lemma 6.5 have rank 7 and a linear dependence with positive coefficients. So no direction keeps all eight constraints satisfied, and $\Pi(Z_7) = \{c\}$. Theorem 6.10 gives an integer map for kZ_7 exactly when $8 \mid k$, and $k_{\min}(Z_7) = 8$. By Lemma 6.8, $8Z_7$ is a sum of the chains (6.7). The value $k_{\min}(Z_7) = 8$ is also certified independently of the theorem: kZ_7 has no integer map for each $k = 1, \ldots, 7$ (DRAT proofs), and a map for $8Z_7$ is checked exactly.

	N	m	point of $\Pi(Q)$	$k_{\min}$	certificates
D_0	7	4	$\frac{1}{4}(1,1,1,1)$	4	no map for $k \leq 3$; map for $k = 4$
D_1	11	6	$\frac{1}{8}(1,1,1,1,2,2)$	8	no map for $k \leq 2$; map for $k = 8$
D_2	15	8	$\frac{1}{16}(1,1,1,1,2,2,4,4)$	16	none beyond Theorem 6.17
Z_7	7	7	$\frac{1}{8}(2,1,1,6,3,1,2)$	8	no map for $k \leq 7$; map for $k = 8$
G_9	9	9	$\frac{1}{32}(2,25,28,31,10,8,1,19,4)$	32	no map for $k \leq 31$; map for $k = 32$
H_9	9	8	$\frac{1}{18}(1,1,2,2,2,3,3,4)$	18	no map for $k \leq 17$; map for $k = 18$
H_{11}	11	8	$\frac{1}{26}(1,2,2,3,3,4,5,6)$	26	no map for $k \leq 25$; map for $k = 26$
H_{15}	15	8	$\frac{1}{36}(1,2,3,4,5,6,7,8)$	36	no map for $k \leq 35$; map for $k = 36$
H_{17}	17	9	$\frac{1}{32}(6,4,6,5,2,3,4,1,1)$	32	no map for $k \leq 31$; map for $k = 32$

TABLE 3. Star inequalities with large least scaling. The point is the unique point of the star polytope, in the coin order of the construction for D_j, in the order of the RHS terms of (6.20) and (C.6) for Z_7 and G_9, and in the coin order of Appendix C.2 for $H_9, \ldots, H_{17}$, and $k_{\min}$ is the least common denominator of its coordinates (Theorem 6.10). "No map" means no integer map on the full cube, certified by a DRAT proof for each listed k. "Map" means an integer map on the full cube, checked exactly (Appendix A).

Table 3 also lists five star inequalities on 9 to 17 parties, written out in Appendix C.2, whose least scaling exceeds $2^{(N+1)/4}$ and is certified independently of Theorems 6.10 and 6.14. Like Z_7, the inequality G_9 has no complementary pair of words, and its polytope is a single point because all ten constraints are tight at the listed point, their rows have rank 9, and they have a linear dependence with positive coefficients. The others are of the form $Q_{\mathcal{T}}$. The lists of H_9 and H_{11} are half-regular lists of 8 and 10 weighings of 8 coins, that of H_{15} is paired with coin weights $1, 2, \ldots, 8$, and that of H_{17} is paired and comes from an 8×9 matrix of the construction of Alon and Vũ (Section 6.4).

6.4. **Superexponential least scaling of star inequalities.** Let B be an $n \times (n+1)$ matrix of rank n. In this subsection and in the proofs of Appendix B.7, the letter n denotes a matrix dimension. The kernel of B is a line $\mathbb{R}w$, and Alon and Vũ measure how unequal the entries of w are by

$$\nu(B) = \max\left\{|w_K / w_{K'}| \colon K, K' \in [n+1],\ w_{K'} \neq 0\right\}, \tag{6.21}$$

which does not depend on the choice of w [26], where it is denoted $\xi(B)$. If B has entries ± 1, each row is a weighing of $n+1$ coins, with the coins of entry -1 on one pan, and the weight vectors that balance all n weighings are the multiples of w. Alon and Vũ construct such matrices with $\nu(B)$ superexponentially large in n, and Theorem 6.14 turns them into star inequalities.

Corollary 6.22 (Superexponential least scaling of star inequalities)**.**

(i) *For every $n \times (n+1)$ matrix B with entries ± 1 and rank n there is a valid star inequality Q on $2n+1$ parties with $\nu(B) \le k_{\min}(Q) < \infty$.*

(ii) *For every $N \ge 3$ there is a valid star inequality Q_N on N parties, implied by subadditivity and not balanced, with $k_{\min}(Q_N) < \infty$ and*

$$\log_2 k_{\min}(Q_N) \;\ge\; \frac{N}{4}\log_2 N - \Big(\frac{5}{4} + o(1)\Big)N \qquad (N \to \infty)\ . \tag{6.23}$$

Part (ii) rests on a proposition of Alon and Vũ [26], which gives for every n a matrix B as in (i) with $\nu(B) \ge 2^{\frac{1}{2}n\log_2 n - n(2+o(1))}$, and through it on the main theorem of [26], which constructs $(-1,1)$ matrices of every order whose inverses have superexponentially large entries. The proof is in Appendix B.7. After the signs of some columns of B are changed and the coins of weight 0 are removed, the rows of B and their negatives form a paired list that satisfies (W1) and (W2), and $k_{\min} = q \ge \max_K \omega_K \ge \nu(B)$. Here q is the number of coins of the weighing process that Alon and Vũ build from B, in which the coin K is taken ω_K times, and $q \ge \nu(B)$ is their estimate for this number [26]. For even N one weighing is repeated, which keeps the inequality a star inequality. These inequalities have $\rho_O = 1$, and for a star inequality with $\rho_O > 0$ the point z of the proof of Theorem 6.10, which has only the bit of $S([N])$ set, lies in $\mathcal{A}(Q)$ together with 0 and the occurrence words, so the direction ($\Rightarrow$) of that proof applies on $\mathcal{A}(Q)$, $k^{\mathcal{A}}_{\min}(Q) = k_{\min}(Q)$, and (ii) holds for $k^{\mathcal{A}}_{\min}$ as well.

The least scaling of a star inequality with a nonempty polytope is also bounded from above. Let $\Delta(n)$ be the largest absolute value of the determinant of an $n \times n$ matrix with entries in $\{0,1\}$, with $\Delta(0) = 1$. We call a star inequality *complement-closed* if, for the word η of every party and of the purifier, the complementary word $\beta - \eta$ is also the word of a party or of the purifier. The inequalities $Q_{\mathcal{T}}$ with a paired list are complement-closed, since the word $\mathbf{1}$ of the party N is the complement of the purifier's word 0.

Proposition 6.24 (An upper bound for star inequalities)**.** *Let Q be a star inequality on N parties with $\Pi(Q) \ne \emptyset$. Then*

$$k_{\min}(Q) \;\le\; 2\Delta(N) \;\le\; 2(N+1)^{(N+1)/2}2^{-N}\ . \tag{6.25}$$

If Q is complement-closed, then $k_{\min}(Q) \le 2\Delta(\lfloor (N-1)/2 \rfloor)$.

The proof is in Appendix B.7. By Theorem 6.10, $k_{\min}(Q)$ is at most the common denominator of the coordinates of a vertex v of $\Pi(Q)$, and the non-integer coordinates of v solve a nonsingular linear system with coefficients ± 1 and integer right-hand sides, whose denominators the determinant bounds. The second inequality of (6.25) is Hadamard's inequality. Star inequalities have $N \ge 3$, since $2 \le |J_K| \le N-1$, and for $N \ge 3$ the right-hand side of (6.25) is below $N^{N/2}$, because its ratio to $N^{N/2}$ is $2(1+\frac{1}{N})^{N/2}(N+1)^{1/2}2^{-N} \le 2e^{1/2}(N+1)^{1/2}2^{-N} < 1$. By (6.25) and Corollary 6.22, the largest finite value of $k_{\min}$ over N-party star inequalities lies between $N^{(1/4-o(1))N}$ and $N^{(1/2+o(1))N}$. For complement-closed star inequalities, which include those of Corollary 6.22, the second bound of the proposition is $N^{(1/4+o(1))N}$, so within this class the exponent $\frac{1}{4}$ of Corollary 6.22 cannot be improved.

Adding parties that lie in no term changes neither validity [6, 19] nor the maps: a graph model for the larger set of parties is one for the smaller set once the boundary vertices of the new parties are labeled O, and the occurrence word of a new party is 0, so its pin is that of the purifier. So the supremum of the finite values of $k_{\min}$ over valid N-party inequalities does not decrease with N. By Table 3 it is at least 8 for $N = 7, 8$, at least 32 for $9 \le N \le 14$ and at least 36 for $N \ge 15$, certified at $N = 7$, 9 and 15 and derived by padding Z_7, G_9 and H_{15} for the other N, and by Corollary 6.22 it grows at least as $N^{(1/4-o(1))N}$, through star inequalities. A padded star inequality is not a star

inequality, because its term $S([N])$ no longer contains all parties, which is why Corollary 6.22 treats even N by a repeated weighing.

6.5. **Scope of the star results.** By Lemma 6.8, every star inequality with a nonempty polytope, in particular every one in Sections 6.2–6.4 and every one behind the superexponential bound of Corollary 6.22, is implied by subadditivity. So none of the values of $k_{\min}$ of star inequalities in Sections 6.2–6.4 says anything about how much scaling the facets of HEC_N need, nor do they concern balanced inequalities. For a star inequality, (2.5) gives $Q \cdot r_{iO} = \rho_i + \rho_O - |\eta^{(i)}|$. For $Q_{\mathcal{T}}$ this is $|T_i| + 1 - |T_i| = 1$ for each party $i \leq N-1$, so $Q_{\mathcal{T}}$ is not balanced, and in particular neither D_j nor the inequalities of Corollary 6.22 are, and neither are Z_7 and Q'. In fact, if a star inequality Q is balanced, then $|\eta^{(i)}| = \rho_i + \rho_O$ for every party i, so for $c \in \Pi(Q)$ the triangle inequality $|\eta^{(i)}| \leq \|c\|_1 + \|\eta^{(i)} - c\|_1 \leq \rho_O + \rho_i$ holds with equality in every coordinate, which gives $c_K = 0$ whenever $i \notin J_K$. Every J_K misses some party, so $\Pi(Q) \subseteq \{0\}$, and Theorem 6.10 gives $k_{\min}(Q) \in \{1, \infty\}$. So no balanced star inequality has $1 < k_{\min} < \infty$. The failures of Sections 3 and 4 are not star inequalities, and the star results do not cover them. Their least scalings, where we know them, are determined in Sections 3 and 6.1. More generally, none of our inequalities without an unscaled map is known to define a facet, so our results do not bear on the empirical remark of Grimaldi, Headrick, Hubeny and Shteyner that rescaling does not create maps for the fully split presentation, made in their discussion of primitive, that is facet-defining, inequalities [16]. The one facet we treat, F_7, has $k^{\mathcal{A}}_{\min}(F_7) = 1$.

7. Conclusions

The question was whether valid holographic entropy inequalities that are balanced or superbalanced have contraction proofs without rescaling. In Sections 3 and 4 it was shown that neither balance nor superbalance guarantees an unscaled map: a balanced seven-party inequality and a superbalanced six-party inequality fail at a single point of the cube, and for superbalanced eight- and seven-party inequalities the failure survives five splittings by instances of MMI. In Section 5 one failure was followed to a facet F_7 of HEC_7, which has an unscaled map on its realizable domain, but only maps that break its symmetry. In Section 6 the least scaling of star inequalities was described by the lattice points of a polytope, and coin weighings gave star inequalities whose least scaling grows faster than exponentially with N.

Multiple lines of generalization lie open. First, the least scaling of the superbalanced failures. For E_0, E_1, E_3 and E_4 it is 2, so by the additivity noted after Definition 2.10, kQ has a map for every even k. For U_6, E_2, E_5, Q_8, and the inequalities Q'_8 and Q''_8 of Section 4.3, we do not know $k_{\min}$ on the full cube. The cases we could decide are consistent with the possibility that kQ has a map exactly when k is even, which a map for an odd multiple of any of the inequalities of Section 4 would rule out. Second, whether some facet of HEC_N has no unscaled map. The inequalities closest to our failures are the summands V_7, V'_7 and P_7 of the decompositions of E_0 in Section 5. We do not know whether they are facets, and the half-sum $\frac{1}{2}(V'_7 + P_7)$ has no unscaled map although V'_7 and P_7 have one on their realizable domains (Remark 5.9). Third, whether F_7 has an unscaled map on the full cube $\{0,1\}^{20}$. Fourth, whether a valid balanced or superbalanced inequality without an unscaled map exists with $N \leq 5$. Fifth, the growth of the least scaling. For star inequalities the lower bound $N^{(1/4-o(1))N}$ of Theorem 1.1(d) and the upper bound $N^{(1/2+o(1))N}$ that follows from it leave a gap, which closes at exponent $\frac{1}{4}$ for the complement-closed star inequalities of Section 6.4, and for valid inequalities that are not star inequalities we give no upper bound. In work to be reported separately we construct valid balanced inequalities whose finite least scaling grows faster than any power of N, and valid superbalanced inequalities that have no map for any scaling below the same bound. Whether the least scaling of balanced or superbalanced inequalities can grow exponentially with N, and how large the least scaling of a facet can be, we do not know. We relegate such questions to future work.

Appendix A. The certificates

Every statement of this paper that rests on a computation is of one of four kinds: that an inequality has no integer map on a given domain (possibly with a symmetry imposed), that it has an explicit rational or integer map, that a set of explicit graph models has a given rank, or that a graph model violates a given inequality. For each kind we say what is computed, why it proves the statement that the text needs, and in which arithmetic. The certificates and the programs that check them are in the code repository, and Tables 4 and 5 list, for each statement, the domain, the certificate and the number of independent implementations that produced or checked it.

A.1. **Non-existence of maps: SAT encodings and DRAT proofs.** Fix an inequality Q, a scaling $k \geq 1$ and a domain Ω, which is the full cube $\{0,1\}^L$, the realizable domain $\mathcal{A}(Q)$ or the subdomain $\mathcal{U}(Q)$ of (2.14). Whether kQ has an integer map on Ω is a finite constraint satisfaction problem [13], which we write as a Boolean formula in conjunctive normal form (CNF). Earlier searches for contraction maps used a greedy extension algorithm [6] and deterministic rules [13], and integer programs were used to test invalidity [19]. By Lemma 2.12, the contraction condition needs to be imposed only on pairs irreducible in Ω, and a pair with $d_\alpha(u,v) \geq |\beta|$ can be dropped, because $\sum_r |n_r(u) - n_r(v)| \leq k|\beta|$ holds for any two values in the ranges. The encoders are of two types. Encoders counted as independent in Table 4 were implemented separately.

Both encodings have, for each pair u, v that is kept, auxiliary variables that mark where the values at u and v differ, and a cardinality constraint, encoded by totalizers [32], sequential counters [33] or cardinality networks [34] depending on the implementation, partly through PySAT [35], that allows at most $kd_\alpha(u,v)$ of them to be true. The *split encoding* uses Lemma 2.9, with one variable for each point $u \in \Omega$ and each of the $k|\beta|$ unit copies of the RHS terms. The *order encoding* represents each count $n_r(u) \in \{0, \dots, k\beta_r\}$ by the variables $[n_r(u) \geq s]$, $1 \leq s \leq k\beta_r$, with the clauses $[n_r(u) \geq s+1] \Rightarrow [n_r(u) \geq s]$, so that the number of s at which the variables of u and v differ is $|n_r(u) - n_r(v)|$. In both, the pins are unit clauses, and a map gives a satisfying assignment, with each auxiliary variable set to the truth value of the difference it marks, so if the CNF is unsatisfiable, no map exists. For an existence statement we never rely on the solver: a map read off a satisfying assignment is checked as in Section A.2.

We decided the formulas with the SAT solvers CaDiCaL [36] and Kissat [37]. For an unsatisfiable formula the solver writes a proof in the DRAT format, a sequence of clauses each of which can be checked by unit propagation or by the resolution-asymmetric-tautology criterion, ending with the empty clause. Every unsatisfiability claim in this paper comes with such a proof, accepted by the checker `drat-trim` [27], which does not share code with the solvers. For the star inequalities G_9, H_9, H_{11}, H_{15} and H_{17} of Table 3 the domain is the set of the $N+2$ points 0, z and $x^{(1)}, \dots, x^{(N)}$, where z is the point of the proof of Theorem 6.10 with only the bit of $S([N])$ set. Every map on the full cube restricts to this set, so a DRAT proof on it excludes maps on the full cube without using Theorem 6.10. A DRAT proof does not certify that the CNF encodes the map problem. That step rests on the encoder. Every non-existence statement was obtained from at least two independent encoders, except those for E_5, for $3Q_8$ and for the absence of an equivariant map for V_7, whose checked DRAT proofs come from one encoder (Table 4).

A.2. **Validity: exact maps.** A validity certificate for Q is a table of values $n_r(u)$ for all points u of the full cube or of $\mathcal{A}(Q)$, given as integers or as rational numbers with a common denominator. It proves validity by Lemma 2.13 (on the full cube also by the theorem of Bao et al. applied to MQ, with M the common denominator). A checker works from the list of terms of Q alone: it rebuilds the domain from Definition 2.11 (or takes the full cube) and verifies that the set of points of the certificate is exactly this domain, verifies the pins (2.7) and the ranges $0 \leq n_r(u) \leq \beta_r$, and verifies the contraction condition (2.8) on every pair of points of $\mathcal{A}(Q)$, or on every edge of the full cube. All comparisons are made in exact integer arithmetic after multiplication by the common

denominator. Some checkers first screen all pairs in floating point with a rigorous error bound and then check exactly every pair that the screen does not decide. Every validity certificate was checked by at least one program written independently of the code that produced the map.

The integer maps for $2E_0$, $2E_1$, $2E_3$ and $2E_4$ on the full cube are tables of integers, one value $n_r(u)$ for each point u of the cube and each RHS term r. The proof uses only these four tables. Each was checked as described above, in exact integer arithmetic, on all edges of the cube by two checkers independent of the program that found it, and on all pairs of points by at least one of them.

A.3. **Facets and invalidity: graph models.** A graph-model certificate is a finite graph with integer edge weights and labeled boundary vertices. The checker recomputes its entropy vector S_G by its own exact max-flow computations (with SciPy [38] and NetworkX), one minimum cut per nonempty $I \subseteq [N]$. A graph with $Q \cdot S_G = 0$ and $X \cdot S_G > 0$ shows that $Q - \varepsilon X$ is invalid for every $\varepsilon > 0$. This is how we bound the steps of the splittings in Section 4.

For a facet, the certificate is a list of $2^N - 2$ graph models with $Q \cdot S_G = 0$, and the checker computes the rank of the matrix of their entropy vectors modulo a prime p. Reduction modulo p does not increase the rank of an integer matrix, so rank $2^N - 2$ modulo p implies rank $2^N - 2$ over $\mathbb{Q}$, the vectors lying in $Q^\perp$. With the validity of Q and the full-dimensionality of HEC_N, this proves that Q defines a facet. For F_7 we have two such lists, of 126 graphs each, built independently. For each list, the entropy vectors were recomputed by two max-flow implementations independent of the code that built the graphs, and the rank was computed modulo three different primes.

A.4. **Equivariant maps.** A relabeling $g \in \mathrm{Stab}(Q)$ that fixes the purifier permutes the LHS terms, the RHS terms (preserving the coefficients) and the points of $\mathcal{A}(Q)$, and maps pins to pins. A map n is *equivariant* if $n_{g(r)}(g \cdot u) = n_r(u)$ for all g, r and u. The equivariant encoding is the order encoding with the variables of (u, r) and $(g \cdot u, g(r))$ identified and the contraction condition imposed on one pair from each orbit of pairs, which suffices because the conditions on the pairs of an orbit coincide after the identification. Every equivariant map gives a satisfying assignment, so a DRAT proof of unsatisfiability shows that no equivariant integer map exists. The equivariant non-existence statement for F_7 rests on two independent encoders, and that for V_7 rests on one.

Appendix B. Omitted proofs

This appendix proves the lemmas of Section 2 and the results on star inequalities of Section 6.

B.1. **Proofs of Lemmas 2.9 and 2.12.**

Proof of Lemma 2.9. Given f, put $n_r(u) = \sum_{s=1}^{\beta_r} f(u)_{r,s}$. The pins hold because $f(x^{(i)})_{r,s} = [i \in J_r]$ for every copy s, and $\sum_r |n_r(u) - n_r(v)| \le \sum_{r,s} |f(u)_{r,s} - f(v)_{r,s}| \le d_\alpha(u, v)$. Conversely, given an integer map n, put $f(u)_{r,s} = [s \le n_r(u)]$. The copies of J_r at u and v differ exactly for s between $\min(n_r(u), n_r(v)) + 1$ and $\max(n_r(u), n_r(v))$, so $d_1(f(u), f(v)) = \sum_r |n_r(u) - n_r(v)| \le d_\alpha(u, v)$, and the pins of n give $f(x^{(i)}) = \overline{y}^{(i)}$ and $f(0) = 0$. □

Proof of Lemma 2.12. By induction on $d_\alpha(u, v)$. If u, v is reducible through z, then $d_\alpha(u, v) = d_\alpha(u, z) + d_\alpha(z, v)$, and both terms are smaller than $d_\alpha(u, v)$ because $z \neq u, v$ and all $\alpha_l > 0$. By induction and the triangle inequality, $\sum_r |n_r(u) - n_r(v)| \le d_\alpha(u, z) + d_\alpha(z, v) = d_\alpha(u, v)$. □

B.2. **Proof of Lemma 2.13.** The words of the graph-model proof lie in $\mathcal{A}(Q)$ because of the following properties of minimal min cuts, as stated by Avis and Hernández-Cuenca [19]. We include the short proof for completeness.

Lemma B.1 (Minimal min cuts)**.** *Fix a graph model and nonempty $I, I' \subseteq [N]$. (a) The intersection of all min cuts of I is a min cut W_I of I, the* minimal min cut*. (b) If $I \subseteq I'$, then $W_I \subseteq W_{I'}$. (c) If $I \cap I' = \emptyset$, then $W_I \cap W_{I'} = \emptyset$.*

statement	domain (points)	certificate	independent checks
Balanced inequalities (Section 3)			
B_7^* valid	—	identity with 43 instances of SA and MMI	exact, 2 programs
$2B_7^*$ has a map	full cube (256)	integer map	2 checkers
B_7^* has no map	full cube (256)	CNF and DRAT proof	3 encoders
B_7 valid; $2B_7$ has a map	— ; full cube (256)	identity with 46 instances; integer map	exact check
B_7 has no map	full cube (256)	CNF and DRAT proof	2 encoders
Q_8^A has no map; $2Q_8^A$ has one	full cube (128)	CNF and DRAT proof; integer map	3 encoders; all-pairs check
Superbalanced inequalities (Section 4)			
U_6 valid	—	identity with instances of MMI	exact, 2 programs
U_6 has no map	$\mathcal{A}$ (580)	CNF and DRAT proofs	2 encoders, and the hand proof
$2U_6$ has a map	$\mathcal{A}$ (580)	integer map	2 checkers
Q_8 valid	full cube (16,384)	rational map	2 checkers
Q_8 has no map	$\mathcal{U}$ (135)	CNF and DRAT proof	3 encoders
Q_8' valid	full cube (16,384)	rational map	3 checkers
Q_8' has no map	$\mathcal{U}$ (135)	CNF and DRAT proof	2 encoders
Q_8'' valid	full cube (16,384)	rational map	2 checkers
Q_8'' has no map	$\mathcal{U}$ (135)	CNF and DRAT proof	3 encoders
E_0, E_1 have no map	$\mathcal{U}$ (134)	CNF and DRAT proofs	3 encoders each
E_1 valid	$\mathcal{A}$ (2,187)	rational map	2 checkers
E_2, E_3, E_4 valid	$\mathcal{A}$ (2,916 each)	rational maps	2 checkers each
E_2, E_3, E_4 have no map	$\mathcal{U}$ (134)	CNF and DRAT proofs	2 encoders each
E_5 valid	$\mathcal{A}$ (6,912)	rational map	2 checkers
E_5 has no map	$\mathcal{U}$ (134)	CNF and DRAT proof	1 encoder
bounds on the steps $Q_8 \to Q_8'$, $E_0 \to E_1$, $E_1 \to E_2$, $E_4 \to E_5$	—	37 graph models for the first; for the others graphs with 138, 168 and 10 vertices, the last a lift of an extreme ray of HEC_5 [7]	entropy vectors by 2 max-flow codes for the first, ≥ 3 for the others
The facet F_7 (Section 5)			
F_7 valid	$\mathcal{A}$ (11,046)	two rational maps, for F_7 and $2F_7$	4 checkers each
F_7 defines a facet	—	two lists of 126 tight graph models; rank modulo primes	2 max-flow codes, 3 primes
F_7 has a map	$\mathcal{A}$ (11,046)	integer map	2 checkers
F_7 has no equivariant map	$\mathcal{A}$ (11,046)	CNFs and DRAT proofs	2 encoders
V_7, V_7' valid; each has a map	$\mathcal{A}$ (10,032 each)	rational maps; integer maps	2 checkers each
V_7 has no equivariant map	$\mathcal{A}$ (10,032)	CNF and DRAT proof	1 encoder
X_7 valid	$\mathcal{A}$ (6,912)	rational map	2 checkers
decomposition of $24E_0$	—	integer identity	2 programs
P_7 valid; has a map	$\mathcal{A}$ (6,912)	integer map	3 checkers
identities (5.10)	—	integer identities	3 programs
$\frac{1}{2}(V_7' + P_7)$ has no map	$\mathcal{U}$ (134)	CNF and DRAT proofs	2 encoders

TABLE 4. The certificates. "Map" means an integer map for the inequality as stated. Domains: the full cube, $\mathcal{A}$ and $\mathcal{U}$, with their numbers of points. The last column counts encoders or checkers that are separate programs. For every map at least one checker shares no code with the program that found it, and every DRAT proof is checked by `drat-trim`. The validity of Y_7 is derived in the text from (5.5). The certificates of Section 6 are listed in Table 5.

Proof. The cut weight is submodular, $C(X \cap Y) + C(X \cup Y) \leq C(X) + C(Y)$ for $X, Y \subseteq V$: an edge cut by both $X \cap Y$ and $X \cup Y$ has one endpoint in $X \cap Y$ and the other outside $X \cup Y$, so it is cut by X and by Y, and an edge cut by one of $X \cap Y$, $X \cup Y$ is cut by X or by Y.

statement	domain (points)	certificate	independent checks
$3Q_8$ has no map	$\mathcal{U}$ (135)	CNF and DRAT proof	1 encoder
$2E_s$ has a map (E_s valid), $s = 0, 1, 3, 4$	full cube (16,384, 32,768)	integer maps	2 checkers each
$2E_4$ has a map	$\mathcal{A}$ (2,916)	integer map, equivariant	3 checkers
Q': no map for $k = 1, 3$; maps for $k = 2, 4$	full cube (16)	CNF and DRAT proofs; integer maps	3 encoders and a hand proof; 2 checkers
D_0: no map for $k \leq 3$; map for $k = 4$	full cube (256)	CNF and DRAT proofs; integer map	3 encoders and the proof in Section 6; 2 checkers
D_1: no map for $k \leq 2$; map for $k = 8$	full cube (4,096)	CNF and DRAT proofs; integer map	3 encoders and the proof in Section 6; 2 checkers
Z_7: no map for $k \leq 7$; map for $k = 8$	full cube (256)	CNF and DRAT proofs; integer map	3 encoders and an exact lattice-point computation; 2 checkers
G_9: no map for $k \leq 31$; map for $k = 32$	0, z, $x^{(i)}$ (11); full cube (1,024)	CNF and DRAT proofs; integer map	2 encoders; 3 checkers
H_9: no map for $k \leq 17$; map for $k = 18$	0, z, $x^{(i)}$ (11); full cube (1,024)	CNF and DRAT proofs; integer map	2 encoders; 3 checkers
H_{11}: no map for $k \leq 25$; map for $k = 26$	0, z, $x^{(i)}$ (13); full cube (4,096)	CNF and DRAT proofs; integer map	2 encoders; 2 checkers
H_{15}: no map for $k \leq 35$; map for $k = 36$	0, z, $x^{(i)}$ (17); full cube (65,536)	CNF and DRAT proofs; integer map	2 encoders; 2 checkers
H_{17}: no map for $k \leq 31$; map for $k = 32$	0, z, $x^{(i)}$ (19); full cube (262,144)	CNF and DRAT proofs; integer map	2 encoders; 2 checkers

TABLE 5. The certificates of Section 6, in the format of Table 4. The domain 0, z, $x^{(i)}$ is the set of $N + 2$ points of Section A.1. The validity of the star inequalities is derived in the text from identities with instances of subadditivity.

(a) If W, W' are min cuts of I, then $W \cap W'$ and $W \cup W'$ are I-cuts, so $2S(I) \leq C(W \cap W') + C(W \cup W') \leq C(W) + C(W') = 2S(I)$, and $W \cap W'$ is a min cut. There are finitely many min cuts, so their intersection is a min cut.

(b) The set $W_I \cap W_{I'}$ is an I-cut and $W_I \cup W_{I'}$ an I'-cut, because $b^{-1}(I) \cap b^{-1}(I') = b^{-1}(I)$ and $b^{-1}(I) \cup b^{-1}(I') = b^{-1}(I')$. Submodularity of the cut function [39] gives $C(W_I \cap W_{I'}) + C(W_I \cup W_{I'}) \leq S(I) + S(I')$, so $W_I \cap W_{I'}$ is a min cut of I. By (a) it contains W_I, that is, $W_I \subseteq W_{I'}$.

(c) Let $X = W_I \cap W_{I'}$. It contains no boundary vertex, since $b^{-1}(I) \cap b^{-1}(I') = \emptyset$. Suppose $X \neq \emptyset$, and let a_1, a_2 and a_3 be the total weights of the edges from X to $W_I \setminus W_{I'}$, to $W_{I'} \setminus W_I$ and to $V \setminus (W_I \cup W_{I'})$. The set $W_I \setminus X$ is an I-cut, and removing X from W_I uncuts the edges from X to the outside of W_I and cuts those from X to $W_I \setminus W_{I'}$, so $C(W_I \setminus X) = S(I) + a_1 - a_2 - a_3$. A proper subset of W_I is not a min cut of I by (a), hence $a_1 - a_2 - a_3 > 0$. In the same way $a_2 - a_1 - a_3 > 0$. Adding the two gives $a_3 < 0$, which is impossible. Minimality is needed in (c): maximal min cuts of disjoint regions can share a bulk vertex. □

Proof of Lemma 2.13. It suffices to prove $Q \cdot S \geq 0$ for a fixed graph model. Let $W_l = W_{I_l}$ be the minimal min cut of the l-th LHS term, and give each vertex v the word $x(v) \in \{0, 1\}^L$ with $x(v)_l = [v \in W_l]$.

Step 1: the words lie in $\mathcal{A}(Q)$, and boundary vertices carry the occurrence words. If $x(v)_l = x(v)_{l'} = 1$, then $W_l \cap W_{l'} \neq \emptyset$, so $I_l \cap I_{l'} \neq \emptyset$ by Lemma B.1(c): this is (R1). If $x(v)_l = 1$ and $I_l \subseteq I_{l'}$, then $v \in W_l \subseteq W_{l'}$ by Lemma B.1(b), so $x(v)_{l'} = 1$: this is (R2). A boundary vertex of party i lies in an I_l-cut exactly when $i \in I_l$, so its word is $x^{(i)}$, and a boundary vertex of O lies in no I_l-cut, so its word is 0. Put $n_r(v) = n_r(x(v))$ for every vertex v.

Step 2: layer cake. Fix r and $t \in [0, \beta_r)$, and let $U_r(t) = \{v \in V : n_r(v) > t\}$. By the pins, a boundary vertex of a party i has $n_r(v) = \beta_r[i \in J_r]$, which exceeds t exactly when $i \in J_r$, and a boundary vertex of O has $n_r(v) = 0 \le t$. So $U_r(t)$ is a J_r-cut and $C(U_r(t)) \ge S(J_r)$. An edge e with endpoints u_e, v_e is cut by $U_r(t)$ exactly when t lies in the interval $[\min(n_r(u_e), n_r(v_e)), \max(n_r(u_e), n_r(v_e)))$, which is contained in $[0, \beta_r]$ and has length $|n_r(u_e) - n_r(v_e)|$. The function $t \mapsto C(U_r(t))$ takes finitely many values, and integrating it over $[0, \beta_r)$ gives

$$\beta_r S(J_r) \le \int_0^{\beta_r} C\big(U_r(t)\big)dt = \sum_{e\in E} w_e \big|n_r(u_e) - n_r(v_e)\big| \ . \tag{B.2}$$

Step 3: summing. By Step 1 the words of both endpoints of every edge lie in $\mathcal{A}(Q)$, so the contraction condition applies to them. Summing (B.2) over r,

$$\begin{aligned}\sum_{r=1}^{R} \beta_r S(J_r) &\le \sum_{e\in E} w_e \sum_{r=1}^{R} \big|n_r(u_e) - n_r(v_e)\big| \le \sum_{e\in E} w_e d_\alpha\big(x(u_e), x(v_e)\big) \\ &= \sum_{l=1}^{L} \alpha_l \sum_{e\in E} w_e \big|x(u_e)_l - x(v_e)_l\big| \ .\end{aligned}$$

The inner sum on the right is the weight of the edges cut by W_l, which is $S(I_l)$. Hence $\sum_r \beta_r S(J_r) \le \sum_l \alpha_l S(I_l)$. □

B.3. **Star inequalities.** This part of the appendix proves Theorems 6.10, 6.14 and 6.17, Corollary 6.22, Proposition 6.24 and Lemmas 6.5 and 6.8. Notation is that of Section 6: Q is a star inequality (6.3) with RHS terms $J_1, \dots, J_m$, coefficients β_K, radii ρ_i, words $\eta^{(i)}$ and polytope $\Pi(Q)$.

B.4. **Proof of Theorem 6.10.** *The cube.* The LHS terms of Q are the singletons $\{i\}$ with $\rho_i > 0$ and the set $[N]$ if $\rho_O > 0$. It is convenient to use one bit for every singleton and one bit, labeled $\star$, for $[N]$, also when the corresponding radius is 0. So let $\mathcal{B} = \{0,1\}^{[N]\cup\{\star\}}$, give bit i the weight ρ_i and bit $\star$ the weight ρ_O, and let $d(x, x') = \sum_{i\in[N]} \rho_i |x_i - x'_i| + \rho_O |x_\star - x'_\star|$. A party i lies in the LHS terms $\{i\}$ and $[N]$ and in no other, so its occurrence word is $e_i + e_\star$, and the purifier's is 0. We claim that, for every $k \ge 1$, kQ has an integer map on the full cube in the sense of Definition 2.6 if and only if there is a map $n\colon \mathcal{B} \to \prod_{K=1}^{m} \{0, 1, \dots, k\beta_K\}$ with

$$n(e_i + e_\star) = k\eta^{(i)} \ \ (i \in [N]) \ , \qquad n(0) = 0 \ , \qquad \|n(x) - n(x')\|_1 \le k\, d(x, x') \ \ (x, x' \in \mathcal{B}) \ . \tag{B.3}$$

If all radii are positive, $\mathcal{B}$ is the full cube of Q and (B.3) is Definition 2.6 for kQ. Otherwise let π delete the bits of weight 0. It maps $\mathcal{B}$ onto the full cube of Q, maps $e_i + e_\star$ to the occurrence word of i and 0 to 0, and satisfies $d(x, x') = d_\alpha(\pi x, \pi x')$. So an integer map n' for kQ gives the map $n = n' \circ \pi$ satisfying (B.3). Conversely, if n satisfies (B.3) and bit l has weight 0, then $\|n(x) - n(x \oplus e_l)\|_1 \le 0$, so n does not depend on the bits of weight 0 and equals $n' \circ \pi$ for a map n' on the full cube of Q, which has the pins and the contraction condition of an integer map for kQ.

Edges suffice. If x and x' differ in the bits $l_1, \dots, l_s$, the walk $x = x_0, x_1, \dots, x_s = x'$ that flips these bits one at a time has $d(x, x') = \sum_t d(x_{t-1}, x_t)$. By the triangle inequality for $\|\cdot\|_1$, the contraction condition in (B.3) holds for all pairs once it holds for all pairs x, $x \oplus e_l$, that is, once

$$\|n(x) - n(x \oplus e_l)\|_1 \le k\rho_l \qquad \text{for all } x \in \mathcal{B} \text{ and } l \in [N] \cup \{\star\} \ , \tag{B.4}$$

where $\rho_\star = \rho_O$.

(⇒) Let n satisfy (B.3), and let $z = e_\star$. The point z differs from $e_i + e_\star$ only in bit i, and from 0 only in bit $\star$. So $d(z, e_i + e_\star) = \rho_i$ for $i \in [N]$ and $d(z, 0) = \rho_O$, and the contraction condition at these $N + 1$ pairs gives

$$\|n(z) - k\eta^{(i)}\|_1 \le k\rho_i \quad (i \in [N]) \ , \qquad \|n(z)\|_1 \le k\rho_O \ .$$

Dividing by k, the point $c = n(z)/k$ lies in $\prod_K [0, \beta_K]$, satisfies all constraints of (6.4), and has $kc = n(z) \in \mathbb{Z}^m$. So $c \in \Pi(Q) \cap \frac{1}{k}\mathbb{Z}^m$.

($\Leftarrow$) Let $c \in \Pi(Q)$ with $kc \in \mathbb{Z}^m$. Then $a_K = k(\beta_K - c_K)$ and $b_K = kc_K$ are nonnegative integers with $a_K + b_K = k\beta_K$. For each K define two functions $\mathcal{B} \to \{0,1\}$,

$$\phi^1_K(x) = [x_i = 1 \text{ for some } i \in J_K] \ , \qquad \phi^2_K(x) = [x_\star = 1 \text{ and } x_i = 0 \text{ for all } i \in [N] \setminus J_K] \ ,$$

and put

$$n_K(x) = a_K\, \phi^1_K(x) + b_K\, \phi^2_K(x) \ . \tag{B.5}$$

Then $n_K(x) \in \{0, a_K, b_K, k\beta_K\} \subseteq \{0, \dots, k\beta_K\}$. *Pins.* At $x = e_i + e_\star$, the only singleton bit that is set is i, so $\phi^1_K = [i \in J_K]$, and $\phi^2_K = [i \notin [N] \setminus J_K] = [i \in J_K]$. Hence $n_K(e_i + e_\star) = k\beta_K[i \in J_K] = k\eta^{(i)}_K$. At $x = 0$ both functions vanish, so $n(0) = 0$. *Edges.* The function ϕ^1_K depends only on the bits $i \in J_K$, and ϕ^2_K only on the bits $i \in [N] \setminus J_K$ and $\star$. Flipping one bit changes each function by at most 1. So flipping a bit $i \in [N]$ changes n_K by at most a_K if $i \in J_K$ and by at most b_K if $i \notin J_K$, and

$$\|n(x) - n(x \oplus e_i)\|_1 \le \sum_{K\colon i \in J_K} k(\beta_K - c_K) + \sum_{K\colon i \notin J_K} kc_K = k \sum_K \left|\eta^{(i)}_K - c_K\right| = k\|c - \eta^{(i)}\|_1 \le k\rho_i \ .$$

Flipping $\star$ changes only the ϕ^2_K, so $\|n(x) - n(x \oplus e_\star)\|_1 \le \sum_K kc_K = k\|c - \eta^{(O)}\|_1 \le k\rho_O$. This is (B.4), so n satisfies (B.3), and kQ has an integer map on the full cube.

In the language of Lemma 2.9, ϕ^1_K and ϕ^2_K are the maps that prove SA^1_K and SA^2_K for one copy of J_K, and (B.5) uses them on a_K and b_K copies, the multiplicities of the decomposition (6.9) of kQ.

Consequences. If $\Pi(Q) \neq \emptyset$, then Q is valid by Lemma 6.8. By Lemma 6.5, $\Pi(Q)$ is a nonempty bounded polyhedron defined by linear inequalities with integer coefficients and integer right-hand sides, so it has a vertex. A vertex is the unique solution of a system of linear equations with integer coefficients, so it is rational, and it lies in $\frac{1}{k}\mathbb{Z}^m$ for k the common denominator of its coordinates. Hence the minimum in (6.11) exists, and by the equivalence it equals $k_{\min}(Q)$. If $\Pi(Q) = \emptyset$, the direction ($\Rightarrow$) shows that no kQ has an integer map. □

B.5. **Proof of Theorem 6.14.** We use the notation of Section 6.3. For $\xi \in \mathbb{R}^m$ and $T \subseteq [m]$ we have $(\mathbf{1} - 2 \cdot \mathbf{1}_T) \cdot \xi = \xi(\bar{T}) - \xi(T)$, so ξ balances T exactly when this number vanishes.

(a), $Q_\mathcal{T}$ is a star inequality. The vector ω balances every T_i by (W1), and its entries are positive, so $\omega(T_i) = \omega(\bar{T_i}) = q/2 > 0$, and $T_i \neq \emptyset$, $T_i \neq [m]$. Hence the radii $|T_i|$ and $m - 1 \ge 1$ are positive integers. The set J_K consists of the party N and the parties $i \le N - 1$ with $K \in T_i$. Coordinate K of the identity in (W2) reads

$$\sum_{i\colon K \notin T_i} \gamma_i = \sum_{i\colon K \in T_i} \gamma_i \ .$$

If K lay in no T_i, the right-hand side would vanish while the left-hand side is positive. So K lies in some T_i, and $|J_K| \ge 2$. If K lay in every T_i, the left-hand side would vanish. So some party $i \le N - 1$ is not in J_K, and $|J_K| \le N - 1$. If $J_K = J_{K'}$ for coins $K \neq K'$, then K and K' lie in the same sets T_i, so the vector $\mathbf{1}_{\{K\}} - \mathbf{1}_{\{K'\}}$ balances every T_i, and by (W1) it is a multiple of ω, which is impossible because ω has positive entries. So the J_K are distinct, and $Q_\mathcal{T}$ is a star inequality on N parties with m RHS terms.

(a), all radii are attained at $c = \omega/q$. Since $m \ge 2$ and the entries of ω are positive, $c \in (0,1)^m$, and $\|c\|_1 = 1$ and $c(T_i) = c(\bar{T_i}) = \frac{1}{2}$. By (6.13), $\|c - \mathbf{1}_{T_i}\|_1 = 1 + |T_i| - 1 = |T_i|$, $\|c - \mathbf{1}\|_1 = 1 + m - 2 = m - 1$ and $\|c - 0\|_1 = 1$. So $c \in \Pi(Q_\mathcal{T})$, and every constraint holds with equality.

(a), $\Pi(Q_\mathcal{T}) = \{c\}$. Let $\xi \in \Pi(Q_\mathcal{T})$. The words 0 of the purifier and $\mathbf{1}$ of the party N are complementary, with radii adding up to $m = |\beta|$, so $\|\xi\|_1 = 1$ by Lemma 6.5. By (6.13) the

constraint of a party $i \le N-1$ reads $\|\xi\|_1 + |T_i| - 2\xi(T_i) \le |T_i|$, that is, $\xi(\bar{T}_i) - \xi(T_i) \le 0$. The scalar product of the identity in (W2) with ξ gives

$$\sum_{i=1}^{N-1} \gamma_i \big(\xi(\bar{T}_i) - \xi(T_i)\big) = 0 \ ,$$

a sum of nonpositive terms with positive coefficients, so every term vanishes and ξ balances every T_i. By (W1), $\xi = t\,\omega$ for some $t \in \mathbb{R}$, and $\|\xi\|_1 = 1$ with $\xi \ge 0$ gives $t = 1/q$, so $\xi = c$.

(b). Lemma 6.8 at c gives (6.9) with all positivity coefficients equal to 0, because every radius is attained at c. Multiplying by q and using $qc_K = \omega_K$ and $\beta_K = 1$ gives the identity of (b), a combination with nonnegative integer coefficients of the inequalities (6.7), which are valid for all quantum states.

(c). By Theorem 6.10 and (a), $kQ_{\mathcal{T}}$ has an integer map on the full cube if and only if $kc \in \mathbb{Z}^m$, that is, q divides $k\omega_K$ for every K. Since the ω_K have no common factor, some integer combination of them equals 1, so this holds exactly when q divides k. Hence $k_{\min}(Q_{\mathcal{T}}) = q$. □

B.6. **Proof of Theorem 6.17.** We use the weights (6.15), the halves (6.16) and the parties of D_j as defined in Section 6.3, and write $\omega(T) = \sum_{K \in T} \omega_K$.

Lemma B.6. *For every $j \ge 0$, $q_j = 2^{j+2}$, $\omega_1^{(j)} = 1$, the list $\mathcal{T}_j$ has $m_j - 1$ members, and every $T \in \mathcal{T}_j$ has $\omega^{(j)}(T) = q_j/2$.*

Proof. By induction on j. For $j = 0$, $q_0 = 4$, $\omega_1^{(0)} = 1$, $\mathcal{T}_0$ has $3 = m_0 - 1$ members, and each has two coins of weight 1. Assume the statement for j. The total weight at level $j+1$ is $q_{j+1} = q_j + q_j/2 + q_j/2 = 2q_j = 2^{j+3}$, and coin 1 keeps the weight 1. The list $\mathcal{T}_{j+1}$ has $(m_j - 1) + 2 = m_{j+1} - 1$ members. A lifted half $T \cup \{u_j\}$ and the half $T_1^{(j)} \cup \{v_j\}$ weigh $q_j/2 + q_j/2 = q_j = q_{j+1}/2$, and $\{u_j, v_j\}$ weighs $q_j = q_{j+1}/2$. □

The inequality D_j is $Q_{\mathcal{T}}$ for the paired list $\mathcal{T}$ of the halves $T \in \mathcal{T}_j$ and their complements, with $m = m_j \ge 4$ coins and $N = 2(m_j - 1) + 1 = N_j \ge 7$ parties. Condition (W2) holds with all $\gamma_i = 1$, because $(\mathbf{1} - 2 \cdot \mathbf{1}_T) + (\mathbf{1} - 2 \cdot \mathbf{1}_{\bar{T}}) = 0$. A vector balances $\bar{T}$ exactly when it balances T, so (W1) says that the solutions $\xi \in \mathbb{R}^{m_j}$ of

$$\xi(T) = \xi(\bar{T}) \qquad \text{for every } T \in \mathcal{T}_j \tag{B.7}$$

form the line $\mathbb{R}\omega^{(j)}$, and the entries of $\omega^{(j)}$ are positive and have no common factor because $\omega_1^{(j)} = 1$. Let $\Lambda_j \subseteq \mathbb{R}^{m_j}$ be the space of solutions of (B.7). We show by induction on j that $\Lambda_j = \mathbb{R}\omega^{(j)}$. The vector $\omega^{(j)}$ lies in Λ_j by Lemma B.6, since $\omega^{(j)}(T) = q_j/2 = \omega^{(j)}(\bar{T})$. For $j = 0$ the equations are

$$\xi_1 + \xi_2 = \xi_3 + \xi_4 \ , \qquad \xi_1 + \xi_3 = \xi_2 + \xi_4 \ , \qquad \xi_1 + \xi_4 = \xi_2 + \xi_3 \ .$$

Adding the first two gives $2\xi_1 = 2\xi_4$, adding the first and the third gives $2\xi_1 = 2\xi_3$, and adding the last two gives $2\xi_1 = 2\xi_2$. So $\xi = \xi_1(1,1,1,1)$. Now assume $\Lambda_j = \mathbb{R}\omega^{(j)}$ and let $\xi = (\xi', \xi_u, \xi_v) \in \Lambda_{j+1}$, with $\xi' \in \mathbb{R}^{m_j}$ the old coordinates and ξ_u, ξ_v those of u_j, v_j. For $T \subseteq [m_j]$ write $\bar{T}' = [m_j] \setminus T$ for its complement among the old coins, and put $T_1 = T_1^{(j)}$. The complement of $T \cup \{u_j\}$ in $[m_{j+1}]$ is $\bar{T}' \cup \{v_j\}$, and that of $T_1 \cup \{v_j\}$ is $\bar{T}_1' \cup \{u_j\}$. So the equations for $T_1 \cup \{u_j\}$ and $T_1 \cup \{v_j\}$ read

$$\xi'(T_1) + \xi_u = \xi'(\bar{T}_1') + \xi_v \ , \qquad \xi'(T_1) + \xi_v = \xi'(\bar{T}_1') + \xi_u \ .$$

Subtracting the second from the first gives $2\xi_u = 2\xi_v$, so $\xi_u = \xi_v$. Then the equation for $T \cup \{u_j\}$ becomes $\xi'(T) = \xi'(\bar{T}')$ for every $T \in \mathcal{T}_j$, so $\xi' \in \Lambda_j$ and $\xi' = t\,\omega^{(j)}$ for some $t \in \mathbb{R}$. Finally, the complement of $\{u_j, v_j\}$ is the set of all old coins, and its equation reads $\xi_u + \xi_v = \xi'([m_j]) = t\,q_j$, so $\xi_u = \xi_v = t\,q_j/2$ and $\xi = t\,(\omega^{(j)}, q_j/2, q_j/2) = t\,\omega^{(j+1)}$. Hence $\Lambda_{j+1} = \mathbb{R}\omega^{(j+1)}$. So (W1) holds with $\omega = \omega^{(j)}$, and $q = q_j = 2^{j+2}$ by Lemma B.6. Theorem 6.14 now gives (a), (b) and (c), and $2^{j+2} = 2^{(N_j+1)/4}$ because $N_j + 1 = 4j + 8$. □

B.7. **Proofs of Corollary 6.22 and Proposition 6.24.**

Proof of Corollary 6.22. (i) Let w span the kernel of B. Changing the sign of a column K of B and of w_K keeps $Bw = 0$, the entries ±1, the rank and $\nu(B)$, so we may assume $w \geq 0$. Let $F = \{K\colon w_K > 0\}$. If $F = \{K\}$, then $Bw = w_K \cdot (\text{column } K) = 0$, which is impossible for a column with entries ±1, so $|F| \geq 2$. Let B_F be the $n \times |F|$ matrix of the columns of B in F. A solution y of $B_F y = 0$ extends by zeros to a solution of $By = 0$, which is a multiple of w, so the solutions of $B_F y = 0$ form the line $\mathbb{R}w_F$. Number the coins of F by $[m]$, $m = |F|$, and let ω be the vector with positive integer entries without a common factor on the ray of w_F. For each row b of B_F let $T_b = \{K\colon b_K = -1\}$, so that $b = \mathbf{1} - 2 \cdot \mathbf{1}_{T_b}$ and $b \cdot \xi = \xi(\bar{T}_b) - \xi(T_b)$. The list of the n sets T_b, each followed by its complement, has $N - 1 = 2n$ members. A vector balances T_b and $\bar{T}_b$ exactly when $b \cdot \xi = 0$, so (W1) holds, and (W2) holds with all $\gamma_i = 1$ because the list is paired. Theorem 6.14 gives a valid star inequality on $2n+1$ parties with $k_{\min} = q = \sum_K \omega_K$. Since $\nu(B) = \max_K \omega_K / \min_K \omega_K$ and $\min_K \omega_K \geq 1$, we get $k_{\min} \geq \max_K \omega_K \geq \nu(B)$.

(ii) Let $N = 2n + 1$ be odd, $n \geq 1$. By a proposition of Alon and Vũ [26] there is an $n \times (n+1)$ matrix B with entries ±1 and rank n with $\log_2 \nu(B) \geq \frac{1}{2} n \log_2 n - n(2 + o(1))$, and (i) gives Q_N with $\log_2 k_{\min}(Q_N) \geq \log_2 \nu(B)$. With $n = (N-1)/2$ we have $\frac{1}{2} n \log_2 n = \frac{N}{4} \log_2 N - \frac{N}{4} + O(\log N)$ and $n(2 + o(1)) = (1 + o(1))N$, which gives (6.23). Let $N = 2n + 2$ be even. Add to the list of the case $2n + 1$ a second copy of its first set T_1. The balance conditions are unchanged, so (W1) still holds, and (W2) holds with $\gamma = \frac{1}{2}$ on the two copies of T_1 and $\gamma = 1$ on the other sets. Theorem 6.14 gives a star inequality Q_N on N parties with the same q, and the same estimate with $n = (N-2)/2$ gives (6.23). □

Proof of Proposition 6.24. By Lemma 6.5, for each $i \in [N] \cup \{O\}$ the constraint $\|c - \eta^{(i)}\|_1 \leq \rho_i$ reads $s^{(i)} \cdot c \leq b_i$ on the box, with $s^{(i)}_K = 1$ if $\eta^{(i)}_K = 0$, $s^{(i)}_K = -1$ if $\eta^{(i)}_K = \beta_K$, and $b_i = \rho_i - \sum_{K\colon \eta^{(i)}_K = \beta_K} \beta_K \in \mathbb{Z}$. As in the proof of Theorem 6.10, $\Pi(Q)$ has a vertex v, and $k_{\min}(Q)$ is at most the least common denominator of the coordinates of v. Let $F = \{K\colon 0 < v_K < \beta_K\}$ and $t = |F|$. The coordinates of v outside F are 0 or β_K, so they are integers. Since v is a vertex, the constraints that hold with equality at v have rank m. The box constraints among them involve only coordinates outside F, so the equalities $s^{(i)} \cdot v = b_i$ restricted to the coordinates in F have rank t, and t of them determine v_F: $Mv_F = b'$, with M a nonsingular $t \times t$ matrix with entries ±1 formed by the restrictions of t of the vectors $s^{(i)}$, and $b' \in \mathbb{Z}^t$. In particular $t \leq N + 1$. If $t \leq 1$, then v is integral and $k_{\min}(Q) = 1$. Let $t \geq 2$.

For a square matrix of order $\ell \geq 1$ with entries ±1, subtracting the first row from the others gives rows with entries in $\{0, \pm2\}$, so its determinant is divisible by $2^{\ell-1}$. By Cramer's rule $v_F = \operatorname{adj}(M) b' / \det M$, and the entries of $\operatorname{adj}(M)$ are minors of M of order $t-1$, divisible by 2^{t-2}. So the least common denominator of v divides $|\det M| / 2^{t-2}$. Changing the signs of some rows and columns of M does not change $|\det M|$ and makes its first row and first column consist of 1s. Subtracting the first row from the others then leaves the first column $(1, 0, \ldots, 0)$ and the block $-2B'$ in the other rows and columns, with B' a $(t-1) \times (t-1)$ matrix with entries in $\{0, 1\}$, so $|\det M| = 2^{t-1} |\det B'|$. Hence

$$k_{\min}(Q) \;\leq\; \frac{|\det M|}{2^{t-2}} \;=\; 2|\det B'| \;\leq\; 2\Delta(t-1) \;\leq\; 2\Delta(N)\,,$$

since Δ does not decrease, as bordering a matrix by a row and a column that are zero except for a diagonal 1 keeps its determinant. For the second inequality of (6.25), let B'' be an $n \times n$ matrix with entries in $\{0, 1\}$, and let M'' be the matrix of order $n+1$ whose first row and first column consist of 1s and whose remaining block is $\mathbf{1}\mathbf{1}^{\mathsf{T}} - 2B''$. Its entries are ±1, and subtracting the first row from the others gives $|\det M''| = 2^n |\det B''|$, while Hadamard's inequality bounds

$|\det M''|$ by the product of the Euclidean lengths of the rows, $(n+1)^{(n+1)/2}$ (see e.g. [26]). So $\Delta(n) \le (n+1)^{(n+1)/2} 2^{-n}$.

If Q is complement-closed, then $s^{(i)} = -s^{(i')}$ whenever $\eta^{(i')} = \beta - \eta^{(i)}$, and no word is its own complement because $\beta_K \ge 1$. The at most $N+1$ distinct words therefore fall into at most $\lfloor (N+1)/2 \rfloor$ complementary pairs, and two words of the same pair, or two equal words, give rows of M that are equal or opposite, which a nonsingular M does not contain. So $t \le \lfloor (N+1)/2 \rfloor$, and the bound above gives $k_{\min}(Q) \le 2\Delta(t-1) \le 2\Delta(\lfloor (N-1)/2 \rfloor)$. □

B.8. **The remaining proofs.**

Proof of Lemma 6.5. On the box $|c_K - 0| = c_K$ and $|c_K - \beta_K| = \beta_K - c_K$, which gives (6.6). Each coordinate contributes c_K to one of the two distances and $\beta_K - c_K$ to the other. Two numbers bounded by two radii whose sum equals the sum of the radii equal the radii. □

Proof of Lemma 6.8. Compare coefficients on the two sides of (6.9). The term $S(J_K)$ has coefficient $-(\beta_K - c_K) - c_K = -\beta_K$ on the right. The singleton $S(i)$ appears in SA^1_K for $i \in J_K$ and in SA^2_K for $i \notin J_K$, so the chains give it the coefficient $\sum_{K:\, i \in J_K} (\beta_K - c_K) + \sum_{K:\, i \notin J_K} c_K = \|c - \eta^{(i)}\|_1$, and the positivity term brings it to ρ_i. The term $S([N])$ appears only in the SA^2_K, with total coefficient $\sum_K c_K = \|c - \eta^{(O)}\|_1$, and the last term brings it to ρ_O. No other term appears. The coefficients are nonnegative because $0 \le c_K \le \beta_K$ and $c \in \Pi(Q)$. If $kc \in \mathbb{Z}^m$, all coefficients of kQ in (6.9) are nonnegative integers, and each chain is a sum of instances of subadditivity. □

Appendix C. The inequalities written out

This appendix lists the inequalities that the main text uses but does not print and that no identity in the text fixes, and it compares F_7 with the seven-party inequalities in the literature. Terms are written in the notation of Section 2: $3ABC$ stands for $3S(ABC)$, and terms containing the purifier are rewritten by purification, so that every term is a subset of the parties. Within each side, terms are ordered by size and then lexicographically.

The other inequalities of the text are fixed by identities printed there. These are $Q'_8 = 3Q_8 - 2R_8$ and $Q''_8 = Q'_8 - 3\,\mathrm{MMI}(A{:}H{:}DE)$, with R_8 as defined in Section 4.3, the identities (4.10) for $E_1, \dots, E_5$, and $6E_5 = 2X_7 + Y_7 + 3\mu_5$ for Y_7, with X_7 of (C.2). Their full coefficient lists, and the identity behind Proposition 3.2(i), are in the code repository `https://github.com/ahippelainen/unscaled-contraction-public`.

C.1. **The summands of Section 5.** Besides $\mu_0, \dots, \mu_4$ of (4.9), Section 5 uses the orbit sum

$$\mu_5 = \mathrm{MMI}(C{:}DF{:}EG) + \mathrm{MMI}(E{:}BF{:}CG) + \mathrm{MMI}(G{:}BD{:}CE)\ , \tag{C.1}$$

and the seven-party inequalities

(C.2)

$$\begin{aligned} X_7 :\quad & 7ABC + 7ADE + 7AFG + 5BDF + 5BDG + 5BEF + 5BEG + 5CDF + 5CDG \\ & + 5CEF + 6CEG + 7ABDF + 7ABEG + 7ACDG + 7ACEF \\ & + 4BCDE + 4BCFG + 4DEFG \\ & \ge\ 4B + 5C + 4D + 5E + 4F + 5G + 4AB + 3AC + 4AD \\ & + 3AE + 4AF + 3AG + 3BC + 4BD + 4BE + 4BF + 4BG + 4CD \\ & + 4CE + 4CF + 4CG + 3DE + 4DF + 4DG + 4EF + 4EG \\ & + 3FG + 5ABDG + 5ABEF + 5ACDF + 6ACEG + 5BCDEFG + 7ABCDEFG\ , \end{aligned}$$

$$
\begin{aligned}
V_7: \quad & 2ABC + 2ADE + 2AFG + BDF + 2BDG + 2BEF + BEG \\
& + 2CDF + CDG + CEF + 2CEG + 2ABDF + 2ABEG + 2ACDG \\
& + 2ACEF + 3BCFG + BCDEF + 2BCDEG + BDEFG + 2CDEFG \\
& \geq\ B + 2C + 3D + 4E + F + 2G + 2AB + 2AD \\
& + 2AF + BE + 2BF + 3BG + CD + 3CF + 2CG + DG \\
& + EF + ABDG + ABEF + ACDF + 3ACEG + BCDE + DEFG + BCDFG \\
& + 4BCDEFG + 2ABCDEFG\ ,
\end{aligned}
\tag{C.3}
$$

$$
\begin{aligned}
V_7': \quad & 2ABC + 2ADE + 2AFG + BDF + BDG + BEF + BEG \\
& + CDF + CDG + CEF + 3CEG + 2ABDF + 2ABEG + 2ACDG \\
& + 2ACEF + 2DEFG + BCDEF + BCDEG + 2BCDFG + 2BCEFG \\
& \geq\ 2B + 4C + 2E + F + 3G + 2AB + 2AD + 2AF \\
& + BD + BE + CD + CE + 2DF + 2DG + 2EF + 2EG \\
& + ABDG + ABEF + ACDF + 3ACEG + 2BCFG + 4BCDEFG + 2ABCDEFG\ ,
\end{aligned}
\tag{C.4}
$$

and, for Remark 5.9,

$$
\begin{aligned}
P_7: \quad & 4ABC + 4ADE + 4AFG + 3BDF + 3BDG + 3BEF + 3BEG + 3CDF + 3CDG \\
& + 3CEF + 3CEG + 4ABDF + 4ABEG + 4ACDG + 4ACEF \\
& + 4BCDE + 2BCFG + 2DEFG \\
& \geq\ 2B + 2C + 2D + 2E + 3F + 3G + 2AB + 2AC + 2AD \\
& + 2AE + 2AF + 2AG + 2BC + 3BD + 3BE + 2BF + 2BG + 3CD \\
& + 3CE + 2CF + 2CG + 2DE + 2DF + 2DG + 2EF + 2EG \\
& + 2FG + 3ABDG + 3ABEF + 3ACDF + 3ACEG + BCDEF + BCDEG \\
& + 2BCDEFG + 4ABCDEFG\ .
\end{aligned}
\tag{C.5}
$$

C.2. **The star inequalities of Table 3.** The nine-party star inequality of Table 3 without complementary pairs of words is

$$
\begin{aligned}
G_9: \quad & 3A + 5B + 5C + 5D + 3E + 4F + 6G + 6H + 4I + 4ABCDEFGHI \\
& \geq\ ABCDG + ABCHI + ABEFG + ACDEF + ACEGH \\
& + BCDEH + BCFGH + CDEGI + DFGHI\ .
\end{aligned}
\tag{C.6}
$$

With c the point of Table 3, Lemma 6.8 gives $32G_9 = \sum_{K=1}^{9} \left[(32 - 32c_K)\,\mathrm{SA}_K^1 + 32c_K\,\mathrm{SA}_K^2\right]$ with $32c = (2, 25, 28, 31, 10, 8, 1, 19, 4)$.

The inequalities H_9, H_{11}, H_{15} and H_{17} are the inequalities $Q_{\mathcal{T}}$ of Theorem 6.14 for the following lists of subsets of $[m]$, with the coins numbered as in Table 3 and the party i carrying the word $\mathbf{1}_{T_i}$. For H_9, $m = 8$ and

$$
\mathcal{T} = \big(\{5,7,8\}, \{1,2,3,5,7\}, \{2,4,5,8\}, \{1,4,6,7\}, \{1,3,4,8\}, \{2,3,6,7\}, \{1,2,6,8\}, \{3,4,5,6\}\big)\ .
$$

For H_{11}, $m = 8$ and

$$
\begin{aligned}
\mathcal{T} = \big(&\{1,3,4,5,6\}, \{1,2,3,5,7\}, \{1,4,6,7\}, \{2,3,4,8\}, \{2,3,6,7\}, \{1,2,6,8\}, \\
&\{2,4,5,7\}, \{3,7,8\}, \{5,6,8\}, \{1,4,5,8\}\big)\ .
\end{aligned}
$$

In both lists every coin lies in exactly half of the sets, so (W2) holds with all $\gamma_i = 1$. For H_{15}, $m = 8$ and $\mathcal{T} = (T_1, \ldots, T_7, \bar{T}_1, \ldots, \bar{T}_7)$ with

$$
(T_1, \ldots, T_7) = \big(\{3,7,8\}, \{4,6,8\}, \{2,3,5,8\}, \{5,6,7\}, \{2,3,6,7\}, \{2,4,5,7\}, \{3,4,5,6\}\big)\ .
$$

For H_{17}, $m = 9$ and $\mathcal{T} = (T_1, \ldots, T_8, \bar{T}_1, \ldots, \bar{T}_8)$ with

$$(T_1, \ldots, T_8) = \big(\{2,3,4,8\}, \{3,4,5,6\}, \{2,4,6,7\}, \{2,3,5,7\}, \{3,4,6,8,9\}, \{2,4,5,7,9\}, \{2,3,5,6,9\}, \{3,4,7,9\}\big) ,$$

the rows of an 8×9 matrix of the construction of Alon and Vũ [26], read as weighings as in the proof of Corollary 6.22. In each case the vector ω of (W1) is q times the point of Table 3.

C.3. **Seven-party inequalities compared with F_7.** Members of the toric, projective-plane and cyclic families with more than seven parties give seven-party inequalities when parties are merged or deleted. Both operations keep every coefficient on its side and can only combine, cancel or remove terms, so they do not increase $|\alpha|$ or $|\beta|$, while an integer multiple of F_7 has $|\alpha| \geq 39$ and $|\beta| \geq 54$. Every member of the three families with at most 14 parties has $|\beta| \leq 50$ or $|\alpha| < 39$, so none of them yields a multiple of F_7 in this way.

The table contains every seven-party inequality that we found in the works on holographic entropy inequalities cited here and in the papers that cite [30, 8, 40, 41], screened by their abstracts. We did not compare F_7 with the conjectured generalizations of the toric inequalities in [40], whose seven-party instances are not written out there, with the averaged inequalities of the symmetrized cones [42, 43], which constrain only sums over relabelings, with family members with more than 14 parties, or with works outside this search. We found no published list of facets of HEC_7. The facet property of the seven-party members of the toric and projective-plane families was shown in [9].

source	inequalities	number	max. coefficient
Czech, Wang [30]	the seven-region inequality, both printed forms	2	1
Akers, Hernández-Cuenca, Rath [11]	eq. (1.5)	1	1
Bao, Naskar [13]	eqs. (A.11), (A.12)	2	1
Bao, Furuya, Naskar [14]	eq. (3.2)	1	1
Chou, Lao, Yang [44]	eq. (5.15)	1	1
toric family [8, 45]	the seven-party members	4	1
projective-plane family [8, 45]	the seven-party member	1	3
cyclic family [6]	the nonzero seven-party members	31	≤ 2

TABLE 6. The 43 seven-party inequalities with which F_7 was compared. Inequalities that agree up to relabeling are counted separately, so several rows contain the same orbit. The largest coefficient is taken after division by the greatest common divisor of the coefficients.

REFERENCES

[1] S. Ryu and T. Takayanagi, Holographic Derivation of Entanglement Entropy from the anti–de Sitter Space/Conformal Field Theory Correspondence, Phys. Rev. Lett. 96 (2006), 181602, https://doi.org/10.1103/PhysRevLett.96.181602; arXiv:hep-th/0603001.

[2] V. E. Hubeny, M. Rangamani, and T. Takayanagi, A Covariant holographic entanglement entropy proposal, JHEP 07 (2007), 062, https://doi.org/10.1088/1126-6708/2007/07/062; arXiv:0705.0016.

[3] M. Headrick and T. Takayanagi, A Holographic proof of the strong subadditivity of entanglement entropy, Phys. Rev. D 76 (2007), 106013, https://doi.org/10.1103/PhysRevD.76.106013; arXiv:0704.3719.

[4] P. Hayden, M. Headrick, and A. Maloney, Holographic Mutual Information is Monogamous, Phys. Rev. D 87 (2013), no. 4, 046003, https://doi.org/10.1103/PhysRevD.87.046003; arXiv:1107.2940.

[5] A. C. Wall, Maximin Surfaces, and the Strong Subadditivity of the Covariant Holographic Entanglement Entropy, Class. Quant. Grav. 31 (2014), no. 22, 225007, https://doi.org/10.1088/0264-9381/31/22/225007; arXiv:1211.3494.

[6] N. Bao, S. Nezami, H. Ooguri, B. Stoica, J. Sully, and M. Walter, The Holographic Entropy Cone, JHEP 09 (2015), 130, https://doi.org/10.1007/JHEP09(2015)130; arXiv:1505.07839.

[7] S. Hernández Cuenca, Holographic entropy cone for five regions, Phys. Rev. D 100 (2019), no. 2, 026004, https://doi.org/10.1103/PhysRevD.100.026004; arXiv:1903.09148.

[8] B. Czech, S. Shuai, Y. Wang, and D. Zhang, Holographic entropy inequalities and the topology of entanglement wedge nesting, Phys. Rev. D 109 (2024), no. 10, L101903, https://doi.org/10.1103/PhysRevD.109.L101903; arXiv:2309.15145.

[9] B. Czech, Y. Liu, and B. Yu, Two infinite families of facets of the holographic entropy cone, SciPost Phys. 17 (2024), 084, https://doi.org/10.21468/SciPostPhys.17.3.084; arXiv:2401.13029.

[10] B. Czech and X. Dong, Holographic Entropy Cone with Time Dependence in Two Dimensions, JHEP 10 (2019), 177, https://doi.org/10.1007/JHEP10(2019)177; arXiv:1905.03787.

[11] C. Akers, S. Hernández-Cuenca, and P. Rath, Quantum Extremal Surfaces and the Holographic Entropy Cone, JHEP 11 (2021), 177, https://doi.org/10.1007/JHEP11(2021)177; arXiv:2108.07280.

[12] N. Li, C.-S. Dong, D.-H. Du, and F.-W. Shu, Improved proof-by-contraction method and relative homologous entropy inequalities, JHEP 06 (2022), 153, https://doi.org/10.1007/JHEP06(2022)153; arXiv:2204.03192.

[13] N. Bao and J. Naskar, Properties of the contraction map for holographic entanglement entropy inequalities, JHEP 06 (2024), 039, https://doi.org/10.1007/JHEP06(2024)039; arXiv:2403.13283.

[14] N. Bao, K. Furuya, and J. Naskar, Towards a complete classification of holographic entropy inequalities, JHEP 03 (2025), 117, https://doi.org/10.1007/JHEP03(2025)117; arXiv:2409.17317.

[15] B. Czech and S. Shuai, Nesting is not contracting, JHEP 06 (2025), 122, https://doi.org/10.1007/JHEP06(2025)122; arXiv:2501.17222.

[16] G. Grimaldi, M. Headrick, V. E. Hubeny, and P. Shteyner, Combinatorial properties of holographic entropy inequalities, preprint (2026), arXiv:2601.09987.

[17] X.-X. Ju and Y. Zhao, The holographic multi-entropy cone, JHEP 09 (2026), 253, https://doi.org/10.1007/JHEP09(2026)253; arXiv:2606.15173.

[18] N. Bao, N. Cheng, S. Hernández-Cuenca, and V. P. Su, The Quantum Entropy Cone of Hypergraphs, SciPost Phys. 9 (2020), no. 5, 067, https://doi.org/10.21468/SciPostPhys.9.5.067; arXiv:2002.05317.

[19] D. Avis and S. Hernández-Cuenca, On the foundations and extremal structure of the holographic entropy cone, Discrete Appl. Math. 328 (2023), 16–39, https://doi.org/10.1016/j.dam.2022.11.016; arXiv:2102.07535.

[20] N. Bao, K. Furuya, and J. Naskar, On the completeness of contraction map proof method for holographic entropy inequalities, JHEP 12 (2025), 140, https://doi.org/10.1007/JHEP12(2025)140; arXiv:2506.18086v3.

[21] J. Naskar, Multipartite correlations in holography, Ph.D. thesis, Northeastern U., 2026.

[22] B. Czech and S. Shuai, Renormalization group is the principle behind the holographic entropy cone, JHEP 07 (2026), 165, https://doi.org/10.1007/JHEP07(2026)165; arXiv:2601.02472.

[23] B. Czech, Y. Feng, X. Wu, and M. Xie, Holographic entropy inequalities pass the majorization test, SciPost Phys. 21 (2026), no. 2, 048, https://doi.org/10.21468/scipostphys.21.2.048; arXiv:2601.09989.

[24] V. E. Hubeny, M. Rangamani, and M. Rota, The holographic entropy arrangement, Fortsch. Phys. 67 (2019), no. 4, 1900011, https://doi.org/10.1002/prop.201900011; arXiv:1812.08133.

[25] T. He, V. E. Hubeny, and M. Rangamani, Superbalance of Holographic Entropy Inequalities, JHEP 07 (2020), 245, https://doi.org/10.1007/JHEP07(2020)245; arXiv:2002.04558.

[26] N. Alon and V. H. Vũ, Anti-Hadamard Matrices, Coin Weighing, Threshold Gates, and Indecomposable Hypergraphs, J. Combin. Theory Ser. A 79 (1997), no. 1, 133–160, https://doi.org/10.1006/jcta.1997.2780.

[27] N. Wetzler, M. J. H. Heule, and W. A. Hunt, DRAT-trim: Efficient checking and trimming using expressive clausal proofs, in: Theory and Applications of Satisfiability Testing – SAT 2014, Lecture Notes in Computer

Science, vol. 8561, Springer International Publishing, 2014, pp. 422–429, https://doi.org/10.1007/978-3-319-09284-3_31.

[28] D. N. Kozlov and V. H. Vu, Coins and Cones, J. Combin. Theory Ser. A 78 (1997), no. 1, 1–14, https://doi.org/10.1006/jcta.1996.2724.

[29] S. X. Cui, P. Hayden, T. He, M. Headrick, B. Stoica, and M. Walter, Bit Threads and Holographic Monogamy, Commun. Math. Phys. 376 (2019), no. 1, 609–648, https://doi.org/10.1007/s00220-019-03510-8; arXiv:1808.05234.

[30] B. Czech and Y. Wang, A holographic inequality for N = 7 regions, JHEP 01 (2023), 101, https://doi.org/10.1007/JHEP01(2023)101; arXiv:2209.10547.

[31] H. Araki and E. H. Lieb, Entropy inequalities, Commun. Math. Phys. 18 (1970), 160–170, https://doi.org/10.1007/BF01646092.

[32] O. Bailleux and Y. Boufkhad, Efficient CNF Encoding of Boolean Cardinality Constraints, in: Principles and Practice of Constraint Programming – CP 2003, Lecture Notes in Computer Science, vol. 2833, Springer Berlin Heidelberg, 2003, pp. 108–122, https://doi.org/10.1007/978-3-540-45193-8_8.

[33] C. Sinz, Towards an Optimal CNF Encoding of Boolean Cardinality Constraints, in: Principles and Practice of Constraint Programming - CP 2005, Lecture Notes in Computer Science, vol. 3709, Springer Berlin Heidelberg, 2005, pp. 827–831, https://doi.org/10.1007/11564751_73.

[34] R. Asín, R. Nieuwenhuis, A. Oliveras, and E. Rodríguez-Carbonell, Cardinality Networks and Their Applications, in: Theory and Applications of Satisfiability Testing - SAT 2009, Lecture Notes in Computer Science, vol. 5584, Springer Berlin Heidelberg, 2009, pp. 167–180, https://doi.org/10.1007/978-3-642-02777-2_18.

[35] A. Ignatiev, A. Morgado, and J. Marques-Silva, PySAT: A Python toolkit for prototyping with SAT oracles, in: Theory and Applications of Satisfiability Testing – SAT 2018, Lecture Notes in Computer Science, vol. 10929, Springer International Publishing, 2018, pp. 428–437, https://doi.org/10.1007/978-3-319-94144-8_26.

[36] A. Biere, T. Faller, K. Fazekas, M. Fleury, N. Froleyks, and F. Pollitt, CaDiCaL 2.0, in: Computer Aided Verification, Lecture Notes in Computer Science, vol. 14681, Springer Nature Switzerland, 2024, pp. 133–152, https://doi.org/10.1007/978-3-031-65627-9_7.

[37] A. Biere, K. Fazekas, M. Fleury, and M. Heisinger, CaDiCaL, Kissat, Paracooba, Plingeling and Treengeling entering the SAT competition 2020, in: Proceedings of SAT Competition 2020: Solver and Benchmark Descriptions (Helsinki) (T. Balyo, N. Froleyks, M. J. H. Heule, M. Iser, M. Järvisalo, and M. Suda, eds.), Department of Computer Science Report Series B, vol. B-2020-1, Department of Computer Science, University of Helsinki, 2020, pp. 50–53, https://researchportal.helsinki.fi/en/publications/proceedings-of-sat-competition-2020-solver-and-benchmark-descript/.

[38] P. Virtanen et al., SciPy 1.0–Fundamental Algorithms for Scientific Computing in Python, Nature Meth. 17 (2020), 261, https://doi.org/10.1038/s41592-019-0686-2; arXiv:1907.10121.

[39] W. H. Cunningham, On submodular function minimization, Combinatorica 5 (1985), no. 3, 185–192, https://doi.org/10.1007/BF02579361.

[40] N. Bao, K. Furuya, and J. Naskar, A framework for generalizing toric inequalities for holographic entanglement entropy, JHEP 10 (2024), 251, https://doi.org/10.1007/JHEP10(2024)251; arXiv:2408.04741.

[41] S. Hernández-Cuenca, V. E. Hubeny, and H. F. Jia, Holographic entropy inequalities and multipartite entanglement, JHEP 08 (2024), 238, https://doi.org/10.1007/JHEP08(2024)238; arXiv:2309.06296.

[42] B. Czech and S. Shuai, Holographic Cone of Average Entropies, Commun. Phys. 5 (2022), 244, https://doi.org/10.1038/s42005-022-01019-6; arXiv:2112.00763.

[43] M. Fadel and S. Hernández-Cuenca, Symmetrized holographic entropy cone, Phys. Rev. D 105 (2022), no. 8, 086008, https://doi.org/10.1103/PhysRevD.105.086008; arXiv:2112.03862.

[44] C.-J. Chou, H. B. Lao, and Y. Yang, A graphical framework for proving holographic entanglement entropy inequalities in multipartite systems, preprint (2025), arXiv:2512.18726.

[45] V. E. Hubeny and Y. Liu, Tripartite Form Universality in Holographic Entropy Inequalities, preprint (2026), arXiv:2609.22504.