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Nonuniqueness of site percolation on nonamenable quasi-transitive graphs

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Abstract

Benjamini and Schramm conjectured in 1996 that site percolation on every nonamenable quasi-transitive graph has $p_c < p_u$. We prove this in Hutchcroft's operator form: for Bernoulli site percolation on every connected, locally finite, nonamenable quasi-transitive graph, the critical two-point operator is bounded on ℓ^2, $p_c < p_{2\to 2} \le p_u$, and in the standard monotone coupling infinitely many infinite clusters coexist at every parameter of a nondegenerate interval. Since bond percolation is site percolation on the line graph, this contains OpenAI's recent bond theorem. For unimodular automorphism groups we adapt OpenAI's proof: cut vertices replace bridges, a pruning lemma handles retained contact vertices, a rejection step controls the junctions created by opening a vertex, and pivotal corridors become induced paths. For nonunimodular groups we extend Tran and Nguyen's site argument from vertex-transitive to quasi-transitive graphs.

1 Introduction

1.1 The site form of the Benjamini–Schramm conjecture

Let $G = (V, E)$ be an infinite, connected, locally finite graph. In *Bernoulli site percolation* with parameter $p \in [0, 1]$, each vertex is *open* independently with probability p and *closed* otherwise; $\mathbb{P}_p$ denotes the law. The open cluster C_x of a vertex x is the set of vertices joined to x by a path all of whose vertices, endpoints included, are open; in particular $C_x = \varnothing$ when x is closed. Write $x \leftrightarrow y$ when $y \in C_x$, and let N_∞ be the number of infinite open clusters. The two thresholds of the phase diagram are

$$p_c(G) = \inf\{p \in [0,1] : \mathbb{P}_p(N_\infty \ge 1) > 0\}, \qquad p_u(G) = \inf\{p \in [0,1] : \mathbb{P}_p(N_\infty = 1) = 1\}.$$

The graph is *quasi-transitive* if $\mathrm{Aut}(G)$ has finitely many orbits on V, and then its degrees are bounded; d denotes the maximal degree. For finite nonempty $A \subset V$ let $\partial A = \{v \in V \setminus A : v \sim a \text{ for some } a \in A\}$ be the outer vertex boundary, and put

$$h_V(G) = \inf_{0<|A|<\infty} \frac{|\partial A|}{|A|}.$$

The graph is *nonamenable* if $h_V(G) > 0$; for bounded degree this is equivalent to positivity of the edge isoperimetric constant.

Benjamini and Schramm's paper on percolation beyond $\mathbb{Z}^d$ is written for site percolation: "Throughout the paper, site percolation is discussed. Except when dealing with planar graphs, the

results and questions remain equally valid, with only minor modifications, for bond percolation" [2, p. 72]. Their Cheeger constant is h_V, defined with the vertex boundary [2, p. 73], and their nonuniqueness conjecture reads

> "Conjecture 6 Assume that the almost transitive graph G has positive Cheeger constant. Then $p_c(G) < p_u(G)$." [2, p. 76]

where almost transitive means quasi-transitive [2, p. 72]. On amenable quasi-transitive graphs the infinite cluster is unique whenever it exists, by the argument of Burton and Keane [4], as noted in [2, p. 76], so $p_c = p_u$ there; the conjecture asserts that nonamenability alone produces a nonuniqueness phase.

We study the stronger operator form of Hutchcroft [9, Proposition 2.3 and Conjecture 6.1]. Let $T_p(x, y) = \mathbb{P}_p(x \leftrightarrow y)$ be the two-point kernel; for site percolation $T_p(x, x) = p$. Equip V with counting measure, let T_p act on finitely supported functions by $(T_p f)(x) = \sum_y T_p(x, y) f(y)$, and put

$$p_{2\to 2}(G) = \sup\{p \in [0, 1] : T_p \text{ extends to a bounded operator on } \ell^2(V)\}.$$

For bond percolation the conjecture, together with the strict operator gap $p_c < p_{2\to 2}$, was recently proved on every nonamenable quasi-transitive graph in a manuscript released by OpenAI [15]. That theorem does not give the site form. The site form is the more general one: bond percolation on a graph H is site percolation on its line graph, which is quasi-transitive and nonamenable when H is (Appendix A.2), so a site theorem implies the corresponding bond theorem; no reduction in the other direction is available.

1.2 Results

Theorem 1.1. *Let G be an infinite, connected, locally finite, quasi-transitive graph with $h_V(G) > 0$. For Bernoulli site percolation on G:*

(1) $\sup_{0\le p<p_c} \|T_p\|_{2\to 2} = \|T_{p_c}\|_{2\to 2} < \infty$;

(2) $p_c(G) < p_{2\to 2}(G) \le p_u(G)$;

(3) *there are $p_c < p_1 < p_2 < 1$ such that, in the coupling by independent uniform vertex labels, almost surely there are infinitely many infinite clusters simultaneously for every $p \in [p_1, p_2]$.*

Loops and multiple edges are allowed: site percolation depends only on adjacency (Section 2.1). In the coupling of (3), each vertex v carries an independent uniform label $U_v \in [0, 1]$, and v is open at parameter p when $U_v \le p$.

Corollary 1.2. *Let G be an infinite, connected, locally finite, quasi-transitive graph with $h_V(G) > 0$. Then the conclusions* (1)–(3) *of Theorem 1.1 hold for Bernoulli bond percolation on G, with uniform labels on the edges in* (3)*. That is, Theorem 1.1 contains the bond theorem [15, Theorem 1.1].*

Corollary 1.2 follows from Theorem 1.1 applied to the line graph $L(G)$. With d the maximal number of edges at a vertex of G, the graph $L(G)$ is quasi-transitive with $h_V(L(G)) \ge h_V(G)/d$; the open site clusters of $L(G)$ are the edge sets of the bond clusters of G with at least one edge; and, with the vertex–edge incidence kernel B, $T_p^{\mathrm{bond}} \le I + B\, T_p^{\mathrm{site},L(G)} B^*$ entrywise and $\|B\|^2 \le 2d$. The proof is in Appendix A.2.

1.3 Relation to earlier work

Site percolation. The site inequality $p_c < p_u$ was known under hypotheses beyond nonamenability. Benjamini and Schramm proved $0 < p_c < p_u < 1$ for site and bond percolation on one-ended nonamenable transitive planar graphs [3, Theorem 1.1], after Lalley's work on site percolation on Fuchsian groups [11]. Schonmann's theorem [18] gives $p_c^{\mathrm{site}} < p_u^{\mathrm{site}}$ on transitive graphs of degree d with $h_V(G)/d \geq 1/\sqrt{2}$ [14, Corollary 7.39(ii)]; the vertex-isoperimetric constant Φ_V of [14, p. 180], with unit weights, is h_V. Czajkowski proved a nonuniqueness phase for bond and site percolation on reflection groups of some polyhedra in $\mathbb{H}^3$ [5]. Li proved site nonuniqueness for properly embedded planar graphs of minimum degree at least seven [12] and under an end-structure hypothesis for planar graphs [13]. For vertex-transitive graphs with nonunimodular automorphism group, Tran and Nguyen proved the site tilted-threshold separation $p_c < p_c(\lambda)$ [20, Proposition 5.5], where $p_c(\lambda)$ is the supremum of the parameters at which the tilted susceptibility $\sum_x T_p(u,x)\Delta(u,x)^\lambda$, with Δ the modular function, is bounded in u (Section 3). Together with Hutchcroft's operator argument this gives the conclusions (1)–(2) of Theorem 1.1 for those graphs (Proposition E.1). We are not aware of an earlier statement of $p_c < p_u$ for site percolation on all nonamenable quasi-transitive graphs.

The bond theorem. The unimodular case of Theorem 1.1 follows the architecture of [15], described in Section 1.4. Every result whose proof changes for sites is stated here and each changed step is proved; steps that carry over are cited by their number in [15], and Appendix G lists the correspondence. In the nonunimodular case, [15] uses Hutchcroft's bond theorem $p_c < p_{q\to q}$ [9, Theorem 2.9], where $p_{q\to q}$ is the ℓ^q analogue of $p_{2\to 2}$; it rests on the bond tilted-susceptibility theory of [10]. For sites we replace it by a quasi-transitive form of Tran and Nguyen's site argument [20, Section 5].

Known components. Three of the conclusions are known or routine. The inequality $p_{2\to 2} \leq p_u$ holds because uniqueness forces $T_p(x,y) \geq \theta^2$ by Harris's inequality [15, Lemma 2.4], [14, p. 269], [12, Proposition 1.3]. The coupled interval is a consequence of the simultaneous phase theorem of Häggström, Peres and Schonmann [7, Theorem 1.3], whose results are stated for bond percolation with the remark "All our results and proofs may be adapted to site percolation as well" [7, Section 1]; we include the direct proof of [15, Proposition 9.3] for sites (Proposition 2.11). The supremum identity in (1) is monotone convergence. Critical finiteness, the absence of an infinite cluster at p_c, is the site theorem of the companion paper [16, Theorem 1.1(2)], whose nonunimodular part [16, Theorem 1.2] is also used in the nonunimodular case (Appendix F).

Unrefereed inputs. The nonunimodular case of Theorem 1.1 depends on Tran and Nguyen's arXiv preprint [20], posted in September 2026 and not refereed. We reuse the arguments of its Section 5, change the steps listed in Appendix F and prove each replacement, and use the other steps as written. The unimodular case does not use [20]. Both cases use [15], which is also unrefereed, and the companion paper [16], also a preprint.

1.4 Ideas of the proof

The theorem reduces to bounding T_{p_c}: sprinkling then gives $p_c < p_{2\to 2}$, and the remaining conclusions follow from the known components above (Section 3). After the passage to the underlying simple graph, we split according to whether $\mathrm{Aut}(G)$ is unimodular.

The unimodular case: the architecture of [15]. We first describe the bond argument in its own terms. Fix a cap M; the open cluster of a vertex x is *capped* if it has at most M vertices. In a connected piece of a cluster, the *pivotal distance* between two vertices is the number of bridges separating them, and a capped cluster is *good* if, whenever a set of its vertices is deleted, every remaining piece has pivotal diameter at most KM^{τ}, where K is the number of open edges from the piece to the deleted set and τ is slightly larger than $1/2$. A *local endpoint map* is a randomized rule that reads the configuration near x and either fails or outputs a vertex y of the capped good cluster of x in a fixed orbit; it is measured by its success probability u and by the norm r of its *endpoint kernel* $K(x,y) = \mathbb{P}(\text{output } y \mid \text{success at } x)$. Suppose that the critical moment $\mathbb{E}_{p_c}|C_o|^{\alpha}$ at a vertex o, averaged over the finitely many vertex types, is infinite for some $0 < \alpha < 1/2$. Then an *amplification* combines k independent copies of a map into a new one. A chain of k old steps runs from a vertex z into the cluster A of the root x, and the *switching* splices the chain into A step by step: each step takes the piece of its own cluster, in its own independent configuration, that hangs off the current root set, and joins it to that set through a contact edge, the *seam*, by exchanging the bits of the two configurations on a mask. The *inverse* of the switching locates the seams among the separating bridges of the output cluster by a code of gaps, which goodness keeps short, and undoes the exchanges; a second-moment bound undoes two inverses at once. Iterating the amplification produces maps whose endpoint kernels become dispersive much faster than their success probability decays or their cap grows. Run just below p_c in a *deleted domain*, the complement of an independent cluster, such a map contradicts the divergent moment. The resulting fractional moments give an almost linear susceptibility bound and a subpower bound for the triangle trace, and a deformation of T_p, which penalizes connections forced through a *corridor* of pivotal edges, the loop erasure of a random-walk path, shows that T_p cannot blow up at p_c.

What changes for sites. Each bond-specific object changes for sites, and five changes carry new ideas.

(i) *Cut vertices in place of bridges.* For sites an open edge cannot be closed independently of its endpoints, so the separating objects are the cut vertices of a piece, its pivotal sites, and pivotal distance counts separating cut vertices. Its triangle inequality carries an extra $+1$, which costs only constants (Section 4).

(ii) *Pruning a retained contact.* To prove goodness, [15] closes all but one contact edge of a bad piece B, so that B hangs from the rest of the cluster by a single edge and the pivotal distances inside B persist in the modified cluster. For sites the retained object is a contact vertex a, which may have several neighbours in B; if B is a path and a is adjacent to both ends, $B \cup \{a\}$ is a cycle with no cut vertices at all. We also close all but one neighbour b^* of a in B, and a Voronoi-cell argument chooses b^* so that the pivotal distance drops by at most a factor $2d$ (Lemma 4.6).

(iii) *Junction rejection.* For bonds, opening a contact edge joins exactly two clusters. For sites the seam is a closed vertex on the boundary of the root set, and opening it attaches every open component adjacent to it, up to $d-1$ foreign branches. We reject the splice unless every neighbour of the seam outside the closed neighbourhoods of the current root set and of the inserted piece is closed (Section 5.2). These rejection bits have not been read before, so each step costs at most a factor $(1-p_c)^{d-1}$; this cost is linear in the number of steps in the exponent and is absorbed by the amplification, while the inverse of the switching stays unambiguous.

(iv) *Joint restoration in the second moment.* The second-moment bound of [15] undoes two switchings whose inserted pieces are far apart, and uses that removing both inserted parts, the

sets hanging at the two first seams, leaves a connected set containing the root. For bonds the two first seams are bridges, and this holds. For sites they are cut vertices, which can lie on one cycle through the root, and removing both hanging sets can disconnect the root's set. We record the two seams and their neighbourhoods in the remaining set, at a polynomial cost, and replay the switching from a possibly disconnected set (Lemma 5.8). The root cluster of the remaining set can then be much smaller than that set, so a lower size window used in [15] is lost; a second-moment bound at the scales selected in Lemma 5.1 replaces it.

(v) *Induced corridors.* The deformation penalizes connections for which every vertex of a prescribed corridor is pivotal. For sites a corridor must be an induced path with closed blockers around it, since a chord or an open neighbour would bypass its pivotal vertices. We take the greedy shortcut of the loop erasure of the random-walk path, which is induced, and its interior is entered and left only through its two endpoints (Lemma 6.5). When a connection through a corridor is split into two legs at the corridor, the corridors whose interior meets the first one or its endpoints form a new class, of bounded total weight (Lemma 6.11).

The other changes are local: in the site Russo identity in operator norm, the derivative of T_p is a sum over closed pivotal sites, which splits into a contact kernel $|\partial C_x \cap \partial C_y|$ and a remainder bounded by $(1-p)I + \mathrm{Adj}\, T_p + T_p \mathrm{Adj}$ (Lemma 2.8); deleted domains become $V \setminus \mathrm{Nb}[A]$ with $\mathrm{Nb}[A] = A \cup \partial A$; and several constants depend on the degree.

The nonunimodular case. Hutchcroft's operator argument bounds $\|T_p\|$ by the tilted susceptibility $\sup_u \sum_x T_p(u,x)\Delta(u,x)^{1/2}$; the argument is model-free (Section 7.1). It therefore suffices to show that the site tilted susceptibility is finite slightly above p_c. Tran and Nguyen proved this for vertex-transitive graphs [20, Proposition 5.5]. Their argument uses transitivity in many places: neighbours one layer up or down in height, where the height of z relative to v is $\log \Delta(v,z)$; constants uniform in the root; and transports at a fixed root. On a quasi-transitive graph such neighbours need not exist, for instance at pendant vertices, and lower bounds uniform in the root fail at *plateau* vertices, whose neighbours all have the same height. We replace these uses by monotone paths with length bounds, connectors that end in a prescribed orbit, a tilted mass transport averaged over a random root, comparisons between orbits, and a transfer of peak estimates to a *sloped* vertex (Section 7.3). The decisive step is the quasi-transitive form of the surgery inequality [20, (5.76)], the site analogue of [10, Lemma 6.4], which we apply at a sloped vertex (Proposition 7.10).

1.5 Organization

Section 2 fixes conventions and states the site tools, whose proofs are in Appendix A. Section 3 proves Theorem 1.1 from two case theorems, Theorem 3.1 (unimodular) and Theorem 3.2 (nonunimodular); the proof of Corollary 1.2 is in Appendix A.2. Sections 4–6 prove Theorem 3.1: good critical clusters with the pruning lemma, the amplification with junction rejection and joint restoration in the second moment, and the critical bound with induced corridors. The routine site adaptations of [15] used there are proved in Appendices B, C and D. Section 7 proves Theorem 3.2; the vertex-transitive deduction is in Appendix E, and the proofs of the devices and the change list against [20, Section 5] are in Appendix F. Appendix G lists the correspondence with the numbered statements of [15]. Section 8 records where the hypotheses enter.

2 Preliminaries

2.1 Conventions

Reduction to a simple graph. Site percolation on a graph with loops or multiple edges is site percolation on its underlying simple graph G_s, which has the same vertex set and in which $x \sim y$ if and only if $x \neq y$ are joined by at least one edge. The graph G_s is infinite, connected and locally finite; $\mathrm{Aut}(G)$ acts on V through a subgroup of $\mathrm{Aut}(G_s)$, so G_s is quasi-transitive; and vertex boundaries, hence h_V, are unchanged. All site quantities (p_c, p_u, $p_{2\to2}$, T_p and the label coupling) coincide for G and G_s. Unless stated otherwise, *graphs are simple from now on*, $d < \infty$ bounds the degrees, and unimodularity refers to $\mathrm{Aut}(G_s)$. In contrast with [15, Lemma 2.1], which subdivides multiple edges for bond percolation, simplification adds no new sites.

Site notation. A configuration is $\omega \in \{0,1\}^V$, with 1 meaning open, and $\mathbb{P}_p$ is the product Bernoulli(p) law. Domains are vertex sets $D \subseteq V$ with their induced graphs. The cluster C_x^D is the set of vertices joined to x by an open path inside D; it is *empty when x is closed or $x \notin D$*. Put $C_x = C_x^V$, $s_x = |C_x|$ and $T_p^D(x,y) = \mathbb{P}_p(y \in C_x^D)$, so $T_p(x,x) = p$. For $A \subseteq V$, ∂A is the outer vertex boundary, $\mathrm{Nb}[A] = A \cup \partial A$, $N(z)$ is the set of neighbours of z, and $\mathrm{Nb}[z] = \mathrm{Nb}[\{z\}] = N(z) \cup \{z\}$. Inside a domain D, for $W \subseteq D$, put $\partial_D W = \partial W \cap D$ and $\mathrm{Nb}_D[W] = W \cup \partial_D W$. Every vertex of ∂C_x is closed, and adjacent open vertices share a cluster. Adj is the adjacency kernel. For a closed vertex v put

$$\widehat{C}_v = \{v\} \cup \bigcup_{a \sim v} C_a,$$

the cluster of v after opening it. If $C^{(1)}, \dots, C^{(m)}$, $m \le d$, are the distinct open clusters adjacent to a closed v, with sizes n_i, put $N_v = \sum_i n_i = |\widehat{C}_v| - 1$ and $Q_v = \sum_{i \neq j} n_i n_j$, a sum over ordered pairs.

We use the constants

$$\kappa_p = \Big(\frac{1+d}{p}\Big)^2, \qquad C_{\mathrm{del}}(p) = \frac{1}{p} + d, \qquad c_{\mathrm{nb}}(p) = 1 + \frac{d}{p} + \frac{d^2}{p^2}, \tag{2.1}$$

all bounded on $p \ge p_c/2$, and two consequences of Harris's inequality. (H1) If $z \sim a$ with $a, z \in D$, then $p\,T_p^D(v,z) \le T_p^D(v,a)$, since $\{v \leftrightarrow z \text{ in } D\} \cap \{a \text{ open}\} \subseteq \{v \leftrightarrow a \text{ in } D\}$; iterating, $p^l T_p^D(v,z) \le T_p^D(v,a)$ when z and a are joined by a path of length l in D. (H2) $\delta_{wy} \le T_p(w,y)/p$, since $T_p(y,y) = p$. Consequently, entrywise,

$$I + \mathrm{Adj}\, T_p \le \frac{1+d}{p}\, T_p, \qquad I + T_p \mathrm{Adj} \le \frac{1+d}{p}\, T_p. \tag{2.2}$$

Kernel facts from [15, Section 2]: if $|A(x,y)| \le B(x,y)$ with B nonnegative and bounded, then $\|A\| \le \|B\|$; if all row and column sums of $B \ge 0$ are at most b, then $\|B\| \le b$.

2.2 Types, mass transport, trace and walk

In the unimodular case we use the type averages of [15, Section 2.3]. Let $\Gamma = \mathrm{Aut}(G)$ be unimodular, with vertex orbits $O_1, \dots, O_{h_0}$ and representatives $o_1, \dots, o_{h_0}$. With a fixed Haar measure on Γ, put

$$\sigma_i = \frac{\mathrm{vol}(\Gamma_{o_i})^{-1}}{\sum_j \mathrm{vol}(\Gamma_{o_j})^{-1}}, \qquad [F(o)] = \sum_{i=1}^{h_0} \sigma_i F(o_i), \qquad c_\sigma = (\min_i \sigma_i)^{-1}.$$

A bracket $[F(o)]$ around a function of the root o always denotes this finite type average; brackets after $\mathbb{E}_p$ are ordinary expectation brackets. Operator norms, adjoints and positivity refer to

counting-measure $\ell^2(V)$. Let P be the symmetric lazy walk, $P(x,y) = 1/(2d)$ for $x \sim y$ and $P(x,x) = 1 - \deg(x)/(2d)$. We use the following statements of the unrefereed preprint [15] (Section 1.3) as stated; they concern only the graph, invariant operators, or finite product measures.

Lemma 2.1 ([15, Lemma 2.7])**.** *For every invariant nonnegative function m on ordered vertex pairs, $[\sum_y m(o,y)] = [\sum_x m(x,o)]$. The identity also applies to the expectation of a covariant random mass and to absolutely summable signed or complex masses. In particular, an invariant stochastic kernel on a single orbit has column sums one.*

Lemma 2.2 ([15, Lemma 2.8])**.** *For a bounded invariant operator S, $\operatorname{tr} S = [S(o,o)]$ is a positive normalized trace: $\operatorname{tr} I = 1$ and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for bounded invariant A, B. If A is positive semidefinite, invariant and bounded, and $B = B^*$ is invariant and bounded, then $|\operatorname{tr}(BA)| \le \|B\| \operatorname{tr} A$ and $A(x,x) \le c_\sigma \operatorname{tr} A$.*

Lemma 2.3 ([15, Lemma 2.9])**.** *P is positive semidefinite and $\rho := \|P\| < 1$. Its projection onto the vertex types is an irreducible lazy finite Markov chain, and there are $c_* > 0$ and $l_0 < \infty$ with $\sum_{y \in O_j} P^l(x,y) \ge c_*$ for every $x \in V$, $j \le h_0$ and $l \ge l_0$.*

The bound $\rho < 1$ is where nonamenability enters the unimodular argument; Section 8 lists its uses.

2.3 Site tools

The following site forms of the preliminary lemmas of [15, Section 2] are proved in Appendix A. They are routine adaptations, except that the Russo identity acquires a remainder term. Lemma 2.4 holds on every graph of Theorem 1.1; the others are stated where they are used. Put $\chi_x(p) = \mathbb{E}_p s_x$ and $\chi_{\max}(p) = \sup_x \chi_x(p)$, and, when $\mathrm{Aut}(G)$ is unimodular, $\chi(p) = [\chi_o(p)]$.

Lemma 2.4 (Critical finiteness)**.** *Let G be as in Theorem 1.1, simple. Then $0 < p_c < 1$, and at p_c all open clusters are finite almost surely, simultaneously at every vertex and in every induced subgraph under restriction of the full configuration.*

The lower bound is path counting, the upper bound a Peierls argument with the closed outer vertex boundary, and finiteness at p_c is [16, Theorem 1.1(2)], since $h_V > 0$ forces exponential growth.

Lemma 2.5 (Association, disjoint witnesses, tree bound)**.** *Increasing nonnegative functions of the site bits are positively correlated, and increasing events with disjoint finite open witnesses satisfy the van den Berg–Kesten inequality. For all x, y, z and every domain D,*

$$\mathbb{P}_p(x \leftrightarrow y,\ x \leftrightarrow z \text{ in } D) \le \kappa_p \sum_w T_p^D(x,w) T_p^D(w,y) T_p^D(w,z). \tag{2.3}$$

In particular $\mathbb{E}_p s_x^2 \le \kappa_p \chi_{\max}(p)^3$.

Lemma 2.6 (Threshold order; site form of [15, Lemma 2.4])**.** *For every infinite, connected, locally finite, quasi-transitive graph, $p_{2\to 2} \le p_u$ for site percolation.*

Lemma 2.7 (Subcritical finiteness and type comparison; site form of [15, Lemma 2.10])**.** *Let $\mathrm{Aut}(G)$ be unimodular. For $p < p_c$: $\chi_{\max}(p) < \infty$, $\sup_x \mathbb{E}_p s_x^2 \le \kappa_p \chi_{\max}(p)^3$ and $\chi_{\max} \le c_\sigma \chi$. There is $c > 0$ with $\chi_x(p) \ge c\chi(p)$ for all x and $p_c/2 \le p < p_c$. The kernel T_p is symmetric, positive semidefinite and bounded, with $\|T_p\| \le \chi_{\max}(p)$.*

Finiteness of $\chi_{\max}$ below p_c is site sharpness on quasi-transitive graphs [1, Theorem 2], where [15, Lemma 2.10] uses the bond argument of [6]; the other assertions are proved in Appendix A.

Lemma 2.8 (Russo identities in operator norm; site form of [15, Lemma 2.11])**.** *Let* $\mathrm{Aut}(G)$ *be unimodular. The map* $p \mapsto T_p$ *is locally absolutely continuous in operator norm on* $(0, p_c)$*, and so is* χ*. For almost every* $p \in (0, p_c)$*,*

$$(1-p)\,T_p'(x,y) = \sum_v \mathbb{P}_p(v \text{ closed},\ x \in \widehat{C}_v,\ y \in \widehat{C}_v,\ x \not\leftrightarrow y), \tag{2.4}$$

equivalently $(1-p)T_p' = K_p^c + R_p$ *with the nonnegative symmetric invariant kernels*

$$K_p^c(x,y) = \mathbb{E}_p[k_s(C_x, C_y);\ x, y \text{ open},\ C_x \neq C_y], \qquad k_s(A,B) = |\partial A \cap \partial B|,$$

$R_p(x,y) = \mathbb{P}_p(x \text{ closed},\ y \in \widehat{C}_x) + \mathbb{P}_p(y \text{ closed},\ x \in \widehat{C}_y)$ *for* $x \neq y$*, and* $R_p(x,x) = 1-p$*. Moreover, entrywise,*

$$0 \le R_p \le (1-p)I + \mathrm{Adj}\,T_p + T_p\mathrm{Adj}, \qquad (1-p)T_p' \le (I + T_p\mathrm{Adj})(I + \mathrm{Adj}\,T_p), \tag{2.5}$$

and

$$(1-p)\chi' = [\mathbb{E}_p \mathbf{1}\{o \text{ closed}\}(1 + 2N_o + Q_o)]. \tag{2.6}$$

With $\Psi(p) = \chi^{-2}[\mathbb{E}_p \mathbf{1}\{o \text{ closed}\}Q_o]$*,*

$$(1-p)\chi' \ge \chi^2\Psi(p), \qquad (1-p)\chi' \le \chi^2\Psi(p) + 1 + 2d\chi_{\max}. \tag{2.7}$$

The pointwise identity (2.4) is the site Russo formula combined with the identity

$$\mathbb{P}(v \text{ pivotal}) = (1-p)^{-1}\mathbb{P}(v \text{ closed and pivotal}),$$

as in the proof of [1, Proposition 19] and in [20, Auxiliary Lemma 5.A]. The new feature is the remainder R_p, from the pivotal sites $v \in \{x, y\}$. In [15] the Russo identity enters only the proof of Proposition 6.1, which needs the lower bound in (2.7) and local absolute continuity, and the proof of Proposition 6.3, which needs (2.4)–(2.5); there the remainder contributes at most a constant multiple of the triangle trace (Section 6.1). In [15], $D(p)$ denotes the bond analogue of $\Psi(p)$.

Lemma 2.9 (Sprinkling; site form of [15, Lemma 2.12])**.** *For* $0 \le p, a \le 1$*, entrywise,*

$$T_{p+(1-p)a} \le T_p + \sum_{j \ge 1} a^j (I + T_p\mathrm{Adj})\,[\mathrm{Adj}(I + T_p\mathrm{Adj})]^{j-1}(I + \mathrm{Adj}\,T_p). \tag{2.8}$$

If T_p *is bounded and* $ad(1 + d\|T_p\|) < 1$*, the left side is bounded. In particular, if* $p_c < 1$*, boundedness of* T_{p_c} *implies* $p_c < p_{2\to 2}$*.*

Lemma 2.10 (Monotone passage; site form of [15, Lemma 9.2])**.** *Let* G *be an infinite locally finite graph, let* $p_* \in (0, 1]$*, and let* T_p *be the Bernoulli site two-point kernel of* G *at parameter* p*. Then* T_{p_*} *is bounded on counting-measure* $\ell^2(V)$ *if and only if* $\sup_{0 \le p < p_*} \|T_p\|_{2\to 2} < \infty$*. In that case the supremum equals* $\|T_{p_*}\|_{2\to 2}$*.*

Proposition 2.11 (One coupling for an interval; site form of [15, Proposition 9.3])**.** *Let* G *be infinite, connected, locally finite and quasi-transitive. Suppose that* $p_c < p_1 < p_2 < 1$ *and that* T_{p_2} *is bounded on* $\ell^2(V)$*. In the coupling by independent uniform labels on the vertices, almost surely there are infinitely many infinite clusters simultaneously at every* $p \in [p_1, p_2]$*.*

As noted in Section 1.3, Proposition 2.11 also follows from [7, Theorem 1.3] with its remark on site percolation, once $p_c < p_u$ is known.

3 Proof of Theorem 1.1 from the two cases

The proof rests on two theorems, one for each type of automorphism group. The first is the site form of [15, Theorem 2.6], proved in Sections 4–6.

Theorem 3.1 (Critical boundedness for a unimodular action)**.** *Let G be an infinite, connected, locally finite, simple, quasi-transitive graph with $h_V(G) > 0$. If $\mathrm{Aut}(G)$ is unimodular, then for Bernoulli site percolation $\|T_{p_c}\|_{2\to 2} < \infty$.*

The second is proved in Section 7. Let Δ be Hutchcroft's modular function of $\Gamma = \mathrm{Aut}(G)$ [10, Section 2.1], and for Bernoulli site percolation put

$$\chi_{p,\lambda}(u) = \sum_x T_p(u,x)\Delta(u,x)^\lambda, \qquad p_c(\lambda) = \sup\Big\{p : \sup_u \chi_{p,\lambda}(u) < \infty\Big\},$$

and $p_t(G,\Gamma) = \sup\{p : \sup_u \chi_{p,\lambda}(u) < \infty \text{ for some } \lambda \in \mathbb{R}\}$.

Theorem 3.2. *Let G be an infinite, connected, locally finite, quasi-transitive graph such that $\Gamma = \mathrm{Aut}(G)$ is nonunimodular. For Bernoulli site percolation:*

(1) *$p_c < p_c(\lambda) \le p_c(1/2) = p_t(G,\Gamma)$ for every $0 < \lambda < 1$;*

(2) *$\|T_p\|_{2\to 2} \le \sup_u \chi_{p,\lambda}(u)$ for every p and λ; hence $p_c < p_t(G,\Gamma) \le p_{2\to 2} \le p_u$;*

(3) *$\|T_{p_c}\|_{2\to 2} \le \sup_u \chi_{p_c,1/2}(u) < \infty$, and $\sup_{p<p_c}\|T_p\|_{2\to 2} = \|T_{p_c}\|_{2\to 2}$.*

Vertex-transitivity is neither assumed nor excluded.

For vertex-transitive G, Theorem 3.2 is the combination of [20, Proposition 5.5] with Hutchcroft's operator bound (Proposition E.1). The nonunimodular case uses no nonamenability hypothesis, since a nonunimodular automorphism group forces nonamenability (Section 7.2).

Proof of Theorem 1.1. By Section 2.1 we may assume that $G = G_s$ is simple. We split according to whether $\mathrm{Aut}(G)$ is unimodular; the two cases are exhaustive.

$\mathrm{Aut}(G)$ *unimodular.* Theorem 3.1 gives $\|T_{p_c}\| < \infty$. By Lemma 2.4, $p_c < 1$, so Lemma 2.9 at $p = p_c$ with $0 < a < (d(1 + d\|T_{p_c}\|))^{-1}$ gives a bounded T_{p_2} at $p_2 := p_c + (1 - p_c)a \in (p_c, 1)$; hence $p_c < p_{2\to 2}$. Lemma 2.6 gives $p_{2\to 2} \le p_u$, and Lemma 2.10 with $p_* = p_c$ gives the supremum identity. This proves (1) and (2). For (3), choose $p_1 \in (p_c, p_2)$ and apply Proposition 2.11.

$\mathrm{Aut}(G)$ *nonunimodular.* Theorem 3.2(2)–(3) gives $p_c < p_t(G, \mathrm{Aut}\, G) \le p_{2\to 2} \le p_u$, $\|T_{p_c}\| < \infty$ and $\sup_{p<p_c}\|T_p\| = \|T_{p_c}\|$, which are (1) and (2). For (3), pick $p_2 \in (p_c, p_{2\to 2})$. Then T_{p_2} is bounded, because the parameters with bounded T_p form a downward interval by entrywise domination [15, Lemma 2.4]; and $p_2 < p_{2\to 2} \le p_u \le 1$. Take $p_1 \in (p_c, p_2)$ and apply Proposition 2.11. □

4 Good critical clusters

The local endpoint maps of Section 5 splice pieces cut from capped critical clusters, and the inverse of each splice must locate its seam among the separating vertices of the output cluster. This section proves the site form of [15, Proposition 3.2]: outside a stretched-exponentially small event, every piece left after a deletion from a capped critical cluster has few separating cut vertices relative to its number of contacts. Throughout, G is simple, infinite, connected, locally finite and quasi-transitive with $h_V(G) > 0$, and $p = p_c$. No unimodularity is used in this section.

4.1 Cut vertices and good clusters

For sites, an open edge between two open vertices cannot be closed independently, so the bridges of [15, Definition 3.1] are replaced by cut vertices.

Definition 4.1 (Site pivotal distance and good clusters)**.** Let H be a finite connected set of open vertices, with its induced graph. A *cut vertex separating x from y* is a vertex $z \in H \setminus \{x, y\}$ such that x and y lie in different components of $H \setminus \{z\}$. Let $\mathrm{piv}_H(x, y)$ be the number of such z, and put $\mathrm{diam}_{\mathrm{piv}}(H) = \max_{x,y \in H} \mathrm{piv}_H(x, y)$ and $\mathrm{rad}_{\mathrm{piv}}(H, x) = \max_{y \in H} \mathrm{piv}_H(x, y)$.

Fix $0 < \varepsilon < 1/2$ and put $\tau = 1/2 + \varepsilon$. A finite open cluster C with $|C| \le M$ is *good for cap M* if the following holds for every $A \subseteq C$: for every open component B of $C \setminus A$, let K be the number of graph edges between B and A; whenever $K \ge 1$,

$$\mathrm{diam}_{\mathrm{piv}}(B) \le K\, M^{\tau},$$

with the pivotal distance computed inside B.

An open component is a connected component of the induced graph on $C \setminus A$. Since C is open, every edge between B and A joins two open vertices, so K is the site reading of the number of open edges from B to A in [15, Definition 3.1]. Call a vertex of $C \setminus B$ adjacent to B a *contact* of B. Every contact lies in A, the set $\partial B \cap C$ is exactly the set of contacts, every vertex of $\partial B \setminus C$ is closed, and with k the number of contacts, $k \le K \le dk$. If C is connected and $A \neq \varnothing$, then $K \ge 1$ for every component B, and $C_x = \varnothing$ is good.

Lemma 4.2 (Elementary facts)**.** *Let H be finite and connected, and $x, y, w \in H$.*

(1) (Order.) *The cut vertices separating x from y lie on every x–y path in H, and they occur in the same order on every simple x–y path. More generally, for x and a nonempty set $Y \subseteq H \setminus \{x\}$, the vertices $z \in H \setminus \{x\}$ that meet every path from x to Y ($z \in Y$ allowed) occur in one order along every simple path from x to a fixed $y \in Y$.*

(2) (Triangle inequality.) $\mathrm{piv}_H(x, y) \le \mathrm{piv}_H(x, w) + \mathrm{piv}_H(w, y) + 1$.

(3) (Deletion.) *If $R \subseteq H$ and x, y lie in one component H'' of $H \setminus R$, then every cut vertex of H separating x from y that is not in R lies in H'' and separates x from y in H''. Hence* $\mathrm{piv}_{H''}(x, y) \ge \mathrm{piv}_H(x, y) - |R|$.

(4) (Separated indices.) *If $z_1, \dots, z_m$ are the cut vertices separating x from y, listed in the order of* (1)*, then* $\mathrm{piv}_H(z_t, z_{t'}) \ge |t - t'| - 1$.

(5) $\mathrm{piv}_H(x, y) \le |H| - 2$ *when* $|H| \ge 2$*, so* $\mathrm{diam}_{\mathrm{piv}}(H) \le |H| - 1$.

The proofs are short graph arguments (Appendix B.1). The $+1$ in (2) is absent for bridges, since an edge separating x from y separates x from w or w from y; it costs only constants below.

Proposition 4.3 (Good critical clusters; site form of [15, Proposition 3.2])**.** *For each $\varepsilon \in (0, 1/2)$, uniformly in x and for all sufficiently large M,*

$$\mathbb{P}_{p_c}(s_x \le M,\ C_x \text{ not good for cap } M) \le e^{-M^{\varepsilon}}.$$

The event that $s_x \le M$ and C_x is good for cap M is determined by the site bits in the deterministic ball $B(x, M)$. The constants depend only on d, p_c and ε.

4.2 Marked pivotal moments and the radius tail

The probabilistic input is the site form of the marked-cluster estimate of [15, Lemmas 3.3 and 3.4]. In a deterministic domain $D \ni x$, mark vertices independently with probability $h \in (0, 1/2]$, independently of ω, and let $E = E(D, x, h)$ be the event that C_x^D contains a mark; it requires x to be open. A vertex is *pivotal for* E if changing its state changes $\mathbf{1}_E$, with the marks and all other bits fixed. Let N_{piv} be the number of open pivotal vertices *other than* x. On E the root x is always an open pivotal, since closing it empties C_x^D; excluding it makes the recursion below a recursion of the same quantity. On E, an open $v \neq x$ is pivotal if and only if every path in C_x^D from x to a mark meets v, so by Lemma 4.2(1), with Y the set of marks, the open non-root pivotals are linearly ordered from x toward the marks. (If x itself is marked there are no open non-root pivotals, since closing any $v \neq x$ leaves x marked, and the ordering is vacuous.)

Lemma 4.4 (Site marked pivotal moments; site form of [15, Lemma 3.3])**.** *There is a constant C, depending only on d and p_c, such that, with $A_h = C\sqrt{\log(1/h)/h} \geq 1$, for every domain D, every $x \in D$, every integer $i \geq 0$ and every $r \geq 0$,*

$$\mathbb{E}\Big[\binom{N_{\mathrm{piv}}}{i}\mathbf{1}_E\Big] \leq A_h^i\,\mathbb{P}(E), \qquad \mathbb{P}(E,\ N_{\mathrm{piv}} \geq r) \leq 2\mathbb{P}(E)e^{-r/(4A_h)}.$$

The proof (Appendix B.2) follows [15]: an exploration score bounds the first moment, through Hoeffding's lemma and the identity $\mathbb{E}[Z\mathbf{1}_E] = (1-p_c)\mathbb{E}[(N_{\mathrm{piv}}+\mathbf{1}_E)\mathbf{1}_E]$, and an exact recursion over the first pivotal gives the higher binomial moments. For sites the recursion removes the root cluster with the first pivotal v closed, $W = C_x^{D\setminus\{v\}}$, and continues in the exterior domain $(D\setminus \mathrm{Nb}_D[W])\cup\{v\}$ rooted at v; there is no separate factor p_c, since the openness of v is part of the continued event. Also $\mathbb{P}(E) \geq p_c h$ in place of h, as x must be open.

Lemma 4.5 (Site radius tail; site form of [15, Lemma 3.4])**.** *Fix $\varepsilon \in (0, 1/2)$ and $\kappa_0 \in (0, 1]$. There are $c > 0$ and M_0, depending only on d, p_c, ε and κ_0, such that, uniformly in x, for all $M \geq M_0$ and all real κ with $\kappa_0 \leq \kappa \leq M^{1-\tau}$,*

$$\mathbb{P}_{p_c}(s_x \leq M,\ \mathrm{rad}_{\mathrm{piv}}(C_x, x) \geq \kappa M^{\tau}/2) \leq \exp(-c\kappa^2 M^{2\varepsilon}/\log M).$$

For integers $1 \leq \kappa \leq M^{1-\tau}$ this is the bound of [15, Lemma 3.4], with the bridge replaced by a cut vertex; it follows from Lemma 4.4 with a single mark at a witness y (Appendix B.2). The real range $\kappa \geq \kappa_0$, used with $\kappa_0 = 1/(2d)$ below, changes only M_0 and c.

4.3 Pruning at a retained contact

The bond proof keeps one open contact edge from the rest of the root cluster into a bad piece B; then B has one attachment, and its bridge tree embeds isometrically [15, Section 3]. For sites the retained object is a contact vertex a, which may have several neighbours in B and so short-circuit the cut vertices of B: if B is a path and a is adjacent to both of its ends, then $B \cup \{a\}$ is a cycle, with no cut vertices at all. We therefore also close all but one neighbour of a in B. The next lemma shows that the kept neighbour can be chosen so that the pivotal distance drops by at most a factor $2d$.

Lemma 4.6 (Pruning)**.** *Let B be a finite connected graph with $\mathrm{diam}_{\mathrm{piv}}(B) = \mathsf{D}$, and let $N \subseteq B$ be nonempty with $|N| \leq d$. Then there are $b^* \in N$ and $v \in B$ with the following property. Put $R = N \setminus \{b^*\}$, and let B'' be the component of b^* in $B \setminus R$. Then $v \in B''$ and*

$$\mathrm{piv}_{B''}(b^*, v) \geq \frac{\mathsf{D}}{2d} - d - \frac{1}{2}.$$

Proof. For $u \in B$ let $n(u)$ be a vertex of N nearest to u in the graph distance of B, ties broken by a fixed order, and put $V_b = \{u \in B : n(u) = b\}$ for $b \in N$. These Voronoi cells partition B.

(i) *Each $u \in V_b$ lies in the component of b in $B \setminus (N \setminus \{b\})$.* If $u \in N$ then $n(u) = u$, so $u = b$. Otherwise take a geodesic in B from u to b, of length $\mathrm{dist}(u, N)$. A vertex $q \neq b$ on it lying in N would satisfy $\mathrm{dist}(u, q) < \mathrm{dist}(u, b)$, which is impossible. So the geodesic avoids $N \setminus \{b\}$.

(ii) *Some cell contains a far vertex.* Put $L = \max_{b \in N} \max_{u \in V_b} \mathrm{piv}_B(b, u)$. Choose a pair (u_0, w_0) with $\mathrm{piv}_B(u_0, w_0) = \mathsf{D}$, and list its separating cut vertices $z_1, \dots, z_{\mathsf{D}}$ in the order of Lemma 4.2(1). If $z_t, z_{t'} \in V_b$, then by Lemma 4.2(4) and (2),

$$|t - t'| - 1 \leq \mathrm{piv}_B(z_t, z_{t'}) \leq \mathrm{piv}_B(z_t, b) + \mathrm{piv}_B(b, z_{t'}) + 1 \leq 2L + 1.$$

So the indices t with $z_t \in V_b$ lie in an interval of length at most $2L + 2$, and each cell contains at most $2L + 3$ of the z_t. The at most d cells contain all D of them, so $\mathsf{D} \leq d(2L + 3)$, that is, $L \geq \mathsf{D}/(2d) - 3/2$.

(iii) *Deleting R.* Choose (b^*, v) with $v \in V_{b^*}$ and $\mathrm{piv}_B(b^*, v) = L$, by a fixed order. By (i), $v \in B''$. By Lemma 4.2(3) with $|R| \leq d - 1$, $\mathrm{piv}_{B''}(b^*, v) \geq L - (d - 1) \geq \mathsf{D}/(2d) - d - 1/2$. □

The lemma is deterministic, so (b^*, v, R) is a fixed function of (B, N); it uses no randomness, no independence and no group action.

4.4 Proof of Proposition 4.3: the modified configuration

Fix $\varepsilon \in (0, 1/2)$; constants depend only on d, p_c and ε. Take M so large that $M^\tau \geq 4d^2$. Let $\mathrm{Bad} = \{s_x \leq M,\ C_x \text{ not good for cap } M\}$; on Bad, x is open. Choose a witnessing pair (A, B) by a fixed finite ordering: $A \subseteq C_x$, B an open component of $C_x \setminus A$ with $K \geq 1$ graph edges to A, and $\mathsf{D} := \mathrm{diam}_{\mathrm{piv}}(B) > KM^\tau$. Since $\mathsf{D} \leq |B| - 1 < M$ (Lemma 4.2(5)), $K < M^{1-\tau}$. The proof closes a set J of open vertices of C_x, chosen by fixed rules, and shows that the resulting configuration ω' has a pivotal radius of order D; a Peierls-type count then compares with Lemma 4.5.

- *Case 1, $x \in B$.* J is the set of all contacts of B. In ω' every vertex of ∂B is closed, so $C'_x = B$. A diameter pair of B and Lemma 4.2(2) give $\mathrm{rad}_{\mathrm{piv}}(C'_x, x) \geq (\mathsf{D} - 1)/2$.
- *Case 2, $x \notin B$.* Take a simple open path γ in C_x from x to B, by a fixed ordering. Its vertex just before its first vertex in B is a contact, so γ meets the set of contacts; let a be the *first* contact on γ ($a = x$ is possible). Put $N = N(a) \cap B$, which is nonempty with $|N| \leq d$, and apply Lemma 4.6 to (B, N), obtaining b^*, v, $R = N \setminus \{b^*\}$ and B''. Let J consist of all contacts other than a, together with R.

In Case 2 the following hold (Appendix B.3).

(a) γ from x to a is open in ω'.

(b) In ω', the only open vertex outside B'' with a neighbour in B'' is a, and the only neighbour of a in B'' is b^*.

(c) Hence every cut vertex of B'' separating b^* from v, and b^* itself, separate x from v in C'_x, and Lemma 4.6 gives $\mathrm{rad}_{\mathrm{piv}}(C'_x, x) \geq \mathsf{D}/(2d) - d + 1/2$.

Closing only shrinks clusters, and $\mathsf{D} > KM^\tau \geq 4d^2$, so in both cases ω' lies in the radius-tail event

$$O_K := \{s_x \leq M,\ \mathrm{rad}_{\mathrm{piv}}(C_x, x) \geq \kappa M^\tau/2\}, \qquad \kappa = K/(2d), \quad 1/(2d) \leq \kappa \leq M^{1-\tau}.$$

Appendix B.3 completes the proof by counting. Let Bad_K be the part of Bad whose witness has count K. The configuration ω is ω' with at most $K + d$ vertices of C_x reopened, and these

can be listed one at a time as neighbours of the growing root cluster, so each ω' has at most $(K+d+1)(dM)^{K+d}$ preimages in Bad_K. With $\vartheta = \max\{1, p_c/(1-p_c)\}$, a pointwise comparison on the finite product of the bits in $B(x,M)$ gives $\mathbb{P}(\mathrm{Bad}_K) \le (K+d+1)(dM)^{K+d}\vartheta^{K+d}\,\mathbb{P}(O_K)$. Lemma 4.5 with $\kappa_0 = 1/(2d)$ and a sum over the fewer than M values of K give $\mathbb{P}(\mathrm{Bad}) \le e^{-M^{\varepsilon}}$ for all sufficiently large M, uniformly in x, and a breadth-first exploration of C_x stopped after more than M open vertices gives locality.

Compared with [15], Case 1 closes the $k \le K$ contact vertices in place of the open contact edges. Case 2 replaces retention of the first contact edge entering B by retention of the first contact vertex a on γ, closure of a's other neighbours in B, and the choice of b^* by Lemma 4.6. This loses the factor $2d$ in the radius, absorbed by the real range of Lemma 4.5, and adds at most $d-1$ changed vertices, absorbed in the summation. The sequential listing of the reopened vertices replaces the observation of [15, Section 3] that every changed edge is incident to $B \subseteq C'_x$: after pruning, $B \setminus B''$ and some contacts can leave C'_x.

5 Amplification of local endpoint maps

This section proves the site form of [15, Proposition 4.3]: a heavy tail of the critical cluster size yields local rules whose endpoint kernels are dispersive much faster than their success probability decays or their cap grows. Throughout, G is simple, infinite, connected, locally finite and quasi-transitive with $h_V(G) > 0$, $\Gamma = \mathrm{Aut}(G)$ is unimodular, and percolation has parameter p_c, with $0 < p_c < 1$ by Lemma 2.4. Put

$$c_0 := (1-p_c)^{d-1} \in (0,1).$$

Suppose that

$$[\mathbb{E}_{p_c} s_o^{\alpha}] = \infty \qquad \text{for some } 0 < \alpha < \tfrac{1}{2}, \tag{5.1}$$

and choose $\beta > \alpha$ and $\varepsilon > 0$ so that, with $\tau = \frac{1}{2} + \varepsilon$, $\beta + \tau < 1$, as in [15, Section 4]. Goodness (Definition 4.1) always uses this τ, and enters only through the inequality $\mathrm{diam}_{\mathrm{piv}}(B) \le KM^{\tau}$, in Lemma 5.6.

5.1 Local endpoint maps

Lemma 5.1 (Selection of an orbit; site form of [15, Lemma 4.1])**.** *There are a vertex orbit O, a constant $c > 0$ and an unbounded sequence $\mathcal{N}$ of dyadic integers such that, for $n = 2^j \in \mathcal{N}$ and $\pi = [\mathbb{P}_{p_c}(n \le s_o < 2n)]$,*

$$\begin{aligned}
&\text{(a)} && \pi \ge n^{-\beta}, && \\
&\text{(b)} && [\mathbb{P}_{p_c}(n/2 \le s_o < 2n)] \le (1+2^{\beta})\pi, && \\
&\text{(c)} && \mathbb{E}_{p_c}\big[|C_x \cap O| \mathbf{1}\{n \le s_x < 2n\}\big] \ge cn\pi && (x \in O), \\
&\text{(d)} && [\mathbb{P}_{p_c}(2^i \le s_o < 2^{i+1})] \le 2^{\beta(j-i)}\pi && (0 \le i \le j), \\
&\text{(e)} && \mathbb{E}_{p_c}[s_x^2;\ 1 \le s_x < 2n] \le C_O n^2 \pi && (x \in O),
\end{aligned} \tag{5.2}$$

where (e) *follows from* (d)*, with $C_O = 4/(\sigma_O(1-2^{\beta-2}))$ and σ_O the type weight of O. The orbit and the sequence can be kept fixed throughout the amplification.*

Bounds (a)–(c) are those of [15, Lemma 4.1], with $s_o = 0$ when o is closed. Bounds (d) and (e) are new. The proof selects $\mathcal{N}$ among the record indices j of $j \mapsto 2^{\beta j}\pi_j$, where $\pi_j = [\mathbb{P}_{p_c}(2^j \le s_o < 2^{j+1})]$, that is, $2^{\beta i}\pi_i < 2^{\beta j}\pi_j$ for all $i < j$; this gives (d), whose case $i = j-1$ gives (b). We call the $n \in \mathcal{N}$ *record scales*. The second-moment bound (e) at record scales is used only in Lemma 5.8, to replace a lower size window that the site argument loses. Both are proved in Appendix C.1.

Definition 5.2 (Local endpoint map; site form of [15, Definition 4.2])**.** A *local endpoint map* on an orbit O is a randomized rule at each $x \in O$ that reads site bits in a deterministic finite ball about x and uses auxiliary randomness independent of those bits. Its conditional rules are covariant under $\mathrm{Aut}(G)$. The rule either fails or outputs $y \in C_x \cap O$. A map with integer cap M is required, on every successful input pattern, to have $s_x \le M$ and C_x good for cap M. Write $u > 0$ for its success probability and

$$K(x,y) = \mathbb{P}(\text{output } y \mid \text{success at } x), \quad r = \|K\|, \quad U = \log(1/u), \quad m = \log M, \quad H = \log(1/r).$$

All probabilities include the auxiliary randomness, and the norm is on counting-measure $\ell^2(O)$.

The definition is read with two conventions, which hold for every map constructed below. (i) A successful input pattern is a pattern of the read bits together with an auxiliary outcome on which the rule succeeds, and the guarantees $y \in C_x \cap O$, $s_x \le M$ and goodness hold for every configuration agreeing with the pattern on the read ball; they are properties of the pattern, since $\{s_x \le M,\ C_x \text{ good},\ y \in C_x\}$ is determined by the bits within distance M of x. (ii) The rule is a fixed probability kernel from read-bit patterns to outcomes, realized with auxiliary uniforms whose law does not depend on p. Invariance makes u independent of $x \in O$; K has row sums one and, by Lemma 2.1 restricted to $O \times O$, column sums one; so $0 < r \le 1$.

Proposition 5.3 (Efficient site endpoint maps; site form of [15, Proposition 4.3])**.** *Under (5.1), for every $\eta > 0$ there is a local endpoint map on the orbit O of Lemma 5.1, with $M \ge 2$ and $H > 1$, such that*

$$\frac{U+m}{H} < \eta.$$

Moreover the maps can be chosen so that $H \to \infty$. Their radii are finite, but need not be bounded uniformly over the construction.

The proof follows [15, Section 4]: a starting map from a random walk with H as large as desired (Lemma 5.4), an amplification step built from k old steps by splicing (Lemma 5.9), and an iteration whose parameter order makes $(U+m)/H$ decrease geometrically. The site changes are in the splicing step, Sections 5.2–5.5; the starting map, the step and the iteration are adapted in Appendix C.1.

Lemma 5.4 (Starting map; site form of [15, Lemma 4.4])**.** *For every H_0 there is a site local endpoint map on O with $H > H_0$, with cap as large as prescribed.*

5.2 The experiment and the site switching procedure

Fix an old map with parameters (u, M, r), an integer $k \ge 1$, and $n \in \mathcal{N}$ with $kM \le n/2$. The seeds are as in [15, Section 4.2]: old-map seeds, independent continuous priorities on the edges of G ordering contact lists, and a uniform $V_i \in [0,1]$ for each index $1 \le i \le k$. Ties have probability zero and are rejected.

For fixed $x, z \in O$, require in a distinguished site field ω that $A = C_x(\omega)$ satisfies $n \le |A| < 2n$, so that x is ω-open. Independently, start at z and take k old-map steps, with new independent site fields and new seeds for each step. List a successful chain in reverse order, $y_0, y_1, \dots, y_k = z$; let E_i be the field used for the step from y_i, with output y_{i-1}, and $C_i = C_{y_i}(E_i)$, which contains y_i and y_{i-1}, has $|C_i| \le M$ and is good for cap M. Require $y_0 \in A$. The experiment has mass $1/n$ if all this holds *and all rejection tests below pass*, and zero otherwise.

Start with $S = A$ and no index selected. An index is *eligible* if it exceeds every index already selected. While $z \notin S$:

(1) *Selection.* Select the largest eligible i with $C_i \cap \mathrm{Nb}[S] \neq \varnothing$.

(2) *Piece.* In the *standard case* $y_i \notin \mathrm{Nb}[S]$, let Y_i be the component of y_i in the E_i-open subgraph induced by $V \setminus \mathrm{Nb}[S]$; let W_i be the set of E_i-open vertices of ∂S adjacent to Y_i; and let the *contact list* be the K_i graph edges between Y_i and W_i, ordered by priority. Choose the contact edge of rank $\lceil K_i V_i \rceil$, write it $\{w_i, b_i\}$ with $w_i \in W_i$ and $b_i \in Y_i$, and call w_i the *seam*. In the *degenerate case* $y_i \in \partial S$, which is possible only for $i = k$ (Lemma 5.5), put $w_k = z$ and $Y_k = \varnothing$; no uniform is used.

(3) *Junction rejection.* Put $R_i = N(w_i) \setminus (\mathrm{Nb}[S] \cup \mathrm{Nb}[Y_i])$, with $\mathrm{Nb}[\varnothing] = \varnothing$. Reject, with mass zero, unless every vertex of R_i is closed in the current ω.

(4) *Exchange.* Exchange the current ω and E_i bits on the mask $\mathrm{Mask}_i = \{w_i\} \cup (\mathrm{Nb}[Y_i] \setminus \mathrm{Nb}[S])$.

(5) Set $S \leftarrow S \cup \{w_i\} \cup Y_i$, and call $P_i = \{w_i\} \cup Y_i$ the *inserted piece*.

Opening a closed seam w attaches every currently open component adjacent to w, up to $d-1$ foreign branches; this is the junction problem. Steps (3)–(4) force the new root cluster to be exactly $S \cup P_i$: the mask makes Y_i open and its outer boundary off $\mathrm{Nb}[S]$ closed, both from E_i, and the rejection set R_i covers every other neighbour of w_i off $\mathrm{Nb}[S]$, whatever its E_i-state. The rejection set must include E_i-closed neighbours of w_i off $\mathrm{Nb}[S] \cup \mathrm{Nb}[Y_i]$ too, since they are outside the mask and an ω-open one would attach. The set R_i is not exchanged at its own step; it meets later masks only at the next seam, whose input bit the inverse restores by undoing that mask, so the inverse has no ambiguity there.

Lemma 5.5 (Geometry of the site switching procedure; site form of [15, Lemma 4.5])**.** *Every step is well defined, with $K_i \geq 1$ in the standard case. If the step's rejection test passes, its effect on the distinguished root cluster is exactly $S \mapsto S \cup \{w_i\} \cup Y_i$, and w_i is a cut vertex of the new root cluster separating S from Y_i. The selected indices increase. Let $i_1 < \cdots < i_m$ be the selected indices and S_j the set before the j-th selection, $S_1 = A$. Then every later piece avoids the closed neighbourhood of every earlier set: $P_{i_{j'}} \cap \mathrm{Nb}[S_j] = \varnothing$ for $j' > j$, and $Y_{i_j} \cap \mathrm{Nb}[S_j] = \varnothing$. Consequently w_{i_j} is the only vertex of $C' \setminus S_j$ adjacent to S_j, where $C' = C_x(\omega')$ is the final root cluster, so*

$$S_j = \textit{the component of } x \textit{ in } C' \setminus \{w_{i_j}\};$$

for $j < m$, and for $j = m$ in the standard case, w_{i_j} separates x from z in C'. In the degenerate case $w_{i_m} = z$. Finally

$$z \in C_x(\omega'), \qquad s_x(\omega') \leq M' = 2n + kM. \tag{5.3}$$

Lemma 5.5 is proved in Appendix C.2, by induction over the steps. Before each step the current root set S is the open cluster of x in the current distinguished field. After a passed rejection test and the exchange, the outer boundary of $S \cup \{w_i\} \cup Y_i$ lies in $(\partial S \setminus \{w_i\}) \cup (\partial Y_i \setminus \mathrm{Nb}[S]) \cup R_i$, which is closed, so the new root cluster is exactly $S \cup \{w_i\} \cup Y_i$. Maximality of the selected index keeps every later piece outside the closed neighbourhood of every earlier set. The same induction shows that the rejection bits are fresh: $R_i \subseteq V \setminus \mathrm{Nb}[S]$, while every earlier mask and rejection set lies in $\mathrm{Nb}[S]$, so no ω-bit of $V \setminus \mathrm{Nb}[S]$ has been read.

Thus C' is a chain $A - w_{i_1} - Y_{i_1} - w_{i_2} - \cdots - w_{i_m} - Y_{i_m}$ in which each set is adjacent to the earlier ones only through the next seam; $w_{i_{j+1}}$ may be adjacent to w_{i_j} or to vertices of Y_{i_j}, but to nothing in S_j.

5.3 Inverse codes

For $c \in C' \setminus \{x\}$ let $X(c)$ be the component of x in $C' \setminus \{c\}$ and $\mathrm{Hang}(c) = C' \setminus X(c)$, the *hanging set* at c. It contains c, and c is its only vertex adjacent to $X(c)$, so c is determined by $\mathrm{Hang}(c)$.

Lemma 5.6 (Inverse codes and contact probabilities; site form of [15, Lemma 4.6]). *Fix the output fields, x, z, the old-map seeds and the priorities. Let $C' = C_x(\omega')$, and let $\Lambda = (\lambda_0, \lambda_1, \dots, \lambda_q)$ be the list with $\lambda_0 = z$ followed by the cut vertices of C' separating x from z, ordered from z toward x. A valid switching input is specified by its set of selected indices and positive gaps locating the selected seams in Λ. Each fixed code determines at most one input bit tuple, without using the uniforms V_i. After integrating those uniforms, the total mass of valid codes is at most*

$$W^k, \qquad W = 1 + \sum_{l=1}^{M} \min\Big\{1, \frac{4dM^\tau}{l}\Big\} \le CM^\tau \log M \quad (M \ge 2). \tag{5.4}$$

The proof (Appendix C.2) reads the seams off the hanging sets. By Lemma 5.5, $\mathrm{Hang}(w_{i_j}) = C' \setminus S_j$, so the seams occur in Λ in the order $w_{i_m}, \dots, w_{i_1}$, and from a code one recovers the sets S_j, the pieces and the masks, undoes the exchanges in reverse order and replays the procedure. Goodness of the old cluster C_{i_j}, with deleted set $C_{i_j} \cap \mathrm{Nb}[S_j]$, gives the gap bound $l_j \le 3 + K_{i_j} M^\tau$; this is the only place where goodness enters. The prescribed seam is an endpoint of at most $\min\{d, K_{i_j}\}$ contact edges, so the uniform V_{i_j} lies in a set of measure at most $\min\{1, d/K_{i_j}\} \le \min\{1, 4dM^\tau/l_j\}$, and summing over codes gives (5.4).

5.4 The load and its mean

All comparisons are made on finite bit spaces, by conditioning on the bits outside a deterministic ball about x that contains every query, mask and rejection set of every successful experiment at (x, z), as in [15, Section 4.3] (Appendix C.2). At fixed seeds an exchange swaps bits between equally distributed i.i.d. fields on a common mask, and rejection changes no bit; so the number of open bits is preserved, and each successful input tuple and its output have the same product probability. With the injective inverse at a fixed code (Lemma 5.6), define, for a single reference i.i.d. distinguished site field ω', $L_x(z, \omega')$ as $1/n$ times the sum over codes of the valid-code indicators, integrated over reference i.i.d. auxiliary output fields and all seeds, including the V_i. This is the density of the switched mass at z relative to the law of ω', and by (5.4), $0 \le L_x(z, \omega') \le W^k/n$. It vanishes unless (5.3) holds, and it is a covariant function of ω' in the ball of radius M' about x (Appendix C.2). Put $L = \sum_{z \in O} L_x(z, \omega')$ and $b_0 = \mathbb{E}L$. As in [15], the reference product law is not a claim about the law of the actual output of a switch.

Lemma 5.7 (Mean load with junction rejection; site form of [15, Lemma 4.7]). *Let $F(x, z) = \mathbb{E}L_x(z, \omega')$, $B_n(x, y) = \mathbb{P}_{p_c}(y \in C_x,\ n \le s_x < 2n)$ and $F_0(x, z) = (u^k/n) \sum_{y \in O} B_n(x, y) K^k(z, y)$, the bond formula of [15, Lemma 4.7]. Then, entrywise and for the row sum b_0 of F, which is constant on O,*

$$c_0^k F_0 \le F \le F_0, \qquad c c_0^k \pi u^k \le b_0 \le C \pi u^k, \qquad \|F/b_0\| \le c_0^{-k} r^k.$$

Dropping the rejection tests gives the bond formula, so $F \le F_0$. The lower bound $F \ge c_0^k F_0$ is new: conditionally on the root cluster A, the old chain with its fields and the seeds, and on the event that the earlier tests pass, the rejection sets of the selected steps are determined, lie outside $\mathrm{Nb}[A]$ and outside all earlier masks, and have at most $d - 1$ vertices each, so all tests pass with conditional probability at least $(1 - p_c)^{(d-1)k} = c_0^k$. The row-sum and norm bounds then follow as in [15, Lemma 4.7]. The proof is in Appendix C.2.

5.5 The second moment and joint restoration

The second-moment bound of [15, Lemma 4.8] undoes two switching inverses at once. For two inverses whose inserted pieces are far apart, it uses that removing two disjoint hanging components

from the root cluster leaves a connected set containing x. For bonds the first seams are bridges and this holds. For sites it fails: if c^1 and c^2 lie far apart on a cycle of C' through x, each carrying its pieces, removing both hanging sets cuts the cycle. We record the two seams and their neighbourhoods in the remaining set, which costs a polynomial factor, and replay the switching from a set rather than a connected cluster.

Lemma 5.8 (Second moment; site form of [15, Lemma 4.8])**.** *There is a fixed polynomial $\mathcal{Q}(k, M) \geq 1$, for example a large graph-dependent multiple of $1 + \nu + \nu^2 + d^2\nu^3 + 4^d\nu^4$ with $\nu = kM$, such that*

$$\mathbb{E}L^2 \leq \mathcal{Q}(k, M)\, \pi u^{2k} \qquad \text{whenever } kM \leq n/2 \text{ and } n \geq (W/u)^k. \tag{5.5}$$

Outline of the proof of Lemma 5.8. Expand L^2 with a common reference field ω' and two independent families of auxiliary output fields and seeds; a summand is a pair of valid coded inverses with endpoints z_1, z_2. For a switching inverse put $B = \mathrm{Hang}(w_{i_1}) = C' \setminus A = P_{i_1} \cup \cdots \cup P_{i_m}$ (Lemma 5.5); it contains its endpoint and has $|B| \leq \nu$. There are three classes.

Class 1 (at least one inverse makes no switch) is verbatim from [15, Lemma 4.8] and contributes at most $C\pi u^{2k}$. *Class 2* (both switch, and B^1, B^2 are nested or at graph distance at most 2) is a counting bound: hanging sets are nested or disjoint, so, given the first inverse, the second endpoint takes at most $\nu + \nu^2 + d^2\nu^3$ values, each of integrated mass at most W^k/n, and the class contributes at most $C(\nu + \nu^2 + d^2\nu^3)\pi u^{2k}$. Both are proved in Appendix C.3. The site change is in Class 3.

Class 3: both switch, and $\mathrm{dist}(B^1, B^2) \geq 3$*, that is,* $\mathrm{Nb}[B^1] \cap \mathrm{Nb}[B^2] = \varnothing$.

Locality of each inverse. Every vertex that inverse a exchanges, $\mathrm{Mask}^a := \bigcup_j \mathrm{Mask}^a_{i_j}$, every rejection set, every contact set W^a_j and every contact edge lies in $\mathrm{Nb}[B^a]$, because $w^a_j, Y^a_j \subseteq B^a$. So the two mask collections are disjoint, including their seams and rejection sets.

Joint restoration. Let ω^a, with restored auxiliary family E^a, be the input of inverse a alone, so that $C_x(\omega^a) = A^a = C' \setminus B^a$, $n \leq |A^a| < 2n$, and all of ∂A^a is ω^a-closed. Write $c^a = w^a_{i_1}$ for the first seam of inverse a, so that $B^a = \mathrm{Hang}(c^a)$ and $c^a \in \partial A^a$. Undo both inverses (the masks are disjoint, so the order is irrelevant), and call the result ω_{00}; it equals ω^1 off Mask^2 and ω^2 off Mask^1. Put $A_{00} := C' \setminus (B^1 \cup B^2) = A^1 \setminus B^2$, so $|A_{00}| < 2n$. The set A_{00} need not be connected. What survives is the following.

(J1) *A_{00} is a union of ω_{00}-clusters; $A_x := C_x(\omega_{00})$ is its component containing x, and every other component is adjacent to c^1 or c^2.* We have $\mathrm{Mask}^a \cap A^a = \varnothing$, since each $\mathrm{Mask}^a_{i_j}$ misses $S^a_j \supseteq A^a$; so A_{00} is ω_{00}-open, as $\omega_{00} = \omega'$ there. A vertex of ∂A_{00} inside C' is c^1 or c^2, the only vertex of B^a adjacent to $X(c^a) \supseteq A_{00}$; and $c^a \in \partial A^a$ is ω^a-closed and not in Mask^b for $b \neq a$, so it is ω_{00}-closed. A vertex $y \in \partial A_{00} \setminus C'$ is in $\partial A^1 \cap \partial A^2$; if $y \in \mathrm{Mask}^a$, then $y \notin \mathrm{Mask}^b$ and $\omega_{00}(y) = \omega^a(y)$ is closed; if y is in neither mask, $\omega_{00}(y) = \omega'(y)$ is closed. So ∂A_{00} is ω_{00}-closed. A component of A_{00} not containing x has a neighbour in $C' \setminus A_{00} = B^1 \cup B^2$, as C' is connected, hence is adjacent to c^1 or c^2.

(J2) $c^a \in \partial A_x$. Since $\mathrm{Hang}(c^1) \cap \mathrm{Hang}(c^2) = \varnothing$, $c^1 \notin \mathrm{Hang}(c^2)$, so some x–c^1 path γ in C' avoids c^2; it stays in $X(c^2)$, and before its first visit to c^1 it stays in $X(c^1)$. So the part of γ before c^1 is a connected subset of $X(c^1) \cap X(c^2) = A_{00}$ containing x; it lies in A_x, and its last vertex is adjacent to c^1. Likewise for c^2.

(J3) *Recovering A_{00} with bounded extra data.* By (J1), $A_{00} = A_x \cup \bigcup_{y \in N^1 \cup N^2} C_y(\omega_{00})$ with $N^a := N(c^a) \cap A_{00}$. Record the extra datum $\xi = (c^1, c^2, N^1, N^2)$. The seam c^a lies in the first selected old cluster of chain a, hence in its trajectory union T^a, with $|T^a| \leq \nu$, which is computable from the restored chain, and $N^a \subseteq N(c^a)$ has at most 2^d values. So, given the restored tuple, ξ takes at most $4^d\nu^2$ values, and (ω_{00}, ξ) determines A_{00}.

Appendix C.3 completes Class 3. Run virtually from the *set* $S_0 = A_{00}$ with the field ω_{00}, the switching procedure reproduces both original runs step by step, because, by distance 3, a vertex of $\mathrm{Nb}[B^1]$ is neither in B^2 nor adjacent to it. So the joint undo, together with the datum ξ of (J3), is injective, and the map to (ω_{00}, E^1, E^2) is at most $4^d\nu^2$-to-one. By (J2) each restored trajectory hits $\mathrm{Nb}[C_x(\omega_{00})]$, and the trajectory-hit bound of [15, Lemma 4.8], with the record-scale bound (e) of Lemma 5.1 in place of the lost lower size window, bounds Class 3 by $C4^d\nu^4\pi u^{2k}$. Summing the three classes gives (5.5) with $\mathcal{Q}(k, M) = C(1 + \nu + \nu^2 + d^2\nu^3 + 4^d\nu^4)$.

The extra factor $4^d\nu^4$ is polynomial in $\nu = kM$, hence $O_M(\log(k+1))$ after taking logarithms, and it is absorbed exactly as the bond term ν^3 is.

5.6 One amplification step

Lemma 5.9 (One amplification step; site form of [15, Lemma 4.9])**.** *Fix an old map. For $n \in \mathcal{N}$ tending to infinity and $k = O(\log n)$ with $kM \le n/2$ and $n \ge (W/u)^k$, there is, for all sufficiently large such n, a new site local endpoint map with*

$$u' \ge c' c_0^{2k}\pi/\mathcal{Q}(k, M), \qquad r' \le 2c_0^{-k} r^k, \qquad M' = 2n + kM.$$

The threshold for n may depend on the fixed old map.

Lemma 5.9 truncates L at a multiple of $\mathcal{Q}(k, M)u^k c_0^{-k}$ using Lemmas 5.7 and 5.8, discards root clusters that are not good for cap M', which costs relative mass $\mathcal{Q}(k, M)c_0^{-2k} n^\beta e^{-(M')^\varepsilon} \to 0$ because $c_0^{-2k} = n^{O(1)}$ for $k = O(\log n)$, and outputs z with probability $L_x(z, \omega')$ divided by the truncation level. The junction cost enters as c_0^{2k} in u' and c_0^{-k} in r'; both are linear in k in the exponent, and in the iteration of Proposition 5.3 they are absorbed by keeping $t = U + (\tau + \xi)m$ and H above fixed multiples of $\log(1/c_0)$. Neither touches $\beta + \tau < 1$. The proofs of Lemma 5.9 and Proposition 5.3 are in Appendix C.1.

6 Critical boundedness for a unimodular action

This section proves Theorem 3.1. It follows [15, Sections 5–9]. The efficient maps of Proposition 5.3 give critical fractional moments, hence an almost linear susceptibility bound and a subpower bound for the triangle trace; a deformation of T_p by pivotal corridors then contradicts unboundedness of T_{p_c}. Section 6.1 states the site forms of the moment and trace estimates, whose proofs are routine adaptations given in Appendix D. The new ingredient is the site corridor, Sections 6.2 and 6.3. Throughout, G is simple, infinite, connected, locally finite and quasi-transitive with $h_V(G) > 0$, and $\mathrm{Aut}(G)$ is unimodular.

6.1 Moments, susceptibility and the triangle trace

Proposition 6.1 (Critical fractional moments; site form of [15, Proposition 5.3])**.** *For every $0 < \alpha < 1/2$, $\sup_v \mathbb{E}_{p_c} s_v^\alpha < \infty$.*

The proof is by contradiction with (5.1), and uses Proposition 5.3 only through four properties of its maps: success implies an output $y \in C_x \cap O$ with $s_x \le M$; locality; covariance; and the efficiency inequality. A map is run at p slightly below p_c, in the deleted domain $V \setminus \mathrm{Nb}[A]$ of an independent size-biased cluster $A = C_x$. Its small norm sends endpoint mass outside $\mathrm{Nb}[A]$, and comparing the susceptibility of that domain with χ bounds $\Psi(p)$ from below by $c_\gamma \chi^{-\gamma}$ for every $\gamma > 0$, so that the inequality $(1 - p)\chi' \ge \chi^2\Psi(p)$ of Lemma 2.8 gives $\chi' \ge c\chi^{2-\gamma}$; an exploration

comparison with critical percolation then contradicts (5.1). For sites, removing $\mathrm{Nb}[A]$ rather than A makes the exterior bits fresh, and the upper bound for the deleted-domain potential costs a factor d, from the at most d clusters adjacent to a closed vertex of ∂A (Appendix D.1).

Proposition 6.2 (Almost linear susceptibility; site form of [15, Proposition 5.5])**.** *For every $\eta > 0$ there is C_η with $\chi_{\max}(p) \le C_\eta (p_c - p)^{-1-\eta}$ for $0 \le p < p_c$.*

This is the decision-tree argument of [15, Proposition 5.5], with the covariance bound [15, Lemma 5.4] for an arbitrary finite product law and an exploration by site queries, whose revealment is at most $(d+2)C_a n^{-a}$ by Proposition 6.1.

Write $T = T_p$, $m_p = \|T_p\|$ and $a_j = \operatorname{tr} T^j$.

Proposition 6.3 (Subpower triangle growth; site form of [15, Proposition 6.2])**.** *For every $\eta > 0$, $a_3(p) \le C_\eta (p_c - p)^{-\eta}$ for $p_c/2 \le p < p_c$.*

The proof (Appendix D.2) differentiates a_3 by Lemma 2.8. The contact term is controlled, as in [15, Lemma 6.1], by comparing a full and an exterior exploration score of one cluster; for sites their difference is exactly $-p\, k_s(A, C_v)$, that is, $-p$ times the number of common boundary vertices, all of which are closed. The remainder contributes $3 \operatorname{tr}(T^2 R_p) \le C\, a_3$, by Lemma 2.2 and $a_2 \le a_3/p$, and adds a bounded coefficient to the differential inequality; and $a_3 \ge p^3$ replaces $a_3 \ge 1$.

6.2 Induced corridors

Fix $p_c/2 \le p < p_c$, $q = 1 - p$ and $k \ge 3$, and write $T_0 = T_p$, $m_0 = \|T_0\|$, $\chi_0 = \chi(p)$ and $\chi_0^+ = \max_x \mathbb{E}_p s_x$, as in [15, Section 7]. The deformation of [15] penalizes a connection whenever it must traverse an isolated open corridor, a path specified by the loop erasure of a raw walk path whose edges are all pivotal. For sites the corridor is a set of pivotal vertices. A chord between two of its vertices, or an open neighbour of one of them, would let a connection bypass the vertices in between, so the corridor must be an induced path surrounded by closed vertices. We obtain an induced path by shortcutting the loop erasure greedily.

Definition 6.4 (Site templates; replaces [15, Definition 7.1])**.**

- A *raw path* $w = (w_0, \dots, w_k)$ is a k-step path of P, with base $u = w_0$, endpoint $v = w_k$ and probability $\mu_u(w) = \prod_{i<k} P(w_i, w_{i+1})$, as in [15, Definition 7.1].
- Let λ_w be its chronological loop erasure, with holding steps removed. Its *greedy shortcut* $\gamma_w = (g_0 = u, g_1, \dots, g_\ell = v)$ is defined by $i_0 = 0$ and $i_{t+1} = \max\{j > i_t : \lambda_w(j) \sim \lambda_w(i_t)\}$, $g_t = \lambda_w(i_t)$. Then γ_w is an *induced* path: if $g_s \sim g_t$ with $t \ge s+2$, the maximality at step s is contradicted. Its vertices form a subset of the vertex set $\mathcal{W}(w)$ of the raw path.
- ℓ_w is the length of γ_w and $I_w = \{g_1, \dots, g_{\ell-1}\}$ its interior, so $u, v \notin I_w$. The raw path is retained as a *template* if and only if $\ell_w \ge 3$.
- The *blockers* are $B_w = N(I_w) \setminus \gamma_w$, with $b_w = |B_w|$. The weight is $c(w) = \mu_u(w)\, p^{-|I_w|} q^{-b_w}$.
- A template is *eligible* in a domain D if $\gamma_w \subseteq D$. A blocker outside D counts as closed (absent).
- For $x, y \in D$ connected in D, N_{xy}^D is the sum of $c(w)$ over eligible templates such that $x, y \notin I_w$, every vertex of I_w is open and pivotal for $x \leftrightarrow y$ in D (closing it destroys the connection), and every blocker is closed or absent. Put $N_{xy}^D = 0$ on disconnection.
- As in [15, Definition 7.1], $W_t^D(x,y) = \mathbf{1}\{x \leftrightarrow y \text{ in } D\} e^{-tN_{xy}^D}$, $X_t^D(x) = \sum_{y \in D} W_t^D(x,y)$, $s_t^D(x) = \mathbb{E}_p X_t^D(x)$, $T_t(x,y) = \mathbb{E}_p W_t^V(x,y)$, $\chi_t = [s_t^V(o)]$ and $m_t = \|T_t\|$. Now $W_t^D(x,x) = \mathbf{1}\{x \text{ open}\}$.

Lemma 6.5 (Isolation)**.** *Let a template be counted for $x \leftrightarrow y$ in D. Then every simple open x–y path in D contains u, then all of I_w consecutively in corridor order, then v, or the same in reverse order.*

Proof. Pivotality puts every interior vertex on the path. An open, present neighbour of g_i outside I_w is u (only if $i = 1$) or v (only if $i = \ell - 1$): there is no chord, by inducedness, and blockers are closed or absent. The path starts and ends outside I_w, so it enters I_w through one of u, v and leaves through the other, and it cannot re-enter. □

Lemma 6.6 (Basic bounds and monotonicity; site form of [15, Lemma 7.2])**.** *There is a fixed C with $c_k = e^{Ck}$ such that $\sum_{w:z\in I_w} c(w) \le c_k/k$ for every vertex z. Moreover $N^D_{xy} \le c_k|C^D_x|$; $W^D_t(x, y)$ is increasing under site openings and domain enlargement; T_t is invariant and symmetric with $0 \le T_t \le T_0$; and $m_t \le \sup_x s^V_t(x) \le c_\sigma \chi_t$. Differentiability in t holds as in [15].*

The weight bound combines $p^{-|I_w|}q^{-b_w} \le ((2/p_c)(1 - p_c)^{-d})^k$ with the fact that the raw paths through z have total walk probability at most $k + 1$. Monotonicity follows from Lemma 6.5: a template counted after an enlargement contains u, I_w and v on every old simple open path, so it was eligible, open and pivotal before. The proof is in Appendix D.3.

Lemma 6.7 (Corridor calibration; site form of [15, Lemma 7.3])**.** *Let $Q_w = \{I_w$ open, B_w closed$\}$ and $D_w = V \setminus (I_w \cup B_w) = (V \setminus \mathrm{Nb}[I_w]) \cup \{u, v\}$. Then Q_w is independent of the percolation in D_w and $c(w)\mathbb{P}_p(Q_w) = \mu_u(w)$. For $x, y \notin I_w$, on Q_w the template is counted for $x \leftrightarrow y$ in V if and only if $x \in C^{D_w}_u$, $y \in C^{D_w}_v$ and $C^{D_w}_u \ne C^{D_w}_v$, or the same with u, v interchanged.*

Proof. Independence and calibration. Q_w is determined by the bits of $I_w \cup B_w$, which is disjoint from D_w, and $\mathbb{P}(Q_w) = p^{|I_w|}q^{b_w}$. *(⇐)* On Q_w the open set is $\mathrm{open}(D_w) \cup I_w$, and I_w is joined to the rest of the open set only through u and v, by inducedness and the closed blockers. So $x \leftrightarrow y$ via u, the corridor and v, and there is no D_w-bypass; every interior vertex is pivotal. *(⇒)* By Lemma 6.5, the path splits into an open x–u path and an open v–y path. These avoid I_w, which is traversed once, and B_w, which is closed, so they lie in D_w. If $C^{D_w}_u = C^{D_w}_v$, there is a bypass, contradicting pivotality. *Terminals.* If $x \in B_w$, then x is closed on Q_w and lies outside D_w, and both sides are false. □

Proposition 6.8 (Small change of operator norm; site form of [15, Proposition 7.4])**.** *For every $t \ge 0$, $0 \le T_0 - T_t \le 2t\, T_0 P^k T_0$ entrywise, and $m_t \ge m_0 - 2t\rho^k m_0^2$.*

Proof. "Counted" implies Q_w. Lemma 6.7, independence, and the van den Berg–Kesten inequality for the two distinct exterior clusters in D_w, with $T^{D_w} \le T_0$, bound the term of w by $\mu_u(w)(T_0(x, u)T_0(v, y) + T_0(x, v)T_0(u, y))$. The rest is as in [15, Proposition 7.4]. □

6.3 Two legs and the decrease of the susceptibility

The deformation decreases χ_t at a rate bounded below by the deformed susceptibilities at the two corridor endpoints, in the domain with the corridor deleted. Two estimates of [15, Section 7] carry over with site constants; their proofs are in Appendix D.3.

Lemma 6.9 (Deletion estimate; site form of [15, Lemma 7.6])**.** *For $A' \subseteq D$ and every v, $s^{D\setminus A'}_t(v) \ge s^D_t(v) - C_{\mathrm{del}}(p)\, \chi^+_0 \sum_{z\in \mathrm{Nb}[A']} T_0(v, z)$.*

Lemma 6.10 (Two distinct exterior clusters; site form of [15, Lemma 7.7])**.** *There is C', depending only on G and uniform in D, $p \in [p_c/2, p_c)$, k and t, such that for every domain D and $u, v \in D$,*

$$\mathbb{E}_p(X^D_t(u)X^D_t(v);\ C^D_u \ne C^D_v) \ge s^D_t(u)s^D_t(v) - C'(\chi^+_0)^2 T^3_0(u, v).$$

In both, the site change is that a template whose interior meets a deleted set can be newly counted after the deletion, either because an interior vertex was not pivotal before or because a blocker in the deleted set was open before; in the second case every surviving path visits $\mathrm{Nb}[A']$, which is why the deleted neighbourhood appears. Lemma 6.10 uses Lemma 6.9 with $A' = \mathrm{Nb}[C_u^D] \cap D$, (H1) and the tree bound (2.3), with $C' = C_{\mathrm{del}} c_{\mathrm{nb}} \kappa_p$ at $p = p_c/2$.

The two-leg decomposition is where the induced corridor changes the bookkeeping.

Lemma 6.11 (Two legs; site form of [15, Lemma 7.8]). *On Q_w, if $x \in C_u^{D_w}$, $y \in C_v^{D_w}$ and $C_u^{D_w} \neq C_v^{D_w}$, then*

$$N_{xy}^V \le 2c_k + N_{xu}^{D_w} + N_{vy}^{D_w}.$$

Proof. Templates near the corridor. Templates w' with $I_{w'} \cap (I_w \cup \{u, v\}) \neq \varnothing$ have total weight at most $(k+1)c_k/k \le 2c_k$, by Lemma 6.6. This class is new for sites: it contains templates whose interior passes through u and whose endpoint lies in I_w. Such templates are counted in V but are eligible in no leg.

The remaining templates lie in one leg. Take a counted w' with $I_{w'}$ disjoint from $I_w \cup \{u, v\}$. Then $I_{w'}$ is a connected set of open vertices on the x–y path outside $I_w \cup \{u, v\}$, so it lies in one exterior cluster, say $C_u^{D_w}$. Its endpoints u', v' are open and adjacent to $I_{w'}$. If $v' \in I_w$, its open neighbour in $I_{w'}$ would be u or v, which is excluded. So $u', v' \notin I_w \cup B_w$, and $\gamma_{w'}$ is eligible in D_w.

It is counted for the leg. We have $x, u \notin I_{w'}$. Pivotality for $x \leftrightarrow u$ in D_w holds: a bypass followed by the corridor and a v–y path, which lies in the other cluster and hence avoids $I_{w'}$, would bypass in V. Blockers closed in V are closed or absent in D_w. So w' is counted in $N_{xu}^{D_w}$ with the same weight. The case of the cluster of v is symmetric. □

Let $\mathbb{E}_{\mathrm{path}}$ denote averaging over the type-weighted root $u = o$ and a length-k P-path from u, with endpoint v, as in [15, Section 7], and put $a_3 = \mathrm{tr}(T_0^3)$.

Proposition 6.12 (Decrease of the averaged susceptibility; site form of [15, Proposition 7.5]). *For all $t \ge 0$,*

$$-\chi_t' \ge e^{-2tc_k} \Big(\mathbb{E}_{\mathrm{path}} \big[\mathbf{1}\{\ell_w \ge 3\} s_t^{D_w}(u) s_t^{D_w}(v) \big] - C' (\chi_0^+)^2 \rho^k a_3 \Big).$$

Proof. This is the proof of [15, Proposition 7.5], with Lemma 6.7 for the calibration, Lemma 6.11 in place of the bond two-leg bound, which turns the factor e^{-tc_k} into e^{-2tc_k}, Lemma 6.10, and $\mathbb{E}_{\mathrm{path}} T_0^3(u, v) = \mathrm{tr}(P^k T_0^3) \le \rho^k a_3$. □

6.4 Endpoint escapes and the proof of Theorem 3.1

The last ingredients bound the deformed row sums from below near the corridor ends. Their proofs are in Appendix D.3.

Lemma 6.13 (Comparison along a path; site form of [15, Lemma 7.9]). *If a simple path in D joins b to z and has length l, then $s_t^D(b) \ge e^{-t(l+1)c_k} p^l s_t^D(z)$.*

Lemma 6.14 (Smoothing; site form of [15, Lemma 8.2]). *$(P^l T_0)(a, b) \le \sqrt{c_\sigma p}\, \rho^l \le \sqrt{c_\sigma}\, \rho^l$ for all vertices a, b and $l \ge 0$.*

Lemma 6.15 (Two endpoint escapes; site form of [15, Lemma 8.1]). *For every $\tau > 0$ there are $J, C_1, k_0, b_* > 0$, depending only on G and τ, such that for every $p_c/2 \le p < p_c$, $k \ge k_0$ and $R \ge 1$, a set of raw paths of $\mathbb{E}_{\mathrm{path}}$-probability at least b_* has $\ell_w \ge 3$ and admits, from each of u and v, a simple path in D_w of length at most $j = J + \lceil C_1 \log((k+1)R) \rceil$ to a vertex z with*

$$\sum_{b \in \mathrm{Nb}[\mathrm{Nb}[I_w]]} T_0(z, b) \le \tau / R.$$

For Lemma 6.15 the escape must stay in $D_w = V \setminus (\mathrm{Nb}[I_w] \setminus \{u, v\})$, so the avoidance event of [15, Section 8] is replaced by a site avoidance event: every escape vertex other than the start is at graph distance at least 2 from every raw-path vertex other than the start. Each point collision in the bond estimates becomes membership in a neighbourhood $\mathrm{Nb}[x]$, at a cost of a factor at most $d+1$, and the remaining choices of constants are those of [15].

The proof of Theorem 3.1 is that of [15, Theorem 2.6] in [15, Section 9], with the site lemmas above; it is given in Appendix D.4. Suppose that T_{p_c} is unbounded, and let $\delta = p_c - p$. Sprinkling (Lemma 2.9) forces $m_0 \geq c\delta^{-1}$, and Propositions 6.2 and 6.3 make $R = \chi_0^+/m_0$ and a_3 subpower in δ. A row comparison from Lemma 6.13, the escapes of Lemma 6.15 and the deletion estimate keep $s_t^{D_w}(u)s_t^{D_w}(v)$ of order χ_t^2 on a positive proportion of raw paths, up to a factor polynomial in kR. For a corridor length $k = o(\log(1/\delta))$ chosen from $\rho < 1$, Proposition 6.8 keeps $m_t \geq m_0/2$ up to time $t_* = (kR)^{C_2+2}/m_0$, while Proposition 6.12 gives $-\chi_t' \geq c(kR)^{-C_2}\chi_t^2$; integrating, χ_{t_*} would be smaller than the bound $m_{t_*} \leq c_\sigma \chi_{t_*}$ allows. The site changes in this deduction are the constant C_{del} in the choice of τ, the factor e^{-2tc_k} of Proposition 6.12, the lower bound $a_3 \geq (p_c/2)^3$ in place of $a_3 \geq 1$, and the constant $1/d^2$ in the sprinkling bound.

7 Nonunimodular automorphism groups

This section proves Theorem 3.2. Parts (2) and (3) follow from part (1) by Hutchcroft's operator bound, which is model-free (Section 7.1). Part (1), the strict inequality $p_c < p_c(\lambda)$ for the site tilted susceptibility, is the quasi-transitive form of Tran and Nguyen's [20, Proposition 5.5]. We obtain it by running [20, Section 5] on a quasi-transitive graph. The steps that use vertex-transitivity are replaced with the help of the devices of Section 7.3; the full list of replacements is in Appendix F. Every other step of [20, Section 5] is used as written, so this case inherits the correctness of that unrefereed preprint.

7.1 The operator bound

Let $\Gamma \leq \mathrm{Aut}(G)$ be quasi-transitive, with Hutchcroft's modular function $\Delta = \Delta_\Gamma$ [10, Section 2.1]. For a nonnegative kernel τ and $\lambda \in \mathbb{R}$ put $\chi_\lambda(u) = \sum_x \tau(u,x)\Delta(u,x)^\lambda$ and $\chi_\lambda^* = \sup_u \chi_\lambda(u)$.

Lemma 7.1 (Hutchcroft [10, Lemma 7.1], [9, proof of Theorem 2.9])**.** *Let G be connected and locally finite and $\Gamma \leq \mathrm{Aut}(G)$ quasi-transitive. For every nonnegative kernel τ, every $\lambda \in \mathbb{R}$, $n \geq 1$ and $v \in V$, $(\tau^n)(v,v) \leq (\chi_\lambda^*)^n$. If τ is also symmetric, then $\|\tau\|_{2\to 2} \leq \chi_\lambda^*$.*

For $\tau = T_p$ this is the first step of the proof of [9, Theorem 2.9], which obtains $\sup_v T_p^n(v,v) \leq \chi_{p,\lambda}^n$ from the proof of [10, Lemma 7.1] and then the norm bound from [9, Proposition 2.2].[1] The telescoping of [10, (7.1)–(7.2)] uses only the cocycle identity $\Delta(u,v)\Delta(v,w) = \Delta(u,w)$, valid for quasi-transitive Γ [10, Lemma 2.3], and $\tau \geq 0$, so it holds for every nonnegative kernel and every n. For symmetric τ, the Schur test with weight $w = \Delta(o,\cdot)^\lambda$, for which $\sum_y \tau(x,y)w(y) = w(x)\chi_\lambda(x) \leq \chi_\lambda^* w(x)$ by the cocycle identity, gives the norm bound directly. Applied to the site kernel T_p, Lemma 7.1 gives $\|T_p\| \leq \sup_u \chi_{p,\lambda}(u)$.

For vertex-transitive G, Theorem 3.2 follows from this bound and Tran and Nguyen's tilted-threshold separation [20, Proposition 5.5]; Appendix E records the deduction (Proposition E.1). The proof below does not use it, since it covers vertex-transitive graphs too.

[1] [10, Lemma 7.1] is written for transitive Γ and $n = 3$. The reference in [9] to [10, Theorem 1.11] corresponds to Theorem 1.8 in arXiv:1711.02590v3, which is stated for transitive Γ, although Sections 3–6 there are written for quasi-transitive Γ.

7.2 The quasi-transitive setting

From now on G is infinite, connected, locally finite and quasi-transitive, $\Gamma = \mathrm{Aut}(G)$ is nonunimodular, and d is the maximal degree; every use of the common degree in [20, Section 5] is an upper bound and becomes d. A nonunimodular quasi-transitive automorphism group forces nonamenability, since every quasi-transitive group of automorphisms of an amenable graph is unimodular [19, 17], [16, Section 2.2]; so $h_V(G) > 0$, and $p_c < 1$ (Appendix F.2, item on $p_c < 1$).

(S1) *Modular function.* Δ is diagonally Γ-invariant, satisfies $\Delta(u,v)\Delta(v,w) = \Delta(u,w)$, and $\Delta(x,\cdot)$ is harmonic for simple random walk [10, Lemma 2.3]; Γ is unimodular if and only if $\Delta \equiv 1$. Write $h_v(z) = \log \Delta(v,z)$ for the raw height of z relative to v, and $h(z) = h_o(z)$ for a fixed o, so that $h_v(z) = h(z) - h(v)$. (The companion paper normalizes heights by t_0; here they are raw.)

(S2) *The constants t_0 and δ_0.* $t_0 = \max_{x\sim y} \log \Delta(x,y) > 0$ and $\delta_0 = \min\{|\log \Delta(x,y)| : x \sim y,\ \Delta(x,y) \neq 1\} > 0$; both are well defined because there are finitely many edge orbits and $\Delta \not\equiv 1$, that is, Γ is nonunimodular. An edge changes h by at most t_0.

(S3) *Sloped and plateau vertices.* A vertex y is *sloped* if some neighbour z has $h(z) < h(y)$, and a *plateau* vertex otherwise. By harmonicity a plateau vertex has all its neighbours at its own height, and a sloped vertex also has a neighbour of strictly larger height. Slopedness is Γ-invariant. Sloped vertices exist, since some edge has $\Delta \neq 1$ and both its endpoints are sloped. By quasi-transitivity $R_{\mathrm{sl}} := \sup_y \mathrm{dist}(y, \text{sloped set}) < \infty$, and $R^* := \max_O \sup_y \mathrm{dist}(y, O) < \infty$ over the orbits O.

(S4) *Orbit weights.* μ is the measure of [10, Section 2.1] on the finite set of orbits, and $\mu_{\min} = \min_O \mu(O) > 0$.

(S5) *Layers.* As in [20, (5.6)], an independent uniform offset $R_o \in [0,1)$ is transported equivariantly by $R_v = R_o - h_o(v)/t_0 \bmod 1$, and $L_j(v) = \{x : j + R_v - 1 < h_v(x)/t_0 \le j + R_v\}$, $L_{m,n}(v) = \bigcup_{m\le j\le n} L_j(v)$. The exact shift rule $u \in L_k(v) \Rightarrow L_j(u) = L_{j+k}(v)$ holds by the cocycle identity, and the joint law of the configuration and the offset field is Γ-invariant. A path cannot skip a layer, and a path along which h is weakly monotone, going from $L_a(v)$ to height beyond $L_b(v)$, visits $L_b(v)$.

We use the notation of [20, Section 5]:

- $K_v = C_v$ is the open cluster of v, and $|K_v|_{v,\lambda} = \sum_{u\in K_v} \Delta(v,u)^\lambda$ for $\lambda \in \mathbb{R}$;
- $\widehat{\mathbb{P}}_p^v = \mathbb{P}_p(\cdot \mid v \text{ open})$, and $\widehat{\mathbb{P}}_p^{v,x}$ also conditions on $R_v = x$;
- $X_j^{m,n}(v) = |\{x \in L_j(v) : v \leftrightarrow x \text{ in } L_{m,n}(v)\}|$;
- a vertex z is the *peak* of a set $A \ni z$ if $h_z(u) < 0$ for all $u \in A \setminus \{z\}$; P_z is the event that z is the peak of its open cluster, and $P_v(s)$ the event that v is the peak of the set joined to it inside $L_{-s,\infty}(v)$;
- $Q_{v;k,\ell,r} = \{v \not\leftrightarrow L_{\ell-1}(v) \text{ in } L_{-k-r-2,\infty}(v)\}$;
- $\widehat{\chi}_{p,\lambda} = \chi_{p,\lambda}/p$ is the conditional tilted susceptibility of [20, (5.13)].

The supremum $\widehat{\mathbb{E}}_p^{m,n}(j;a)$ of [20, (5.31)] is used only in Appendix F and is defined there.

The two obstructions to running [20, Section 5] on G are these. First, the argument there moves between layers along neighbours $z^\pm$ with $\log \Delta(z, z^\pm) = \pm t_0$; on a quasi-transitive graph such neighbours need not exist, for instance at pendant vertices. Second, several estimates are used uniformly in the root, and transports are applied at a fixed root; on a quasi-transitive graph

the lower bounds for peak events fail at plateau vertices, where P_v forces $K_v = \{v\}$, so that $\widehat{\mathbb{P}}(v \leftrightarrow L_{-k}(v),\ P_v) = 0$ for $k \geq 1$.

7.3 Devices

Lemma 7.2 (Finite modifications [16, (F.1)])**.** *Let $0 < p < 1$, let $B \subseteq V$ be finite, and let Φ be a measurable map of configurations with $\Phi(\omega) = \omega$ off B. Then $\mathbb{P}_p(A) \geq \min(p, 1-p)^{|B|}\mathbb{P}_p(\Phi(\omega) \in A)$ for every event A. The same holds with independent extra randomness, such as the offset field, that Φ does not change.*

The first device replaces the neighbours $z^{\pm}$ by monotone paths of bounded length.

Lemma 7.3 (Monotone paths with a length bound; [16, Lemma F.1] with lengths)**.** *For every vertex y and every $t > 0$ there is a path from y to a vertex z, of length at most $L(t) := (\lceil t/\delta_0 \rceil + 1)(R_{\mathrm{sl}} + 1)$, such that $h(z) \leq h(y) - t$, h is nonincreasing along the path, and h strictly decreases at its last step. The same holds with all inequalities between heights reversed.*

The proof (Appendix F.1) walks along a shortest path to a nearest sloped vertex, at distance at most R_{sl}; that path is level, since a step from a plateau vertex keeps h unchanged. It then steps to a strictly lower neighbour, which lowers h by at least δ_0, and repeats $\lceil t/\delta_0 \rceil$ times.

The second steers a path into a prescribed orbit while controlling its heights; it replaces the uses of transitivity that move a root into a given position.

Lemma 7.4 (Orbit-targeted connectors)**.** *There are constants M_* and H_* such that for every vertex y and every orbit O^*:*

- (down) *there is a path $y = y_0, \dots, y_m = q$ with $m \leq M_*$, $q \in O^*$, $h(y_j) \leq h(y)$ for all j, and $h(y) - H_* \leq h(q) \leq h(y) - t_0$;*
- (up) *there is a path with the mirrored properties: $q \in O^*$, $h(y_j) \geq h(y)$ for all j, and $h(y) + t_0 \leq h(q) \leq h(y) + H_*$.*

The proof (Appendix F.1) applies Lemma 7.3 with $t = (R^* + 1)t_0$ and appends a shortest path to O^*, of length at most R^*, along which h changes by at most R^*t_0; so $M_* = L((R^* + 1)t_0) + R^*$ and $H_* = M_*t_0$ work.

The third replaces the one-layer buffer of [20, (5.1)]: an exploration below a layer leaves the layer above untested.

Lemma 7.5 (First crossing)**.** *Let $W \subseteq V$ be deterministic once the offset field is fixed, let $v \in W$, and explore the open cluster K of v in the induced graph on W, testing boundary sites one at a time, as in the cluster kernel [20, (5.36)].*

(1) *No site of $V \setminus W$ is tested, and conditionally on the exploration the labels of $V \setminus W$ are i.i.d. uniform, by [20, (5.36)–(5.38)] with $Q_{\mathrm{old}} = \varnothing$.*

(2) *If $W = L_{-\infty,k}(v)$, every neighbour of K outside W lies in $L_{k+1}(v)$ and is adjacent to $K \cap L_k(v)$.*

(3) *If $W = L_{-\ell,\infty}(v)$, every neighbour of K outside W lies in $L_{-\ell-1}(v)$ and is adjacent to $K \cap L_{-\ell}(v)$.*

Proof. The exploration queries only sites of W. For (2)–(3), an edge changes h by at most t_0, and the layers have thickness exactly t_0. □

The fourth replaces the fixed-root tilted transport [20, (5.30)] by an average over a random root.

Lemma 7.6 (Random-root conditioned tilted mass transport)**.** *Let $F(x, y, \omega, R) \geq 0$ be measurable and diagonally Γ-invariant, jointly with the configuration ω and the equivariant offset field R, and suppose that F vanishes unless both x and y are open. Let $\rho \sim \mu$ be independent, and let $\widehat{\mathbb{E}}_\mu$ denote expectation over ρ, the offset field, and ω under $\widehat{\mathbb{P}}^\rho$. Then*

$$\widehat{\mathbb{E}}_\mu \sum_x F(\rho, x) = \widehat{\mathbb{E}}_\mu \sum_x F(x, \rho)\Delta(\rho, x). \tag{7.1}$$

Moreover, $\sup_v Q(v) \leq \mu_{\min}^{-1} \mathbb{E}_\mu Q(\rho)$ *for every orbit-invariant* $Q : V \to [0, \infty]$.

Proof. Multiply both sides of (7.1) by p. Each summand forces ρ open, so the identity becomes $\mathbb{E}\sum_x F(\rho, x) = \mathbb{E}\sum_x F(x, \rho)\Delta(\rho, x)$, which is the tilted mass-transport principle [10, Proposition 2.2] applied to the diagonally invariant function $(x, y) \mapsto \mathbb{E}F(x, y, \omega, R)$. This is the argument of [20] for (5.30) there, with the fixed root replaced by ρ. For the last assertion, a maximizing orbit has weight at least $\mu_{\min}$. □

The fifth compares quantities at different roots, so that finiteness or divergence at one root transfers to all.

Lemma 7.7 (Orbit comparability)**.** *There is $c_5 = c_5(G, p, \lambda) > 0$ such that for all vertices u, v:*

(a) $\chi_{p,\lambda}(u) \geq c_5 \chi_{p,\lambda}(v)$*;*

(b) $\widehat{\mathbb{E}}^u[|K_u|^a_{u,\lambda}] \geq c_5 \widehat{\mathbb{E}}^v[|K_v|^a_{v,\lambda}]$ *for* $0 < a \leq 1$*;*

(c) $\widehat{M}_{p,\lambda,\theta}(u) \geq c_5 \widehat{M}_{p,\lambda,\theta}(v)$ *for all* $\theta > 0$*, where* $\widehat{M}_{p,\lambda,\theta}(v) = \widehat{\mathbb{E}}^v[1 - e^{-\theta|K_v|_{v,\lambda}}]$ *is the tilted magnetization.*

In particular, finiteness or divergence of each quantity does not depend on the root, and its supremum over roots is a maximum over orbits.

All three quantities are orbit-invariant. The proof (Appendix F.1) moves the second root into the orbit of v at distance at most R^* from u and uses a path π of length at most R^* between them. Part (a) is Hutchcroft's inequality $\chi(u) \geq T_p(u, v')\Delta^\lambda(u, v')\chi(v')$ [10, proof of Proposition 3.1]. For (b) and (c), on $\{\pi \text{ open}\}$ the two clusters coincide and the cocycle identity relates their tilted sizes, and Harris's inequality gives the comparison; $c_5 = p^{R^*+1}e^{-|\lambda|t_0R^*}$ works, since $\min\{\Delta(u, v')^\lambda, 1\} \geq e^{-|\lambda|t_0R^*}$.

The sixth transfers peak estimates to a sloped vertex. Put $F_j(v, x) = \widehat{\mathbb{E}}^{v,x}[X_j^{-\infty,\infty}(v);\ P_v]$, as in [20, Auxiliary Lemma 5.G].

Lemma 7.8 (Peak transfer)**.** *Let v_0 be a sloped vertex and $0 < p \leq p_c$. There are C_6 and J_6 such that for every vertex u and every $k \geq 1$,*

$$\int_0^1 F_{-k}(u, x)\, dx \leq C_6 \sum_{j=0}^{J_6} \int_0^1 F_{-k-j}(v_0, y)\, dy.$$

Proof. The connector σ. Since v_0 is sloped, there is $\sigma = (v_0 = \sigma_0, \sigma_1, \ldots, \sigma_s)$: first a step to a strictly lower neighbour, then the path of Lemma 7.3 (down) with $t = (2R^* + 1)t_0$. So $h(\sigma_i) < h(v_0)$ for $i \geq 1$, $h(\sigma_s) \leq h(v_0) - t$, and $s \leq 1 + L(t)$.

The path π_u. For a vertex u, choose $g \in \Gamma$ with $\mathrm{dist}(u, g\sigma_s) \leq R^*$, and let π_u be a shortest path from u to $g\sigma_s$ followed by the reversal of $g\sigma$; its endpoint is $w = gv_0$, in the orbit of v_0. Along the first segment $h \leq h(u) + R^*t_0$; $h(g\sigma_s) \geq h(u) - R^*t_0$, so $h(w) \geq h(u) - R^*t_0 + t > h(u) + R^*t_0$; and

along the reversed $g\sigma$ every vertex except w lies strictly below $h(w)$. Hence w is the unique highest vertex of π_u, $0 < h(w) - h(u) \le H_6 := (R^* + s)t_0$, and $|\pi_u| \le M_6 := R^* + s$.

The map. Fix u and x. Let $z \in K_u \cap L_{-k}(u)$. On $E_z = \{u \text{ open},\ P_u,\ z \in K_u \cap L^x_{-k}(u)\}$, let Φ open every site of π_u and close every site of $N(\pi_u) \setminus (K_u \cup \pi_u)$. The set $B = \pi_u \cup N(\pi_u)$ has at most $(d+1)(M_6+1)$ sites. After Φ, the open cluster of w is exactly $K_u \cup \pi_u$: every open neighbour of π_u lies in $K_u \cup \pi_u$, and neighbours of K_u off π_u keep their states and were closed, being boundary sites of K_u.

The peak. The unique peak of the image cluster is w: K_u lies at heights at most $h(u) < h(w)$, and $\pi_u \setminus \{w\}$ lies strictly below $h(w)$. So $\Phi(E_z) \subseteq E'_z := \{w \text{ open},\ P_w,\ z \in K_w \cap L_{-k-j}(w)\}$, where $j \in [0, J_6]$ is defined by $u \in L_{-j}(w)$; the exact shift rule gives $L_{-k}(u) = L_{-k-j}(w)$, and $J_6 := \lceil H_6/t_0 \rceil + 1$.

The bound. Lemma 7.2 gives $\mathbb{P}(E_z) \le \min(p, 1-p)^{-|B|}\mathbb{P}(E'_z)$. Sum over z and divide by p on both sides; both events force their root open. Integrate over x: the induced offset at w is $R_w = x - h_u(w)/t_0 = x + h_w(u)/t_0 \bmod 1$, a translation, which preserves Lebesgue measure. Orbit invariance gives $F(w,\cdot) = F(v_0,\cdot)$ with the equivariant offsets. This proves the claim with $C_6 = \min(p, 1-p)^{-(d+1)(M_6+1)}$. □

At a sloped vertex the one-layer comparison of peak moments of [20, Auxiliary Lemma 5.G] holds in its full range; at a plateau vertex it can fail at $j = -1$.

Lemma 7.9 (Quasi-transitive form of [20, Auxiliary Lemma 5.G])**.** *At a sloped vertex v, for every $j \le -1$, $F_{j-1}(v,x) \le CF_j(v,x)$ and $F_{j+1}(v,x) \le CF_j(v,x)$. At an arbitrary vertex v, the first inequality holds for $j \le -1$ and the second for $j \le -2$. Consequently [20, (5.67)] holds at sloped v for $j \le -1$, and at every v for $j \le -2$.*

The proof (Appendix F.1) follows [20], with the neighbours $z^{\pm}$ replaced by paths of Lemma 7.3 truncated at their first visit to the target layer; each path is opened, its neighbours outside K_v and the path are closed, and Lemma 7.2 is applied to each marked vertex. At a plateau vertex P_v forces $K_v = \{v\}$, so $F_j = 0$ for $j \le -1$ while $F_0 > 0$; this is why the case $j = -1$ needs a sloped vertex.

7.4 The surgery inequality at a sloped vertex

The decisive step of the critical analysis in [20, Section 5] is the surgery inequality [20, (5.76)], the site analogue of [10, Lemma 6.4]. It bounds a restricted first moment, uniformly over roots, by a ratio of moments at one root. Its proof in [20] uses transitivity in four places: the maximal-decrement neighbours used to leave the explored region, the independence of a supremum from the root of the residual cluster, the fixed-root transport in the downstream step (5.113), and a denominator bound uniform in the root. We replace the first by Lemmas 7.5 and 7.4, the second by steering the residual root into a maximizing orbit, the third by Lemma 7.6, and the fourth by evaluating at one sloped vertex.

Proposition 7.10 (Quasi-transitive form of [20, (5.76)])**.** *There are integers $a_* \ge 2$ and $C_p < \infty$, depending only on G and p, such that the following holds for every $0 < p \le p_c$, every $k, r \ge 0$ and every $\ell \ge 2$. Put $S = k + \ell + r + a_* + 1$ and let $N_\ell(v)$ be the event that the open cluster A of v in the induced graph on $L_{-\ell,\infty}(v)$ has a neighbour in $L_{-\ell-1}(v)$. For every (v,x) with $\widehat{\mathbb{P}}^{v,x}(N_\ell(v) \cap P_v(S)) > 0$,*

$$\sup_{v',x'} \widehat{\mathbb{E}}^{v',x'}\big[X^{-k-r,0}_{-k}(v');\ Q_{v';k,\ell+1,r+a_*}\big] \le C_p \,\frac{\sum_{i=1}^{a_*+1} \widehat{\mathbb{E}}^{v,x}\big[X^{-S,0}_{-k-\ell-i}(v) \mid P_v(S)\big]}{\widehat{\mathbb{P}}^{v,x}\big(N_\ell(v) \mid P_v(S)\big)}.$$

Compared with [20, (5.76)], the left side has the slack parameters $(\ell+1, r+a_*)$, the right side sums $i \in [1, a_*+1]$ at depth S, $N_\ell(v)$ replaces $\{v \leftrightarrow L_{-\ell}(v)\}$, and the infimum over (v,x) is restricted to positive denominators. The first three are bounded shifts, absorbed downstream; the last is harmless, because the downstream use needs one (v,x). These are the quasi-transitive features of [10, (6.19)].

The two new steps. The proof (Appendix F.3) follows [20] with the new indices, except for two steps. Let O^* be an orbit attaining the left side, which is a maximum over the finitely many orbits. First, the proof explores the open cluster A of v in $L_{-\ell,\infty}(v)$ with its tested boundary, keeps the outcomes on which v is the peak of A and A has a neighbour $q_1 \in L_{-\ell-1}(v)$, and descends from q_1 along a connector of Lemma 7.4 (down) to a vertex $q \in O^*$. Every site of the connector lies at height at most $h(q_1)$, hence in the untested region $L_{-\infty,-\ell-1}(v)$ (Lemma 7.5), and $q \in L_{-\ell-a}(v)$ for some $a \in [2, a_*]$; this replaces the maximal-decrement neighbours of [20, (5.77)]. Second, the surgery opens the connector and closes its neighbours outside A, the tested boundary, the residual cluster C_q of q and the connector, changing at most $(d+1)M_*$ coordinates. As the connector lies strictly below the explored region, opening it attaches A only to C_q, which on the event considered does not reach the heights at least $h(v)$, so v stays the peak. Because $q \in O^*$, the residual moment at q dominates the moment $\widehat{\mathbb{E}}^{q,y}[X_{-k}^{-k-r,0}(q);\ Q_{q;k,\ell+1,r+a_*}]$ for every offset y, and the supremum of these over y is the left side itself; this replaces the transitivity that makes the residual supremum independent of the root in [20].

Downstream, Proposition 7.10 is applied at a sloped vertex v_0, where, for every $\zeta > 0$ and some $c_\zeta > 0$, the critical peak-survival bound

$$\inf_x \widehat{\mathbb{P}}_{p_c}^{v_0,x}\big(v_0 \leftrightarrow L_{-k}(v_0),\ P_{v_0}\big) \ge c_\zeta e^{-t_0(\alpha_{p_c}^s - 1+\zeta)k} \tag{7.2}$$

holds by the argument of [20, (5.85)–(5.91)] at v_0, whose subcritical input is supplied at sloped vertices by Lemmas 7.8 and 7.9. Here α_p^s is the *height exponent*: the exponential decay rate in t of the probability that an open root is joined to the vertices at height at least t above it inside the vertices at height at least its own (Appendix F.2).

7.5 Proof of Theorem 3.2

Part (1). Appendix F runs [20, Section 5] in the present setting. With the devices above, it proves the inputs of [20, Proposition 5.5] on G: the divergence of tilted moments at $p_c(\lambda)$ at every root, the critical fractional estimate [20, (5.34)] uniformly in the root and offset, the first-moment identities (5.59)–(5.60), and the lower bound (5.73) there. The proof of [20, Proposition 5.5] then gives $p_c < p_c(\lambda)$ for $0 < \lambda < 1/2$, and, as there, at $\lambda = 1/2$. In place of the fixed-root symmetry used there, the averaged symmetry of Lemma 7.6 together with Lemma 7.7(a) gives $p_c(\lambda) = p_c(1-\lambda)$, and Cauchy–Schwarz gives $p_c(\lambda) \le p_c(1/2)$; so $p_c < p_c(\lambda) \le p_c(1/2)$ for $0 < \lambda < 1$. Finally $p_t(G,\Gamma) = p_c(1/2)$: if $\sup_u \chi_{p,\lambda}(u) < \infty$ for some λ, the averaged symmetry gives finiteness of $\chi_{p,1-\lambda}$ at ρ, hence at every root by Lemma 7.7(a), and Cauchy–Schwarz bounds $\chi_{p,1/2}(u)$ by $(\chi_{p,\lambda}(u)\chi_{p,1-\lambda}(u))^{1/2}$; the reverse inequality is trivial.

Part (2). The kernel T_p is nonnegative and symmetric, so Lemma 7.1 gives $\|T_p\| \le \sup_u \chi_{p,\lambda}(u)$. For $p < p_t$, $\chi_{p,1/2}$ is finite at every root and its supremum is a maximum over orbits (Lemma 7.7(a)), so T_p is bounded and $p_{2\to2} \ge p_t$. The bound $p_{2\to2} \le p_u$ is Lemma 2.6.

Part (3). By part (1) there is $p' > p_c$ with $\sup_u \chi_{p',1/2}(u) < \infty$, and monotonicity in p gives $\sup_u \chi_{p_c,1/2}(u) < \infty$; part (2) bounds $\|T_{p_c}\|$. The supremum identity is Lemma 2.10. □

8 Discussion

Where nonamenability enters the unimodular case. On an amenable quasi-transitive graph $p_c = p_u$, so an argument that ran unchanged there would prove a false statement. In the unimodular case nonamenability enters through $\rho = \|P\| < 1$ (Lemma 2.3) and through $p_c < 1$. The decisive uses are those of $\rho < 1$: the starting map of Lemma 5.4 needs $H > 1$, that is, an endpoint kernel of norm less than one; the norm loss $2t\rho^k m_0^2$ of Proposition 6.8 and the error $\rho^k a_3$ of Proposition 6.12 need decay in k; the smoothing and escape estimates of Lemmas 6.14–6.15 need $\rho^l \to 0$; and the choice of k in the proof of Theorem 3.1 uses $\log(1/\rho) > 0$. On an amenable unimodular quasi-transitive graph every invariant finite-range stochastic kernel on an orbit has norm one, by testing against Følner sets with column sums one (Lemma 2.1), so $H = 0$ for every local endpoint map and Proposition 5.3 fails there; the moment argument then has no input. None of the site changes, namely the Russo remainder, the degree-dependent constants, the deleted domains $V \setminus \mathrm{Nb}[A]$, junction rejection, the pruning lemma, joint restoration in the second moment, the induced corridors and the site avoidance event of Lemma 6.15, touches these uses; each changes only graph-dependent constants or the deleted set.

Where nonunimodularity enters. If $\mathrm{Aut}(G)$ is unimodular, then $\Delta \equiv 1$, $\chi_{p,\lambda} = \chi_p$ for every λ, and $p_t = p_c$, so the conclusion $p_c < p_t$ of Theorem 3.2(1) is false; for amenable graphs it must be, since $p_c = p_{2\to 2} = p_u$ there. The argument of Section 7 uses nonunimodularity in four places: $t_0 > 0$ and $\delta_0 > 0$, for every connector and layer argument; the existence of sloped vertices, for Lemma 7.8; the tilt in [20, Proposition 5.5], which compares the divergence of a tilted moment at $p_c(\lambda)$ with the upward decay $e^{-(t_0-\zeta)k}$ of [20, (5.34)]; and the lower bound $\alpha^s_{p_c} \geq 1$ of [20, (5.73)] for the height exponent of (7.2). This bound comes from the tilted symmetry $\widehat{\chi}_{p,1} = \widehat{\chi}_{p,0}$ and is the source of the asymmetry between upward and downward crossings. The argument does not need G to be non-vertex-transitive: for vertex-transitive G every vertex is sloped, μ is a point mass, and the argument reduces to [20, Section 5].

The full nonuniqueness interval. Theorem 1.1(3) gives one interval $[p_1, p_2]$. Häggström, Peres and Schonmann prove that, for bond percolation on every infinite, locally finite, connected, quasi-transitive graph, in the standard coupling almost surely $N(p) = \infty$ for all $p \in (p_c, p_u)$ [7, Theorem 1.3], and they state that their results and proofs may be adapted to site percolation [7, Section 1]. With that adaptation, which we have not checked, Theorem 1.1(2) gives infinitely many infinite site clusters simultaneously for all $p \in (p_c, p_u)$.

Scope. The bond paper also deduces mean-field critical behaviour and exponential connectivity below $p_{2\to 2}$ from the operator bound [15, Corollary 1.3]. We have not checked the site forms of those deductions, and claim none of them.

A Proofs of the site tools and of the bond corollary

This appendix proves the statements of Section 2.3, which are site adaptations of the corresponding lemmas of [15, Section 2], writing out every changed step, and then Corollary 1.2.

A.1 The site tools

Proof of Lemma 2.4. An open self-avoiding path with $n+1$ sites from x has probability at most $d^n p^{n+1}$, so there is no infinite cluster when $pd < 1$. For the upper bound, the number of connected

sets of size n containing x is at most d^{2n} [15, Lemma 2.2]. If $C_x = A$ then ∂A is closed and $|\partial A| \geq h_V n$, so $\mathbb{P}_p(|C_x| < \infty) \leq (1-p) + \sum_{n \geq 1} d^{2n}(1-p)^{h_V n} < 1$ for p near 1. As in [15, Lemma 2.2], $|B(x, r+1)| \geq (1+h_V)|B(x,r)|$, so G has exponential growth, and with $p_c < 1$ the site theorem [16, Theorem 1.1(2)] gives no infinite cluster at p_c. Countability gives all vertices at once, and a cluster in an induced subgraph lies inside a full-graph cluster. □

Proof of Lemma 2.5. Harris's inequality [8] and the van den Berg–Kesten inequality [21]: the proofs quoted in [15, Lemma 2.3] are for an arbitrary finite product of Bernoulli bits followed by exhaustion, and apply verbatim to vertex bits. For the tree bound, on the event take a simple open path π from x to y in D and a simple open path from z to x, and let w be its first vertex on π. The three sets $\pi[x,w]$ (which contains w), $\pi(w,y]$ and the part of the z-path strictly before w are pairwise disjoint sets of open vertices. The second is empty if and only if $w = y$, and otherwise it is an open path from a neighbour of w to y; likewise for z. A union over w and the two neighbours, and the van den Berg–Kesten inequality, give $\mathbb{P} \leq \sum_w T(x,w)(I + \mathrm{Adj}T)(w,y)(I + \mathrm{Adj}T)(w,z)$, and (2.2) gives (2.3). Summing over y, z gives $\mathbb{E}_p s_x^2 \leq \kappa_p \chi_{\max}^3$. □

Proof of Lemma 2.6. The proof of [15, Lemma 2.4] applies verbatim. Under almost sure uniqueness at p, some vertex has $\mathbb{P}_p(|C_x| = \infty) > 0$, by countability. Harris's inequality with the event that the finitely many sites of a fixed path from x to y are open makes this positive at every vertex, and the finitely many types give $\inf_x \theta_x = \theta > 0$. Uniqueness and Harris's inequality give $T_p(x,y) \geq \mathbb{P}_p(x \leftrightarrow \infty,\ y \leftrightarrow \infty) \geq \theta^2$, so $\langle \mathbf{1}_F, T_p \mathbf{1}_F \rangle \geq \theta^2 |F|^2$ for finite F, and T_p is unbounded. The site kernel T_p is entrywise increasing in p, so the parameters with bounded T_p form a downward interval, and $p_{2\to 2} \leq p_u$. □

Proof of Lemma 2.7. Finiteness of $\chi_x(p)$ for $p < p_c$ is site sharpness on quasi-transitive graphs [1, Theorem 2], [16, Corollary 8.3]; the abstract of [1] states that its results apply to site as well as bond percolation. Since $\chi_x(p)$ depends only on the orbit of x, $\chi_{\max}(p)$ is a maximum over finitely many orbits and is finite. The second moment is Lemma 2.5. For the lower comparison, along a path $x = z_0, \dots, z_l = y$, $\chi_x \geq \mathbb{E}[|C_y| \mathbf{1}\{z_0, \dots, z_{l-1} \text{ open}\}] \geq p^l \chi_y$ by Harris's inequality. Positive semidefiniteness: $\sum_{x,y} \bar{f}(x) T(x,y) f(y) = \mathbb{E} \sum_C |\sum_{x \in C} f(x)|^2$, the sum over open clusters, which is consistent with $T(x,x) = \mathbb{P}(x \text{ open})$. The rest is as in [15, Lemma 2.10]. □

Proof of Lemma 2.8. Finite domain. In a finite induced Λ the event $\{x \leftrightarrow y \text{ in } \Lambda\}$ is increasing in finitely many site bits, and finite-product differentiation gives $\frac{d}{dp}\mathbb{P} = \sum_v \mathbb{P}(v \text{ pivotal})$. Pivotality of v does not depend on ω_v, so $\mathbb{P}(v \text{ pivotal}) = (1-p)^{-1}\mathbb{P}(v \text{ closed},\ v \text{ pivotal})$. A closed v is pivotal if and only if $x \not\leftrightarrow y$ and both x, y lie in the cluster of v after opening it, that is, $x, y \in \widehat{C}_v^{\Lambda}$. This is (2.4) in Λ, with each closed pivotal site counted once.

Splitting. Consider first $v \notin \{x, y\}$. Then $x \in C_a$ and $y \in C_b$ with $a, b \sim v$, so x, y are open, $C_x \neq C_y$ and $v \in \partial C_x \cap \partial C_y$. Conversely, every $v \in \partial C_x \cap \partial C_y$ with $C_x \neq C_y$ nonempty is closed and pivotal. This gives K^c. The terms $v = x$ and $v = y$ give R; when $x = y$ they are one term, equal to $\mathbb{P}(x \text{ closed}) = 1 - p$. For $x \neq y$, the event $\{x \text{ closed},\ y \in \widehat{C}_x\}$ requires $y \in C_a$ for some $a \sim x$, so its probability is at most $(\mathrm{Adj}T)(x,y)$; hence the bound on R.

Majorant. $\mathbf{1}\{x, y \in \widehat{C}_v,\ x \not\leftrightarrow y\} \leq (\delta_{xv} + \sum_{a \sim v} \mathbf{1}\{x \in C_a\})(\delta_{yv} + \sum_{b \sim v} \mathbf{1}\{y \in C_b\})$, and when $x \in C_a$, $y \in C_b$ and $C_a \neq C_b$ the two clusters are disjoint witnesses, so the van den Berg–Kesten inequality gives $(1-p)T'(x,y) \leq \sum_v (I + T\mathrm{Adj})(x,v)(I + \mathrm{Adj}T)(v,y)$.

Exhaustion. Fix $[p', p''] \subset (0, p_1]$ with $p_1 < p_c$. For $\Lambda_n \uparrow V$, the indicator $\mathbf{1}\{v \text{ closed},\ x, y \in \widehat{C}_v^{\Lambda_n},\ x \not\leftrightarrow y \text{ in } \Lambda_n\}$ converges pointwise to its full-graph version: finite open paths witness membership in $\widehat{C}_v$, and connection in V is eventually witnessed in Λ_n. The summands are dominated

by $(I + T_{p_1}\mathrm{Adj})(x,v)(I + \mathrm{Adj}T_{p_1})(v,y)$, which is summable in v and y, with row sums at most $(1 + d\chi_{\max}(p_1))^2$. Dominated convergence passes the integrated identity to V.

Operator norm. From here the proof of [15, Lemma 2.11] applies verbatim with the majorant $(1-p_1)^{-1}(I + T_{p_1}\mathrm{Adj})(I + \mathrm{Adj}T_{p_1})$ in place of the bond majorant: row differentiation in ℓ^1, finitely many row types, invariance, symmetry and the row and column bound.

The identity (2.6). Summing (2.4) over y, which the same domination justifies, gives $(1-p)\chi'_x = \sum_v \mathbb{E}[\mathbf{1}\{v \text{ closed},\ x \in \widehat{C}_v\}(|\widehat{C}_v| - s_x)]$. If $x \in C_a \subseteq \widehat{C}_v$, the $y \in \widehat{C}_v$ with $y \not\leftrightarrow x$ are those of $\widehat{C}_v \setminus C_x$; if $x = v$ is closed, $s_x = 0$. Lemma 2.1 with the nonnegative mass $m(x,v) = \mathbb{E}[\mathbf{1}\{v \text{ closed},\ x \in \widehat{C}_v\}(|\widehat{C}_v| - s_x)]$ moves the root to v: $(1-p)\chi' = [\mathbb{E}\mathbf{1}\{o \text{ closed}\}\sum_{x\in\widehat{C}_o}(|\widehat{C}_o| - s_x)] = [\mathbb{E}\mathbf{1}\{o \text{ closed}\}(|\widehat{C}_o|^2 - \sum_i n_i^2)]$. Finally $(1+\sum_i n_i)^2 - \sum_i n_i^2 = 1 + 2N_o + Q_o$.

The bounds (2.7). $Q_o \geq 0$, and $\mathbb{E}[\mathbf{1}\{o \text{ closed}\}N_o] \leq \sum_{a\sim o}\chi_a \leq d\chi_{\max}$. □

For comparison with the bond quantity, $Q_o \leq \sum_{b,c\sim o} s_b s_c \mathbf{1}\{C_b \neq C_c\} \leq (d^2/4)Q_o$, since each cluster i is met by m_i neighbours of o and $m_i m_j \leq d^2/4$.

Proof of Lemma 2.9. Take independent site fields $\omega_1 \sim \mathbb{P}_p$ and $\omega_2 \sim \mathbb{P}_a$; their union has law $\mathbb{P}_{p+(1-p)a}$. On a simple union-open path $x = z_0, \dots, z_n = y$, let $s_1, \dots, s_j$ be its ω_1-closed vertices in order; they are ω_2-open. The maximal ω_1-open segments between them are pairwise disjoint ω_1-open witnesses. The segment before s_1 is empty ($x = s_1$) or joins x to a neighbour of s_1, with kernel $I + T_p\mathrm{Adj}$. A segment between s_r and s_{r+1} is empty ($s_r \sim s_{r+1}$) or joins a neighbour of s_r to a neighbour of s_{r+1}, with kernel $\mathrm{Adj} + \mathrm{Adj}T_p\mathrm{Adj} = \mathrm{Adj}(I + T_p\mathrm{Adj})$. The final segment gives $I + \mathrm{Adj}T_p$. Independence of ω_2 at the distinct s_r gives a^j, and the van den Berg–Kesten inequality in ω_1 gives the product; extra terms only increase the bound. For the norms, $\|I + T_p\mathrm{Adj}\| \leq 1 + d\|T_p\|$ and $\|\mathrm{Adj}(I + T_p\mathrm{Adj})\| \leq d(1 + d\|T_p\|)$, so the series converges when $ad(1 + d\|T_p\|) < 1$. If $p_c < 1$, a positive such a exists at $p = p_c$. □

Proof of Lemma 2.10. The proof of [15, Lemma 9.2] applies verbatim: finite-domain site connection probabilities are polynomials in p, and connectivity is an increasing union of finite-domain events, so $T_p(x,y) \uparrow T_{p_*}(x,y)$ as $p \uparrow p_*$, and the bilinear-form argument there is model-free. □

Proof of Proposition 2.11. We follow [15, Proposition 9.3]. Let N_{12} be the number of p_2-clusters containing an infinite p_1-cluster. Every automorphism-invariant event of the label field has probability zero or one, by approximation with finite-coordinate events of vertex labels and an automorphism moving their support off itself; so N_{12} is almost surely a constant, which is positive since $p_1 > p_c$.

$N_{12} \neq 1$. Put $\theta(x) = \mathbb{P}_{p_1}(|C_x| = \infty)$. Some $\theta(x)$ is positive, and $\theta(x) \geq p_1^l\theta(y)$ along an open path of length l by Harris's inequality, so $\theta_* = \inf_x \theta(x) > 0$ by the finitely many types. If $N_{12} = 1$, Harris's inequality gives $T_{p_2}(x,y) \geq \theta_*^2$, contradicting boundedness as in Lemma 2.6.

N_{12} is not a finite $n \geq 2$. By countability there are fixed x, y which, with positive probability, lie in infinite p_1-clusters in distinct p_2-clusters. Replace the labels of the vertices of a fixed finite path from x to y, endpoints included, by independent uniforms on $[0, p_1]$. The modified law is absolutely continuous with respect to the original, and the modification only opens sites at both parameters. Opening finitely many sites, each adjacent to at most d old clusters, joins only finitely many old clusters into any new cluster, so every infinite new p_1-cluster contains an infinite old one; every old counted p_2-cluster lies in a counted new one; and old p_2-clusters only merge. The two distinguished counted clusters merge, so the new count is at most $n - 1$, contradicting absolute continuity. Hence $N_{12} = \infty$ almost surely.

Simultaneity. On this event choose an infinite p_1-cluster in each of infinitely many distinct counted p_2-clusters. For every $p \in [p_1, p_2]$ their containing p-clusters are infinite and distinct, because their p_2-clusters are distinct. □

A.2 Bond percolation: proof of Corollary 1.2

Proof of Corollary 1.2. Let $L = L(G)$ be the line graph: its vertices are the edges of G, loops and parallel copies included, and two distinct edges are adjacent when they share an endpoint. Let d be the maximal number of edges at a vertex of G. Then L is infinite, connected and locally finite, and automorphisms of G act on L with finitely many orbits, because every edge contains a vertex from a finite set of orbit representatives, each of which lies in at most d edges. For a finite nonempty $F \subseteq V(L)$, let W be the set of endpoints of edges in F. Each vertex $y \in \partial W$ is joined by an edge g to some $w \in W$; the edge g is not in F, since its endpoint y is not in W, and g shares w with an edge of F. Distinct y give distinct g, so $|\partial_L F| \geq |\partial W| \geq h_V(G)|W| \geq (h_V(G)/d)|F|$, as $|F| \leq d|W|$. Hence $h_V(L) > 0$.

Bond percolation on G with edge labels is site percolation on L with the same labels. The open clusters of L are the edge sets of the bond clusters that contain an edge, and a bond cluster is infinite if and only if its edge set is, by local finiteness. Hence the numbers of infinite clusters agree for every configuration, so p_c and p_u agree, and (3) for bond percolation on G is (3) of Theorem 1.1 for L. For the two-point kernels, $x \leftrightarrow y$ in bond percolation on G means $x = y$, or some open edge $e \ni x$ and some open edge $f \ni y$ lie in one open cluster of L. With the incidence kernel $B(x, e) = \mathbf{1}\{x \in e\}$ this gives, entrywise,

$$T_p^{\mathrm{bond}} \leq I + B\, T_p^{\mathrm{site},L} B^*, \qquad \|B\|^2 = \|BB^*\| \leq 2d,$$

since $\sum_y (BB^*)(x, y) = \sum_{e \ni x} |e| \leq 2d$. Theorem 1.1(1) for L therefore bounds $T_{p_c}^{\mathrm{bond}}$, and (2) for L gives $p_{2\to 2}^{\mathrm{bond}}(G) \geq p_{2\to 2}^{\mathrm{site}}(L) > p_c$. The inequality $p_{2\to 2}^{\mathrm{bond}} \leq p_u^{\mathrm{bond}}$ is [15, Lemma 2.4], and the supremum identity is [15, Lemma 9.2]. □

B Proofs for Section 4

B.1 Proof of Lemma 4.2

(1) A path from x to y avoiding z would put x and y in one component of $H \setminus \{z\}$. For the order, let u, v both meet every path from x to Y, and suppose that u precedes v on a simple path γ from x to $y \in Y$ and v precedes u on a simple path γ' from x to $y' \in Y$. Then γ from x to u avoids v, and γ' from u to y' avoids v, since v comes before u on γ'. Their concatenation is a walk from x to Y avoiding v, which contains a path from x to Y avoiding v, a contradiction. Equivalently, the cut vertices separating x from y are the cut-vertex nodes strictly inside the path from x to y in the block–cut tree.

(2) Let z separate x from y. If $z = w$ it is the $+1$. Otherwise, if z separated neither x from w nor w from y, paths $x \to w$ and $w \to y$ avoiding z would give an x–y walk avoiding z. Since $z \notin \{x, y, w\}$, it is counted in $\mathrm{piv}_H(x, w)$ or in $\mathrm{piv}_H(w, y)$.

(3) Let $z \notin R$ separate x from y in H. Every x–y path in H'' is an x–y path in H, so it contains z; since H'' is connected and contains x and y, such a path exists, so $z \in H''$, and z separates x from y in H''. At most $|R|$ of the separating vertices lie in R.

(4) In the block–cut tree, the path from z_t to $z_{t'}$ is a subpath of the path from x to y, and the cut-vertex nodes strictly inside it are $z_{t+1}, \ldots, z_{t'-1}$ for $t < t'$. These separate z_t from $z_{t'}$.

(5) Separating vertices are distinct and exclude x and y. □

B.2 Proofs of Lemmas 4.4 and 4.5

Proof of Lemma 4.4. First moment. Explore C_x^D in a fixed, mark-independent order: query ω_x; while some discovered open vertex has an unqueried neighbour in D, query the first such neighbour

in a fixed order. Test the mark of each newly discovered open vertex immediately, and stop at the first mark or when the cluster is exhausted (x closed counts as exhausted). No vertex is queried twice. The procedure terminates almost surely: if C_x^D is infinite, its discovery sequence is infinite and the independent marks give a mark almost surely. Let T be the number of site queries and N the number of open vertices discovered when it stops. Conditionally on ω, the discovery order of C_x^D is fixed, and N is a geometric(h) variable stopped at $|C_x^D|$; on E it is that variable conditioned to be at most $|C_x^D|$. In both cases its mean is at most $1/h$. Each discovered open vertex causes at most d further queries, so $T \le 1 + dN$, and $\mathbb{E}T \le (d+1)/h$ and $\mathbb{E}[T \mid E] \le (d+1)/h$.

Let $Z = \sum_{v \text{ queried}} (\omega_v - p_c)$ and $Q_v = \mathbf{1}\{v \text{ queried}\}$. The variable Q_v is a function of the marks and of the bits queried before v, so it is measurable without ω_v. If v is pivotal, set $\omega_v = 1$: then E holds, the procedure stops at a mark m, and the discovered set is connected and contains x and m, hence an x–m path, which meets v; so v is queried with $\omega_v = 1$, and since Q_v does not depend on ω_v, $Q_v = 1$. If v is not pivotal, $\mathbf{1}_E$ does not depend on ω_v. Conditioning on all other bits and the marks, $\mathbb{E}[(\omega_v - p_c) Q_v \mathbf{1}_E \mid \text{rest}]$ is 0 if v is not pivotal, and $\mathbb{E}[(\omega_v - p_c)\omega_v \mid \text{rest}] = p_c(1-p_c)$ if v is pivotal, since then $Q_v = 1$ and $\mathbf{1}_E = \omega_v$. Hence $\mathbb{E}[(\omega_v - p_c) Q_v \mathbf{1}_E] = p_c(1-p_c)\mathbb{P}(v \text{ pivotal})$; summation is allowed since $\sum_v \mathbb{E}|\omega_v - p_c| Q_v \le \mathbb{E}T < \infty$. An open pivotal lies in E, and ω_v is independent of the pivotality of v, so $p_c \mathbb{P}(v \text{ pivotal}) = \mathbb{P}(v \text{ open pivotal})$. With $N_{\text{piv}}^+ = N_{\text{piv}} + \mathbf{1}_E$ the count including the root,

$$\mathbb{E}[Z \mathbf{1}_E] = (1-p_c)\mathbb{E}[N_{\text{piv}}^+ \mathbf{1}_E] \ge (1-p_c)\mathbb{E}[N_{\text{piv}} \mathbf{1}_E]. \tag{B.1}$$

Each queried bit is a fresh Bernoulli(p_c) variable given the transcript; the marks are independent coins and enter neither Z nor T. Hoeffding's lemma at the bounded stopping times $T \wedge n$ and Fatou's lemma give $\mathbb{E}\exp(\lambda Z - \lambda^2 T/8) \le 1$. Now $\mathbb{P}(E) \ge p_c h$, since x open and marked suffices. With $c_{\text{m}} = 1 + \log(1/p_c)/\log 2$ and $h \le 1/2$, $\log(1/(p_c h)) \le c_{\text{m}} \log(1/h)$. Conditional Jensen gives

$$\lambda \mathbb{E}[Z \mid E] \le \log(1/\mathbb{P}(E)) + (\lambda^2/8)\mathbb{E}[T \mid E] \le c_{\text{m}} \log(1/h) + \lambda^2 (d+1)/(8h).$$

With $\lambda = \sqrt{8 c_{\text{m}} h \log(1/h)/(d+1)}$ and (B.1), $\mathbb{E}[N_{\text{piv}} \mid E] \le (1-p_c)^{-1} \sqrt{c_{\text{m}}(d+1)/2}\, \sqrt{\log(1/h)/h}$. Put $C = \max\{1, (1-p_c)^{-1}\sqrt{c_{\text{m}}(d+1)/2}\}$. Since $\sqrt{\log(1/h)/h} \ge \sqrt{2\log 2} > 1$ for $h \le 1/2$, $A_h \ge 1$, and the case $i = 1$ holds uniformly over (D, x). The case $i = 0$ is trivial.

Higher binomial moments. Put $b_i(D,x) = \mathbb{E}[\binom{N_{\text{piv}}}{i} \mathbf{1}_E]$ and $b_0(D,x) = \mathbb{P}(E)$. We claim that, for $i \ge 1$,

$$b_i(D,x) = \sum_{v \in D \setminus \{x\}} \sum_{\substack{W \subset D \text{ finite} \\ x \in W,\ v \in \partial_D W}} \mathbb{P}(C_x^{D \setminus \{v\}} = W)(1-h)^{|W|}\, b_{i-1}(D'_{W,v}, v), \tag{B.2}$$

with $D'_{W,v} = (D \setminus \text{Nb}_D[W]) \cup \{v\}$, where ∂_D and Nb_D are taken in D. There is no separate factor p_c: the openness of v is part of the event E' below, hence of $b_{i-1}(D', v)$. Decompose a selected i-set of open non-root pivotals by its first element v in the linear order.

- Given v, put $W = C_x^{D \setminus \{v\}}$, the root cluster with v forced closed. On E with v an open non-root pivotal, W contains x, has no marks, and is almost surely finite, since an infinite W carries a mark almost surely; and $v \in \partial_D W$, since C_x^D is connected and W is the component of x in $C_x^D \setminus \{v\} \ne C_x^D$. The event $\{C_x^{D \setminus \{v\}} = W\}$ is determined by the bits of W (open) and of $\partial_D W \setminus \{v\}$ (closed), and involves neither ω_v nor any bit of $D' \setminus \{v\}$.
- Given that event, if v is open then $C_x^D = W \cup C_v^{D'}$. Indeed $C_x^D \setminus W$ consists of v and the other components of $C_x^D \setminus \{v\}$, each adjacent to v; it is connected, contains v, and avoids W and the closed set $\partial_D W \setminus \{v\}$, so it lies in D' and in $C_v^{D'}$; conversely $C_v^{D'} \subseteq C_x^D$. No vertex of $C_v^{D'} \setminus \{v\}$ is adjacent to W, since it would lie in the closed set $\partial_D W \setminus \{v\}$. So W is attached to $C_v^{D'}$ only through v.

- W carries no mark, which gives the factor $(1-h)^{|W|}$, so E holds if and only if $E' := \{C_v^{D'}$ contains a mark$\}$ holds, and E' requires v open. Conversely, if $C_x^{D\setminus\{v\}} = W$, W is unmarked and E' holds, then v is an open pivotal for E: closing it gives $C_x^D = W$.
- Every path from x to a mark enters $C_v^{D'}$ only through v, and a vertex $u \in C_v^{D'} \setminus \{v\}$ meets every x–mark path if and only if it meets every v–mark path inside $C_v^{D'}$. So the open non-root pivotals of E after v are exactly the open non-root pivotals of E' in the domain D' with root v, and the earlier ones lie in W.
- The bits and marks of $W \cup (\partial_D W \setminus \{v\})$ and of D' are independent, and the law of (ω, marks) on D' is the domain-D' law. So the conditional contribution is $b_{i-1}(D', v)$.
- If the open non-root pivotals are $v_1 < \cdots < v_n$, the subsets with first element v_j number $\binom{n-j}{i-1}$, and $\sum_{j=1}^{n}\binom{n-j}{i-1} = \binom{n}{i}$. All terms are nonnegative, so Tonelli's theorem justifies the decomposition without a priori moment bounds.

The case $i = 1$ of (B.2) expresses b_1 through b_0. If $b_{i-1} \le A_h^{i-1} b_0$ for every domain and root, then (B.2) gives $b_i(D,x) \le A_h^{i-1} b_1(D,x) \le A_h^i b_0(D,x)$, uniformly over all (D,x). Then $\mathbb{E}[(1+(2A_h)^{-1})^{N_{\text{piv}}}\mathbf{1}_E] = \sum_i (2A_h)^{-i} b_i \le 2\mathbb{P}(E)$, and $\log(1+(2A_h)^{-1}) \ge (4A_h)^{-1}$ for $A_h \ge 1$, so Markov's inequality gives the tail bound. □

Compared with [15, Lemma 3.3], the changes are vertex queries ($T \le 1+dN$, so d becomes $d+1$), $\mathbb{P}(E) \ge p_c h$ in place of h (the constant c_{m}), the root excluded from N_{piv}, and the recursion (B.2) with exterior $(D \setminus \text{Nb}[W]) \cup \{v\}$ and no factor p_c.

Proof of Lemma 4.5. On the event, choose a witness $y \in C_x$ with $\text{piv}_{C_x}(x,y) \ge R := \kappa M^\tau/2$ by a fixed ordering. Mark vertices independently with probability h. The conditional probability that y is the only mark in C_x is $h(1-h)^{|C_x|-1} \ge h(1-h)^M$. In that case every cut vertex $z \in C_x \setminus \{x,y\}$ separating x from y is an open non-root pivotal for $E = E(V,x,h)$, since closing z separates y, the only mark, from x. So $N_{\text{piv}} \ge R$, and by Lemma 4.4,

$$\mathbb{P}(s_x \le M,\ \text{rad}_{\text{piv}} \ge R) \le \frac{2}{h(1-h)^M}\exp\Big(-\frac{R}{4A_h}\Big).$$

Set $X = \kappa^2 M^{2\varepsilon}/\log M$ and $h = X/(bM) = \kappa^2 M^{-1+2\varepsilon}/(b\log M)$, with b a large constant fixed below. Then $\kappa \le M^{1-\tau}$ gives $h \le 1/(b\log M) \le 1/2$; $\kappa \ge \kappa_0$ gives $h \ge \kappa_0^2 M^{-1+2\varepsilon}/(b\log M)$, so $\log(1/h) \le 2\log M$ for $M \ge M_0(\kappa_0, b, \varepsilon)$; hence $A_h = C\sqrt{\log(1/h)/h} \le C\sqrt{2b}\, M^{1/2-\varepsilon}\log M/\kappa$ and $R/(4A_h) \ge X/(8C\sqrt{2b})$; and $-\log(1-h) \le 2h$ gives $M(-\log(1-h)) \le 2X/b$. The logarithm of the bound is therefore at most $O(\log M) + 2X/b - X/(8C\sqrt{2b})$. Choose b so that the last two terms sum to at most $-c_1 X$. Since $X \ge \kappa_0^2 M^{2\varepsilon}/\log M$ dominates $\log M$, the bound holds for large M with $c = c_1/2$. □

B.3 Proof of Proposition 4.3

We use the event Bad, the witness (A,B) with count K and $\mathsf{D} = \text{diam}_{\text{piv}}(B)$, the two cases and the set J of Section 4.4, and ω', the configuration ω with J closed. Let Bad_K be the part of Bad whose chosen witness has count K, and let $k \le K$ be the number of contacts of B.

We prove the claims of Case 2.

(a) *γ from x to a is open in ω'.* Its vertices before a are neither in B (γ has not yet entered B, since the contact preceding the first B-vertex of γ is at or after a) nor contacts (a is the first). Also $R \subseteq B$ and $a \notin J$.

(b) *In ω', the only open vertex outside B'' with a neighbour in B'' is a, and the only neighbour of a in B'' is b^*.* Let $y \notin B''$ be adjacent to B''. If $y \in B$, then $y \in R$, since a vertex of $B \setminus R$ adjacent to the component B'' of $B \setminus R$ lies in B''; so y is closed in ω'. If $y \notin B$, then $y \in \partial B$; vertices of $\partial B \cap C_x$ are contacts, closed in ω' except a, and vertices of $\partial B \setminus C_x$ are closed in ω and unchanged. Finally $N(a) \cap B'' \subseteq N \setminus R = \{b^*\}$.

(c) $\mathrm{rad}_{\mathrm{piv}}(C'_x, x) \geq \mathsf{D}/(2d) - d + 1/2$. By (a), $C'_x \supseteq \gamma[x, a] \cup \{b^*\} \cup B''$, since $a \sim b^*$ and B'' is open and connected in ω'. Let π be a simple path in C'_x from x to v. As $x \notin B''$, by (b) the path enters B'' only along the edge $a \to b^*$ and cannot leave B'' except from b^*; so it visits b^* once, and its part after b^* is a b^*–v path inside B''. Hence every cut vertex of B'' separating b^* from v, and b^* itself, separate x from v in C'_x; none equals x or v. Here $v \neq b^*$, because $M^\tau \geq 4d^2$ and $\mathsf{D} > M^\tau$ give $\mathsf{D}/(2d) - d - 1/2 > d - 1/2 > 0$. By Lemma 4.6, $\mathrm{rad}_{\mathrm{piv}}(C'_x, x) \geq \mathrm{piv}_{B''}(b^*, v) + 1 \geq \mathsf{D}/(2d) - d + 1/2$.

Both cases land in a radius-tail event. Closing only shrinks clusters, so $s'_x \leq s_x \leq M$. Since $\mathsf{D} > KM^\tau \geq M^\tau \geq 4d^2$: in Case 1, $(\mathsf{D} - 1)/2 \geq KM^\tau/2 - 1/2 \geq KM^\tau/(4d)$; in Case 2, $\mathsf{D}/(2d) - d + 1/2 \geq KM^\tau/(2d) - d \geq KM^\tau/(4d)$, because $KM^\tau/(4d) \geq d$. So ω' lies in

$$O_K := \{s_x \leq M,\ \mathrm{rad}_{\mathrm{piv}}(C_x, x) \geq \kappa M^\tau/2\}, \qquad \kappa = K/(2d), \quad 1/(2d) \leq \kappa \leq M^{1-\tau}.$$

Cost of the change. No independence of the input-dependent choices is needed. We have $|J| = m \leq k \leq K$ in Case 1 and $m \leq (k-1) + (d-1) \leq K + d$ in Case 2. Every vertex of J is open in ω and lies in C_x (contacts lie in C_x, and $R \subseteq B \subseteq C_x$), and ω is ω' with J opened.

- *Reconstruction count.* Put $Q_0 = C_x(\omega') \ni x$; here $x \notin J$, since $x \in B$ in Case 1 and a contact equal to x is a in Case 2. List J as $j_1, \ldots, j_m$, where j_i is the first vertex, in a fixed order, of $J \setminus \{j_1, \ldots, j_{i-1}\}$ adjacent to $Q_{i-1} := C_x(\omega'$ with $j_1, \ldots, j_{i-1}$ opened). It exists: $Q_{i-1} \subseteq C_x(\omega)$, and if $J \not\subseteq Q_{i-1}$ then $Q_{i-1} \neq C_x(\omega)$, so an edge of $C_x(\omega)$ leaves Q_{i-1} to a vertex that is open in ω but not in Q_{i-1}, hence lies in J and has not been listed. Since $|Q_{i-1}| \leq M$, each j_i is one of at most dM vertices of ∂Q_{i-1}, which ω' and $j_1, \ldots, j_{i-1}$ determine. So an output ω' has at most $\sum_{m=0}^{K+d} (dM)^m \leq (K + d + 1)(dM)^{K+d}$ preimages in Bad_K.
- *Probability ratio.* $\mathbb{P}(\omega)/\mathbb{P}(\omega') = (p_c/(1-p_c))^m \leq \vartheta^{K+d}$ with $\vartheta = \max\{1, p_c/(1-p_c)\}$.
- *Finite product.* On Bad_K every vertex of C_x lies within distance $M - 1$ of x, and $J \subseteq C_x$. The events Bad_K and O_K are determined by the bits in $B(x, M)$ (locality below), and $\omega \mapsto \omega'$ changes only those bits. Conditioning on the bits outside $B(x, M)$ and comparing pointwise on the finite product gives

$$\mathbb{P}(\mathrm{Bad}_K) \leq (K + d + 1)(dM)^{K+d} \vartheta^{K+d}\, \mathbb{P}(O_K).$$

The sequential reconstruction replaces the observation of [15, Section 3] that every changed edge is incident to $B \subseteq C'_x$: after pruning, $B \setminus B''$ and some contacts can leave C'_x.

Summation. By Lemma 4.5 with $\kappa_0 = 1/(2d)$ and $\kappa = K/(2d)$,

$$\mathbb{P}(\mathrm{Bad}_K) \leq \exp\Big(C_1(K + d)\log M - \frac{cK^2 M^{2\varepsilon}}{4d^2 \log M}\Big),$$

with C_1 depending on d and p_c. As $K \geq 1$, $K + d \leq (d+1)K$, and the exponent is at most $-cK^2 M^{2\varepsilon}/(8d^2 \log M)$ as soon as $M^{2\varepsilon}/(\log M)^2 \geq 8d^2(d+1)C_1/c$, uniformly in $K \geq 1$. Summing

over the fewer than M values of K,

$$\mathbb{P}(\text{Bad}) \le M \exp(-cM^{2\varepsilon}/(8d^2 \log M)) \le e^{-M^{\varepsilon}}$$

for all sufficiently large M, uniformly in x.

Locality. Explore C_x breadth-first by site queries: query x, then unqueried neighbours of discovered open vertices; stop when the cluster is exhausted or more than M (that is, $\lfloor M \rfloor + 1$) open vertices are found. While at most M open vertices are known they form a connected set containing x, so they lie in $B(x, M-1)$ and every queried vertex lies in $B(x, M)$. If x is closed, $C_x = \varnothing$, which is good. If the cluster is exhausted, its vertex set is known, and goodness depends only on that set and on G. If more than M open vertices are found, $s_x \le M$ fails. So $\{s_x \le M,\ C_x \text{ good for cap } M\}$ is determined by the bits in $B(x, M)$. □

C Proofs for Section 5

C.1 Scales, the starting map, one step and the iteration

Proof of Lemma 5.1. Bounds (a)–(c) are [15, Lemma 4.1] with site clusters, and its proof applies verbatim: $s_o = 0$ when o is closed, which contributes nothing to $[\mathbb{E}s_o^{\alpha}]$ or to any bin $2^j \le s_o < 2^{j+1}$ with $j \ge 0$, and the transport is uniform within each finite cluster in the bin. For (d), put $\pi_j = [\mathbb{P}_{p_c}(2^j \le s_o < 2^{j+1})]$; at a record index j of $2^{\beta j}\pi_j$, $2^{\beta i}\pi_i < 2^{\beta j}\pi_j$ for every $i < j$, and (b) is the case $i = j-1$. Passing to a subsequence of record indices, as in the type selection of [15, Lemma 4.1], keeps this. Then for $x \in O$, $\mathbb{P}(2^i \le s_x < 2^{i+1}) \le \pi_i/\sigma_O$, and

$$\mathbb{E}[s_x^2;\ 1 \le s_x < 2n] \le \sum_{i \le j} 4^{i+1} \frac{\pi_i}{\sigma_O} \le \frac{4}{\sigma_O}\, \pi \sum_{i \le j} 4^i 2^{\beta(j-i)} \le C_O 4^j \pi. \qquad \square$$

Proof of Lemma 5.4. From $x \in O$ sample a length-ℓ walk with kernel P, independently of percolation. Require its endpoint to lie in O and every distinct vertex visited by the walk, *including* x, to be open. If e is the number of distinct visited vertices ($1 \le e \le \ell + 1$; holding steps add none), accept also an independent coin of probability $p_c^{\ell+1-e}$. Each walk path then succeeds with probability exactly $p_c^{\ell+1}$, so before any cap the endpoint mass kernel is $p_c^{\ell+1} P^{\ell}|_{O \times O}$. The output lies in $C_x \cap O$, being joined to x by an open walk. By Lemma 2.3 its row sum is at least $c_* p_c^{\ell+1}$ for $\ell \ge l_0$, and its norm is at most $p_c^{\ell+1}\rho^{\ell}$. Fix ℓ with $2\rho^{\ell}/c_* < e^{-H_0}$. Critical finiteness (Lemma 2.4) and Proposition 4.3 allow any sufficiently large M, rejecting unless $s_x \le M$ and C_x is good for cap M, at a loss of at most half the row sum, uniformly on O by invariance. Entrywise domination gives $r \le 2\rho^{\ell}/c_* < e^{-H_0}$. The capped cluster and its goodness are determined by the site bits in a deterministic finite ball (Proposition 4.3), and the walk visits only the ball of radius ℓ. Covariance holds because P is defined from the graph. □

Proof of Lemma 5.9. Put $b = C_0 \mathcal{Q}(k, M) u^k c_0^{-k}$ with a fixed large C_0. By Lemmas 5.7 and 5.8, $\mathbb{E}[L;\ L > b] \le \mathbb{E}L^2/b \le \pi u^k c_0^k / C_0 \le b_0/4$ once $C_0 \ge 4/c$. On the remaining fields discard also those whose root cluster is not good for cap M'; since L is supported on $s_x \le M'$ by (5.3), this loses at most $be^{-(M')^{\varepsilon}}$ by Proposition 4.3 at cap M', which relative to $b_0 \ge cc_0^k \pi u^k$ is at most $C\mathcal{Q}(k, M) c_0^{-2k} n^{\beta} e^{-(M')^{\varepsilon}} \to 0$, because $c_0^{-2k} = n^{O(1)}$ for $k = O(\log n)$ and $M' \ge 2n$. For large n it is at most $b_0/4$. On a retained field output z with probability $L_x(z, \omega')/b$ and fail otherwise, which is legitimate as $L \le b$. The rule is local and covariant, and every successful input has $z \in C_x(\omega') \cap O$, $s_x(\omega') \le M'$ and $C_x(\omega')$ good for cap M'. The retained unnormalized kernel F'

satisfies $0 \le F' \le F$ entrywise and has row sum $b_0' \ge b_0/2$. Hence $u' = b_0'/b \ge cc_0^k \pi u^k/(2C_0 \mathcal{Q} u^k c_0^{-k})$ and $r' = \|F'/b_0'\| \le 2\|F\|/b_0 \le 2c_0^{-k} r^k$. The probabilities $L_x(z,\omega')/b$ are fixed numbers computed at p_c, so the rule is a fixed kernel from the read site bits to outputs; it can be applied at another parameter unchanged, and its success conditions still hold. □

Proof of Proposition 5.3. We fix the parameters in the following order.

(1) Choose $\xi > 0$ with $c_\xi = (\beta + \xi + (\tau+\xi)(1+\xi))/(1-\xi) < 1$, possible as $\beta + \tau < 1$, as in [15, Section 4].

(2) Put $H_* = \max\{1, 2\log(1/c_0)/\xi\}$ and $T_* = 4\log(1/c_0)/\xi$.

(3) By Lemma 5.4, start with $H > H_*$ and M so large that $(\tau+\xi)\log M \ge T_*$, that the estimate in (4) giving $\log a \le t$ holds, and that Proposition 4.3 applies at every cap at least M.

(4) For a fixed old map put $t = U + (\tau+\xi)m$ and $a = \lceil CM^\tau \log M/u \rceil$ with C large enough that $a \ge \max\{2, W/u\}$ (Lemma 5.6). For large M, uniformly in $0 < u \le 1$, $\log a \le U + \tau m + \log\log M + O(1) \le t$.

(5) Only then choose $n \in \mathcal{N}$ large and $k = \lfloor \log n/\log a \rfloor$. Then $k \to \infty$, $k = O(\log n)$, $n \ge a^k \ge (W/u)^k$, $kM \le n/2$ and every goodness and truncation threshold hold eventually, and $k\log a \le \log n < (k+1)t$.

(6) For n large and the old map fixed, Lemma 5.9 gives $U' \le \beta\log n + \log \mathcal{Q}(k,M) + 2k\log(1/c_0) + O(1)$. Since $t \ge (\tau+\xi)m \ge T_*$, $2k\log(1/c_0) \le k\xi t/2$; also $\beta\log n < \beta k t + \beta t$ and $\log \mathcal{Q}(k,M) = O_M(\log(k+1))$. Hence $U' \le k(\beta+\xi)t$. Next $M' \le (5/2)n$ gives $m' \le k(1+\xi)t$. From $r' \le 2(r/c_0)^k$, $H' \ge k(H - \log(1/c_0)) - \log 2 \ge k(1-\xi/2)H - \log 2 \ge k(1-\xi)H$, using $\log(1/c_0) \le \xi H/2$ (as $H \ge H_*$) and $k \ge 2\log 2/(\xi H)$. Consequently, with $t' = U' + (\tau+\xi)m'$, $t'/H' \le c_\xi t/H$.

(7) Choose n larger if necessary so that $H' \ge 2H$, which holds once $k(1-\xi) \ge 2$. The invariants persist: $H' \ge 2H > H_*$ and $M' \ge M$, so $t' \ge (\tau+\xi)\log M' \ge T_*$ and (4) applies to the new map.

(8) Iterate with the same O and $\mathcal{N}$. Then $t/H \to 0$ geometrically and $H \to \infty$. Since $U + m \le \max\{1, (\tau+\xi)^{-1}\}t$, eventually $(U+m)/H < \eta$, with $M \ge 2$ and $H > 1$. Each map has a finite radius.

The maps are covariant under $\mathrm{Aut}(G)$, since every rule (the walk P, the switching rules, rejection sets, masks, the load, truncation and the goodness test) is defined from G and seeds whose joint law is invariant. On success the output satisfies $y \in C_x \cap O$, $s_x \le M$ and goodness for every configuration agreeing on the read ball, and the auxiliary randomness (the walk, a coin of fixed probability $p_c^{\ell+1-e}$, or one uniform for a categorical draw with fixed probabilities $L_x(z,\omega')/b$) does not depend on p. □

C.2 The switching procedure, the inverse codes and the mean load

Proof of Lemma 5.5. Existence of a selection. Initially $C_1 \ni y_0 \in A$, so index 1 is eligible with $C_1 \cap \mathrm{Nb}[A] \ne \varnothing$. After a selection at $i < k$ the new S contains y_i (Step (5), and $y_i \in Y_i$ in the standard case), and $y_i \in C_{i+1}$, so the eligible index $i+1$ meets $\mathrm{Nb}[S]$ while $z \notin S$. Hence the procedure runs until $z \in S$, the indices increase, and there are at most k steps.

Position of y_i. Let i be selected with current S. If $i < k$, the index $i+1$ is eligible and larger, so by maximality $C_{i+1} \cap \mathrm{Nb}[S] = \varnothing$; as $y_i \in C_{i+1}$, $y_i \notin \mathrm{Nb}[S]$, which is the standard case. If $i = k$, then $y_k = z \notin S$ by the stopping rule, so either $z \notin \mathrm{Nb}[S]$ (standard) or $z \in \partial S$ (degenerate). In

the degenerate case z is E_k-open, since success of the old step at z gives $y_{k-1} \in C_z(E_k)$, which is nonempty only if z is open.

$K_i \geq 1$ in the standard case. C_i is connected, contains $y_i \notin \mathrm{Nb}[S]$ and meets $\mathrm{Nb}[S]$. Along a path in C_i from y_i to $\mathrm{Nb}[S]$, the vertices before the first vertex $u \in \mathrm{Nb}[S]$ lie in $C_i \setminus \mathrm{Nb}[S]$ and are connected to y_i, hence lie in Y_i; u is adjacent to a vertex outside $\mathrm{Nb}[S]$, so $u \notin S$, that is, $u \in \partial S$; and u is E_i-open. So $u \in W_i$ and $K_i \geq 1$.

Induction hypothesis. Before each step, $S = C_x(\text{current } \omega)$; in particular all of ∂S is closed in the current ω. This holds for $S = A$. Each field E_i is exchanged at most once, at its own selection, so it is unmodified before index i is selected.

One step. Let the rejection test pass. After the exchange on Mask_i:

- S is unchanged and open, since $\mathrm{Mask}_i \cap S = \varnothing$ ($w_i \notin S$, and $\mathrm{Nb}[Y_i] \setminus \mathrm{Nb}[S]$ misses S);
- w_i and $Y_i \subseteq \mathrm{Nb}[Y_i] \setminus \mathrm{Nb}[S]$ are open, being E_i-open;
- $\partial Y_i \setminus \mathrm{Nb}[S]$ is closed: a vertex there is E_i-closed, since otherwise it would be an E_i-open vertex of $V \setminus \mathrm{Nb}[S]$ adjacent to Y_i, hence in Y_i;
- $\partial S \setminus \{w_i\}$ is closed: it was closed, and $\mathrm{Mask}_i \cap \mathrm{Nb}[S] = \{w_i\}$;
- R_i is closed, by the rejection test and $R_i \cap \mathrm{Mask}_i = \varnothing$.

The set $S \cup \{w_i\} \cup Y_i$ is connected ($w_i \in \partial S$, $w_i \sim b_i \in Y_i$ and Y_i is connected; in the degenerate case $w_k \in \partial S$). Its outer boundary is contained in $(\partial S \setminus \{w_i\}) \cup (\partial Y_i \setminus \mathrm{Nb}[S]) \cup R_i$: a vertex of ∂Y_i is not in S, as $Y_i \cap \mathrm{Nb}[S] = \varnothing$, so it lies in ∂S or in $\partial Y_i \setminus \mathrm{Nb}[S]$; and a neighbour $v \notin S \cup Y_i \cup \{w_i\}$ of w_i lies in $\partial S \setminus \{w_i\}$ if $v \in \mathrm{Nb}[S]$, in $\partial Y_i \setminus \mathrm{Nb}[S]$ if $v \in \mathrm{Nb}[Y_i] \setminus \mathrm{Nb}[S]$, and in R_i otherwise. All of these are closed, so the new root cluster is exactly $S \cup \{w_i\} \cup Y_i$, which re-establishes the induction hypothesis. No vertex of Y_i is adjacent to S, so S and Y_i lie in different components after deleting w_i.

Freshness of the rejection bits. $R_i \subseteq V \setminus \mathrm{Nb}[S]$. Every earlier mask and rejection set lies in $\mathrm{Nb}[S_{j+1}] \subseteq \mathrm{Nb}[S]$, because $\mathrm{Mask}_{i_j} \subseteq \mathrm{Nb}[P_{i_j}] \subseteq \mathrm{Nb}[S_{j+1}]$ and $R_{i_j} \subseteq N(w_{i_j}) \subseteq \mathrm{Nb}[S_{j+1}]$; and the bin test reads only $\mathrm{Nb}[A]$ on success. So the current ω on R_i is the original ω there, and no ω-bit of $V \setminus \mathrm{Nb}[S]$ has been read.

Chain structure. Let $j < j'$. At the selection of $i_{j'-1}$, every larger eligible index, in particular $i_{j'}$, has $C_{i_{j'}} \cap \mathrm{Nb}[S_{j'-1}] = \varnothing$. Since $P_{i_{j'}} \subseteq C_{i_{j'}}$ ($w_{i_{j'}} \in W_{i_{j'}} \subseteq C_{i_{j'}}$, being E-open and adjacent to $Y_{i_{j'}} \subseteq C_{i_{j'}}$; in the degenerate case $w = z \in C_k$) and $S_j \subseteq S_{j'-1}$, we get $P_{i_{j'}} \cap \mathrm{Nb}[S_j] = \varnothing$. By definition $Y_{i_j} \cap \mathrm{Nb}[S_j] = \varnothing$. So in $C' = S_j \cup P_{i_j} \cup \cdots \cup P_{i_m}$ the only vertex outside S_j adjacent to S_j is w_{i_j}; as S_j is connected and contains x, it is the component of x in $C' \setminus \{w_{i_j}\}$. For $j < m$, $z \notin S_j$ (no stop) and $z \neq w_{i_j}$ ($z = y_k \in C_k$ with k eligible at step j, so $C_k \cap \mathrm{Nb}[S_j] = \varnothing$, while $w_{i_j} \in \partial S_j$), so w_{i_j} separates x from z. For $j = m$ in the standard case, $z \in Y_{i_m}$, since the stop occurs when $z \in S$ and $z \notin S_m \cup \{w_{i_m}\}$; again w_{i_m} separates.

Size. $|A| < 2n$, and there are at most k pieces, each inside some C_i, of size at most M. □

Proof of Lemma 5.6. Step 1 (order). If $v \in \mathrm{Hang}(c)$, every x–v path in C' meets c. Hanging sets are nested or disjoint: if $c' \in \mathrm{Hang}(c)$ then $\mathrm{Hang}(c') \subseteq \mathrm{Hang}(c)$, since an x–v path meets c' and its initial segment to c' meets c; and if neither $c' \in \mathrm{Hang}(c)$ nor $c \in \mathrm{Hang}(c')$, then $\mathrm{Hang}(c) \cap \mathrm{Hang}(c') = \varnothing$, for if v lay in both, a simple x–v path would meet both, say c first, its segment from c' to v would avoid c, and if some x–c' path avoided c the concatenation would be an x–v walk avoiding c, so $c' \in \mathrm{Hang}(c)$, a contradiction. Every entry of Λ has z in its hanging set, so the hanging sets of the entries of Λ are strictly nested, in the order of Λ: $\lambda_a \in \mathrm{Hang}(\lambda_b)$ if and only if $a \leq b$.

Step 2 (the code). By Lemma 5.5, $\mathrm{Hang}(w_{i_j}) = C' \setminus S_j$. Hence $w_{i_{j+1}} \in \mathrm{Hang}(w_{i_j})$, and the seams appear in Λ in the order $w_{i_m}, \ldots, w_{i_1}$, at positions $p_m < p_{m-1} < \cdots < p_1$, with $p_m = 0$ exactly in

the degenerate case ($w_{i_m} = z = \lambda_0$; in the standard case w_{i_m} is a separating cut vertex, so $p_m \geq 1$). The code is the index set $\{i_1 < \cdots < i_m\}$ together with the gaps $l_m = p_m + 1$ and $l_j = p_j - p_{j+1}$ for $j < m$, all at least one.

Step 3 (at most one input per code). From the code read off the seams; put $S_j = X(w_{i_j})$, $S_{m+1} = C'$, $Y_{i_j} = S_{j+1} \setminus (S_j \cup \{w_{i_j}\})$ (Lemma 5.5), and $\mathrm{Mask}_{i_j} = \{w_{i_j}\} \cup (\mathrm{Nb}[Y_{i_j}] \setminus \mathrm{Nb}[S_j])$. Undo the exchanges in reverse chronological order, $j = m, \ldots, 1$, exchanging the current distinguished field with E_{i_j} on Mask_{i_j}. The candidate input tuple is now fixed, and no V_i has been used. Replay the old rules from z with the restored fields and seeds (successes, endpoints y_i, clusters C_i), the bin test, $y_0 \in A$ and the switching procedure, and discard the code if any test fails or the procedure does not reproduce the given output and code. The degenerate or standard case is read off from $p_m = 0$.

Step 4 (gap bounds; goodness enters only here). Consider a standard step j. Apply Definition 4.1 to the old cluster $C = C_{i_j}$ with $A_{\mathrm{set}} = C_{i_j} \cap \mathrm{Nb}[S_j]$. The component of y_{i_j} in $C_{i_j} \setminus A_{\mathrm{set}}$ is Y_{i_j}, since an E-open path from y_{i_j} avoiding $\mathrm{Nb}[S_j]$ stays in the full cluster C_{i_j}. The neighbours of Y_{i_j} in $\mathrm{Nb}[S_j]$ lie in ∂S_j, as $Y_{i_j} \cap \mathrm{Nb}[S_j] = \varnothing$, and those in C_{i_j} are exactly the E-open ones, that is, W_{i_j}. So the number of graph edges between Y_{i_j} and A_{set} is $K_{i_j} \geq 1$, and goodness gives

$$\mathrm{diam}_{\mathrm{piv}}(Y_{i_j}) \leq K_{i_j} M^\tau.$$

Fix $a \in N(w_{i_j}) \cap Y_{i_j}$, for example b_{i_j}.

- $j < m$. The entries of Λ strictly between $w_{i_{j+1}}$ and w_{i_j} lie in $\mathrm{Hang}(w_{i_j}) \setminus \{w_{i_j}\}$ and outside $\mathrm{Hang}(w_{i_{j+1}})$, that is, in $(C' \setminus S_j \setminus \{w_{i_j}\}) \cap S_{j+1} = Y_{i_j}$. By Lemma 5.5, $w_{i_{j+1}}$ is adjacent to P_{i_j} and not to S_j. If it has no neighbour in Y_{i_j}, it is adjacent to w_{i_j}; then a path from x through $S_j \cup \{w_{i_j}\}$ to $w_{i_{j+1}}$, continued through the connected set $P_{i_{j+1}} \cup \cdots \cup P_{i_m}$ to z, avoids Y_{i_j}, so no entry lies strictly between and $l_j = 1$. Otherwise fix $b \in N(w_{i_{j+1}}) \cap Y_{i_j}$. For every a–b path π in Y_{i_j}, the concatenation $x \rightsquigarrow w_{i_j} \to a \rightsquigarrow_\pi b \to w_{i_{j+1}} \rightsquigarrow z$ is an x–z path in C', so every separating cut vertex in Y_{i_j} lies on every such π: it is a, b, or a cut vertex of Y_{i_j} separating a from b. Hence $l_j - 1 \leq 2 + \mathrm{piv}_{Y_{i_j}}(a, b) \leq 2 + K_{i_j} M^\tau$.
- $j = m$, *standard.* Then $z \in Y_{i_m}$, and the entries of Λ before w_{i_m} are z and separating cut vertices in $\mathrm{Hang}(w_{i_m}) \setminus \{w_{i_m}\} = Y_{i_m}$; each of the latter lies on every a–z path in Y_{i_m}, so it is a or separates a from z in Y_{i_m}. Hence $p_m \leq 1 + (1 + K_{i_m} M^\tau)$, that is, $l_m \leq 3 + K_{i_m} M^\tau$.

In both cases $l_j \leq 3 + K_{i_j} M^\tau$, and $l_j \leq |P_{i_j}| \leq M$, since the counted entries (and z when $j = m$) are distinct vertices of $P_{i_j} \subseteq C_{i_j}$. A degenerate step has $l_m = 1 \leq M$.

Step 5 (contact probability). In the reconstructed input each contact list and its priority order are fixed without any V_i. If the prescribed seam w_{i_j} is an endpoint of $\kappa_j \leq \min\{d, K_{i_j}\}$ contact edges, the uniform V_{i_j} must lie in the union of their rank intervals $((a-1)/K_{i_j}, a/K_{i_j}]$, of Lebesgue measure $\kappa_j / K_{i_j} \leq \min\{1, d/K_{i_j}\}$. A degenerate step imposes no condition. These sets are determined without the uniforms, so the measure of valid (V_i) is at most $\prod_j \min\{1, d/K_{i_j}\}$, and further validity restrictions only decrease it. From $l \leq 3 + KM^\tau$: if $l \leq 3$, then $\min\{1, d/K\} \leq 1 = \min\{1, 4dM^\tau/l\}$; if $l \geq 4$, then $K \geq (l-3)/M^\tau \geq l/(4M^\tau)$, so $d/K \leq 4dM^\tau/l$. Thus each selected index contributes at most $\min\{1, 4dM^\tau/l_j\}$. The priorities are integrated once under their law, not counted as inverse choices.

Step 6 (sum). Summing over a skipped index or a gap $1 \leq l \leq M$ at each of the k indices gives W^k, and $W \leq 1 + 4dM^\tau(1 + \log M) \leq CM^\tau \log M$ for $M \geq 2$. □

The finite bit space and the load. All comparisons are made on finite bit spaces, as in [15, Section 4.3]. In a successful experiment the old clusters $C_k \ni y_{k-1} \in C_{k-1} \ni \cdots \ni y_0 \in A$ join z to A with $\mathrm{dist}(y_i, y_{i-1}) < M$, and A is connected with $|A| < 2n$, so $\mathrm{dist}(x, z) \leq 2n + kM$. A capped

whole-cluster site test explores the cluster and its closed outer vertex boundary, or stops as soon as the cap is exceeded; the old rule has a fixed query radius and its successful steps move at most M; and every mask and rejection set lies in $\mathrm{Nb}[C']$. So a deterministic larger ball about x contains every query, mask and rejection set of every successful experiment at (x, z); there are finitely many endpoints and codes; and checking an inverse stops at the first failed test. Condition on all bits outside this ball.

At fixed seeds an exchange swaps bits between equally distributed i.i.d. fields on a common mask, and rejection changes no bit; so the number of open bits is preserved, and each successful input tuple and its output have the same product probability. With the injective inverse at a fixed code (Lemma 5.6), define, for a single reference i.i.d. distinguished site field ω', $L_x(z, \omega')$ as $1/n$ times the sum over codes of the valid-code indicators, integrated over reference i.i.d. auxiliary output fields and all seeds, including the V_i. This is the density of the switched mass at z relative to the law of ω', and by (5.4), $0 \le L_x(z, \omega') \le W^k/n$. The function $L_x(z, \cdot)$ vanishes unless (5.3) holds, and on that event it is a function of ω' on $\mathrm{Nb}[C_x(\omega')]$, since the cluster, masks, rejection sets, contacts and the restored bin test all lie there; hence it is a function of ω' in the ball of radius M' about x, and it is covariant. Put $L = \sum_{z \in O} L_x(z, \omega')$ and $b_0 = \mathbb{E}L$. As in [15], the reference product law is not a claim about the law of the actual output of a switch.

Proof of Lemma 5.7. The density comparison preserves total mass, so $nF(x, z)$ is the probability that the experiment at (x, z) succeeds. Without the rejection tests the success probability is that of [15]: the k independent old steps have unconditional endpoint kernel $u^k K^k$. Dropping the tests gives $F \le F_0$.

Lower bound. Condition on $\{C_x(\omega) = A\}$ for a fixed finite A in the bin, an event measurable with respect to ω on $\mathrm{Nb}[A]$, on all E_i and on all seeds, and assume that the chain succeeds with $y_0 \in A$. On the event that the tests of steps $1, \dots, j-1$ pass, the j-th selection, piece, seam and R_{i_j} are functions of the conditioning data only: apart from the rejection tests the procedure reads ω only through S, which on passing equals $A \cup P_{i_1} \cup \cdots \cup P_{i_{j-1}}$, and contact sets are read in E_i, not in ω. The sets R_{i_j} miss all earlier masks, by the freshness part of Lemma 5.5, so each test reads original ω-bits. So there are sets $R_{i_1}, \dots, R_{i_m}$, determined by the conditioning, with $\{\text{all tests pass}\} = \{\omega \text{ closed on } R_{i_1} \cup \cdots \cup R_{i_m}\}$. They are disjoint from $\mathrm{Nb}[A]$, as $R_{i_j} \subseteq V \setminus \mathrm{Nb}[S_j]$, so under the conditioning ω is i.i.d. Bernoulli(p_c) on them, and

$$\mathbb{P}(\text{all tests pass} \mid \text{conditioning}) = (1-p_c)^{|\bigcup_j R_{i_j}|} \ge (1-p_c)^{(d-1)k} = c_0^k,$$

since $|R_i| \le \deg(w_i) - 1$ (w_i has a neighbour in S). Integrating gives $F \ge c_0^k F_0$.

Row sums and norm. As in [15, Lemma 4.7]: B_n is symmetric (on $y \in C_x$ the clusters coincide, and both are empty unless x and y are open), its O-row sum b_n satisfies $cn\pi \le b_n \le 2n\pi/\sigma_O$ by (5.2), and K^k has column sums one, so the row sum of F_0 is $b_0^{(0)} = u^k b_n/n$ and $F_0/b_0^{(0)} = (B_n/b_n)(K^k)^*$ has norm at most r^k. Then $c_0^k b_0^{(0)} \le b_0 \le b_0^{(0)}$, and since $0 \le F \le F_0$ entrywise, $\|F\| \le \|F_0\| \le b_0^{(0)} r^k \le c_0^{-k} b_0 r^k$. □

C.3 Proof of Lemma 5.8

Proof of Lemma 5.8. Expand L^2 with a common reference field ω' and two independent families of auxiliary output fields and seeds; a summand is a pair of valid coded inverses with endpoints z_1, z_2. For a switching inverse put $B = \mathrm{Hang}(w_{i_1}) = C' \setminus A = P_{i_1} \cup \cdots \cup P_{i_m}$ (Lemma 5.5); it contains its endpoint and has $|B| \le \nu$. There are three classes.

Class 1: at least one inverse makes no switch. This is verbatim from [15, Lemma 4.8]. A no-switch inverse at z is valid only if $A = C_x(\omega')$ is in the bin, $z \in A$, and the chain from z succeeds with

$y_0 \in A$; so its integrated part satisfies $L_0 \le (u^k/n)\mathbf{1}\{n \le |A| < 2n\} \sum_{z\in A\cap O} \sum_{y\in A\cap O} K^k(z,y) \le 2u^k$ by the column sums. These summands contribute at most $2\mathbb{E}[L_0 L] \le 4u^k b_0 \le C\pi u^{2k}$ by Lemma 5.7.

Class 2: both switch, and B^1, B^2 are nested or at graph distance at most 2. Hanging sets are nested or disjoint (Lemma 5.6, Step 1). Fix ω' and the first inverse, hence $B^1 = \mathrm{Hang}(c^1)$ and z_1. The second inverse has $B^2 = \mathrm{Hang}(c^2)$, where c^2 is z_2 or a cut vertex separating x from z_2, with $z_2 \in B^2$ and $|B^2| \le \nu$. The number of possible z_2 is at most $\nu + \nu^2 + d^2\nu^3$:

- if $B^2 \subseteq B^1$, then $z_2 \in B^1$: at most ν;
- if $B^1 \subsetneq B^2$, then $c^1 \in \mathrm{Hang}(c^2)$ and $c^2 \ne c^1$, so c^2 separates x from c^1; the cut vertices separating x from c^1, followed by c^1, are linearly ordered by nesting of hanging sets, and those from c^2 onward lie in $\mathrm{Hang}(c^2)$, of size at most ν; so c^2 is one of the last ν entries, and each determines B^2 and at most ν endpoints: at most ν^2;
- if they are disjoint with $1 \le \mathrm{dist}(B^1, B^2) \le 2$, choose $b \in B^2$ within distance 2 of B^1, in at most $d^2\nu$ ways ($d + d(d-1) = d^2$ vertices at distance 1 or 2 from a vertex); c^2 is b or a cut vertex separating x from b, and lies among the last ν entries of that ordered list, since all later entries and b lie in $\mathrm{Hang}(c^2)$; then at most ν endpoints: at most $d^2\nu^3$.

For each endpoint, Lemma 5.6 bounds the integrated second-inverse mass by W^k/n. Integrating the first inverse, this class is at most $b_0(\nu + \nu^2 + d^2\nu^3)W^k/n \le C(\nu + \nu^2 + d^2\nu^3)\pi u^{2k}$, by Lemma 5.7 and $n \ge (W/u)^k$.

Class 3: both switch, and $\mathrm{dist}(B^1, B^2) \ge 3$, that is, $\mathrm{Nb}[B^1] \cap \mathrm{Nb}[B^2] = \varnothing$.

Locality of each inverse, the joint restoration and the claims (J1)–(J3) are as in Section 5.5.

Injectivity with the extra datum. Fix z_1, z_2 and all seeds. The restored auxiliary fields and old-map seeds recover both old chains by replay from their endpoints, and ξ gives A_{00}. Run the switching procedure of Section 5.2 virtually from the *set* $S_0 = A_{00}$ with field ω_{00} and the restored first chain, ignoring only the bin test and the test $y_0 \in A$; the rules use S only as a set, and connectivity of S_0 is never used. Let S denote the current set of the original first forward run, from A^1 with ω^1. Throughout, $B^2 \subseteq A^1 \subseteq S$, and initially $S_0 = A_{00} = A^1 \setminus B^2$. The key fact is that for $U \subseteq \mathrm{Nb}[B^1]$,

$$U \cap S = U \cap S_0, \qquad U \cap \mathrm{Nb}[S] = U \cap \mathrm{Nb}[S_0], \qquad U \cap \partial S = U \cap \partial S_0 \qquad \text{whenever } S_0 = S \setminus B^2,$$

since a vertex of $\mathrm{Nb}[B^1]$ is neither in B^2 nor adjacent to it. We show by induction that $S_0 = S \setminus B^2$, that the current distinguished fields of the two runs agree off Mask^2 and their auxiliary fields agree, and that each step selects the same index, piece, contacts, seam, rejection set and mask.

(1) *Index.* The original seam satisfies $w_i \in C_i \cap \partial S$ and $w_i \in B^1$, so $w_i \in C_i \cap \mathrm{Nb}[S_0]$. Every larger eligible index misses $\mathrm{Nb}[S] \supseteq \mathrm{Nb}[S_0]$. Maximality selects the same i, also when unused clusters meet B^2 or $y_0 \in B^2$. The case, standard or degenerate, agrees: $y_i \in B^1$, and the key fact applies.

(2) *Piece.* $Y_i^0 \supseteq Y_i$, since $\mathrm{Nb}[S_0] \subseteq \mathrm{Nb}[S]$. If it were larger, some E_i-open $u \notin Y_i \cup \mathrm{Nb}[S_0]$ would be adjacent to Y_i; then $u \in \mathrm{Nb}[B^1]$, so $u \notin \mathrm{Nb}[S]$ by the key fact, and u would belong to Y_i. So $Y_i^0 = Y_i$. This is where distance 3 is needed: at distance 2 the slit domain could grow through a vertex of $\partial S \cap \partial B^2$ adjacent to Y_i.

(3) *Contacts and seam.* $W_i^0 = W_i$, by the key fact on $N(Y_i)$, so the contact list, its priority order and, with the same V_i, the seam agree.

(4) *Rejection.* $R_i^0 = N(w_i) \setminus (\mathrm{Nb}[S_0] \cup \mathrm{Nb}[Y_i]) = R_i$, by the key fact on $N(w_i)$; the two current fields agree on $R_i \subseteq \mathrm{Nb}[B^1]$, which is disjoint from Mask^2, so the test passes in the virtual run.

(5) *Mask and update.* $\mathrm{Mask}_i^0 = \mathrm{Mask}_i$, the fields agree there, so after the exchange they still agree off Mask^2, and $S_0 \cup P_i = (S \cup P_i) \setminus B^2$.

(6) *Stopping.* $z_1 \in S_0$ if and only if $z_1 \in S$, as $z_1 \in B^1$.

The same holds for the second virtual run. Running both separately from the jointly restored tuple and combining their disjoint distinguished-field changes reconstructs ω', both auxiliary output families and all seams, hence both codes. Thus, at fixed z_1, z_2 and seeds, $(\omega', E'^1, E'^2, \mathrm{code}^1, \mathrm{code}^2) \mapsto (\omega_{00}, E^1, E^2, \xi)$ is injective, so $(\omega', E'^1, E'^2, \mathrm{code}^1, \mathrm{code}^2) \mapsto (\omega_{00}, E^1, E^2)$ is at most $4^d\nu^2$-to-one, by (J3). By (J2), each restored successful trajectory, the union of its old-map whole clusters, contains $c^a \in \partial A_x$, so it hits $\mathrm{Nb}[C_x(\omega_{00})]$.

Mass. As in [15, Lemma 4.8], the joint undo preserves the weight $p_c^a(1-p_c)^{N-a}$ of the full finite bit tuple, with a open bits among N. With the multiplicity bound, summing weights gives at most $4^d\nu^2$ times the mass of the image in an unrestricted reference product experiment, weighted by $1/n^2$, with an i.i.d. root field and two independent old-chain families; integrate the seeds by Tonelli's theorem. Retain only $s_x = |C_x| < 2n$ (from $A_x \subseteq A_{00}$), success of both chains, and that each trajectory hits $\mathrm{Nb}[C_x]$. The lower size window of [15, Lemma 4.8] is not available, since A_x can be much smaller than A_{00}; it is replaced by the record-scale bound (e) of (5.2).

Trajectory hits, as in [15, Lemma 4.8] with site clusters. For finite A, let $h_z(A)$ be the probability that a successful k-step old trajectory from $z \in O$ hits A. Then $\sum_{z\in O} h_z(A) \le c_\sigma|A|\nu u^k$, by transporting unit mass from z to each vertex of its trajectory union on success (expected outgoing mass at most νu^k) and Lemma 2.1. Apply it to $\mathrm{Nb}[A]$, with $|\mathrm{Nb}[A]| \le (d+1)|A|$. Conditional on the reference root cluster the two chain families are independent, so this class is at most

$$4^d\nu^2 \cdot \frac{C\nu^2 u^{2k}}{n^2}\mathbb{E}_{p_c}[s_x^2;\ 1 \le s_x < 2n] \le C4^d\nu^4\pi u^{2k},$$

by the record-scale bound $\mathbb{E}_{p_c}[s_x^2;\ 1 \le s_x < 2n] \le C_O n^2\pi$ for $x \in O$ (Lemma 5.1). Summing the three classes proves (5.5) with $\mathcal{Q}(k, M) = C(1 + \nu + \nu^2 + d^2\nu^3 + 4^d\nu^4)$. □

D Proofs for Section 6

Throughout, G is simple, infinite, connected, locally finite and quasi-transitive with $h_V(G) > 0$, and $\mathrm{Aut}(G)$ is unimodular. Steps described as verbatim are those of [15], read with site bits; each was checked for site percolation.

D.1 Moments

Lemma D.1 (Maximal susceptibility in an intrinsic ball; site form of [15, Lemma 5.1])**.** *For a deterministic domain D put $f_D(v) = \mathbb{E}_p|C_v^D|$, which is zero off D. Let $B^D(v, M)$ be the set of vertices reachable from v by an open path in D of length at most M, empty if v is closed; a maximum over the empty set is 0. If $v \in D$ and $0 < p < p_c$, then*

$$\mathbb{E}_p \max_{z\in B^D(v,M)} f_D(z) \le \Big(1 + \frac{(1-p)M}{p}\Big) f_D(v).$$

Proof. Take D finite first. *Resampling.* For $w \in D$, expose $B = C_w^D(\omega)$, and resample every site of $\mathrm{Nb}[B] \cap D$ (resample $\{w\}$ if $B = \varnothing$). The event $\{C_w^D = B\}$ is measurable with respect to these sites, so the resampled $\tilde\omega$ has product law independent of B, and $\sum_w \mathbb{P}(Z \in B) = \mathbb{E}\max f_D$ as in [15, Lemma 5.1], where Z is the maximizer for $\tilde\omega$. *First boundary site.* Suppose $v \notin B$ and

$Z \in B$. Along a simple $\tilde{\omega}$-open path $v = z_0, \dots, z_l = Z$ with $l \le M$, the first vertex in $\mathrm{Nb}[B]$ is some $y = z_i \in \partial B$ with $i \le l - 1 \le M - 1$, because the path must cross ∂B to enter B; this y is ω-closed, and if $i \ge 1$ then $z_0, \dots, z_{i-1}$ lie in $D \setminus \mathrm{Nb}[B]$, where $\tilde{\omega} = \omega$. Let N_w count the $y \in \partial B$ such that $y = v$, or v is joined to $N(y)$ in $D \setminus \mathrm{Nb}[B]$ by an ω-open path of length at most $M - 2$; set $N_w = 0$ if $v \in B$. Then $\mathbb{P}(Z \in B) \le T^D(v, w) + \mathbb{E}N_w$. *Opening y.* Open one counted y. In ω^y, $v \leftrightarrow w$, and y is a pivotal open site, since closing it returns ω, where $v \not\leftrightarrow w$. All pivotal sites of $v \leftrightarrow w$, including the terminal v, lie in a fixed order on every simple open v–w path. The prefix $z_0, \dots, z_i = y$ has $i + 1 \le M$ vertices, so y is among the first M pivotal sites from v. For a fixed output and distinguished site, closing it recovers the input, and comparing the two values of that bit gives $\mathbb{E}N_w \le ((1-p)/p)MT^D(v, w)$. *Infinite domains and $M = 0$.* Exhaustion is as in [15, Lemma 5.1]; for $M = 0$, $\mathbb{E}[\mathbf{1}\{v \text{ open}\} f_D(v)] \le f_D(v)$. □

Lemma D.2 (A local map in a deleted domain; site form of [15, Lemma 5.2])**.** *Fix a site map from Proposition 5.3, with critical success probability $u > 0$, cap $M \ge 2$ and $H > 1$, and run it at parameter p. On a map-dependent interval $p_0 \le p < p_c$ with $p_0 \ge p_c/2$, $u_p \ge u/2$ and $\|K_p\| \le e^{-(H-1)}$. For deterministic $A \subseteq V$ put*

$$F_A = f_{V \setminus \mathrm{Nb}[A]}/\chi \quad (\textit{zero on } \mathrm{Nb}[A]), \qquad h_A(v) = \sum_{a \in \mathrm{Nb}[A],\ \mathrm{dist}(v,a) \le M} \ \sum_{b \sim a} F_A(b).$$

There are Z and C_{map}, uniform on the interval, with $2 \le Z \le CM/u$ and, for every $v \in O$, $(K_p F_A)(v) \le Z F_A(v) + C_{\mathrm{map}} h_A(v)$.

Proof. The continuity paragraph of [15, Lemma 5.2] is verbatim with site bits; it uses that success, with output $Y \in C_v \cap O$ and $s_v \le M$, is determined by the read site bits, which follows from Definition 5.2 since $\{s_v \le M,\ Y \in C_v\}$ is determined by the bits of $B(v, M)$, and that the auxiliary randomness has a law not depending on p. On success at p there is a simple open path of length at most $M - 1$ from v to Y, because $Y \in C_v$ and $s_v \le M$. If the path avoids $\mathrm{Nb}[A]$, then $Y \in B^{V \setminus \mathrm{Nb}[A]}(v, M)$, and Lemma D.1 with $D = V \setminus \mathrm{Nb}[A]$ bounds the contribution by $(1 + (1-p)M/p)F_A(v)$; this case is empty if $v \in \mathrm{Nb}[A]$. If $Y \in \mathrm{Nb}[A]$, the contribution is zero. Otherwise let b be the first vertex after the last visit to $\mathrm{Nb}[A]$ and a its predecessor; then $a \in \partial A$, and the terminal segment from b to Y lies in $V \setminus \mathrm{Nb}[A]$ with length $l \le M$. The events that the segment is open and $|C_Y^{V \setminus \mathrm{Nb}[A]}|$ are increasing, and on the first $C_b^{V \setminus \mathrm{Nb}[A]} = C_Y^{V \setminus \mathrm{Nb}[A]}$; Harris's inequality gives $f_{V \setminus \mathrm{Nb}[A]}(b) \ge p^{l+1} f_{V \setminus \mathrm{Nb}[A]}(Y)$, so $F_A(Y) \le p^{-M-1} F_A(b)$. Summing over last exits with $\mathrm{dist}(v, a) \le M$ bounds this contribution by $p^{-M-1} h_A(v)$. Dividing by $u_p \ge u/2$ gives $Z = \max(2, (2/u)(1 + 2M/p_c)) \le CM/u$ and $C_{\mathrm{map}} = 2(2/p_c)^{M+1}/u$. □

Proof of Proposition 6.1. Assume (5.1), so that Proposition 5.3 applies, and let Ψ be as in Lemma 2.8, so $(1-p)\chi' \ge \chi^2 \Psi(p)$.

Upper bound. Expose $A = C_x$, $x \in O$, in an independent p-field and put $\widehat{\mathbb{E}}[G(A)] = \mathbb{E}_p[(s_x/\chi)G(C_x)]$. For fixed a: if $a \in C_x$, all neighbours of a lie in $\mathrm{Nb}[C_x]$, where $F_A = 0$. If $a \in \partial C_x$, then a is closed and C_x is one of its adjacent clusters, say cluster i, with $n_i = s_x$. Conditionally on C_x the bits outside $\mathrm{Nb}[C_x]$ are fresh; for $b \sim a$ with $b \notin \mathrm{Nb}[C_x]$, in the same configuration $|C_b^{V \setminus \mathrm{Nb}[C_x]}| = s_b$, and if b is open its cluster is some $j \ne i$. Hence

$$\widehat{\mathbb{E}}\Big[\mathbf{1}\{a \in \mathrm{Nb}[A]\} \sum_{b \sim a} F_A(b)\Big] = \chi^{-2} \mathbb{E}\Big[s_x \mathbf{1}\{a \in \partial C_x\} \sum_{b \sim a,\ b \notin \mathrm{Nb}[C_x]} s_b\Big] \le \chi^{-2} \mathbb{E}\Big[\mathbf{1}\{a \text{ closed}\} n_i\, d \sum_{j \ne i} n_j\Big]$$
$$\le d\chi^{-2} \mathbb{E}[\mathbf{1}\{a \text{ closed}\} Q_a] \le d c_\sigma \Psi(p),$$

the last step because $\mathbb{E}[\mathbf{1}\{a \text{ closed}\}Q_a]$ is constant on types with type average $\chi^2\Psi$. Bounded degree gives $\widehat{\mathbb{E}}h_A(v) \le C'_{\mathrm{map}}\Psi(p)$ uniformly in v, and iterating Lemma D.2 with $F_A(x) = 0$ and stochasticity gives $\widehat{\mathbb{E}}(K_p^j F_A)(x) \le C_{\mathrm{map}} Z^j \Psi(p)$, with a constant renamed.

Lower bound identity. Fresh exterior exposure gives $\mathbb{E}[s_x f_{V\setminus \mathrm{Nb}[C_x]}(v)] = \mathbb{E}[s_x s_v \mathbf{1}\{v \notin \mathrm{Nb}[C_x]\}] = \mathbb{E}(s_x s_v;\ C_x \ne C_v)$, since when both are open, $v \notin \mathrm{Nb}[C_x]$ if and only if $C_v \ne C_x$. So $\widehat{\mathbb{E}}(K_p^j F_A)(x) = \chi^{-2}\sum_{v\in O} K_p^j(x,v)\mathbb{E}_p(s_x s_v;\ C_x \ne C_v)$, as in [15, Proposition 5.3].

Diagonal loss. Harris's inequality and Lemma 2.7 give the lower bound c_0 without distinctness, and the diagonal-loss identity of [15, Proposition 5.3] holds verbatim (if x is closed, $s_x = 0$), with $\mathbb{E}s_x^2 \le \kappa_p \chi_{\max}^3$ (Lemma 2.5).

Conclusion. The choice of j, the resulting lower bound $\Psi(p) \ge c_\gamma \chi^{-\gamma}$, the inequality $\chi' \ge c\chi^{2-\gamma}$ (now from the lower bound in (2.7)) and its integration are verbatim. The transcript comparison is verbatim with site exploration, which makes at most $(d+1)N$ site queries, so $\mathbb{E}_p L_N^b \le e^{(d+1)C_b}$. The final summation is verbatim. □

Proof of Proposition 6.2. As in [15, Proposition 5.5], with these changes. The algorithm reveals the marks, then explores each marked cluster by site queries: the marked vertex, then the neighbours of each discovered open vertex. A non-marked vertex z is queried only if some $y \in \mathrm{Nb}[z]$ is open with C_y containing a mark, so its revealment is at most $1/n + (d+1)\sup_y \mathbb{E}_p \min(1, s_y/n) \le (d+2)C_a n^{-a}$, uniformly, by Proposition 6.1, the restriction coupling and monotonicity in $p \le p_c$. Increasing site functions have resampling influence $2p(1-p)\mathbb{P}(\text{pivotal})$, and $F'(p) = \sum_z \mathbb{P}(z \text{ pivotal})$ by finite-product differentiation. The rest, including the covariance bound [15, Lemma 5.4], is verbatim. □

D.2 The triangle trace

Write $T = T_p$, $m_p = \|T_p\|$ and $a_j = \operatorname{tr} T^j$. For a nonnegative invariant function F of an ordered pair of distinct *nonempty* open clusters put $\mathcal{D}(F) = [\mathbb{E}_p \mathbf{1}\{o \text{ open}\} s_o^{-1} \sum_{B\ne C_o} F(C_o, B)]$, the sum over open clusters B, and $k_J(A,B) = \sum_{x\in A, y\in B} J(x,y)$.

Rerooting. $\mathcal{D}(k_J F) = [\mathbb{E}_p \sum_v J(o,v)F(C_o,C_v)\mathbf{1}\{o,v \text{ open},\ C_o \ne C_v\}]$ and $\mathcal{D}(F(A,B)) = \mathcal{D}(F(B,A))$, by transport from an open x to each $y \in C_x$, as in [15, Section 6].

Contacts. For distinct open clusters A, B, $k_s(A,B) \le k_{\mathrm{Adj}^2}(A,B) \le d^2 k_s(A,B)$. Each common boundary vertex z gives a path $x \sim z \sim y$; conversely the middle vertex of such a path with $x \in A$ and $y \in B$ is not in $A \cup B$, since distinct open clusters are not adjacent, so it lies in $\partial A \cap \partial B$, and each z carries at most d^2 such paths.

Lemma D.3 (Contacts of two large clusters; site form of [15, Lemma 6.1])**.** *Fix $0 < \alpha < 1/2$ and $\varepsilon_1 > 0$ with $r = 1/2 + \varepsilon_1 - 2\alpha < 0$. Uniformly for $p_c/2 \le p < p_c$ and $L \ge 1$, $\mathcal{D}(k_s^2 \mathbf{1}\{\min(|A|,|B|) \ge L\}) \le CL^r$.*

Proof. Fix x, v and $H_n = \{x, v \text{ open},\ C_x \ne C_v,\ n \le \min(s_x, s_v) < 2n\}$. The van den Berg–Kesten inequality, monotone coupling and Proposition 6.1 give $\mathbb{P}(H_n) \le Cn^{-2\alpha}$. *Full score.* Explore C_v by site queries: query v, then every unqueried neighbour of each discovered open vertex, with cap $N = \lceil 2(d+1)n \rceil$. When the exploration completes, the queried set is exactly $\mathrm{Nb}[C_v]$, and $S_{\mathrm{full}} = \sum_{\mathrm{queried}}(\omega_z - p)$ satisfies $\mathbb{P}(|S_{\mathrm{full}}| > t) \le 2e^{-2t^2/N}$. *Exterior score.* Expose $A = C_x$ with ∂A, that is, the bits of $\mathrm{Nb}[A]$. If $v \notin \mathrm{Nb}[A]$, explore the cluster of v in $V \setminus \mathrm{Nb}[A]$ with the same rule and cap; the queried bits are fresh, so the same tail bound holds. If x is closed, set $S_{\mathrm{ext}} = 0$; this case never enters the next step, since $H_n \subseteq \{x \text{ open}\}$. *Difference.* On $H_n \cap \{s_v \le s_x\}$, v is open and $C_v \ne C_x$, so $v \notin \mathrm{Nb}[A]$ and $C_v \cap \mathrm{Nb}[A] = \varnothing$. Both explorations finish, since $|\mathrm{Nb}[C_v]| \le (d+1)s_v < N$, with queried sets $\mathrm{Nb}[C_v]$ and $C_v \cup (\partial C_v \setminus \partial A)$; note $\partial C_v \cap A = \varnothing$ because the clusters are not adjacent. The difference is $\partial A \cap \partial C_v$, which is all closed, so exactly $S_{\mathrm{full}} - S_{\mathrm{ext}} = -p\, k_s(A, C_v)$.

Conclusion. Since $k_s \le |\partial C_v| \le ds_v < 2dn$ on this event, the rest of [15, Lemma 6.1] is verbatim and gives $\mathbb{E}[k_s \mathbf{1}_{H_n}] \le Cn^r$. Finally $\mathcal{D}(k_s^2 \mathbf{1}\{\cdot\}) \le \mathcal{D}(k_{\mathrm{Adj}^2} k_s \mathbf{1}\{\cdot\})$; apply the rerooting with $J = \mathrm{Adj}^2$. There are at most d^2 vertices v with $\mathrm{Adj}^2(o,v) > 0$, each with weight at most d. The dyadic bins are as in [15]. □

Proof of Proposition 6.3. Russo for the trace. By Lemma 2.8 and cyclicity, $a_3' = 3\,\mathrm{tr}(T^2 T')$ almost everywhere. With $J = T^2$, Lemma 2.8 and the rerooting give

$$(1-p)a_3' = 3\mathcal{D}(k_s k_J) + 3\,\mathrm{tr}(JR_p).$$

Terminal terms. $\mathrm{tr}(JR_p) \le (1-p)a_2 + \mathrm{tr}(T^2 \mathrm{Adj} T) + \mathrm{tr}(T^3 \mathrm{Adj}) = (1-p)a_2 + 2\,\mathrm{tr}(\mathrm{Adj} T^3)$. By Lemma 2.2 with $A = T^3$ (positive semidefinite, invariant and bounded) and $B = \mathrm{Adj}$ (self-adjoint, invariant and bounded), $\mathrm{tr}(\mathrm{Adj} T^3) \le \|\mathrm{Adj}\|\,\mathrm{tr}\, T^3 \le d a_3$. Also $a_3 \ge [\sum_y T(o,y)T(y,y)T(y,o)] = p a_2$, so $a_2 \le a_3/p$. Hence $0 \le \mathrm{tr}(JR_p) \le ((1-p)/p + 2d)a_3 \le C a_3$.

Square bound and the small part. The square bound of [15, Proposition 6.2] is verbatim, since distinct clusters are disjoint witnesses. For the small-A part, $k_s \le k_{\mathrm{Adj}^2}$; reroot with $J' = \mathrm{Adj}^2$, expose $\mathrm{Nb}[A]$ with $A = C_o$, and use $v \notin \mathrm{Nb}[A]$, so that the exterior cluster is fresh. This gives $\mathcal{D}(k_s k_J \mathbf{1}\{|A| < L\}) \le d^2 c_\sigma a_3 [\mathbb{E} s_o \mathbf{1}\{s_o < L\}] \le C L^{1-\alpha} a_3$. The large part is Cauchy–Schwarz with Lemma D.3, as in [15].

Differential inequality. $a_3 \ge p^3 \ge (p_c/2)^3$ replaces $a_3 \ge 1$. So $b = \sqrt{a_3} \ge (p_c/2)^{3/2}$ and $b' \le C L^{1-\alpha} b + Cb + C L^{1/4+\varepsilon_1/2-\alpha} m_p^{3/2}$; the added term Cb has a bounded, hence integrable, coefficient. The exponent choices and the integration are verbatim, using Proposition 6.2. □

D.3 Deformation and escapes

Proof of Lemma 6.6. Weight bound. $p^{-|I|} q^{-b} \le ((2/p_c)(1-p_c)^{-d})^k$, and $\sum_u P(w_i = z) = \sum_u P^i(u,z) = 1$ for each step i, so $\sum_u \sum_{w \ni z} \mu_u(w) \le k+1$; choose C accordingly. *Penalty bound.* Charge each counted template to an interior vertex of one fixed simple open path. *Monotonicity.* For an enlargement $(D', \omega') \ge (D, \omega)$ of a pair connected before, take a template counted after. Every old simple open path is a new one, so it contains I_w and, by Lemma 6.5, u and v. Hence the template was eligible in D, and open and pivotal before. Blockers closed or absent after were closed or absent before, and the condition $x, y \notin I_w$ is deterministic. *Exhaustion.* The rest is as in [15, Lemma 7.2], using $\mathbb{E}|C_x|^2 < \infty$ (Lemmas 2.5 and 2.7): once an exhaustion contains C_x^D, all counted templates agree, since their blockers lie in $\mathrm{Nb}[C_x^D]$ and are closed or absent both before and after they enter. □

Proof of Lemma 6.9. Reduction to a visit. Suppose that $W_t^D(v,y) > W_t^{D \setminus A'}(v,y)$. If the connection is destroyed, every old simple path visits A'. Otherwise a template is newly counted after deletion. It is eligible in D, its interior is open, and the terminal condition is unchanged. Either an interior vertex was not pivotal before, in which case an old simple path avoiding it visits A'; or a blocker $\beta \in A'$ was open before, in which case $\beta \sim z$ for some interior $z \notin A'$, and every surviving simple path, being an old one, visits $z \in \mathrm{Nb}[A']$. So the whole loss is at most the indicator that some old simple open v–y path in D visits $\mathrm{Nb}[A']$. *Splitting.* Split at the visited z: a prefix containing z, and a suffix that is empty ($z = y$) or runs from $N(z)$ to y; the two are disjoint witnesses, and the van den Berg–Kesten inequality gives $\mathbb{E}(W^D - W^{D \setminus A'}) \le \sum_{z \in \mathrm{Nb}[A']} T_0(v,z)(I + \mathrm{Adj} T_0)(z,y)$. *Summing* over y, $\sum_y (I + \mathrm{Adj} T_0)(z,y) \le 1 + d\chi_0^+ \le (1/p + d)\chi_0^+$, since $\chi_0^+ \ge p$. □

Proof of Lemma 6.10. Exposure. Expose $A = C_u^D$ with its closed boundary, that is, the bits of $\mathrm{Nb}[A] \cap D$. Then $X_t^D(u)$ is determined: counted templates have interior and terminals in A and

blockers in $\mathrm{Nb}[A]$, and a blocker in A would be open, so the template would not be counted. The bits of $D \setminus \mathrm{Nb}[A]$ are fresh. *Deleting* $\mathrm{Nb}[A]$ *changes nothing outside.* For $v \notin \mathrm{Nb}[A]$, which is forced when v is open and $C_v \neq A$, $C_v^D \cap \mathrm{Nb}[A] = \varnothing$, and weighted connections from v in D and in $D \setminus \mathrm{Nb}[A]$ agree exactly: eligibility and pivotality are cluster-internal, and a blocker of a template inside C_v cannot lie in A, since it would be open and adjacent to C_v, so it is closed or absent in both domains. Including $v \in \mathrm{Nb}[A]$ by the zero convention, $\mathbb{E}(X(u)X(v);\ C_u \neq C_v) = \mathbb{E}[X_t^D(u) s_t^{D\setminus \mathrm{Nb}[A]}(v)]$. *Error.* Apply Lemma 6.9 with $A' = \mathrm{Nb}[A] \cap D$, and note $\mathrm{Nb}[A'] \subseteq \{z : \mathrm{dist}(z, A) \leq 2\}$. By (H1), $\sum_{z\in \mathrm{Nb}[A']} T_0(v,z) \leq c_{\mathrm{nb}}(p) \sum_{a\in A} T_0(v,a)$, and by Lemma 2.5, $\mathbb{E}(|A|\mathbf{1}\{a \in A\}) \leq \kappa_p \chi_0^+ T_0^2(u,a)$. So $C' = C_{\mathrm{del}} c_{\mathrm{nb}} \kappa_p$, evaluated at $p = p_c/2$, works. □

Proof of Lemma 6.13. Let $F = \{\pi_0, \dots, \pi_{l-1} \text{ open}\}$, so $\mathbb{P}(F) = p^l$. Templates counted for $b \leftrightarrow y$ whose interior meets the path have total weight at most $(l+1)c_k/k$. For the others, interior pivotality for $z \leftrightarrow y$ holds by concatenation, and $z \notin I_{w'}$ because z lies on the path; blockers and eligibility are unchanged. Harris's inequality for $\mathbf{1}_F$ and $W_t^D(z,y)$, both increasing by Lemma 6.6, finishes the proof. □

Proof of Lemma 6.14. The proof of [15, Lemma 8.2] applies verbatim with $T_0(b,b) = p$ and $\mathrm{tr}\, T_0 = p$. □

Proof of Lemma 6.15. Use the labelled experiment of [15, Lemma 8.1]. Replace the event that the escape avoids every raw-path vertex other than the start by the *site avoidance event*: every escape vertex other than the start is at graph distance at least 2 from every raw-path vertex other than the start. Since $I_w \subseteq \mathcal{W}(w) \setminus \{u, v\}$, this keeps the escape inside $D_w = V \setminus (\mathrm{Nb}[I_w] \setminus \{u, v\})$. The event refers only to the vertex set of the raw path and the escape, so rooted reversal transfers its bound from u to v unchanged.

- *Deterministic part.* $\mathrm{dist}(g_{-i}, g_j) = i + j \geq 2$ for $i, j \geq 1$ along the geodesic segments, so the negative half never violates avoidance against the positive half.
- *Near-collision bounds.* Replace each point collision by membership in $\mathrm{Nb}[x]$, which costs a factor at most $d + 1$: a middle time r near a negative-segment vertex g_{-i}, $i \geq 1$, has probability at most $(d+1)\rho^r$, and zero for $r < J + i - 1$; a tail time l near a positive-segment vertex has probability at most $(d+1)\rho^l$, and zero for $l < J$; a tail–middle collision has $\sum_x P^l(g_{-J}, x)((I + \mathrm{Adj})P^r)(x, g_J) \leq (1+d)\rho^{l+r}$, and zero for $l + r < 2J - 1$. So $E_J^s \leq (d+1)c_*^{-1}(2(J+1)\sum_{r\geq J-1} \rho^r + \sum_{n\geq 2J-1}(n+1)\rho^n) \to 0$.
- *Suffix collisions.* $\|(I + \mathrm{Adj})\nu H_{J,i}\| \leq (1+d)\sqrt{c_\sigma} c_*^{-1} \rho^L$, so $F_J^s = (d+1)F_J$, in the notation of [15, Lemma 8.1].
- $\mathrm{dist}(u, v) \geq 3$ is unchanged, and any u–v path has length at least $\mathrm{dist}(u, v) \geq 3$, so $\ell_w \geq 3$.
- *Potential test.* Over $S = \mathrm{Nb}[\mathrm{Nb}[I_w]]$, with $|S| \leq (1 + d + d^2)(k+1)$, using Lemma 6.14.
- *Constants.* The order of choice (J, then L_0, then k_0), the domination and $b_* = \vartheta_0/(2D_J)$ are as in [15, Lemma 8.1]. □

D.4 Proof of Theorem 3.1

We follow the proof of [15, Theorem 2.6] in [15, Section 9], with these changes, in its order.

(1) *Setup.* Suppose that T_{p_c} is unbounded. Put $p_0 = p_c/2$ and, for $p_0 \leq p < p_c$, $\delta = p_c - p$, $S = \log(1/\delta)$, $m_0 = \|T_p\|$, $R = \chi_0^+/m_0 \geq 1$ and $a_3 = \mathrm{tr}(T_p^3)$; here $R \geq 1$ since $m_0 \leq \chi_0^+$.

(2) *Lower bound on m_0.* Sprinkling (Lemma 2.9) with $a = \delta/(1-p)$ would bound T_{p_c} if $ad(1+dm_0) < 1$. Hence $m_0 \geq (1-p)/(d^2\delta) - 1/d \geq c\delta^{-1}$ for small δ. With Propositions 6.2 and 6.3, and $(p_c/2)^3 \leq a_3 \leq C_\eta \delta^{-\eta}$, this gives $\log R = o(S)$ and $|\log a_3| = o(S)$.

(3) *Row comparison.* $s_t^V(z) \geq c_f \chi_t$ whenever $tc_k \leq 1$, with $c_f = e^{-(L_f+1)} p_0^{L_f}$, by Lemma 6.13 along a path of length at most L_f to an orbit maximizing s_t^V, where every vertex is within distance L_f of every orbit.

(4) *Fixing τ.* Choose $0 < \tau < c_f/(4c_\sigma C_{\mathrm{del}}(p_0))$ and the constants of Lemma 6.15 for this τ. Then $p^{2j} \geq c(kR)^{-C_2}$ for $k \geq k_0$ and $R \geq 1$, as in [15, Section 9]. With $\lambda = -\log \rho > 0$ choose K_* with $\lambda K_* > C_2 + 4$, and put $A_\delta = \log(2R) + \log(2+a_3) + \log S$, $k = \lceil K_* A_\delta \rceil$, $B_* = (kR)^{C_2+2}$ and $t_* = B_*/m_0$. Then $k \to \infty$, $k = o(S)$, and $B_* \rho^k = o(1)$, $R^2 a_3 \rho^k = o((kR)^{-C_2})$ and $t_* j c_k = o(1)$, as in [15, Section 9]; the last estimate also gives $t_*(j+1)c_k = o(1)$ and $t_* c_k = o(1)$.

(5) *Retained norm.* Proposition 6.8 and $B_* \rho^k = o(1)$ give $m_t \geq m_0/2$ and $\chi_t \geq m_0/(2c_\sigma)$ on $0 \leq t \leq t_*$, using $m_t \leq c_\sigma \chi_t$ (Lemma 6.6).

(6) *Positive term.* For a raw path selected by Lemma 6.15, with escape endpoint z, Lemma 6.9 with $A' = I_w \cup B_w$ (so $\mathrm{Nb}[A'] \subseteq \mathrm{Nb}[\mathrm{Nb}[I_w]]$), the escape bound, the row comparison and the retained norm give

$$s_t^{D_w}(z) \geq c_f \chi_t - C_{\mathrm{del}} \chi_0^+ \tau / R = c_f \chi_t - C_{\mathrm{del}} \tau m_0 \geq (c_f - 2c_\sigma C_{\mathrm{del}} \tau)\chi_t \geq (c_f/2)\chi_t.$$

Lemma 6.13 along the two escapes gives $s_t^{D_w}(u) s_t^{D_w}(v) \geq e^{-2t(j+1)c_k} p^{2j} (c_f \chi_t/2)^2$, and since such paths have $\mathbb{E}_{\mathrm{path}}$-probability at least b_*, $\mathbb{E}_{\mathrm{path}}[\mathbf{1}\{\ell_w \geq 3\} s_t^{D_w}(u) s_t^{D_w}(v)] \geq c_1 (kR)^{-C_2} \chi_t^2$ on $[0, t_*]$.

(7) *Error and contradiction.* The error in Proposition 6.12 is $C'(\chi_0^+)^2 \rho^k a_3 = C' R^2 m_0^2 \rho^k a_3 \leq 4C' c_\sigma^2 R^2 \rho^k a_3 \chi_t^2 = o((kR)^{-C_2} \chi_t^2)$, and the prefactor e^{-2tc_k} is bounded below on $[0, t_*]$. So $-\chi_t' \geq c_2 (kR)^{-C_2} \chi_t^2$ on $[0, t_*]$. Integrating, $1/\chi_{t_*} \geq c_2 (kR)^{-C_2} t_* = c_2 (kR)^2/m_0$, while the retained norm gives $1/\chi_{t_*} \leq 2c_\sigma/m_0$. These are incompatible as $kR \to \infty$. Hence T_{p_c} is bounded. □

E The vertex-transitive case

For vertex-transitive graphs, Theorem 3.2 combines Lemma 7.1 with the site tilted-threshold separation of Tran and Nguyen. This deduction is not needed for Theorem 1.1, since Theorem 3.2 covers vertex-transitive graphs too.

Proposition E.1 (Tran–Nguyen [20, Proposition 5.5] and Hutchcroft)**.** *Let G be an infinite, connected, locally finite, vertex-transitive graph whose automorphism group is nonunimodular. For Bernoulli site percolation:*

(1) $\|T_p\|_{2\to 2} \leq \chi_{p,\lambda}(v)$ *for every* $p \in [0,1]$ *and* $\lambda \in \mathbb{R}$*;*

(2) $p_c < p_t = p_c(1/2) \leq p_{2\to 2} \leq p_u$*, so in particular* $p_c < p_{2\to 2} \leq p_u$*;*

(3) $\|T_{p_c}\|_{2\to 2} \leq \chi_{p_c,1/2}(v) < \infty$*, and* $\sup_{0 \leq p < p_c} \|T_p\|_{2\to 2} = \|T_{p_c}\|_{2\to 2}$.

Here $\chi_{p,\lambda}(v)$ does not depend on v. The proof identifies the objects of [20, Section 5] with Hutchcroft's for the site kernel: the same modular function, the same normalization in λ, and $p_{st} = p_c(1/2) = p_t$. Neither source states the combined inequality.

The objects of [20, Section 5]. The standing hypothesis of [20, Section 5] is that G is infinite, connected, locally finite and vertex-transitive, with nonunimodular canonical action, and $\Delta = \Delta_{\mathrm{Aut}(G)}$ is taken from [10, Section 2.1]; for transitive Γ this is $\Delta(x,y) = |\operatorname{Stab}_y x|/|\operatorname{Stab}_x y|$. The fixed-root transport [20, (5.30)] is the tilted mass-transport principle [10, Proposition 2.2] with the random root replaced by v, which is legitimate for transitive Γ. The site tilted susceptibility $\chi_{p,\lambda}(v) = \sum_u \mathbb{P}_p(v \leftrightarrow u)\Delta(v,u)^\lambda$ of [20, Auxiliary Lemma 5.A] is Hutchcroft's $X_{p,\lambda}(v)$ for the site kernel $\tau_p = T_p$, with the same exponent convention, and $p_{sc}(\lambda) = \sup\{p : \chi_{p,\lambda}(v) < \infty\}$ is Hutchcroft's $p_c(\lambda)$ for τ_p. The conditional version $\widehat{\chi}_{p,\lambda} = \chi_{p,\lambda}/p$ of [20, (5.13)] has the same finiteness set for $p > 0$. The $u = v$ term is $\mathbb{P}_p(v \leftrightarrow v) = p$ in both. Finally, [20] defines $p_{st} = p_{sc}(1/2)$, while Hutchcroft defines $p_t = \sup\{p : X_{p,\lambda} < \infty \text{ for some } \lambda\}$; for the site kernel these agree. Indeed $p_t \geq p_{st}$ is trivial; conversely, if $\chi_{p,\lambda} < \infty$, then [20, (5.30)] applied to $F(a,b,\omega) = \mathbf{1}\{a \leftrightarrow b \text{ in } \omega\}\Delta(a,b)^\lambda$, which vanishes unless a and b are open, and $\Delta(x,v) = \Delta(v,x)^{-1}$ give $\chi_{p,\lambda}(v) = \chi_{p,1-\lambda}(v)$, and Cauchy–Schwarz against $\mathbb{P}_p(v \leftrightarrow \cdot)$ gives $\chi_{p,1/2} \leq (\chi_{p,\lambda}\chi_{p,1-\lambda})^{1/2} < \infty$, so $p \leq p_{sc}(1/2)$, as in the proof of [20, Proposition 5.5].

The inputs of [20, Proposition 5.5]. Its hypotheses, the estimates (5.23)–(5.26), (5.34), (5.59)–(5.60) and (5.73) there, are each derived inside [20, Section 5] from earlier material of that section and from the external results cited there, including the tilted mass-transport principle [10, Proposition 2.2] and site sharpness [1]. Only (5.26) and (5.34) enter the inequality $p_{sc} < p_{sc}(\lambda) \leq p_{st}$; its proof applies (5.34) with $1 - \varepsilon = 3/4$ and (5.26) with $\eta = 1/2$, and both exponents match. We use [20, Lemma 5.2(2)–(3)] as a cited input; its proofs have not been checked line by line here.

Proof of Proposition E.1. (1) T_p is nonnegative and symmetric, and $\chi_{p,\lambda}(u)$ does not depend on u, by invariance and transitivity; apply Lemma 7.1. (2) The hypothesis of [20, Section 5] is the present one, so [20, Proposition 5.5] gives $p_c < p_{sc}(\lambda) \leq p_{st}$ for $0 < \lambda < 1$; at $\lambda = 1/2$, $p_c < p_{st} = p_t$. Since $\chi_{p,\lambda}(v)$ is nondecreasing in p, every $p < p_{st}$ has $\chi_{p,1/2} < \infty$, hence $\|T_p\| < \infty$ by (1), and $p_{2\to 2} \geq p_t$. The bound $p_{2\to 2} \leq p_u$ is Lemma 2.6. (3) By (2) there is $p' > p_c$ with $\chi_{p',1/2} < \infty$, hence $\chi_{p_c,1/2} < \infty$, and (1) bounds $\|T_{p_c}\|$. The supremum identity is Lemma 2.10. □

F The quasi-transitive form of Tran–Nguyen's argument

This appendix proves Theorem 3.2(1). We run [20, Section 5] in the setting of Section 7.2. Appendix F.1 proves the devices of Section 7.3 whose proofs are deferred, Appendix F.2 lists the changes, and Appendix F.3 proves Proposition 7.10. We write TN(5.∗), TN Lemma 5.2, TN Proposition 5.5 and so on for the equations and numbered statements of [20]; unprefixed numbers refer to this paper. Every statement of [20, Section 5] not mentioned below is used as written; every statement that changes is listed in the order of [20], either as "as written except" or with a replacement proof. The common degree of [20] becomes the maximal degree d; every use of it is an upper bound on neighbour counts, boundary sizes, connector costs or path counts. "Root" means a vertex v with the law $\widehat{\mathbb{P}}^{v,x} = \mathbb{P}_p(\cdot \mid v \text{ open},\ R_v = x)$, and $\widehat{\mathbb{E}}_p^{m,n}(k;a) = \sup_{v,x} \widehat{\mathbb{E}}_p^{v,x}[(X_k^{m,n}(v))^a]$ keeps its definition TN(5.31) as a supremum over all $v \in V$ and $x \in [0,1)$.

F.1 Proofs of the devices

Proof of Lemma 7.3. Let y' be a sloped vertex at minimal distance from y, so $\operatorname{dist}(y,y') \leq R_{\mathrm{sl}}$. On a shortest path from y to y' every vertex before y' is a plateau vertex, by minimality, and a step from a plateau vertex keeps h unchanged by (S3); so the path is level. Step from y' to a strictly lower neighbour, which lowers h by at least δ_0, and repeat from the new vertex. After $\lceil t/\delta_0 \rceil$ rounds,

each of length at most $R_{\mathrm{sl}}+1$, the height has dropped by at least t, and the last step is strict. The reversed statement is identical, with a strictly higher neighbour of a sloped vertex. □

Proof of Lemma 7.4. Apply Lemma 7.3 with $t=(R^*+1)t_0$; this gives a nonincreasing path from y to y' with $h(y')\le h(y)-(R^*+1)t_0$, of length at most $L(t)$. Append a shortest path from y' to O^*, of length at most R^*; along it, $h\le h(y')+R^*t_0\le h(y)-t_0$. The endpoint q satisfies $h(q)\ge h(y')-R^*t_0\ge h(y)-L(t)t_0-R^*t_0$. Take $M_*=L(t)+R^*$ and $H_*=(L(t)+R^*)t_0$, and loop-erase if needed. The case (up) is the mirror image. □

Proof of Lemma 7.7. All three quantities are orbit-invariant, so we may replace v by a vertex v' in its orbit with $\mathrm{dist}(u,v')\le R^*$, and fix a path π from u to v' of length at most R^*. Let $c_\Delta=\min\{\Delta(u,v')^\lambda,1\}\ge e^{-|\lambda|t_0R^*}$. Part (a) is Hutchcroft's inequality $\chi(u)\ge\tau_p(u,v')\Delta^\lambda(u,v')\chi(v')$ [10, proof of Proposition 3.1], with $\tau_p(u,v')\ge p^{R^*+1}$. For (b) and (c), let $f\ge0$ be increasing with $f(0)=0$. On $\{\pi\text{ open}\}$, $K_u=K_{v'}$ and $|K_u|_{u,\lambda}=\Delta(u,v')^\lambda|K_{v'}|_{v',\lambda}$, by the cocycle identity. Under $\widehat{\mathbb{P}}^u$, a product measure on $V\setminus\{u\}$, Harris's inequality gives $\widehat{\mathbb{E}}^u[f(|K_u|_{u,\lambda})]\ge p^{R^*}\widehat{\mathbb{E}}^u[f(\Delta(u,v')^\lambda|K_{v'}|_{v',\lambda})]$, and Harris's inequality for the increasing event $\{u\text{ open}\}$ gives $\widehat{\mathbb{E}}^u[f(\cdot)]\ge\mathbb{E}_p[f(\cdot)]$. Since $f(0)=0$ and $K_{v'}=\varnothing$ when v' is closed, $\mathbb{E}_p[f(\cdot)]=p\,\widehat{\mathbb{E}}^{v'}[f(\cdot)]$. Take $f(s)=s^a$ for (b), where $\Delta^{\lambda a}\ge c_\Delta$; and $f(s)=1-e^{-\theta s}$ for (c), where concavity gives $1-e^{-\theta cs}\ge\min(c,1)(1-e^{-\theta s})$. So $c_5=p^{R^*+1}c_\Delta$ works. □

Proof of Lemma 7.9. We follow [20], with $z^\pm$ replaced as follows. For the first inequality, mark $z\in K_v\cap L_{j-1}(v)$ and take the path of Lemma 7.3 (up) from z with $t=t_0$, truncated at its first visit w to $L_j(v)$ (S5); its sites lie in $L_{j-1,j}(v)$, at heights at most $t_0(j+x)<0=h_v(v)$. For the second, with $z\neq v$, mark $z\in K_v\cap L_{j+1}(v)$ and take the path of Lemma 7.3 (down) from z, truncated at its first visit to $L_j(v)$; its heights are at most $h_v(z)<0$, since v is the unique peak. The case $z=v$ is possible only when $j=-1$, since $v\in L_0(v)$. If v is sloped, first step to a strictly lower neighbour and continue as above. If v is a plateau vertex, P_v forces $K_v=\{v\}$, so $F_j=0$ for $j\le-1$ while $F_0>0$; this is why the case $j=-1$ needs sloped v. The map opens the path and closes its neighbours outside K_v and the path; v stays the unique peak and w is counted by $X_j(v)$. For fixed (v,x) the path depends only on z, at most $(d+1)(L(t_0)+2)$ coordinates change, Lemma 7.2 applies to each marked z, and the map $z\mapsto w$ is at most $d^{L(t_0)+2}$-to-one. The proof of TN(5.67) uses TN(5.66) at $j-1$, j and $j+1$, so it inherits the stated ranges. □

F.2 The change list

Standing hypothesis. Replace it by: G infinite, connected, locally finite and quasi-transitive, and $\Gamma=\mathrm{Aut}(G)$ nonunimodular; Δ is the quasi-transitive modular function (S1).

t_0, $z^\pm$, H_t, A^s_p, TN(5.1)–TN(5.4) and left continuity.

- Delete the choice of neighbours $z^\pm$ with $\log\Delta(z,z^\pm)=\pm t_0$; such neighbours need not exist, for instance at pendant vertices.
- For each vertex v put $H_t(v)=\{z:h_v(z)\ge t\}$, $A^s_{p,v}(t)=\mathbb{P}_p(v\leftrightarrow H_t(v)\text{ inside }H_0(v))$ and $\widehat{A}^s_{p,v}=A^s_{p,v}/p$, and let $a(t)=\inf_v\widehat{A}^s_{p,v}(t)$, a minimum over orbits.
- *Replacement for* TN(5.1)*:*

$$\widehat{A}^s_{p,v}(t+s)\ge p\,\widehat{A}^s_{p,v}(t)\,a(s),\qquad\text{hence}\qquad a(t+s)\ge p\,a(t)a(s)\qquad(s,t\ge0).\tag{F.1}$$

Let $W = H_0(v) \setminus H_t(v)$, explore the open cluster K of v in W, and let E_1 be the event that K has a neighbour in $H_t(v)$. If $v \leftrightarrow H_t(v)$ inside $H_0(v)$, the first vertex of a witnessing path in $H_t(v)$ is preceded by a vertex of K, so E_1 contains this event. On E_1 choose a neighbour $w \in H_t(v)$ of K measurably from the exploration. By Lemma 7.5 the labels of $H_t(v)$ are untested. The event

$$E_2(w) = \{w \text{ open},\ w \leftrightarrow H_{t+s}(v) \text{ inside } H_t(v)\}$$

depends only on them and contains $\{w \leftrightarrow H_s(w) \text{ inside } H_0(w)\}$, because $h_v(w) \geq t$; so conditionally $\mathbb{P}(E_2(w)) \geq A^s_{p,w}(s) \geq pa(s)$. Finally $E_1 \cap E_2(w)$ is contained in $\{v \leftrightarrow H_{t+s}(v) \text{ inside } H_0(v)\}$; divide by p. This replaces the one-layer buffer t_0 of TN(5.1), and no site tested on the first interface is used.

- *Replacement for* TN(5.2)–TN(5.4). With $c_p = -\log p$ and $f(t) = -\log a(t)$, (F.1) gives $f(t+s) \leq f(t) + f(s) + c_p$, so $b(n) = f(nt_0) + c_p$ is subadditive. By Fekete's lemma and monotonicity of a, $\alpha^s_p = \lim_t -t^{-1}\log a(t)$ exists, $\alpha^s_p = \inf_n b(n)/(nt_0)$, and $a(t) \leq p^{-1}e^{t_0\alpha^s_p}e^{-\alpha^s_p t}$. The factor $e^{t_0\alpha^s_p}$ is finite: forcing a path of Lemma 7.3 (up) gives $a(t) \geq p^{L(t)}$ up to a constant, so $\alpha^s_p \leq |\log p| \limsup_t L(t)/t = |\log p|(R_{\mathrm{sl}}+1)/\delta_0 < \infty$.
- *Uniformity in the root.* $\widehat{A}^s_{p,v}(t) \geq p^{M_*+1}\widehat{A}^s_{p,w}(t)$ for all v, w, t: Lemma 7.4 (up) from v, targeting the orbit of w, gives a path π to w' in that orbit with all heights at least $h(v)$ and $h(w') > h(v)$, so $H_0(w') \subseteq H_0(v)$ and $H_t(w') \subseteq H_t(v)$, and Harris's inequality for $\{\pi \text{ open}\}$ with the endpoint change of Lemma 7.7 gives the claim. Hence $\sup_v \widehat{A}^s_{p,v}(t) \leq p^{-M_*-1}a(t) \leq p^{-M_*-2}e^{t_0\alpha^s_p}e^{-\alpha^s_p t}$, which is TN(5.4) uniformly in the root, and $-t^{-1}\log \widehat{A}^s_{p,v}(t) \to \alpha^s_p$ for every v.
- *Left continuity.* As written: $a(t)$ is a minimum of finitely many orbit functions $\widehat{A}^s_{p,O}(t)$, each left-continuous and increasing in p, as in [10, Lemma 5.5], so $a(t)$ is left-continuous and increasing, and $-\log a(t) + c_p$ is left-continuous and decreasing. The normalization remarks TN(5.2)–TN(5.3) are as written with $\widehat{A}^s_p$ replaced by a.

TN Theorem 5.1 and the layer set-up. TN Theorem 5.1, $\alpha^s_{p_c} = 1$, is neither needed nor claimed. TN(5.6) is as written (S5). Delete TN(5.7), the implication $u \in L_k(v) \Rightarrow u^+ \in L_{k+1}(v)$; its uses are replaced below. The conditional laws and TN(5.8)–TN(5.9) are as written; they are site BK and tree bounds and use no transitivity.

$p_c(\lambda)$, TN Auxiliary Lemma 5.A, TN(5.12)–TN(5.13) and p_{st}.

- $p_c(\lambda) = \sup\{p : \chi_{p,\lambda}(v) < \infty\}$ does not depend on v by Lemma 7.7(a), and $\sup_u \chi_{p,\lambda}(u) = \chi^*_{p,\lambda}$ is a maximum over orbits. Positivity is as written.
- $p_c(\lambda) < 1$ for $0 \leq \lambda < 1/2$: for $\lambda = 0$ this is $p_c < 1$, below; for $\lambda > 0$ the paragraph of [20] is as written except that the path and its endpoint γo come from Lemma 7.4 (up) with target the orbit of o, giving $\Delta(o, \gamma o) \geq e^{t_0} > 1$ and a path of length at most M_*; the translates argument, $\Delta(o, \gamma^n o) = \Delta(o, \gamma o)^n$, is as written.
- TN Auxiliary Lemma 5.A: "Transitivity makes the supremum equal to $\chi_{p,\lambda}(v)$" becomes "the supremum is at most $\chi^*_{p,\lambda}$", so $\frac{d}{dp}\chi_{p,\lambda}(v) \leq C(\chi^*_{p,\lambda})^2$; and $Y_{\Lambda_R}(s) \leq \chi_{s,\lambda}(v)$ becomes $Y_{\Lambda_R}(s) \leq \chi^*_{s,\lambda}$. The divergence at the threshold and TN(5.12) are then as written for χ^*, and Lemma 7.7(a) transfers TN(5.12) to every root: $\widehat{\chi}_{q-\varepsilon,\lambda}(v) \geq c\varepsilon^{-1}$ for every v.
- TN(5.13) is as written. The fixed-root symmetry $\widehat{\chi}_{p,\lambda} = \widehat{\chi}_{p,1-\lambda}$ becomes the averaged symmetry $\widehat{\mathbb{E}}_\mu \widehat{\chi}_{p,\lambda}(\rho) = \widehat{\mathbb{E}}_\mu \widehat{\chi}_{p,1-\lambda}(\rho)$, which is Lemma 7.6 with $F(a,b) = \mathbf{1}\{a \leftrightarrow b\}\Delta(a,b)^\lambda$; by Lemma 7.7(a), $p_c(\lambda) = p_c(1-\lambda)$. $p_{st} = p_c(1/2)$ as written.

Magnetization, TN(5.14)–TN(5.26) and TN Auxiliary Lemmas 5.B–5.D. This replaces the averaging; every root-local step is as written.

- With $q = p_c(\lambda)$ and $0 \le \lambda < 1/2$, the temporary hypothesis $\widehat{\mathbb{E}}_q^v[|K_v|_{v,\lambda}^{(1+\eta)/2}] < \infty$ is taken at one root; by Lemma 7.7(b) it holds at every root, with finite supremum.
- TN Auxiliary Lemmas 5.B and 5.C, TN(5.14), the conditioning construction and TN(5.15) are as written at a fixed root. In TN Auxiliary Lemma 5.D, delete the sentence asserting that transitivity makes the last moment independent of z; $M_a = \sup_z \widehat{\mathbb{E}}^z[|K_z|_{z,\lambda}^a]$ is finite by Lemma 7.7(b), and TN(5.16) is as written, with M_Δ a maximum over finitely many edge orbits. TN(5.17) is as written at a fixed root.
- TN(5.17)→TN(5.18): take $\widehat{\mathbb{E}}_\mu$, that is, set $v = \rho \sim \mu$, before splitting according to $\Delta(\rho, u) \le 1$ or > 1, and apply Lemma 7.6 in place of TN(5.30), after integrating out the ghosts, to

$$F(a,b) = \mathbf{1}\{a \leftrightarrow b,\ \Delta(a,b) > 1\} e^{-\theta |K_a|_{a,\lambda}} \Delta(a,b)^{\lambda(1+\delta)}.$$

  The transported sum is

$$\widehat{\mathbb{E}}_\mu \sum_{u \in K_\rho} \mathbf{1}\{\Delta(\rho,u) < 1\} e^{-\theta |K_u|_{u,\lambda}} \Delta(\rho,u)^{1-\lambda(1+\delta)},$$

  which is TN(5.18) at ρ; the following estimates are pointwise.
- TN(5.19), averaged: $\widehat{\mathbb{E}}_\mu \widehat{\mathbb{P}}^\rho(G_{\rho,\theta}(K_\rho) \ge 2,\ D_\rho^c) \le C(p/(1-p))\theta^{1+\delta}\chi_\theta$ with $\chi_\theta = \widehat{\mathbb{E}}_\mu \widehat{\chi}_{p,\lambda,\theta}(\rho)$. TN(5.20)–TN(5.21) are root-local; with $M_\theta = \widehat{\mathbb{E}}_\mu \widehat{M}_{p,\lambda,\theta}(\rho)$, $\widehat{\mathbb{E}}_\mu \widehat{\mathbb{P}}^\rho(T_\rho = 1) = \theta \partial_\theta M_\theta$ and $\widehat{\mathbb{E}}_\mu \widehat{\mathbb{P}}^\rho(D_\rho) \le (\max_v \widehat{M}_\theta(v))^2 \le c_5^{-2} M_\theta^2$ by Lemma 7.7(c), uniformly in $\theta > 0$.
- TN(5.22), averaged: $M_\theta \le C M_\theta^2 + \theta \partial_\theta M_\theta + C(p/(1-p))\theta^{1+\delta}\partial_\theta M_\theta$, the averaged form of [10, Lemma 4.3] with the site constants.
- TN(5.23)–TN(5.25) are as written with $\varphi(\theta) = M_\theta$ at $p = q$: $\varphi'(0+) = \infty$ by TN(5.12) at every root; $\psi(t) \le -\log(1-t)$ since $X \ge 1$ at every root; and TN(5.25) uses the averaged moment, finite by Lemma 7.7(b).
- TN(5.26): the induction and contradiction are as written, and give $\widehat{\mathbb{E}}_\mu \widehat{\mathbb{E}}^\rho_{p_c(\lambda)}[|K_\rho|_{\rho,\lambda}^{(1+\eta)/2}] = \infty$. Every orbit has positive weight, so by Lemma 7.7(b),

$$\widehat{\mathbb{E}}^v_{p_c(\lambda)}[|K_v|_{v,\lambda}^{(1+\eta)/2}] = \infty \qquad \text{for every } v,\ 0 \le \lambda < 1/2,\ \eta > 0. \tag{F.2}$$

TN Auxiliary Lemmas 5.E and 5.F, $p_c < 1$ and the root-peak denominator. TN Auxiliary Lemma 5.E and the Reimer remarks are general and as written; TN(5.27) is as written; delete TN(5.28), whose role is taken by the surgery in the subcritical-peak item below. TN Auxiliary Lemma 5.F is replaced by the site theorem for nonunimodular groups [16, Theorem 1.2]: if $\mathrm{Aut}(G)$ is nonunimodular and $p_c < 1$, critical site percolation has almost surely no infinite cluster; so at every $0 < p \le p_c$ every open cluster is finite almost surely. The bound $p_c < 1$ is as written with root o: a nonunimodular Γ forces nonamenability (Section 7.2), and the count of connected sets needs only bounded degree. In the root-peak denominator, $(1-p)^d$ becomes $(1-p)^{\deg(v)} \ge (1-p)^d$.

TN(5.30) and the transports. TN(5.30) is replaced by Lemma 7.6. Every transport of [20] is used only through Lemma 7.6 with $\rho \sim \mu$: the susceptibility symmetry, the large-Δ ghost half, the up/down identity TN(5.49), the peak-to-layer transports TN(5.68) and TN(5.113), and the Hölder transport TN(5.92).

TN(5.31)–TN(5.33), TN Lemma 5.2, the kernels and the first moments.

- TN(5.31)–TN(5.33) are as written, with suprema over all $v \in V$ and $x \in [0,1)$.
- In [20, Lemma 5.2], replace the hypothesis as above; clauses 1–3 are claimed and clause 4 is dropped. The uniformity of clause 1 in the root and the offset holds for the supremum-type bounds, the first-moment and probability estimates TN(5.41)–TN(5.62) and TN(5.65), and TN(5.34). It does not hold for the peak lower bounds TN(5.69), TN(5.70), TN(5.72), TN(5.86), TN(5.90) and TN(5.91), which hold at every sloped vertex, uniformly in the offset and over the finitely many sloped orbits; at a plateau vertex P_v forces $K_v = \{v\}$. Theorem 3.2 never uses root-uniformity of the peak lower bounds.
- The cluster kernel TN(5.36)–TN(5.38), the two-level boundary kernel and the first-hit residual kernel are as written; they hold on any countable graph.
- TN(5.39), replaced: $\widehat{\mathbb{P}}_p(m+n) \geq p\,\widehat{\mathbb{P}}_p(m)\widehat{\mathbb{P}}_p(n)$ for $m, n \geq 0$. Explore the open cluster of v in $L_{0,m-1}(v)$; by Lemma 7.5, a connection to $L_{m,\infty}(v)$ passes through an untested neighbour $w \in L_m(v)$ of that cluster; $L_{0,\infty}(w) = L_{m,\infty}(v)$ by the shift rule, with untested labels; continue from w as in [20]. Then TN(5.40) has $a_p(n) + c_p$ subadditive without the index shift, and the limit identification is as written.
- TN(5.41), replaced. The first inequality, $a(t_0(n+1)) \leq \widehat{\mathbb{P}}^{v,x}(v \leftrightarrow L_n(v) \text{ in } L_{0,\infty}(v))$, is the inclusion argument of [20] at v with $\widehat{A}^s_{p,v} \geq a$. The second, $\widehat{\mathbb{P}}^{v,x}(v \leftrightarrow L_n(v) \text{ in } L_{0,\infty}(v)) \leq p^{-L(t_0)} \sup_w \widehat{A}^s_{p,w}(t_0 n)$: replace v^- by the endpoint v' of a path π of Lemma 7.3 (down) from v with $t = t_0$; relative to v', $L_{0,\infty}(v)$ lies at raw height at least $h_{v'}(v) - t_0 \geq 0$, π lies at heights at least 0, and $L_n(v)$ lies above raw height $t_0 n$; the Harris and endpoint-change step is as written with the event $\{\pi \text{ open}\}$ and the factor $p^{|\pi|}$. Both inequalities are uniform in (v, x, n) by the uniformity in the root above.
- TN(5.42), replaced. Let u be the endpoint of a path of Lemma 7.3 (down) from v with $t = (r+1)t_0$, of length at most $L((r+1)t_0) \leq C_L(r+1)$. Then $v \in L_{r+1+i}(u)$ for some $i \in [0, \lceil L((r+1)t_0)\rceil]$, the shift rule gives $L_{-r,\infty}(v) = L_{1+i,\infty}(u) \subseteq L_{1,\infty}(u)$ and $L_n(v) = L_{n+r+1+i}(u)$, and the reversed path lies in $L_{0,r+1+i}(u)$. The rest is as written with p^{r+1} replaced by $p^{|\text{path}|}$, and gives $\sup_{v,x} \widehat{\mathbb{P}}^{v,x}(v \leftrightarrow L_n(v) \text{ in } L_{-r,\infty}(v)) \leq C_p e^{C_0(r+1)} e^{-t_0 \alpha^s_p n}$ with $C_0 = C_L |\log p|$.
- The path decompositions TN(5.43)–TN(5.45) and the submultiplicativity of U_p and D_p are as written.
- TN(5.46), replaced: $\sup_{v,y} \widehat{\mathbb{E}}^{v,y} X^{-\infty,k+1}_{k+1}(v) \leq B_p \inf_{w,x} \widehat{\mathbb{E}}^{w,x} X^{-\infty,k}_k(w)$. The inclusion of [20] and TN(5.43) give, at each fixed v, $\sup_y \widehat{\mathbb{E}}^{v,y} X_{k+1}(v) \leq B^{(0)}_p \inf_{y'} \widehat{\mathbb{E}}^{v,y'} X_k(v)$; the proof in [20] uses transitivity to pass between roots. Across orbits, fix (v, y) and (w, x). Lemma 7.4 (up) from w, targeting the orbit of v, gives v' in that orbit with $v' \in L_j(w)$, $j \in [1, J_2]$, $J_2 = \lceil H_*/t_0 \rceil + 1$, by a path with heights in $[h(w), h(w)+H_*]$, hence inside $L_{0,J_2}(w)$. With $R' = R_{v'}$ induced by $R_w = x$, orbit invariance and the per-vertex inequality give $\widehat{\mathbb{E}}^{v,y} X_{k+1}(v) \leq B^{(0)}_p \widehat{\mathbb{E}}^{v',R'} X_k(v')$; first-visit submultiplicativity, the first line of TN(5.43), gives $\widehat{\mathbb{E}}^{v',R'} X_k(v') \leq U_p(j) \widehat{\mathbb{E}}^{v',R'} X_{k-j}(v')$; and for $k \geq J_2$ Harris's inequality for the path and the endpoint change give $\widehat{\mathbb{E}}^{w,x} X^{-\infty,k}_k(w) \geq p^{M_*+1} \widehat{\mathbb{E}}^{v',R'} X^{-\infty,k-j}_{k-j}(v')$, because $L_{-\infty,k-j}(v') = L_{-\infty,k}(w)$, $L_{k-j}(v') = L_k(w)$ and the path lies in $L_{0,J_2}(w) \subseteq L_{-\infty,k}(w)$. Chaining gives $B_p = B^{(0)}_p \max_{j \leq J_2} U_p(j) p^{-M_*-1}$; the finitely many $k < J_2$ have positive finite values and are absorbed.

- TN(5.47), replaced: with $D_y^v(k) = \widehat{\mathbb{E}}^{v,y} X_{-k}^{-\infty,0}(v)$,

$$\sup_{v,y} D_y^v(k+1) \le B'_p \inf_{w',x'} D_{x'}^{w'}(k) \qquad (k \ge 0). \tag{F.3}$$

Offsets are treated as in [20]: layer sets and counts may be evaluated with any deterministic offset at any vertex, since the configuration is independent of the offset field. Put $G_1 = L(t_0)$ and $J = G_1 + 3$.

  - *Step A (upward connector, in place of $w = v^+$).* Fix (v, y), let π be the path of Lemma 7.3 (up) from v with $t = t_0$, and let w be its endpoint; π is nondecreasing, $|\pi| \le G_1$ and $g := h_v(w) \in [t_0, G_1 t_0]$. For every offset x at w: $L_{-\infty,0}(v; y) \subseteq L_{-\infty,0}(w; x)$, since the upper boundaries are $t_0 y < t_0 \le g + t_0 x$ in raw height from v; $\pi \subseteq L_{-\infty,0}(w; x)$, since heights along π are at most $h(w)$; and $L_{-k-1}(v; y) \subseteq \bigcup_{j=1}^{J} L_{-k-j}(w; x)$, since for z on the left $h_w(z)/t_0 = h_v(z)/t_0 - g/t_0 \in (-k-3-G_1, -k-2+y] \subseteq (-k-3-G_1, -k-1+x)$. For u counted on the left, with connection event E_u in $L_{-\infty,0}(v; y)$, Harris's inequality gives $p^{|\pi|}\widehat{\mathbb{P}}^{v,y}(E_u) \le \widehat{\mathbb{P}}^{v,y}(E_u \cap \{\pi \setminus \{v\} \text{ open}\})$; the right side forces v and w open, so it equals $\widehat{\mathbb{P}}^{w,x}(E_u \cap \{\pi \setminus \{w\} \text{ open}\})$, the endpoint change of [20]; and on that event u is joined to w inside $L_{-\infty,0}(w; x)$ with $u \in L_{-k-j}(w; x)$ for some $j \in [1, J]$. Summing over u and applying the last-visit decomposition at $L_{-k}(w; x)$, the second line of TN(5.44),

$$D_y^v(k+1) \le p^{-G_1} \Big(\sum_{j=1}^{J} D_p(j) \Big) D_x^w(k) \qquad \text{for every } x \in [0, 1). \tag{F.4}$$

  - *Step B (cross-orbit step, in place of the transitivity that makes $D_x^w(k)$ independent of w).* Fix (w', x'). Lemma 7.4 (down) from w', targeting the orbit of w, gives a path σ to w'' in that orbit with $|\sigma| \le M_*$, heights along σ at most $h(w')$, and $h(w'') \in [h(w') - H_*, h(w') - t_0]$. With R'' the offset at w'' induced by x', $w'' \in L_{-j'}(w'; x')$ for some $j' \in [1, J_2]$, and $L_i(w''; R'') = L_{i-j'}(w'; x')$. Also $\sigma \subseteq L_{-\infty,0}(w'; x')$, and on $\{\sigma \text{ open}\}$ every vertex of $L_{-k}(w'; x') = L_{-k+j'}(w''; R'')$ joined to w'' inside $L_{-\infty,j'}(w''; R'') = L_{-\infty,0}(w'; x')$ is counted by $X_{-k}^{-\infty,0}(w')$. Domain monotonicity, Harris's inequality and the endpoint change give, for $k \ge J_2$, $D_{x'}^{w'}(k) \ge p^{M_*+1} D_{R''}^{w''}(k - j')$, and the last-visit decomposition gives $D_{R''}^{w''}(k) \le D_{R''}^{w''}(k - j') D_p(j')$. Hence $D_{R''}^{w''}(k) \le p^{-M_*-1} D_p(j') D_{x'}^{w'}(k)$ for $k \ge J_2$.
  - *Step C (combination).* As w'' lies in the orbit of w and the joint law of the configuration and the equivariant offsets is Γ-invariant, $D_x^w(k) = D_x^{w''}(k)$ for every x. Apply (F.4) with $x = R''$, then Step B: for $k \ge J_2$, $D_y^v(k+1) \le B'_p D_{x'}^{w'}(k)$ with $B'_p = p^{-G_1 - M_* - 1} (\max_{j' \le J_2} D_p(j')) \sum_{j=1}^{J} D_p(j)$, finite for $p \in I_h^s$.
  - *Small k.* For $0 \le k < J_2$, $\sup D(k+1) \le D_p(k+1) < \infty$, and $\inf_{w',x'} D_{x'}^{w'}(k) \ge p^{L((k+2)t_0)} > 0$ by forcing a path of Lemma 7.3 (down) from w' of depth $(k+2)t_0$, truncated at its first visit to $L_{-k}(w')$. These cases are absorbed into B'_p.

  Every downstream use of TN(5.47) needs the full supremum-infimum form (F.3): the chain after TN(5.47), the pointwise lower bound in the second line of TN(5.51), the pointwise lower bounds for $k < 0$ in TN(5.54), and the summation for TN(5.60) at every root.

- TN(5.48)–TN(5.49) and the sentence that the quantities do not depend on v, replaced. Define $\bar{U}_p(n) = \widehat{\mathbb{E}}_\mu \int_0^1 \widehat{\mathbb{E}}^{\rho,x} X_n^{-\infty,n}(\rho)\, dx$ and $\bar{D}_p(n)$ likewise. Lemma 7.6 applied to the transport of [20] gives $\bar{U}_p(n) \asymp e^{-t_0 n} \bar{D}_p(n)$, which is TN(5.49) with $\rho \sim \mu$. Then $\inf_{v,x} \le$ average $\le \sup_{v,x}$, and TN(5.46)–(F.3) put every fixed-(v, x) moment within constants and a one-layer shift of the averages.

- One-layer comparison $U_p(k) \le C_U U_p(k+1)$, replaced: mark z, take the path of Lemma 7.3 (up) from z with $t = t_0$, truncated at its first visit w to $L_{k+1}(v)$, and force it open; the image is counted by $X_{k+1}^{-\infty,k+1}(v)$. Lemma 7.2 is applied per mark, and $z \mapsto w$ is at most $d^{L(t_0)+1}$-to-one.
- One-layer comparison for D_p, replaced; [20] obtains it from the maximal-decrement map and TN(5.47). For every (v,x) and $k \ge 0$, $D_x^v(k) \le C_{Dp} D_x^v(k+1)$ with $C_{Dp} = d^{L(t_0)+1} \min(p, 1-p)^{-(L(t_0)+1)}$, hence $D_p(k) \le C_{Dp} D_p(k+1)$. Mark $z \in L_{-k}(v)$ joined to v inside $L_{-\infty,0}(v)$, and let π_z be the path of Lemma 7.3 (down) from z with $t = t_0$, truncated at its first visit w to $L_{-k-1}(v)$; it is nonincreasing, so $\pi_z \subseteq L_{-k-1,-k}(v) \subseteq L_{-\infty,0}(v)$ for $k \ge 0$. For $k \ge 1$ its heights are negative, so $v \notin \pi_z \setminus \{z\}$; for $k = 0$ it may start at $z = v$, which is open. The map opens π_z, changes at most $L(t_0)+1$ coordinates, does not change the root's state, and maps $\{z$ joined to v in $L_{-\infty,0}(v)\}$ into $\{w$ joined to v in $L_{-\infty,0}(v)\}$. Lemma 7.2 is applied per mark, and $z \mapsto w$ is at most $d^{L(t_0)+1}$-to-one since $\mathrm{dist}(z,w) \le L(t_0)$. The other direction, $D_p(k+1) \le D_p(k) D_p(1)$, is last-visit submultiplicativity, as written.
- TN(5.50), rechecked. First chain: $D^{\sup}(k) \le B_p' \bar{D}(k-1)$ by (F.3), then $\le C_p e^{t_0(k-1)} \bar{U}(k-1)$ by the averaged TN(5.49), then $\le C_p e^{t_0(k-1)} U^{\sup}(k-1) \le C_p C_U^2 e^{t_0 k} U^{\sup}(k+1)$ by the U comparison twice. Reverse chain: $U^{\sup}(k) \le B_p \bar{U}(k-1)$ by TN(5.46), then $\le C_p e^{-t_0(k-1)} \bar{D}(k-1)$, then $\le C_p e^{-t_0(k-1)} D^{\sup}(k-1) \le C_p C_{Dp}^2 e^{-t_0 k} D^{\sup}(k+1)$ by the D comparison twice. Both lines of TN(5.50) hold with a new C_p, and the deductions after TN(5.50) are as written.
- TN(5.50)–TN(5.62) are as written; they manipulate suprema and per-(v,x) inequalities. At TN(5.56)–TN(5.58) the comparison becomes $\sup_w \widehat{A}_{p,w}^s(t_0 k) \ge p^{L(t_0)} \sup_{v,x} \widehat{\mathbb{P}}^{v,x}(\cdots)$, by the second inequality of TN(5.41). The lower inequality of TN(5.41) at fixed (v,x) gives $\widehat{\mathbb{P}}^{v,x}(\cdots) \ge a(t_0(k+1)) \ge p^{M_*+1} \sup_w \widehat{A}_{p,w}^s(t_0(k+1))$ by the uniformity in the root, so the lower bound in TN(5.59) holds at every (v,x), and TN(5.59)–TN(5.60) follow as written. Where [20] uses $\widehat{\chi}_{p,1/2} < \infty$ to place p in I_h^s, it needs $\sup_v \widehat{\chi}_{p,1/2}(v) < \infty$, a maximum over orbits finite by Lemma 7.7(a); the lower bounds of TN(5.51) used there hold at every (v,x) by TN(5.46)–(F.3). Site sharpness on quasi-transitive graphs [1] identifies $\widehat{\chi}_{p,0}(v) < \infty$ with $p < p_c$ at every root.

TN(5.73). As written, except that the tilted symmetry $\widehat{\chi}_{p,1} = \widehat{\chi}_{p,0}$ becomes the averaged symmetry together with Lemma 7.7(a): $\widehat{\chi}_{p,1}(v) < \infty$ for every v when $p < p_c$. Log-convexity in the tilt is pointwise. So $\alpha_{p_c}^s \ge 1$, with α^s as above.

Subcritical peaks, TN(5.63)–TN(5.72).

- TN(5.63)–TN(5.65), replaced by a first-crossing surgery. Fix $p < p_c$ and (v,x). Explore the open cluster K of v in $W = L_{-\infty,k}(v)$, which is finite almost surely; let $N = |K \cap L_k(v)|$ and $H_k' = \{K$ has a neighbour in $L_{k+1}(v)\}$. By Lemma 7.5(2) and the first-entry argument, $\{v \leftrightarrow L_{k+1}(v)\} \subseteq H_k' \subseteq \{v \leftrightarrow L_k(v)\}$, so by TN(5.59), $\widehat{\mathbb{P}}^{v,x}(H_k') \asymp_p e^{-t_0 \beta_p^s k}$ uniformly in (v,x). Since $N \le X_k^{-\infty,k}(v)$, TN(5.51) gives $\widehat{\mathbb{E}}^{v,x} N \le C_p e^{-t_0 \beta_p^s k}$, hence $\widehat{\mathbb{E}}^{v,x}[N \mid H_k'] \le C_p'$; this replaces TN(5.63). On H_k' choose a neighbour $w \in L_{k+1}(v)$ of K measurably; by Lemma 7.5, w and all neighbours of K outside W are untested, at most dN of them, all in $L_{k+1}(v)$. Prescribe w open, and prescribe closed every other untested neighbour of K and every untested neighbour of w; a tested neighbour of w is in K, if tested open, or closed. So at most $d(N+1)$ sites are prescribed closed, and by TN(5.36) the conditional probability is at least $p(1-p)^{d(N+1)}$. On the prescribed event the full cluster is $K \cup \{w\}$, and since $h_v(w) > t_0(k + R_v) \ge \max_W h_v$, w is its unique peak. TN(5.64) is as written with Z_k replaced by $K \cap L_k$: conditional Jensen

gives $c'_p > 0$ and $\widehat{\mathbb{P}}^{v,x}(\mathrm{peak}(K_v) \in L_{k+1}(v)) \geq c'_p \widehat{\mathbb{P}}^{v,x}(H'_k) \geq ce^{-t_0\beta_p^s k}$. The upper bound is as written, which gives TN(5.65) uniformly in (v, x) at every vertex.

- TN Auxiliary Lemma 5.G and TN(5.67) are replaced by Lemma 7.9.
- TN(5.68), replaced. Lemma 7.6 applied to $T_k(a, b) = \mathbf{1}\{P_a,\ b \in K_a \cap L_{-(k+1)}(a)\}$ gives $p\,\widehat{\mathbb{E}}_\mu \int_0^1 F_{-(k+1)}(\rho, y)\,dy = \widehat{\mathbb{E}}_\mu \mathbb{E} \sum_a \mathbf{1}\{a = \mathrm{peak}(K_\rho),\ a \in L_{k+1}(\rho)\}\Delta(\rho, a)$, with weight $\Delta(\rho, a) \asymp e^{t_0 k}$, and TN(5.65) at every vertex gives that the average $\sum_O \mu(O) \int F_{-(k+1)}(O, y)\,dy$ is $\asymp_p e^{-t_0(\beta_p^s - 1)k}$.
- TN(5.69), replaced. The upper bound holds at every v, with $\int F \leq \mu_{\min}^{-1}$ times the average. At a sloped v_0, Lemma 7.8 and Lemma 7.9 iterated J_6 times give $\int F_{-(k+1)}(v_0, y)\,dy \geq c \max_O \int F_{-(k+1)}(O, y)\,dy \geq ce^{-t_0(\beta-1)k}$, and TN(5.67) at v_0 replaces the offset average by any fixed x; the case $k = 0$ is as written, using $(1-p)^d \leq F_0 \leq \widehat{\chi}_{p,0}$. So TN(5.69) holds at every sloped vertex, uniformly in the offset and over the finitely many sloped orbits:

$$\widehat{\mathbb{E}}^{v_0,x}[X_{-k}^{-\infty,\infty};\ P_{v_0}] \asymp_p e^{-t_0(\beta_p^s - 1)k}.$$

- TN(5.70)–TN(5.72) are as written at a sloped v_0, and TN(5.61) holds at every vertex; so TN(5.72) holds at every sloped vertex, uniformly in x, k, ℓ. This is the site form of [10, Lemma 5.17], which asserts the existence of a suitable vertex.

Reimer, the surgery map, TN(5.76) and critical peak survival, TN(5.74)–TN(5.91).

- TN(5.74)–TN(5.75) are as written, with $M = (d+1)s$.
- TN(5.76)–TN(5.83) are replaced by Proposition 7.10, proved in Section F.3.
- Denominator positivity is needed only at a sloped v_0: a path of Lemma 7.3 (down) from v_0 with first step strict, as for σ in Lemma 7.8, of depth at least $(\ell+1)t_0$, together with its closed exterior, gives $\widehat{\mathbb{P}}(N_\ell \cap P_{v_0}(S)) \geq p^L(1-p)^{d(L+1)} > 0$.
- TN(5.84) is as written, except that the uniformity in the root becomes a maximum over the finitely many orbits of sequences tending to zero; each tends to zero since clusters at $p \leq p_c$ are finite, hence of bounded height. The one-layer offset enlargement is as written.
- TN(5.85)–TN(5.91) are as written at a sloped v_0: TN(5.86) is TN(5.72) at v_0 at level p_-; TN(5.87)–TN(5.89) are conditional on A_k^- at v_0; and the residual-kernel bound uses TN(5.84), a supremum over all roots. The conclusion is (7.2).

TN Proposition 5.3. Every offset-averaged quantity in TN(5.93), TN(5.94)–TN(5.95) and TN(5.106) becomes the average over $\rho \sim \mu$ and the offset, $\widehat{\mathbb{E}}_\mu \int dx\, \widehat{\mathbb{E}}^{\rho,x}$, and "the root is fixed and the offset is averaged" becomes "the root $\rho \sim \mu$ and the offset are averaged". TN(5.92)–TN(5.93) are as written with Lemma 7.6; the Hölder step TN(5.94)–TN(5.95) is as written under the joint average; and the up/down recursions TN(5.96)–TN(5.105) are as written, per fixed (v, x), with each entry's cluster bounded by the supremum over all roots.

TN Proposition 5.4.

- TN(5.107), replaced: $\widehat{\mathbb{E}}_p^{m,n}(k; a) \leq \mu_{\min}^{-1} \sum_{j=-1}^{1} \widehat{\mathbb{E}}_\mu \int dy\, \widehat{\mathbb{E}}^{\rho,y}[(X_{k+j}^{m-1,n+1}(\rho))^a]$, by the per-vertex offset inclusion of [20] and the last assertion of Lemma 7.6. TN(5.108) is as written with C_p replaced by $C_p\mu_{\min}^{-1}$.

- TN(5.109), replaced: $\widehat{\mathbb{E}}_p^{m+r,n+r}(k+r;a) \le C_{sh,0}e^{C_{sh}|r|}\widehat{\mathbb{E}}_p^{m,n}(k;a)$. Take γ to be the path of Lemma 7.3 from v (up if $r>0$, down if $r<0$) with $t=|r|t_0$, truncated at its first visit to $L_r(v)$; its length is at most $L(|r|t_0) \le (|r|t_0/\delta_0+2)(R_{\mathrm{sl}}+1)$, and it lies in $L_{0,r}(v)$, respectively $L_{r,0}(v)$, hence in $L_{m+r,n+r}(v)$ under the stated hypotheses. The Harris and endpoint-change argument TN(5.110)–TN(5.111) is as written with $p^{|r|}$ replaced by $p^{|\gamma|}$; here $C_{sh} = |\log p|(R_{\mathrm{sl}}+1)t_0/\delta_0$ and $C_{sh,0} = p^{-2(R_{\mathrm{sl}}+1)}$.
- TN(5.112) is proved in Section F.3, through Proposition 7.10 and TN(5.113)–TN(5.117) at a sloped v_0.
- TN(5.118)–TN(5.125) are as written, with the constant prefactors $C_{sh,0}$ and $\mu_{\min}^{-1}$, which enter finitely many times, and the choices of δ in TN(5.119) and TN(5.122) with the new C_{sh}. The order of choice of constants, after ε and ζ and before v,x,k, is unaffected, since every new constant depends only on (G,p). So TN(5.34) holds as stated, uniformly in the root and offset.

TN Proposition 5.5. The statement is as written with the hypotheses (F.2), TN(5.34), TN(5.59)–TN(5.60) and TN(5.73), all proved above. In the proof, TN(5.126) is (F.2) at a fixed root v, and TN(5.127) is as written at that v, using TN(5.34), which is a supremum over roots; this gives $p_c < p_c(\lambda)$ for $0<\lambda<1/2$. The fixed-root symmetry is replaced by the averaged symmetry and Lemma 7.7(a); Cauchy–Schwarz is pointwise; and the strict lower bound at $\lambda=1/2$ is as written. The conclusion is $p_c < p_c(\lambda) \le p_{st} = p_c(1/2)$ for $0<\lambda<1$. The final paragraph of the proof in [20], which proves $\alpha_{p_c}^s = \beta_{p_c}^s = 1$, is not needed.

F.3 Proof of Proposition 7.10

Fix p,k,ℓ,r and (v,x), and write J for the residual coordinates. Let O^* be an orbit attaining $\max_O \sup_y \widehat{\mathbb{E}}^{O,y}[X_{-k}^{-k-r,0}(O);\ Q_{O;k,\ell+1,r+a_*}]$; the maximum exists because there are finitely many orbits, the quantity depends only on the orbit by Γ-invariance of the joint law of the configuration and the equivariant offsets, and the maximum equals the left side of the proposition.

Step 1: exploration, as in [20] except for the retained event. Expose the open cluster A of v in $L_{-\ell,\infty}(v)$ with its tested external boundary Θ in that graph. Retain only outcomes (A,Θ) on which v is the peak of A and A has a neighbour in $L_{-\ell-1}(v)$, that is, on which $N_\ell(v)$ holds; call them admissible. Every site of $L_{-\infty,-\ell-1}(v)$ is untested (Lemma 7.5). Select by a fixed enumeration an edge $\{u,q_1\}$ with $u\in A$ and $q_1 \in L_{-\ell-1}(v)$; since an edge changes height by at most t_0, $u \in L_{-\ell}(v)$.

Step 2: connector, in place of the maximal-decrement neighbours of TN(5.77). Lemma 7.4 (down) with target O^* gives a path $\gamma' = (q_1,\dots,q_m=q)$ with $m\le M_*$ and $q\in O^*$, along which $h(q_j)\le h(q_1)$, with $h(q_1)-H_* \le h(q) \le h(q_1)-t_0$. Choose γ' deterministically from (q_1,O^*) and loop-erase it. Every site of γ' has height at most $h(q_1)$, so $\gamma' \subset L_{-\infty,-\ell-1}(v)$, and γ' is untested and disjoint from $A\cup\Theta$. Since $h(q)\le h(q_1)-t_0$ and layers have thickness t_0, $q\in L_{-\ell-a}(v)$ for some integer $a\in[2,a_*]$ with $a_* := 2+\lceil H_*/t_0\rceil$, and $L_j(q) = L_{j-\ell-a}(v)$ for every j. Put $\gamma = (u,q_1,\dots,q)$; its non-root sites lie strictly below the exploration domain, which is the hypothesis of the map TN(5.75).

Step 3: the residual events, as in [20] with the new indices. Put $\Theta^* = L_{-S,\infty}(v)\setminus(A\cup\Theta)$. For $w\in L_{-k-1,-k+1}(q)$, let $E_{q,w}^{A,\Theta}$ be the event that q is open, that q is joined to w in $L_{-S,-\ell-1}(v)\setminus(A\cup\Theta)$, and that the open cluster C_q of q in Θ^* does not meet $H^+(v) := \{z : h_v(z)\ge 0\}\setminus\{v\}$. The event $B^{A,\Theta}$ is defined as in [20]: an open path leaves A through an untested site below $L_{-\ell}(v)$, uses no site of Θ, and returns to $H^+(v)$ inside Θ^*. The slab used for w is $L_{-k-1,-k+1}(q)$, and the q–w path domain is $L_{-k-r-1,1}(q) = L_{-k-r-1-\ell-a,1-\ell-a}(v)$; since $a\ge 2$ its top index is at most $-\ell-1$, and

since $a \le a_*$ its bottom index is at least $-S$. So the domain lies in $L_{-S,-\ell-1}(v)$, strictly below the explored region.

Step 4: the Reimer step, TN(5.78)–TN(5.79), as written. TN Auxiliary Lemma 5.E and its finite exhaustion use no transitivity; the disjoint-witness argument uses only that a B-path meeting C_q would join q to $H^+(v)$ inside Θ^*, and that $\partial_{\Theta^*} C_q$ is closed. All clusters are finite at $p \le p_c$, by definition for $p < p_c$ and by [16, Theorem 1.2] at $p = p_c$. Hence $\widehat{\mathbb{P}}(E^{A,\Theta}_{q,w} \cap B^{A,\Theta} \mid A, \Theta) \le \widehat{\mathbb{P}}(E^{A,\Theta}_{q,w} \mid A, \Theta)\widehat{\mathbb{P}}(B^{A,\Theta} \mid A, \Theta)$, and TN(5.80) gives $\mathbb{P}(R \mid B^c) \ge \mathbb{P}(R)$ for $R = E^{A,\Theta}_{q,w}$.

Step 5: identification of B^c, as written. Given an admissible (A, Θ), the event $P_v(S)$ is exactly B^c: the set joined to v inside $L_{-S,\infty}(v)$ is A together with whatever is reached by leaving A; every neighbour of A inside $L_{-\ell,\infty}(v)$ is in Θ and closed, so leaving A means entering an untested site below $L_{-\ell}(v)$; hence v stays the peak if and only if no such excursion reaches $H^+(v)$.

Step 6: the surgery, TN(5.75) with γ from Step 2. The map Φ_γ opens $q_1, \dots, q_{m-1}$ (q is already open on R), closes every site adjacent to $\gamma \setminus \{u\}$ outside $A \cup \Theta \cup C_q \cup \gamma$, and changes nothing else. Only coordinates in a set $S_\gamma \subset J$ with $|S_\gamma| \le (d+1)M_*$ change, so $\widehat{\mathbb{P}}(E \mid A, \Theta) \le 2^{(d+1)M_*}[p(1-p)]^{-(d+1)M_*}\widehat{\mathbb{P}}(\Phi_\gamma(E) \mid A, \Theta)$. Preservation is as in [20], with $\gamma \setminus \{u\} \subset L_{-\infty,-\ell-1}(v)$ disjoint from $H^+(v)$: opening γ only attaches A to $C_q \cup (\gamma \setminus A)$, and $C_q \cap H^+(v) = \varnothing$ on R; every other new exit from γ is closed; and a B-path in the image avoiding γ was a B-path before, because the map only closes sites off γ. So Φ_γ maps $R \cap B^c$ into $B^c \cap \{u \text{ joined to } w \text{ through } \gamma\}$. On the image, $A \cup \gamma \cup (\text{the } q\text{–}w \text{ path})$ is an open connected set in $L_{-S,0}(v)$, since $A \subset L_{-\ell,0}(v)$ (v is its peak) and γ and the q–w path lie in $L_{-S,-\ell-1}(v)$. A site of $L_{-k+j}(q)$, $j \in \{-1, 0, 1\}$, lies in $L_{-k-\ell-(a-j)}(v)$ with $a - j \in [1, a_* + 1]$. Summing over w gives, for every admissible (A, Θ),

$$\sum_{i=1}^{a_*+1} \widehat{\mathbb{E}}^{v,x}\big[X^{-S,0}_{-k-\ell-i}(v)\mathbf{1}_{B^c} \mid A, \Theta\big] \ge c_{d,p,M_*}\, \widehat{\mathbb{P}}(B^c \mid A, \Theta)\, p\, \mathbb{E}^{q;A,\Theta}_{p,J}[Z_q;\ E_q], \tag{F.5}$$

with $Z_q = \sum_{j=-1}^{1} X^{-k-r-1,1}_{-k+j}(q)$ counted in the residual domain, $E_q = \{q \not\leftrightarrow H^+(v) \text{ in } \Theta^*\}$, and $\mathbb{E}^{q;A,\Theta}_{p,J}$ the residual product law on J with q conditioned open. The factor $\widehat{\mathbb{P}}(B^c \mid A, \Theta)$ comes from TN(5.80) summed over w, and (F.5) is TN(5.81) multiplied by it, with the new index ranges.

Step 7: removing the residual conditioning, as in [20] with the new parameters. Fix $y \in [0, 1)$. The half-open layer definition gives $L^y_{-k}(q) \subseteq L^{R_q}_{-k-1,-k+1}(q)$ and $L^y_{-k-r,0}(q) \subseteq L^{R_q}_{-k-r-1,1}(q)$, which is the first part of TN(5.82), so $X^{-k-r,0}_{-k}(q; y) \le Z_q$ pointwise. For the events: first, $\Theta^* \subseteq L^{R_q}_{-S+\ell+a,\infty}(q) = L^{R_q}_{-k-r-1-(a_*-a),\infty}(q) \subseteq L^y_{-k-(r+a_*)-2,\infty}(q)$. Second, every $z \in H^+(v)$ has $h_q(z) = h_v(z) - h_v(q) > t_0(\ell + a - 1) \ge t_0(\ell + 1)$, using $a \ge 2$; a path from q, at normalized height 0, to normalized height above $\ell + 1 \ge \ell + y$ moves in steps of at most 1, and its first vertex above $\ell + y - 1$, which is at least $1 > 0$, lies in $L^y_\ell(q)$. So every path from q to $H^+(v)$ meets $L^y_\ell(q) = L^y_{(\ell+1)-1}(q)$. The nonconnection in $Q^y_{q;k,\ell+1,r+a_*}$ holds in $L^y_{-k-r-a_*-2,\infty}(q) \supseteq \Theta^*$, hence in Θ^*, and $Q^y_{q;k,\ell+1,r+a_*} \subseteq E_q$. So $X^{-k-r,0}_{-k}(q; y)\mathbf{1}_{Q^y_{q;k,\ell+1,r+a_*}} \le Z_q \mathbf{1}_{E_q}$, which is TN(5.83) with the new parameters. The variable $Z_q\mathbf{1}_{E_q}$ uses only J-coordinates, since the slab of Z_q lies in $L_{-S,-\ell-1}(v)$ and E_q is defined in $\Theta^* \subseteq J$. Extending the residual law by fresh Bernoulli(p) variables on $A \cup \Theta$, as in [20], gives $\mathbb{E}^{q;A,\Theta}_{p,J}[Z_q;\ E_q] \ge \widehat{\mathbb{E}}^{q,y}[X^{-k-r,0}_{-k}(q);\ Q_{q;k,\ell+1,r+a_*}]$ for every y. In place of the transitivity used at this point in [20]: $q \in O^*$, so the supremum over y of the right side equals the left side of the proposition, by the choice of O^*.

Step 8: averaging, as written. Sum (F.5) against $\widehat{\mathbb{P}}^{v,x}(A, \Theta)$ over admissible (A, Θ). By Step 5, $\sum_{(A,\Theta)} \widehat{\mathbb{P}}(A, \Theta)\widehat{\mathbb{P}}(B^c \mid A, \Theta) = \widehat{\mathbb{P}}^{v,x}(N_\ell(v) \cap P_v(S))$, since $N_\ell(v) \cap P_v(S)$ forces v to be the peak of A, which lies in the set joined inside $L_{-S,\infty}(v)$. This gives $\sum_i \widehat{\mathbb{E}}^{v,x}[X^{-S,0}_{-k-\ell-i}(v);\ P_v(S)] \ge$

$c'\widehat{\mathbb{P}}^{v,x}(N_\ell(v)\cap P_v(S))\cdot\mathrm{LHS}$. Divide by $\widehat{\mathbb{P}}^{v,x}(P_v(S))$, which is at least $(1-p)^d>0$ because it contains the isolated-root event, and rearrange; this proves the proposition with $C_p=1/c'$. □

Downstream: TN(5.113)–TN(5.117), hence TN(5.112).

- TN(5.113), *with a random root.* For $i\in[1,a_*+1]$ put $K=k+\ell$ and $F_i(a,b)=\mathbf{1}\{b\in L_{-K-i}(a),\ a\leftrightarrow b \text{ in } L_{-S,0}(a),\ P_a(S)\}$. It is diagonally invariant with the equivariant offsets and vanishes unless a and b are open, so Lemma 7.6 gives

$$p\sum_O\mu(O)\int_0^1\sum_i\widehat{\mathbb{E}}^{O,y}[X^{-S,0}_{-K-i}(O);\ P_O(S)]\,dy=\mathbb{E}_\mu\mathbb{E}\sum_i\sum_a F_i(a,\rho)\Delta(\rho,a).$$

  If $F_i(a,v)=1$, then $a\in L_{K+i}(v)$ and v is joined to a in $L_{K+i-S,K+i}(v)=L_{-(r+a_*+1-i),K+i}(v)$, with lower depth at most $r+a_*$; there is at most one such a, as in [20], and $\Delta(v,a)\le e^{t_0(K+i+1)}$. The quasi-transitive TN(5.42), uniform in (v,x), bounds the right side by $Ce^{-t_0(\alpha-1)K+C_0r}$. Every orbit has weight at least $\mu_{\min}$, so for every vertex v_0, $\int\sum_i\widehat{\mathbb{E}}^{v_0,y}[X^{-S,0}_{-K-i}(v_0);\ P_{v_0}(S)]\,dy\le(p\mu_{\min})^{-1}Ce^{-t_0(\alpha-1)(k+\ell)+C_0r}$, and some offset x_{v_0} attains at most this value.

- TN(5.114). Take v_0 sloped, so that (7.2) holds for every offset. At (v_0,x_{v_0}) the denominator of Proposition 7.10 is at least $\widehat{\mathbb{P}}^{v_0,x}(v_0\leftrightarrow L_{-\ell-1}(v_0),\ P_{v_0})$: a path from v_0 to $L_{-\ell-1}(v_0)$, stopped at its first entry, stays in $L_{-\ell-1,\infty}(v_0)$ and witnesses $N_\ell(v_0)$, and P_{v_0} implies $P_{v_0}(S)$. By (7.2) this is at least $c_\xi e^{-t_0(\alpha-1+\xi)(\ell+1)}$, and the numerator, divided by $\widehat{\mathbb{P}}(P_{v_0}(S))\ge(1-p)^d$, is at most $Ce^{-t_0(\alpha-1)(k+\ell)+C_0r}$. Hence, for every $\xi>0$, $\sup_{v,x}\widehat{\mathbb{E}}^{v,x}_{p_c}[X^{-k-r,0}_{-k}(v);\ Q_{v;k,\ell+1,r+a_*}]\le C_\xi e^{C_0r-t_0(\alpha-1)k+t_0\xi\ell}$; the $(\alpha-1)\ell$ terms cancel, as in [20].

- TN(5.115)–TN(5.117). The complement of $Q_{v;k,\ell+1,r+a_*}$ is $\{v\leftrightarrow L_\ell(v) \text{ in } L_{-k-r-a_*-2,\infty}(v)\}$. The quasi-transitive TN(5.42) and $\alpha^s_{p_c}\ge1$ give $\sup\widehat{\mathbb{P}}(Q^c)\le Ce^{C_0(k+r)-t_0\ell}$, the factor $e^{C_0a_*}$ being absorbed. TN(5.116)–TN(5.117) are then as written with $\ell=\max\{2,\lceil t_0^{-1}\log N+C_1(k+r)\rceil\}$, where $C_1>C_0/t_0$ is the constant of [20]; the event is now $Q_{v;k,\ell+1,r+a_*}$, and the shift $\ell\to\ell+1$ is absorbed into constants. This proves TN(5.112).

G Correspondence with the bond argument

Tables 1 and 2 list each numbered statement of [15, Sections 2–9], its site form here, and the change. "As stated" means that the statement of [15] is used without change; "verbatim" means that its proof carries over with site bits. We use as stated only the graph, operator and product-measure statements of [15], namely the weighted mass transport, the invariant trace, the walk with a spectral gap, and the decision-tree covariance bound [15, Lemmas 2.7, 2.8, 2.9 and 5.4], and we check their hypotheses at each use. We have checked for site percolation every step of the proofs of [15] that we cite as carrying over.

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[15]	Here	Change for sites
Lemma 2.1	Section 2.1	simplification to G_s instead of subdivision
Lemma 2.2	Lemma 2.4	Peierls with the closed vertex boundary; [16, Theorem 1.1(2)]
Lemma 2.3	Lemma 2.5	vertex bits; tree bound with κ_p
Lemma 2.4	Lemma 2.6	verbatim
Proposition 2.5	Theorem 3.2	replaced: [9, Theorem 2.9] is a bond theorem
Theorem 2.6	Theorem 3.1	site proof, Sections 4–6
Lemmas 2.7–2.9	Lemmas 2.1–2.3	as stated
Lemma 2.10	Lemma 2.7	site sharpness [1]; $T_p(x,x) = p$
Lemma 2.11	Lemma 2.8	closed pivotal sites; remainder R_p
Lemma 2.12	Lemma 2.9	kernels $I + T_p\mathrm{Adj}$ and $\mathrm{Adj}(I + T_p\mathrm{Adj})$
Definition 3.1	Definition 4.1	cut vertices in place of bridges
Proposition 3.2	Proposition 4.3	pruning (Lemma 4.6); sequential reconstruction
Lemma 3.3	Lemma 4.4	root excluded; exterior $(D \setminus \mathrm{Nb}[W]) \cup \{v\}$; $\mathbb{P}(E) \geq p_c h$
Lemma 3.4	Lemma 4.5	real range $\kappa \geq \kappa_0$
—	Lemma 4.6	new
Lemma 4.1	Lemma 5.1	new bounds (d)–(e): bins and second moment at record scales
Definition 4.2	Definition 5.2	site bits; conventions (i)–(ii)
Proposition 4.3	Proposition 5.3	junction costs absorbed in the iteration
Lemma 4.4	Lemma 5.4	visited vertices, including x, open; coin $p_c^{\ell+1-e}$
Lemma 4.5	Lemma 5.5	junction rejection; seams are cut vertices
Lemma 4.6	Lemma 5.6	hanging sets; gap bound $l \leq 3 + KM^\tau$
Lemma 4.7	Lemma 5.7	factor c_0^k
Lemma 4.8	Lemma 5.8	joint restoration at a site obstruction (the jointly restored root set can be disconnected); extra factor $4^d\nu^2$
Lemma 4.9	Lemma 5.9	factors c_0^{2k} and c_0^{-k}

Table 1: Sections 2–4 of [15].

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[15]	Here	Change for sites
Lemma 5.1	Lemma D.1	resampling of $\mathrm{Nb}[B]$; first boundary site
Lemma 5.2	Lemma D.2	deleted domain $V \setminus \mathrm{Nb}[A]$
Proposition 5.3	Proposition 6.1	factor d in the upper bound
Lemma 5.4	—	as stated
Proposition 5.5	Proposition 6.2	revealment $(d+2)C_a n^{-a}$
Lemma 6.1	Lemma D.3	score difference $-p\,k_s(A, C_v)$
Proposition 6.2	Proposition 6.3	remainder at most Ca_3; $a_3 \geq p^3$
Definition 7.1	Definition 6.4	greedy shortcut; vertex blockers
—	Lemma 6.5	new
Lemma 7.2	Lemma 6.6	monotonicity through Lemma 6.5
Lemma 7.3	Lemma 6.7	$D_w = V \setminus (I_w \cup B_w)$
Proposition 7.4	Proposition 6.8	verbatim after calibration
Proposition 7.5	Proposition 6.12	factor e^{-2tc_k}
Lemma 7.6	Lemma 6.9	visits of $\mathrm{Nb}[A']$; constant C_{del}
Lemma 7.7	Lemma 6.10	$C' = C_{\mathrm{del}} c_{\mathrm{nb}} \kappa_p$
Lemma 7.8	Lemma 6.11	new template class of weight $2c_k$
Lemma 7.9	Lemma 6.13	factor $e^{-t(l+1)c_k}$
Lemma 8.1	Lemma 6.15	site avoidance; factors $d+1$
Lemma 8.2	Lemma 6.14	$T_0(b, b) = p$
Claim 9.1	Appendix D.4, step (3)	$c_f = e^{-(L_f+1)} p_0^{L_f}$
Lemma 9.2	Lemma 2.10	verbatim
Proposition 9.3	Proposition 2.11	vertex labels; at most d clusters joined per site

Table 2: Sections 5–9 of [15].

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