Seven Research Directions in Differential Geometry and Differential Topology Enabled by AI

GPT 6 Astra

prompted be Jianqiao Shang

Abstract

This article presents seven AI-generated research directions in differential geometry and differential topology, each centered on a proposed research object or formulation. For each direction, we provide basic definitions and theorems, discuss connections with existing mathematical problems and theories, and outline questions for further study. The definitions introduced in these seven directions are relatively new. The article is intended as a collection of AI-generated research guides rather than a conventional research paper.

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Contents

1 Introduction **2**

2 Critical kernels **3**

3 Uniformly Morse linear systems **10**

4 Presymplectic dependency algebras **16**

5 Diffusion transport defects **22**

6 Tangent shapes and knots **29**

7 Finite-scale bundle comparisons **43**

8 Harmonic closure of differential forms **49**

References **54**

1 Introduction

1.1 Background

The idea behind this article was to let AI explore geometry and topology freely and see what conclusions it could reach by focusing on "heuristic" mathematics.

Using the chat interface of GPT 6 Astra Pro, the prompter started fifteen separate conversations and asked the AI to propose new mathematical objects worth studying. The prompter then merged related topics, discarded problems that did not seem worth pursuing, and collected the directions he himself found interesting into this research guide. These questions are relatively new overall, but they are closely tied to existing human mathematical research.

According to the prompter, further work has shown clear mathematical structure in some of these theories and has even solved open problems in other areas. This gives the prompter reason to believe that at least some of these directions are promising. The prompter will discuss further work on these directions in later papers.

The prompter claims no credit for any of these theories and welcomes human-led mathematical discussion with anyone interested in them. The hope is that AI will create more interesting mathematical problems, rather than just kill existing ones.

References may be incomplete, since the prompter has limited knowledge of each of these areas and AI may have missed relevant works. The prompter apologizes for any omissions of works that should have been cited as sources of the ideas presented here and will correct them as soon as they are reported.

1.2 Critical kernels

Section 2 studies symmetric kernels whose Morse slice functions define finite critical equivalence classes. The associated coverings give strong topological restrictions when the Euler characteristic is nonzero. First-level Morse–Bott attachments yield quadratic cohomology extensions, and additional generation hypotheses lead to recognition by reflection groups. This connects the construction with invariant theory and rational homotopy theory. Further questions concern geometric realizability and what remains true without the generation hypotheses.

1.3 Uniformly Morse linear systems

Section 3 studies finite-dimensional spaces of functions modulo constants whose nonzero elements are Morse. On a closed connected n-manifold, the rank is at most $n + 1$, with equality possible precisely on the standard smooth sphere. Critical coverings, Hessian bundles, and critical-point configurations furnish extension obstructions. A rank-two system on S^3 with linked minima admits an odd nonlinear extension but no rank-three linear extension. Connections with parametrized Morse theory and round fold maps motivate further questions about existence, deformation classes, and maximal systems.

1.4 Presymplectic dependency algebras

Section 4 studies positive systems of closed two-forms of constant rank zero or two and their finite wedge algebras. Restriction to characteristic foliations realizes matroid contraction. The algebras distinguish systems on a fixed torus with identical cohomology classes, matroids, and volume polynomials. These constructions relate multilinear matroid representations to symplectic pairs and polysymplectic geometry. Further questions concern local moduli and the extent to which the algebra captures the geometry and deformation behavior of the chosen system.

1.5 Diffusion transport defects

Section 5 studies rank defects of connection heat kernels and coverings by their invertibility domains. Characteristic classes give lower bounds for the covering number. Flat torus con-

structions exhibit arbitrarily many distinct covering levels as time varies, keeping the metric and connection fixed in each example while allowing dimension to grow. The framework relates vector diffusion methods, characteristic-class obstructions, and theta-function geometry. Further questions concern transversality and the possible time profiles of the covering number.

1.6 Tangent shapes and knots

Section 6 studies families of compact shapes embedded in punctured tangent spaces and oriented knots in spheres of tangent directions. Euler-class and holonomy arguments yield obstructions and rigidity results for shape fields. For a prescribed positive nontrivial torus-knot type on a closed, connected, oriented smooth four-manifold, existence is characterized by a weighted circle reduction of the tangent bundle and explicit characteristic-class conditions. These results connect embedding spaces, classical G-structures, and four-manifold topology. Further questions concern deformation classes, additional differential conditions, and the geometry of linear incidence among tangent directions.

1.7 Finite-scale bundle comparisons

Section 7 organizes smooth fiber identifications by their distance from the diagonal, using an extension radius and a system of restriction maps. We establish metric stability and line-bundle criteria, including an exact half-systole formula for nontrivial real line bundles and computations on flat tori. For closed negatively curved oriented surfaces, the complete family of labelled radii on finite covers recovers the marked length spectrum. This links direct connections with systolic geometry. Further questions concern higher-rank obstructions, critical radii, and information retained by the full comparison system.

1.8 Harmonic closure of differential forms

Section 8 considers the algebra generated by harmonic forms under wedge product, exterior differentiation, and Hodge star on a closed oriented Riemannian manifold. Finite closure produces a finite de Rham model and a canonical Lie algebra. For compact nilmanifolds with descended left-invariant metrics, the construction recovers the metric Lie algebra. Other results include conformal rigidity in the presence of intermediate cohomology and a three-dimensional classification. Connections with geometric formality motivate further study of higher-dimensional finite-closure metrics and their associated Lie algebra actions.

2 Critical kernels

A critical kernel is a symmetric two-point function whose slice critical points form finite equivalence classes. The resulting covering structure imposes restrictions on the topology of the underlying manifold.

Throughout this section, manifolds are smooth, closed (compact without boundary), connected, and of positive dimension. Cohomology is singular cohomology. Put $P_M(t) = \sum_j \dim H^j(M;\mathbb{Q})t^j$. A rational homotopy equivalence is denoted by $\simeq_{\mathbb{Q}}$. For a left action on M, the left action on cohomology is $g \cdot a = (g^{-1})^* a$; formulas involving g^* use ordinary pullback.

2.1 Definition and covering geometry

Definition 2.1 (Critical kernel)**.** A *critical kernel* on M is a smooth function $F : M \times M \to [0, \infty)$ such that:

(i) $F(x, y) = F(y, x)$, and $F(x, y) = 0$ if and only if $x = y$;

(ii) for every $x \in M$, the slice $F_x(y) = F(x, y)$ is Morse;

(iii) the relation $x \sim_F y \iff d_2F(x,y) = 0$ is an equivalence relation.

Write $C_F = \{(x,y) : d_2F(x,y) = 0\}$ and $B_F = M/\!\sim_F$. An isomorphism of kernels is a diffeomorphism $\phi : M \to N$ satisfying $F_N(\phi x, \phi y) = F_M(x,y)$.

Proposition 2.2 (Critical covering)**.** *Let $n = \dim M$. Then C_F is a closed embedded n-manifold; both projections $C_F \to M$ are finite coverings of the same degree $r \geq 2$. The quotient B_F has a natural closed smooth manifold structure, and*

$$q : M \longrightarrow B_F, \qquad C_F = M \times_{B_F} M,$$

is an r-sheeted smooth covering. In particular every slice has exactly r critical points. We call r the critical degree*.*

Proof. The section d_2F of $\pi_2^* T^*M$ is transverse to zero: its vertical derivative at a zero is the nondegenerate Hessian of F_x. The implicit function theorem makes C_F locally a graph $y = \phi(x)$. Thus its first projection is a proper local diffeomorphism, hence a finite covering. The diagonal makes it surjective. Reciprocity of the equivalence relation allows coordinate interchange, proving the same assertion for the second projection; each local branch ϕ is a local diffeomorphism.

For a class $[x] = \{x_1, \dots, x_r\}$, shrink a neighborhood U of $x = x_1$ so that all branches are diffeomorphisms $\phi_i : U \to U_i$, with $\phi_1 = \mathrm{id}$ and the U_i disjoint. Transitivity gives $[\phi_i(u)] = \{\phi_1(u), \dots, \phi_r(u)\}$. Therefore $\bigcup_i U_i$ is saturated, and each $q|_{U_i}$ is a homeomorphism onto the same open subset of B_F. These are quotient charts, with smooth branch maps as transitions. The quotient is Hausdorff because the equivalence relation is closed on a compact Hausdorff space, and second countable because q is open. Finally, a Morse function on a positive-dimensional closed manifold has a minimum and a distinct maximum, so $r \geq 2$. □

Proposition 2.3 (Morse–Bott structure and metric)**.** *The full function F is Morse–Bott with critical manifold C_F. Its normal index at (x,y) equals the index of y for F_x. Also*

$$g_x(u,v) = \mathrm{Hess}_{y=x} F_x(u,v)$$

defines a Riemannian metric, natural under kernel isomorphisms.

Proof. If $d_2F(x,y) = 0$, reciprocity and symmetry imply $d_1F(x,y) = 0$, so $\mathrm{Crit}(F) = C_F$. On a local branch $y = \phi(x)$, set $A = D\phi_x$ and $H = D^2_{yy}F(x,\phi(x))$. Differentiating the critical equations along the branch gives

$$D^2F = \begin{pmatrix} A^\mathsf{T}HA & -A^\mathsf{T}H \\ -HA & H \end{pmatrix}, \qquad D^2F((v,w),(v,w)) = H(w - Av, w - Av).$$

Its kernel is exactly the tangent space to the graph, and its normal inertia is that of H. On the diagonal, x is a nondegenerate minimum of F_x, so g_x is positive definite. Smoothness and naturality follow from the intrinsic Hessian at a critical point. □

Proposition 2.4 (Products)**.** *If F and K are critical kernels on M and N, then*

$$(F \oplus K)((x,u),(y,v)) = F(x,y) + K(u,v)$$

is a critical kernel on $M \times N$. Critical relations and quotient coverings are products, degrees multiply, indices add, and the induced metric is the product metric.

Proof. Slice gradients and Hessians split, and criticality is the product of the two equivalence relations. The other axioms follow directly. □

2.2 Euler rigidity

Theorem 2.5 (Euler rigidity)**.** *If $\chi(M) \neq 0$, then*

$$\chi(M) = r > 0, \qquad \chi(B_F) = 1.$$

Every slice has only even-index critical points and is perfect over $\mathbb{Z}$. The manifold M is simply connected, of even dimension, and its integral homology and cohomology are free and concentrated in even degrees. The covering q is universal; its deck group $G \cong \pi_1(B_F)$ acts freely and satisfies

$$|G| = \chi(M) = \dim_{\mathbb{Q}} H^*(M;\mathbb{Q}), \qquad H^*(M;\mathbb{Q})^G = \mathbb{Q}. \tag{2.1}$$

The quotient B_F is rationally acyclic and nonorientable, and

$$p_i(TM) = 0 \in H^{4i}(M;\mathbb{Z}) \qquad (i > 0). \tag{2.2}$$

Proof. The covering formula gives $\chi(M) = r\chi(B_F)$. For any slice, the Morse formula gives $\chi(M) = \sum_{j=1}^{r} (-1)^{\lambda_j}$. The diagonal contributes a minimum, so $-r < \chi(M) \leq r$. A nonzero integer multiple of r in this interval is r. Hence every λ_j is even.

The Morse CW decomposition has only even cells. Connectedness forces one zero-cell; there are no one-cells, so M is simply connected. Every cellular boundary map vanishes. This proves integral perfection and freeness, and also orientability and even dimension. The universal covering has $|G| = r$.

Transfer injects $H^*(B_F;\mathbb{Q})$ into $H^*(M;\mathbb{Q})$ and identifies it with the invariants. Thus the quotient has no odd rational cohomology. Its Euler characteristic is 1, so its only rational cohomology is $H^0 = \mathbb{Q}$. Since it has positive dimension, it is nonorientable. Finally $TM \cong q^*TB_F$; naturality gives rational vanishing of all positive Pontryagin classes of TM. Torsion-freeness upgrades this to integral vanishing. □

Remark 2.6. Until the end of Subsection 2.5, assume $\chi(M) \neq 0$ and put $A = H^*(M;\mathbb{Q})$. The critical set decomposes as

$$C_F = \bigsqcup_{g \in G} \Gamma_g, \qquad \Gamma_g = \{(x, gx) : x \in M\}.$$

Write c_g and λ_g for its constant critical value and index. Then

$$c_1 = 0, \quad \lambda_1 = 0, \quad c_g > 0 \ (g \neq 1), \quad c_g = c_{g^{-1}}, \quad \lambda_g = \lambda_{g^{-1}}.$$

The last two equalities follow from coordinate interchange. Integral perfection gives

$$P_M(t) = \sum_{g \in G} t^{\lambda_g}. \tag{2.3}$$

In particular, there is one index-zero branch and one index-n branch. Vanishing Pontryagin classes alone does not imply stable parallelizability or spin.

2.3 Examples

The first two examples also cover $\chi(M) = 0$; the restriction in Remark 2.6 is not imposed in this subsection.

Example 2.7 (Spheres and their products)**.** On the unit sphere $S^n \subset \mathbb{R}^{n+1}$, $n \geq 1$, set $F(x, y) = 1 - x \cdot y$. Its critical partners are x and $-x$, with indices 0 and n. The quotient is $S^n \to \mathbb{RP}^n$, of degree 2. Products give

$$F(x, y) = \sum_{i=1}^{k} (1 - x_i \cdot y_i) \qquad \text{on} \qquad \prod_{i=1}^{k} S^{n_i}.$$

The critical group is $(\mathbb{Z}/2)^k$ acting by coordinate antipodes, and a branch has index $\sum_{i\in I} n_i$ when exactly the coordinates in I are reversed. If all $n_i = d$ are even, cohomology is generated in degree d, and the first positive critical branches generate the entire group.

Example 2.8 (Tori)**.** On $T^m = (\mathbb{R}/2\pi\mathbb{Z})^m$, set

$$F(x, y) = \sum_{i=1}^{m} (1 - \cos(y_i - x_i)).$$

Critical partners differ by 0 or π in each coordinate. The quotient covering is $x \mapsto 2x$, of degree 2^m, and the number of index-j points is $\binom{m}{j}$. Here $\chi(T^m) = 0$.

Example 2.9 (Complex full flags)**.** Fix $m \geq 2$ and $a_1 < \cdots < a_m$. Let M be the manifold of Hermitian matrices with this spectrum, so $M \cong U(m)/T^m$. Define

$$F(X, Y) = \tfrac{1}{2} \operatorname{tr}((X - Y)^2).$$

Since tangent vectors at Y are commutators $[K, Y]$ with K skew-Hermitian, the critical equation is $[X, Y] = 0$. Simplicity of the spectrum implies that the partners are the $m!$ permutations of the eigenvalues in a common eigenbasis. Commuting is transitive on this orbit: if X and Z both commute with the simple-spectrum matrix Y, they are diagonal in its eigenbasis.

In that basis, the Hessian on the real two-dimensional (i, j)-mixing space has sign $(a_i - a_j)(a_{\sigma(i)} - a_{\sigma(j)})$. Thus each slice is Morse, with index $2\operatorname{inv}(\sigma)$. The critical group is S_m.

For $a_i = i$, the values are $c_\sigma = \frac{1}{2}\sum_i (i - \sigma(i))^2$. The smallest positive value, 1, occurs exactly at adjacent transpositions, which generate S_m. The classical cohomology presentation is

$$H^*(M; \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_m]/(\sigma_1, \ldots, \sigma_m), \qquad |x_i| = 2,$$

where σ_j are elementary symmetric polynomials; see [1]. This family satisfies both hypotheses of Theorem 2.14 below.

2.4 First critical level

Assume again that $\chi(M) \neq 0$. Define

$$c = \min_{g \neq 1} c_g, \qquad S = \{g \in G : c_g = c\}.$$

For $s \in S$, let $E_s^- \to \Gamma_s \cong M$ be the negative normal bundle, of rank λ_s, and choose its orientation. This is possible because M is simply connected. Write $e_s = e(E_s^-)$. We use the same notation for its integral and rational Euler classes.

Lemma 2.10 (First-level divisibility)**.** *For $s \in S$ and every k,*

$$(1 - s^*)H^k(M; \mathbb{Z}) \subseteq e_s H^{k-\lambda_s}(M; \mathbb{Z}). \tag{2.4}$$

Moreover $e_s \neq 0$ rationally, and every rational cohomology class on a regular sublevel of F extends to $M \times M$.

Proof. Let X_- and X_+ be regular sublevels immediately below and above c. Then $X_- \simeq \Delta$. By Morse–Bott attachment and the Thom isomorphism [2],

$$H^k(X_+, X_-; \mathbb{Z}) \cong \bigoplus_{h \in S} H^{k-\lambda_h}(M; \mathbb{Z}).$$

For $a \in H^k(M; \mathbb{Z})$, $\pi_1^* a - \pi_2^* a$ vanishes on X_-, so its restriction to X_+ lifts relatively. Restriction to Γ_s kills the other summands and multiplies the s-summand by e_s, proving (2.4).

The full normal bundle of Γ_s is isomorphic to s^*TM through the vertical directions. Its negative and positive summands satisfy $e(E_s^-)e(E_s^+) = \pm e(TM) \neq 0$, so $e_s \neq 0$. At every critical level, the relative cohomology groups are concentrated in even degrees. Starting with the diagonal sublevel, the long exact sequences show that all regular sublevels have only even cohomology and that restriction from a later one is surjective. This proves the extension assertion. □

Theorem 2.11 (Quadratic extension at the first level)**.** *For every $s \in S$, write $\lambda = \lambda_s$ and $B = A^{s^*}$. Then*

$$s^2 = 1, \qquad s^* e_s = -e_s, \qquad A = B \oplus e_s B, \tag{2.5}$$

where multiplication by e_s identifies B^j with the anti-invariant part of $A^{j+\lambda}$. In particular,

$$P_M(t) = (1 + t^\lambda) P_B(t). \tag{2.6}$$

The transformation s reverses orientation, and B is a Poincaré-duality algebra of formal dimension $n - \lambda$.

Proof. By (2.4), s^* fixes all degrees below λ, and $s^* - 1$ has rank at most one in degree λ. Take the relative Thom unit for the s-attachment, with zero components for all other first-level attachments, and extend its absolute image to $u \in H^\lambda(M \times M; \mathbb{Q})$ using Lemma 2.10. Its restrictions are 0 on Δ, e_s on Γ_s, and 0 on every other first-level graph.

Write the Künneth decomposition as

$$u = \pi_1^* a + \pi_2^* b + \sum_i \pi_1^* a_i \pi_2^* b_i, \qquad 0 < |a_i|, |b_i| < \lambda.$$

Restriction to Δ is zero; since s^* fixes the lower-degree terms, restriction to Γ_s gives $e_s = (s^* - 1)b \neq 0$. Thus $s^* - 1$ has rank exactly one in degree λ. A finite-order real transformation with this property has one nontrivial eigenvalue, -1. Consequently $s^* = (s^{-1})^*$ in this degree, and both fix all lower degrees. If $s \neq s^{-1}$, the same computation gives $u|_{\Gamma_{s^{-1}}} = e_s$, contradicting the prescribed zero restriction. Hence $s^2 = 1$, and $s^* e_s = -e_s$.

Let A^+ and A^- denote its two eigenspaces. Divisibility gives $A^- = e_s A^+$: for an anti-invariant class, divide by e_s as in (2.4) over $\mathbb{Q}$, then average the resulting factor to make it invariant. Since s acts freely, its Lefschetz number is zero. All cohomology is even, so $\dim A^+ = \dim A^-$. The surjection $e_s \cdot - : A^+ \to A^-$ is therefore an isomorphism. This proves (2.5) and (2.6).

If m is the highest nonzero degree of B, then $n = m + \lambda$ and $A^n = e_s B^m$, so s reverses orientation and $B^m \cong \mathbb{Q}$. For a nonzero $a \in B^i$, Poincaré duality in A gives a partner with nonzero product in top degree. Its anti-invariant part still pairs nontrivially with a and is $e_s b$ for some $b \in B^{m-i}$. Hence $ab \neq 0$, proving duality in B. □

Remark 2.12 (Divided differences)*.* There is a unique rational operator $\partial_s : A \to B$, of degree $-\lambda$, with

$$e_s \partial_s(a) = a - s^* a.$$

It satisfies $\partial_s^2 = 0$, $\partial_s(e_s) = 2$, and

$$\partial_s(ab) = \partial_s(a) b + s^*(a) \partial_s(b).$$

Indeed, multiplication by e_s is injective on B; the right-hand side of the last identity is invariant and becomes $ab - s^*(ab)$ after multiplication by e_s. The extension is equivalently $A \cong B[u]/(u^2 - e_s^2)$; one must not assume $e_s^2 = 0$. The quotient manifold $M/\langle s \rangle$ still has geometric dimension n, even though its rational cohomology algebra has formal dimension $n - \lambda$.

Corollary 2.13 (Topological and group obstructions)**.** (i) *If $P_M(t) = 1 + kt^d + t^{2d}$ with $d > 0$ even and $k > 0$, then $k = 2$. In particular, $\#_k(S^{2m} \times S^{2m})$ admits no critical kernel for $m \geq 1$ and $k \geq 2$.*

(ii) *If G has exactly one nonidentity involution, then $G \cong \mathbb{Z}/2$ and M is a homotopy sphere. Thus cyclic groups of order greater than two and generalized quaternion groups cannot occur in this sector.*

Proof. For (i), the first positive index is d or $2d$. If it were $2d$, Theorem 2.11 would force $P_M = 1 + t^{2d}$. Hence it is d, and $1 + t^d$ divides $1 + kt^d + t^{2d}$, forcing $k = 2$. For the stated connected sum, the middle Betti number is $2k$.

For (ii), the unique index-n branch is an involution because inversion preserves index. Every first-level branch is also an involution, so it must be that same branch. Now $\lambda = n$, and (2.5) gives $P_M = 1 + t^n$ and $|G| = 2$. Simple connectivity and integral perfection make M an integral homology sphere, hence a homotopy sphere by Hurewicz and Whitehead. Odd-order groups have no involution at all and are already excluded by Theorem 2.11. □

2.5 Reflection groups and the $2, 4, 8$ restriction

Theorem 2.14 (Conditional recognition)**.** *Assume, in addition to $\chi(M) \neq 0$, that*

(a) *$A = H^*(M; \mathbb{Q})$ is generated as an algebra by $V = H^d(M; \mathbb{Q})$ for some $d > 0$;*

(b) *S, the transformations at the first positive critical level, generates G.*

Set $\ell = \dim_{\mathbb{Q}} V$. Then d is even, G acts faithfully on V as an essential crystallographic reflection group, and

$$A \cong \mathrm{Sym}_{\mathbb{Q}}(V)/\langle \mathrm{Sym}_{\mathbb{Q}}(V)^G_+ \rangle, \qquad |V| = d. \tag{2.7}$$

The manifold is formal. If $m_1, \dots, m_\ell$ are the ordinary polynomial degrees of basic invariants, then

$$P_M(t) = \prod_{i=1}^{\ell} (1 + t^d + \cdots + t^{d(m_i - 1)}), \qquad \dim M = d \sum_{i=1}^{\ell} (m_i - 1). \tag{2.8}$$

Furthermore:

(i) *If G is nonabelian, then $d \in \{2, 4, 8\}$.*

(ii) *If $d \notin \{2, 4, 8\}$, then $G \cong (\mathbb{Z}/2)^\ell$ and $M \simeq_{\mathbb{Q}} (S^d)^\ell$.*

(iii) *If $d = 2$, then $M \simeq_{\mathbb{Q}} K/T$ for a compact connected semisimple Lie group K with Weyl group G and maximal torus T.*

Proof. Reflections. If a deck transformation fixes V, it fixes the generated algebra A. Its Lefschetz number is then $\chi(M) > 0$, which contradicts freeness unless the transformation is the identity. Thus the action on V is faithful. By integral perfection there are no positive indices below d. If $s \in S$ had $\lambda_s > d$, (2.4) would make it fix V. Therefore $\lambda_s = d$.

In degree d, the integral version of (2.4) gives

$$a - s^* a = e_s D_s(a), \qquad D_s : H^d(M; \mathbb{Z}) \longrightarrow \mathbb{Z}. \tag{2.9}$$

The integer coefficient is unique because $e_s \neq 0$ and $H^d(M; \mathbb{Z})$ is free, and is additive in a. Thus $1 - s^*$ has rank one on V; finite order makes s^* a reflection. It preserves the integral lattice. The generation hypothesis makes G a finite crystallographic reflection group, hence a Weyl group [3]. It is essential because $V^G = 0$ by (2.1).

The spherical Euler obstruction. Choose a G-invariant positive inner product on $V_{\mathbb{R}}$. Since e_s spans the negative eigenspace,

$$D_s(e_s) = 2, \qquad D_s(v) = \frac{2\langle v, e_s \rangle}{\langle e_s, e_s \rangle}.$$

If G is nonabelian, some $s, t \in S$ do not commute. The integers $a = D_s(e_t)$ and $b = D_t(e_s)$ obey

$$ab = \frac{4\langle e_s, e_t\rangle^2}{\langle e_s, e_s\rangle\langle e_t, e_t\rangle} \in \{1, 2, 3\}.$$

Indeed, noncommuting reflection normals are neither parallel nor perpendicular, so $0 < ab < 4$. At least one entry is odd; relabel so that $D_s(e_t)$ is odd.

Integral homology vanishes in degrees $0 < i < d$. Simple connectivity and Hurewicz imply that M is $(d-1)$-connected. The universal coefficient theorem and freeness identify the functional D_s with a class $z_s \in H_d(M; \mathbb{Z})$. Hurewicz represents it by $f : S^d \to M$. The oriented rank-d bundle $f^*E_t^-$ has Euler number

$$\langle e(f^*E_t^-), [S^d]\rangle = D_s(e_t),$$

which is odd. Its Thom space has cells in dimensions d and $2d$, with an attaching map of odd Hopf invariant. Adams's theorem forces $d \in \{2, 4, 8\}$; see [4, 5]. This proves (i).

Coinvariants. The surjection $\mathrm{Sym}(V) \to A$ kills every positive-degree invariant because $A^G = \mathbb{Q}$. It therefore factors through the coinvariant algebra. Chevalley's theorem gives dimension $|G|$ for that algebra [6], while $\dim A = |G|$. The induced surjection is an isomorphism, proving (2.7). The standard Hilbert series of the coinvariant complete intersection gives (2.8).

Rational homotopy. Let $f_1, \dots, f_\ell$ be homogeneous basic invariants, of polynomial degrees $m_i \geq 2$. They form a regular sequence. The free graded-commutative algebra

$$(\Lambda(x_1, \dots, x_\ell, y_1, \dots, y_\ell), D), \qquad \begin{cases} |x_i| = d, & Dx_i = 0, \\ |y_i| = dm_i - 1, & Dy_i = f_i(x), \end{cases} \tag{2.10}$$

has cohomology A by the Koszul calculation. Choose closed representatives of the x_i in $A_{\mathrm{PL}}(M)$ and primitives of their exact relations $f_i(x)$. They define a quasi-isomorphism from (2.10) to $A_{\mathrm{PL}}(M)$. The same algebra maps by a quasi-isomorphism to $(A, 0)$, proving formality in Sullivan's sense [7]. For $d = 2$, the full flag manifold with Weyl group G has this model, proving (iii).

If $d \notin \{2, 4, 8\}$, the first reflections commute. Their distinct normal lines are orthogonal and span V, since the action is essential. Thus $G = (\mathbb{Z}/2)^\ell$ and (2.7) is $\mathbb{Q}[x_1, \dots, x_\ell]/(x_1^2, \dots, x_\ell^2)$. Model (2.10) is then the model of $(S^d)^\ell$, proving (ii). □

Remark 2.15 (Limits of recognition). The generation assumptions are additional hypotheses, not consequences of the kernel axioms. The conclusion for $d = 2$ is rational; it gives no smooth classification. For $d = 4$ or 8, the theorem specifies the coinvariant algebra and rational model, but makes no assertion that every Weyl type is realizable. Without (b), the same proof still shows that two noncommuting first-level transformations force $d \in \{2, 4, 8\}$.

2.6 A twisted product

Example 2.16 (A nonabelian group in degree six)**.** Let $N = (S^6)^4$, let $F_0(x, y) = \sum_{i=1}^4 (1 - x_i \cdot y_i)$, and let σ interchange factors $1, 2$ and factors $3, 4$. On $M = N \times S^6$, with $t = p \cdot q$, define

$$\Phi((x, p), (y, q)) = 10(1 - t) + \frac{1 + t}{2} F_0(x, y) + \frac{1 - t}{2} F_0(x, \sigma y). \tag{2.11}$$

This is a critical kernel whose critical group is the nonabelian group

$$G = (\mathbb{Z}/2)^4 \rtimes \mathbb{Z}/2,$$

where the last involution acts on the four sign changes by the double interchange. Nevertheless $H^*(M; \mathbb{Q})$ is generated in degree 6.

Symmetry follows because σ is an involutive isometry of F_0; positivity and the diagonal zero set are immediate. The q-derivative is

$$\left[-10 + \tfrac{1}{2}\big(F_0(x,y) - F_0(x,\sigma y)\big)\right] d_q t.$$

Since $0 \le F_0 \le 8$, the coefficient is strictly negative, so criticality forces $q = \pm p$. The critical partners are exactly

$$(kx, p) \quad \text{or} \quad (\sigma k x, -p), \qquad k \in (\mathbb{Z}/2)^4.$$

They are the orbits of the stated free action: the additional involution is $(x,p) \mapsto (\sigma x, -p)$. At these points the slice Hessian is block diagonal, with nondegenerate product-sphere and definite q blocks. Thus every slice is Morse.

If k reverses j coordinates, the two branches have respectively

$$(\text{value}, \text{index}) = (2j, 6j), \quad (20 + 2j, 6j + 6).$$

The first positive level contains only the four individual sign changes on N. They generate the proper subgroup $(\mathbb{Z}/2)^4$. Thus hypothesis (b) in Theorem 2.14 cannot be omitted.

2.7 Related work

The geometric tool is Morse–Bott theory [2]; the algebraic and homotopical inputs are Chevalley's reflection invariant theory [6], Adams's Hopf invariant theorem [4], and Sullivan's rational homotopy theory [7]. The odd-Euler-number formulation used above is explained in [5, Section 3.2].

The closest geometric antecedent is Hsiang–Palais–Terng [8]: Corollary 2.11 gives a $2, 4, 8$ restriction for common even multiplicities under their irreducibility and rank hypotheses, and Theorem 6.12 describes cohomology by Weyl coinvariants generated by Euler classes of curvature distributions. Mare [1, Theorem 1.2 and Lemma 3.3] gives closely related formulas for real flag manifolds, including Euler-class reflection relations. These conclusions and divided-difference patterns are established mathematics. Under the hypotheses of Theorem 2.14, the critical-kernel axioms and first-level Thom attachments yield an intrinsic recognition criterion. This does not give a complete classification.

3 Uniformly Morse linear systems

We consider finite-dimensional linear spaces of functions modulo constants whose nonzero elements are Morse. The questions concern their possible ranks and their extension to larger such spaces.

3.1 Definitions and openness

Throughout this section, $M = M^n$ is a closed, connected, smooth manifold with $n \ge 1$. A function is *Morse* if all its critical points are nondegenerate; critical values may coincide. Write $\mathcal{M}(M) \subset C^\infty(M, \mathbb{R})$ for the space of Morse functions, with its C^∞ topology.

Definition 3.1. A *uniformly Morse linear system* of rank $r \ge 1$ is an r-dimensional real subspace $V \subset C^\infty(M, \mathbb{R})/\mathbb{R}$ such that every nonzero class in V is Morse.

Define $\mu_{\text{lin}}(M)$ to be the largest possible rank. A system is *maximal by inclusion* if it is contained in no strictly larger such system.

Adding constants does not affect Morseness. To discuss actual families of functions, choose representatives linearly, for instance by requiring zero mean with respect to a fixed positive smooth density. A basis $[f_1], \dots, [f_r]$ represents V by

$$F = (f_1, \dots, f_r) : M \longrightarrow \mathbb{R}^r, \qquad f_a = a \cdot F \quad (a \in \mathbb{R}^r).$$

The condition is precisely $f_a \in \mathcal{M}(M)$ for every $a \neq 0$. It is enough to test $a \in S^{r-1}$. We do *not* require F to be an immersion or a stable map. Replacing F by $AF + b$, with $A \in \mathrm{GL}(r, \mathbb{R})$, only changes the presentation; pullback by a diffeomorphism preserves the condition. Every nonzero linear subspace of V is again a system.

Proposition 3.2 (Openness)**.** *For fixed r, the admissible ordered tuples $(f_1, \dots, f_r)$ form a C^2-open set among smooth tuples.*

Proof. Choose a Riemannian metric and its covariant Hessian. On the compact set $S^{r-1} \times M$,

$$\|df_a(x)\|^2 + \sigma_{\min}\big(\nabla^2 f_a(x)\big)^2 > 0,$$

where $\sigma_{\min}$ is the smallest singular value. Positivity follows because a zero differential has a nonsingular Hessian. The minimum is positive and stays positive under a sufficiently small C^2 perturbation of the tuple. □

3.2 Critical incidence and the rank bound

Define

$$C(V) = \{(a, x) \in S^{r-1} \times M : df_a(x) = 0\}, \qquad C_i(V) = \{(a, x) \in C(V) : \mathrm{ind}_x(f_a) = i\}.$$

Let $\pi(a, x) = a$ and $\mathrm{ev}(a, x) = x$.

Proposition 3.3 (Critical covering)**.** *The space $C(V)$ is a compact smooth $(r-1)$-manifold, and π is a finite covering. Each $C_i(V)$ is a union of covering components. If $r \geq 2$, the numbers m_i of index-i critical points are independent of a and satisfy*

$$m_i = m_{n-i}, \qquad \chi(M) = \sum_{i=0}^{n} (-1)^i m_i.$$

If $r \geq 3$, every covering component is a copy of S^{r-1}.

Proof. At a zero of df_a, its derivative in the M direction is the invertible Hessian. The implicit function theorem makes π a local diffeomorphism. Its domain is closed in a compact product, so π is a finite covering. The index is locally constant. For $r \geq 2$, connectedness of the sphere makes the counts constant, while $f_{-a} = -f_a$ exchanges indices i and $n - i$. The Euler formula is the Morse equality. Finally S^{r-1} is simply connected when $r \geq 3$. □

In rank two, monodromy can permute critical points: a covering of S^1 need not split into globally labeled branches. Whenever a branch $c : S^{r-1} \to M$ is available, its Hessians define negative and positive eigenspace bundles

$$E_c^- \oplus E_c^+ \cong c^*TM, \qquad \mathrm{rank}\, E_c^- = \mathrm{ind}(c).$$

More generally these bundles are defined over $C_i(V)$. A metric chooses the eigenspaces, but their bundle isomorphism classes do not depend on that choice. Critical coverings and Hessian bundles are standard constructions in parametrized Morse theory [9, §4.1].

Lemma 3.4 (Minimum embedding)**.** *The map $\mathrm{ev} : C_0(V) \to M$ is an embedding. At a minimum x of f_a,*

$$\ker(dF_x^*) = \mathbb{R}a, \qquad \mathrm{rank}\, dF_x = r - 1. \tag{3.1}$$

Proof. Put $K_x = \ker(dF_x^*)$ and $H_b = \mathrm{Hess}_x f_b$ for $b \in K_x$. The form H_a is positive definite. If $b \in K_x$ is independent of a, the real symmetric matrix $H_a^{-1/2} H_b H_a^{-1/2}$ has a real eigenvalue

λ. Thus $H_b - \lambda H_a$ is singular, making the nonzero function class $[f_{b-\lambda a}]$ non-Morse. This proves (3.1).

Two minimum directions at the same point must now be proportional, hence equal: opposite directions exchange minima and maxima. Therefore ev is injective. A tangent vector (b, v) to $C_0(V)$ satisfies

$$b \perp a, \qquad H_a(v, \cdot) + dF_x^* b = 0. \tag{3.2}$$

If $d\,\mathrm{ev}(b, v) = v = 0$, then $b \in \mathbb{R}a \cap a^\perp = 0$. Thus ev is an immersion; compactness makes it an embedding. □

The lemma applies to all *local* minima. The maximum locus has the same image in M, via $(a, x) \mapsto (-a, x)$. Intermediate-index incidence need not embed.

Theorem 3.5 (Sharp bound and sphere rigidity)**.** *For every closed connected n-manifold,*

$$1 \le \mu_{\mathrm{lin}}(M) \le n + 1, \qquad \mu_{\mathrm{lin}}(M) = n + 1 \quad \Longleftrightarrow \quad M \cong S^n$$

where $\cong$ denotes diffeomorphism. If $n \ge 2$ and a system has rank $n+1$, every nonzero member has exactly two critical points, and its representing map is a strictly convex hypersurface embedding in $\mathbb{R}^{n+1}$.

Proof. A Morse function gives rank one. At a minimum, (3.1) gives $r - 1 \le n$. If $r = n + 1$, the embedding $C_0(V) \to M$ has equal dimensions. Every component has open and closed image. Since M is connected and the images are disjoint, there is exactly one component, and the embedding is a diffeomorphism onto M. This component is a connected finite cover of S^n: it is S^n for $n \ge 2$, and a circle for $n = 1$. Conversely, coordinate heights on $S^n \subset \mathbb{R}^{n+1}$ give rank $n + 1$.

For $n \ge 2$, the minimum covering has one sheet; so does the maximum covering. Every point of M lies in the minimum image. Equation (3.1) then implies that any critical height at that point is a multiple of a definite one. There are no saddle points. Also dF has rank n everywhere. If $F(x) = F(y)$, choose a height uniquely minimized at x; then $y = x$. Thus F is an embedding. Every point is a unique support point, and every support direction has one such point; hence $F(M)$ is the boundary of its convex hull. Its second fundamental form is definite, giving the asserted strict convexity. □

Remark 3.6. The final assertion requires $n \ge 2$. On S^1, the rank-two system generated by $\cos(k\theta), \sin(k\theta)$ has k minima and k maxima per nonzero member and may multiply cover a circle.

3.3 Obstructions to linear extension

A *linear extension* of V is a uniformly Morse system $W \supset V$ of rank $r + 1$. Choose compatible representatives. Then the sphere family

$$\alpha_V : S^{r-1} \longrightarrow \mathcal{M}(M), \qquad a \longmapsto f_a,$$

is the equatorial restriction of $\alpha_W : S^r \to \mathcal{M}(M)$. These maps are odd: $\alpha(-a) = -\alpha(a)$.

Proposition 3.7 (Family obstructions)**.** *If a rank-r system extends linearly, α_V extends over D^r. If $r \ge 2$, its critical covering is trivial, every labeled critical branch $S^{r-1} \to M$ is nullhomotopic, and both its Hessian bundles are trivial. Moreover the joint ordered critical-point map*

$$S^{r-1} \longrightarrow \mathrm{Conf}_N(M), \qquad a \longmapsto (c_1(a), \dots, c_N(a)),$$

extends over D^r, where $\mathrm{Conf}_N(M)$ is the space of ordered N-tuples of distinct points.

Proof. Restrict the extended family to a closed hemisphere. For $r \geq 2$, the critical cover over S^r is trivial. Its labeled branches and their Hessian bundles restrict to the given ones. On a hemisphere, each branch extends to a disk, each vector bundle is trivial, and critical points remain distinct. These give all the conclusions. □

These necessary conditions do not suffice for linear extension, even when a smooth odd nonlinear extension exists, as Theorem 3.10 shows.

Proposition 3.8 (An unlink obstruction on S^3)**.** *For a rank-two system on S^3, the image*

$$L_0(V) = \mathrm{ev}(C_0(V)) \subset S^3$$

is a smooth link. If V extends to rank three, the minimum covering of S^1 has trivial monodromy and $L_0(V)$ is an unlink: its components bound pairwise disjoint embedded disks in S^3.

Proof. Lemma 3.4 embeds the finite union of covering circles. In an extension, minimum incidence is a disjoint union of copies of S^2. The lemma embeds these as pairwise disjoint spheres in S^3. Their equators are the original minimum circles, and images of hemispheres give the required disjoint disks. □

3.4 Examples and Hessian bundles

Spheres and tori

On S^n, coordinate heights give $\mu_{\mathrm{lin}}(S^n) = n+1$. On the torus $T^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, fix $c_j > 0$ and let

$$F(\theta_1, \ldots, \theta_n) = \sum_{j=1}^{n} c_j e^{i\theta_j} \in \mathbb{C}.$$

The height $\mathrm{Re}(e^{-i\phi}F) = \sum_j c_j \cos(\theta_j - \phi)$ has 2^n critical points, a diagonal nonsingular Hessian, and $m_i = \binom{n}{i}$. Thus it defines rank two. In particular $\mu_{\mathrm{lin}}(T^2) = 2$, by Theorem 3.5. No value of $\mu_{\mathrm{lin}}(T^n)$ for $n > 2$ is asserted here.

A maximal-rank system on $\mathbb{CP}^2$

For $a = (a_1, a_2) \in \mathbb{C}^2 \cong \mathbb{R}^4$, set

$$A(a) = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ \bar{a}_1 & \bar{a}_2 & 0 \end{pmatrix}, \qquad f_a([z]) = \frac{z^* A(a) z}{z^* z}. \tag{3.3}$$

For $a \neq 0$, the eigenvalues are $-\|a\|, 0, \|a\|$. A projective Rayleigh function with simple eigenvalues has precisely the eigenlines as critical points. In an eigenbasis, its quadratic term near the jth eigenline is

$$\sum_{\ell \neq j} (\lambda_\ell - \lambda_j)|w_\ell|^2.$$

Hence (3.3) is a rank-four system with indices $0, 2, 4$. Since $\mathbb{CP}^2$ is not diffeomorphic to S^4, Theorem 3.5 gives

$$\mu_{\mathrm{lin}}(\mathbb{CP}^2) = 4. \tag{3.4}$$

Its representing map is, up to a constant factor, the first four coordinates of the smooth collapse $\mathbb{CP}^2 \to S^4$:

$$[z : w] \longmapsto \frac{(2z\bar{w},\ |z|^2 - |w|^2)}{|z|^2 + |w|^2}, \qquad z \in \mathbb{C}^2,\ w \in \mathbb{C}.$$

This collapses the hyperplane $w = 0$; see [10, §16] for this construction.

Infinitely many Hessian-bundle types on $S^2 \times S^2$

Let $h : S^2 \to S^2$ be smooth of degree d, choose $0 < \varepsilon < 1$, and define

$$F_h(x, y) = (1 + \varepsilon\langle h(x), y\rangle)x \in \mathbb{R}^3. \tag{3.5}$$

For a unit vector u, the four critical points of $u \cdot F_h$ are

$$x = su, \qquad y = \tau h(su), \qquad s, \tau \in \{1, -1\}.$$

Indeed $u \cdot x = 0$ cannot be critical, since the x-derivative there is a positive multiple of $d(u \cdot x)$. Otherwise the y-equation forces $y = \tau h(x)$, after which the x-equation forces $x = su$.

Use local fiber coordinates centered on $y = \tau h(x)$. In the resulting splitting $T_x S^2 \oplus T_{h(x)} S^2$, the Hessian is

$$-s(1 + \varepsilon\tau)g_{S^2} \oplus (-s\varepsilon\tau)g_{S^2}. \tag{3.6}$$

Thus (3.5) gives a rank-three system with indices $0, 2, 2, 4$. On the two index-two branches, the negative bundles have absolute Euler numbers

$$\{2,\ 2|d|\}. \tag{3.7}$$

For $(s, \tau) = (1, -1)$ the negative bundle is TS^2 over u; for $(-1, -1)$ it is $(h \circ (-\mathrm{id}))^* TS^2$, whose Euler number has absolute value $2|d|$.

In any smooth deformation through admissible triples, critical branches continue and their Hessian bundles extend over $S^2 \times [0, 1]$. Hence the unordered pair (3.7) is invariant. Taking $d = 0, 1, 2, \ldots$ produces infinitely many deformation classes with the same critical-point counts. These maps lie in the round-fold setting of [11, Theorem 5.7(3), Remark 5.8], where the associated self-intersection data also appear.

3.5 Joint critical-point configurations

For the unit-parameter S^3 family (3.3), the maximum and minimum branches are

$$c_+(a) = [a_1 : a_2 : 1], \qquad c_-(a) = [-a_1 : -a_2 : 1].$$

The middle branch is $[-\bar{a}_2 : \bar{a}_1 : 0]$. Every individual branch is nullhomotopic, since the Hopf fibration $S^1 \to S^5 \to \mathbb{CP}^2$ gives $\pi_3(\mathbb{CP}^2) = 0$. All positive and negative Hessian bundles over S^3 are trivial: the middle ones are oriented real two-plane bundles, classified by $H^2(S^3; \mathbb{Z}) = 0$, and the nonzero extreme bundles are pullbacks of $T\mathbb{CP}^2$ along nullhomotopic maps.

Proposition 3.9. *Nevertheless the joint extrema map $(c_+, c_-) : S^3 \to \mathrm{Conf}_2(\mathbb{CP}^2)$ represents a generator, up to sign, of $\pi_3 \mathrm{Conf}_2(\mathbb{CP}^2) \cong \mathbb{Z}$. The family* (3.3) *has no disk filling through Morse functions.*

Proof. Use the affine chart $[z_1 : z_2 : 1] \leftrightarrow (z_1, z_2)$. The pair $(a, -a)$ is homotopic through distinct pairs to $(0, -a)$ by

$$(a, t) \longmapsto ((1 - t)a, -a), \qquad 0 \le t \le 1.$$

The Fadell–Neuwirth fibration [12] is

$$\mathbb{CP}^2 \setminus \{p\} \longrightarrow \mathrm{Conf}_2(\mathbb{CP}^2) \xrightarrow{\mathrm{pr}_1} \mathbb{CP}^2, \qquad p = [0 : 0 : 1].$$

Its fiber deformation retracts to $\mathbb{CP}^1$ by $[z_1 : z_2 : w] \mapsto [z_1 : z_2 : 0]$. Since $\pi_4(\mathbb{CP}^2) = \pi_3(\mathbb{CP}^2) = 0$, fiber inclusion induces an isomorphism on π_3. Under the retraction, $a \mapsto (0, -a)$ becomes $a \mapsto [a_1 : a_2]$, the Hopf map, which generates $\pi_3(S^2)$. A Morse disk filling would retain exactly one minimum and one maximum and would extend their joint configuration, a contradiction. □

The preceding obstruction concerns fillings by general Morse families. The next result separates nonlinear extension from linear extension.

3.6 Linear and nonlinear extensions

Theorem 3.10 (Linked minima)**.** *There is a rank-two uniformly Morse system V on S^3 such that its normalized circle extends to a smooth odd family of Morse functions on a parameter S^2 (indeed S^3), but V has no linear extension to rank three. Every nonzero member has Morse numbers $(2,3,3,2)$.*

Proof. Write $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ and put

$$F(z_1, z_2) = z_1 \left[1 + |z_2|^2 + \frac{1}{10}\operatorname{Re}\big((z_1\bar{z}_2)^2\big)\right], \qquad f = \operatorname{Re} F, \quad g = \operatorname{Im} F. \tag{3.8}$$

On $z_2 = 0$, $F = z_1$, so f, g are independent modulo constants. The bracket is positive and invariant under simultaneous phase multiplication. Hence

$$F(e^{i\theta}z_1, e^{i\theta}z_2) = e^{i\theta}F(z_1, z_2), \qquad f_\theta := \cos\theta f + \sin\theta g = f \circ R_{-\theta}, \tag{3.9}$$

where $R_\theta(z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2)$.

Step 1: the Morse calculation. Away from the coordinate circles use $t = |z_1|^2$, $\psi = \arg(z_1\bar{z}_2)$, and $\phi = \arg z_1$. With $\varepsilon = 1/10$,

$$f = \rho(t, \psi)\cos\phi, \qquad \rho = \sqrt{t}\,[2 - t + \varepsilon t(1-t)\cos 2\psi] > 0.$$

Interior critical points have $\phi = 0, \pi$ and $d\rho = 0$. The angular equation gives $s = \cos 2\psi \in \{1, -1\}$; the radial equation is

$$P_s(t) = 2 - 3t + \varepsilon s(3t - 5t^2) = 0.$$

For both signs, P_s is strictly decreasing on $[0, 1]$, positive at zero and negative at one. Let t_s denote its unique root. There,

$$\rho_{tt} = \frac{P_s'(t_s)}{2\sqrt{t_s}} < 0, \qquad \rho_{\psi\psi} = -4\varepsilon s t_s^{3/2}(1 - t_s), \qquad \rho_{t\psi} = 0.$$

Thus the base amplitude has two nondegenerate maxima ($\psi = 0, \pi$) and two saddles ($\psi = \pi/2, 3\pi/2$). Near $z_2 = 0$, with $\zeta = z_2/z_1 = x + iy$,

$$\rho = 1 + (1/2 + \varepsilon)x^2 + (1/2 - \varepsilon)y^2 + O(|\zeta|^4),$$

giving one base minimum. At $z_1 = 0$, $dF = 2\,dz_1$, so no height is critical there. At $\phi = 0, \pi$, respectively, the Hessians are

$$\operatorname{Hess}\rho \oplus (-\rho), \qquad -\operatorname{Hess}\rho \oplus \rho.$$

All are nonsingular. The base maxima give indices $3, 0$, the saddles $2, 1$, and the minimum $1, 2$, yielding $(2, 3, 3, 2)$. Equation (3.9) proves uniform Morseness.

Step 2: an odd nonlinear extension. Identify S^3 with unit quaternions $q = z_1 + z_2 j$. The function f is odd. For $a = (a_1, a_2, a_3) \in S^2$, define

$$u(a) = a_1 + ia_2 + ja_3, \qquad \Phi_a(q) = f\big(u(a)^{-1}q\big). \tag{3.10}$$

Every Φ_a is an isometric pullback of f, so is Morse, and $\Phi_{-a} = -\Phi_a$. On the equator $a = (\cos\theta, \sin\theta, 0)$, it is f_θ. The same formula works for every $u \in \mathrm{Sp}(1) = S^3$. In particular, a hemisphere gives a disk filling through Morse functions.

Step 3: the linear obstruction. The two local minima of f_θ trace the circles

$$L_\pm = \left\{e^{i\theta}(\sqrt{t_+},\ \pm\sqrt{1 - t_+}) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\right\}.$$

These are two distinct Hopf fibers, so $|\operatorname{lk}(L_+, L_-)| = 1$. Directly, their spanning complex lines are independent; an invertible complex-linear transformation followed by radial normalization sends the pair to the two coordinate circles, the standard Hopf link. Proposition 3.8 would force a zero linking number if a rank-three linear extension existed. This contradiction proves the theorem. □

Corollary 3.11 (Maximality and affine unavoidability)**.** *The system in Theorem 3.10 is maximal by inclusion although* $\mu_{\mathrm{lin}}(S^3) = 4$*. For every* $h \in C^\infty(S^3)$ *there are* $a, b \in \mathbb{R}$ *such that* $h + af + bg$ *is not Morse.*

Proof. Any larger containing system would contain a rank-three extension. If $[h] \in V$, some $h + af + bg$ is constant. Otherwise failure of uniform Morseness for $\mathrm{span}\{[f], [g], [h]\}$ supplies a non-Morse nonzero combination. Its h-coefficient cannot vanish, so divide by it. □

Corollary 3.12 (Deformation obstruction and an infinite family)**.** *The system of Theorem 3.10 cannot be deformed through admissible ordered pairs to a linearly extendible system. Moreover, for every* $k \geq 2$ *there is such a nonextendible system with Morse numbers* $(k, k+1, k+1, k)$*, still admitting an odd nonlinear extension.*

Proof. In a smooth deformation of admissible pairs, the minimum covering extends over $S^1 \times [0, 1]$. Its initially trivial monodromy stays trivial. Lemma 3.4 gives a smooth isotopy of the minimum link, preserving its linking numbers.

For the infinite family replace the square in (3.8) by the kth power, with a sufficiently small positive coefficient ε_k. The unperturbed amplitude $\rho_0(t) = \sqrt{t}(2 - t)$ has a nondegenerate radial maximum at $t = 2/3$ and is independent of ψ. The perturbation is

$$\varepsilon_k t^{(k+1)/2}(1 - t)^{k/2} \cos(k\psi).$$

Its angular critical equation is $\sin(k\psi) = 0$. The radial equation has one root near $2/3$ for each angular solution, with negative radial second derivative; for small ε_k it has no other interior roots. The angular second derivative alternates sign, producing k maxima and k saddles. The north minimum persists, and the south fiber remains regular. Hence the stated Morse counts hold. The k minimum circles are distinct Hopf fibers, so every pair links nontrivially. The same quaternionic formula gives an odd nonlinear extension, while Proposition 3.8 prohibits a linear one. □

A general Morse-family filling gives critical-point maps of disks into M; their images may intersect at different parameter values. A linear extension forces the *entire* minimum incidence to embed in M. The linked-minima example detects precisely this stronger geometric requirement, even though the original loop is nullhomotopic in $\mathcal{M}(S^3)$.

3.7 Related work

The basic language belongs to parametrized Morse theory: critical coverings and Hessian bundles are explicit in Igusa's framing principle [9]. Product maps of two functions and their discriminants are studied by Takao [13]. The sphere-bundle examples and the circular singular-value geometry are closely related to round fold maps [14, 11]; the degree data in Subsection 3.4 already have precedents in that classification. The projective collapse used above is described in [10, §16], and Proposition 3.9 uses the classical configuration-space fibration [12]. Linking also has established roles in Morse theory, including Usher's results for a single Morse function [15].

These results do not provide a general classification or existence theory for uniformly Morse linear systems.

4 Presymplectic dependency algebras

We study positive families of closed rank-two forms through their finite wedge algebras and geometric deletion and contraction.

4.1 Definitions

Throughout this section, manifolds are nonempty, smooth, connected and without boundary. A *closed manifold* is compact without boundary. Kernels of two-forms are taken as kernels of the maps $v \mapsto \iota_v\omega$; products of forms mean wedge products.

Definition 4.1 (Positive rank-two system)**.** Let E be a finite label set. A positive rank-two system on M is a tuple $\boldsymbol{\omega} = (\omega_e)_{e\in E}$ of closed two-forms such that:

(i) each ω_e has constant rank zero or two;

(ii) $K_S := \bigcap_{e\in S} \ker \omega_e$ has constant rank for every $S \subseteq E$, with $K_\varnothing = TM$;

(iii) positive sums introduce no additional kernel directions:

$$\ker\left(\sum_{e\in S} t_e\omega_e\right) = K_S \qquad (t_e > 0). \tag{4.1}$$

The system is *effective* if $K_E = 0$. Its *dependency algebra* is the unital graded real algebra

$$A_\omega := \mathbb{R}[\omega_e : e \in E] \subseteq \Omega^{\mathrm{even}}(M), \qquad \deg \omega_e = 2. \tag{4.2}$$

The coefficients in A_ω are constant real numbers, not arbitrary smooth functions. The full structured object retains the labelled generators and their realization as forms. An isomorphism of labelled systems is a diffeomorphism pulling back each labelled form to its counterpart.

Each rank-two form is pointwise decomposable, so $\omega_e^2 = 0$. Hence A_ω is finite-dimensional, spanned by the squarefree monomials

$$\omega_S := \bigwedge_{e\in S} \omega_e, \qquad \omega_\varnothing = 1, \qquad \dim_{\mathbb{R}} A_\omega \le 2^{|E|}.$$

If the system is effective, $\sum_e \omega_e$ is symplectic; in particular $\dim M$ is even. Zero generators are retained because restriction can create them.

4.2 Matroid structure and foliations

Proposition 4.2 (Matroid rank)**.** *The function*

$$r(S) = \tfrac{1}{2} \operatorname{codim} K_S$$

is the rank function of an ordinary matroid on E. Furthermore,

$$r(S) = |S| \quad \Longleftrightarrow \quad \omega_S(p) \neq 0 \text{ for every } p \in M. \tag{4.3}$$

Proof. At p, put $U_e = (\ker \omega_e(p))^\circ \subseteq T_p^*M$ and $U_S = \sum_{e\in S} U_e$. Then $U_S = K_S(p)^\circ$. By (4.1), $\dim U_S$ is the rank of a skew form and is even. Thus $r(S)$ is an integer, independent of p. Normalization and monotonicity are immediate; adding one U_e increases dimension by at most two. Also

$$\dim U_A + \dim U_B = \dim U_{A\cup B} + \dim(U_A \cap U_B) \ge \dim U_{A\cup B} + \dim U_{A\cap B}.$$

After division by two these are the matroid rank axioms. Finally, a squarefree wedge is nonzero exactly when its two-dimensional support planes sum directly, equivalently when $r(S) = |S|$. □

A subset satisfying (4.3) is *independent*; a *basis* of S is an independent subset of S of size $r(S)$. Pointwise, the U_e form a multilinear representation of the matroid; this linear-algebraic notion is established in the literature [16].

Lemma 4.3 (Coherent orientation of bases)**.** *For any $U \subseteq E$, orient $T_pM/K_U(p)$ by the symplectic form $\Omega_U = \sum_{e\in U} \omega_e$. For every basis B of U, ω_B is positive in this orientation.*

Proof. The path $\Omega_B + s\sum_{e\in U\setminus B} \omega_e$, $0 \le s \le 1$, is nondegenerate on $T_pM/K_U(p)$ by (4.1), since $K_B = K_U$. Its orientation is constant, and $\Omega_B^{r(U)} = r(U)!\omega_B$. □

Proposition 4.4 (Foliations and basis coordinates)**.** *Every K_S is integrable. If the system is effective on a $2n$-manifold and $B = \{b_1, \dots, b_n\}$ is a basis, then locally there are coordinates $(p_1, q_1, \dots, p_n, q_n)$ such that*

$$\omega_{b_i} = dp_i \wedge dq_i \qquad (1 \le i \le n).$$

Proof. The closed form Ω_S has constant-rank kernel K_S. For $X, Y \in \Gamma(K_S)$, Cartan's formula gives $\mathcal{L}_X\Omega_S = 0$ and then $\iota_{[X,Y]}\Omega_S = 0$. Frobenius applies.

Each nonzero ω_{b_i} descends locally to an area form on a surface transverse to its kernel: it annihilates the kernel and is invariant along it. Choose area coordinates p_i, q_i on that surface. By independence, the resulting $2n$ differentials form a coframe, so the combined functions are coordinates. □

4.3 Deletion and contraction

Deleting labels preserves Definition 4.1; algebraically it takes the subalgebra generated by the retained forms. Contraction uses characteristic leaves, rather than an arbitrary submanifold.

Theorem 4.5 (Hereditary compatibility and geometric contraction)**.** *For $S, T \subseteq E$ and $t_e > 0$,*

$$\ker\left(\sum_{e\in T} t_e\omega_e\Big|_{K_S}\right) = K_{S\cup T}. \tag{4.4}$$

Consequently, on every connected leaf L of K_S, the forms $\omega_e|_L$, $e \in E \setminus S$, form a positive rank-two system with rank function

$$r/S(T) = r(S \cup T) - r(S), \qquad T \subseteq E \setminus S. \tag{4.5}$$

This is matroid contraction.

Proof. Put $U = S \cup T$, $s = r(S)$ and $q = r(U) - s$, and choose a basis B of S. Write $\beta = \omega_B$ and $\eta = \sum_{e\in T} t_e\omega_e$. All these forms descend to TM/K_U. If $q > 0$, expand $\beta \wedge \eta^q$. Its nonzero terms are positive multiples of basis wedges for U, by Lemma 4.3. At least one term is nonzero because B extends to a basis of U using labels in T. Hence $\beta \wedge \eta^q \neq 0$.

Since β is a nonzero top form on the annihilator of K_S, this implies $(\eta|_{K_S})^q \neq 0$. Meanwhile K_U is contained in the kernel of $\eta|_{K_S}$ and has codimension $2q$ there. Equality follows. For $q = 0$, $K_U = K_S$ and the restriction is zero.

Taking T to be a singleton proves that each restricted generator has constant rank zero or two and kernel $K_{S\cup\{e\}}$. The general case proves positive compatibility and the common-kernel rank formula. Closedness is preserved by restriction to a leaf. □

Theorem 4.6 (Algebraic contraction)**.** *Choose a basis B_S of S and set $\beta_S = \omega_{B_S}$. The image of A_ω under restriction to the distribution K_S is canonically isomorphic to*

$$A_{\omega/S} := A_\omega / \operatorname{Ann}_{A_\omega}(\beta_S), \qquad \operatorname{Ann}_{A_\omega}(\beta_S) = \{a \in A_\omega : \beta_S \wedge a = 0\}. \tag{4.6}$$

The ideal in (4.6) *is independent of the chosen basis B_S.*

Proof. Pointwise, for any exterior form α,

$$\beta_S \wedge \alpha = 0 \quad \Longleftrightarrow \quad \alpha|_{K_S} = 0.$$

Indeed, choose a cotangent splitting with K_S° as one summand. Wedging by its top form kills exactly the components containing a factor in K_S°; the remaining component is the restriction to K_S. Thus the kernel of the algebra homomorphism

$$A_\omega \longrightarrow \Gamma(\Lambda^{\text{even}} K_S^*)$$

is $\mathrm{Ann}_{A_\omega}(\beta_S)$. Different basis wedges are sections of the same line bundle and differ by a nowhere-zero smooth function, so they have the same annihilator. □

Remark 4.7. The quotient describes the *image along the whole foliation*, not the full algebra of leafwise forms. Restriction to a single leaf can have additional kernel. The rank formula (4.5), however, holds on every leaf. Multiplication by β_S identifies the quotient with $\beta_S A_\omega$ only as a shifted vector space, not as an algebra.

Products supply a second elementary construction. On $M \times N$, combine pullbacks of two systems. Their kernels split, so (4.1) holds, ranks add, and the matroid is the direct sum. Their algebras satisfy

$$A_{M\times N} \cong A_M \otimes_{\mathbb{R}} A_N.$$

The external-product map is injective: evaluations of forms on tangent multivectors separate any finite-dimensional space of forms, so applying such evaluations on one factor detects every coefficient on the other.

4.4 Cohomology and volume

Throughout this subsection assume that M is closed and the system is effective, with $\dim M = 2n$. Orient M by $\sum_e \omega_e$.

Theorem 4.8 (Nonvanishing of independent monomials)**.** *For every $S \subseteq E$,*

$$\prod_{e\in S} [\omega_e] \neq 0 \quad \Longleftrightarrow \quad S \text{ is independent.} \tag{4.7}$$

The classes are in real de Rham cohomology.

Proof. For dependent S, $\omega_S = 0$ by Proposition 4.2. For independent S, extend it to a basis B of E. Lemma 4.3 gives $\int_M \omega_B > 0$. If $\omega_S = d\alpha$, the remaining generators are closed, so $\omega_B = d(\alpha \wedge \omega_{B\setminus S})$, contradicting Stokes' theorem. □

Corollary 4.9 (Obstructions)**.** *The following assertions hold.*

(i) $b_{2k}(M) \geq \binom{n}{k}$ *for $0 \leq k \leq n$. In particular $b_2(M) \geq n$, and each nonzero generator determines a nonzero square-zero class in $H^2(M;\mathbb{R})$.*

(ii) *If $n > 1$, $\mathbb{CP}^n$ admits no effective system.*

(iii) *A system of three pairwise independent generators on a closed four-manifold requires $b_2(M) \geq 3$. In particular, $S^2 \times S^2$ admits no such triple.*

Proof. For (i), fix a basis B. In a relation among $[\omega_S]$ with $S \subseteq B$ and $|S| = k$, wedge with $\omega_{B\setminus S_0}$. Every term except that indexed by S_0 contains a squared generator and vanishes. The positive top integral forces the remaining coefficient to be zero. For (ii), every nonzero class in $H^2(\mathbb{CP}^n;\mathbb{R})$ has nonzero square. For (iii), the intersection matrix of the three classes is

$$\begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}, \qquad a, b, c > 0,$$

whose determinant is $2abc > 0$. Their classes are linearly independent. □

The associated volume polynomial is

$$P_\omega(t) = \frac{1}{n!}\int_M \left(\sum_e t_e\omega_e\right)^n = \sum_{B \text{ a basis}} c_B \prod_{e\in B} t_e, \qquad c_B = \int_M \omega_B > 0. \tag{4.8}$$

Thus it is multiaffine, with support exactly the matroid bases. Theorem 4.8 is a statement about monomials; the algebra map $A_\omega \to H^{\text{even}}(M;\mathbb{R})$ need not be injective on their linear combinations.

4.5 Examples and separation

Product and linear models

Example 4.10 (Products of surfaces)**.** Let $M = \Sigma_1 \times \cdots \times \Sigma_n$, with oriented surfaces carrying area forms σ_i. The forms $\omega_i = \pi_i^*\sigma_i$ give an effective system with every subset independent and

$$A_\omega \cong \mathbb{R}[x_1,\dots,x_n]/(x_1^2,\dots,x_n^2), \qquad \deg x_i = 2.$$

The squarefree products are linearly independent by their distinct degrees in the individual factors.

Example 4.11 (Compact realizations of complex-representable matroids)**.** Let $(c_e)_{e\in E}$ be a rank-n vector configuration in $\mathbb{C}^n$. On $\mathbb{T}^{2n}$ define constant complex one-forms and real two-forms by

$$\alpha_e = \sum_{j=1}^n c_{ej}(dx_j + i\,dy_j), \qquad \omega_e = \frac{i}{2}\alpha_e \wedge \overline{\alpha_e}.$$

The system is effective and its matroid is that of the vectors c_e. Indeed, for the standard complex structure J,

$$\left(\sum_e t_e\omega_e\right)(v, Jv) = \sum_e t_e|\alpha_e(v)|^2,$$

so (4.1) holds; common complex kernels have the required real codimensions. Thus every complex-representable matroid has a compact model. No rationality assumption on the coefficients is required.

A four-generator torus family

On $\mathbb{T}^4 = (\mathbb{R}/\mathbb{Z})^4$, use coordinates x, y, u, v and their globally defined angular one-forms. Set

$$V = dx\wedge dy\wedge du\wedge dv, \qquad a_\varepsilon(u) = 2 + \varepsilon\sin(2\pi u), \qquad 0 \le \varepsilon < 1,$$

and define

$$\begin{aligned} \omega_1 &= dx\wedge dy, & \omega_2 &= du\wedge dv,\\ \omega_3 &= (dx+du)\wedge(dy+dv), & \omega_4 &= (dx + a_\varepsilon du)\wedge(dy + 4\,dv). \end{aligned} \tag{4.9}$$

All four forms are closed and of rank two. Writing $a = a_\varepsilon$, their products are

(i,j)	$(1,2)$	$(1,3)$	$(1,4)$	$(2,3)$	$(2,4)$	$(3,4)$
$(\omega_i\wedge\omega_j)/V$	1	1	$4a$	1	1	$3(a-1)$

Every coefficient is positive. Hence every positive sum involving at least two labels is symplectic, while a singleton has rank two. The system is effective and has matroid $U_{2,4}$: precisely the subsets of size at most two are independent.

Theorem 4.12 (Same cohomology data, different finite algebras)**.** *For the family* (4.9)*, all labelled de Rham classes, the matroid, and the volume polynomial are independent of* ε*. Nevertheless, the algebra dimensions in form degrees* $0, 2, 4$ *are*

	$\dim A^0$	$\dim A^2$	$\dim A^4$
$\varepsilon = 0$	1	4	1
$0 < \varepsilon < 1$	1	4	2

.

Consequently the constant system is not isomorphic to any nonconstant member, even after relabelling.

Proof. Only ω_4 changes. For $F(u) = -\varepsilon \cos(2\pi u)/(2\pi)$,

$$\omega_4^\varepsilon - \omega_4^0 = (a_\varepsilon - 2)du \wedge (dy + 4\,dv) = d\big(F(u)(dy + 4\,dv)\big).$$

Thus the labelled classes agree, and (4.8) gives the same volume polynomial.

The four degree-two generators are linearly independent: the coefficients of $dx \wedge dv$ and $du \wedge dy$ in a relation give $c_3 + 4c_4 = 0$ and $c_3 + ac_4 = 0$. Since $a \neq 4$, $c_3 = c_4 = 0$, followed by $c_1 = c_2 = 0$. The product table gives

$$A^4 = \operatorname{span}_{\mathbb{R}}\{V, aV\},$$

which has dimension one exactly when a is constant. There are no higher nonzero degrees. Any isomorphism of systems induces an isomorphism of these algebras. □

Proposition 4.13 (Local moduli and absence of a common complex structure)**.** *For* (4.9)*:*

(i) *where* $da \neq 0$*, no coordinates make all four forms constant simultaneously;*

(ii) *no almost complex structure preserves all four kernels. In particular the forms cannot all be of type* $(1, 1)$ *for a common almost complex structure.*

Proof. For (i), the intrinsic ratio of top forms is

$$\frac{\omega_1 \wedge \omega_4}{\omega_1 \wedge \omega_2} = 4a(u).$$

It would be constant in any simultaneous constant-coefficient model.

For (ii), write $P = \ker \omega_2 = \operatorname{span}_{\mathbb{R}}\{\partial_x, \partial_y\}$ and $Q = \ker \omega_1 = \operatorname{span}_{\mathbb{R}}\{\partial_u, \partial_v\}$. The remaining kernels are graphs $Q \to P$ of $-I$ and $-D$, where $D = \operatorname{diag}(a, 4)$. A complex structure preserving P, Q is block diagonal. Preservation of the graph of $-I$ identifies its two blocks; preservation of the graph of $-D$ forces the common block to commute with D. Since $a \neq 4$, that block is real diagonal, so it cannot square to $-I$. A $(1, 1)$-form has a complex-invariant kernel. □

Remark 4.14 (Scope of the separation). Theorem 4.12 distinguishes chosen structures on the same manifold; it does not yet distinguish underlying manifolds with the same homotopy type. The abstract algebra also does not classify all local moduli: for nonconstant a the displayed multiplication table, expressed using V, aV, does not record the full function a.

4.6 Related work

The pointwise support planes form 2-arrangements, hence multilinear matroid representations [16]. Complementary two-form examples in dimension four are symplectic pairs [17]. An effective tuple, viewed as an $\mathbb{R}^E$-valued two-form, is polysymplectic [18]; its kernel foliations also connect it to symplectic webs [19].

The finite wedge algebra is retained before passage to cohomology; positive compatibility supplies hereditary restriction and annihilator contraction along characteristic foliations. The raw algebra is an invariant of the structured manifold and may jump under small deformations, as Theorem 4.12 shows.

5 Diffusion transport defects

We study the rank defects of connection heat kernels and coverings by the open sets on which these kernels trivialize a bundle.

5.1 Heat kernels and defects

Throughout this section, (M^n, g) is a smooth, closed, connected Riemannian manifold, and $E \to M$ is a smooth bundle of rank $r \geq 1$ over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, equipped with a Euclidean or Hermitian metric and a compatible connection ∇. The connection Laplacian is

$$L = \nabla^* \nabla \geq 0.$$

Its heat kernel, normalized against Riemannian volume, is

$$H_t(x,p) \in \mathrm{Hom}(E_p, E_x), \qquad (e^{-tL}s)(x) = \int_M H_t(x,p)s(p)\, d\,\mathrm{vol}_g(p), \qquad t > 0.$$

Thus a source vector $v \in E_p$ produces the smooth section $x \mapsto H_t(x,p)v$.

Definition 5.1 (Defect correspondence)**.** For $1 \leq k \leq r$, define

$$\mathcal{D}^{(k)} = \{(t,x,p) \in (0,\infty) \times M \times M : \dim_{\mathbb{F}} \ker H_t(x,p) \geq k\}.$$

Write $\mathcal{D}_t^{(k)} = \{(x,p) : (t,x,p) \in \mathcal{D}^{(k)}\}$ and $D_t(p) = \{x : (x,p) \in \mathcal{D}_t^{(1)}\}$. The associated lost-direction incidence is

$$\mathcal{I} = \{(t,x,p,[v]) : [v] \in \mathbb{P}(E_p),\ H_t(x,p)v = 0\}.$$

Here $\mathbb{P}$ denotes real or complex projectivization as appropriate.

Definition 5.2 (Heat covering number)**.** Set

$$U_t(p) = M \setminus D_t(p) = \{x : H_t(x,p) \text{ is invertible}\},$$

and define

$$\mathfrak{h}_t(M,g,E,\nabla) = \min\left\{ q \geq 1 : \exists p_1, \ldots, p_q \in M, \quad M = \bigcup_{j=1}^{q} U_t(p_j) \right\}. \tag{5.1}$$

When the data are fixed, write simply $\mathfrak{h}_t$.

Finiteness is proved below. The sets $U_t(p)$ are domains of bundle trivializations, not necessarily coordinate neighborhoods. The loci $\mathcal{D}^{(k)}$ are closed in positive time; no smoothness or Whitney stratification is assumed. On any smooth submanifold on which the rank is constant, the kernels form a smooth vector bundle, and $\mathcal{I}$ restricts to its projectivization.

5.2 Basic properties

Proposition 5.3 (Symmetry, positivity, and invariance)**.** *For every $t > 0$,*

$$H_t(x,p)^* = H_t(p,x), \qquad H_t(p,p) > 0.$$

Consequently, $\mathcal{D}_t^{(k)}$ is invariant under $(x,p) \mapsto (p,x)$ and is disjoint from the diagonal. These loci and $\mathfrak{h}_t$ are invariant under metric-preserving bundle isomorphisms intertwining the connections and covering isometries of the base.

Proof. Let (ϕ_j) be an orthonormal eigenbasis of L, with eigenvalues $\lambda_j \geq 0$. The smooth spectral expansion is

$$H_t(x,p) = \sum_j e^{-t\lambda_j} \phi_j(x)\phi_j(p)^*.$$

It gives symmetry and

$$\langle H_t(p,p)v, v\rangle = \sum_j e^{-t\lambda_j} |\langle \phi_j(p), v\rangle|^2.$$

If this vanishes, v is orthogonal to every $\phi_j(p)$ and hence to every smooth section evaluated at p, since smooth spectral expansions converge in C^∞. Thus $v = 0$. Covariance of L gives covariance of its heat kernel, so ranks and covering numbers are preserved. □

Proposition 5.4 (Trivialization and topological obstruction)**.** *For all $t > 0$, $1 \leq \mathfrak{h}_t < \infty$. Each $U_t(p)$ trivializes E. In particular, if E is nontrivial, then $D_t(p) \neq \varnothing$ for every p and t.*

More generally, let $\gamma_1(E), \ldots, \gamma_q(E)$ be positive-degree characteristic classes, with coefficients in a fixed commutative ring, each vanishing on a trivial bundle. If

$$\gamma_1(E) \smile \cdots \smile \gamma_q(E) \neq 0, \qquad \textit{then} \qquad \mathfrak{h}_t \geq q+1.$$

Proof. The map $(x,v) \mapsto H_t(x,p)v$ identifies $U_t(p) \times E_p$ with $E|_{U_t(p)}$. By Proposition 5.3, $p \in U_t(p)$; these open sets cover M, and compactness gives a finite subcover. A cover by one such set would trivialize E globally.

If q heat domains covered M, each $\gamma_i(E)$ would restrict to zero on the ith domain and would lift to relative cohomology for that domain. Their relative cup product lies in $H^*(M,M) = 0$, contradicting the displayed nonzero product. A cover by fewer domains can be padded by repetitions. This is the usual cup-product obstruction to trivializing covers; see [5]. □

Proposition 5.5 (Stability and rescaling)**.** *The integer-valued function $t \mapsto \mathfrak{h}_t$ is upper semicontinuous on $(0,\infty)$. The same assertion holds along smooth families of metrics and compatible connections on a fixed bundle. For a constant $s > 0$,*

$$H_t^{\nabla, sg}(x,p) = s^{-n/2} H_{t/s}^{\nabla, g}(x,p), \qquad \mathfrak{h}_t(M, sg, E, \nabla) = \mathfrak{h}_{t/s}(M, g, E, \nabla). \tag{5.2}$$

Proof. Choose a covering tuple $p_1, \ldots, p_q$ at t_0. Compactness gives

$$\min_{x \in M} \max_{1 \leq i \leq q} \sigma_{\min}(H_{t_0}(x, p_i)) > 0.$$

Continuity of the heat kernel preserves this inequality nearby, so $\mathfrak{h}_t \leq \mathfrak{h}_{t_0}$. Smooth parameter dependence of heat kernels gives the family statement by the same argument. Finally, $L_{sg} = s^{-1} L_g$ and $d\,\mathrm{vol}_{sg} = s^{n/2} d\,\mathrm{vol}_g$, which prove (5.2). □

Theorem 5.6 (Small-time localization)**.** *Fix $p \in M$. For every open neighborhood V of $\mathrm{Cut}(p)$, there is $\varepsilon > 0$ such that*

$$D_t(p) \subset V \qquad (0 < t < \varepsilon).$$

Equivalently, if $t_j \downarrow 0$, $x_j \in D_{t_j}(p)$, and $x_j \to x$, then $x \in \mathrm{Cut}(p)$.

Proof. On a compact set $K \subset M \setminus \mathrm{Cut}(p)$, the uniform relative heat-kernel expansion [20, Theorem 1.1] gives

$$H_t(x,p) = (4\pi t)^{-n/2} e^{-d(x,p)^2/(4t)} \big(a(x,p) P_{x \leftarrow p} + O_K(t)\big), \tag{5.3}$$

where $a > 0$ and $P_{x \leftarrow p}$ is parallel transport along the unique minimizing geodesic. The leading map is uniformly invertible on K. Thus $H_t(x,p)$ is invertible there for sufficiently small t. Apply this to $K = M \setminus V$. □

This is a one-sided localization statement: it does not assert that every cut point is a limit of defects.

5.3 Examples

Example 5.7 (A global parallel frame)**.** If E admits a global parallel orthonormal frame, then

$$H_t(x,p) = h_t(x,p)I_r$$

in that frame, where $h_t > 0$ is the scalar heat kernel. Hence $\mathcal{D}^{(1)} = \varnothing$ and $\mathfrak{h}_t = 1$ for all $t > 0$.

Proposition 5.8 (Rectangular tori with sign holonomy)**.** *On $\mathbb{T}^m = (\mathbb{R}/2\pi\mathbb{Z})^m$ with the product flat metric, consider the trivial complex line bundle with*

$$\nabla = d + \frac{i}{2}\sum_{j=1}^{k} d\theta_j, \qquad 0 \le k \le m.$$

Then, for every $t > 0$,

$$U_t(p) = \prod_{j=1}^{k} (S^1 \setminus \{p_j + \pi\}) \times (S^1)^{m-k}, \qquad \mathfrak{h}_t = k + 1. \tag{5.4}$$

Proof. For one twisted circle, a kernel representative on the universal cover, up to a nowhere-zero phase, is

$$k_t^-(u) = \frac{1}{\sqrt{4\pi t}} \sum_{\ell\in\mathbb{Z}} (-1)^\ell e^{-(u+2\pi\ell)^2/(4t)}.$$

On $(-\pi,\pi)$ this is the Dirichlet heat kernel of that interval with source 0, by the method of images. It is strictly positive in the interior and vanishes at the endpoints. Its only zero modulo 2π is therefore the antipode. The product heat kernel gives the first formula in (5.4).

Given $q \le k$ sources $p^{(1)}, \dots, p^{(q)}$, choose a target with $\theta_j = p_j^{(j)} + \pi$ for $1 \le j \le q$. It lies in none of their heat domains. Conversely, choose $k+1$ sources whose first k coordinates are $(a_\ell, \dots, a_\ell)$, with all a_ℓ distinct. Each coordinate excludes at most one source, so the k coordinates cannot exclude all $k+1$. If $k = 0$, the kernel is everywhere nonzero. □

Thus even a topologically trivial line bundle with zero curvature can have $\mathfrak{h}_t > 1$. Holonomy affects the invariant.

Example 5.9 (Antipodal defects on the round sphere)**.** For the Levi-Civita connection on TS^2, with the standard round metric,

$$H_t(-p,p) = 0 \qquad (t > 0).$$

Indeed, regard the oriented tangent planes as complex lines, with $J_p v = p \times v$. The connection Laplacian commutes with J, so its heat kernel is complex linear. Rotations fixing p and $-p$ act on these two complex tangent lines by opposite characters $e^{i\theta}$ and $e^{-i\theta}$. Equivariance forces their complex-linear intertwiner $H_t(-p,p)$ to vanish. This argument does not determine the entire zero set.

5.4 Line bundles on surfaces

Let Σ be a closed, connected, oriented surface, and let $L \to \Sigma$ be a Hermitian line bundle with a unitary connection. Put $d = \langle c_1(L), [\Sigma]\rangle$.

Proposition 5.10 (Fixed-source index formula)**.** *Fix $t > 0$, $p \in \Sigma$, and $0 \ne v \in L_p$. If the zeros of $s(x) = H_t(x,p)v$ are isolated, then*

$$\sum_{x : s(x) = 0} \operatorname{ind}_x(s) = d.$$

If all zeros are transverse, there are at least $|d|$ of them. For $L = T\Sigma$ with its oriented complex-line structure and Levi-Civita connection, the sum equals $\chi(\Sigma)$.

Proof. The local index is the winding number of the local section around a zero. The sum of these indices is the Euler number of the underlying oriented real plane bundle, which is its first Chern number [5]. A transverse zero has index $+1$ or -1. For the tangent bundle, $c_1(T\Sigma)[\Sigma] = \chi(\Sigma)$. □

Theorem 5.11 (The two-point defect cycle)**.** *Suppose, at a fixed $t > 0$, that H_t is transverse to the zero section as a section of*

$$F = \pi_1^* L \otimes \pi_2^* L^* \longrightarrow \Sigma \times \Sigma.$$

Then $\mathcal{D}_t^{(1)}$ is a smooth closed oriented surface, possibly empty, disjoint from the diagonal. Moreover,

$$\mathrm{PD}[\mathcal{D}_t^{(1)}] = \pi_1^* c_1(L) - \pi_2^* c_1(L), \tag{5.5}$$

$$[\mathcal{D}_t^{(1)}] \cdot [\mathcal{D}_t^{(1)}] = -2d^2. \tag{5.6}$$

Interchanging the two factors restricts to a free orientation-reversing involution on the defect surface.

Proof. A transverse zero set of a complex line-bundle section represents its first Chern class [5]. Since $c_1(F) = \pi_1^* c_1(L) - \pi_2^* c_1(L)$, this gives (5.5). Squaring and integrating over the product gives (5.6); the squares of classes pulled back from either surface vanish, and the mixed product integrates to d^2.

The heat-kernel adjoint identity preserves the zero set. Factor interchange preserves the orientation of the four-dimensional ambient product but induces complex conjugation on the normal complex line, reversing its orientation. It therefore reverses the induced orientation of the zero surface. It has no fixed points there by diagonal positivity. □

Transversality is a hypothesis, not a genericity result proved here. A general smooth heat-kernel section is not holomorphic; its local indices need not be positive.

5.5 A transition on a flat torus

The following example changes with t while the metric, bundle, and connection remain fixed.

Theorem 5.12 (A two-dimensional transition)**.** *Let*

$$a = (1, 0), \quad b = (1/2, \sqrt{3}/2), \quad \Lambda = \mathbb{Z}a + \mathbb{Z}b, \quad M_0 = \mathbb{R}^2/\Lambda,$$

with the Euclidean quotient metric g_0. Write $x = q_1 a + q_2 b$, with $q_j \in \mathbb{R}/\mathbb{Z}$, and equip the trivial Hermitian line bundle with

$$\nabla_0 = d + 2\pi i \left(\frac{5}{12}\, dq_1 - \frac{1}{3}\, dq_2 \right). \tag{5.7}$$

There exists $\varepsilon \in (0, 1)$ such that

$$\mathfrak{h}_t = 2 \quad (0 < t < \varepsilon), \qquad \mathfrak{h}_t = 1 \quad (t \geq 1).$$

For each source and every sufficiently small t, the kernel has exactly two simple zeros, of opposite local index.

Proof. **1. Two useful representations.** Put $\alpha = (5/12, -1/3)$ and

$$\rho(ma + nb) = e^{2\pi i(m\alpha_1 + n\alpha_2)}.$$

The image sum

$$K_t(x) = \frac{1}{4\pi t} \sum_{\lambda \in \Lambda} \rho(\lambda) e^{-|x-\lambda|^2/(4t)} \tag{5.8}$$

obeys $K_t(x+\lambda) = \rho(\lambda)K_t(x)$. In the global periodic frame,

$$H_t(x,0) = e^{-2\pi i\alpha\cdot q}K_t(x).$$

This follows either by gauging the Euclidean heat equation or by Fourier expansion. In particular, K_t and $H_t(\cdot,0)$ have the same zeros and indices. Translation invariance gives $H_t(x,p) = H_t(x-p,0)$ in this frame. Also,

$$K_t(-x) = \overline{K_t(x)}. \tag{5.9}$$

2. A nondegenerate small-time cancellation. At the Voronoi vertex $v = (a+b)/3$, the three nearest lattice points are $0, a, b$. Their phases are

$$A = \rho(a) = -\frac{\sqrt{3}}{2} + \frac{i}{2}, \qquad B = \rho(b) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i,$$

and satisfy

$$1 + \frac{\sqrt{3}}{2}A + \frac{1}{2}B = 0. \tag{5.10}$$

For $x = v + tz$, divide (5.8) by $(4\pi t)^{-1}e^{-|x|^2/(4t)}$. On every bounded z-set the result is

$$F(z) + R_t(z), \qquad F(z) = 1 + Ae^{a\cdot z/2} + Be^{b\cdot z/2}, \qquad \|R_t\|_{C^1} = O(e^{-c/t}) \tag{5.11}$$

for some $c > 0$. The error estimate follows from the positive squared-distance gap to all other lattice points and the quadratic growth of lattice distances.

Since a, b are independent, (5.10) gives the unique zero of F:

$$e^{a\cdot z_0/2} = \frac{\sqrt{3}}{2}, \qquad e^{b\cdot z_0/2} = \frac{1}{2}, \qquad z_0 = \left(\log\frac{3}{4}, -\frac{\log 12}{\sqrt{3}}\right).$$

With the usual orientations of $\mathbb{R}^2$ and $\mathbb{C}$,

$$\det_{\mathbb{R}} DF(z_0) = \frac{3}{32} > 0.$$

The implicit function theorem applied to (5.11) gives a simple zero

$$x_+(t) = v + tz_0 + O(te^{-c/t}). \tag{5.12}$$

The exponentially small remainder extends smoothly to $t = 0$ on bounded z-sets, so the usual parameter version of that theorem applies. Equation (5.9) gives a second zero $x_-(t) = -x_+(t)$ modulo Λ, with opposite index.

3. Exclusion of all other small-time zeros. There are precisely two Voronoi-vertex classes modulo Λ, represented by v and $-v$. Away from small neighborhoods of these vertices, a finite collection of neighborhoods has either one leading lattice point or two leading adjacent lattice points, with a uniform squared-distance gap to all the others. The ratios of phases for adjacent points belong to

$$\{\rho(a), \rho(b), \rho(a-b), \rho(-a), \rho(-b), \rho(b-a)\}.$$

None is -1. Therefore, for each relevant pair of unit phases u, w, there is $c_0 > 0$ such that

$$|ru + sw| \geq c_0(r+s) \qquad (r, s \geq 0).$$

The single leading term or the sum of the two leading terms consequently dominates the exponentially smaller tail. There are no zeros in these neighborhoods for sufficiently small t.

For completeness, uniqueness near v requires control beyond bounded rescaled coordinates. In a sufficiently small fixed neighborhood of v, put

$$w_0 = e^{-|x|^2/(4t)}, \quad w_a = e^{-|x-a|^2/(4t)}, \quad w_b = e^{-|x-b|^2/(4t)}, \quad W = w_0 + w_a + w_b.$$

The sum of all omitted absolute values is at most $Ce^{-c/t}W$, uniformly in that neighborhood. This again follows from a uniform distance gap, together with the convergent Gaussian tail. Hence, at any zero,

$$|q_0 + Aq_a + Bq_b| \le Ce^{-c/t}, \qquad (q_0, q_a, q_b) = W^{-1}(w_0, w_a, w_b).$$

The affine map from $q_0 + q_a + q_b = 1$ to $q_0 + Aq_a + Bq_b \in \mathbb{C}$ is an isomorphism. Its unique zero is the interior point proportional to $(1, \sqrt{3}/2, 1/2)$. Thus (q_0, q_a, q_b) is exponentially close to that point. Taking logarithms of the two weight ratios forces

$$\frac{x - v}{t} = z_0 + O(e^{-c/t}).$$

Every zero near v therefore lies in the neighborhood where the implicit-function zero is unique. The same argument applies near $-v$. This proves that there are exactly two simple zeros.

4. Large-time nonvanishing. In the global periodic frame, the eigenfunctions are Fourier characters and the eigenvalues are

$$\lambda_k = 4\pi^2 Q(k + \alpha), \qquad Q(u, v) = \frac{4}{3}(u^2 - uv + v^2), \qquad k \in \mathbb{Z}^2.$$

Let $k_0 = (-1, 0)$. For $\ell = (u, v) \in \mathbb{Z}^2$ and $B_0 = u^2 - uv + v^2$, direct subtraction gives

$$\lambda_{k_0+\ell} - \lambda_{k_0} = \frac{4\pi^2}{9}(12B_0 - 10u - v) \ge \frac{2\pi^2}{9}(u^2 + v^2). \tag{5.13}$$

Indeed, for integer u, v, $B_0 \ge \max(|u|, |v|)$, and always $B_0 \ge (u^2 + v^2)/2$. For the first inequality, use $B_0 \ge 3u^2/4$, with the cases $|u| \le 1$ checked directly, and likewise for v.

Thus k_0 is the unique ground frequency. Factoring its nonzero contribution out of the spectral expansion leaves $1 + S_t(q)$, where

$$\sup_q |S_t(q)| \le \sum_{(u,v)\neq(0,0)} e^{-(2\pi^2/9)t(u^2+v^2)}.$$

For $t \ge 1$, put $r = e^{-2\pi^2/9} < 1/8$. Since $u^2 \ge |u|$ for integers u,

$$\sup_q |S_t(q)| \le \left(\sum_{u\in\mathbb{Z}} e^{-(2\pi^2/9)u^2}\right)^2 - 1 < \left(\frac{9}{7}\right)^2 - 1 = \frac{32}{49} < 1.$$

Consequently, the heat kernel is nowhere zero for every $t \ge 1$.

5. Covering numbers. At small times, let $Z_t = \{x_+(t), x_-(t)\}$. Every source has the nonempty defect set $p + Z_t$, so one source cannot cover. Choose $p_2 - p_1 \notin Z_t - Z_t$; then the two defect sets are disjoint, and their complementary heat domains cover M_0. Thus $\mathfrak{h}_t = 2$. Large-time nonvanishing gives $\mathfrak{h}_t = 1$. □

5.6 Products and covering levels

Lemma 5.13 (Finite defects in product factors)**.** *Consider a product of flat tori, trivial Hermitian line bundles with constant unitary connections, and the product connection and metric. Fix a time. Suppose that exactly k factors have nonempty finite fixed-source zero sets, and every other factor has a nowhere-zero kernel. Then the product satisfies $\mathfrak{h}_t = k + 1$.*

Proof. The product heat kernel is the product of the factor kernels. For each of $q \le k$ proposed sources, assign a different defective factor and choose that target coordinate to be a zero for the assigned source. The resulting target is missed by all q heat domains.

For the upper bound, choose $k+1$ source coordinates in each defective factor so that the corresponding translates of its finite zero set are pairwise disjoint. This is possible by successively avoiding finitely many differences. A given target coordinate then blocks at most one source in each defective factor. The k factors cannot block all $k+1$ sources. For $k=0$, one source suffices. □

Theorem 5.14 (Arbitrarily many distinct levels)**.** *For each $N \ge 1$, there is a fixed flat torus of dimension $2N$, with a fixed flat unitary connection on a trivial complex line bundle, and nonempty ordered open intervals*

$$I_0 < I_1 < \cdots < I_N$$

such that

$$\mathfrak{h}_t = N + 1 - j \qquad (t \in I_j,\ 0 \le j \le N).$$

The ordering means that every time in I_j precedes every time in I_{j+1}.

Proof. Take N copies of Theorem 5.12, with metrics $s_j g_0$, and the exterior tensor-product line bundle and connection. Choose $s_1 > 0$ and

$$s_{j+1} > s_j/\varepsilon \qquad (1 \le j < N).$$

The effective time in factor j is t/s_j, by (5.2). Define

$$I_0 = (0, \varepsilon s_1), \qquad I_j = (s_j, \varepsilon s_{j+1}) \quad (1 \le j < N), \qquad I_N = (s_N, \infty).$$

On I_j, the first j factors are in their nowhere-zero regime, and the remaining $N-j$ factors each have two simple zeros. Lemma 5.13 gives the result. All the geometric data are chosen once and held fixed. □

This theorem determines values on the displayed intervals. It makes no claim about the intervening times or monotonicity of the complete profile. The dimension grows with N.

Corollary 5.15 (Persistence of a two-time distinction)**.** *For the two-dimensional example, choose $t_s \in (0, \varepsilon)$ and $t_\ell \ge 1$. Every sufficiently small smooth perturbation of its metric and unitary connection on the same trivial line bundle still satisfies*

$$\mathfrak{h}_{t_s} = 2, \qquad \mathfrak{h}_{t_\ell} = 1.$$

Proof. At t_s, translation invariance gives two simple zeros for every source, with a uniform nondegeneracy bound. The parameter implicit function theorem and compactness of the source manifold preserve a zero for every source under small perturbations. Hence $\mathfrak{h}_{t_s} \ge 2$. A fixed two-source cover persists by the positive singular-value margin in Proposition 5.5, giving equality. At t_ℓ, the absolute value of the unperturbed kernel has a positive minimum on $M_0 \times M_0$, so nonvanishing persists. □

5.7 Related work

Connection heat kernels already underlie vector diffusion maps [21] and the vector heat method [22]. The off-cut-locus estimate used here is the standard expansion in the precise uniform form proved by Ludewig [20]. The topological arguments apply classical bundle and characteristic-class theory [5].

Orthogonality loci of coherent states and their polar divisors provide a close structural precedent for kernel-zero geometry [23]. Knöppel studies zero currents of random line-bundle

sections distributed using heat flow [24]; here the sections are deterministic and have a fixed source. The lattice sums in Subsection 5.5 belong to the established theta-function description of torus heat kernels; see, for scalar rectangular and hexagonal tori, Faulhuber [25].

The results concern the rank-defect correspondence and covering by heat trivializations; they do not include a general transversality theorem or a classification of possible time profiles.

6 Tangent shapes and knots

We study embedded shapes in tangent spaces and knots in spheres of tangent directions, with particular attention to oriented four-manifolds.

6.1 Tangent shapes

Throughout this section, manifolds are smooth, Hausdorff, and second countable. The base M^n has no boundary. The model F^m is nonempty and compact, possibly disconnected or with boundary, with $m \le n$. "Closed" means compact without boundary.

Definition 6.1 (Tangent shape)**.** A *tangent F-shape field* is an embedded submanifold, with boundary when appropriate,

$$\mathcal{S} \subset TM \setminus 0_M, \qquad p = \pi|_{\mathcal{S}} : \mathcal{S} \longrightarrow M,$$

such that p is a smooth locally trivial bundle with fibre F. Thus

$$\mathcal{S}_x = p^{-1}(x)$$

is an embedded copy of F in $T_xM \setminus \{0\}$. Here $\pi : TM \to M$ is the tangent-bundle projection and 0_M is its zero section.

A diffeomorphism $f : M \to N$ is an *isomorphism* of shape fields when

$$df(\mathcal{S}) = \mathcal{S}'.$$

A *deformation* is a smooth family of such fields over a fixed base. More precisely, it is an F-bundle embedded fibrewise in

$$\mathrm{pr}_M^* TM \setminus 0 \longrightarrow M \times [0,1].$$

A *parametrized* field additionally specifies a smooth map

$$\iota : M \times F \longrightarrow TM, \qquad \pi(\iota(x,u)) = x,$$

such that each ι_x is an embedding avoiding zero. This is extra data: a general shape bundle need not be trivial.

The tautological section

The pullback p^*TM has a canonical nowhere-zero section:

$$s(x,v) = v \in T_xM, \qquad (x,v) \in \mathcal{S}. \tag{6.1}$$

With a metric, this gives a splitting

$$p^*TM \cong \underline{\mathbb{R}} \oplus s^{\perp}.$$

Consequently,

$$p^*e(TM) = 0$$

whenever TM is oriented [26]. This does *not*, in general, give a nowhere-zero vector field on M.

The space of structures

Set

$$\mathcal{Q}_F^n = \mathrm{Emb}(F, \mathbb{R}^n \setminus \{0\}) / \mathrm{Diff}(F).$$

Here embeddings are identified by reparametrization, and $\mathrm{GL}(n, \mathbb{R})$ acts by its linear action on the ambient space.

Shape fields are precisely smooth sections of the associated bundle

$$\mathrm{Fr}(TM) \times_{\mathrm{GL}(n,\mathbb{R})} \mathcal{Q}_F^n \longrightarrow M. \tag{6.2}$$

Indeed, a frame identifies a fibre shape with an element of $\mathcal{Q}_F^n$, and changing frames gives the indicated linear action. Smoothness can be defined through local smooth families of embeddings.

This is a nonlinear-Grassmannian construction [27].

Basic examples

A one-point shape is a nowhere-zero vector field.

Every Riemannian metric gives the unit-sphere field

$$\mathcal{S}_x = \{v \in T_x M : |v|_g = 1\}$$

and the shell field

$$\mathcal{S}_x = \{v \in T_x M : 1 \le |v|_g \le 2\},$$

with fibres S^{n-1} and $S^{n-1} \times [1, 2]$, respectively.

If TM is trivial, any fixed compact embedded

$$F \subset \mathbb{R}^n \setminus \{0\}$$

gives a constant field in a chosen framing.

No convexity, radiality, or distinguished point of F is required.

6.2 Euler obstruction and surfaces

Theorem 6.2 (Euler obstruction)**.** *Let M^n be closed, connected, and oriented. If M admits a tangent F-shape field, then*

$$\chi(F) e(TM) = 0 \quad \text{in } H^n(M; \mathbb{Z})$$

and therefore

$$\chi(F)\chi(M) = 0. \tag{6.3}$$

The numerical conclusion also holds for closed nonorientable M.

Proof. Equation (6.1) implies

$$p^* e(TM) = 0.$$

The Becker–Gottlieb transfer for the compact-fibre bundle p induces a homomorphism

$$\mathrm{tr}_p : H^*(\mathcal{S}; \mathbb{Z}) \longrightarrow H^*(M; \mathbb{Z})$$

satisfying

$$\mathrm{tr}_p \circ p^* = \chi(F)\,\mathrm{id}$$

[28]. Compact smooth manifolds with boundary have finite CW type, so this applies to the allowed fibres.

Applying this identity to $e(TM)$ gives

$$0 = \mathrm{tr}_p(p^* e(TM)) = \chi(F) e(TM).$$

Pairing with the fundamental class and using

$$\langle e(TM), [M] \rangle = \chi(M)$$

proves the numerical assertion [26].

For nonorientable M, pull the shape field back to the orientation double cover $\widetilde{M}$ and use

$$\chi(\widetilde{M}) = 2\chi(M).$$

□

Thus a base with nonzero Euler characteristic cannot support a shape with nonzero Euler characteristic. This obstruction uses the topology of the fibre, not its knotting.

Theorem 6.3 (Planar domains in tangent planes)**.** *Let M be a closed connected surface, and let F be a compact connected surface admitting a smooth embedding in $\mathbb{R}^2$. Then*

$$M \text{ admits a tangent } F\text{-shape field} \quad \Longleftrightarrow \quad \chi(M)\chi(F) = 0. \tag{6.4}$$

Proof. Necessity is Theorem 6.2. Since F embeds in the plane, it is a planar domain with $b \geq 1$ boundary components and

$$\chi(F) = 2 - b.$$

Suppose first that $\chi(F) = 0$. Then $b = 2$, so F is an annulus. The metric shell

$$1 \leq |v|_g \leq 2$$

constructs the required field on every surface.

Suppose instead that $\chi(M) = 0$. A closed connected surface with zero Euler characteristic is a torus or a Klein bottle, and in either case it admits a nowhere-zero vector field. Normalize such a field to a unit field V, and write

$$TM = \mathbb{R}V \oplus L, \qquad L = V^{\perp}.$$

Choose a planar model A of F in the half-plane $a > 0$ that is invariant under reflection

$$(a, b) \longmapsto (a, -b).$$

Such a model is a disk centred on the positive horizontal axis with $b - 1$ disjoint smaller open disks removed, all centred on that axis.

In a local unit frame (V, W) of $\mathbb{R}V \oplus L$, define

$$\mathcal{S}_x = \{aV_x + bW_x : (a, b) \in A\}.$$

Replacing W by $-W$ leaves this set unchanged. The local definitions therefore glue. Since $a > 0$, every fibre avoids zero. □

Base	Disk	Annulus	Pair of pants
S^2	No	Yes	No
$\mathbb{R}P^2$	No	Yes	No
Orientable surface of genus ≥ 2	No	Yes	No
T^2	Yes	Yes	Yes
Klein bottle	Yes	Yes	Yes

Equation (6.4) classifies *existence*. It does not classify deformation classes or the embeddings of the individual fibres.

6.3 Three-dimensional knot fields

Throughout this subsection, $\dim M = 3$ and $F = S^1$.

Knot type means ordinary unparametrized ambient-isotopy type in $\mathbb{R}^3$. The distinguished origin is not part of this knot-type classification.

A knot type K is *chiral* if it is not isotopic to its mirror image, even after forgetting the orientation of the knot.

A field of *unoriented ambient type* K means that, in arbitrary linear coordinates on each tangent space, its type is K or its mirror.

Theorem 6.4 (Chirality detects orientability)**.** *Fix a chiral knot type K. A closed connected smooth three-manifold admits a tangent knot field of unoriented ambient type K if and only if it is orientable.*

Proof. Fix a handedness of K. For each x, exactly one orientation of T_xM makes $\mathcal{K}_x$ have that handedness.

Knot type is locally constant in a smooth family of compact embeddings: nearby embeddings are isotopic, and isotopy extension supplies an ambient isotopy. Hence the selected orientations vary continuously and orient M.

Conversely, a closed orientable three-manifold is parallelizable [29]. Choose a trivialization

$$TM \cong M \times \mathbb{R}^3$$

and place a representative

$$K \subset \mathbb{R}^3 \setminus \{0\}$$

in every fibre. □

Tangent trefoils

Every closed orientable three-manifold admits tangent trefoils. In a framing of T^3 or S^3, one may use the standard $(2,3)$-torus-knot parametrization

$$\kappa(t) = \big((2+\cos 3t)\cos 2t,\ (2+\cos 3t)\sin 2t,\ \sin 3t\big), \qquad t \in \mathbb{R}/2\pi\mathbb{Z}. \tag{6.5}$$

Its distance from the vertical axis is at least 1, so it avoids the origin.

Holonomy rigidity

Definition 6.5 (Parallel field)**.** On a Riemannian manifold (M, g), a shape field is *parallel* if Levi–Civita parallel transport satisfies

$$P_\gamma(\mathcal{S}_{\gamma(0)}) = \mathcal{S}_{\gamma(1)}$$

for every piecewise smooth path γ.

Lemma 6.6. *For every nontrivial smooth knot $K \subset \mathbb{R}^3$, the orthogonal stabilizer*

$$\mathrm{Stab}_{O(3)}(K) = \{A \in O(3) : A(K) = K\}$$

is finite.

Proof. The stabilizer is a closed subgroup of the compact Lie group $O(3)$.

If its identity component had positive dimension, it would contain a circle of rotations about an axis. The knot cannot lie entirely in that axis, so it contains the circular orbit of an off-axis point.

This orbit is an embedded circle in the connected one-manifold K. Its inclusion is locally a diffeomorphism, so it is open in K; compactness makes it closed. It therefore equals K.

Thus K would be a round circle and hence an unknot, contrary to hypothesis. The stabilizer is consequently a zero-dimensional compact Lie group, and is finite. □

Theorem 6.7 (Parallel-knot rigidity)**.** *If a connected Riemannian three-manifold admits a parallel field whose fibres are nontrivial knots, then*

$$\mathrm{Rm}_g \equiv 0.$$

Proof. At every $x \in M$,

$$\mathrm{Hol}_x(g) \subset \mathrm{Stab}_{O(T_xM,g_x)}(\mathcal{K}_x).$$

The right-hand side is finite by the lemma. The restricted holonomy group is connected, hence trivial.

By the Ambrose–Singer theorem, curvature endomorphisms belong to the restricted holonomy Lie algebra. That Lie algebra is zero, so the curvature vanishes [30]. □

The same argument works in any dimension when a parallel shape has finite orthogonal stabilizer.

The standard flat three-torus has parallel knot fields. Although S^3 has tangent trefoils, no such field can be parallel for any Riemannian metric: a compact simply connected flat manifold of positive dimension cannot exist, because a complete simply connected flat manifold is Euclidean space.

6.4 Deformation and normal variation

Parametrized deformation classes

Theorem 6.8 (Infinitely many parametrized deformation classes)**.** *For every smooth knot type K, there are infinitely many deformation classes of parametrized tangent knot fields of type K on S^3.*

Proof. Fix a framing

$$\tau_x : \mathbb{R}^3 \longrightarrow T_xS^3,$$

an orientation of S^3, and an embedding

$$\kappa : S^1 \longrightarrow \mathbb{R}^3 \setminus \{0\}$$

of type K.

Choose smooth maps

$$R_k : S^3 \longrightarrow \mathrm{SO}(3), \qquad k \in \mathbb{Z},$$

representing the elements of

$$\pi_3(\mathrm{SO}(3)) \cong \mathbb{Z}.$$

Define

$$\gamma_k(x,t) = \tau_x\big(R_k(x)\kappa(t)\big). \tag{6.6}$$

For any parametrized field γ, differentiation at a fixed $t_0 \in S^1$ defines a map

$$u_\gamma(x) = \frac{\tau_x^{-1}(\partial_t\gamma(x,t_0))}{\left|\tau_x^{-1}(\partial_t\gamma(x,t_0))\right|} \in S^2. \tag{6.7}$$

The denominator is nonzero because each fibre map is an embedding. A deformation of parametrized fields induces a homotopy of these maps.

For (6.6),

$$u_{\gamma_k}(x) = R_k(x)a, \qquad a = \frac{\kappa'(t_0)}{|\kappa'(t_0)|}.$$

The fibration

$$\mathrm{SO}(2) \longrightarrow \mathrm{SO}(3) \xrightarrow{R \mapsto Ra} S^2$$

induces an isomorphism on π_3, because

$$\pi_3(\mathrm{SO}(2)) = \pi_2(\mathrm{SO}(2)) = 0.$$

Therefore the maps u_{γ_k} represent distinct elements of

$$\pi_3(S^2) \cong \mathbb{Z}.$$

They are detected by the Hopf invariant [31]. With compatible choices of generators,

$$H(u_{\gamma_k}) = k.$$

Thus the fields γ_k are pairwise inequivalent under deformation through parametrized fields. □

The integer labels use the chosen framing. This proves neither a classification of *all* parametrized fields nor a classification after forgetting parametrizations or quotienting by diffeomorphisms of the base.

Normal variation

Fix a Riemannian metric g and assume now that

$$\partial F = \varnothing.$$

Let

$$\iota : U \times F \longrightarrow TM$$

be a local parametrization of a shape field. For $v = \iota_x(u)$ and $X \in T_xM$, define

$$\mathcal{B}_v(X) = (\nabla_X \iota(\,\cdot\,, u))^{\perp_{T_v \mathcal{S}_x}}. \tag{6.8}$$

Here ∇ is the Levi–Civita connection. The tangent space $T_v\mathcal{S}_x$ is viewed as a subspace of T_xM using the canonical identification

$$T_v(T_xM) \cong T_xM,$$

and the orthogonal projection uses g_x.

Thus $\mathcal{B}$ is a section of

$$p^*T^*M \otimes \nu_{\mathrm{vert}}\mathcal{S},$$

where $\nu_{\mathrm{vert}}\mathcal{S}$ is the normal bundle of the shapes inside the tangent fibres.

Proposition 6.9. *For compact boundaryless fibres,* (6.8) *is independent of local parametrization, and*

$$\mathcal{B} = 0 \quad \Longleftrightarrow \quad \textit{the shape field is parallel.}$$

Proof. An x-dependent reparametrization changes $\nabla_X \iota$ by a vector tangent to $\mathcal{S}_x$. Its normal projection is therefore unchanged. Parallel fields plainly give $\mathcal{B} = 0$.

Conversely, consider a smooth path in the base and trivialize TM along that path by parallel transport. The shape bundle over the path is also trivial, so it can be represented by a smooth family of embeddings ι_t of F into a fixed vector space.

If $\mathcal{B} = 0$, then

$$\partial_t \iota_t = d\iota_t(Z_t)$$

for a smooth time-dependent vector field Z_t on F. Let ϕ_t solve

$$\partial_t \phi_t = -Z_t \circ \phi_t, \qquad \phi_0 = \mathrm{id}\,.$$

Compactness and absence of boundary ensure that this flow exists throughout each compact time interval. The chain rule gives

$$\frac{d}{dt}(\iota_t \circ \phi_t) = 0.$$

Hence the images of the ι_t are constant in the parallel trivialization.

The same conclusion holds along piecewise smooth paths by concatenation. Thus parallel transport preserves the shape field. □

For fibres with boundary, (6.8) alone is insufficient: it does not detect boundary motion tangent to the fibre. For example, an expanding interval in a one-dimensional vector space has identically zero normal variation. No corresponding equivalence is asserted in that case.

6.5 Four-dimensional direction knots

In Subsections 6.5–6.11, unless otherwise specified, M is connected, oriented, four-dimensional, and without boundary. Integral cohomology is used unless coefficients are displayed; $\bar{a}$ denotes reduction of $a \in H^2(M;\mathbb{Z})$ modulo 2.

The *oriented direction bundle* is

$$S(TM) = (TM \setminus 0)/\mathbb{R}_{>0}, \qquad \pi : S(TM) \longrightarrow M.$$

Its fiber is an oriented 3-sphere. A Riemannian metric identifies it with the unit tangent sphere bundle by radial normalization, but no metric is part of the definition.

Definition 6.10 (Tangent-knot field)**.** An *oriented tangent-knot field* is a smooth embedded submanifold $\mathcal{K} \subset S(TM)$ such that $\pi|_{\mathcal{K}}$ is a proper surjective submersion with circle fibers, together with an orientation of its vertical tangent line bundle. A field has *type* K if each fiber $\mathcal{K}_x \subset S(T_xM)$ has the oriented ambient isotopy type of a fixed oriented knot $K \subset S^3$.

The fibers are *unparametrized*: the data do not include a map $M \times S^1 \to S(TM)$. Two fields are equivalent if an orientation-preserving diffeomorphism of their base manifolds carries one to the other under its derivative on directions, preserving fiber orientations. A deformation is a smooth one-parameter family of fields of the fixed type.

Proposition 6.11 (Basic properties)**.** *The total space $\mathcal{K}$ has dimension 5. The map $\mathcal{K} \to M$ is an oriented circle bundle, and its fiber knot type is locally constant. Fields exist locally for every knot type; every parallelizable oriented four-manifold admits a field of every type.*

Proof. Ehresmann's theorem makes the proper submersion a smooth circle bundle. Local trivializations and isotopy extension identify nearby embedded fibers by ambient isotopies. An oriented frame on an open set identifies its direction bundle with $U \times S^3$, where $U \times K$ defines a field. A global frame gives the final assertion. □

Define the *charge* of a field by

$$a(\mathcal{K}) = e(\mathcal{K} \to M) \in H^2(M;\mathbb{Z}).$$

Here the Euler class of an oriented circle bundle is defined using $\mathrm{Diff}^+(S^1) \simeq \mathrm{SO}(2)$, with the convention $e(S(L)) = c_1(L)$ for the unit circle bundle of a complex line bundle L. The charge is preserved by deformation and is natural under equivalences.

For a chosen metric, let $P \to M$ be the oriented orthonormal frame bundle. Write $\mathscr{C}_K$ for the component containing K in

$$\mathrm{Emb}(S^1, S^3)/\,\mathrm{Diff}^+(S^1),$$

with its smooth topology. A field is equivalently a smooth family of points in the associated bundle

$$P \times_{\mathrm{SO}(4)} \mathscr{C}_K \longrightarrow M.$$

This description is the bridge to obstruction theory.

Relation to tangent shapes

The direction construction is a restricted variant of the tangent-shape construction. A metric identifies each positive ray with its unit vector, so a direction-knot field lifts to a tangent S^1-shape field, with the given fiber orientation. Conversely, a tangent circle field with a chosen fiber orientation determines a direction-knot field only when its radial projection is an embedding on each fiber. An arbitrary embedded circle in $T_xM \setminus \{0\}$ need not satisfy this condition. Positive radial lengths are additional data and are forgotten in the direction description.

The orientation of the circle fibers does not specify a global parametrization. The latter would trivialize the circle bundle. Also, the Euler obstruction in Theorem 6.2 is vacuous for $F = S^1$; the four-dimensional restrictions below use the topology of the space of knots in S^3.

6.6 Torus knots and existence

Fix relatively prime integers $p, q \geq 2$, and put

$$K_{p,q} = \left\{2^{-1/2}(z^p, z^q) : |z| = 1\right\} \subset S^3 \subset \mathbb{C}^2,$$

$$H_{p,q} = \{\mathrm{diag}(u^p, u^q) : |u| = 1\} \subset \mathrm{SO}(4).$$

The knot has the orientation induced by $z = e^{it}$. Its orientation-preserving rotational stabilizer is $H_{p,q} \cong S^1$. Assertions concerning these torus knots use the positive type and the specified ambient orientation.

Lemma 6.12 (External knot-space input: Hatcher)**.** *The natural rotation-orbit inclusion*

$$\mathrm{SO}(4)/H_{p,q} \longrightarrow \mathscr{C}_{K_{p,q}} \tag{6.9}$$

is an $\mathrm{SO}(4)$*-equivariant map and an underlying homotopy equivalence.*

This is the oriented version of Hatcher's theorem [32]: the unoriented stabilizer is a weighted copy of $O(2)$; passing to the orientation double cover gives (6.9). The map, not just the abstract homotopy type of its target, is important.

Theorem 6.13 (Weighted circle reduction)**.** *A field of type $K_{p,q}$ and charge a exists on M if and only if there is a complex line bundle L with $c_1(L) = a$ and an oriented real bundle isomorphism*

$$TM \cong \left(L^{\otimes p} \oplus L^{\otimes q}\right)_{\mathbb{R}}. \tag{6.10}$$

Moreover, every such field is homotopic through sections of the knot-space bundle to one arising from an $H_{p,q}$-reduction.

Proof. The equivariant map (6.9) induces a fiberwise homotopy equivalence

$$P/H_{p,q} \longrightarrow P \times_{\mathrm{SO}(4)} \mathscr{C}_{K_{p,q}}.$$

Thus a section on the right determines, up to homotopy, a section on the left; this is the standard homotopy lifting argument over a CW base. Such a section is an $H_{p,q}$-reduction of P, which gives (6.10). Its associated circle bundle is $S(L)$, since $u \in H_{p,q}$ acts on the knot parameter by $z \mapsto uz$. The universal fiber-circle bundle shows that the homotopy preserves a.

Conversely, choose a Hermitian metric on L. In the bundle on the right of (6.10), the map

$$S(L) \longrightarrow S\big((L^{\otimes p} \oplus L^{\otimes q})_{\mathbb{R}}\big), \qquad v \longmapsto [\, v^{\otimes p} \oplus v^{\otimes q}\,]$$

is fiberwise an embedded $K_{p,q}$ because $\gcd(p,q) = 1$. Transport it by the bundle isomorphism. Continuous reductions can be smoothed, so this constructs a smooth field of charge a. □

Remark 6.14. The theorem is a statement about existence and homotopy, not a linear normal form for every given fiber and not a classification up to diffeomorphism. The class a alone need not classify fields. A global fiber parametrization would force $a = 0$, hence (6.10) would force TM to be trivial.

For any such field, the characteristic-class identities are

$$\begin{aligned} e(TM) &= pq\, a^2, \\ p_1(TM) &= (p^2 + q^2)a^2, \\ w_2(TM) &= (p+q)\bar{a}. \end{aligned} \tag{6.11}$$

They follow directly from the Chern classes of the two line summands. In particular, the reduction induces an almost complex structure with $c_1 = (p+q)a$.

Theorem 6.15 (Existence on a closed four-manifold)**.** *Let M be closed and $a \in H^2(M;\mathbb{Z})$. A field of type $K_{p,q}$ with charge a exists if and only if, writing $N = \langle a^2, [M]\rangle$, one has*

$$\begin{aligned} (p+q)\bar{a} &= w_2(TM), \\ pqN &= \chi(M), \\ (p^2+q^2)N &= 3\sigma(M). \end{aligned} \tag{6.12}$$

Proof. Necessity follows from (6.11), the Euler formula and the signature theorem. For sufficiency, choose L with $c_1(L) = a$, and set

$$E = (L^{\otimes p} \oplus L^{\otimes q})_{\mathbb{R}}.$$

Equations (6.12) give equal w_2, e, p_1 for E and TM; equality of the degree-four numbers is equality of classes because $H^4(M;\mathbb{Z}) \cong \mathbb{Z}$. Oriented real rank-four bundles over this base are determined by these classes, by Dold–Whitney [33]. Hence $E \cong TM$ as oriented bundles, and Theorem 6.13 applies. □

The same argument can be expressed through the Hirzebruch–Hopf criterion [34]: $c = (p+q)a$ satisfies $\bar{c} = w_2$ and

$$c^2[M] = 2\chi + 3\sigma.$$

6.7 Topology and arithmetic

Corollary 6.16 (Slope identity)**.** *A closed manifold carrying a $K_{p,q}$-field satisfies*

$$\begin{aligned} 3pq\,\sigma &= (p^2+q^2)\chi, \\ 2\chi - 3\sigma &= -(p-q)^2 N. \end{aligned} \tag{6.13}$$

If M is simply connected, then $N > 0$, $\sigma > 0$, and

$$b_2 = pqN - 2, \qquad \sigma = \frac{p^2+q^2}{3}N.$$

If p and q are both odd, every such M is spin.

Proof. Use (6.12), $\chi = 2 + b_2$ in the simply connected case, and $p + q \equiv 0 \pmod 2$ in the odd–odd case. □

Corollary 6.17 (Trefoil divisibility)**.** *For a closed manifold with a $K_{2,3}$-field,*

$$N \in 12\mathbb{Z}, \qquad (\chi, \sigma) = (72k, 52k) \quad \text{for some } k \in \mathbb{Z}.$$

If it is simply connected, $\chi \geq 72$. If it is also spin, then $N \in 48\mathbb{Z}$ and $\chi \geq 288$. Both simply connected lower bounds are attained by the examples in Subsection 6.9.

Proof. The identities are $\chi = 6N$, $3\sigma = 13N$, and $\bar{a} = w_2$. Thus $N = 3m$ and $\sigma = 13m$. The class a is characteristic for the unimodular intersection form on $H^2(M;\mathbb{Z})/\text{torsion}$, so van der Blij's congruence [35] gives $N \equiv \sigma \pmod 8$. Hence

$$3m \equiv 13m \pmod 8,$$

so $4 \mid m$. In the spin case, Rokhlin's theorem gives $16 \mid \sigma$ (see [36], introduction), whence $16 \mid m$. Simply connectedness gives $N > 0$. □

The slope identity already excludes S^4, $\mathbb{CP}^2$, and $S^2 \times S^2$ from carrying a trefoil field in their standard orientations: their respective pairs (χ, σ) are $(2, 0)$, $(3, 1)$, and $(4, 0)$.

Proposition 6.18 (Asymmetric hyperbolic type)**.** *Let $K \subset S^3$ be a hyperbolic knot whose complement has trivial orientation-preserving isometry group. An oriented four-manifold admits a field of this type if and only if it is parallelizable. Consequently, in the closed case, $\chi = \sigma = 0$ and all positive-degree characteristic classes of TM vanish.*

Proof. For this asymmetric case Hatcher's orbit theorem [32] gives a homotopy equivalence

$$\mathrm{SO}(4) \longrightarrow \mathscr{C}_K.$$

The associated bundle map identifies the existence of a section with that of a section of

$$P \times_{\mathrm{SO}(4)} \mathrm{SO}(4) \cong P.$$

This is a framing. The converse follows by placing a fixed copy of K in every framed fiber. □

6.8 Einstein metrics

Let g be a smooth Riemannian metric on a closed oriented four-manifold. Write

$$E = \mathrm{Ric}_g - \tfrac{1}{4} s_g g$$

for traceless Ricci curvature, and $W^{\pm}$ for the self-dual and anti-self-dual Weyl curvature operators. Put

$$\|T\|_2^2 = \int_M |T|^2 \, d\mu_g.$$

We use the Hilbert–Schmidt Weyl norm convention

$$|W|^2 = \tfrac{1}{4} W_{ijkl} W^{ijkl}, \qquad |E|^2 = E_{ij} E^{ij}.$$

Theorem 6.19 (Curvature identity and Einstein rigidity)**.** *If M carries a $K_{p,q}$-field of charge square N, then every metric g satisfies*

$$\|E\|_2^2 = 8\pi^2 (p-q)^2 N + 4\|W^-\|_2^2 + \frac{1}{12}\|s_g\|_2^2. \tag{6.14}$$

For $N > 0$, this gives the strictly positive lower bound

$$\|E\|_2^2 \geq 8\pi^2 (p-q)^2 N.$$

Every Einstein metric on M, if one exists, is flat.

Proof. The Chern–Gauss–Bonnet and signature formulas, with these conventions, give

$$8\pi^2\chi = \|W^+\|_2^2 + \|W^-\|_2^2 + \frac{1}{24}\|s_g\|_2^2 - \frac{1}{2}\|E\|_2^2,$$
$$12\pi^2\sigma = \|W^+\|_2^2 - \|W^-\|_2^2$$

(see [37], Section 2, in the smooth case). Subtracting yields

$$4\pi^2(2\chi - 3\sigma) = 2\|W^-\|_2^2 + \frac{1}{24}\|s_g\|_2^2 - \frac{1}{2}\|E\|_2^2.$$

Substitute (6.13). Since $p \neq q$, $N > 0$ excludes $E = 0$. If $N < 0$, then $\chi = pqN < 0$, also impossible for an Einstein metric by the first formula. If $N = 0$ and $E = 0$, that formula forces $W^+ = W^- = 0$ and $s_g = 0$, so the full curvature vanishes. □

This is a consequence of classical four-dimensional curvature identities, not a new curvature tensor attached to the knot field. The metric need not be compatible with the field.

6.9 Four-dimensional examples

In a hyperbolic summand of an intersection form, use generators u, v with

$$u^2 = v^2 = 0, \qquad u \cdot v = 1.$$

Then

$$(ru + sv)^2[M] = 2rs.$$

We write $\overline{K3}$ for a K3 surface with reversed complex orientation; it is spin and has

$$b_2 = 22, \qquad \chi = 24, \qquad \sigma = 16,$$

with hyperbolic summands in its intersection form (cf. [36, 38]).

Example 6.20 (A field of every knot type)**.** A global frame on T^4 gives

$$S(TT^4) \cong T^4 \times S^3.$$

For every knot K, the product $T^4 \times K$ is a field. For $K_{2,3}$ this is the explicit family

$$t \longmapsto 2^{-1/2}(e^{2it}, e^{3it}),$$

with $a = 0$.

Example 6.21 (Nonparallelizable with zero characteristic numbers)**.** Let

$$M_0 = (\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2) \# 2(S^1 \times S^3).$$

Its form is $\langle 1\rangle \oplus \langle -1\rangle$ and $\chi = \sigma = 0$. For diagonal generators h, e, set $a = h + e$. Then

$$a^2[M_0] = 0, \qquad \bar{a} = w_2(TM_0) \neq 0.$$

The criterion gives a trefoil field, although M_0 is not parallelizable.

Example 6.22 (Sharp simply connected trefoil bound)**.** The manifold

$$M_1 = \#^{61}\mathbb{CP}^2 \# \#^9\overline{\mathbb{CP}}^2$$

has $(\chi, \sigma) = (72, 52)$. Choose a with coefficient 1 on every positive generator, coefficient 3 on five negative generators, and coefficient 1 on the remaining four. Then

$$a^2[M_1] = 61 - 5 \cdot 9 - 4 = 12, \qquad \bar{a} = w_2(TM_1).$$

Theorem 6.15 gives a trefoil field, attaining $\chi = 72$.

Example 6.23 (Sharp simply connected spin trefoil bound)**.** Let

$$M_2 = \#^{13}\overline{K3}.$$

Then $(\chi, \sigma) = (288, 208)$. In one hyperbolic summand take $a = 2u + 12v$, so

$$a^2[M_2] = 48, \qquad \bar{a} = 0.$$

The trefoil criterion holds, attaining the spin bound $\chi = 288$.

Example 6.24 (An odd–odd torus-knot field)**.** Let

$$M_3 = \#^{74}\overline{K3} \,\#\, \#^{25}(S^2 \times S^2).$$

It is spin, with $(\chi, \sigma) = (1680, 1184)$. Choose $a = u + 24v$, so $a^2[M_3] = 48$. For $(p, q) = (5, 7)$, one has

$$pq \cdot 48 = 1680$$

and

$$(p^2 + q^2)48 = 3552 = 3 \cdot 1184.$$

Thus M_3 admits a $K_{5,7}$-field.

All connected-sum numbers follow from

$$\chi(X \,\#\, Y) = \chi(X) + \chi(Y) - 2$$

and

$$\sigma(X \,\#\, Y) = \sigma(X) + \sigma(Y).$$

The last three examples admit no Einstein metric by Theorem 6.19.

6.10 Spin geography and the 11/8 conjecture

Here $p, q \geq 3$ are coprime and odd. Abbreviate

$$A = p^2 + q^2, \qquad B = pq, \qquad \rho_{p,q} = \frac{A}{3B}.$$

Let $\mathsf{C}_{11/8}$ be the assertion that every closed, smooth, simply connected spin four-manifold X satisfies

$$b_2(X) \geq \frac{11}{8}|\sigma(X)|.$$

This is the simply connected form of the 11/8 conjecture discussed in [36, 38]; the reformulation below does not prove or disprove it.

Proposition 6.25 (Existence and nonexistence regions)**.** *If $\rho_{p,q} < 8/11$, some closed simply connected spin four-manifold admits a $K_{p,q}$-field. If $\rho_{p,q} \geq 4/5$, none does. More precisely, any such field on a closed simply connected manifold obeys*

$$(12B - 5A)N \geq 48. \tag{6.15}$$

Proof. For existence, put

$$\Delta = 24B - 11A > 0.$$

Choose an integer $t \geq 1$ with $m = \Delta t - 1 \geq 1$, and set $n = At$. On

$$M = \#^{n}\overline{K3} \,\#\, \#^{m}(S^2 \times S^2)$$

one has

$$\chi = 48Bt, \qquad \sigma = 16At,$$

and a hyperbolic-summand class $a = u + 24tv$ with

$$a^2[M] = 48t.$$

Since $p + q$ is even, the criterion applies.

For nonexistence, any candidate is spin and has nonzero b_2 and positive signature. Donaldson's theorem excludes a nonzero definite even form; Furuta's theorem [36] therefore applies:

$$BN - 2 = b_2 \geq \frac{5}{4}\sigma + 2 = \frac{5A}{12}N + 2.$$

This is (6.15). In particular, $12B - 5A > 0$, or $\rho_{p,q} < 4/5$. □

Theorem 6.26 (11/8 detection by torus-knot fields)**.** *The assertion* $\mathsf{C}_{11/8}$ *is false if and only if there are coprime odd* $p, q \geq 3$ *and a closed, smooth, simply connected four-manifold carrying a* $K_{p,q}$*-field with*

$$11(p^2 + q^2) > 24pq, \qquad \textit{or equivalently} \qquad \rho_{p,q} > \frac{8}{11}. \tag{6.16}$$

Proof. Given such a field, the manifold is spin, $N > 0$, and

$$\frac{b_2}{\sigma} = \frac{3B}{A} - \frac{6}{AN} < \frac{3B}{A} < \frac{11}{8}.$$

Thus it violates $\mathsf{C}_{11/8}$.

Conversely, let X be a counterexample, oriented so that $s = \sigma(X) > 0$, and put $b = b_2(X)$. Choose coprime odd $p, q \geq 3$ such that

$$\frac{8}{11} < \frac{A}{3B} < \frac{s}{b}. \tag{6.17}$$

This is possible: ratios of coprime odd integers $q/p > 1$ are dense in $(1, \infty)$, and

$$r \longmapsto \frac{r + r^{-1}}{3}$$

maps that interval continuously and strictly increasingly onto $(2/3, \infty)$.

Set

$$D = 3Bs - Ab > 0.$$

For an integer t large enough that $\ell = Dt - 1 \geq 1$, let $k = 2At$ and form

$$M = \#^k X \,\#\, \#^\ell(S^2 \times S^2).$$

This manifold is smooth, simply connected and spin. Direct calculation gives

$$\chi(M) = kb + 2\ell + 2 = 6Bst, \qquad \sigma(M) = ks = 2Ast.$$

In a hyperbolic summand choose $a = u + 3st\, v$, so that

$$a^2[M] = 6st.$$

These numbers satisfy (6.12), including $w_2 = (p + q)\bar{a} = 0$. Thus M carries the required field, and (6.17) gives (6.16). □

Remark 6.27 (Meaning of the equivalence). The theorem translates an existing geography problem into a geometric existence problem; it supplies no new inequality. The range

$$\frac{8}{11} < \rho < \frac{4}{5}$$

is not decided by the arguments here. In particular, $\rho_{3,5} = 34/45 > 8/11$, so a simply connected $K_{3,5}$ example would violate $\mathsf{C}_{11/8}$.

Its charge square must be a positive multiple of 24 by Rokhlin's theorem; the smallest numerical candidate is

$$(\chi, \sigma) = (360, 272).$$

This is not an existence claim. No integral pair has $\rho = 8/11$, since the required ratio

$$q/p = (12 \pm \sqrt{23})/11$$

is irrational.

6.11 Incidence data

Let $\mathrm{Conf}_4(\mathcal{K}/M)$ be the bundle of ordered quadruples of pairwise distinct points in a common fiber. For $\xi_i = [v_i] \in \mathcal{K}_x$, choose any positively oriented volume form ω_x and positive-ray representatives v_i. Define

$$\mathfrak{I}_{\mathcal{K}}(x; \xi_1, \xi_2, \xi_3, \xi_4) = \mathrm{sgn}\big(\omega_x(v_1, v_2, v_3, v_4)\big) \in \{-1, 0, 1\}.$$

Positive rescaling of any v_i or of ω_x leaves this sign unchanged. Hence it is intrinsic to the oriented linear geometry and is natural under orientation-preserving diffeomorphisms. Its zero locus

$$\mathcal{D}_{\mathcal{K}} = \{\mathfrak{I}_{\mathcal{K}} = 0\} \subset \mathrm{Conf}_4(\mathcal{K}/M)$$

is the set of linearly dependent quadruples. After choosing a metric and volume form it is the zero locus of a smooth determinant function, but it need not be a smooth hypersurface.

This data is not a knot-isotopy invariant: an ambient isotopy of a fiber can change linear incidence. No additional characteristic-class obstruction follows merely from writing down $\mathcal{D}_{\mathcal{K}}$. We leave the study of its singularities, differential variation, and possible refinements for future work.

For torus knots, *existence* is already exhausted by Theorem 6.15. To obtain information beyond that criterion, one must study deformation or diffeomorphism classes, impose extra differential conditions, or pass to knot/link types with richer embedding spaces. One cannot obtain a stronger obstruction to unrestricted torus-knot-field existence while keeping the same definition and all of (6.12) satisfied.

6.12 Related work

The quotient $\mathcal{Q}_F^n$ in (6.2) is a standard type of nonlinear Grassmannian. Gay-Balmaz and Vizman study its principal-bundle and smooth-manifold foundations, including compact source manifolds with boundary [27]. Fibrewise shapes therefore belong naturally to the established geometry of embedding spaces.

Suppose all shapes are locally smooth linear images of a fixed model

$$A \subset \mathbb{R}^n \setminus \{0\}.$$

Then the adapted frames form a reduction of $\mathrm{Fr}(TM)$ to

$$G = \mathrm{Stab}_{\mathrm{GL}(n,\mathbb{R})}(A).$$

This is an ordinary G-structure. Allowing shapes to vary outside a single linear orbit is the additional freedom in the present formulation. Riemannian unit ellipsoids are a special case; the general definition discards their convexity and radial restrictions.

Theorem 6.2 is an application of the Becker–Gottlieb transfer [28] and Euler-class theory [26]. Theorem 6.4 uses classical three-manifold parallelizability [29]. Theorem 6.7 is a finite-stabilizer application of Ambrose–Singer holonomy theory [30].

Hatcher and Budney study homotopy types of spaces of knots [32, 39]. Their work is a natural starting point for analyzing the section space in (6.2).

Results for long knots or knots in S^3 do not automatically classify fields in punctured tangent fibres: the ambient space, parametrization conventions, and twisting by TM must be accounted for. Theorem 6.8 uses only the classical Hopf invariant [31].

A further problem is to compute components and obstruction classes of the section space in (6.2) for a prescribed knot or higher-dimensional embedding type.

The question is whether coupling the embedding space to TM yields information beyond that captured by a fixed-model G-structure, ordinary characteristic classes, or known embedding-space invariants. The results here do not settle this question.

Havens–Koytcheff [40] treat related link and solid-torus embedding spaces.

The tangent-bundle reduction belongs to classical G-structure and characteristic-class theory: Dold–Whitney [33] and Hirzebruch–Hopf [34] provide the relevant four-dimensional classification results.

The arithmetic uses van der Blij [35] and Rokhlin; the spin inequalities come from Furuta [36], with stronger related results in Hopkins–Lin–Shi–Xu [38].

The Einstein obstruction uses the classical Hitchin–Thorpe mechanism, with curvature conventions recorded in Atiyah–LeBrun [37].

7 Finite-scale bundle comparisons

A bundle comparison identifies fibers over pairs of points. We study its extension radius from the diagonal and the restriction maps between spaces of comparisons.

7.1 Definitions and basic properties

Throughout this section, M is a closed, connected smooth manifold, g is a smooth Riemannian metric, and $E \to M$ is a smooth real or complex vector bundle of positive rank. Write $p_i : M \times M \to M$ for the projections and put

$$\Delta = \{(x, x) : x \in M\}, \qquad U_r(g) = \{(x, y) \in M^2 : d_g(x, y) < r\} \qquad (r > 0).$$

The strict inequality in this definition matters at critical radii.

Definition 7.1 (Comparisons and their extension radius)**.** A *comparison at scale* r is a smooth bundle isomorphism

$$P : p_1^* E|_{U_r(g)} \longrightarrow p_2^* E|_{U_r(g)}, \qquad P_{xx} = \mathrm{id}_{E_x}\,.$$

Thus $P_{xy} : E_x \to E_y$ is invertible and depends smoothly on (x, y). Let $\mathcal{C}_r(E, g)$ be the space of such maps, with the compact-open C^∞ topology. Define

$$R_E(g) = \sup\{r > 0 : \mathcal{C}_r(E, g) \neq \varnothing\} \in (0, \infty].$$

For $0 < s < r$, restriction gives $\mathrm{res}_{r,s} : \mathcal{C}_r(E, g) \to \mathcal{C}_s(E, g)$. The spaces together with these maps form the *comparison system* $\mathfrak{C}(M, g, E)$.

We impose neither $P_{yx} = P_{xy}^{-1}$ nor $P_{yz}P_{xy} = P_{xz}$. We do not fix an ordinary connection, and we do not take a quotient by gauge transformations. These conventions specify the object being studied.

Proposition 7.2 (Local and global existence)**.** *The following statements hold.*

(i) $\mathcal{C}_r(E,g) \neq \varnothing$ *if and only if* $p_1^*E|_{U_r(g)} \cong p_2^*E|_{U_r(g)}$.

(ii) *If* $0 < r < \operatorname{inj}(M,g)$, *the space* $\mathcal{C}_r(E,g)$ *is contractible.*

(iii) $R_E(g) = \infty$ *if and only if* E *is trivial. If* E *is nontrivial, then*

$$\operatorname{inj}(M,g) \leq R_E(g) \leq \operatorname{rad}(M,g) \leq \operatorname{diam}(M,g), \qquad \operatorname{rad}(M,g) := \min_x \max_y d_g(x,y).$$

Proof. (i) Given an unnormalized isomorphism Q, set $P_{xy} = Q_{xy}(Q_{xx})^{-1}$.

(ii) Choose a connection on E and let P^0_{xy} be parallel transport along the unique minimizing geodesic. Write $P_{xy} = P^0_{xy}A_{xy}$, where $A_{xy} \in \operatorname{Aut}(E_x)$ and $A_{xx} = \mathrm{id}$. For $y_t = \exp_x(t \exp_x^{-1} y)$, the formula

$$H_t(P)_{xy} = P^0_{xy}A_{x,y_t}, \qquad 0 \leq t \leq 1,$$

is a continuous contraction: $H_1(P) = P$ and $H_0(P) = P^0$.

(iii) A trivialization defines a comparison on all of M^2. Conversely, if $r > \operatorname{rad}(M,g)$, some x_0 satisfies $\{x_0\} \times M \subset U_r(g)$; then $P_{x_0,y}$ trivializes E. Part (ii) gives the lower bound. □

For $E = TM$, infinite radius is therefore equivalent to parallelizability. For a general nontrivial bundle, existence at the supremum $r = R_E(g)$ is a separate question.

7.2 Stability and functoriality

Proposition 7.3 (Metric stability)**.** *For two metrics* g, h *on the same* M *and a nontrivial bundle* E, *let*

$$\epsilon = \sup_{x,y \in M} |d_g(x,y) - d_h(x,y)|.$$

Then $|R_E(g) - R_E(h)| \leq \epsilon$. *For every* $a > 0$, *one also has* $R_E(a^2 g) = aR_E(g)$, *allowing the value* ∞.

Proof. For $r > \epsilon$, the inclusion $U_{r-\epsilon}(h) \subset U_r(g)$ gives a restriction map. Taking suprema yields $R_E(h) \geq R_E(g) - \epsilon$ when the right side is positive; otherwise that inequality is automatic. Interchange g, h. Scaling follows from $d_{a^2 g} = a\, d_g$. □

Proposition 7.4 (Pullback and products)**.** *If* $f : (M,g) \to (N,h)$ *is smooth and* 1*-Lipschitz and* $F \to N$ *is a bundle, then*

$$R_{f^*F}(g) \geq R_F(h).$$

In particular, for a product metric on $X \times Y$,

$$R_{p_X^*F}(g_X \oplus g_Y) = R_F(g_X).$$

Proof. The map $f \times f$ carries $U_r(g)$ into $U_r(h)$, so comparisons pull back. Projection to X gives one product inequality. Restriction to the isometric slice $X \times \{y_0\}$ gives the reverse inequality. □

Proposition 7.5 (Relative gauge description)**.** *If* $P^0 \in \mathcal{C}_r(E,g)$, *the map* $A \mapsto P^0 A$ *is a homeomorphism*

$$\operatorname{Gau}_\Delta(p_1^*E|_{U_r}) \cong \mathcal{C}_r(E,g),$$

where the group on the left consists of smooth bundle automorphisms equal to the identity on Δ.

Proof. The inverse sends P to $A_{xy} = (P^0_{xy})^{-1}P_{xy}$. Multiplication and inversion are continuous in the stated topology. □

Thus a nonempty comparison space is a torsor: choosing one comparison identifies it with a group. Without choosing an origin, it has the intrinsic ternary operation

$$[P, Q, R]_{xy} = P_{xy}Q^{-1}_{xy}R_{xy},$$

which is preserved by restriction. Both the spaces and the restriction maps carry information beyond the single number $R_E(g)$.

7.3 Characteristic obstructions and line bundles

For a natural characteristic class c, define

$$\delta_r c(E) := (p_2^* c(E) - p_1^* c(E))|_{U_r(g)}.$$

An isomorphism of the two pullbacks implies $\delta_r c(E) = 0$. This applies, with the appropriate coefficients, to Stiefel–Whitney, Pontryagin and Chern classes. In higher rank these necessary conditions need not be sufficient.

Theorem 7.6 (Line-bundle obstruction and topology)**.** *For a complex line bundle L,*

$$\mathcal{C}_r(L, g) \neq \varnothing \quad \Longleftrightarrow \quad \delta_r c_1(L) = 0 \text{ in } H^2(U_r(g); \mathbb{Z}).$$

For a real line bundle, the corresponding criterion is $\delta_r w_1(L) = 0$ in $H^1(U_r(g); \mathbb{F}_2)$.

If nonempty, $\pi_0\mathcal{C}_r(L, g)$ for a complex line is a torsor for $H^1(U_r(g), \Delta; \mathbb{Z})$, and each path component is contractible. For a real line, the entire nonempty space $\mathcal{C}_r(L, g)$ is contractible.

Proof. A comparison is a nowhere-zero section of the line bundle $p_1^*L^\vee \otimes p_2^*L$, followed by diagonal normalization. Classification of complex and real line bundles by c_1 and w_1 gives the criteria; see [5, Proposition 3.10].

Fix P^0. In the complex case all other comparisons are uniquely $P^0 f$, with $f : U_r \to \mathbb{C}^\times$ smooth and $f|_\Delta = 1$. Relative homotopy classes of these maps are $H^1(U_r, \Delta; \mathbb{Z})$, since $\mathbb{C}^\times \simeq K(\mathbb{Z}, 1)$. Relative smooth approximation identifies smooth and continuous classes. The inclusion $\Delta \subset U_r$ is a cofibration, as it has a tubular neighborhood.

The space U_r is connected: join (x, y) to (x, x) along a minimizing geodesic and then move within Δ. In the component of $f = 1$, every f has a unique smooth logarithm $u : U_r \to \mathbb{C}$ with $u|_\Delta = 0$. Lifting through the exponential covering is continuous in the compact-open smooth topology. Thus this component is homeomorphic to the vector space of such u, and is contractible. Translation gives the other components. For real lines, a normalized ratio is positive on connected U_r, so the real logarithm contracts the whole space. □

Remark 7.7 (What a finite range of scales can detect). If a complex line has a comparison at r_m, choose it as origin and restrict it to $r_1 < \cdots < r_m$. The resulting diagram of comparison spaces is isomorphic to the diagram for the trivial complex line. Over any such finite range, bundle dependence is therefore confined to existence; the normalized scalar mapping spaces themselves depend only on (M, g).

7.4 The real-line radius

Let L be a nontrivial real line bundle and $\alpha = w_1(L) \in H^1(M; \mathbb{F}_2)$. Define its mod-2 systole by

$$\sigma_g(\alpha) = \inf_{\substack{c \text{ closed and piecewise smooth} \\ \langle \alpha, [c] \rangle = 1}} \mathrm{Length}_g(c).$$

It is positive, since sufficiently short loops lie in a contractible metric ball, and finite, since $\alpha \neq 0$ evaluates nontrivially on some loop.

Theorem 7.8 (Half-systole formula)**.** *For every nontrivial real line bundle,*

$$R_L(g) = \tfrac{1}{2}\sigma_g(w_1(L)).$$

Moreover, a comparison exists at $r = \sigma_g(w_1(L))/2$.

Proof. Choose a fiber metric. The two unit vectors in each fiber form a double cover of M, so a path has a canonical isometric transport on L, depending only on its endpoint-fixed homotopy class.

Suppose $0 < r \le \sigma_g(\alpha)/2$. For $d_g(x, y) < r$, choose any path from x to y of length less than r. Two such paths give the same transport: otherwise their concatenation would be an α-odd loop of length less than $2r \le \sigma_g(\alpha)$. Define P_{xy} by this transport. To see smoothness, fix one such path and perturb its endpoints within small coordinate balls. The resulting path lengths remain less than r, and the double-cover transport varies smoothly in local trivializations. These local descriptions agree by independence of the path. Constant paths give $P_{xx} = \mathrm{id}$.

Conversely, if $r > \sigma_g(\alpha)/2$, choose an α-odd loop c based at x of length $\ell < 2r$. Each point of c has distance at most $\ell/2 < r$ from x, by using the shorter of its two arcs. If a comparison existed, $P_{x,c(t)}v$ for $0 \ne v \in L_x$ would be a nowhere-zero section of c^*L. This contradicts $\langle \alpha, [c] \rangle = 1$. □

Example 7.9 (A circle and products)**.** The Möbius line over a circle of circumference a has radius $a/2$. Pulling it back to a product with any closed Riemannian manifold leaves the radius unchanged by Proposition 7.4. In particular, shrinking the extra factor can send the product volume to zero while keeping this radius fixed.

7.5 Flat tori

Let $T = V/\Lambda$ be a flat torus, where V is a Euclidean vector space and Λ a full lattice. Under $H^2(T; \mathbb{Z}) \cong \mathrm{Alt}^2(\Lambda, \mathbb{Z})$, a complex line bundle L has an alternating form $B = c_1(L)$. Set

$$\Lambda_0 = \{\lambda \in \Lambda : B(\lambda, \mu) = 0 \text{ for all } \mu \in \Lambda\}.$$

Theorem 7.10 (The radical-lattice formula)**.** *With the convention* $\min \varnothing = \infty$*,*

$$R_L(g) = \frac{1}{2} \min_{\lambda \in \Lambda \setminus \Lambda_0} \|\lambda\|.$$

If the value is finite, a comparison exists at that radius.

Proof. If $B = 0$, the line is trivial. Otherwise let ℓ denote the minimum in the formula.

Upper bound. Choose $\lambda \notin \Lambda_0$ of length ℓ and $\mu \in \Lambda$ with $B(\mu, \lambda) \ne 0$. For $r > \ell/2$, the map

$$F : (\mathbb{R}/\mathbb{Z})^2 \longrightarrow U_r(g), \qquad F(t, s) = (t\mu, t\mu + s\lambda)$$

is well defined: the two points are at distance at most $\min(s, 1 - s)\ell \le \ell/2$, for $0 \le s \le 1$. The first projection of F has rank at most one and the second traces the two lattice directions. Hence

$$\langle F^* \delta_r c_1(L), [(\mathbb{R}/\mathbb{Z})^2] \rangle = B(\mu, \lambda) \ne 0.$$

Theorem 7.6 rules out a comparison.

Lower bound. Put $W = \mathrm{span}_{\mathbb{R}} \Lambda_0$. Since B is integral, W is the radical of its real extension and is rational with respect to Λ. Thus $q(\Lambda)$ is a lattice in V/W, and

$$q : T \longrightarrow S := (V/W)/q(\Lambda)$$

is a quotient torus. The alternating form descends to Λ/Λ_0, so line-bundle classification gives $L \cong q^*L'$ for a complex line $L' \to S$.

Fix $r \le \ell/2$ and let $B_r^T(0)$ be the open metric ball in T. A point z in this ball has a representative $v \in V$ with $\|v\| < r$. The map

$$a_r : B_r^T(0) \longrightarrow V/W, \qquad a_r(z) = v + W,$$

is well defined: two such representatives differ by $\nu \in \Lambda$ with $\|\nu\| < 2r \le \ell$, hence $\nu \in \Lambda_0$. It is smooth by local lifts and satisfies $a_r(0) = 0$. For $(x, y) \in U_r(g)$, use the path

$$t \longmapsto q(x) + t\, a_r(y - x) \quad \text{in } S, \qquad 0 \le t \le 1.$$

Its endpoints are $q(x), q(y)$, and it is constant on the diagonal. Parallel transport for any connection on L', followed by the fixed identification $L \cong q^*L'$, gives the required comparison. □

Example 7.11 (Trivial on every ball does not suffice)**.** Let $T^2 = \mathbb{R}^2/(a\mathbb{Z} \oplus b\mathbb{Z})$ with $0 < a < b$, and let L have nonzero degree. Its alternating form has zero radical, so $R_L = a/2$. Yet for $a/2 < r < b/2$, every open metric ball is a cylinder and L restricts trivially to each one. Thus triviality on each individual ball does not guarantee a comparison depending jointly and smoothly on the center and endpoint.

7.6 Examples beyond the extension radius

Rectangular tangent tori

For the same rectangular T^2, the tangent bundle is trivial, so $R_{TT^2} = \infty$. Nevertheless the comparison system changes with scale. Translation gives

$$(U_r, \Delta) \cong (T^2 \times B_r^T(0), T^2 \times \{0\}).$$

Using the translation frame, comparisons become smooth maps to $\mathrm{GL}_2(\mathbb{R})$ equal to I on the diagonal. Their determinants are positive because U_r is connected. Polar decomposition retracts $\mathrm{GL}_2^+(\mathbb{R})$ to $\mathrm{SO}(2) \cong S^1$, preserving the identity. Consequently

$$\pi_0 \mathcal{C}_r(TT^2, g) \cong H^1(U_r, \Delta; \mathbb{Z}) \cong H^1(B_r^T(0); \mathbb{Z}) \cong \begin{cases} 0, & 0 < r \le a/2, \\ \mathbb{Z}, & a/2 < r \le b/2, \\ \mathbb{Z}^2, & r > b/2. \end{cases}$$

Each path component is contractible: the positive-definite factor in polar decomposition is contractible, and the normalized S^1 mapping-space components are contracted by logarithms as in Theorem 7.6.

For completeness, the ball is a disk for $r \le a/2$ and a cylinder for $a/2 < r \le b/2$. Above $b/2$ and below the covering radius $\sqrt{a^2 + b^2}/2$, it is a torus with a closed disk removed; at that radius it is a punctured torus, and above it the whole torus. These descriptions give the cohomology above.

The identifications as groups use the translation frame as origin. Across the second transition, restriction induces the projection $\mathbb{Z}^2 \to \mathbb{Z}$. This map, or the natural torsor structure, is essential: as bare sets, $\mathbb{Z}$ and $\mathbb{Z}^2$ are both countably infinite, so their discrete homotopy types alone do not distinguish the two stages.

An unstable sphere obstruction

On the unit round S^5, $\mathrm{inj} = \mathrm{diam} = \pi$. The tangent bundle is nontrivial by Adams's vector-field theorem [41], while

$$TS^5 \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^6.$$

Proposition 7.2 therefore gives

$$R_{TS^5} = \pi, \qquad R_{TS^5 \oplus \underline{\mathbb{R}}} = \infty.$$

All positive-degree Stiefel–Whitney and Pontryagin classes of TS^5 vanish by stable triviality. Its Euler class also vanishes: viewing $S^5 \subset \mathbb{C}^3$, the vector field $x \mapsto ix$ is nowhere zero. Thus characteristic-class differences do not supply complete existence obstructions in higher rank, and stabilization can destroy finite-radius information.

7.7 Surface reconstruction from finite covers

Let S be a closed oriented surface of genus at least two, g a smooth metric with strictly negative Gaussian curvature, and $\Gamma = \pi_1(S)$ with a fixed marking and basepoint. For every finite-index subgroup $H \leq \Gamma$, let $p_H : S_H \to S$ be the corresponding connected based cover. For $0 \neq \alpha \in H^1(S_H; \mathbb{F}_2) = \mathrm{Hom}(H, \mathbb{F}_2)$, let L_α be the real line with $w_1(L_\alpha) = \alpha$, and set

$$\mathcal{R}_g(H, \alpha) = R_{L_\alpha}(p_H^* g).$$

These positive finite numbers are *labelled* by (H, α); the labels are part of the data. Write $\ell_g(\gamma)$ for the length of the unique closed geodesic in the free homotopy class of $1 \neq \gamma \in \Gamma$.

The reconstruction uses two covering results. Scott's theorem, in the form quantified by Patel [42], gives a finite cover in which a primitive closed geodesic has a degree-one simple lift. Theorem C of Alvarez–Brum–Martínez–Potrie, with Wolff [43], states that a simple closed geodesic on a closed hyperbolic surface has, for every $K > 0$, a nonseparating degree-one lift to a finite cover in which every other simple closed geodesic has length greater than K.

Theorem 7.12 (Reconstruction from labelled radii)**.** *For a primitive $\gamma \neq 1$, define the metric-independent index set*

$$I_\gamma = \{(H, \alpha) : [\Gamma : H] < \infty,\ \gamma \in H,\ \alpha(\gamma) = 1\}.$$

Then

$$\ell_g(\gamma) = 2 \sup_{(H,\alpha) \in I_\gamma} \mathcal{R}_g(H, \alpha).$$

The supremum is attained. Thus all the labelled radii determine the marked length spectrum. In particular, if g, h have identical labelled data, there is a diffeomorphism ϕ isotopic to the identity such that $\phi^* h = g$.

Proof. If $(H, \alpha) \in I_\gamma$, the lift of the geodesic representing γ is α-odd and has length $\ell_g(\gamma)$. By Theorem 7.8,

$$2\mathcal{R}_g(H, \alpha) = \sigma_{p_H^* g}(\alpha) \leq \ell_g(\gamma).$$

For the opposite inequality, first pass to a finite cover S_1 where γ has a simple degree-one lift. Choose a hyperbolic metric q on S_1 and a constant C such that $\mathrm{Length}_q(c) \leq C\,\mathrm{Length}_g(c)$ for every curve c; here g denotes the lifted metric. The same inequality holds on every further cover. Apply Theorem C of [43] to the q-geodesic in that simple class with $K > C\ell_g(\gamma)$.

In the resulting cover there is a nonseparating degree-one lift $\widehat{\gamma}$. Every other simple free homotopy class has q-geodesic length greater than K, hence g-geodesic length greater than $K/C > \ell_g(\gamma)$. A shortest noncontractible closed geodesic is simple: splitting at a self-intersection produces a shorter noncontractible loop. It follows that $\widehat{\gamma}$ realizes the g-systole of this cover. Choose α with $\alpha(\widehat{\gamma}) = 1$, possible because $\widehat{\gamma}$ is nonseparating. Its odd systole is then exactly $\ell_g(\gamma)$. Choose the basepoint lift so that the corresponding subgroup contains γ. This gives a pair in I_γ attaining equality.

Every nontrivial surface-group element is a power of a primitive element, and $\ell_g(\eta^k) = |k|\ell_g(\eta)$. Thus the data recover all marked lengths. The last assertion is the marked length spectrum rigidity theorem for negatively curved surfaces [44]; see also the stronger Anosov-surface result [45]. □

Corollary 7.13 (Exact multiplicative comparison)**.** *For two such metrics on the same marked surface,*

$$\sup_{H,\ \alpha\neq 0}\left|\log\frac{\mathcal{R}_g(H,\alpha)}{\mathcal{R}_h(H,\alpha)}\right| = \sup_{\gamma\neq 1}\left|\log\frac{\ell_g(\gamma)}{\ell_h(\gamma)}\right|.$$

Both suprema are finite.

Proof. A uniform multiplicative bound for marked lengths passes to the infimum over α-odd classes, hence to the radii by Theorem 7.8. Conversely, the same bound for every labelled radius passes to the supremum over I_γ in Theorem 7.12. The metrics are bi-Lipschitz on compact S, which makes both bounds finite. □

Remark 7.14 (Scope of the reconstruction result)*.* The covering and rigidity mechanisms are supplied by the cited work. Indeed, the same isolation argument proves

$$\ell_g(\gamma) = \sup_{\substack{[\Gamma:H]<\infty\\ \gamma\in H}} \operatorname{sys}(S_H, p_H^* g) \quad \text{for primitive } \gamma.$$

Here sys is the length of a shortest noncontractible closed curve. The comparison-radius formulation packages these mechanisms through line bundles; it does not assert rigidity from an unordered collection of radii, or from finitely many covers.

7.8 Related work

The individual maps P_{xy} are the linear direct connections studied by Kubarski and Teleman [46], formerly called quasi-connections. Discrete principal-bundle connections provide a related framework [47]. The organizing emphasis here is the metric family of domains U_r, extension obstructions, and the full restriction system.

The line-bundle criteria use classical classification [5, Proposition 3.10]. Persistent characteristic classes of line bundles have been studied by Tinarrage [48]. Homotopic distance between maps [49] is adjacent to the comparison of the two projections, although it measures a different invariant.

The real-line formula translates extension into the length of loops detected by a mod-2 class. Subsection 7.7 combines that formula with finite-cover isolation and marked-length rigidity results. Further questions concern higher-rank unstable obstructions, behavior at critical radii, and examples where the restriction system detects geometric features missed by ordinary characteristic classes.

8 Harmonic closure of differential forms

We study the real algebra generated by harmonic forms under wedge product, exterior differentiation and Hodge star. All closure operations are algebraic, not topological.

8.1 Definitions and basic properties

Throughout this section, M^n is a smooth, connected, closed, oriented manifold, $n \geq 2$, and g is a smooth Riemannian metric. Write $\Omega^\bullet(M)$ for real smooth differential forms, δ_g for the formal L^2-adjoint of d, and

$$\Delta_g = d\delta_g + \delta_g d, \qquad \mathcal{H}_g^k = \ker(\Delta_g|_{\Omega^k(M)}).$$

Hodge theory gives $\mathcal{H}_g^k \cong H^k_{\mathrm{dR}}(M)$; set $b_k = \dim H^k_{\mathrm{dR}}(M)$ and $B = \sum_{k=0}^n b_k$.

Definition 8.1 (Harmonic closure)**.** The *harmonic closure* of g is the smallest graded real subalgebra

$$\mathcal{C}_g = \langle \mathcal{H}_g^\bullet \rangle_{\wedge, d, *_g} \subseteq \Omega^\bullet(M)$$

containing every harmonic form and preserved by d and $*_g$. The metric has *finite harmonic closure* if $\dim_{\mathbb{R}} \mathcal{C}_g < \infty$.

Only real scalar coefficients are allowed: $\mathcal{C}_g$ is not a $C^\infty(M)$-module. Equivalently, put $V_0 = \mathcal{H}_g^\bullet$ and

$$V_{j+1} = \operatorname{span}_{\mathbb{R}}(V_j \cup dV_j \cup *_g V_j \cup \{\alpha \wedge \beta : \alpha, \beta \in V_j\}).$$

Then each V_j is finite-dimensional and $\mathcal{C}_g = \bigcup_{j \geq 0} V_j$. In particular, $1, \operatorname{vol}_g \in \mathcal{C}_g$. Since $\delta_g = \pm *_g d *_g$, both δ_g and Δ_g preserve $\mathcal{C}_g$.

For any diffeomorphism $\phi : N \to M$, $\mathcal{C}_{\phi^* g} = \phi^* \mathcal{C}_g$; orientation reversal only changes the sign of the Hodge star. Constant rescaling does not change the embedded algebra: $\mathcal{C}_{c^2 g} = \mathcal{C}_g$ for $c > 0$, because the harmonic spaces are unchanged and $*_{c^2 g}|_{\Omega^k} = c^{n-2k} *_g$.

Proposition 8.2 (Scalar criterion and pointwise bounds)**.** *The following conditions are equivalent:*

$$\dim \mathcal{C}_g < \infty \quad \Longleftrightarrow \quad \mathcal{C}_g^0 = \mathbb{R} \cdot 1.$$

If they hold, same-degree elements of $\mathcal{C}_g$ have constant pointwise inner products, and evaluation at every $x \in M$ is injective:

$$\operatorname{ev}_x : \mathcal{C}_g^k \hookrightarrow \Lambda^k T_x^* M.$$

Consequently, with $c_k = \dim \mathcal{C}_g^k$,

$$b_k \leq c_k \leq \binom{n}{k}, \qquad c_k = c_{n-k}, \qquad \dim \mathcal{C}_g \leq 2^n.$$

Every nonzero element of $\mathcal{C}_g^k$ has constant positive norm. If $\mathcal{C}_g$ is infinite, it contains a nonconstant function f, and $1, f, f^2, \ldots$ are linearly independent.

Proof. In a finite-dimensional algebra of functions, the powers of any f satisfy a nonzero polynomial relation $P(f) = 0$. Its image is finite, so connectedness makes f constant. Conversely, if $\mathcal{C}_g^0 = \mathbb{R}$, then

$$\langle \alpha, \beta \rangle_g = *_g(\alpha \wedge *_g \beta) \in \mathbb{R} \qquad (\alpha, \beta \in \mathcal{C}_g^k).$$

A form vanishing at one point therefore vanishes everywhere. This proves evaluation injectivity and the bounds, without initially assuming finite dimensionality. Hodge star gives $c_k = c_{n-k}$. Finally, a nonconstant smooth function on connected M has an interval in its image, so its powers satisfy no nonzero polynomial relation. □

Proposition 8.3 (Finite Hodge model)**.** *If $\mathcal{C}_g$ is finite, its inclusion in $\Omega^\bullet(M)$ is a quasi-isomorphism of differential graded algebras. There is an orthogonal decomposition*

$$\mathcal{C}_g = \mathcal{H}_g^\bullet \oplus d\mathcal{C}_g \oplus \delta_g \mathcal{C}_g.$$

Writing $r_k = \operatorname{rank}(d : \mathcal{C}_g^k \to \mathcal{C}_g^{k+1})$, with $r_{-1} = r_n = 0$, one has

$$c_k = b_k + r_{k-1} + r_k, \qquad \varepsilon(g) := \frac{\dim \mathcal{C}_g - B}{2} = \sum_{k=0}^{n-1} r_k \in \mathbb{Z}_{\geq 0}.$$

Moreover, $\varepsilon(g) = 0$ exactly when g is geometrically formal.

Proof. The restrictions of d and δ_g are adjoint and preserve this finite-dimensional space. Finite-dimensional Hodge decomposition applies. Its harmonic subspace equals $\mathcal{H}_g^\bullet$, since it is defined by the same equations $d\alpha = \delta_g \alpha = 0$ and contains all ambient harmonic forms. Thus inclusion identifies cohomology through the same harmonic representatives. The dimension formulas follow degreewise. Finally, $\varepsilon(g) = 0$ is equivalent to $\mathcal{C}_g = \mathcal{H}_g^\bullet$, equivalently to closure of harmonic forms under wedge products, the definition of geometric formality [50]. □

The profile $(c_0, \dots, c_n)$ and $\varepsilon(g)$ are scale-invariant metric invariants. One may set $\varepsilon(g) = \infty$ for infinite closure. A finite real de Rham model does not by itself imply rational ellipticity or determine integral cohomology.

8.2 Examples

Example 8.4 (Cohomology spheres and flat tori)**.** If M is a rational homology sphere, then for every metric

$$\mathcal{C}_g = \mathcal{H}^\bullet_g = \operatorname{span}_{\mathbb{R}}\{1, \operatorname{vol}_g\}, \qquad \dim \mathcal{C}_g = 2.$$

On a flat n-torus all harmonic forms are parallel; they are precisely the constant forms on the universal cover. Hence $\mathcal{C}_g = \mathcal{H}^\bullet_g$ has dimension 2^n. Both examples have $\varepsilon(g) = 0$.

Example 8.5 (The Heisenberg three-manifold)**.** Let M be a compact Heisenberg quotient with global coframe

$$d\alpha = d\beta = 0, \qquad d\gamma = \alpha \wedge \beta.$$

Make this coframe orthonormal and orient by $\alpha \wedge \beta \wedge \gamma$. The harmonic forms are

$$\mathcal{H}^\bullet_g = \operatorname{span}_{\mathbb{R}}\{1, \alpha, \beta, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \wedge \beta \wedge \gamma\}.$$

Indeed, these forms are closed and coclosed; invariant cohomology and Nomizu's theorem [51] show that they account for all cohomology classes. Since $*_g(\alpha \wedge \beta) = \gamma$,

$$\mathcal{C}_g = \Lambda^\bullet \operatorname{span}_{\mathbb{R}}\{\alpha, \beta, \gamma\}, \qquad \dim \mathcal{C}_g = 8, \qquad \varepsilon(g) = 1.$$

This metric is not geometrically formal: $\alpha \wedge \beta = d\gamma$ is nonzero and exact. The triple Massey product $\langle [\alpha], [\alpha], [\beta] \rangle$ is represented by $\pm\alpha \wedge \gamma$, with zero indeterminacy and nonzero value.

Example 8.6 (An infinite closure on the two-torus)**.** On $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, take $g = e^{2u(x,y)}(dx^2 + dy^2)$ with u smooth and nonconstant. Hodge star on one-forms is conformally invariant, so dx, dy remain harmonic, while

$$*_g(dx \wedge dy) = e^{-2u} \in \mathcal{C}^0_g.$$

Proposition 8.2 gives $\dim \mathcal{C}_g = \infty$. Also, an oriented surface of genus $q \geq 2$ has $b_1 = 2q > 2$, so it admits no finite-closure metric.

Proposition 8.7 (Homogeneous examples)**.** *Every compact oriented Riemannian manifold admitting a connected transitive group of isometries has finite harmonic closure.*

Proof. Each group element is isotopic to the identity and hence acts trivially on cohomology. It preserves harmonic forms, so uniqueness of harmonic representatives makes them invariant. Invariant forms constitute a finite-dimensional algebra preserved by d and $*_g$; it contains $\mathcal{C}_g$. □

8.3 Lie algebra and reconstruction

Theorem 8.8 (Canonical infinitesimal action)**.** *Suppose $\mathcal{C}_g$ is finite, and set*

$$\mathfrak{l}_g = \{\alpha^\sharp : \alpha \in \mathcal{C}^1_g\}, \qquad r = \dim \mathcal{C}^1_g.$$

Then $\mathfrak{l}_g$ is an r-dimensional Lie algebra of vector fields whose evaluation has rank r at every point. Its fields preserve volume and integrate to a locally free action of a simply connected Lie group. Contraction and Lie derivative by these fields preserve $\mathcal{C}_g$. The action need not be isometric.

Proof. For $\alpha \in \mathcal{C}_g^1$, $\delta_g \alpha$ is constant and $\langle \delta_g \alpha, 1 \rangle_{L^2} = \langle \alpha, d1 \rangle_{L^2} = 0$; thus $\delta_g \alpha = 0$. For one-forms one has

$$\delta_g(\alpha \wedge \beta) = (\delta_g \alpha)\beta - (\delta_g \beta)\alpha - [\alpha^\sharp, \beta^\sharp]^\flat.$$

Hence $\mathfrak{l}_g$ is bracket-closed. Evaluation injectivity follows from Proposition 8.2; coclosedness is equivalent to zero divergence. Compactness makes all fields complete, and the integration theorem for complete finite-dimensional Lie algebra actions gives the locally free action. Finally, $\iota_{\alpha^\sharp}\omega = \pm *_g(\alpha \wedge *_g \omega)$ preserves $\mathcal{C}_g$, and Cartan's formula gives the assertion for Lie derivatives. □

Corollary 8.9 (Full rank)**.** *If $r = n$, then $\mathcal{C}_g = \Lambda^\bullet(\mathcal{C}_g^1)$ and (M, g) is isometric to $\Gamma\backslash G$ with a metric descended from a left-invariant metric on a simply connected Lie group G, where Γ is discrete and cocompact. Furthermore, finite closure never has $r = n - 1$.*

Proof. Constant inner products give a global orthonormal coframe $\theta^1, \dots, \theta^n$ in $\mathcal{C}_g^1$. Its wedges, together with evaluation injectivity, exhaust $\mathcal{C}_g$; thus

$$d\theta^i = \sum_{j<k} c^i_{jk} \theta^j \wedge \theta^k \qquad \text{with constant } c^i_{jk}.$$

These are Maurer–Cartan equations. Equivalently, the complete action of Theorem 8.8, taken as a right action, is transitive with discrete stabilizer Γ, giving $M = \Gamma\backslash G$ and the stated metric. If $r = n - 1$, the star of the wedge of an orthonormal basis of $\mathcal{C}_g^1$ supplies an additional orthogonal unit one-form in $\mathcal{C}_g^1$, a contradiction. □

Theorem 8.10 (Nilmanifold reconstruction)**.** *Let G be simply connected and nilpotent, let $\Gamma < G$ be discrete and cocompact, and give $M = \Gamma\backslash G$ a metric descended from any left-invariant metric. Then, as an embedded algebra of forms,*

$$\mathcal{C}_g = \Lambda^\bullet \mathfrak{g}^*, \qquad \dim \mathcal{C}_g = 2^n.$$

Thus $(\mathcal{C}_g, \wedge, d, *_g)$ determines the metric Lie algebra and the simply connected metric universal cover, but not the lattice Γ.

Proof. The invariant algebra $A = \Lambda^\bullet \mathfrak{g}^*$ is preserved by d and $*_g$. Nomizu's theorem [51] makes $A \hookrightarrow \Omega^\bullet(M)$ a quasi-isomorphism. Internal Hodge decomposition in A shows that its harmonic space is the whole ambient harmonic space. Hence $\mathcal{C}_g \subseteq A$, so $\mathcal{C}_g$ is finite.

Choose a coframe adapted to the nilpotent structure, with $d\theta^i \in \Lambda^2 \operatorname{span}_{\mathbb{R}}\{\theta^1, \dots, \theta^{i-1}\}$. Inductively, if the earlier forms lie in $\mathcal{C}_g$, then $d\theta^i \in \mathcal{C}_g$ and is globally exact. Proposition 8.3 supplies $\eta \in \mathcal{C}_g^1$ with $d\eta = d\theta^i$. The invariant one-form $\theta^i - \eta$ is closed and coclosed: its codifferential is a constant whose integral is zero. It is therefore harmonic, so $\theta^i \in \mathcal{C}_g$. Induction proves equality. On degree one, d recovers the Lie bracket and $*_g(\alpha \wedge *_g \beta)$ recovers the inner product. □

These quotient metrics are locally homogeneous; they need not be globally homogeneous as Riemannian manifolds.

8.4 Conformal rigidity

Theorem 8.11 (Rigidity from one cohomology class)**.** *Let $0 < k < n$ and $0 \neq a \in H^k_{\mathrm{dR}}(M)$. Suppose the harmonic representatives of a for g and $h = e^{2u}g$ each have constant pointwise length in their respective metrics. Then the representatives coincide and u is constant.*

Proof. Put $p = n/k > 1$ and, on smooth closed representatives of a, define

$$E_g(\omega) = \int_M |\omega|_g^p \, dV_g.$$

If α is harmonic with $|\alpha|_g \equiv c > 0$, then for $\omega = \alpha + d\eta$, strict convexity gives

$$E_g(\omega) \geq E_g(\alpha) + pc^{p-2} \int_M \langle \alpha, d\eta \rangle_g \, dV_g = E_g(\alpha),$$

with equality only for $\omega = \alpha$. Thus α uniquely minimizes E_g. Since $|\omega|_h = e^{-ku}|\omega|_g$ and $dV_h = e^{nu} dV_g$, one has $E_h = E_g$. The two harmonic representatives are therefore equal. Their constant positive norms satisfy $c_h = e^{-ku} c_g$, forcing u to be constant. □

Corollary 8.12 (Conformal uniqueness and symmetries)**.** *Suppose M is not a rational homology sphere. Then each conformal class contains at most one finite-closure metric up to constant scaling. Every nonconstant conformal rescaling of a finite-closure metric has infinite closure. Every conformal self-diffeomorphism of a finite-closure metric is an isometry.*

Proof. Some intermediate cohomology class is nonzero. Proposition 8.2 and Theorem 8.11 apply to its harmonic representatives. For a conformal self-map ϕ, naturality gives finite closure for $\phi^* g$, so $\phi^* g = c^2 g$. Equality of total volumes, viewed as densities, gives $c = 1$. □

Theorem 8.13 (Closedness and generic infinitude)**.** *The finite-closure locus*

$$\mathcal{F}(M) = \{g \in \mathrm{Met}^\infty(M) : \dim \mathcal{C}_g < \infty\}$$

is closed in the C^∞ topology. If M is not a rational homology sphere, $\mathcal{F}(M)$ is nowhere dense and its complement is open dense. For a rational homology sphere, $\mathcal{F}(M) = \mathrm{Met}^\infty(M)$.

Proof. Suppose $g_j \to g$ smoothly with $g_j \in \mathcal{F}(M)$. Fix a basis of cohomology. Standard elliptic dependence of harmonic projections implies that its g_j-harmonic representatives converge smoothly to the g-harmonic representatives. Every fixed finite expression in these forms using $d, \wedge, *_{g_j}$ therefore converges smoothly. If $\mathcal{C}_g$ were infinite, $2^n + 1$ such expressions could be chosen independently. Their Gram determinant in a fixed background L^2 inner product stays positive for large j, contradicting $\dim \mathcal{C}_{g_j} \leq 2^n$. This proves closedness; the C^∞ topology here is metrizable. When intermediate cohomology is nonzero, arbitrarily small nonconstant conformal perturbations leave $\mathcal{F}(M)$ by Corollary 8.12, so it has empty interior. The remaining case was computed in the first example. □

8.5 Low-dimensional consequences

Proposition 8.14 (Dimension three)**.** *For a closed, connected, oriented three-manifold, the finite-closure metrics are exactly as follows.*

$b_1(M)$	*Condition on g*	$\dim \mathcal{C}_g$
0	*Every metric.*	2
1	*A nonzero harmonic one-form has constant length.*	4
2	*A descended left-invariant metric on a compact Heisenberg quotient, up to isometry.*	8
3	*A flat torus metric.*	8
> 3	*No such metric.*	—

Proof. The bound gives $b_1 \leq 3$. By Hodge star, $\mathcal{C}_g^2 = *\mathcal{C}_g^1$; the excluded rank $n - 1$ gives $r = \dim \mathcal{C}_g^1 \in \{0, 1, 3\}$. If $b_1 = 0$, only degrees zero and three have harmonic forms. If $b_1 = 1$, write its generator as α; constant length makes $\mathrm{span}_{\mathbb{R}}\{1, \alpha, *\alpha, \mathrm{vol}_g\}$ an algebra, proving both directions.

If $b_1 = 2$, choose pointwise orthonormal harmonic forms α, β, and put $\gamma = *(\alpha \wedge \beta)$. Corollary 8.9 makes their structure coefficients constant. The equations $d\alpha = d\beta = 0$ and

$d * \alpha = d * \beta = 0$ force $d\gamma = c\,\alpha \wedge \beta$. Here $c \neq 0$, since otherwise all three forms would be harmonic. These are Heisenberg structure equations; the converse follows from Theorem 8.10. If $b_1 = 3$, the harmonic orthonormal coframe is closed, giving a flat torus. Its converse is immediate. □

Proposition 8.15 (A four-dimensional reduction)**.** *If $n = 4$ and $\chi(M) \neq 0$, then g has finite harmonic closure if and only if it is geometrically formal. In particular this applies to closed simply connected four-manifolds.*

Proof. A nonzero element of $\mathcal{C}_g^1$ would have nowhere-zero metric dual, contradicting $\chi(M) \neq 0$. Thus $\mathcal{C}_g^1 = \mathcal{C}_g^3 = 0$. The differential vanishes on $\mathcal{C}_g$, and Proposition 8.3 gives geometric formality. The converse is automatic. For a simply connected closed four-manifold, $\chi(M) = 2 + b_2(M) > 0$. □

8.6 Related work

Geometric formality [50] is exactly $\mathcal{C}_g = \mathcal{H}_g^\bullet$. Constant-length harmonic forms and nilmanifold rigidity were studied in [52]; the high-b_1 conclusions above overlap that theory. Nomizu's theorem [51] supplies the invariant cohomology model. The conformal argument is closely related to Ornea–Pilca's middle-degree uniqueness theorem [53, Proposition 2.4] and the nonlinear conformal n/k-Hodge theory of Kangasniemi [54]. Hájek's small subalgebras are generated by harmonic data and preserved by a special propagator [55, Definition 3.19]; ABC-geometric formality admits a Hodge-star-stable bidifferential algebra formulation with additional complex hypotheses [56, Definition 4.4]. The construction considered here is the real canonical $(\wedge, d, *)$-closure.

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