On the Hodge, Tate, and standard conjectures for abelian varieties

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To the memory of my teacher, John Tate, with regret that he did not see this support for his vision.

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This is a report on two preprints released by OpenAI on October 6, 2026,

- Milne's rationality conjecture for abelian varieties,
- The rational Hodge conjecture for CM abelian varieties.

placing them in context.[1]

1 The rationality conjecture

NOTATION 1.1. Everything is $\otimes\mathbb{Q}$. For example, abelian varieties are objects in the category in which Homs are $\mathrm{Hom}^0(A,B) = \mathrm{Hom}(A,B)\otimes\mathbb{Q}$. The Lefschetz classes are the classes in the $\mathbb{Q}$-algebra generated by divisor classes. An abelian variety in characteristic zero is CM if its endomorphism algebra is largest possible.

Statement of the rationality conjecture

Let $\mathbb{Q}^{\mathrm{al}}$ be an algebraic closure of $\mathbb{Q}$ and w a place of $\mathbb{Q}^{\mathrm{al}}$ lying over p. The residue field $\mathbb{F}$ at w is an algebraic closure of $\mathbb{F}_p$.

1.2. Let A be an abelian variety of dimension n over $\mathbb{Q}^{\mathrm{al}}$ with good reduction to an abelian variety A_0 over $\mathbb{F}_p$. Let γ be an algebraic class on A and δ a Lefschetz class on A_0 of complementary codimension,

$$\begin{array}{ccccc} A & & H^{2*}(A,\mathbb{Q}_\ell(*)) & & \gamma \longmapsto \gamma_\ell \\ \Big| & & \Big\downarrow\simeq & & \\ A_0 & & H^{2*}(A_0,\mathbb{Q}_\ell(*)) & & \delta \longmapsto \delta_\ell,\ (\gamma_\ell)_0. \end{array}$$

Then

$$\langle (\gamma_\ell)_0 \cdot \delta_\ell \rangle \in H^{2n}_{\mathrm{et}}(A_0,\mathbb{Q}_\ell(n)) \simeq \mathbb{Q}_\ell,$$

[1] Expanded version of my notes for a talk at the University of Michigan, Ann Arbor, November 5, 2026. This article was written without the use of AI.

and we can ask whether this ℓ-adic number is a rational number independent of ℓ. In this case, the answer is obviously yes, because γ reduces to an algebraic class γ_0 on A_0 and $\langle(\gamma_\ell)_0 \cdot \delta_\ell\rangle = \langle\gamma_0 \cdot \delta\rangle \in \mathbb{Q}$,

$$\gamma \longmapsto \gamma_\ell$$

$$\delta, \gamma_0 \longmapsto \delta_\ell, (\gamma_0)_\ell = (\gamma_\ell)_0.$$

The conjecture says that the same should be true if γ is an absolute Hodge class.

CONJECTURE 1.3. *Let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ with good reduction at w. If γ is an absolute Hodge class on A and δ is a Lefschetz class on A_0 of complementary dimension, then $\langle\gamma_0 \cdot \delta\rangle$, which a priori lies in*[2] *$\prod \mathbb{Q}_\ell$, actually lies in $\mathbb{Q} \subset \prod \mathbb{Q}_\ell$.*

1.4. What is an absolute Hodge class? Let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$. Deligne's theorem (1982, 2.11) says that there exists a $\mathbb{Q}$-subspace $\mathcal{B}^r(A)$ of $\prod_\ell H^{2r}_{\mathrm{et}}(A, \mathbb{Q}_\ell(r))$ that is mapped isomorphically onto $\mathcal{B}^r(A_{\mathbb{C}}) \overset{\text{def}}{=} H^{2r}(A_{\mathbb{C}}, \mathbb{Q}(r)) \cap H^{p,p} \subset \prod_\ell H^{2r}_{\mathrm{et}}(A, \mathbb{Q}_\ell(r))$ by every homomorphism $\mathbb{Q}^{\mathrm{al}} \to \mathbb{C}$,

$$\begin{array}{ccc} \prod_\ell H^{2r}(A, \mathbb{Q}_\ell(r)) & \xrightarrow{\simeq} & \prod_\ell H^{2r}(A_{\mathbb{C}}, \mathbb{Q}_\ell(r)) \\ \uparrow & & \uparrow \\ \mathcal{B}^r(A) & \xrightarrow{\simeq} & H^{2r}(A_{\mathbb{C}}, \mathbb{Q}(r)) \cap H^{r,r}. \end{array}$$

The classes in $\mathcal{B}^r(A)$ are the absolute Hodge classes on A.

The condition for a class in $\prod_\ell H^{2r}_{\mathrm{et}}(A, \mathbb{Q}_\ell(r))$ to be absolutely Hodge is transcendental, and there is no rational γ_0. Whence the problem.

Proof of the rationality conjecture.

THEOREM 1.5 (OPENAI). *The rationality conjecture is true for all abelian varieties.*

1.6. There is an obvious approach to proving the conjecture. Let A, γ, δ be as in the statement of the conjecture. If we can lift (A_0, γ_0, δ) to (A', γ', δ') in characteristic zero with γ' an absolute Hodge class and δ' a Lefschetz class,

$$\begin{array}{ccc} (A, \gamma) & & (A', \gamma', \delta'), \\ & \searrow \quad \swarrow & \\ & (A_0, \gamma_0, \delta) & \end{array}$$

then

$$\langle\gamma_0 \cdot \delta\rangle = \langle\gamma'_0 \cdot \delta'_0\rangle = \langle\gamma' \cdot \delta'\rangle \in \mathbb{Q},$$

because γ' and δ' are both absolute Hodge classes. Sometimes it is possible to lift (A_0, γ_0, δ) and sometimes it isn't. The problem is to show that it is possible to lift it often enough to force rationality.

[2] Implicitly, $\ell \neq p$. There is also a p-factor, which I'm ignoring for simplicity.

1.7. First, two observations:

- ◇ an induction argument shows that it suffices to lift enough divisors (not Lefschetz classes);
- ◇ if A is a polarized abelian variety, then

$$\mathrm{NS}(A)_{\mathbb{Q}} \simeq \{u \in \mathrm{End}^0(A) \mid u^\dagger = u\}, \quad D_u \leftrightarrow u,$$

where $\dagger$ is the Rosati involution defined by the polarization, so lifting a divisor class is much the same as lifting an endomorphism.

1.8. Lifting theorems for abelian varieties have a long history. In the 1930s, Deuring showed that it is always possible to lift an elliptic curve and an endomorphism from characteristic p to characteristic 0. In the proof of his part of the Honda–Tate theorem, Honda showed that it is always possible to lift an abelian variety to a CM abelian variety in characteristic zero.[3] The conjecture of Langlands and Rapoport predicts that the reduction of any point on a Shimura variety lifts to a CM-point in the variety…The most recent lifting result is Theorem 2.2.7 of Kisin and Zhou 2025.

1.9. It has long been known[4] that it suffices prove the rationality conjecture for Weil classes on CM abelian varieties. This means that we have the following.

- ◇ An abelian variety A of CM-type over $\mathbb{Q}^{\mathrm{al}}$ (of dimension g).
- ◇ An action $\nu : E \to \mathrm{End}^0(A)$ of a CM field E on A such that the tangent space to A at 0 is a free $E \otimes_{\mathbb{Q}} \mathbb{Q}^{\mathrm{al}}$-module. This condition implies that the subspace

$$\bigwedge_E^{2r} H^1(A_{\mathbb{C}}, \mathbb{Q}) \subset H^{2r}(A_{\mathbb{C}}, \mathbb{Q}), \quad 2r = \dim_E H^1(A_{\mathbb{C}}, \mathbb{Q}),$$

consists of Hodge classes (i.e., classes of type (r,r)). Correspondingly, we get a $\mathbb{Q}$-vector space $W_E^r(A)$ of absolute Hodge classes on A, whose elements are the Weil classes on A.
- ◇ A polarization λ of A whose Rosati involution involution acts as complex conjugation on E.

Given such a triple, we have to prove that $\langle \gamma_0 \cdot \delta \rangle \in \mathbb{Q}$ for all Weil classes γ on A and Lefschetz classes of codimension $g - r$ on A_0.

1.10. We seek a set S of Lefschetz classes on A_0 such that

(a) S is big enough:

$$\langle \gamma_0 \cdot \delta \rangle \in \mathbb{Q} \text{ for all Weil classes } \gamma \text{ and } \delta \in S$$
$$\Rightarrow \ \langle \gamma_0 \cdot \delta \rangle \in \mathbb{Q} \text{ for all Weil classes } \gamma \text{ and } \textit{all } \delta;$$

(b) S is not too big: it is possible to lift $(A_0, \nu_0, \lambda_0, \delta)$ to characteristic zero.

[3] It is not always possible to lift an abelian variety with an endomorphism (even up to isogeny). Let E be a field of degree $2g$ over $\mathbb{Q}$ acting on an abelian variety A of dimension g. In characteristic zero, E must be a CM-field, but in characteristic p it need not be.

[4] See, for example, Milne 2008, 10.8.

OpenAI gives the following answer. Let

$$U = \{u \in \mathrm{End}^0(A) \mid u = u^\dagger, \quad ue = \bar{e}u \text{ for all } e \in E\};$$

then

$$S = \{D_u^r \cdot \xi^{g-2r} \mid u \text{ is an invertible element of } U\}$$

works. Here ξ is the divisor class of the polarization.

Given S, it only takes a page to prove that it satisfies (a) and a couple of pages to deduce (b) from the lifting theorem of Kisin and Zhou. See Madapusi 2026.

1.11. There are precedents for this argument. For example, there is an old theorem (Zink 1983, 2.7) stating that, under some conditions, it is possible to lift all semisimple subalgebras of $\mathrm{End}_E^0(A_0)$, which allows you to prove rationality for some Weil classes.

Applications

1.12. There are many heuristic arguments proving results assuming a sufficiently good theory of motives. Deligne's theorem makes it possible to define a category of abelian motives over a field of characteristic zero with all the properties Grothendieck's category would have if the Hodge conjecture were known. The rationality conjecture makes it possible to extend this to mixed characteristic (Milne 2009),

$$\begin{array}{ccc} \mathsf{Mot}(\mathbb{Q}^{\mathrm{al}}) & \hookleftarrow & \mathsf{Mot}'(\mathbb{Q}^{\mathrm{al}}) \\ & & \downarrow \\ & & \mathsf{Mot}(\mathbb{F}). \end{array}$$

This was, in fact, the motivation for the conjecture. With its proof, some heuristic arguments become proofs. Here, we give only one.

1.13. Let A be an abelian variety over a number field k, and let G be the Mumford–Tate group of $A_{\mathbb{C}}$. It follows from Deligne's theorem that, after a finite extension of k, there are well-defined Frobenius conjugacy classes in $G(\mathbb{Q}_\ell)$, ℓ a prime. It is an immediate consequence of the above theory of motives in mixed characteristic that these are rational, i.e., arise from a single conjugacy class in $G(\mathbb{Q})$. Cf. Serre 1994, 12.6.

I was not surprised that my conjecture was proved — indeed, for the last twenty years I've been trying to persuade everyone that it was accessible. However, as were most mathematicians, I was surprised by the next result.

2 The Hodge conjecture for CM abelian varieties

THEOREM 2.1 (OPENAI). *The Hodge conjecture is true for abelian varieties of CM type.*

Why is this so important? After all, we know the Hodge conjecture for many abelian varieties. We give three reasons.

2.2. We know the Hodge conjecture for any algebraic variety for which the "known" algebraic classes exhaust the $\mathbb{Q}$-algebra of Hodge classes. For example, since we know the Hodge conjecture for the $(1,1)$ Hodge classes, we know the conjecture for any variety for which these classes generate the $\mathbb{Q}$-algebra of Hodge classes. Results of this nature, while interesting, tell us nothing about whether the Hodge conjecture is true. For that, we have to prove the conjecture for a class of algebraic varieties for which the known algebraic classes do not exhaust the Hodge classes. The abelian varieties of CM type form such a class.

2.3. The Hodge conjecture for CM abelian varieties is not known to imply the Hodge conjecture for all abelian varieties, but it would be strange if this wasn't true. The CM-points are dense in the moduli space of abelian varieties. If the $\mathbb{Q}$-algebra of algebraic classes differed from the $\mathbb{Q}$-algebra of Hodge classes at some point, it would be very odd if this doesn't persist over a CM point.

2.4. Finally, as we shall see, once the Hodge conjecture is known for CM abelian varieties, it is possible to prove other very interesting results.

We now sketch the proof of the conjecture. Note that this all takes place over $\mathbb{C}$, so much of it is complex analysis.

2.5 (Hazama 2000, 2003). *It suffices to prove the conjecture in codimension* 2.

The proof is combinatorial. Fix a (large) CM-field K finite and Galois over $\mathbb{Q}$ with Galois group G. To give a CM abelian variety over $\mathbb{C}$ split by K, i.e., such that $\mathrm{End}^0(A)\otimes_{\mathbb{Q}} K$ is a product of matrix algebras over K, amounts to giving a finite G-set S and a subset S^+ of S such that $S = S^+ \sqcup \iota S^+$. Here ι is complex conjugation. For example, if A has complex multiplication by a CM-algebra E, then $S = \mathrm{Hom}(E,K)$ and S^+ is the CM type. If A corresponds to (S,S^+), then it is possible to read off $H^*(A,\mathbb{Q})$ with its Hodge structure from (S,S^+), and even $H^*(A^n,\mathbb{Q})$ because A^n corresponds to the disjoint union of n copies of (S,S^+). Let $[K:\mathbb{Q}] = 2m$. Let $H_1,\dots,H_m$ be the coordinate hyperplanes in $\mathbb{R}^m$, and take S to be the set of connected regions of $\mathbb{R}^m \smallsetminus \bigcup H_i$ and S^+ the subset on which $x_1 > 0$. For a suitable action of G on $\mathbb{R}^m$, hence on (S,S^+), there is a CM abelian variety $A(S,S^+)$ over $\mathbb{C}$ and Hazama proves:

(a) for every $n \geq 0$, the $\mathbb{Q}$-algebra $\mathcal{B}^*(A^n)$ of Hodge classes is generated by the classes of degree ≤ 2;

(b) every CM abelian variety split by K is isogenous to an isogeny factor of A^n for some n.

Thus, if the Hodge conjecture holds in codimension ≤ 2 for abelian varieties split by K, then it holds in all codimensions for for $A(S,S^+)^n$ by (a), and hence for every CM abelian variety split by K by (b).

2.6. *It suffices to prove the conjecture in dimension* 2.

The standard conjecture of Lefschetz type is known for abelian varieties: the map on algebraic cycles

$$L^{n-4}: \mathcal{A}^2(A) \xrightarrow{\simeq} \mathcal{A}^{n-2}(A)$$

is an isomorphism. Here L is the Lefschetz operator defined by a smooth hyperplane section.

2.7. *Proof of the conjecture in dimension* 2.

Thus, the problem comes down to embedding algebraic surfaces[5] into A. Let S be a be such a surface. If the rational Hodge structure $H^1(A,\mathbb{Q})$ occurs in $H^1(S,\mathbb{Q})$, then A is an isogeny factor of the Albanese variety of S, and so we get a map $f: S \to A$.

For S we take a Picard surface. These are quotients of the 2-dimensional complex ball

$$\mathbb{B}^2 = \{(z_1,z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}$$

by a discrete subgroup of the group of automorphisms of a three-dimensional vector space equipped with a hermitian form

$$\psi: V \times V \to E$$

over a CM-field E such that ψ has signature $(2,1)$ at one embedding of the totally real subfield of E and signature $(3,0)$ at all other embeddings. From Liu 2014, 2021, we find that the Hodge structure $H^1(A,\mathbb{Q})$ occurs in $H^1(S,\mathbb{Q})$ for S a Picard surface. Now comes the hard part. We have to show that the cycle $f(S)$ is not numerically equivalent to zero. For this, we have to prove that $Z \cdot f(S) \neq 0$ for some Hodge class Z on A of codimension 2. Here, we refer the reader to van Hoften and Sempliner 2026, which in addition to the original OpenAI manuscript draws on the manuscript Bergeron et al. 2016 (which proves the Hodge conjecture in small degrees for some arithmetic quotients of complex balls).

3 The Hodge, Tate, and standard conjectures

The Hodge, Tate, and Grothendieck standard conjectures were stated in the 1950s and 1960s, and have haunted mathematicians ever since. Except for divisors on abelian varieties (Tate, Zarhin, Faltings), little progress has been made in proving the conjectures. Thus, the recent proofs for abelian fourfolds have attracted considerable attention.

Conjecture	Abelian fourfolds	General abelian varieties
Hodge 1950	Markman 2025[6]	CM abelian varieties 2026 (OpenAI)
Tate 1963	Broe 2026	Divisors on AVs over finite fields 1966 (Tate)
		All AVs over finite fields 1999* (Milne)
Grothendieck 1968	Ancona 2021	All abelian varieties 2002* (Milne)

Now we have the statements at right. The asterisk means that the results were conditional until 2026. Since the proofs of the asterisked results have suddenly become much more interesting (because unconditional), I shall explain them. Both are applications of the theory of tannakian categories.[7]

[5] In fact, instead of A, the proof works with a CM abelian variety of Weil type, as in (c) of the proof of Proposition 5.1, Deligne 1982.

[6] In his 2026 ICM talk, Markman said he had a program for proving that Weil classes on all abelian varieties are algebraic. This would imply OpenAI's result and be much stronger.

[6] On the last occasion that I saw Tate, I asked him if he believed the Tate conjecture. He said he did for divisors, but otherwise he had no idea. He would have been interested in this result.

[7] The lesson of tannakian categories is that, as Galois taught us 200 years ago, we should use groups to try to understand complicated situations. See my book, *Tannakian Categories*, a near-final copy of which is available on my website.

The Tate conjecture for abelian varieties over finite fields

THEOREM 3.1 (Milne 1999a,b). *The full Tate conjecture holds for all abelian varieties over finite fields.*

By contrast, the Tate conjecture is known for very few surfaces over finite fields — essentially only for $K3$-surfaces, and then only after fifty years of struggle.

3.2. For a smooth projective variety X over $\mathbb{F}_q$, the full Tate conjecture says that

$$Z(X,t) \sim c(X)\cdot(1-q^r t)^{-\rho_r} \quad \text{as } t \to q^{-r},$$

where $c(X) \neq 0$ and ρ_r is the rank of the group of numerical equivalence classes of algebraic cycles of codimension r. Let $H^i(X_{Wet},\mathbb{Z}(r))$ denote the Weil-étale cohomology of complex of sheaves defined by Bloch's higher Chow groups, let $e: H^{2r}(X_{Wet},\mathbb{Z}(r)) \to H^{2r+1}(X_{Wet},\mathbb{Z}(r))$ denote cup-product with the canonical element of $H^1(k_{Wet},\mathbb{Z})$, and let

$$\chi(X_{Wet},\mathbb{Z}(r)) = \prod_{i\neq 2r,2r+1}[H^i(X_{Wet},\mathbb{Z}(r))]^{(-1)^i}\frac{[\operatorname{Ker}(e)]}{[\operatorname{Coker}(e)]}.$$

The conjecture of Lichtenbaum and Milne[8] asserts that

$$c(X) = \pm\chi(X_{Wet},\mathbb{Z}(r))\cdot q^{\chi(X,\mathcal{O}_X,r)},$$

where $\chi(X,\mathcal{O}_X,r) = \sum_{i\leq r,j}(-1)^{i+j}(r-i)\dim H^j(X,\Omega^i_{X/k})$. This is now a theorem when X is an abelian variety. See Milne 2015, Theorem 11 and Remark 12.

3.3. Theorem 3.1 implies that the generalized Tate conjecture holds for all abelian varieties over finite fields Milne and Ramachandran 2009, 1.10 (and even more, loc. cit.).

In addition to Theorem 2.1, the proof of Theorem 3.1 relies on the magical properties of Lefschetz classes on abelian varieties and a serendipitous equality.

3.4 (LEFSCHETZ CLASSES). Let k be an algebraically closed field. The Lefschetz classes on abelian varieties have remarkable properties.[9] For example, the Lefschetz standard conjecture holds for them

$$L^{g-2r}: \mathcal{D}(A)^r \xrightarrow{\simeq} \mathcal{D}(A)^{g-r}.$$

For any morphism $f: A \to B$ of abelian varieties, f_* maps Lefschetz classes to Lefschetz classes.[10] Numerical and homological equivalence agree on Lefschetz classes for every Weil cohomology theory. Using Lefschetz classes on abelian varieties, it is possible to define a category $\mathsf{LMot}(k)$ of "Lefschetz motives". This is a canonically polarized tannakian category over $\mathbb{Q}$ with the expected fibre functors, and it is possible to describe its fundamental group.

[8] Lichtenbaum defined $\chi(X_{Wet},\mathbb{Z}(r))$ and conjectured that it gives the non-p part of $c(X)$. The author added the term $q^{\chi(X,\mathcal{O}_X,r)}$ to take care of p and simplified the definition of $\chi(X_{Wet},\mathbb{Z}(r))$.

[9] Hazama also contributed to the theory of Lefschetz classes on abelian varieties. For an abelian variety A over $\mathbb{C}$, he and Kumar Murty showed that the Lefschetz classes on A are sometimes, and sometimes not, the fixed classes of a connected algebraic group. The author modified their argument to show that they are *always* the fixed classes of a *not necessarily connected* algebraic group, and that this is true more generally for abelian varieties over an arbitrary field and a Weil cohomology theory. From this the magical properties of Lefschetz classes on abelian varieties follow.

[10] This should surprise the reader, because it is (very) false for maps $f: X \to Y$ of algebraic varieties. For example, let Z be an algebraic cycle on Y that is not a Lefschetz class, and choose a subvariety X of Y in which Z has codimension 1.

3.5 (A SERENDIPITOUS EQUALITY). As before, let $\mathbb{Q}^{\mathrm{al}}$ denote an algebraic closure of $\mathbb{Q}$, let w be a prime of $\mathbb{Q}^{\mathrm{al}}$ lying over p, and let $\mathbb{F}$ be the residue field at w. We have a diagram of tannakian categories at left, and the corresponding diagram of tannakian fundamental groups at right,[11]

$$\begin{array}{ccc} \mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) & \longleftarrow & \mathsf{LCM}(\mathbb{Q}^{\mathrm{al}}) \\ \downarrow & & \downarrow \\ \mathsf{Mot}(\mathbb{F}) & \longleftarrow & \mathsf{LMot}(\mathbb{F}) \end{array} \qquad \begin{array}{ccc} S & \hookrightarrow & T \\ \uparrow & & \uparrow \\ G & \hookrightarrow & L \end{array}$$

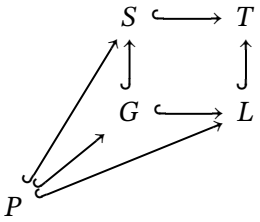
where

Category	Based on	Correspondences
$\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$	CM abelian varieties over $\mathbb{Q}^{\mathrm{al}}$	algebraic classes
$\mathsf{LCM}(\mathbb{Q}^{\mathrm{al}})$	CM abelian varieties over $\mathbb{Q}^{\mathrm{al}}$	Lefschetz classes
$\mathsf{Mot}(\mathbb{F})$	abelian varieties over $\mathbb{F}$	algebraic classes
$\mathsf{LMot}(\mathbb{F})$	abelian varieties over $\mathbb{F}$	Lefschetz classes.

The groups at right are affine group schemes of multiplicative type over $\mathbb{Q}$, hence described by their character groups.

We know nothing about G except that it is contained in S (because of Theorem 2.1) and in L (because Lefschetz classes are algebraic).

Let P be the affine group scheme attached to the category of motives over $\mathbb{F}$ when we assume the Tate and standard conjectures (see Milne 1994a). It is called the Weil-number pro-torus, and it fits into the above diagram as follows,



The outside maps are explicitly known. For any abelian variety A, there are natural actions of both G and P on the cohomology of A. The first fixes the algebraic classes and the second the Tate classes, so

$$\text{the Tate conjecture holds for all abelian varieties over } \mathbb{F} \iff P = G.$$

Looking at the diagram, we see that we could prove that $P = G$ if S and L intersect in P. A priori, there is no reason to expect that this should be true, but serendipitously[12] it is,

$$P = L \cap S \quad \text{(intersection inside } T\text{)}.$$

This completes the sketch of the proof of Theorem 3.1.[13]

This section summarizes parts of Milne 1999a,b.

[11] The fundamental group of a tannakian category is an affine group scheme *in* the category. For the categories in question, the fundamental groups are commutative, hence affine group schemes in the usual sense.

[12] Of course, we could have $P = G$ without the serendipitous equality — we just wouldn't be able to prove it. It is the serendipitous equality that makes possible this beautifully simple proof.

[13] This sketches a proof only of the Tate conjecture, not the full Tate conjecture. To obtain the full Tate conjecture, one can use Clozel's theorem that ℓ-adic homological equivalence equals numerical equivalence for some ℓ and the folklore theorem in Tate 1994.

The Hodge standard conjecture for abelian varieties

THEOREM 3.6 (MILNE 2002). *The standard conjecture of Hodge type holds for all abelian varieties.*

3.7. Let A be a polarized abelian variety of dimension g. As the Lefschetz standard conjecture holds for abelian varieties, we have an isomorphism

$$L^{g-2r} : \mathcal{A}^r(A) \to \mathcal{A}^{g-r}(A)$$

of spaces of algebraic classes. Here L is the Lefschetz operator defined by a smooth hyperplane section. Let

$$P^r(A) = \operatorname{Ker}\Big(\mathcal{A}^r(A) \xrightarrow{L^{g-2r+1}} \mathcal{A}^{g-r+1}(A))\Big).$$

The Hodge standard conjecture says that the pairing

$$P^r(A) \times P^r(A) \to \mathbb{Q}, \quad a, b \mapsto L^{g-2r} a \cdot b$$

is positive definite.

3.8. By a specialization argument, it suffices to prove the conjecture over finite fields: if A specializes to A_0, then $\mathcal{A}^r(A) \subset \mathcal{A}^r(A_0)$, and the pairing for A_0 restricts to the pairing for A.

3.9. The Hodge standard conjecture has a tannakian interpretation. It holds for a collection of algebraic varieties over a field k if and only if the category of motives generated by the varieties admits a polarization for which the pairings coming from geometry are positive.

3.10. Roughly speaking, a polarization on a tannakian category is a compatible family of positive hermitian forms on the objects. There is an elaborate, and deep, theory of polarizations on tannakian categories. For example, if G is an algebraic group over $\mathbb{R}$, then the (tannakian) category of finite-dimensional representations of G admits a polarization if and only if G is an inner form of a compact group.

3.11. Hodge theory shows that the standard conjecture of Hodge type holds in characteristic zero. Therefore $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ admits a canonical polarization for which the geometric pairings are positive. The proof of Theorem 3.6 uses the diagram

$$\begin{array}{ccc} \mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) & \longleftarrow & \mathsf{LCM}(\mathbb{Q}^{\mathrm{al}}) \\ \downarrow & & \downarrow \\ \mathsf{Mot}(\mathbb{F}) & \longleftarrow & \mathsf{LMot}(\mathbb{F}) \end{array}$$

and the theory of polarizations on tannakian categories to show that the polarization on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ passes to the quotient tannakian category $\mathsf{Mot}(\mathbb{F})$.[14]

This section summarizes parts of Milne 2002.

[14] Alongside the problem of resolution of singularities, Grothendieck considered the proof of the standard conjectures to be the most urgent task in algebraic geometry. He once told Spencer Bloch that he hoped to see the standard conjectures proved in his lifetime. In this, he was disappointed. At least, we now know that the conjectures hold for all abelian varieties, and so we can construct the "abelian" part of Grothendieck's motivic paradise.

Applications

3.12. We now know that the category $\mathsf{Mot}(\mathbb{F})$ in 1.12 is defined by algebraic cycles and the category $\mathsf{Mot}(\mathbb{Q}^{\mathrm{al}})$ is defined by almost-algebraic cycles.

3.13. In their 1987 paper, Langlands and Rapoport prove their conjecture for some simple Shimura varieties of PEL-type under the assumption of

- the Hodge conjecture for CM abelian varieties,
- the Tate conjecture for abelian varieties over finite fields, and
- the Hodge standard conjecture for abelian varieties.

In 1999 and 2002 I proved that the first of these conjectures implied the remaining two. Now that OpenAI has proved the first of the conjectures, we know everything.

3.14. In Milne 1994b, I proved that all Shimura varieties of abelian type with rational weight can be realized as moduli varieties for motives in characteristic 0. I leave it as a challenge to the reader to extend this to mixed characteristic, and to prove the conjecture of Langlands and Rapoport (1987) following the heuristic argument they used to derive it.

References

BERGERON, N., MILLSON, J., AND MOEGLIN, C. 2016. The Hodge conjecture and arithmetic quotients of complex balls. *Acta Math.* 216:1–125.

DELIGNE, P. 1982. Hodge cycles on abelian varieties (notes by J.S. Milne), pp. 9–100. *In* Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics. Springer-Verlag, Berlin.

HAZAMA, F. 2000. Hodge cycles on abelian varieties of S_n-type. *J. Algebraic Geom.* 9:711–753.

HAZAMA, F. 2003. On the general Hodge conjecture for abelian varieties of CM-type. *Publ. Res. Inst. Math. Sci.* 39:625–655.

KISIN, M. AND ZHOU, R. 2025. Strongly compatible systems associated to semistable abelian varieties. arXiv:2505.02165.

LANGLANDS, R. P. AND RAPOPORT, M. 1987. Shimuravarietäten und Gerben. *J. Reine Angew. Math.* 378:113–220.

LIU, Y. 2014. Relative trace formulae toward Bessel and Fourier-Jacobi periods on unitary groups. *Manuscripta Math.* 145:1–69.

LIU, Y. 2021. Fourier-Jacobi cycles and arithmetic relative trace formula (with an appendix by Chao Li and Yihang Zhu). 9:1–147.

MADAPUSI, K. 2026. Milne's rationality conjecture via simultaneous lifting of Weil classes and endomorphisms. hexagon:2610.00226v1.

MILNE, J. S. 1994a. Motives over finite fields, pp. 401–459. *In* Motives (Seattle, WA, 1991), volume 55, Part 1 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI.

MILNE, J. S. 1994b. Shimura varieties and motives, pp. 447–523. *In* Motives (Seattle, WA, 1991), volume 55, Part 2 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI.

MILNE, J. S. 1999a. Lefschetz classes on abelian varieties. *Duke Math. J.* 96:639–675.

MILNE, J. S. 1999b. Lefschetz motives and the Tate conjecture. *Compositio Math.* 117:45–76.

MILNE, J. S. 2002. Polarizations and Grothendieck's standard conjectures. *Ann. of Math. (2)* 155:599–610.

MILNE, J. S. 2008. The Tate conjecture over finite fields (AIM talk). v2.2, May 7, 2008, available on the author's website under articles; see also arXiv:0709.3040.

MILNE, J. S. 2009. Rational Tate classes. *Mosc. Math. J.* 9:111–141.

MILNE, J. S. 2015. Addendum to "Values of zeta functions of varieties over finite fields". *Amer. J. Math.* 137:1703–1712.

MILNE, J. S. AND RAMACHANDRAN, N. 2009. Motivic complexes over finite fields and the ring of correspondences at the generic point. *Pure Appl. Math. Q.* 5:1219–1252.

SERRE, J.-P. 1994. Propriétés conjecturales des groupes de Galois motiviques et des représentations l-adiques, pp. 377–400. *In* Motives (Seattle, WA, 1991), volume 55, Part 1 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI.

TATE, J. T. 1994. Conjectures on algebraic cycles in l-adic cohomology, pp. 71–83. *In* Motives (Seattle, WA, 1991), volume 55 of *Proc. Sympos. Pure Math.* Amer. Math. Soc.

VAN HOFTEN, P. AND SEMPLINER, J. 2026. The Hodge conjecture for CM abelian varieties. hexagon:2610.00259v2.

ZINK, T. 1983. Isogenieklassen von Punkten von Shimuramannigfaltigkeiten mit Werten in einem endlichen Körper. *Math. Nachr.* 112:103–124.