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Autocorrelation Phase Lattices on Cyclic Groups: Unit Supports, $2pq$ Orbit Profiles, and Binary Reconstruction

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Abstract

For a finite cyclic group and an exact Fourier support S, we study the lattice $\mathcal{L}_{\le d}(S)$ generated by zero-sum relations of length at most d inside the full relation lattice $K(S)$. Its closing degree is both a local separating degree and the order at which every invisible phase ratio becomes a translation character. For the exact unit support of every even cyclic group, we compute

$$\mathbb{Z}^{U_n}/\mathcal{L}_{\le 4}(U_n) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z}, & 3 \nmid n, \\ \mathbb{Z} \oplus \mathbb{Z}/(n/6)\mathbb{Z}, & 6 \mid n. \end{cases}$$

Moreover $\mathcal{L}_{\le 5}(U_n) = \mathcal{L}_{\le 4}(U_n)$. Thus fourth- and fifth-order rational ambiguity coincide: fourth order reconstructs up to translation when $3 \nmid n$, while for $n = 6m$ the complete surviving family has normalized ratios parameterized by norm-one elements of $\mathbb{Q}(\zeta_6)$, modulo the residual translation subgroup μ_6; order six closes it. For $n = 2pq$ with distinct odd primes, we classify every nonempty $\mathrm{Aut}(C_n)$-stable support. The distributions of these 255 supports by closing degree are

$$(1, 2, 154, 80, 4, 14) \quad (p = 3), \qquad (1, 2, 154, 92, 4, 2) \quad (p, q \ge 5).$$

The proof uses seven local CRT presentations, cubic and quartic Skeletons, and explicit depth-five and depth-six obstructions. Galois-equivariant lower-bound ratios show that the lattice, rational-signal and integer-signal closing degrees coincide on every stable support. Finally, for binary indicators, a supportwise crosswalk and a uniform exclusion of all deep supports give $r_{\mathrm{set}}(C_{2pq}) = 4$ under translations. The value is also four for the same oriented analytical deck when outputs are identified up to translation or reflected translation.

Keywords: higher-order autocorrelation; finite cyclic group; zero-sum sequence; separating invariant; phase lattice; Galois descent; rational and integer signals; set reconstruction; analytical deck.

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1 Introduction and exact results

Higher-order autocorrelations are translation invariants built from products of a function along several shifted copies. Their use in pattern recognition and periodic reconstruction has a long history [AK62, MR68, CW70, KI93]. On a finite cyclic group, the Fourier transform converts an order-k autocorrelation into products over zero-sum k-tuples. This makes reconstruction below the classical global order-six line of Grünbaum–Moore [GM95] a problem about the integral generation of zero-sum relations.

The answer depends sharply on the exact Fourier support. For a support $S \subset C_n$, let $\mathcal{L}_{\le d}(S)$ be generated by the nonnegative zero-sum multiplicity vectors of length at most d, and let $K(S)$ be the full integral relation lattice. We call

$$\delta(S) = \min\{d : \mathcal{L}_{\le d}(S) = K(S)\}$$

the local closing degree. Section 2 identifies it with the maximum separating-atom length on S, using the established quotient-group language of Schefler–Zhao–Zhong [SZZ26]. Section 3 explains why the same lattice controls phase ratios invisible to autocorrelations.

We compute this object in two directions. The first is a signal-level theorem and complete ambiguity parametrization for the exact unit support of every even cyclic group. The second is a complete supportwise classification when the group order is $2pq$, together with exact rational and integer thresholds and explicit lower-bound pairs on every stable support of depth greater than one. The sole depth-one support is direct. Their common row is the unit orbit.

1.1 The orbit-stable classification

Let $p < q$ be distinct odd primes. The automorphism orbits in C_{2pq} are the eight element-order classes $\mathcal{O}_d$ for $d \mid 2pq$. Hence a rational Fourier support, which is stable under the Galois action, is a union of these eight orbits and there are $2^8 - 1 = 255$ nonempty possibilities.

Theorem 1.1 (Orbit-stable profiles)**.** *For $n = 2pq$, the numbers of nonempty* $\mathrm{Aut}(C_n)$*-stable supports of local closing degree* $1, \dots, 6$ *are*

$$(1, 2, 154, 80, 4, 14) \qquad \text{if } p = 3,$$

and

$$(1, 2, 154, 92, 4, 2) \qquad \text{if } p, q \ge 5.$$

Equivalently, these are the distributions of the maximum separating-atom length on the stable supports. In both regimes the orbit-stable local maximum is six.

For $A \in \{\mathbb{Q}, \mathbb{Z}\}$, let $\delta_A(S)$ be the least $d \ge 1$ such that any two A-valued signals with exact Fourier support S and equal autocorrelations of orders $1, \dots, d$ differ by a translation. The phase-ratio bridge gives the upper bounds $\delta_{\mathbb{Z}}(S) \le \delta_{\mathbb{Q}}(S) \le \delta(S)$. The reverse bounds require arithmetic realizations of the lattice obstructions and are the content of the following theorem.

Theorem 1.2 (Uniform rational and integer realization)**.** *Let $n = 2pq$, where $p < q$ are distinct odd primes. For every nonempty* $\mathrm{Aut}(C_n)$*-stable exact Fourier support S,*

$$\delta_{\mathbb{Z}}(S) = \delta_{\mathbb{Q}}(S) = \delta(S). \tag{1.0}$$

Thus the rational-signal and integer-signal reconstruction profiles are the two profiles in Theorem 1.1. Equivalence is by translation only.

For a subset $A \subset C_n$, its analytical ordered k-deck is (3.1) with $f = \mathbf{1}_A$. Let $r_{\mathrm{set}}(C_n)$ be the least d for which this d-deck determines every subset up to translation. Indicators are idempotent, so zero shifts make the exact d-deck contain every lower deck; this fact is special to the binary class. Lemma 11.1 shows, including for the empty set and sets of size below d, that this is exactly Pebody's cumulative necklace-deck convention rather than a different analytical reconstruction problem.

Theorem 1.3 (Binary reconstruction in the $2pq$ family)**.** *If $p < q$ are distinct odd primes, then*

$$r_{\mathrm{set}}(C_{2pq}) = 4. \tag{1.0a}$$

The upper bound follows by combining the supportwise phase-lattice criterion with a binary exclusion of every depth-five and depth-six support. The lower bound is the published even-order three-deck obstruction of Keleti–Kolountzakis [KK06].

Pebody determined the multiset reconstruction numbers of finite abelian groups and obtained the general cyclic set bound six [Peb04, Theorems 2.1–2.2 and Lemma 9.3]. Combining his comparison of set and multiset reconstruction with the multiset classification and the even-order lower bound of Keleti–Kolountzakis already gives

$$r_{\mathrm{set}}(C_{2^a}) = 4 \quad (a \geq 4), \qquad r_{\mathrm{set}}(C_{2\ell^k}) = 4 \quad (\ell > 3 \text{ prime},\ 2\ell^k > 10). \tag{1.0b}$$

Pebody later conjectured value four for every even cyclic order greater than ten and reported that his method covered many even orders, including every one below 240 [Peb07]. Theorem 1.3 adds the arithmetically disjoint squarefree family $2pq$, with two distinct odd prime factors, inside that conjectural line; no absolute priority or firstness assertion is made.

The proof is uniform in the primes. It does not arise from interpolation of finite atlases. We first compute the cumulative degree-two, degree-three and degree-four presentations of seven nonzero CRT boxes. Exactly nineteen cubic box-count patterns can occur. Five propagation lemmas convert them into a ten-clause criterion selecting exactly 77 of the 127 nonempty seven-box masks; ten explicit defect families prove the converse. At order four, fourteen connector types reduce the generic branch to complete primary-coordinate graphs and the $p = 3$ branch to a fixed six-coordinate integral presentation. The remaining masks are two depth-five bridges and one or seven depth-six critical bases. Adding the optional zero orbit doubles every base row and gives the two displayed profiles.

The ordinary separating Noether number is a different invariant: it maximizes over modules and supports, and its exact finite-abelian value belongs to the separating-invariant literature [Hua26]. Theorem 1.1 is a supportwise distribution restricted to the eight order orbits of one cyclic group.

There is also a closer general framework. Domokos characterizes separating monomials by integral generation of zero-sum sequence monoids, first for diagonalizable actions and then, over non-closed fields, by imposing the criterion on every Galois-stable character subset [Dom23, Dom26]. That criterion explains why the stable supports considered here are the natural local objects. It does not give the least truncation degree for any prescribed support, the ten-clause Cubic Skeleton, either arithmetic profile, or the exact

unit-support quotient. Theorem 1.1 computes precisely this supportwise layer for the regular representation of C_{2pq}.

1.2 Exact unit supports

Write $U_n = (\mathbb{Z}/n\mathbb{Z})^\times$. The second theorem begins with an integral dichotomy and then performs the Galois descent needed for rational signals.

Theorem 1.4 (Even unit-support dichotomy and signal classification)**.** *Let n be even.*

1. *The fourth-order quotient is*

$$\mathbb{Z}^{U_n}/\mathcal{L}_{\le 4}(U_n) \cong \begin{cases} C_n, & 3 \nmid n, \\ \mathbb{Z} \oplus C_{n/6}, & 6 \mid n. \end{cases} \tag{1.1}$$

   If $6 \mid n$, then $\mathcal{L}_{\le 6}(U_n) = K(U_n)$. In every even modulus, $\mathcal{L}_{\le 5}(U_n) = \mathcal{L}_{\le 4}(U_n)$.

2. *If $3 \nmid n$, every two rational-valued signals with exact Fourier support U_n and equal autocorrelations through order four differ by a translation.*

3. *If $n = 6m$, every pair with exact support U_n and equal data through order four has, after translating the second signal, the form*

$$\widehat{f}(u) = \sigma_u(\alpha), \qquad \widehat{g}(u) = \sigma_u(z\alpha), \tag{1.2}$$

   where $\alpha \in \mathbb{Q}(\zeta_n)^\times$ and $z \in \mathbb{Q}(\zeta_6)^\times$ satisfies $z\overline{z} = 1$. Conversely every pair in (1.2) agrees through order five, and hence through order four. Normalized parameters are translation-equivalent exactly modulo μ_6, and equality at order six is equivalent to $z^6 = 1$.

The integral calculation uses a universal ambient group for the sifted set

$$T_{q,c} = \{t \in C_q : \gcd(1 + ct, q) = 1\}.$$

It is a concrete use of the Freiman-homomorphism framework of Konyagin–Lev and Grynkiewicz [KL00, Gry13]. The exceptional free direction is not an arbitrary cyclotomic effect: it is the second admissible class modulo three. After translation and rational descent, the relative phase lies in the quadratic subfield even though the common amplitude ranges over the full cyclotomic field.

Corollary 1.5 (The common unit row)**.** *For $n = 2pq$, the unit support is the single orbit $U_{2pq} = \mathcal{O}_{2pq}$. Its closing degree is*

$$\delta(U_{2pq}) = \begin{cases} 6, & p = 3, \\ 4, & p, q \ge 5. \end{cases}$$

Theorem 1.1 and Theorem 1.4 therefore agree on their literal common support.

Corollary 1.6 (U_{30})**.** *For exact support U_{30}, formula (1.2) is the complete rational classification through both orders four and five. The larger field $\mathbb{Q}(\zeta_{30})$ enlarges the common amplitude α, while the relative phase remains in the norm-one torus of $\mathbb{Q}(\zeta_6)$.*

This specializes to the question raised by Agulnick–Busick-Warner [ABW26]; Section 5 gives the precise comparison. The Fourier monomial and Galois criteria used in passing from lattices to signals are due to Casper–Orozco [CO26].

1.3 Claim levels and organization

The distinction between the two headline conclusions is binding. On the unit support, all surviving lattice characters are classified and rationally realized. On all 255 supports, we determine the exact rational and integer thresholds; whenever the depth exceeds one, we construct at least one Galois-equivariant lower-bound ratio for the failure immediately below closure. The sole depth-one support is handled directly. We do not classify the complete rational ambiguity torus on every atlas row.

Sections 2–5 establish the common dictionary and the unit-support theorem. Sections 6–9 prove the $2pq$ lattice classification. Section 10 constructs the nontrivial supportwise rational and integer lower-bound pairs and proves Theorem 1.2. Section 11 specializes the classifier to binary indicators, proves Theorem 1.3, and records the oriented-deck translation/reflection corollary. Section 12 returns to autocorrelation and literature boundaries. The appendices record the full connector and Boolean tables and separate computational assurance from proof.

> **Scope boundary.** No absolute priority or firstness assertion is made. The binary phase/deck crosswalk is part of the Pebody–Grünbaum–Moore–Simon lineage, and the even three-deck lower bound is imported from Keleti–Kolountzakis. The generic critical support used later is the sharpness family of Casper–Orozco; the global sixth-order line and the ratio-character method have earlier sources. The general non-closed-field separating criterion is Domokos's. The contribution certified here is the displayed fixed-support theorem chain, its exact supportwise classification, and the explicit Galois-equivariant realization of every nontrivial lower-bound layer.

2 Zero-sum lattices and local separating degree

Let G be a finite abelian group and let $\varnothing \neq S \subset G$. Write

$$\mathcal{B}(S) = \left\{ a \in \mathbb{N}_0^S : \sum_{s \in S} a_s s = 0 \right\}$$

for the zero-sum sequence monoid. Its atoms are denoted by $\mathcal{A}(S)$. For $d \geq 1$, let $\mathcal{B}(S)_{\leq d}$ be the submonoid generated by atoms of length at most d. The integral relation lattice is

$$K(S) = \ker\left(\mathbb{Z}^S \longrightarrow \langle S \rangle\right), \qquad (a_s) \longmapsto \sum_s a_s s,$$

and $\mathcal{L}_{\leq d}(S)$ is generated by all nonnegative zero-sum vectors of length at most d.

Using all rows of length at most d or only atoms of length at most d is equivalent at the integral level. A reducible zero-sum sequence factors as a sum of shorter zero-sum sequences and adds no new generator beyond its atoms. We retain the row language in coordinate proofs because one autocorrelation monomial naturally gives one row; the atom language is the bridge to separating invariants.

For reference, the objects used throughout are separated in Table 1. In particular, $A_d(S)$ contains the finite target coordinates as well as the obstruction; it should not be called the ambiguity group.

For a cancellative monoid M, write $\mathsf{q}(M)$ for its quotient group. The next proposition is the precise bridge between the lattice used here and the separating-atom filtration of Schefler–Zhao–Zhong [SZZ26].

Object	Definition	Role
$\mathcal{B}(S)$	Nonnegative zero-sum sequences	Monoid of Fourier monomial indices
$K(S)$	Full integral kernel in $\mathbb{Z}^S$	All additive relations on the support
$\mathcal{L}_{\le d}(S)$	Span of zero-sum rows through length d	Relations visible to bounded autocorrelations
$A_d(S)$	$\mathbb{Z}^S/\mathcal{L}_{\le d}(S)$	Universal ambient presentation
$K(S)/\mathcal{L}_{\le d}(S)$	Kernel of $A_d(S) \to \langle S\rangle$	Integral obstruction lattice
$\mathrm{Hom}(K/\mathcal{L}_{\le d}, \mathbb{C}^\times)$	Character dual of the obstruction	Complex phase ambiguity modulo translations
$\delta(S)$	Least d with $\mathcal{L}_{\le d} = K$	Local closing/separating degree

Table 1: Notation and logical roles.

Proposition 2.1 (Group-completion dictionary)**.** *For every S and d,*

$$\mathcal{L}_{\le d}(S) = \mathsf{q}(\mathcal{B}(S)_{\le d}), \qquad K(S) = \mathsf{q}(\mathcal{B}(S)), \tag{2.1}$$

where both quotient groups are viewed in $\mathbb{Z}^S$. Consequently

$$\delta(S) = \max\{|A| : A \in \mathcal{A}_{\mathrm{sep}}(S)\}, \tag{2.2}$$

the maximum length of a separating atom over S.

Proof. Every nonnegative zero-sum vector of length at most d factors into atoms, each of length at most d. Conversely, each atom of length at most d is one of the rows generating $\mathcal{L}_{\le d}(S)$. Taking integral differences gives the first identity in (2.1).

The inclusion $\mathsf{q}(\mathcal{B}(S)) \subset K(S)$ is immediate. For the reverse inclusion, let $x = x^+ - x^- \in K(S)$. The two nonnegative parts have the same group sum, say $h \in \langle S\rangle$. Since the finite group $\langle S\rangle$ is generated by S, there is a nonnegative vector $c \in \mathbb{N}_0^S$ with group sum $-h$: begin with an integral expression and add suitable multiples of element orders to make every coefficient nonnegative. Then $x^+ + c$ and $x^- + c$ lie in $\mathcal{B}(S)$ and their difference is x.

By Definition 2.5 of [SZZ26], a separating atom is an atom not contained in the quotient group generated by shorter atoms. The same source proves that separating atoms generate the full quotient group. Thus the least truncation degree generating $K(S)$ is their maximum length, which is (2.2). □

The proposition does not require $S = -S$. This is useful because it separates the algebraic dictionary from Fourier rationality. When S is the exact support of a rational-valued signal, Galois stability later implies $S = -S$, but no symmetry hypothesis is needed in Proposition 2.1.

Domokos's lattice criterion for diagonalizable actions states the same generation principle at the level of coordinate subrepresentations [Dom23, Theorem 1.3]. Its non-closed-field extension requires the criterion on every Galois-stable character subset of the regular representation [Dom26, Theorem 5.4]. We use that result as the general field-theoretic boundary, not as a source of the numerical profiles: our local invariant asks for the *least* d generating the group completion on each prescribed stable support. Sections 6– 9 compute that truncation exactly for all 255 supports of C_{2pq}.

Example 2.2 (The first two profile rows). For $S = \{0\}$, the vector e_0 is a length-one zero sum, so $K(S) = \mathcal{L}_{\le 1}(S)$ and $\delta(S) = 1$. For the nonzero order-two orbit $S = \mathcal{O}_2$, the map $\mathbb{Z}e \to C_2$ has kernel $2\mathbb{Z}e$ and its first nonnegative generator is the pair row $2e$. Hence $\delta(S) = 2$. These are the mechanisms behind the first two entries of both profiles.

2.1 Ambient and obstruction quotients

Define

$$A_d(S) = \mathbb{Z}^S/\mathcal{L}_{\le d}(S).$$

There is an exact sequence

$$0 \longrightarrow K(S)/\mathcal{L}_{\le d}(S) \longrightarrow A_d(S) \longrightarrow \langle S\rangle \longrightarrow 0. \tag{2.3}$$

The middle group is convenient for presentations. The left group is the obstruction to closing at degree d. Its complex character group is the phase ambiguity modulo translations. These three objects will be kept distinct throughout.

If $A_d(S)$ is finite, (2.3) gives the index identity

$$[K(S) : \mathcal{L}_{\le d}(S)] = \frac{|A_d(S)|}{|\langle S\rangle|}. \tag{2.3a}$$

If $A_d(S)$ has a free summand, the obstruction does as well. The later proofs retain explicit presentations rather than using only (2.3a), because a presentation identifies both the obstruction generator and the row that eventually kills it.

For later comparison, define the support-restricted orbit-stable maximum

$$\beta_{\mathrm{orb}}(G) = \max\{\delta(S) : \varnothing \neq S \subset G,\ \alpha S = S \text{ for every } \alpha \in \mathrm{Aut}(G)\}. \tag{2.4}$$

This is only working notation. It is not the ordinary separating Noether number $\beta_{\mathrm{sep}}(G)$ studied in [SZZ26, Hua26]. Theorem 1.1 states that (2.4) equals six for C_{2pq} and gives the complete distribution below that maximum.

The stability restriction is substantial. Without it one may choose small supports whose relation lattices close much later, but such supports cannot be exact Fourier supports of rational signals. Thus (2.4) is adapted to the rational Fourier problem and is not a new name for the unrestricted separating Noether number.

2.2 The zero-sum face picture

Let Δ_S be the coordinate face of the standard simplex supported on S, and let Λ_G impose the integral congruence $\sum_s a_s s = 0$. A zero-sum sequence of length m supported in S is exactly a lattice point of

$$m\Delta_S \cap \Lambda_G. \tag{2.5}$$

Indeed, the simplex equation is $\sum_s a_s = m$, nonnegativity gives the face, and membership in Λ_G is the zero-sum condition. This is the support-face identity used in the zero-sum-polytope theory of Han–Wang–Zhang–Zhang [HWZZ26]. Our question is not to count the points in (2.5), but to ask at which dilation their integral span generates the full relation lattice.

This is generation rather than enumeration. Ehrhart-type data count the zero-sum sequences at each length; the closing degree asks when their integral combinations span

every relation. Many short rows may be redundant, while a single separating atom can delay closure. The two Skeleton arguments below replace large row lists by a small set of functional consequences whose span is proved uniformly.

There is also an equivariance distinction. A setwise $\mathrm{Aut}(G)$-stable face need not consist of fixed lattice points: automorphisms may permute its coordinates. We classify generation on stable supports. We do not require each zero-sum sequence or multiplicity vector to be fixed by the action.

3 Autocorrelation phase lattices

Write $C_n = \mathbb{Z}/n\mathbb{Z}$ additively and fix $\zeta_n = e^{2\pi i/n}$. For $f : C_n \to \mathbb{C}$, use the Fourier convention

$$\widehat{f}(s) = \sum_{x \in C_n} f(x)\zeta_n^{-sx}.$$

The order-k autocorrelation is

$$M_k(f; t_1, \dots, t_{k-1}) = \sum_{x \in C_n} f(x)f(x+t_1)\cdots f(x+t_{k-1}). \tag{3.1}$$

Our convention counts the number of signal factors: $M_1(f) = \sum_x f(x)$, M_2 is the usual periodic autocorrelation, and the bispectrum is the Fourier transform of M_3. This normalization matters when comparing order bounds across the literature.

Taking the Fourier transform of (3.1) in all shift variables gives, up to the fixed normalization power of n,

$$\widehat{M}_k(f; s_1, \dots, s_{k-1}) = \widehat{f}(-s_1 - \cdots - s_{k-1}) \prod_{j=1}^{k-1} \widehat{f}(s_j). \tag{3.2}$$

Thus the relevant monomials are indexed by ordered zero-sum k-tuples. Passing from an ordered tuple to its multiplicity vector loses no equation, because the product is commutative. This elementary identity is the reason the monoid $\mathcal{B}(S)$, rather than merely the set of distinct subset sums, governs the problem.

We use two results of Casper–Orozco [CO26]. First, equality of (3.1) is equivalent to equality of Fourier products over zero-sum k-tuples. Second, f is rational-valued precisely when its Fourier transform is Galois equivariant:

$$\widehat{f}(as) = \sigma_a(\widehat{f}(s)) \qquad (a \in (\mathbb{Z}/n\mathbb{Z})^\times). \tag{3.3}$$

The earlier bispectral literature often numbers the order by one fewer Fourier factor; in particular the convention in [KI93] is shifted relative to (3.1).

For clarity, the rationality assertion can also be seen directly. If $f(x) \in \mathbb{Q}$, applying σ_a termwise to the Fourier sum replaces ζ_n^{-sx} by ζ_n^{-asx} and gives (3.3). Conversely, if a Fourier vector is equivariant and is zero on a Galois-stable complement, the inverse transform

$$f(x) = \frac{1}{n} \sum_{s \in C_n} \widehat{f}(s)\zeta_n^{sx} \tag{3.4}$$

is fixed by every σ_a, hence lies in $\mathbb{Q}$. We will use this converse in Sections 5 and 10.

Lemma 3.1 (Exact support from order two)**.** *If two real-valued signals have equal second autocorrelations, then their Fourier transforms have the same zero set.*

Proof. Reality gives $\widehat{f}(-s) = \overline{\widehat{f}(s)}$. The zero-sum pair $(s, -s)$ therefore contributes $|\widehat{f}(s)|^2$. Equality of all such products determines exactly which coefficients vanish. □

Proposition 3.2 (Phase-ratio bridge)**.** *Let $f, g : C_n \to \mathbb{Q}$ have common exact Fourier support S, and put $r_s = \widehat{g}(s)/\widehat{f}(s)$. The autocorrelations agree through order d if and only if*

$$r^a = \prod_{s \in S} r_s^{a_s} = 1 \qquad \text{for every } a \in \mathcal{L}_{\le d}(S). \tag{3.5}$$

Translation ratios are exactly the characters of $\mathbb{Z}^S$ that kill $K(S)$. Thus $\mathcal{L}_{\le d}(S) = K(S)$ if and only if every complex ratio invisible through order d is a translation ratio.

Proof. By the Fourier monomial criterion, equality through order d means equality of the products over all zero-sum tuples of length at most d. A tuple meeting $C_n \setminus S$ makes both products zero. On a tuple contained in S, every coefficient is nonzero, so division is legitimate and gives (3.5) for its multiplicity vector. The map $a \mapsto r^a$ is a homomorphism $\mathbb{Z}^S \to \mathbb{C}^\times$, hence it kills the bounded nonnegative rows exactly when it kills their integral span.

If $g(x) = f(x + y)$, the Fourier shift rule gives $r_s = \zeta_n^{ys}$. The quotient map $\mathbb{Z}^S \to \langle S \rangle$ has kernel $K(S)$, so the characters killing $K(S)$ are the characters of $\langle S \rangle$. The translation ratios have the same cardinality $|\langle S \rangle|$ and therefore exhaust this character group. □

Proposition 3.3 (Direct Galois realization)**.** *Let $S \subset C_n$ be Galois-stable and let $r : S \to \mathbb{Q}(\zeta_n)^\times$ satisfy*

$$r_{as} = \sigma_a(r_s) \quad (a \in U_n), \qquad r^c = 1 \quad (c \in \mathcal{L}_{\le d}(S)). \tag{3.7}$$

If r is not a translation ratio, then there are rational-valued signals, and also integer-valued signals, with exact Fourier support S, equal autocorrelations through order d, and not related by translation.

Proof. Set $\widehat{f} = 1$ and $\widehat{g} = r$ on S, and set both transforms to zero off S. Stability and the first identity in (3.7) make the two Fourier vectors Galois equivariant, so (3.3)–(3.4) give rational-valued signals. Every displayed coefficient on S is nonzero, hence the Fourier support is exactly S. The second identity and Proposition 3.2 give equal autocorrelations through order d, while the hypothesis on r excludes a translation.

Choose one positive common denominator D for all values of f and g. Then Df, Dg are integer-valued. Their order-k autocorrelations are both multiplied by D^k, their exact Fourier supports are unchanged, and a translation between them would divide by D to a translation between f, g. Thus the same ratio supplies both arithmetic lower bounds. □

There is a useful exact-sequence version of the same argument. Applying $\mathrm{Hom}(-, \mathbb{C}^\times)$ to (2.3) is exact because $\mathbb{C}^\times$ is divisible. It gives

$$1 \longrightarrow \widehat{\langle S \rangle} \longrightarrow \mathrm{Hom}(A_d(S), \mathbb{C}^\times) \longrightarrow \mathrm{Hom}(K(S)/\mathcal{L}_{\le d}(S), \mathbb{C}^\times) \longrightarrow 1. \tag{3.6}$$

The first term is the translation-character subgroup. The last term is the residual complex phase ambiguity. Formula (3.6) also explains why the integral obstruction itself and the phase group are dual objects, not the same object. A free summand in $K/\mathcal{L}$ produces a copy of $\mathbb{C}^\times$; a C_m summand produces μ_m.

Example 3.4 (Why exact support is a hypothesis). Suppose a zero-sum tuple contains a frequency at which both Fourier coefficients vanish. Its monomial equality is then $0 = 0$ and imposes no condition on ratios at the other frequencies. One cannot divide the tuple equation and then silently discard the zero. Lemma 3.1 first identifies the common exact support; Proposition 3.2 divides only after restricting to it.

The propositions separate two arithmetic tasks. The lattice calculation classifies all complex multiplicative ratios. On the unit support, Section 5 classifies the complete rational family. For all stable C_{2pq} supports, Section 10 constructs a Galois-equivariant ratio whenever the lattice has not yet closed. That is enough to determine the exact rational and integer thresholds, without parametrizing every rational ambiguity torus.

Remark 3.5 (Translations versus reflection). The equivalence throughout the paper is translation only. Reflection sends s to $-s$ in frequency and is not quotiented unless it happens to coincide with a translation on a particular signal.

The ratio-character method and CRT patching already occur in the classical cyclic work of Grünbaum–Moore [GM95]. Our later use is a support-sensitive integral classification, not a claim of ownership of that method.

4 The unit-support lattice for every even modulus

We now compute the integral quotient in Theorem 1.4. The main input is a universal ambient group for a sifted arithmetic progression. For $q \geq 1$ and $c \in \mathbb{Z}$, put

$$T_{q,c} = \{t \in C_q : \gcd(1 + ct, q) = 1\}.$$

For $T \subset C_q$, let $\mathrm{UAG}(T)$ be the quotient of $\mathbb{Z}^T$ by the Freiman rows

$$e_a + e_b - e_{a'} - e_{b'} \qquad (a + b = a' + b' \text{ in } C_q).$$

This is the universal target for Freiman homomorphisms on T, in the sense of Konyagin–Lev and the structural additive literature [KL00, Gry13].

Call a prime divisor of q *restricting* if it does not divide c. Let d be the product of the restricting primes below five. In the application $d \in \{1, 2, 3, 6\}$.

Theorem 4.1 (Sifted universal ambient group)**.** *Assume every restricting prime not dividing d is at least five. Then*

$$\mathrm{UAG}(T_{q,c}) \cong \mathbb{Z}^{\varphi(d)} \oplus C_{q/d}. \tag{4.1}$$

Proof. Write $T = T_{q,c}$. If $s, s' \in dC_q$, the Chinese remainder theorem supplies x such that

$$x, x + s \in T, \qquad x, x + s, x + s + s' \in T. \tag{4.2}$$

Indeed, at a prime dividing d these points occupy one residue class and we avoid its unique forbidden class. At every other restricting prime there are at most three forbidden classes, while the prime has size at least five. The choices lift through prime powers and then combine by CRT.

At a restricting prime power $\ell^a \| q$, membership in T excludes one residue class modulo ℓ, not one class modulo ℓ^a. We first choose x modulo ℓ outside the at most three forbidden

classes and then lift it arbitrarily modulo ℓ^a. At a nonrestricting prime, $\ell \mid c$, so $1 + cx$ is never zero modulo ℓ and no avoidance is needed. This explains why d records small primes but not their exponents.

Let W be the attained admissible classes modulo d; then $|W| = \varphi(d)$. Every Freiman relation preserves the multiset of its two classes in W. This point deserves an explicit local check. Modulo two there is only one admissible class, so preservation is automatic; this uniqueness is what makes the cases $d = 2, 6$ work. Modulo three there are two admissible classes, and the three unordered two-element multisets have distinct sums. Combining these statements by CRT proves the assertion modulo d. Consequently the class-count functions descend to $\mathrm{UAG}(T)$.

The multiset conclusion is stronger than preservation of the total class sum. Modulo three, translate the admissible classes to $\{0, 1\}$: the three multisets $\{0, 0\}, \{0, 1\}, \{1, 1\}$ have distinct sums $0, 1, 2$. Modulo two, there is only one admissible class. The latter uniqueness, rather than a generic two-class argument, is the load-bearing fact in the $d = 2, 6$ cases.

For $s \in dC_q$, define

$$\Delta(s) = [e_{x+s}] - [e_x],$$

where $x, x + s \in T$. The first part of (4.2) and $(x + s) + y = x + (y + s)$ make this independent of x. The three-point part gives $\Delta(s + s') = \Delta(s) + \Delta(s')$. Thus $D = \Delta(d)$ has order dividing q/d.

Choose $t_w \in T$ in each $w \in W$. Every generator has the normal form

$$[e_t] = [e_{t_w}] + \Delta(t - t_w).$$

Define

$$\Psi(e_t) = \left(e_w, \frac{t - t_w}{d}\right) \in \mathbb{Z}^W \oplus C_{q/d}. \tag{4.3}$$

Class-multiset preservation cancels the first coordinates of a Freiman row, and equality of the original sums cancels the second. Hence Ψ descends. It sends the chosen representatives onto the free basis and D to 1, so it is surjective. Conversely the displayed normal forms generate $\mathrm{UAG}(T)$; an element in that form with zero image has every free coefficient and the slope coefficient equal to zero. Thus Ψ is injective. □

Equivalently, a Freiman homomorphism $F : T_{q,c} \to A$ is determined by one base value for every admissible class w modulo d and one additive slope D of order dividing q/d:

$$F(t) = F(t_w) + \frac{t - t_w}{d} D. \tag{4.4a}$$

There is no relation among the $\varphi(d)$ base values before inverse pairs are imposed. In the unit application with $d = 3$, the difference of the two base values is precisely the exceptional free direction.

4.1 Fourth-order dichotomy

Write $n = 2N$. The bijection $t \mapsto u(t) = 1 + 2t$ identifies $T_{N,2}$ with U_n. Negation corresponds to the involution $\iota(t) = -1 - t$. There are no odd-length zero-sum rows on U_n: an odd number of odd residues cannot sum to zero modulo the even integer n.

Modulo the inverse-pair rows, every zero-sum quadruple becomes a Freiman relation. More precisely, replacing the last two entries by their inverse partners turns

$$u(a) + u(b) + u(c) + u(d) = 0$$

into $a + b = \iota(c) + \iota(d)$ in C_N, and the converse construction is immediate. Therefore

$$\mathbb{Z}^{U_n}/\mathcal{L}_{\leq 4}(U_n) \cong \mathrm{UAG}(T_{N,2})/\langle \text{inverse rows} \rangle. \tag{4.4}$$

Proposition 4.2 (All-even unit quotient)**.** *For every even n,*

$$\mathbb{Z}^{U_n}/\mathcal{L}_{\leq 4}(U_n) \cong \begin{cases} C_n, & 3 \nmid n, \\ \mathbb{Z} \oplus C_{n/6}, & 6 \mid n. \end{cases} \tag{4.5}$$

Moreover $\mathcal{L}_{\leq 5}(U_n) = \mathcal{L}_{\leq 4}(U_n)$.

Proof. If $3 \nmid n$, Theorem 4.1 gives $\mathbb{Z} \oplus C_N$, with $e_t \mapsto (1, t)$. An inverse row maps to $(2, -1)$. Writing A for the free generator and B for the torsion generator, the quotient has $B = 2A$ and $2NA = 0$, hence is $C_{2N} = C_n$.

If $n = 6m$, the same theorem gives $\mathbb{Z}^2 \oplus C_m$. The free coordinates count the admissible classes $0, 2$ modulo three, which the involution swaps. Every inverse row therefore has image $(1, 1, \kappa)$ for one fixed $\kappa \in C_m$. The vector $(1, 1)$ is primitive. Quotienting eliminates exactly one free generator, and its torsion component can be absorbed into that eliminated generator. The result is $\mathbb{Z} \oplus C_m$. Finally the parity observation preceding (4.4) excludes every odd row, proving the last assertion. □

In the second branch, the torsion absorption can be made explicit. Let A_0, A_2 be the free basis vectors for the two admissible classes modulo three and let B generate C_m. Replace A_2 by $A_2 + \kappa B$. This is an automorphism of $\mathbb{Z}^2 \oplus C_m$, and it changes the inverse row to the primitive vector $A_0 + A_2$. Quotienting leaves $\mathbb{Z}(A_0 - A_2) \oplus C_m$. Thus the apparently nonzero slope component of the inverse row neither kills nor enlarges any torsion factor.

The target map supplies a useful check. In the generic branch the surviving generator maps to $1 \in C_n$, so its order is exactly n. In the exceptional branch the surviving free difference maps to zero in the finite target and is therefore a genuine obstruction, while the C_m coordinate is the residual translation coordinate.

4.2 Degree-six closure

Suppose $n = 6m$ and set

$$v_t = 1 + 6t, \qquad T_m = \{t \in C_m : \gcd(1 + 6t, m) = 1\}.$$

Then $U_{6m} = \{\pm v_t : t \in T_m\}$. After the inverse rows, send

$$e_{v_t} \longmapsto (1, t), \qquad e_{-v_t} \longmapsto (-1, -t) \quad \text{in } \mathbb{Z} \oplus C_m. \tag{4.6}$$

Every row of length at most five maps to zero. Its first coordinate is the difference between the numbers of residues congruent to 1 and -1 modulo six; it is divisible by six and has absolute value at most five. The zero-sum congruence then kills the second coordinate. The translation kernel inside (4.6) is generated by $(6, -1)$.

Lemma 4.3 (Sifted two-sum lemma)**.** $T_m + T_m = C_m$.

Proof. For prescribed $y \in C_m$, choose a prime by prime so that both a and $y-a$ lie in T_m. A prime restricting these conditions divides neither six nor the relevant coefficient, so it is at least five; at most two residue classes are forbidden. Primes dividing six impose no restriction. Lifting through prime powers and applying CRT gives the required a. □

Prime by prime, the two excluded classes are explicitly -6^{-1} and $y + 6^{-1}$ modulo a prime $\ell \nmid 6$; they may coincide, but there are never more than two. Since then $\ell \geq 5$, a class remains. If $\ell = 2$ or 3, $1+6a$ is automatically a unit modulo ℓ, even when ℓ divides m.

Choose three pairs in T_m with sums $-1, 0, 0$. The corresponding six positive units sum to $6 + 6(-1) = 0$ modulo $6m$, and their row maps to $(6, -1)$. Thus degree six kills the last kernel generator.

Every row of length at most five killed the entire presentation in (4.6), whereas $(6, -1)$ is nonzero there. The constructed sixth-order row is therefore a generator of the complete translation kernel, not merely an arbitrary relation appearing at degree six. This supplies both the strict lower bound and the closing upper bound.

Corollary 4.4. *If $6 \mid n$, then $\mathcal{L}_{\leq 6}(U_n) = K(U_n)$, while the free summand in (4.5) proves nonclosure through degree five.*

Proposition 4.2 and Corollary 4.4 prove the lattice part of Theorem 1.4. The global order-six reconstruction theorem of Grünbaum–Moore [GM95] is compatible with this bound; the explicit support quotient and its closing row are the calculations above.

5 Rational signals on the exact unit support

We now add the Galois condition that is absent from the integral calculation. If $3 \nmid n$, Proposition 4.2 says $\mathcal{L}_{\leq 4}(U_n) = K(U_n)$. The phase-ratio bridge then shows that every surviving ratio is a translation, proving part (2) of Theorem 1.4.

Let $n = 6m$. Proposition 4.2 gives $\mathcal{L}_{\leq 5}(U_{6m}) = \mathcal{L}_{\leq 4}(U_{6m})$. Hence the phase-ratio bridge identifies the fourth- and fifth-order invisible ratios with the same character group. In the coordinates of (4.6), such a character has

$$r(v_t) = ab^t, \qquad r(-v_t) = a^{-1}b^{-t}, \qquad b \in \mu_m. \tag{5.1}$$

Indeed, a character of $\mathbb{Z} \oplus C_m$ is freely specified by the image a of $(1, 0)$ and by the image b of $(0, 1)$, with the sole condition $b^m = 1$. The negative formula follows from the inverse-pair row, not from an additional reality assumption. Thus (5.1) is the complete complex ambiguity before translations and Galois descent.

A translation by y multiplies (a, b) by (ζ_n^y, ζ_m^y). Choose y modulo m to make $b = 1$. Define

$$\varepsilon(u) = \begin{cases} 1, & u \equiv 1 \pmod 6, \\ -1, & u \equiv -1 \pmod 6. \end{cases}$$

The normalized ratio is then

$$r_u = z^{\varepsilon(u)}. \tag{5.2}$$

Theorem 5.1 (Exact-unit rational descent)**.** *Let $f, g : C_{6m} \to \mathbb{Q}$ have exact Fourier support U_{6m} and equal autocorrelations through order four. After translating g, there are $\alpha \in \mathbb{Q}(\zeta_{6m})^\times$ and $z \in \mathbb{Q}(\zeta_6)^\times$ such that*

$$\widehat{f}(u) = \sigma_u(\alpha), \qquad \widehat{g}(u) = \sigma_u(z\alpha), \qquad z\overline{z} = 1. \tag{5.3}$$

Conversely every such pair, extended by zero off the unit support, is rational-valued and agrees through order five, hence also through order four. Two normalized parameters are translation-equivalent exactly modulo μ_6, and order six agrees exactly when $z^6 = 1$.

Proof. Rationality gives Galois covariance (3.3). Applying it to the ratio at $u = 1$ yields

$$\sigma_a(z) = z^{\varepsilon(a)} \qquad (a \in U_{6m}). \tag{5.4}$$

The kernel of $\varepsilon : U_{6m} \to \{\pm 1\}$ has index two and fixes ζ_6. Its fixed field is therefore precisely $\mathbb{Q}(\zeta_6)$, so $z \in \mathbb{Q}(\zeta_6)^\times$. Complex conjugation corresponds to $a = -1$, and (5.4) gives $\overline{z} = z^{-1}$.

To justify the fixed field statement without a degree shortcut, restriction of automorphisms gives the surjection

$$\mathrm{Gal}(\mathbb{Q}(\zeta_{6m})/\mathbb{Q}) \longrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_6)/\mathbb{Q}) \cong \{\pm 1\}.$$

It is exactly $a \mapsto \varepsilon(a)$. Hence its kernel has fixed field $\mathbb{Q}(\zeta_6)$. Equation (5.4) says that every element of this kernel fixes z, while the nontrivial coset sends it to its conjugate.

Put $\alpha = \widehat{f}(1)$. Galois covariance gives the first equation in (5.3), and (5.2)–(5.4) give the second. Conversely, for arbitrary nonzero α and norm-one z, the right side of (5.3) is Galois equivariant and therefore defines rational signals with exact support U_{6m}. Every zero-sum row of length at most five has zero ε-sum by the argument around (4.6), so its ratio product is one. Proposition 3.2 proves equality of the autocorrelations.

The exact-support assertion in the converse is immediate but worth recording: every conjugate $\sigma_u(\alpha)$ and $\sigma_u(z\alpha)$ is nonzero, while we set all nonunit coefficients to zero. Galois equivariance preserves this zero pattern. Formula (3.4) then gives rational time-domain values, and the automorphism $u = -1$ gives the conjugate symmetry needed for reality.

Translations preserving the normalization $b = 1$ have $m \mid y$, and ζ_{6m}^y then runs through μ_6. Every degree-six row has ε-sum 0 or ± 6, while the closing row constructed after Lemma 4.3 has sum six. Hence the sixth-order data read exactly z^6. They agree precisely for $z \in \mu_6$, which is exactly the residual translation case. □

The norm-one parameter is a one-dimensional rational torus. Since $\mathbb{Q}(\zeta_6) = \mathbb{Q}(\sqrt{-3})$, every parameter other than -1 can be written

$$z = \frac{a + b\sqrt{-3}}{a - b\sqrt{-3}} qquad (a, b \in \mathbb{Q}), \tag{5.5}$$

with -1 obtained by $a = 0$. This parametrization is optional for the proof, but it makes clear that the common fourth- and fifth-order ambiguity is generally infinite even though the residual translation subgroup μ_6 is finite. The sixth-order test collapses the torus exactly to those six translation points.

This proves all signal-level assertions in Theorem 1.4. The exact-support hypothesis is essential: it licenses every Fourier division in Proposition 3.2 and in (5.1).

5.1 The C_{30} question

At $n = 30$, Theorem 5.1 gives the order-four and order-five classification stated in Corollary 1.6. The displayed construction of Agulnick–Busick-Warner [ABW26] uses a common amplitude in a quadratic subfield. The complete classification permits α to range over all of $\mathbb{Q}(\zeta_{30})^\times$, while the relative phase remains in the same norm-one torus in $\mathbb{Q}(\zeta_6)$. Thus the larger field enlarges the common amplitude, not the ambiguity mechanism. This is an answer to that support-specific question, not a priority assertion.

For example, choose any primitive $\alpha \in \mathbb{Q}(\zeta_{30})^\times$ and any rational point (5.5) with $z^6 \neq 1$. The inverse transforms supplied by (5.3) form a rational pair agreeing through order five, and therefore through order four, but not order six. Changing α changes both signals simultaneously and does not change their ratio. Changing z moves along the actual ambiguity family. This separates the common-amplitude degree of freedom from the relative phase in a way that is invisible if one works only with a single displayed example.

> **Scope boundary.** Theorem 5.1 classifies rational signal pairs only on the exact unit support. Later sections use the same lattice formalism on all 255 stable supports, but do not claim that every integral obstruction there has a rational signal realization.

6 Local CRT box presentations for C_{2pq}

Fix distinct odd primes $p < q$ and identify $C_{2pq} = C_2 \times C_p \times C_q$. In the fixed order

$$(E, P, Q, P1, Q1, F, G) = \big(\mathcal{O}_2, \mathcal{O}_p, \mathcal{O}_q, \mathcal{O}_{2p}, \mathcal{O}_{2q}, \mathcal{O}_{pq}, \mathcal{O}_{2pq}\big), \tag{6.1}$$

these are the seven nonzero automorphism orbits. The orbit $\mathcal{O}_1 = \{0\}$ will be added only after the seven-box classification.

The name "box" records the nonzero CRT coordinates. With $a \in C_p^\times$ and $b \in C_q^\times$, the complete list is

box	points	order	cardinality
E	$(1,0,0)$	2	1
P	$(0,a,0)$	p	$p-1$
Q	$(0,0,b)$	q	$q-1$
$P1$	$(1,a,0)$	$2p$	$p-1$
$Q1$	$(1,0,b)$	$2q$	$q-1$
F	$(0,a,b)$	pq	$(p-1)(q-1)$
G	$(1,a,b)$	$2pq$	$(p-1)(q-1)$.

(6.1a)

Thus a stable support is a choice of boxes and possibly $\mathcal{O}_1$. There are $2^7 - 1 = 127$ nonempty base masks and $2(127) + 1 = 255$ nonempty stable supports.

The quotient $A_d(B)$ is universal for functional equations: a map $h : B \to H$ into any abelian group kills every zero-sum row of length at most d exactly when it factors through $A_d(B)$. The proof below solves these equations and then uses coordinate characters to show that the resulting parameters are independent. This identifies the quotient, not merely an upper bound on it.

For a single box B, the cumulative quotients through degree four are as follows; "same" repeats the preceding entry, and the $p = 3$ rows have $q \geq 5$.

B	$A_2(B)$	$A_3(B)$	$A_4(B)$
E	C_2	C_2	C_2
P	$\mathbb{Z}^{(p-1)/2}$	C_p	C_p
Q	$\mathbb{Z}^{(q-1)/2}$	C_q	C_q
$P1$, $p=3$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$P1$, $p\geq 5$	$\mathbb{Z}^{(p-1)/2}$	same	C_{2p}
$Q1$	$\mathbb{Z}^{(q-1)/2}$	same	C_{2q}
F	$\mathbb{Z}^{(p-1)(q-1)/2}$	$C_p \oplus C_q$	same
G, $p=3$	$\mathbb{Z}^{q-1}$	same	$\mathbb{Z} \oplus C_q$
G, $p\geq 5$	$\mathbb{Z}^{(p-1)(q-1)/2}$	same	$C_2 \oplus C_p \oplus C_q$

Table 2: Cumulative local box presentations.

Equivalently, at degree four a homomorphism from a selected box quotient to an arbitrary abelian target has the universal normal forms

$$\begin{array}{c|l} E & c,\ 2c=0, \\ P & au,\ pu=0, \\ Q & bv,\ qv=0, \\ P1\ (p\geq 5) & c+au,\ 2c=pu=0, \\ P1\ (p=3) & au,\ u \text{ free}, \\ Q1 & c+bv,\ 2c=qv=0, \\ F & au+bv,\ pu=qv=0, \\ G\ (p\geq 5) & c+au+bv,\ 2c=pu=qv=0, \\ G\ (p=3) & au+bv,\ u \text{ free},\ qv=0. \end{array} \tag{6.2}$$

Proposition 6.1 (Local box theorem)**.** *Table 2 and the normal forms in (6.2) hold uniformly for all admissible p, q.*

Proof. Every nontrivial box is stable under negation without fixed points. Pair rows identify opposite generators and give the free ranks in the A_2 column; on E, the row $2e_E$ gives C_2.

On a parity-zero punctured prime line, pairs and triples give partial additivity $f(x+y)=f(x)+f(y)$ whenever the three terms are nonzero. Moving along the punctured line yields $f(x)=xu$, and the full prime cycle gives $ru=0$. This proves the P, Q rows.

To see the punctured-line step explicitly, fix $x\neq 0$. Partial additivity gives $f(kx)=kf(x)$ until the additive cycle returns to zero; the final step gives $rf(x)=0$. Every nonzero point is a multiple of x, so with $u=x^{-1}f(x)$ one obtains $f(a)=au$. Conversely this formula kills every zero-sum row, proving that the C_r factor is exact.

On a parity-one punctured line there are no odd rows. For $r\geq 5$, the quadruples $(x,1,1,-x-2)$ give $f(x+2)=f(x)+2f(1)$. Removing zero from the translation-by-two cycle leaves a connected path, so $f(x)=c+xu$. An opposite pair gives $2c=0$. For $r\geq 7$, the valid row $(2,2,2,r-6)$ gives $4c+ru=0$, hence $ru=0$. At the endpoint $r=5$, the separate zero-sum row $(1,1,1,2)$ gives $4c+5u=0$, hence $5u=0$. The coordinate characters realize both factors. For $r=3$, every quadruple contains two signs of each kind and gives nothing beyond the pair relation, leaving one free generator. This proves both $P1$ branches and the always-generic $Q1$ row, including the endpoint $q=5$.

The exception at three is structural. When $r\geq 5$, deleting zero from the translation-by-two cycle leaves enough overlapping edges to compare all values. For $r=3$, the two

remaining vertices are opposites and every even zero-sum relation uses the two signs equally. The generic C_2 constant and C_3 slope therefore merge into one unrestricted integral sign parameter.

For F with $p, q \geq 5$, fix a horizontal displacement s and compare

$$(x + s, y) + (c, d) = (x, y) + (c + s, d). \tag{6.3}$$

An auxiliary (c, d) can avoid at most four p-classes and three q-classes; at $p = 5$ the endpoint is handled by using the already fixed nonzero endpoints before choosing the auxiliary class. Thus horizontal differences are independent of the base point. A second comparison makes them additive, and the vertical argument is identical. Hence $f(a, b) = c + au + bv$ with $pu = qv = 0$. Pairs give $2c = 0$ and a valid triple gives $3c = 0$, so $c = 0$.

When $p = 3$, write $h(b) = f(1, b)$ and recover the negative sheet by pairs. Same-sheet triples make pair sums with a fixed nonzero total have the same h-value. A bridge point avoiding at most four q-classes covers zero totals, already for $q = 5$. The difference equation gives $h(b) = u + bv$, $3u = 0$, $qv = 0$. Therefore $A_3(F) = C_3 \oplus C_q$.

For G with $p, q \geq 5$, a zero-sum quadruple made from two opposite-sheet pairs makes every horizontal and vertical difference base-point independent. The same avoidance argument gives $f(a, b) = c + au + bv$. Pairs impose $2c = 0$, and the three coordinate characters prove independence of the C_2, C_p, C_q factors.

Finally let $p = 3$. Again write $h(b) = f(1, b)$. Every zero-sum quadruple has two signs of each kind and is a Freiman relation on $C_q^\times$. Its universal solutions are $h(b) = u + bv$ with u free and $qv = 0$. This gives $A_4(G) = \mathbb{Z} \oplus C_q$ and completes the table. □

The avoidance arguments are uniform at the small endpoints. The two points being compared are fixed before the auxiliary point is selected. At most four p-classes and three q-classes are then unavailable. At $p = 5$ the explicit endpoint relations leave the fifth class; no asymptotic density estimate or division by $p - 3$ is used. The separate two-sheet proof handles $p = 3$, precisely where the generic auxiliary point need not exist.

Finally, the three CRT coordinate characters restrict to the parity constant, the p-slope, and the q-slope in (6.2). They certify independence of the three primary factors in every generic row. In the exceptional $P1, G$ rows, the integral sign map has infinite image and proves that no local row through degree four can kill the free generator.

The free sign in the last row is the unique local mechanism capable of surviving through degree five. It is the local form of the exceptional unit support in Section 4.

Corollary 6.2 (Compatibility on the unit box)**.** *For $G = \mathcal{O}_{2pq} = U_{2pq}$,*

$$\delta(G) = 6 \quad (p = 3), \qquad \delta(G) = 4 \quad (p, q \geq 5).$$

Proof. If $p = 3$, the local form is $\mathbb{Z} \oplus C_q$; the critical-family argument in Section 9 shows exact depth six. Independently, Proposition 4.2 and Corollary 4.4 give the same conclusion for $n = 6q$. If $p, q \geq 5$, (6.2) is the ambient CRT character group and degree three has not yet imposed the generic quartic relations, so the exact depth is four; this also follows from $3 \nmid 2pq$ in Proposition 4.2. □

7 The Cubic Skeleton

Represent a nonempty union of the seven boxes (6.1) by its Boolean mask in the order $(E, P, Q, P1, Q1, F, G)$. The mask E is already closed at degree two. For every other

mask, the following formula characterizes cubic closure:

A homomorphism $A_3(S) \to H$ is a function on S annihilating every zero-sum multiset of length at most three. Closure means that each such function is the restriction of one character of the generated CRT subgroup. Writing its coordinates as (η, α, β), a character has

$$2\eta = 0, \qquad p\alpha = 0, \qquad q\beta = 0, \tag{7.0}$$

and the seven restrictions

E	η	$P[a]$	$a\alpha$
$Q[b]$	$b\beta$	$P1[a]$	$\eta + a\alpha$
$Q1[b]$	$\eta + b\beta$	$F[a,b]$	$a\alpha + b\beta$
$G[a,b]$	$\eta + a\alpha + b\beta$		

The local cubic equations already force the displayed linear forms on P, Q, F. Initially the parity-one boxes may carry arbitrary odd functions $x(a), y(b), z(a,b)$. The problem is exactly to decide which selected-box patterns force these odd functions into the common character form (7.0).

$$\begin{aligned} T_3 = {} & (F \wedge G) \vee (P \wedge P1 \wedge F) \vee (Q \wedge Q1 \wedge F) \\ & \vee (E \wedge P \wedge Q \wedge P1) \vee (E \wedge P \wedge Q \wedge Q1) \\ & \vee (P \wedge \neg Q1 \wedge \neg G) \vee (Q \wedge \neg P1 \wedge \neg G) \\ & \vee (F \wedge \neg P1 \wedge \neg Q1) \vee (P \wedge Q \wedge G \wedge \neg E) \\ & \vee (P1 \wedge Q1 \wedge F \wedge \neg E). \end{aligned} \tag{7.1}$$

Theorem 7.1 (Cubic iff theorem)**.** *A nonempty seven-box mask other than E closes at degree three if and only if it satisfies (7.1). Exactly 77 masks do so, and all have first closing degree three.*

7.1 Exhaustive triple types

A zero-sum triple contains zero or two parity-one points. Coordinate feasibility gives exactly the following nineteen box-count patterns:

$$\begin{gathered} PPP,\ QQQ,\ FFF,\ PFF,\ QFF,\ PQF,\ FGG,\ Q1FG,\ P1FG, \\ P1Q1F,\ QGG,\ QQ1Q1,\ QP1G,\ PGG,\ PQ1G,\ PP1P1, \\ EFG,\ EQQ1,\ EPP1. \end{gathered} \tag{7.2}$$

Up to exchanging p and q, every mixed equation is consequently of one of the ten forms

$$\begin{aligned} & P + 2P1, \quad P + 2F, \quad P + Q + F, \quad P + 2G, \quad F + 2G, \\ & E + P + P1, \quad E + F + G, \quad Q + P1 + G, \quad P1 + F + G, \quad P1 + Q1 + F. \end{aligned} \tag{7.3}$$

There is no parity-compatible twentieth pattern.

Here is the coordinate check behind exhaustiveness. The parity coordinate of a triple is zero only when it contains zero or two points from $P1, Q1, G$. With no parity-one point, each prime coordinate must either occur three times in one active box or be shared among P, Q, F, giving the first six patterns in (7.2). With two parity-one points, their nonzero prime coordinates determine which parity-zero box can cancel them. If the

two points lie in the same box this gives $FGG, QGG, PGG, QQ1Q1, PP1P1$ and their mixed-axis versions; if they lie in different boxes it gives the remaining attachments $Q1FG, P1FG, P1Q1F, QP1G, PQ1G, EFG, EQQ1, EPP1$. Every case is realized by nonzero coordinates. Three parity-one points are impossible before the prime coordinates are considered.

7.2 Sufficiency

We use five propagation facts, all valid with values in an arbitrary abelian group.

Lemma 7.2 (Cubic propagation)**.** *The equations in (7.3) have the following consequences.*

1. *If $P, P1$ occur, the relation $P + 2P1$ forces $x(a) = \eta + a\alpha$ with $2\eta = 0$. The exchanged assertion holds.*

2. *$P+2F$ identifies the P and F p-slopes; the exchanged and two-axis versions identify the corresponding slopes.*

3. *If P, G occur, then $z(a, b) = a\alpha + v(b)$ with v odd. If Q, G also occur, then $z(a, b) = \eta + a\alpha + b\beta$, $2\eta = 0$.*

4. *If F, G occur, then $z(a, b) = \eta + a\phi + b\psi$ with $2\eta = 0$.*

5. *The rows $E + P + P1, E + Q + Q1, E + F + G$ identify the parity constant with E. The rows $Q + P1 + G$ and $P + Q1 + G$ give, respectively, $z(a, b) = x(a) + b\beta$ and $z(a, b) = a\alpha + y(b)$. Once $x(a) = \eta + a\phi$, the row $P1 + F + G$ gives $z(a, b) = \eta + a\phi + b\psi$, and similarly after exchanging the primes. Finally $P1 + Q1 + F$ forces $x(a) - a\phi = -(y(b) - b\psi) = \eta$ with $2\eta = 0$.*

Proof. For (1), $P + 2P1$ gives $x(a) + x(c) = (a + c)\alpha$. If $h(a) = x(a) - a\alpha$, taking $c = a$ gives $2h(a) = 0$, while varying c makes h constant. This includes $p = 3$.

For (2), $P + 2F$ gives $2a(\phi - \alpha) = 0$. The difference is p-torsion and p is odd, hence $\phi = \alpha$. The other cases are identical.

For (3), fixing one coordinate in $P + 2G$ and subtracting two instances makes the p-difference independent of the q-base point, producing $a\alpha + v(b)$; opposite pairs make v odd. At $p = 3$ the two apparent sheets are identified by that simultaneous oddness. Applying the exchanged equation and comparing the two forms gives the affine expression and the two-torsion constant.

For (4), subtract the F slopes and set $h(a, b) = z(a, b) - a\phi - b\psi$. Relations $F + 2G$ make h constant on the punctured-rectangle adjacency graph and pairs impose $2h = 0$. The graph is connected for $p \geq 5$. For $p = 3$, simultaneous oddness reduces opposite first coordinates and a third nonzero q-coordinate completes the comparison.

For (5), the three rows containing E compare its value directly with the constant term of the other parity-one box. Substitution in $Q + P1 + G$ and $P + Q1 + G$ gives the two displayed attachment formulas. Substitution in $P1 + F + G$ gives the affine form on G. In $P1 + Q1 + F$, separating the p and q variables makes $x(a) - a\phi$ and $-(y(b) - b\psi)$ one constant; inverse pairs make it two-torsion. □

For each of the ten clauses in (7.1), Appendix B gives one uniform routing scheme with five explicit fields: the optional selected box, the mixed row used, the part of Lemma 7.2 applied, and the resulting normal form. The table includes the cases in which no mixed row is needed because a selected primary coordinate or the sole parity box is an independent character factor. Thus every bit not fixed by a clause is accounted for. Reading any row of that matrix shows that every selected box carries the restriction of one common character (η, α, β); hence the obstruction quotient vanishes. This proves the forward implication of Theorem 7.1 without a finite mask enumeration.

7.3 Necessity

The Boolean complement of (7.1), together with the special mask E, is the union of the ten terms

$$\begin{array}{ll} N_1 = G\neg F\neg P, & N_2 = G\neg F\neg Q, \\ N_3 = P1\neg F\neg P, & N_4 = Q1\neg F\neg Q, \\ N_5 = P1\neg G\neg P\neg Q1, & N_6 = Q1\neg G\neg P1\neg Q, \\ N_7 = \neg E\neg F\neg P\neg Q, & N_8 = EG\neg F\neg P1\neg Q1, \\ N_9 = EQ1\neg G\neg P\neg Q, & N_{10} = P1Q1\neg E\neg F\neg G. \end{array} \tag{7.4}$$

Conjunction signs are suppressed only in this display.

For N_1, N_3, take a nonzero odd function $\sigma_p : C_p^\times \to \mathbb{Z}$ on the selected $P1$ and G boxes. Every available dangerous triple either needs the missing F or P, or pairs opposite p-arguments; the function therefore kills every cubic row. N_2, N_4 use the exchanged witness. For N_5, N_6, use an odd function on the single surviving parity-one axis. For N_7, use any nonzero odd function on a selected box among $P1, Q1, G$: three parity-one points cannot be zero-sum. For N_8, N_9, take target C_2, value one on E, and zero elsewhere. For N_{10}, take target C_2, value one on $P1$, and zero elsewhere. The missing-box hypotheses prevent every one-point attachment, while two occurrences cancel in C_2.

One may choose the integer witness concretely: select a representative pair $\{a, -a\}$, put $\sigma_p(a) = 1$, $\sigma_p(-a) = -1$, and set all other values to zero. For N_1, N_3, the only surviving triple equations involving its support pair opposite p-arguments, so the two values cancel. Because the generated CRT subgroup is finite, every homomorphism from it to $\mathbb{Z}$ is zero; the nonzero witness cannot be an ambient character. This avoids any appeal to dimension counting.

For the C_2 witnesses, an ambient parity character gives the same value on every selected parity-one box. The N_8, N_9, N_{10} assignments deliberately give different values to two selected parity-one boxes, while the absent boxes remove every cubic relation that would compare them. Hence these witnesses are also noncharacters for a structural reason, not merely because a finite matrix has positive nullity.

The integer-valued witnesses cannot be characters of a finite group, and the C_2 witnesses disagree with the unique ambient parity character on the selected boxes. Each is therefore a genuine cubic defect. Appendix B expands the coverage table, proving the reverse implication without a machine-generated Boolean premise.

Finally, degree two contains only inverse-pair rows inside each box. Table 2 shows that E is the only nonempty base mask already closed at two. Direct Boolean reduction of the ten disjointly audited clauses gives 77 positive masks; Appendix B includes an elementary count. Thus all 77 close first at three.

Adjoining the optional zero orbit doubles them, so 154 of the 255 supports have depth

three.

8 The Quartic Skeleton

We next work modulo the local degree-four forms (6.2). Fourteen uniform zero-sum connector types, listed in Appendix A, attach the remaining local parameters. The specialization

$$P1[a] + G[-a, b] + G[c, d] + G[-c, -b - d] = 0 \tag{8.1}$$

and its $p \leftrightarrow q$ exchange are load-bearing. This is the second parameter choice of type X10, not a fifteenth connector: it supplies both the slope edge and the parity/sign edge.

The connector types perform three overlapping jobs:

job	types
axis-to-rectangle slope attachment	$X2, X3, X6$
rectangle and cross-box comparison	$X1, X4, X7, X9, X10B, X13, X14$
parity anchoring	$X5, X8, X10A, X10B, X11, X12.$

(8.0)

Some types necessarily occur in two rows. In particular, merging X10A and X10B merely because they have the same box counts would erase (8.1). Appendix A supplies uniform nonvanishing parameter choices, including the $p = 3, q = 5$ endpoints.

We repeatedly use the following elementary isolation principle. If a connector image is $r+bs$, where s has odd prime order, two admissible values of b whose difference is a unit isolate s and then r. A difference $2s$ also suffices by Bezout. Thus primary components can be separated symbolically, without sampling the primes.

Lemma 8.1 (Primary isolation)**.** *Let s have order dividing an odd prime r. If $x+b_1s = 0$ and $x + b_2s = 0$, with $b_1 - b_2$ a unit modulo r, then $s = x = 0$. The conclusion also holds when $b_1 - b_2 = 2$.*

Proof. Subtraction gives $(b_1 - b_2)s = 0$. Multiplication by a unit is an automorphism of the r-primary group, so $s = 0$ and substitution gives $x = 0$. Two is a unit because r is odd. The argument may be applied after primary projection and therefore does not assume that a splitting was already chosen. □

Theorem 8.2 (Quartic closure theorem)**.** *Among the masks not closed by degree three, the exact degree-four failure sets are*

$$\begin{aligned} p, q \geq 5 &: \ 0x18,\ 0x1a,\ 0x1c, \\ p = 3 &: \ 0x08,\ 0x0c,\ 0x18,\ 0x1a,\ 0x1c,\ 0x40,\ 0x44,\ 0x48,\ 0x4c. \end{aligned} \tag{8.2}$$

Consequently 46 generic and 40 prime-three residual masks close first at degree four.

8.1 Generic branch

Suppose $p, q \geq 5$. All local parameters split into their 2-, p-, and q-primary layers. On the p-layer, the vertices $P, P1, F, G$ have the six edges

$$\begin{aligned} &P - P1 : \text{X2}, \quad P - F : \text{X6}, \quad P - G : \text{X3}, \\ &P1 - F : \text{X13}, \quad P1 - G : \text{X10B}, \quad F - G : \text{X1}. \end{aligned} \tag{8.3}$$

This is a complete graph. The exchanged connectors give the complete graph on $Q, Q1, F, G$ for the q-layer. Therefore each selected prime layer has exactly one ambient coordinate.

An edge in (8.3) means equality of the named local slope parameters. X1, for example, gives twice the $F - G$ difference; Lemma 8.1 removes the factor two. X10B gives the $P1 - G$ difference, while X10A by itself only anchors a parity combination. Since the graph is complete, every nonempty induced subgraph is connected, including supports containing only two of the four slope vertices.

The parity vertices are $E, P1, Q1, G$. Edges from E to the other three and from each odd line to G are available whenever their endpoint boxes occur. The remaining edge $P1 - Q1$ comes from X7 if F occurs, or from X14 if both P, Q occur. Disconnection is possible only if $P1, Q1$ occur, E, G, F do not, and P, Q are not both present. In the bit order (6.1) these are exactly $0x18, 0x1a, 0x1c$. This proves the first line of (8.2).

These masks decode to $P1 \cup Q1$, $P \cup P1 \cup Q1$, and $Q \cup P1 \cup Q1$. The graph already identifies the surviving class as a diagonal parity discrepancy: it is the generic depth-six obstruction on the first mask and the depth-five obstruction on the latter two.

8.2 The prime-three branch

Let $p = 3$. The C_q vertices still form a complete graph. The parity/three part has six generators

$$(e, p, u, c, f, g) = (E, P, P1, Q1, F, G),$$

with $2e = 3p = 2c = 3f = 0$ and u, g free. Their common images in C_6 are $(3, 4, 1, 3, 4, 1)$. The connector types yield the following twelve rows:

$$\begin{array}{c|c} FG : f + 2g & PP1 : p + 2u \\ PG : p + 2g & PF : p + 2f \\ P1F : 2u - 2f & P1G : u - g \\ EG : e + 3g & EP1 : e + 3u \\ EQ1 : e + c & Q1G : c + 3g \\ P1Q1F : u + c - f & PP1Q1 : p - u + c. \end{array} \tag{8.4}$$

Every row maps to zero in C_6, and the isolation principle separates an attached C_q component.

The target map is

$$e, p, u, c, f, g \longmapsto 3, 4, 1, 3, 4, 1 \quad \text{in } C_6. \tag{8.5}$$

Each row of (8.4) visibly lies in its kernel. The free elements u, g are integral lifts of a sign coordinate. An anchor turns such a lift into one element of order six; it is not legitimate to split it in advance as an independent C_2 constant and C_3 slope. The integral presentation is exactly what keeps that distinction visible.

If neither free generator occurs, ordinary connectors give the ambient target. If g occurs without u, one of $E, P, F, Q1$ anchors it to order six; without an anchor the failures are $0x40, 0x44$. If both occur, (8.1) gives $u = g$, and the same anchors close the common generator; the failures are $0x48, 0x4c$. If u occurs without g, one of E, P, F anchors it. Without an anchor the failures are $0x08, 0x0c, 0x18, 0x1c$. When anchored, $Q1$ is attached by the exchanged X12, X7, or X14 except for $0x1a$, where an extra C_2 survives. These are precisely the nine failures in (8.2).

The division is exhaustive because u, g are the only free generators. If both are absent, all remaining generators have order two or three and the ordinary connectors identify them with (8.5). If exactly one is present, the listed anchor set contains every row in (8.4) with a nonzero coefficient on it. If both occur, X10B first reduces them to one common generator and the same anchor analysis applies. The exceptional $Q1$ attachment on $0x1a$ leaves exactly the diagonal C_2 killed by the fifth-order bridge.

Theorem 7.1 accounts for 77 masks. Removing those, E, and the failure sets from the 127 base masks leaves 46 generic and 40 prime-three masks. The local forms show they were not closed earlier, so their first closing degree is four. This proves Theorem 8.2.

Remark 8.3. The hexadecimal masks are only compact names. Appendix B decodes the bit order and lists the routed families. The proof above is uniform in p, q and uses no sampled Smith normal form.

The finite atlases at $n = 30, 42, 70, 110$ were used for discovery and regression only. The proof of Theorem 8.2 consists of the parameterized rows in Appendix A, Lemma 8.1, the complete primary graphs, and the fixed integral presentation (8.4). None of those steps interpolates from a numerical prime sample.

9 Depth five, depth six, and the complete profiles

The failures in (8.2) split into two depth-five bridges and the critical depth-six bases.

For each family we prove exactness in two steps. First we exhibit a bounded quotient with a nonzero restriction to the translation kernel, proving failure at the preceding degree. Then we write a zero-sum row representing a generator of that kernel. A late relation alone would give only an upper bound; a defect alone would give only a lower bound.

Proposition 9.1 (Generic critical pair)**.** *If $p, q \geq 5$, the support $P1 \cup Q1$ has*

$$K(P1 \cup Q1)/\mathcal{L}_{\leq 5}(P1 \cup Q1) \cong C_2 \tag{9.1}$$

and closes at degree six.

Proof. The internal presentations are C_{2p} and C_{2q}. A mixed relation of length at most four must split two-plus-two between the boxes, and each pair is opposite. Odd rows do not exist, so through degree five $A_5(P1 \cup Q1) = C_{2p} \oplus C_{2q}$. The map to the generated CRT subgroup identifies the two parity coordinates, and its kernel is their diagonal element, proving (9.1).

If x, y generate C_{2p}, C_{2q}, the map to the CRT target sends their parity components to the same element of C_2 and their odd-primary components to independent coordinates. The kernel is generated by $px - qy$, an element of order two. Every mixed four-term row is already an opposite pair in each box, and every odd row is forbidden by parity, so no relation through degree five changes that class.

Choose a coordinate-zero triple $(1, 1, -2)$ in each odd line. Each triple has odd parity, while their union is a genuine length-six zero-sum row. It represents the diagonal element, so degree six closes the support. □

This support is the sharpness family appearing in Casper–Orozco [CO26]; the exact quotient and its placement in the full profile are the assertions used here.

Proposition 9.2 (Prime-three critical family)**.** *Let $p = 3$. The seven bases*

$$P1, \quad Q \cup P1, \quad P1 \cup Q1, \quad G, \quad Q \cup G, \quad P1 \cup G, \quad Q \cup P1 \cup G \tag{9.2}$$

have fifth-order obstruction $\mathbb{Z}$ and close at degree six.

Proof. Put $A = P1 = \mathcal{O}_6$, $B = Q1 = \mathcal{O}_{2q}$, $C = Q = \mathcal{O}_q$, and $D = G = \mathcal{O}_{6q}$. Let η be the C_3 sign on A, D and zero on B, C. On each support in

$$A,\ A \cup C,\ A \cup B,\ D,\ C \cup D,\ A \cup D, A \cup C \cup D, \tag{9.3}$$

every zero-sum row of length at most five has $\eta = 0$. Its sign sum is divisible by three. Except on $A \cup B$, parity also makes it even, and absolute value at most five forces zero. On $A \cup B$, a value ± 3 would require three A points and an odd number of B points. A single B point cannot have zero q-coordinate, while three make total length at least six.

The local forms are

$$A : \mathbb{Z}u, \quad C : C_q v, \quad B : C_2 c \oplus C_q v, \quad D : \mathbb{Z}u \oplus C_q v.$$

Available triples glue the q-slopes on $C \cup D$, and available quadruples glue sign parameters on $A \cup D$. The map to the CRT target then has cyclic kernel generated by $6u$, except on $A \cup B$, where it is generated by $3u + c$. The normalized η-value of any additional relation is its coefficient in this kernel, and the preceding visibility argument forces it to vanish through degree five. Hence every obstruction is exactly $\mathbb{Z}$.

The seven presentations may be summarized as

base type	kernel generator	normalized sign
$A,\ A \cup C,\ D,\ C \cup D,\ A \cup D,\ A \cup C \cup D$	$6u$	$\eta/6$
$A \cup B$	$3u + c$	$\eta/3$.

(9.3a)

The last column denotes the integer coefficient on the cyclic kernel; it is not division inside an abelian target. The visibility argument proves that this coefficient is zero on every row of length at most five.

Six positive A or D points realize $6u$. On $A \cup B$, take three positive A points and three B points whose q-coordinates sum to zero; this realizes $3u + c$. Thus all seven bases close at six. □

Proposition 9.3 (The two bridges)**.** *In both arithmetic regimes the bases*

$$P \cup P1 \cup Q1, \qquad Q \cup P1 \cup Q1 \tag{9.4}$$

have exact depth five.

Proof. For the first base, the local forms are xa, $c_p + xa_p$, and $c_q + yb$. A mixed triple gives $a_p = a$. The remaining kernel is generated by $c_p + c_q$. No row of length at most four represents it: an odd number of points is needed in both parity-one boxes. One $Q1$ point cannot cancel its q-coordinate, while three leave only one $P1$ point, which cannot cancel its p-coordinate. A length-five row with one P point, one opposite $P1$ point, and three $Q1$ points with zero q-sum kills the diagonal parity. This proves exact depth five. Exchanging p, q gives the second bridge when $p, q \geq 5$.

The lower bound is uniform in the primes. A representative of $c_p + c_q$ must use an odd number of points from each parity-one box. At total length at most four, the only possible mixed splits are $1+1$ and $1+3$. The first leaves both odd coordinates nonzero; in the second, the single point on one axis cannot be cancelled by points on the other. Thus the diagonal parity class genuinely survives all quartic rows.

If $p = 3$, the mixed triple on $P \cup P1$ gives $\alpha + 2u = 0$; with $3\alpha = 0$ this is a C_6 parameter. Adding $Q1$ again leaves only the diagonal parity kernel, excluded at four and killed by the same five-term row. For the exchanged bridge, the q-slopes glue and the fourth-order kernel is $\mathbb{Z}(3u + c)$. A row of length at most four has zero sign: sign ± 3 would require three $P1$ points and one $Q1$ point, whose q-coordinate cannot vanish. The explicit relation

$$3P1[1] + Q[-y] + Q1[y] = 0$$

represents $3u + c$ at length five. □

9.1 Assembly of the 255 supports

The zero orbit splits off harmlessly. Indeed e_0 is itself a length-one zero-sum row, so adjoining $\mathcal{O}_1$ leaves every obstruction quotient unchanged. Thus $\mathcal{O}_1$ alone has depth one and every nonempty seven-box base has two versions of the same depth.

Indeed, for every $d \geq 1$,

$$\mathbb{Z}^{S\cup\{0\}} = \mathbb{Z}^S \oplus \mathbb{Z}e_0, \qquad \mathcal{L}_{\leq d}(S \cup \{0\}) = \mathcal{L}_{\leq d}(S) \oplus \mathbb{Z}e_0, \tag{9.4a}$$

and the same direct sum holds for K. This proves the doubling rule rather than treating it as a numerical symmetry of the atlas.

Proof of Theorem 1.1. The support E is the only depth-two base and contributes two raw supports. Theorem 7.1 gives 77 depth-three bases and hence 154 supports.

If $p, q \geq 5$, Theorem 8.2 gives 46 depth-four bases, Proposition 9.3 gives two depth-five bases, and Proposition 9.1 gives one depth-six base. Therefore

$$1, \quad 2, \quad 2(77) = 154, \quad 2(46) = 92, \quad 2(2) = 4, \quad 2(1) = 2, \tag{9.5}$$

which sums to 255.

If $p = 3$, the corresponding numbers of bases are $40, 2, 7$, by Theorem 8.2, Proposition 9.3, and Proposition 9.2. Hence

$$1, \quad 2, \quad 154, \quad 2(40) = 80, \quad 4, \quad 2(7) = 14, \tag{9.6}$$

again summing to 255. Every family was accompanied both by closure and by a defect at the preceding degree, so these are first-closing depths rather than upper bounds. Proposition 2.1 identifies them with maximum separating-atom lengths. □

As checks on the assembly,

$$1 + 2 + 154 + 92 + 4 + 2 = 255, \qquad 1 + 2 + 154 + 80 + 4 + 14 = 255. \tag{9.7}$$

The cubic layer is identical in both regimes. The twelve-support decrease in the quartic entry when $p = 3$ is transferred exactly to the depth-six entry: six additional base masks, doubled by the optional zero orbit. This is the profile-level trace of the free prime-three sign directions.

Corollary 6.2 now follows from both theorem lines: the single box G is among the seven critical bases when $p = 3$, and among the generic quartic masks when $3 \nmid 2pq$.

10 Galois realization on every stable support

The preceding sections determine the lattice closing degree on every $\mathrm{Aut}(C_{2pq})$-stable support. We now prove that none of those lower bounds disappears when the time-domain signals are required to be rational or integer-valued. The division of ownership is important: the Cubic and Quartic Skeletons and the depth-five and depth-six presentations supply the already established lattice classifier; on every support of depth greater than one, the argument below supplies a Galois-equivariant, nontranslation phase ratio immediately below its closing degree. The sole depth-one support is direct. Proposition 3.3 then turns each ratio into rational and integer signal pairs.

We continue to write $n = 2pq$ with $p < q$ odd primes and $\sigma_u(\zeta_n) = \zeta_n^u$. The box notation is that of Section 6. If the zero orbit is present, every ratio below is extended by $r_0 = 1$.

10.1 Non-torsion orbit phases and the low-depth layers

The phase families used at depths three and four come from an elementary norm-one construction.

Lemma 10.1 (Non-torsion cyclotomic phase)**.** *For every divisor $m \mid n$ with $m > 2$, there is $z_m \in \mathbb{Q}(\zeta_m)^\times$ such that $z_m\overline{z_m} = 1$ and z_m is not a root of unity. If $s_m = n/m$, then*

$$\psi_m(us_m) = \sigma_u(z_m) \qquad (us_m \in \mathcal{O}_m) \tag{10.1}$$

is well defined, Galois equivariant, and satisfies $\psi_m(-s) = \psi_m(s)^{-1}$. It is not the restriction of a translation ratio.

Proof. For $t \in \mathbb{Q}$ put

$$z_m(t) = \frac{t - \zeta_m}{t - \zeta_m^{-1}}.$$

Conjugation inverts this value. The map $t \mapsto z_m(t)$ is injective on $\mathbb{Q}$, since equality at t, u gives $(t - u)(\zeta_m - \zeta_m^{-1}) = 0$. A number field contains only finitely many roots of unity, so some rational t gives the required non-torsion z_m.

Two units carrying s_m to the same point have the same restriction to $\mathbb{Q}(\zeta_m)$, so (10.1) is well defined. Equivariance is built into the definition, and the unit -1 gives the inverse identity. Every value of a translation ratio is a root of unity, whereas every conjugate of z_m is non-torsion. Hence ψ_m is not a translation. □

The following observation isolates the logical step used repeatedly below. It also makes clear which part comes from the existing lattice classifier and which part is new in this section.

Lemma 10.2 (Closing-degree witness)**.** *Let S be stable and suppose $\mathcal{L}_{\le d-1}(S) \subsetneq \mathcal{L}_{\le d}(S) = K(S)$. If a Galois-equivariant ratio $r = (r_s)_{s \in S}$ satisfies the inverse condition, kills every row of $\mathcal{L}_{\le d-1}(S)$, and is not a translation, then it is detected by a row of degree d. Moreover, it is realized by exact-support rational and integer signal pairs that agree through order $d - 1$ and fail to agree at order d.*

Proof. If r killed $\mathcal{L}_{\le d}(S) = K(S)$, the phase-ratio criterion of Proposition 3.2 would make it a translation, contrary to the hypothesis. Hence some generator introduced at degree

d has nontrivial product. Proposition 3.3 realizes r by taking $\widehat{f} = 1$ and $\widehat{g} = r$ on S; its common-denominator rescaling preserves support, equality of lower autocorrelations, and failure at degree d. □

The zero orbit separates at every degree. Indeed, e_0 is a length-one zero-sum row and, as in (9.4a),

$$K(S \cup \{0\}) = K(S) \oplus \mathbb{Z}e_0, \qquad \mathcal{L}_{\le d}(S \cup \{0\}) = \mathcal{L}_{\le d}(S) \oplus \mathbb{Z}e_0. \tag{10.2}$$

Thus adjoining $\mathcal{O}_1$ changes neither a witness nor its depth. The support $\mathcal{O}_1$ alone has depth one.

For the depth-two base $E = \mathcal{O}_2$, take $r_E = 2$. It is rational and invariant, there is no degree-one relation on E, and a translation takes only the values ± 1 there. The pair row $E + E = 0$ closes at degree two, so Proposition 3.3 proves exact rational and integer depth two, with or without $\mathcal{O}_1$.

Each of the 77 depth-three bases other than E contains an orbit $\mathcal{O}_m$ with $m > 2$. Put $r = \psi_m$ on one such orbit and $r = 1$ on all other boxes. The only relations through degree two are the zero row and inverse pairs, so Lemma 10.1 makes r invisible through degree two and nontranslational. The Cubic Skeleton closes the base at degree three. This realizes all 154 raw depth-three rows after the optional zero orbit is restored. Notice that the construction does not need a choice compatible across different bases: the theorem asks for a witness on each fixed support. Once the orbit $\mathcal{O}_m$ is chosen, however, all of its Fourier values are forced by one cyclotomic number and Galois equivariance, so the resulting inverse DFT is rational rather than merely real.

10.2 The nine depth-four families

For a point in a box with p-coordinate a or q-coordinate b, write $\psi_p(a)$ or $\psi_q(b)$ for the corresponding value in (10.1). On boxes not named in the phase column, the ratio is one. Table 3 is a canonical first-match cover of every base that fails the cubic criterion and closes at degree four. The last column records the generic and prime-three first-match counts; those counts are checks on the routing, not premises of the cancellation proof.

Family	Mask condition	Nontrivial ratio	Cubic types seeing it	$d4$ gen./$p = 3$
$N1$	$G, \neg F, \neg P$	$\psi_p(a)$ on $P1, G$	$QGG, QP1G$	16/12
$N2$	$G, \neg F, \neg Q$	$\psi_q(b)$ on $Q1, G$	$PGG, PQ1G$	8/8
$N3$	$P1, \neg F, \neg P$	$\psi_p(a)$ on $P1, G$	$QGG, QP1G$	6/4
$N4$	$Q1, \neg F, \neg Q$	$\psi_q(b)$ on $Q1, G$	$PGG, PQ1G$	5/5
$N5$	$P1, \neg G, \neg P, \neg Q1$	$\psi_p(a)$ on $P1$	none	4/4
$N6$	$Q1, \neg G, \neg P1, \neg Q$	$\psi_q(b)$ on $Q1$	none	4/4
$N8$	E, G $\neg F, \neg P1, \neg Q1$	-1 on E	none	1/1
$N9$	$E, Q1$ $\neg G, \neg P, \neg Q$	-1 on E	none	1/1
$N10$	$P1, Q1$ $\neg E, \neg F, \neg G$	-1 on $P1$	$PP1P1$; two signs	1/1
Total				46/40

Table 3: The nine canonical depth-four Galois lifts.

We first justify that the table is complete without invoking a finite search. The necessity half of Theorem 7.1 covers the complement of its ten positive clauses by $N1, \ldots, N10$.

The omitted valid family is

$$N7: \quad \neg E, \neg F, \neg P, \neg Q.$$

A nonempty $N7$ mask uses only $P1, Q1, G$. If it contains G, it is in $N1$; if it does not contain G but contains $P1$, it is in $N3$; otherwise it is the $Q1$ case of $N4$. Hence

$$N7 \subset N1 \cup N3 \cup N4, \tag{10.3}$$

and the nine displayed families cover every cubic-negative mask.

Next we verify invisibility through degree three. Inverse pairs cancel by Lemma 10.1. For $N1, N3$, the exhaustive triple list in Theorem 7.1 shows that only QGG and $QP1G$ can see the marked p-phase. Their zero p-coordinate forces the two marked arguments to be $a, -a$. The $N2, N4$ argument is the same on the q-coordinate, using $PGG, PQ1G$. The exclusions in $N5, N6$ remove every triple with an unbalanced marked factor. In $N8, N9$, every triple containing E would require an excluded box. Finally, $N10$ sees only $PP1P1$, where the sign occurs twice. Thus every row of Table 3 kills $\mathcal{L}_{\leq 3}$ and is Galois equivariant.

The six cyclotomic rows are nontranslations by Lemma 10.1. The three sign rows require separate arguments. In $N8$, the ratio is one on the generator $1 \in G$, forcing a putative translation parameter to be zero, but it is -1 on E. In $N9$, the point $p \in Q1$ has ratio one, forcing the translation parameter to be even, whereas -1 on $E = pq$ requires it to be odd. In $N10$, the point $q \in P1$ with ratio -1 forces an odd parameter, while the point $p \in Q1$ with ratio one forces an even parameter.

The Quartic Skeleton says that every base counted in the last column has $\mathcal{L}_{\leq 4} = K$. Lemma 10.2 therefore applies to each displayed ratio: a degree-four row detects it, and the rational and integer realizations have exact depth four. There are 46 generic or 40 prime-three bases, and (10.2) doubles these to 92 or 80 supports. Thus the last column of Table 3 is used only after the symbolic cancellation and nontranslation arguments have been completed; it is never a substitute for either argument.

10.3 Depth five and six

The two depth-five bases are

$$S_p = P \cup P1 \cup Q1, \qquad S_q = Q \cup P1 \cup Q1.$$

On either base put $r = -1$ on $P1$ and $r = 1$ elsewhere. This is the $N10$ sign and is not a translation. If a zero-sum row of length at most four had product -1, it would contain odd numbers of both parity-one boxes. On S_p, if $Q1$ occurs once, its nonzero q-coordinate cannot be cancelled by the P and $P1$ boxes. If $Q1$ occurs three times, the length bound leaves exactly one $P1$ point and no P point, so its nonzero p-coordinate cannot be cancelled. The argument on S_q is symmetric, with $P1, Q1$ and p, q exchanged. Hence the sign is invisible through degree four. It is detected by the five-term rows

$$\begin{aligned} S_p: &\quad P[-a] + P1[a] + Q1[1] + Q1[1] + Q1[-2], \\ S_q: &\quad P1[1] + P1[1] + P1[-2] + Q[-b] + Q1[b]. \end{aligned} \tag{10.4}$$

All shown coordinates are nonzero. When $p = 3$, the first-axis triple $1, 1, -2$ is simply $1, 1, 1$. Thus both bridge bases, and their zero-orbit copies, have exact rational and integer depth five.

Suppose now that $p, q \geq 5$. The only depth-six base is $P1 \cup Q1$. Use the same sign on $P1$. A negative product through degree five would require odd cardinalities in both boxes, so the total length would be even and at most four. The preceding $1+1$ and $3+1$ obstruction again rules this out. The six-term row

$$P1[1] + P1[1] + P1[-2] + Q1[1] + Q1[1] + Q1[-2] \tag{10.5}$$

is zero-sum and contains three negative factors. It detects the ratio and proves exact depth six for the base and its zero-orbit copy.

It remains to realize the seven prime-three critical bases. Put

$$A = P1 = \mathcal{O}_6, \quad B = Q1 = \mathcal{O}_{2q}, \quad C = Q = \mathcal{O}_q, \quad D = G = \mathcal{O}_{6q}. \tag{10.6}$$

They are

$$A, A \cup C, A \cup B, D, C \cup D, A \cup D, A \cup C \cup D. \tag{10.7}$$

Choose a non-torsion norm-one $z \in \mathbb{Q}(\zeta_6)^\times$. On A, D, let $\varepsilon(s) \in \{1, -1\}$ be the nonzero $\mathbb{F}_3$ coordinate and set $r_s = z^{\varepsilon(s)}$; set $r = 1$ on B, C. Every unit modulo $6q$ is 1 or -1 modulo six. It acts on z as the identity or conjugation and on ε by the same sign, so the ratio is Galois equivariant.

For a zero-sum multiset, let η be the sum of the marked exponents. Its $\mathbb{F}_3$ coordinate gives $3 \mid \eta$. Except on $A \cup B$, parity also makes η even. Thus a row of length at most five has $6 \mid \eta$ and $|\eta| \leq 5$, hence $\eta = 0$. On $A \cup B$, a value $\eta = \pm 3$ would need three A points and an odd number of B points. One B point cannot cancel the q-coordinate, while three already make total length six. The ratio is therefore invisible through degree five on all seven bases.

Degree six detects it explicitly. Six positive A points work whenever A is present outside $A \cup B$. If only D is marked, use three pairs $D[1, b], D[1, -b]$. On $A \cup B$, use three positive A points together with $B[1], B[1], B[-2]$. The resulting products are z^6 or z^3, neither one because z is non-torsion. The same non-torsion value also excludes a translation. Hence all seven bases and their zero-orbit copies have exact rational and integer depth six.

For reference, the detector assignment is displayed without suppressing any of the seven cases:

Critical base	Degree-six zero-sum detector	Product
A	six positive A points	z^6
$A \cup C$	six positive A points	z^6
$A \cup B$	three positive A points and $B[1] + B[1] + B[-2]$	z^3
D	$D[1, b] + D[1, -b]$, repeated three times	z^6
$C \cup D$	$D[1, b] + D[1, -b]$, repeated three times	z^6
$A \cup D$	six positive A points	z^6
$A \cup C \cup D$	six positive A points	z^6

The coordinates in the middle column sum to zero in both CRT components. Because z is non-torsion, the last column is nontrivial in every row; because each listed base already has $\mathcal{L}_{\leq 6} = K$, these are genuine closing rows, not merely upper-bound certificates.

10.4 The realized pairs and the proof audit

For every ratio chosen above, the rational pair is completely explicit in Fourier coordinates. Namely,

$$f_r(x) = \frac{1}{n}\sum_{s\in S}\zeta_n^{sx}, \qquad g_r(x) = \frac{1}{n}\sum_{s\in S} r_s\zeta_n^{sx}.$$

Galois equivariance fixes both sums under every element of $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$; hence $f_r, g_r \in \mathbb{Q}^{C_n}$. Their Fourier transforms are respectively 1 and r_s at every point of S and zero off S, so their support is exactly S. If D clears all time-domain denominators, then Df_r, Dg_r are integer signals with the same exact Fourier support. Multiplying both signals by D multiplies each order-k autocorrelation by D^k, so it neither creates nor removes the first separating order. It also leaves the phase ratio unchanged, and therefore cannot turn a nontranslation pair into a translation pair.

The proof obligations are summarized in the following table. Its middle columns cite the human argument that makes the ratio invisible below the claimed depth and the independent lattice statement that closes at that depth. This separation prevents the finite profile enumeration from being used as a surrogate for the new realization argument.

Depth	Nonzero bases	Lower-degree cancellation	Closing input or detector
1	none; zero orbit only	vacuous	the singleton row e_0
2	E	no degree-one row	the inverse pair $E + E$
3	77 Cubic-Skeleton bases	inverse-pair cancellation for one ψ_m	the Cubic Skeleton, $\mathcal{L}_{\le 3} = K$
4	46 generic or 40 prime-three bases	the nine rows of Table 3 against the exhaustive cubic types	the Quartic Skeleton, $\mathcal{L}_{\le 4} = K$
5	S_p, S_q	parity plus the $1+1$ and $3+1$ CRT obstructions	the two rows in (10.4)
6, generic	$P1 \cup Q1$	parity reduces length at most five to the same CRT obstruction	the row in (10.5)
6, $p = 3$	seven bases in (10.7)	$3 \mid \eta$, parity, and the separate $A \cup B$ argument	the seven detectors displayed above

The depth-one row is direct, and the depth-two row uses Proposition 3.3, whose hypotheses here are Galois equivariance, lower-degree cancellation, and nontranslation. Every row from depth three onward has supplied all four ingredients required by Lemma 10.2: Galois equivariance, inverse symmetry, lower-degree cancellation, and nontranslation. The zero-orbit splitting in (10.2) then transfers each nonzero-base construction to its optional-zero copy without altering the applicable ingredients.

10.5 Assembly

We can now prove the signal-level theorem without changing the lattice classifier.

Proof of Theorem 1.2. The inclusion of signal classes and Proposition 3.2 give

$$\delta_{\mathbb{Z}}(S) \le \delta_{\mathbb{Q}}(S) \le \delta(S). \tag{10.8}$$

The depth-one support was handled directly. For every support with $\delta(S) > 1$, the preceding construction gives a Galois-equivariant ratio that is not a translation and kills $\mathcal{L}_{\le \delta(S)-1}(S)$. Proposition 3.3 therefore gives both reverse inequalities. The three depths are equal.

The underlying nonzero bases are partitioned as

	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$
$p=3$	1	77	40	2	7
$p,q\ge 5$	1	77	46	2	1.

(10.9)

Restoring the optional zero orbit doubles every base and adds the sole depth-one support. This gives exactly the two profiles in Theorem 1.1 for both rational and integer signals. □

The theorem establishes thresholds, not a complete moduli description. The exact unit-support torus is classified in Section 5; on a stable support of depth greater than one, the construction above deliberately records only the Galois-equivariant lower-bound ratio needed for exactness.

11 Binary set reconstruction on C_{2pq}

For a subset $A \subset C_n$ write $f_A = \mathbf{1}_A$ and $S_A = \operatorname{supp} \widehat{f}_A$. Its analytical ordered k-deck is exactly the autocorrelation $M_k(f_A)$ in (3.1); order counts factors, and equivalence in this subsection means translation only. Let $\rho_T(A)$ be the least d such that equality of the exact d-deck forces a translate of A. This is the analytical convention used by Keleti–Kolountzakis and, after fixing one argument at the identity, by Simon [KK06, Sim17].

For a nonempty pattern $T \subset C_n$, write $[T]$ for its translation class, $\mu_A([T])$ for the number of distinct subsets of A in that class, and $\operatorname{Stab}(T) = \{h \in C_n : T + h = T\}$. Pebody's 2007 cumulative d-deck is the multiset encoded by the values $\mu_A([T])$ for $1 \le |T| \le d$.

Lemma 11.1 (Pebody/analytical deck equivalence)**.** *Let $1 \le d \le n$. For every $A \subset C_n$, including $A = \varnothing$ and $|A| < d$, the exact ordered analytical d-deck determines Pebody's cumulative combinatorial d-deck and conversely. More precisely, if $T = \{0, t_1, \dots, t_{j-1}\}$ has j distinct elements, then*

$$M_j(f_A; t_1, \dots, t_{j-1}) = |\operatorname{Stab}(T)|\, \mu_A([T]).$$

Pebody's 2004 count-valued set deck records the left-hand side directly; the 2007 necklace convention records the right-hand multiplicity without the known stabilizer factor [Peb04, Peb07].

Proof. The analytical value counts the translations $x \in C_n$ for which $x + T \subset A$. Every distinct translated copy of T contained in A has exactly $|\operatorname{Stab}(T)|$ such translation witnesses, proving the displayed identity. An exact d-deck gives every lower order by repeating the zero shift. Conversely, in any ordered d-tuple, repeated positions collapse to their set of distinct positions because $\mathbf{1}_A^m = \mathbf{1}_A$; the displayed identity then recovers its analytical value from the cumulative combinatorial deck. The same argument gives zero on both sides for every nonempty pattern when $A = \varnothing$, and it requires no lower bound on $|A|$. □

The exact deck becomes cumulative for indicators. If $1 \le k \le d$, then

$$M_k(f_A; t_1, \dots, t_{k-1}) = M_d(f_A; t_1, \dots, t_{k-1}, 0, \dots, 0), \tag{11.1}$$

because $f_A^m = f_A$. This zero-shift identity is false for general weighted signals and is the first load-bearing binary input.

Proposition 11.2 (Supportwise binary crosswalk)**.** *Every nonempty $A \subset C_n$ satisfies*

$$\rho_T(A) \le \delta(S_A). \tag{11.2}$$

Equality need not hold.

Proof. Put $d = \delta(S_A)$. If $d = 1$, then $S_A = \{0\}$: a nonzero frequency has a positive-order kernel relation that degree-one rows cannot generate, and the zero frequency splits off. Hence the nonempty binary function f_A is constant and is reconstructed at order one.

Assume $d \ge 2$ and that B has the same exact d-deck. Formula (11.1) gives equality of the second decks, so Lemma 3.1 gives $S_B = S_A =: S$. On this common exact support set $r_s = \widehat{f}_B(s)/\widehat{f}_A(s)$. The Fourier monomial identity and (11.1) say that r kills $\mathcal{L}_{\le d}(S)$. Since $\mathcal{L}_{\le d}(S) = K(S)$, Proposition 3.2 makes r a translation character. Thus B is a translate of A. □

For strictness, a singleton in C_{30} has full Fourier support, whose closing depth in the present classifier is three, while all singletons form one translation orbit and are determined by their first deck. Hence $\rho_T(A) = 1 < 3 = \delta(S_A)$; only the upper bound in (11.2) is used below.

11.1 The deep-support exclusion

Fix distinct odd primes $p < q$ and use the CRT coordinates of Section 6. The complete inverse-Fourier dictionary, before the constant orbit $\mathcal{O}_1$, is

E	$\mathcal{O}_2$	$(-1)^\varepsilon e$	P	$\mathcal{O}_p$	$U(x)$
Q	$\mathcal{O}_q$	$T(y)$	$P1$	$\mathcal{O}_{2p}$	$(-1)^\varepsilon V(x)$
$Q1$	$\mathcal{O}_{2q}$	$(-1)^\varepsilon W(y)$	F	$\mathcal{O}_{pq}$	$R(x,y)$
G	$\mathcal{O}_{2pq}$	$(-1)^\varepsilon Z(x,y)$			

(11.3)

Here U, T, V, W have zero mean in their displayed prime coordinate, while R, Z have no zero frequency in either prime coordinate. Exact presence of a box means that its displayed component is nonzero. Every nonempty indicator also has $\mathcal{O}_1$, since $\widehat{f}_A(0) = |A| > 0$, and adjoining it does not change depth by (10.2). Moreover, f_A is rational-valued, so (3.3) makes S_A Galois-stable and the profile theorem applies.

Theorem 1.1 leaves exactly the following bases above depth four; the mandatory $\mathcal{O}_1$ is suppressed in the table:

	depth five	depth six
$p, q \geq 5$	$P \cup P1 \cup Q1,\ Q \cup P1 \cup Q1$	$P1 \cup Q1$
$p = 3$	$P \cup P1 \cup Q1,\ Q \cup P1 \cup Q1$	$P1,\ P1 \cup Q,\ P1 \cup Q1,\ G,$ $Q \cup G,\ P1 \cup G,\ P1 \cup Q \cup G$

(11.4)

Thus a binary upper bound four is equivalent to excluding every row of (11.4) as an exact Fourier support.

There is also a useful elementary mask-polynomial obstruction for part of this list. General lower bounds for prescribed cyclotomic mask-polynomial divisors are studied in [KŁMS26], but no such general bound is needed here. For $A \subset C_{2pq}$ put $A(X) = \sum_{a \in A} X^a$. Absence of the Fourier orbit of order s is equivalent to $\Phi_s(X) \mid A(X)$. For $P1 \cup Q1$ the absent-order set is $\{2, p, q, pq, 2pq\}$; when $p = 3$, the corresponding sets for $P1$, G, and $P1 \cup G$ are respectively $\{2, p, q, 2q, pq, 2pq\}$, $\{2, p, q, 2p, 2q, pq\}$, and $\{2, p, q, 2q, pq\}$. Every one of these four sets contains $2, p, q$. The three cyclotomic polynomials are distinct monic irreducibles in $\mathbb{Z}[X]$. By Gauss's lemma their individual divisibilities hold in $\mathbb{Z}[X]$, and pairwise coprimality therefore gives

$$\Phi_2(X)\Phi_p(X)\Phi_q(X) \mid A(X).$$

Evaluation at $X = 1$ gives

$$2pq = \Phi_2(1)\Phi_p(1)\Phi_q(1) \mid A(1) = |A|.$$

An indicator with any of the four exact supports is nonempty and proper: $\mathcal{O}_1$ is mandatory, while the full-group indicator has no nontrivial Fourier orbit. Thus $0 < |A| < 2pq$, contradicting this divisibility. This direct argument does not exclude the two bridges or the remaining prime-three rows in (11.4). The argument below is retained because it excludes the entire list uniformly.

Lemma 11.3 (Two finite-set rigidity facts)**.** *If a nonempty finite $X \subset \mathbb{C}$ satisfies $X + h = X$, then $h = 0$. If $X, Y \subset \mathbb{C}$ are nonempty finite sets and $|X + Y| \leq 2$, then one of X, Y is a singleton.*

Proof. For the first assertion, iteration of a nonzero h would put infinitely many distinct points $x + mh$ in X. For the second, if both sets had at least two points, each would have exactly two. Two distinct translates of, say, Y would then both equal the same two-point sumset, giving a forbidden nonzero translation period of Y. □

Theorem 11.4 (Binary exclusion of every deep support)**.** *No base in (11.4), after restoring $\mathcal{O}_1$, is the exact Fourier support of a binary indicator on C_{2pq}.*

Proof. Assume first $p, q \geq 5$. On the critical base $P1 \cup Q1$ the inverse form is

$$f(\varepsilon, x, y) = c + (-1)^{\varepsilon}\big(V(x) + W(y)\big). \tag{11.5}$$

Both summands are nonconstant. Adding the two parity slices shows that $2c \in \{0, 1, 2\}$. The extreme values force $V + W = 0$ and contradict exact support. Otherwise $c = 1/2$

and $V(C_p)+W(C_q) \subset \{-1/2, 1/2\}$. Lemma 11.3 makes one summand constant; its zero mean makes it zero, again contradicting exact support.

On $P \cup P1 \cup Q1$ one has

$$f(\varepsilon, x, y) = c + U(x) + (-1)^{\varepsilon}(V(x) + W(y)). \tag{11.6}$$

For fixed x, the sum of the parity slices is $2(c+U(x))$. If it were zero or two, both binary slices would be fixed for every y and their difference would make W constant, contrary to exact $Q1$. The sum is therefore one for every x, so U is constant; its zero mean gives $U = 0$, contrary to exact P.

The other bridge has $f = c + T(y) + (-1)^{\varepsilon}(V(x) + W(y))$. Now the parity-slice sum is $2(c + T(y))$, and the same argument, using exact $P1$ to keep V nonconstant, forces the zero-mean term T to vanish. This also proves both bridge exclusions when $p = 3$.

It remains to treat the seven $p = 3$ critical bases. Identify $C_2 \times C_3$ with C_6. If a binary $h : C_6 \to \{0, 1\}$ has Fourier support in $\mathcal{O}_1 \cup \mathcal{O}_6$, vanishing of the order-three frequencies gives

$$h_0 + h_3 = h_1 + h_4 = h_2 + h_5 =: s. \tag{11.7}$$

For $s = 0, 2$ the function is constant. For $s = 1$ it has three ones, whereas vanishing at the order-two frequency would require $h_0 + h_2 + h_4 = h_1 + h_3 + h_5$, impossible because the two integer sides have odd total. Thus every such slice is constant.

Write $\mathsf{A} = P1$, $\mathsf{B} = Q1$, $\mathsf{C} = Q$, and $\mathsf{D} = G$. For each fixed $y \in C_q$, all six nonexceptional branches have slice support contained in $\mathcal{O}_1 \cup \mathcal{O}_6$. To make the completeness check visible, their inverse forms and contradictions are listed separately:

Base	Slice form, with y fixed	Component killed by slice constancy
A	$c + u(t)$	u, hence exact A
$\mathsf{A} \cup \mathsf{C}$	$c + s(y) + u(t)$	u, hence exact A
D	$c + z(t, y)$	z, hence exact D
$\mathsf{C} \cup \mathsf{D}$	$c + s(y) + z(t, y)$	z, hence exact D
$\mathsf{A} \cup \mathsf{D}$	$c + u(t) + z(t, y)$	both A and D
$\mathsf{A} \cup \mathsf{C} \cup \mathsf{D}$	$c + s(y) + u(t) + z(t, y)$	both A and D

In the last two rows, Fourier transformation in y separates the zero y-frequency A component from the nonzero D components. The only branch not in the table is $\mathsf{A} \cup \mathsf{B}$.

On that exceptional base,

$$f(t, y) = c + u(t) + (-1)^t w(y), \tag{11.8}$$

where u has exact support $\mathcal{O}_6 = \{1, 5\}$ and w is nonconstant. For each even t, the nonconstant set $c + u(t) + w(C_q)$ is contained in $\{0, 1\}$ and hence equals it. Comparing two even values and applying the first part of Lemma 11.3 shows that u is constant on even residues; the odd residues give the same conclusion. Thus $u(t + 2) = u(t)$. A period-two function on C_6 has Fourier support in $\{0, 3\} = \mathcal{O}_1 \cup \mathcal{O}_2$, disjoint from $\mathcal{O}_6$, so $u = 0$, the final contradiction. □

11.2 Exact reconstruction and the reflection quotient

Proof of Theorem 1.3. Every nonempty binary exact support has depth at most four by Theorem 11.4 and the complete profile theorem. Proposition 11.2 therefore gives $r_{\mathrm{set}}(C_{2pq}) \leq 4$; the empty set is determined by its first deck.

For the reverse inequality put $k = pq \geq 15$. Keleti–Kolountzakis prove that, in C_{2k},

$$\{0\} \cup \{3, \dots, k-1\} \cup \{k+1, k+2\}, \qquad \{0,1\} \cup \{3, \dots, k-1\} \cup \{k+2\} \tag{11.9}$$

have the same analytical three-deck but are not translates [KK06, Theorem 2.22]. Hence order three is insufficient and the value is four. □

Keep the deck oriented, but now call two outputs equivalent when one is a translation or a reflected translation of the other; denote the resulting number by $r_{\mathrm{set}}^{T/I,\mathrm{or}}$. This changes the output quotient, not the data being measured.

Corollary 11.5 (Oriented deck modulo translations and reflections)**.** *For distinct odd primes $p < q$,*

$$r_{\mathrm{set}}^{T/I,\mathrm{or}}(C_{2pq}) = 4. \tag{11.10}$$

Proof. Theorem 1.3 gives the upper bound. For the lower bound set $m = q$, $\bar{n} = 2p$, and represent C_{2pq} by $qx + i$ with $x \in C_{2p}$ and $i \in C_q$. In C_{2p} put

$$H_A = \{0, 2, \dots, 2p-2\}, \qquad H_B = \{0, 1, \dots, p-1\}.$$

In C_q let $J_B = \{0, 1, -1\}$, $J_A = C_q \setminus J_B$, and set $\ell_{-1} = 1$, $\ell_i = 0$ otherwise. Define $D_i = H_A + \ell_i$ for $i \in J_A$ and $D_i = H_B + \ell_i$ for $i \in J_B$, and

$$\mathcal{A} = \{qx + i : x \in D_i,\ i \in C_q\}, \qquad \mathcal{A}' = \{qx + i : x \in D_i + 2\mathbf{1}_{\{i=0\}},\ i \in C_q\}. \tag{11.11}$$

This is Simon's Proposition 3.11 with $i_0 = 0$ and $\delta = 2$. Its only nonvacuous equation is at $j = 1$: $\ell_{-1} + \ell_1 = 1 = 2\ell_0 + 2 - 1$ in C_{2p}, so the two sets have equal combinatorial three-decks. They have size pq, and Simon's Corollary 1.41 therefore transfers equality to the oriented analytical three-deck. Since q is prime and $|J_B| = 3$, J_B is not a subgroup coset; the consequence of Proposition 3.13 excludes translations. At $j = 2$ its reflection test has $\ell_{-2} + \ell_2 = 0$ but right side 1 modulo two, excluding reflected translations. Thus order three fails even in the coarser output quotient, proving (11.10) [Sim17, Corollary 1.41 and Propositions 3.11, 3.13]. □

11.3 Load-bearing controls and theorem boundary

The binary hypothesis cannot be replaced even by a three-letter alphabet. The function $(0, 0, 1, 2, 2, 1)$ on C_6 has exact Fourier support $\mathcal{O}_1 \cup \mathcal{O}_6$: its coefficients are 6 at zero and $-3 \pm i\sqrt{3}$ at the two order-six frequencies, with every other coefficient zero. Thus the binary slice argument at (11.7) is genuinely an idempotent two-value phenomenon.

Nor does Theorem 11.4 collapse the weighted signal problem. The integer functions

$$(14, 7, -7, -14, -7, 7), \qquad (2, -11, -13, -2, 11, 13) \tag{11.12}$$

on C_6 have exact support $\mathcal{O}_6$, agree at every autocorrelation through order five, and are not translates. Their all-zero-shift sixth moments are 15529668 and 13196868, respectively. Direct Fourier multiplication therefore retains the depth-six obstruction outside the binary class.

Finally, the two Keleti–Kolountzakis sets in (11.9) satisfy $F = (k+2) - E$: they are reflected translates. Their published pair proves the translation-only lower bound but cannot prove Corollary 11.5. The separate Simon specialization (11.11), including the failed reflection test, is therefore logically necessary rather than a second presentation of the same witness.

Scope boundary. The result is restricted to binary indicators on C_{2pq} with $p < q$ distinct odd primes. It asserts neither a ternary or weighted analogue nor all even squarefree orders, orders with more than two odd prime factors, noise stability, or noncyclic reconstruction. The Simon corollary keeps the analytical deck oriented. Pebody's general even-order conjecture remains a separate problem outside this theorem.

12 Interpretation and prior-art boundary

The common invariant of the paper is the filtration

$$\mathcal{L}_{\le d}(S) \subset K(S).$$

It has three readings that should not be conflated. Its quotient-group completion is the supportwise separating-atom invariant of Proposition 2.1; its lattice points are the first dilations of the zero-sum face (2.5); and its complex character group is the phase ambiguity modulo translations in Proposition 3.2. All three readings now give the same closing threshold on every stable C_{2pq} support: Section 10 realizes every positive-order failure below closure by rational and integer signals, while the depth-one support is direct. A complete parametrization of all surviving rational ratios is asserted only on the exact unit support.

The two algebraic readings may be compressed as

$$\begin{aligned} \mathcal{L}_{\le d}(S) = K(S) &\iff \text{all complex invisible ratios are translations,} \\ &\iff \text{all separating atoms over } S \text{ have length } \le d. \end{aligned} \tag{12.1}$$

For rational signals, equality gives sufficiency immediately, but failure initially produces only a complex character obstruction. Realizing it by two rational Fourier vectors requires Galois covariance. Section 10 constructs such a lower-bound ratio on every stable C_{2pq} support of depth greater than one, while Section 5 gives the stronger full ambiguity classification on U_n.

The historical higher-autocorrelation literature established translation invariance, support-generation and ratio-character uniqueness mechanisms, and global finite-cyclic bounds [AK62, MR68, CW70, KI93, GM95]. Konyagin–Lev supply the universal Freiman ambient group used in Section 4 [KL00]; the modern separating-invariant works supply the zero-sum generation criterion, non-closed-field Galois-stable criterion, quotient-group and degree language [Dom17, Dom23, Dom26, SZZ26, Hua26]; and the recent autocorrelation papers supply the exact Fourier and rationality criteria and the support-specific questions [ABW26, CO26]. The zero-sum face reformulation connects the result to the polyhedral counting framework of Han–Wang–Zhang–Zhang [HWZZ26].

For subsets, Pebody supplies the finite-abelian reconstruction framework, the cyclic upper bound six, the multiset classification and the even-order value-four conjecture [Peb04, Peb07]. Together with Keleti–Kolountzakis's even-order three-deck obstruction, that classification already gives the exact infinite families C_{2^a} for $a \ge 4$ and $C_{2\ell^k}$ for primes $\ell > 3$ whenever $2\ell^k > 10$ [KK06]. The family C_{2pq} treated here is disjoint from both. Kiss–Łaba–Marshall–Somlai study general mask-polynomial lower bounds [KŁMS26]. For four of the deep support patterns used here, the needed obstruction instead follows directly by evaluating the three prime-power cyclotomic divisors at 1;

the remaining rows still require the uniform binary argument. The lower bound in Theorem 1.3 is the Keleti–Kolountzakis pair; the translation/reflection lower bound is a specialization of Simon's deck construction and equivalence tests [Sim17]. The binary increment here is the uniform exclusion of all deep $2pq$ supports, not ownership of these general mechanisms.

The overlap with Domokos's 2026 criterion is architectural and direct: Theorem 5.4 there decides separation by testing integral generation on every Galois-stable character subset of a regular representation. Here a support S is fixed first, and the object computed is the least degree at which its block-monoid group completion reaches $K(S)$. The general criterion therefore owns the field and support framework; it does not supply the local CRT presentations, the profiles $(1, 2, 154, 80, 4, 14)$ and $(1, 2, 154, 92, 4, 2)$, or the rational norm-one classification on the unit row.

Within that boundary, the contribution assembled here is one theorem chain. For unit supports of all even cyclic groups, the fourth-order quotient has a sharp dichotomy, the exceptional branch closes at six, and its entire rational phase family is the norm-one torus over $\mathbb{Q}(\zeta_6)$ modulo residual translations. For C_{2pq}, the local presentations, Cubic Skeleton, Quartic Skeleton, bridges, and critical families classify every one of the 255 stable supports; the nine canonical Galois lifts, two bridge signs, and depth-six phases realize every nontrivial first failure over both $\mathbb{Q}$ and $\mathbb{Z}$. Their literal overlap is the unit box, where the two independent derivations agree.

The regime change has a local explanation. If $3 \nmid 2pq$, every parity-one punctured line has split by degree four into a parity constant and an odd-primary slope; only the diagonal parity on $P1 \cup Q1$ survives later. When $p = 3$, the $P1$ and G boxes retain integral sign lifts, and seven masks preserve one through degree five. The difference between the two profiles is therefore propagated from one local change in Table 2 through the same connector system.

Established here	Not asserted here
All-even unit quotient and exact degree-six closure	Arbitrary non-stable support classification
Complete rational ambiguity family on the exact unit support	Rational ambiguity-torus classification on every row of the 255-support atlas
Exact rational and integer thresholds on all 255 stable supports, with explicit lower-bound pairs whenever the depth exceeds one	Signals on non-stable supports, positivity, or fixed alphabets
Exact local separating-degree profile on stable C_{2pq} supports	Ordinary separating Noether number or connector minimality
Uniform symbolic Cubic and Quartic Skeletons	Noise stability, positivity, or finite-alphabet reconstruction
Exact binary set-reconstruction number four on C_{2pq}	Pebody's conjecture for all even cyclic orders
Oriented-deck value four modulo translations/reflections	A reflection-symmetrized deck or a weighted analogue

Table 4: The theorem boundary of the unified paper.

Several stronger-sounding statements are deliberately absent. We do not claim abso-

lute priority, a complete ambiguity-torus parametrization on every atlas row, a classification for arbitrary non-stable supports, or a filtration through all cyclotomic subfields. We also do not claim the full even cyclic set-reconstruction conjecture. Nor do the exact-arithmetic replays in Appendix C replace any uniform proof. They audit the theorem architecture and its finite routing only.

The computations do have a reproducibility role. They check Boolean exhaustiveness, connector images, endpoint branches, and independent Smith presentations. Mutation controls reject missing X10B sign glue, altered failure sets, and corrupted atlases. The direction of dependence remains unchanged: the written lemmas imply the theorem, while the receipts test their implementation and transcription.

The maximum orbit-stable local separating degree on C_{2pq} is therefore six in both arithmetic regimes, but the locus attaining it changes sharply: one generic parity bridge becomes seven prime-three sign families. That change is already visible in the unit-support dichotomy and is explained locally by the free sign directions in Table 2. This is the structural link between the two halves of the paper.

Several separate questions remain natural: a complete rational ambiguity parametrization on every atlas row, analogous Skeletons for higher-rank groups, and robust reconstruction from approximate data. They require, respectively, a finer cocycle/descent analysis, a larger orbit geometry, and quantitative conditioning estimates. The exact cyclic result here is a base case for those directions, not a claim to solve them.

A Local reductions and uniform connector schemata

In CRT coordinates write

$$\begin{aligned} P[a] = (0, a, 0), \quad Q[b] = (0, 0, b), \\ P1[a] = (1, a, 0), \quad Q1[b] = (1, 0, b), \\ F[a, b] = (0, a, b), \quad G[a, b] = (1, a, b), \quad E = (1, 0, 0). \end{aligned}$$

A.1 Expanded local reductions

We record the endpoint details suppressed from Proposition 6.1. Let a target-valued function kill the appropriate rows.

On a parity-zero punctured line, pairs give $f(-x) = -f(x)$ and triples give

$$f(x + y) = f(x) + f(y) \qquad (x, y, x + y \neq 0). \tag{A.1}$$

Fixing a generator of the prime cycle and walking through its nonzero multiples proves $f(x) = xu$, $ru = 0$. On a parity-one line there are no odd rows. For $r \geq 5$, the quadruple $(x, 1, 1, -x - 2)$ and its pair reductions give the recurrence

$$f(x + 2) - f(x) = 2f(1). \tag{A.2}$$

The translation-by-two cycle on C_r becomes a path after deleting zero, so (A.2) determines $f(x) = c + xu$. Opposite pairs give $2c = 0$. For $r \geq 7$, the row $(2, 2, 2, r - 6)$ gives $4c + ru = 0$ and therefore $ru = 0$. For $r = 5$, the separate row $(1, 1, 1, 2)$ gives $4c + 5u = 0$ and therefore $5u = 0$. Conversely $c + xu$ kills every even zero-sum row. For $r = 3$, each quartic row has two points from each sign class, so the quotient remains $\mathbb{Z}$.

For the parity-zero rectangle F, suppose first $p, q \geq 5$. Pairs and triples give partial additivity. For $s \in C_p$ define

$$\Delta_s(x, y) = f(x + s, y) - f(x, y)$$

when both endpoints are in the punctured rectangle. Given two base points, choose c outside

$$\{0, -s, -x - s, -x' - s\}$$

and d outside $\{0, -y, -y'\}$. The choices exist for $p \geq 5, q \geq 7$; at $p = 5$, at most four classes are forbidden and the fifth remains. Applying partial additivity to

$$(x + s, y) + (c, d) = (x, y) + (c + s, d) \tag{A.3}$$

and to the primed version makes Δ_s base-point independent. A second auxiliary point proves $\Delta_{s+t} = \Delta_s + \Delta_t$. Repeating vertically gives $f(a, b) = c + au + bv$, $pu = qv = 0$. An opposite pair gives $2c = 0$ and the triple $(1, 1), (1, 1), (-2, -2)$ gives $3c = 0$, hence $c = 0$.

When $p = 3$, set $h(b) = f(1, b)$ and recover the other sheet from pairs. A triple has zero C_3 coordinate precisely when its three signs agree. Thus pair sums having the same nonzero total have the same h-sum. To cross a zero total, choose a bridge point outside at most four q-classes; this works at $q = 5$. The resulting difference equation yields $h(b) = u + bv$, $3u = qv = 0$, and hence $f(a, b) = au + bv$.

For the parity-one rectangle G in the generic branch, the four points

$$(1, x + s, y),\ (1, -x, -y),\ (1, x', y'),\ (1, -x' - s, -y') \tag{A.4}$$

sum to zero whenever their coordinates are nonzero. Pair reduction of (A.4) makes horizontal differences base-point independent; the vertical argument is identical. This gives $c + au + bv$ with $2c = pu = qv = 0$. If $p = 3$, every quartic row contains two signs of each kind and becomes a Freiman relation on $C_q^\times$. Therefore $h(b) = u + bv$ with u free and $qv = 0$. These arguments prove every row of Table 2, including the smallest generic prime.

A.2 The fourteen structural types

Every displayed parameter is nonzero, and every derived parameter is required to be nonzero. The following table gives the fourteen structural types; an exchanged orientation means $p \leftrightarrow q$ and $P \leftrightarrow Q$, $P1 \leftrightarrow Q1$.

Type	Box count	Uniform zero-sum schema
X1	0000012	$F[-2a, -2b] + G[a, b] + G[a, b]$
X2	0102000	$P[-2a] + 2P1[a]$; exchange
X3	0100002	$P[-2a] + G[a, b] + G[a, -b]$; exchange
X4	0001011	$P1[a] + F[c, b] + G[-a-c, -b]$; choose $a, c, a+c \neq 0$; exchange
X5	1000011	$E + F[a, b] + G[-a, -b]$
X6	0100020	$P[-2a] + F[a, b] + F[a, -b]$; exchange

Type	Box count	Uniform zero-sum schema
X7	0001110	$P1[a] + Q1[b] + F[-a, -b]$
X8	1101000	$E + P[a] + P1[-a]$; exchange
X9	0011001	$Q[b] + P1[a] + G[-a, -b]$; exchange
X10A	0001003	$P1[-c] + G[a, b] + G[-a, b] + G[c, -2b]$; exchange
X10B	0001003	$P1[a]+G[-a, b]+G[c, d]+G[-c, -b-d]$; exchange
X11	1000003	$E + G[a, b] + G[a, b] + G[-2a, -2b]$
X12	1003000	$E + P1[a] + P1[a] + P1[-2a]$; exchange
X13	0002020	$2P1[a] + F[-a, b] + F[-a, -b]$; exchange
X14	0111100	$P[a] + Q[b] + P1[-a] + Q1[-b]$

A count string uses the order $(E, P, Q, P1, Q1, F, G)$. Substitution proves each row coordinate by coordinate. Multiplication by -1 or -2 preserves nonzero coordinates over odd primes. For X4 one may choose $a = c = 1$; then $a + c = 2 \neq 0$. For X10B choose $a = c = b = d = 1$; the derived entries are -1 and -2. Thus every schema is available at the endpoints $p = 3$ and $q = 5$.

The A and B choices of X10 share a box-count type but have different images in the local presentation. X10B is needed for the $P1 - G$ slope edge in (8.3) and for the integral relation $u = g$ in (8.4). This parameter distinction is why a count-pattern catalogue alone is insufficient for the Quartic Skeleton.

For completeness, the connector functions are partitioned as follows. X2, X3, and X6 attach prime-axis slopes to odd lines or rectangles. X1, X4, X7, X9, X10B, X13, and X14 compare rectangle slopes and cross-box parity. X5, X8, X10A/B, X11, and X12 anchor parity constants. Since two is invertible in both odd-primary layers, a doubled slope difference imposes the same equality as an undoubled one. In the $p = 3$ integral sign layer the integer coefficients are retained, producing precisely the rows (8.4).

A.3 Connector images in the prime-three layer

For $p = 3$, discard the already separated C_q component and use the generators (e, p, u, c, f, g) of Section 8. The required images are

type	support	integral image
$X1$	F, G	$f + 2g$
$X2$	$P, P1$	$p + 2u$
$X3$	P, G	$p + 2g$
$X6$	P, F	$p + 2f$
$X13$	$P1, F$	$2u - 2f$
$X10B$	$P1, G$	$u - g$
$X11$	E, G	$e + 3g$
$X12$	$E, P1$	$e + 3u$
$X12^*$	$E, Q1$	$e + c$
$X10B^*$	$Q1, G$	$c + 3g$
$X7$	$P1, Q1, F$	$u + c - f$
$X14$	$P, Q, P1, Q1$	$p - u + c.$

(A.5)

Substitution of the local forms into each parameterized zero-sum row gives this table. Every image vanishes under $(e, p, u, c, f, g) \mapsto (3, 4, 1, 3, 4, 1)$ in C_6. If a chosen schema

also has a C_q-slope component, two parameter values isolate it by Lemma 8.1. Thus (A.5) is an integral presentation, not the result of evaluating at one value of q.

In the generic branch, primary projection turns the same substitutions into graph edges. The p-slope graph on $P, P1, F, G$ and the q-slope graph on $Q, Q1, F, G$ are complete. On parity coordinates, X12 and X11 connect E to $P1, Q1, G$; X10B and its exchange connect the odd lines to G; X7 or X14 provides the remaining $P1 - Q1$ comparison under the support conditions stated in Section 8. These images account for every use of the connector table in the proof.

A.4 Two exact closing rows

The generic critical pair $P1 \cup Q1$ closes by combining the two triples $(1, 1, -2)$, one in each odd coordinate line. For the prime-three family, six positive $P1$ or G points give the $6u$ generator. On $P1 \cup Q1$, three positive $P1$ points and three $Q1$ points with zero q-sum give $3u + c$. These are existence constructions over every admissible prime, not residue samples.

B Boolean routing and exact mask lists

The hexadecimal encoding assigns bit values

$$E = 0x01,\ P = 0x02,\ Q = 0x04,\ P1 = 0x08,\ Q1 = 0x10,\ F = 0x20,\ G = 0x40. \tag{B.1}$$

For example, $0x1a = P \cup P1 \cup Q1$. The zero orbit is not a bit in this encoding; it is adjoined after classifying the 127 nonempty base masks.

B.1 The 77 cubic masks

Evaluating the ten clauses of (7.1) by the eight assignments of (E, P, Q) gives the counts

(E,P,Q)	000	100	010	110	001	101	011	111	total
#	6	5	9	9	9	9	15	15	77.

(B.2)

Here each three-character label lists the truth values in the displayed order (E, P, Q). For each fixed triple, the count is obtained by substituting in (7.1) and enumerating the sixteen assignments of $(P1, Q1, F, G)$; thus (B.2) is a sixteen-case elementary truth-table count, not a computational theorem premise.

The following matrix expands every positive clause. "Optional" means that the indicated box may be either selected or absent under that clause; each line is used only when it is selected. The symbols e, x, y, z denote the values on $E, P1, Q1, G$. A semicolon separates the base normalization from the optional attachments.

In T_6, T_7, T_8, the parenthetical independent case is not a missing relation. If the relevant connector box is absent, the corresponding primary coordinate, or E as the only parity box, is a direct factor of the generated CRT subgroup and may be assigned its own character parameter. Thus all optional bits are covered, including absence as well as presence, and every row of Table 6 ends in the common character normal form (7.0).

The complete positive list is

Clause	Box handled	Cubic row used	Lemma part	Resolved generator or restriction
T_1	base F, G	$F + 2G$	(4)	$z = \eta + a\phi + b\psi$
T_1	optional P	$P + 2F$	(2)	$\alpha = \phi$
T_1	optional Q	$Q + 2F$	(2)	$\beta = \psi$
T_1	optional $P1$	$P1 + F + G$	(5)	$x = \eta + a\phi$
T_1	optional $Q1$	$Q1 + F + G$	(5)	$y = \eta + b\psi$
T_1	optional E	$E + F + G$	(5)	$e = \eta$
T_2	base $P, P1, F$	$P + 2P1$; $P + 2F$	(1), (2)	$x = \eta + a\alpha$ and $\phi = \alpha$
T_2	optional Q	$Q + 2F$	(2)	$\beta = \psi$
T_2	optional $Q1$	$P1 + Q1 + F$	(5)	$y = \eta + b\psi$
T_2	optional G	$P1 + F + G$	(5)	$z = \eta + a\alpha + b\psi$
T_2	optional E	$E + P + P1$	(5)	$e = \eta$
T_3	base $Q, Q1, F$	$Q + 2Q1$; $Q + 2F$	(1), (2)	$y = \eta + b\beta$ and $\psi = \beta$
T_3	optional P	$P + 2F$	(2)	$\alpha = \phi$
T_3	optional $P1$	$P1 + Q1 + F$	(5)	$x = \eta + a\phi$
T_3	optional G	$Q1 + F + G$	(5)	$z = \eta + a\phi + b\beta$
T_3	optional E	$E + Q + Q1$	(5)	$e = \eta$
T_4	base $E, P, Q, P1$	$P + 2P1$; $E + P + P1$	(1), (5)	$x = e + a\alpha$
T_4	optional F	$P + Q + F$	(2)	$(\phi, \psi) = (\alpha, \beta)$
T_4	optional $Q1$	$Q + 2Q1$; $E + Q + Q1$	(1), (5)	$y = e + b\beta$
T_4	optional G	$Q + P1 + G$	(5)	$z = e + a\alpha + b\beta$
T_5	base $E, P, Q, Q1$	$Q + 2Q1$; $E + Q + Q1$	(1), (5)	$y = e + b\beta$
T_5	optional F	$P + Q + F$	(2)	$(\phi, \psi) = (\alpha, \beta)$
T_5	optional $P1$	$P + 2P1$; $E + P + P1$	(1), (5)	$x = e + a\alpha$
T_5	optional G	$P + Q1 + G$	(5)	$z = e + a\alpha + b\beta$

Table 6: Complete cubic sufficiency routing, clauses T_1–T_5: clause, optional box, mixed row, lemma and resolved restriction.

Clause	Box handled	Cubic row used	Lemma part	Resolved generator or restriction
T_6	base P; $Q1, G$ absent	local	local	$P[a] = a\alpha$
T_6	optional $P1$	$P + 2P1$	(1)	$x = \eta + a\alpha$
T_6	optional F	$P + 2F$	(2)	$\phi = \alpha$
T_6	optional Q	$Q + 2F$ if F	(2)	$\beta = \psi$ if linked; otherwise an independent q-character
T_6	optional E	$E + P + P1$ if $P1$	(5)	$e = \eta$ if linked; otherwise the sole parity character
T_7	base Q; $P1, G$ absent	local	local	$Q[b] = b\beta$
T_7	optional $Q1$	$Q + 2Q1$	(1)	$y = \eta + b\beta$
T_7	optional F	$Q + 2F$	(2)	$\psi = \beta$
T_7	optional P	$P + 2F$ if F	(2)	$\alpha = \phi$ if linked; otherwise an independent p-character
T_7	optional E	$E + Q + Q1$ if $Q1$	(5)	$e = \eta$ if linked; otherwise the sole parity character
T_8	base F; $P1, Q1$ absent	local	local	$F[a, b] = a\phi + b\psi$
T_8	optional P	$P + 2F$	(2)	$\alpha = \phi$
T_8	optional Q	$Q + 2F$	(2)	$\beta = \psi$
T_8	optional G	$F + 2G$	(4)	$z = \eta + a\phi + b\psi$
T_8	optional E	$E + F + G$ if G	(5)	$e = \eta$ if linked; otherwise the sole parity character
T_9	base P, Q, G; E absent	$P + 2G$; $Q + 2G$	(3)	$z = \eta + a\alpha + b\beta$
T_9	optional F	$P + Q + F$	(2)	$(\phi, \psi) = (\alpha, \beta)$
T_9	optional $P1$	$Q + P1 + G$	(5)	$x = \eta + a\alpha$
T_9	optional $Q1$	$P + Q1 + G$	(5)	$y = \eta + b\beta$
T_{10}	base $P1, Q1, F$; E absent	$P1 + Q1 + F$	(5)	$x = \eta + a\phi$ and $y = \eta + b\psi$
T_{10}	optional P	$P + 2F$	(2)	$\alpha = \phi$
T_{10}	optional Q	$Q + 2F$	(2)	$\beta = \psi$
T_{10}	optional G	$P1 + F + G$	(5)	$z = \eta + a\phi + b\psi$

Table 7: Complete cubic sufficiency routing, clauses T_6–T_{10}; continuation of Table 6.

```
02 03 04 05 06 07 0a 0b 0e 0f 14 15 16 17 1f 20 21 22 23 24 25 26 27
2a 2b 2e 2f 34 35 36 37 38 3a 3b 3c 3d 3e 3f 46 4e 4f 56 57 5e 5f
60 61 62 63 64 65 66 67 68 69 6a 6b 6c 6d 6e 6f 70 71 72 73 74 75
76 77 78 79 7a 7b 7c 7d 7e 7f.
```

Each entry is hexadecimal and carries the prefix $0x$ understood.

The necessity witnesses in Section 7 may be summarized by

terms	target	defect
N_1, N_3	$\mathbb{Z}$	odd p-function on $P1, G$
N_2, N_4	$\mathbb{Z}$	odd q-function on $Q1, G$
N_5	$\mathbb{Z}$	odd function on $P1$
N_6	$\mathbb{Z}$	odd function on $Q1$
N_7	$\mathbb{Z}$	odd function on a surviving parity-one box
N_8, N_9	C_2	1 on E, 0 elsewhere
N_{10}	C_2	1 on $P1$, 0 elsewhere.

(B.3)

Formula (7.4) is verified by the same fixed truth table: every assignment failing (7.1) satisfies at least one N_i, and each N_i falsifies (7.1). The witnesses in (B.3) then prove mathematical nonclosure for the whole term, not merely for the listed finite masks.

The Boolean verification can also be organized without hexadecimal masks. Split first on whether G is selected. If G occurs without F, failure to attach its p- or q-dependence gives N_1 or N_2. If F occurs, the F, G positive clause applies. Next suppose G is absent. A selected $P1$ without either P or F gives N_3, and the exchanged condition gives N_4. Once both odd lines are considered, absence of their remaining axis and cross attachments gives N_5, N_6, while absence of every parity-zero connector gives N_7. The only residual failures are mismatched parity constants, exactly N_8, N_9, N_{10}. This case split proves that the ten negative terms cover the complement of the ten positive routes.

For the integer defects, choose an odd function supported on one opposite pair. A triple that can see it either contains the opposite argument and cancels, or requires one of the boxes excluded by its N_i term. Since the ambient CRT subgroup is finite, the nonzero map to $\mathbb{Z}$ cannot be a character. For the C_2 defects, every relation that survives the missing-box conditions contains the marked parity box an even number of times. An ambient parity character would have to agree on all parity-one boxes, whereas the witness does not. These observations verify both relation-annihilation and noncharacterhood for each family.

B.2 Quartic routing lists

After removing E and the 77 cubic masks, the generic masks closing at four are

```
08 09 0c 0d 10 11 12 13 19 1b 1d 1e 28 29 2c 2d 30 31 32 33 39 40
41 42 43 44 45 47 48 49 4a 4b 4c 4d 50 51 52 53 54 55 58 59 5a 5b 5c
5d.
```

The three omitted failures are $18, 1a, 1c$. When $p = 3$, the 40 masks closing at four are

```
09 0d 10 11 12 13 19 1b 1d 1e 28 29 2c 2d 30 31 32 33 39 41 42 43
45 47 49 4a 4b 4d 50 51 52 53 54 55 58 59 5a 5b 5c 5d.
```

The nine omitted failures are $08, 0c, 18, 1a, 1c, 40, 44, 48, 4c$.

The bridge masks are $1a = P \cup P1 \cup Q1$ and $1c = Q \cup P1 \cup Q1$. In the generic branch the sole critical mask is $18 = P1 \cup Q1$. In the prime-three branch the critical masks are

$$08,\ 0c,\ 18,\ 40,\ 44,\ 48,\ 4c,$$

which decode exactly to the seven supports in (9.2). This recovers the base counts $77 + 46 + 2 + 1 + 1 = 127$ and $77 + 40 + 2 + 7 + 1 = 127$, where the final one is E.

The list is a routing index, not a second proof. Membership in the cubic list routes to Theorem 7.1; membership in a quartic list routes to the generic graph or prime-three integral case of Theorem 8.2; the remaining nine masks route to Propositions 9.1, 9.2, and 9.3. Each route is uniform in the primes and carries a defect at the preceding order.

C Reproducibility and assurance boundary

All theorem proofs are contained in the preceding sections. The companion repository supplies exact-arithmetic, symbolic, manuscript and package audits. They detect transcription, routing, endpoint, convention and software errors; no finite replay is used to infer a statement for all primes.

Layer	Audited object
Unit and local audits	Unit-support quotients, the UAG presentation, seven CRT boxes, connector coverage and endpoint cases.
Skeleton audits	Symbolic Cubic and Quartic reductions, exact failure sets and nine encoded mutations, with no sampled prime as a proof premise.
Binary audit	The C_6 slice obstruction, all 21 deep-support rows at $n = 30, 42, 70$, the Keleti–Kolountzakis lower-bound pair and four scope mutations. This is bounded assurance, not the uniform proof of Section 11.
Repository gate	Required paths, source and license boundaries, Section 11, the four reconstruction sources and six semantic mutation controls.
Manuscript gate	Title/PDF identity, bibliography closure, deck conventions, claim boundaries, unresolved references and source isolation.
Isolation gates	Manifest-only and synthetic Git-index trees repeat the core or full route without undeclared inputs.

Table 8: Assurance layers and their roles.

From the repository root, the reviewer first selects independently controlled absolute Python and Git executables. In a POSIX shell the standard routes are

```
PYTHON_ABSOLUTE=/absolute/path/to/python
GIT_ABSOLUTE=/absolute/path/to/git
trusted() {
  target=$1
  shift
  "$PYTHON_ABSOLUTE" -I -S -B scripts/runtime_bootstrap.py \
    --git-executable "$GIT_ABSOLUTE" --target "$target" -- "$@"
}
```

```
trusted scripts/verify.py --profile manifest
trusted scripts/verify.py --profile core
trusted scripts/verify.py --profile full
"$PYTHON_ABSOLUTE" -I -S -B -O scripts/runtime_bootstrap.py \
  --git-executable "$GIT_ABSOLUTE" --target scripts/verify.py -- \
  --profile full
trusted scripts/verify_manifest_only.py
trusted scripts/verify_git_checkout.py --profile full
```

The equivalent PowerShell invocation and the exact runtime trust boundary are recorded in `REPRODUCE.md`. The displayed commands are the actual bootstrap route: direct execution of a verifier is intentionally rejected. The core route runs the unit, UAG, CRT, Skeleton and bounded binary checks. Individual scripts write runtime JSON under `results/` and generated presentations under `certificates/`; the stable aggregate is `results/public_package_verification.json`. The source manifest and release attestation bind the distributed inputs and title-named PDF.

C.1 Build and visual verification

The title-named distributed PDF is produced from a clean source-only copy by the repository's locked builder:

```
TOOL_CACHE_PARENT=/absolute/path/to/TectonicProject
TECTONIC_CACHE_ROOT="$TOOL_CACHE_PARENT/Tectonic/bundles"
trusted scripts/build_locked_pdf.py \
  --tectonic-executable /absolute/path/to/tectonic \
  --tectonic-release-archive /absolute/path/to/release.zip \
  --tectonic-cache-root "$TECTONIC_CACHE_ROOT" \
  --source-dir /absolute/path/to/clean/source \
  --output-dir /absolute/path/to/empty/output
```

The resolved cache-root argument must end literally in `TectonicProject/Tectonic/bundles`; the builder rejects any other layout before compilation. The tracked `pdf_toolchain.lock.json` binds the Tectonic release asset, executable and resource-bundle identity. The tracked `tectonic_cache_payloads.lock.json` enumerates a canonical closed cache of 535 payloads by relative path, byte count and SHA-256; it contains every resource read by a clean build, plus a conservatively retained locked superset. The wrapper rejects missing or extra payloads, symlinks, junctions and other reparse points; verifies the closed payload tree both before and after compilation; and then passes the locked bundle explicitly together with `-only-cached`. An empty cache can first be materialized with `scripts/materialize_locked_tectonic_cache.py`; in the current freeze the clean build fetches 393 resources and the materializer explicitly adds the other 142 locked entries. That one online bootstrap must be followed by the complete payload check, while accepted release builds are offline. A different executable, payload tree, bundle or command fails closed. A conventional pdfLaTeX/BibTeX build remains the independent cross-engine lane:

```
job=Autocorrelation_Phase_Lattices_on_Cyclic_Groups
pdflatex -halt-on-error -jobname="$job" main.tex
```

```
bibtex "$job"
pdflatex -halt-on-error -jobname="$job" main.tex
pdflatex -halt-on-error -jobname="$job" main.tex
```

Two clean locked Tectonic builds are compared byte for byte. The final PDF is rendered independently and every page is inspected for clipping, overflow, overlap, broken tables and unintended blank pages. Extracted text is scanned for unresolved references. Every cited key must occur in the repository-local `refs.bib`, every bibliography entry must be cited, and the four reconstruction sources must retain their checked identifiers.

C.2 Package and claim boundary

The repository contains only manuscript source, verification code, generated certificates and review documentation. It redistributes no third-party paper PDFs. The bounded binary audit does not replace the all-prime exclusion proof. The optional translation/reflection corollary keeps the analytical deck oriented and changes only the output quotient. Reproduction establishes author-operated package coherence; it does not establish priority, independent external reproduction, publication, journal acceptance or a DOI.

Data and code availability

The theorem proofs are contained in this manuscript. The companion repository is available at

`https://github.com/aconsciousfractal/FCIG-Autocorrelation-Phase-Lattices-on-Cyclic-Groups`.

It contains the exact-arithmetic checks, symbolic certificates, source-boundary records and deterministic replay described in Appendix C. Those artifacts are assurance layers rather than premises of the proofs. Third-party full texts are not redistributed.

Acknowledgment of prior frameworks

The Fourier monomial and Galois criteria used here are taken from Casper–Orozco; the classical translation-invariant and global sixth-order lines include Adler–Konheim and Grünbaum–Moore. The universal ambient group belongs to the Konyagin–Lev framework. Domokos supplies the zero-sum generation criterion, including its non-closed-field, Galois-stable-support form; the quotient-group/separating-atom language is drawn from the separating-invariant literature. Precise uses are identified at the relevant statements. The supportwise realization step here consists of explicit Galois-equivariant lower-bound ratios and does not re-claim those general criteria.

AI assistance and verification status

This work involved substantial AI assistance using GPT-5.6 Sol via OpenAI Codex, as reported by the author. AI systems contributed to mathematical exploration, candidate

proof arguments, verification code, and manuscript drafting and revision. The author directed the research and is responsible for the claims and source attributions. The computational checks and AI reviews are assurance evidence, not independent human peer review or an end-to-end formal proof. At the time of this Hexagon submission, the author reports that no part of the contribution has been checked and understood by a human.

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