hexagon:2611.00236v1

Sharp thresholds for five-colour cube assemblies

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Abstract

We determine the least pool sizes guaranteeing a frame or a complete $a \times b \times c$ box from unit cubes whose faces use all five colours of a fixed palette, for every $a, b, c \geq 2$. Proper rotations are allowed; the target has monochromatic exterior faces using all five colours. The frame threshold is $\max\{15, 4(a+b+c)-16, 2(a+b+c)+1\}$, and the complete-box threshold is $\max\{abc, 15, 2(a+b+c)+1\}$. The proof combines a certified fifteen-cube corner theorem with elementary Hall arguments and a uniform three-variety obstruction. It completes the five-colour frame column left open by Berkove, Eickemeyer and Schulman (2018). All threshold statements are formalised in Lean. A fixed-size flow formulation decides feasibility and constructs assemblies in time linear in the explicit source list. We also determine the corner thresholds when the number of source varieties is bounded, and classify all fourteen-cube obstructions into 56 classes under common colour relabelling and simultaneous reflection, described by three structural forms. The minimum size of a collection capable of reproducing every $2 \times 2 \times 2$ target is also fifteen.

AI transparency. The closing disclosure describes the AI contributions; Section 11 states the formal verification scope.

1 Introduction

MacMahon's coloured-cube puzzles [1] ask when a collection of face-coloured cubes can form a larger cube with monochromatic exterior faces. Here every source and target uses all five colours of a fixed palette. Proper rotations identify varieties; reflections are not allowed as moves. Collections may contain repeated varieties, and touching interior faces need not have equal colours. For $a, b, c \geq 2$, a *frame* consists of the eight corners and the noncorner positions on the twelve edges of an $a \times b \times c$ box. It has $L = 4(a+b+c) - 16$ positions. Write $F_5(a, b, c)$ for the least pool size guaranteeing a frame and $G_5(a, b, c)$ for the least pool size guaranteeing a complete box. An assembly may use a subset of the pool; its target colouring may depend on the collection.

Theorem 1 (All cuboid thresholds)**.** *For every integer $a, b, c \geq 2$,*

$$F_5(a, b, c) = \max\{15, 4(a+b+c) - 16, 2(a+b+c) + 1\},$$
$$G_5(a, b, c) = \max\{abc, 15, 2(a+b+c) + 1\}.$$

The three frame terms have separate explanations: 15 comes from the fourteen-cube corner obstruction, $4(a+b+c) - 16$ counts frame positions, and $2(a+b+c)+1$ comes from a uniform failing collection of size $2(a+b+c)$. Completion gives the conceptual relation

$$G_5(a, b, c) = \max\{abc, F_5(a, b, c)\} :$$

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once a frame is built, arbitrary unused sources fill the remaining positions, each of which exposes at most one face. The frame threshold depends only on the sum $s = a+b+c$: it is 15 for $s = 6, 7$, 17 for $s = 8$, and the frame's own cardinality $4s - 16$ for $s \geq 9$. Up to permuting sides, only three complete boxes can require spare sources: $2 \times 2 \times 2$, $2 \times 2 \times 3$, and $2 \times 2 \times 4$, whose thresholds are $15, 15, 17$. The distinction between the two thresholds matters: a $2 \times 3 \times 3$ frame has sixteen positions and needs seventeen available sources, whereas its complete box uses any eighteen sources without spares. Table 1 lists the smallest shapes. For $s \geq 9$, $F_5 = L$ and $G_5 = abc$.

Box	s	Frame positions	F_5	Volume	G_5
$2 \times 2 \times 2$	6	8	**15**	8	**15**
$2 \times 2 \times 3$	7	12	**15**	12	**15**
$2 \times 2 \times 4$	8	16	**17**	16	**17**
$2 \times 3 \times 3$	8	16	**17**	18	18
$2 \times 2 \times 5$	9	20	20	20	20
$2 \times 3 \times 4$	9	20	20	24	24
$3 \times 3 \times 3$	9	20	20	27	27

Table 1: Thresholds for the smallest boxes. A bold entry exceeds the number of positions: some collections with no spare sources fail.

For ordinary cubes, abbreviate $F_5(n) = F_5(n, n, n)$ and $G_5(n) = G_5(n, n, n)$. The classification immediately gives the following.

Corollary 2 (Ordinary cubes)**.**

$$F_5(n) = \begin{cases} 15, & n = 2, \\ 12n - 16, & n \geq 3, \end{cases} \qquad G_5(n) = \max\{n^3, 15\} \quad (n \geq 2).$$

In particular, any twenty sources form the frame of a $3 \times 3 \times 3$ *cube, and any 27 form a complete* $3 \times 3 \times 3$ *cube.*

Berkove, Eickemeyer and Schulman [2, Section 8, Table 7] record frame thresholds for palettes of two to six colours, leaving the five-colour column open. Table 2 inserts the values proved here; the cuboid classification goes beyond that table.

Side length	2 colours	3 colours	4 colours	**5 colours**	6 colours
$n = 2$	11	23	11	**15**	24
$n = 3$	20	35	20	**20**	24
$n \geq 4$	$12n - 16$	$16n - 17$	$12n - 16$	$\mathbf{12n - 16}$	$12n - 16$

Table 2: Cube-frame thresholds. The five-colour column is proved here; other columns are as recorded in [2, Table 7].

We first prove the corner threshold independently of the extensions. Write $N_5 = F_5(2) = G_5(2)$.

Theorem 3 (Corner threshold)**.** $N_5 = 15$.

An elementary fourteen-cube obstruction proves the lower bound. For the upper bound, Hall deficiencies and symmetry reduce hypothetical counterexamples to finite problems, excluded by interval certificates (Sections 4–5). For cuboids, Lemma 17 builds corners from eight sources lacking a common pair and any three others. This lets the proof choose targets avoiding scarce edge requirements; compatibility bounds then complete the Hall argument (Sections 6–7). These

cuboid deductions need no further interval certificates. Sections 8–10 give the construction algorithm, support thresholds, extremal obstructions and minimum universal corner collections. The complete extremal classification in Section 9.1 has its own certificate computation and yields a short deletion proof of the corner threshold. The universal minimum has an independent elementary proof. Section 11 describes the verification scope.

The side-length restriction is substantive. With a side of length one, a source can meet two opposite exterior faces and expose four or more faces at once. The corner-and-edge model and the completion by arbitrary unused cubes then need replacement.

The accompanying repository contains the certificates, programs, geometric audits and Lean sources.[1]

The companion paper [5] considers four-colour sources and targets using exactly three colours. It gives the sharp frame threshold $16n - 3$ for every side length $n \geq 2$, with a classification of the largest failing collections. The two developments share the palette-independent geometry in the `colour-cubes-core` library [6]; neither threshold theorem is used to prove the other.

2 Geometry and Hall deficiencies

Represent a cube by its colours on the faces $(+x, -x, +y, -y, +z, -z)$ in a right-handed coordinate frame. A five-colour cube has one repeated colour. There are 75 varieties: for each repeated colour, three have its two faces opposite and twelve have them adjacent. In the opposite case, the other four colours form two opposite pairs, giving three possibilities. In the adjacent case, fix the two equal faces; the 24 orders of the other colours are identified in pairs by the rotation exchanging those equal faces. Thus there are 15 opposite-repeat varieties and 60 adjacent-repeat varieties. Each class is one orbit under colour permutations.

A corner type is the ordered triple of its incident face colours, modulo cyclic permutation. Order the outward face normals positively, equivalently read the triple counterclockwise from outside along the body diagonal. This reverses the clockwise convention in [2], consistently on every cube, and therefore preserves all matching and containment relations. Repeated occurrences of a corner type are retained as distinct target positions.

Lemma 4 (Corner–rotation correspondence)**.** *A source cube fits a target corner by a proper rotation if and only if it has the same cyclic corner type at some corner.*

Proof. A proper rotation preserves the orientation of a corner frame, so it induces a cyclic permutation of its three outward faces. Conversely, any two positively oriented corner frames, with a prescribed cyclic alignment, are related by a determinant-positive signed permutation of the coordinate axes. This is a proper rotation of the cube. □

The palette-independent Lean version of this lemma is reused from the shared `colour-cubes-core` library [6].

Fix a target T. Form a bipartite graph whose left vertices are its eight corner positions and whose right vertices are the individual source copies. A copy is adjacent to a position when it fits there. A matching covering the eight positions is exactly a solution: each chosen copy can be rotated independently, since interior faces are unrestricted. By Hall's theorem [3], failure is equivalent to a nonempty set D of target positions having fewer than $|D|$ neighbouring copies.

Index the varieties by $i = 1, \ldots, 75$, and let x_i be their multiplicities. For a set S of varieties, write $x(S) = \sum_{i \in S} x_i$. Let $N_T(D)$ be the varieties fitting at least one position of D. The 255 nonempty choices of D give 19,125 cuts across the 75 targets. Failure of T is equivalent to the existence of a cut

$$x(N_T(D)) \leq |D| - 1, \qquad \varnothing \neq D \subseteq \{1, \ldots, 8\}. \tag{1}$$

[1] The proof artifact is pinned to Git commit `c4bd4efc2047`. Appendix A.4 gives the full identifier, certificate checksum and reproduction commands.

3 An elementary fourteen-cube obstruction

Consider four copies of A, five of B, and five of C, where

$$A = (0, 0, 1, 2, 3, 4), \quad B = (0, 1, 2, 3, 2, 4), \quad C = (0, 3, 1, 2, 4, 1).$$

Their corner types are shown below. Triples denote cyclic classes; the last column counts positions whose type occurs in neither of the other varieties.

Variety	Corner types	Private positions
A	$013, 014, 023, 024, 031, 032, 041, 042$	5
B	$022, 023, 034, 042, 122, 124, 132, 143$	6
C	$011, 014, 021, 042, 113, 123, 134, 243$	6

Lemma 5. *The collection $4A + 5B + 5C$ has no five-colour solution.*

Proof. A solvable target must have every corner type in the displayed union. Its only two-colour types are $022, 122, 011, 113$. An adjacent-repeat target therefore repeats either colour 2, with types $022, 122$, or colour 1, with types $011, 113$. In either case these two corners determine the colours on the faces at the ends of the repeated-colour edge. The two orders of the remaining colours give a mirror pair: B and its mirror, or C and its mirror. The mirror of B requires 043, and the mirror of C requires 012; both types are absent. Only B, C remain.

An opposite-repeat target requires both cyclic orientations of each corner triple. The unordered triples with both orientations in the union are

$$013, 014, 023, 024, 123, 134.$$

If its repeated colour is r, delete r from the four triples containing it. The remaining pairs must form a $K_{2,2}$ on the other four colours: choose one face from each of their two opposite pairs. Colours 2 and 4 occur in fewer than four listed triples. For $r = 1$ or $r = 3$, the four pairs have degree sequence $(3, 2, 2, 1)$, so do not form $K_{2,2}$. For $r = 0$, the pairs are $13, 14, 23, 24$, forcing opposite pairs $\{1, 2\}$ and $\{3, 4\}$. This is A.

Thus only A, B, C can be targets. Their private positions require respectively at least five, six and six copies of the corresponding variety, exceeding the available four, five and five copies (Figure 1). None is solvable. □

4 Reduction to finite problems

Eight copies of any variety solve that variety, so a solution-free collection has every multiplicity at most seven. Suppose such a collection has size 15.

Lemma 6 (Anchored Hall cover)**.** *Suppose a collection contains m copies of a variety A and does not solve A. There is a set K of its target positions such that*

$$|K| \le 7 - m, \qquad \#\{\textit{other copies fitting some position outside } K\} \le 7 - m - |K|. \tag{2}$$

Conversely, such a set certifies failure of target A.

Proof. Apply (1) to A and put $K = D^c$. Every copy of A is a neighbour of the nonempty set D. Subtract these m copies from the bound $|D| - 1 = 7 - |K|$. The converse is the same Hall inequality read backwards. □

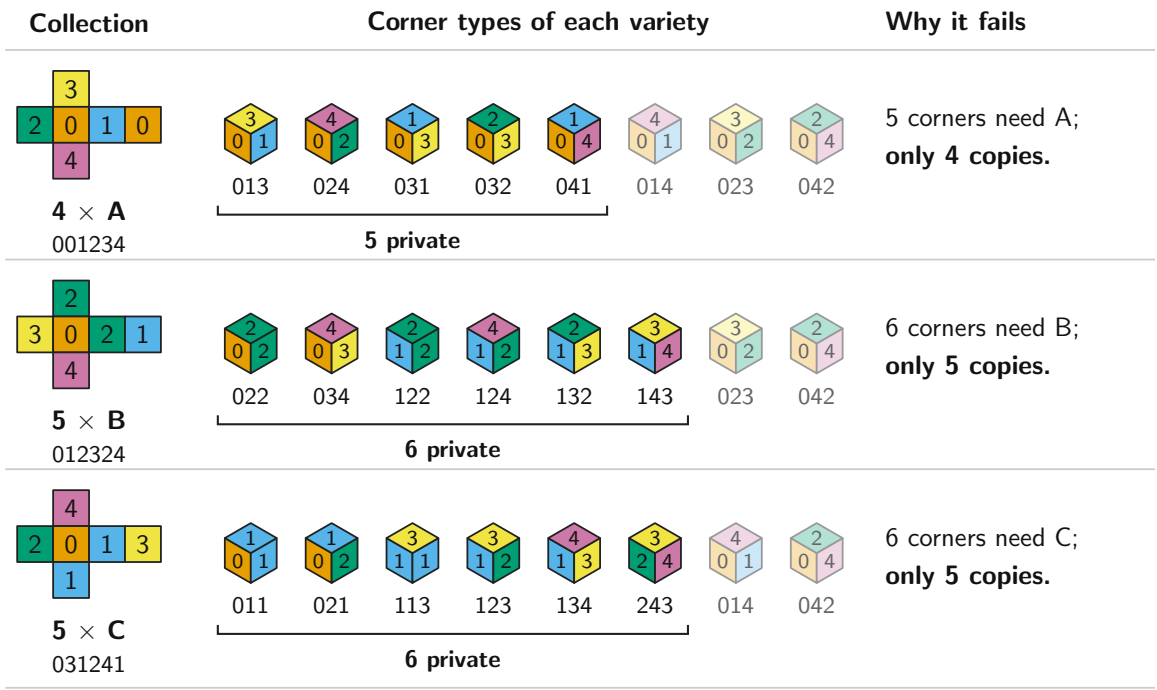


Figure 1: The fourteen-cube obstruction $4A + 5B + 5C$. Each net has $+x$ at its centre. Corner types are read counterclockwise from outside; the highlighted corner types occur on neither of the other two displayed varieties. A target of variety X has these private positions, and only copies of X fit them.

First suppose the maximum multiplicity m is at least two. Choose an adjacent-repeat anchor attaining m if possible; otherwise choose an opposite-repeat anchor attaining m. In the latter case all adjacent-repeat multiplicities are at most $m-1$. Relabel the colours to send the anchor to a fixed representative of its orbit.

For each cover K in Lemma 6, put

$$S(K) = N_A(K^c) \setminus \{A\}, \qquad b(K) = 7 - m - |K|.$$

Write $S = S(K)$ and $b = b(K)$. The case has $x_A = m$, $x(S) \le b$, and the multiplicity bounds just described. Additional upper bounds use two-variety solutions, for each variety $V \neq A$. If j copies of A and $8-j$ copies of V build a target, with $j \le m$, then $x_V \le 7 - j$ in a solution-free collection. These tests are finite: the two varieties must cover all eight target positions, A must fit at least j, and V at least $8-j$. They are also sufficient: at most j positions are unavailable to V, and all of them are available to A. Assign these to A, extend the assignment to j positions that A fits, and assign the remaining $8-j$ positions to V.

Colour permutations and simultaneous reflection of the entire collection preserve solvability. This action has $2 \cdot 5! = 240$ elements; its anchor stabilisers have orders $240/15 = 16$ and $240/60 = 4$ for opposite-repeat and adjacent-repeat varieties, respectively. First identify equal pairs $(S(K), b(K))$, even when they arise from inequivalent position subsets K; then retain one representative per orbit of these pairs under the anchor stabiliser. The counts below refer to these constraint pairs. Transporting a cover also transports its constrained varieties, budget and multiplicities, so no counterexample is lost. Global reflection transports the whole collection; individual cubes still move only by proper rotations. The repeated-variety cases give 244 constraint orbits; the breakdown is in Appendix A.

For distinct collections a different choice of anchor reduces the count.

Lemma 7 (Shared two-colour corner)**.** *Among eleven adjacent-repeat cubes, two share a two-colour corner type. If a distinct collection contains such a pair and one member is chosen as anchor, the set consisting of its two two-colour positions is not Hall-deficient.*

Proof. Each adjacent-repeat cube with repeated colour r has exactly two such types, rrx and rry, with $x \neq y$. There are only $5 \cdot 4 = 20$ types rrx. Eleven cubes contribute 22 incidences, so two cubes share a type. For the two characteristic positions of the chosen anchor, the anchor itself and the other cube are two distinct neighbours. Their neighbourhood therefore has size at least two. □

If an opposite-repeat variety is present, choose it as anchor; this gives 31 constraint orbits with every multiplicity at most one. Otherwise use Lemma 7 to choose an adjacent-repeat anchor. Omit the cover complementary to its two characteristic positions and set all opposite-repeat capacities to zero. Its stabiliser preserves these positions, so the omission is compatible with the orbit reduction. It removes one of the 83 adjacent-repeat constraint orbits, leaving 82 further cases.

A *root case* is an integer feasibility problem specified by an anchor multiplicity, coordinate bounds and an anchored Hall budget.

Proposition 8. *Every solution-free fifteen-cube collection is carried by colour relabelling and, if needed, global reflection into one of the 357 exported root cases.*

Proof. The preceding arguments cover repeated and distinct collections. The finite catalogue, two-variety bounds and coverage of all anchor-cover orbits are checked from physical rotations by the accompanying audit. It regenerates rotations from quarter turns, separately from the signed-axis permutations used by the case generator, and checks every exported bound and every cover orbit. The resulting counts are

$$244 + 31 + 82 = 357.$$

The cases need not be disjoint; completeness of the cover is what is required. □

Each root specifies an anchor A of multiplicity m, coordinate bounds ℓ, u, a set S of non-anchor varieties and a budget b. The bounds fix $\ell_A = u_A = m$, with $\ell_i = 0$ for $i \neq A$. A vector $x \in \mathbb{Z}^{75}_{\geq 0}$ is *admissible* for this root if

$$\ell \leq x \leq u, \qquad \sum_{i=1}^{75} x_i = 15, \qquad x(S) \leq b, \tag{3}$$
$$\text{for every target } T, \text{ there is } \varnothing \neq D \subseteq \{1, \ldots, 8\} \text{ with } x(N_T(D)) \leq |D| - 1.$$

Thus admissibility includes failure of every target, in addition to the anchor constraints. The cuts are alternatives within each target, but at least one cut must hold for every target.

The cover can be reduced without further search. If $S' \subseteq S$ and $b' \geq b$, then $x(S) \leq b$ implies $x(S') \leq b'$ for nonnegative x. Together with coordinatewise inclusion of the root boxes, this identifies 49 cases contained in other cases, leaving 308. Write u for the root upper bounds. In 29 retained cases, $b + u(S^c) < 15$ already contradicts the prescribed total, so only 279 cases need further exclusion proofs. The exact map and retained case numbers are in `data/q5-root-reduction.json`; Lean checks all containments and arithmetic exclusions. All 357 original root definitions remain available.

Lemma 9 (Hall-budget projection)**.** *Suppose a root has anchor multiplicity m, budget $x(S) \leq b$, and total N, with the anchor outside S. For an integer λ with $0 \leq \lambda \leq b$, put $G = N - b + \lambda$ and suppose $G \geq m$. Every admissible vector x has a vector $y \leq x$ with the same anchor multiplicity, total G, budget $y(S) \leq \lambda$, and a Hall deficiency for every target. It obeys the original upper bounds, tightened to $\min(u_i, \lambda)$ on S.*

Proof. First retain only $\min(x(S), \lambda)$ copies in S. This removes at most $b - \lambda$ copies and leaves the anchor untouched, so the remaining total is at least G. Delete further non-anchor copies until the total is G; this is possible because $G \geq m$. Every coordinate decreases, so all upper bounds and Hall deficiency inequalities are preserved. Nonnegativity and $y(S) \leq \lambda$ give $y_i \leq \lambda$ for $i \in S$. □

For our anchored covers, $b = 7 - m - |K|$, so

$$G = m + (N - 7) + |K| + \lambda.$$

At $N = 15$, the reduced collection consists of the m fixed anchor copies and $8 + |K| + \lambda$ other copies; in particular, $G \geq m$ is automatic. Apply the lemma after selecting and normalising the original root, retaining its bounds. No new anchor selection is required for the smaller collection. The resulting 279 projected problems have totals between 9 and 15. Their parameters and exceptional cases are recorded in Appendix A.

5 Interval certificates

For a projected problem, admissibility is defined by (3) with tightened upper bounds and total G. In this section denote its projected budget by b. We certify that no admissible vector exists.

Write a box as $\ell \leq x \leq u$. For a cut $x(S) \leq r$, two necessary conditions for feasibility in the box are

$$\ell(S) \leq r, \qquad G \leq r + u(S^c). \tag{4}$$

The first follows from the lower bounds; the second follows from $G = x(S) + x(S^c)$. When they hold, put $\sigma = r - \ell(S)$ and $\gamma = r + u(S^c) - G$. Every satisfying vector obeys

$$x_i \leq \min(u_i, \ell_i + \sigma) \quad (i \in S), \qquad x_i \geq \max(\ell_i, u_i - \gamma) \quad (i \notin S). \tag{5}$$

For example, the upper bound follows by giving all other coordinates of S their lower bounds; the lower bound follows by giving all other coordinates of S^c their upper bounds.

For a target, retain its cuts satisfying (4). If none remain, no vector in the box can fail that target. Otherwise let I and U be the intersection and union of their neighbourhoods, and let σ_*, γ_* be the largest corresponding σ, γ. Then all vectors in the box that fail this target remain in the refined box

$$u'_i = \begin{cases} \min(u_i, \ell_i + \sigma_*) & i \in I, \\ u_i & i \notin I, \end{cases} \qquad \ell'_i = \begin{cases} \max(\ell_i, u_i - \gamma_*) & i \notin U, \\ \ell_i & i \in U. \end{cases} \tag{6}$$

Indeed, a failing vector satisfies one of the retained cuts, and its bounds in (5) imply (6).

For a concrete example, use the archive's zero-based numbering for both certificates and varieties. Projected certificate 28 starts with two copies of adjacent variety 3, total $G = 14$ and budget zero. All other lower bounds are zero, and the upper bounds sum to 40. At this root, consider a refinement using target 4. Of that target's 255 cuts, 17 satisfy (4); their neighbourhood intersection is $\{4, 10, 30\}$ and their maximum slack is $\sigma_* = 1$. The three corresponding upper bounds therefore decrease from 2 to 1. Here varieties 3 and 4 have face tuples $(0, 1, 0, 2, 3, 4)$ and $(0, 1, 0, 2, 4, 3)$; all indices refer to `data/five-colour-geometry.txt`.

The root budget is treated as a family containing a single cut. The total $\sum_i x_i = G$ gives

$$\ell'_i = \max\left(\ell_i, G - \sum_{j \neq i} u_j\right), \qquad u'_i = \min\left(u_i, G - \sum_{j \neq i} \ell_j\right).$$

A total-bound leaf closes exactly when $\sum_i \ell_i > G$ or $\sum_i u_i < G$. A budget leaf closes when $\ell(S) > b$ or $b + u(S^c) < G$; a target leaf closes when none of its 255 cuts satisfies (4). The checker

thus has 77 closing reasons: one total test, one budget test and 75 target tests. Refinements must tighten at least one bound and leave nonempty coordinate intervals; a split must satisfy $\ell_i \le v < u_i$.

A certificate is a finite tree of refinements and integer splits $x_i \le v$ or $x_i \ge v+1$. Each leaf is excluded by total bounds or by a cut family with no surviving cut. Induction on this tree proves that its root contains no admissible vector. The separate C++ checker reads these instructions and verifies the deductions; it performs no search and does not use the search program's matching implementation.

Proposition 10 (Certificate replay)**.** *The separate checker accepts an exclusion certificate for each of the 279 projected problems.*

Archive sizes and rejection controls are recorded in Appendix A.

Proof of Theorem 3. Lemma 5 gives $N_5 \ge 15$. A solution-free collection of fifteen cubes would satisfy an exported root case by Proposition 8, and would supply a Hall deficiency for every target. Containment and the initial budget tests leave one of 279 roots. Lemma 9 supplies an admissible vector in its projected problem, which Proposition 10 and the sound interval deductions exclude. Hence $N_5 \le 15$. □

6 Compatibility and missing-pair families

An edge type is the unordered pair of its incident face colours, with repetition allowed. A source fits a noncorner edge position exactly when it has an edge of the required type, since the incident faces can be aligned by a proper rotation. A geometric edge parallel to a side of length a contributes $a-2$ frame positions.

The frame arguments depend on how many positions a source can miss. Lemmas 12 and 13 state the shape-independent facts used in the directional bounds of Section 7. The following form of Hall's theorem isolates the matching step.

Lemma 11 (Hall criterion with a matched subset)**.** *Consider L target positions and $N \ge L$ source copies. Suppose every source misses at most d positions. If a set P_0 of positions can be matched to distinct copies, and every position outside P_0 has at least d neighbours, then all L positions can be matched.*

Proof. For a position set $D \subseteq P_0$, the given matching supplies $|D|$ neighbours. If $D \not\subseteq P_0$ and $|D| \le d$, a position in $D \setminus P_0$ alone has at least d neighbours. If $|D| > d$, every source neighbours D, since none can miss all its positions. In each case Hall's inequality holds. □

Universal-copy consequence. In a pool of $N \ge L$ sources, if each source misses at most d of the L positions and at least d source copies each fit all L target positions, a complete matching exists. This is the case $P_0 = \varnothing$ of Lemma 11.

Lemma 12 (Missing pairs and shared corners)**.** *Every five-colour variety has eight distinct cyclic corner types. An adjacent-repeat variety lacks exactly one bichromatic edge type; an opposite-repeat variety lacks exactly two, which are disjoint. Two disjoint missing pairs determine a unique opposite-repeat variety. If either of two varieties is opposite-repeat, they share at least two corner types. A target has at most one monochromatic edge; each bichromatic type occurs on at most two edges, and a doubled type contains the repeated colour. Thus two disjoint bichromatic types account for at most three geometric target edges.*

Proof. A missing bichromatic pair occupies opposite singleton faces: if a colour is repeated, a face of the other colour cannot be opposite both copies. An adjacent-repeat variety has one opposite pair of singleton faces, and an opposite-repeat variety has two. Two disjoint missing

pairs specify two opposite face pairs, leaving the fifth colour on the remaining pair, which determines a unique variety.

For the shared-corner assertion, let the opposite-repeat variety repeat r, with singleton opposite pairs $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$. It contains both cyclic orientations of each rxy with $x \in \{\alpha, \beta\}$ and $y \in \{\gamma, \delta\}$. In the other variety, an r-face with no adjacent r-face has at least three distinct colours around it. Their cyclic order crosses between the two pairs at least twice, giving two shared corners. If its r-faces are adjacent, the other neighbours of each form a path of three distinct colours. Each path crosses between the pairs and gives a shared corner, and these have different types. Corner counts are symmetric: a five-colour variety has eight distinct cyclic corner types, since the only repeated colour multisets occur at the two ends of an edge, in opposite cyclic orders.

Only the repeated colour can form a monochromatic edge. Two singleton faces meet along at most one edge; hence a doubled bichromatic type contains the repeated colour and occurs at most twice. Two disjoint types cannot both contain that colour, proving the final bound. □

Lemma 13 (A transverse edge forces a common corner)**.** *Let source and target both be adjacent-repeat varieties. If the source lacks the type of a target edge not parallel to the target's monochromatic edge, then it shares a corner type with the target.*

Proof. Relabel the colours and rotate the target to $T = (0, 1, 0, 2, 3, 4)$. Its monochromatic edge is parallel to z. The types occurring on edges transverse to the monochromatic edge are $03, 04, 13, 14, 23, 24$. The proper half-turn interchanging the x- and y-directions and reversing z, followed by the colour relabelling $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, fixes T and pairs these types as $(03, 04)$, $(13, 24)$, and $(14, 23)$. Thus three cases suffice. For each missing type the source has its two colours on opposite faces. The other three colours occur around that axis in one of two cyclic orientations. The three adjacent-repeat choices in each orientation share six tricoloured corners. Reading those common corners gives:

Missing type	Shared corners in one orientation	Shared corners in the other orientation
03	$024, 041$	123
13	032	$041, 142$
14	$013, 123$	024

Every entry is nonempty, so the source and target share a corner. □

For a bichromatic pair $\alpha\beta$, let $\mathcal{H}_{\alpha\beta}$ be the family of varieties lacking that edge type. In every such cube, α and β are opposite singleton faces. The four faces around this axis use the other three colours, one of them twice. There are nine varieties in $\mathcal{H}_{\alpha\beta}$: three opposite-repeat varieties O_r, and six adjacent-repeat varieties A_r^+ and A_r^-, where r is the repeated colour. The signs distinguish the two cyclic orientations of the three distinct ring colours. All three A_r^+ varieties share the same six tricoloured corner types; all three A_r^- varieties share the reverse six types.

Lemma 14 (The missing-pair corner array)**.** *The eight corners of an opposite-repeat target $O_r \in \mathcal{H}_{\alpha\beta}$ form a 2×2 array with two positions per cell. Rows correspond to the two unordered ring-colour pairs involving r, and columns to their cyclic orientations. A source O_r fits every cell; the other two opposite-repeat varieties each fit one row; and the positive and negative adjacent-repeat varieties fit the corresponding columns. Consequently, every variety in $\mathcal{H}_{\alpha\beta}$ shares at least four corner types with every opposite-repeat variety in that family.*

Proof. Each cell contains one corner at each end of the $\alpha\beta$ axis. Reading the ring colours gives the stated incidences. Figure 2 shows the example $O_2 = (0, 1, 2, 2, 3, 4)$ in $\mathcal{H}_{01}$, with $+$ chosen to include 023. Relabelling the three ring colours gives the other two targets, and relabelling the palette gives every missing-pair family. Cyclic rotations identify corner labels. In particular, the two positions in each cell have identical neighbourhoods among sources in $\mathcal{H}_{\alpha\beta}$. □

	A_r^+ sources positive column	A_r^- sources negative column
pair 23 O_3 fits this row	$023, 132$	$032, 123$
pair 24 O_4 fits this row	$042, 124$	$024, 142$

O_2 fits all four cells

Figure 2: Corner array of O_2 in the family lacking 01. Each cell contains two target positions. Here $r \in \{2, 3, 4\}$; all three adjacent-repeat sources of a given sign fit the same column.

7 Rectangular boxes

We prove Theorem 1. A prescribed-family corner lemma and a uniform lower obstruction come first, followed by the thin-box upper bounds and the remaining shapes. The only certified input is Theorem 3; the prescribed-family lemma is independent of it.

After ordering the sides as $a \le b \le c$, the upper bound has three exhaustive regimes:

Shape	Main mechanism
$a = 2,\ b = 3$	Exceptional-defect control and retargeting
$a = b = 2$	Concentration and retargeting along a rod
$a + b \ge 6$	Weighted compatibility and reservations

For the quantitative arguments, write $m = a - 2$, $t = b - 2$, $M = c - 2$, and $\tau = m + t + M$. Thus $L = 8 + 4\tau$ is the frame size, and N is the number of available copies. A source's *defect* is the number of target positions it misses. The parameter d is the Hall cutoff; a bichromatic pair is called rare when at most $d - 1$ sources possess it. Its missing family then has at least $N - d + 1$ copies. The main bounds are:

Regime	$N = L$	d	Exceptional Hall sets
$2 \times 3 \times c$, $c \ge 4$	$12 + 4M$	$2M + 6$	Size $d + 1$; Theorem 25
$2 \times 2 \times c$, $c \ge 5$	$8 + 4M$	$2M + 4$	Sizes $d + 1, d + 2$; Theorem 28
$a + b \ge 6$	$8 + 4\tau$	$\tau + M + 6$	Reservations; Theorem 31

In the wide regime every source's defect is at most d. The last column lists the exceptional Hall-set sizes that require additional arguments in the thin regimes; these are distinct from individual source defects and are handled by concentration and retargeting. The smaller boundary shapes are treated separately, with spare sources where necessary. We establish the thin regimes first, then the general weighted bound. In each case an initial corner matching verifies part of Hall's inequalities; the final matching may rearrange those placements.

Lemma 15 (Rare pairs)**.** *In a collection of five-colour cubes, any family of bichromatic pairs each missing from more than half the source copies consists of at most two pairwise disjoint pairs.*

Proof. The missing families of two such pairs intersect. A source missing both pairs forces them to be disjoint, by Lemma 12. There are at most two disjoint pairs in a five-colour palette. □

7.1 Corners from a missing-pair family

Retain the family $\mathcal{H}_{\alpha\beta}$ from Lemma 14, with its opposite-repeat varieties and their 2×2 corner array. Write $K(W)$ for the cyclic corner types of W.

Lemma 16 (Corner projections onto a missing-pair family)**.** *Fix a bichromatic pair $\alpha\beta$. Let C^+ and C^- be the two sets of six tricoloured corner types shared by the three adjacent-repeat varieties of the respective cyclic orientations in $\mathcal{H}_{\alpha\beta}$. An opposite-repeat variety in $\mathcal{H}_{\alpha\beta}$ has four corners in each set. For a source $W \notin \mathcal{H}_{\alpha\beta}$, put*

$$p(W) = |K(W) \cap C^+|, \qquad m(W) = |K(W) \cap C^-|.$$

Then

$$(p(W), m(W)) \in \{(4,0), (2,1), (1,2), (2,2), (0,4)\}.$$

Proof. Relabel the poles as $0, 1$ and the ring colours as $2, 3, 4$. Choose the positive orientation so that

$$C^+ = \{023, 034, 042, 124, 132, 143\}, \qquad C^- = \{024, 032, 043, 123, 134, 142\}.$$

Triples denote cyclic classes; face order is $(+x, -x, +y, -y, +z, -z)$. The following table gives representatives of all geometric cases outside $\mathcal{H}_{01}$, up to a colour permutation preserving the pole pair and a simultaneous reflection, which may exchange C^+ and C^-. Such a reflection is only a symmetry of this incidence calculation, not an allowed assembly move.

W	$p(W)$	$m(W)$
$(0,0,1,2,3,4)$	2	2
$(0,1,0,2,3,4)$	0	4
$(0,2,0,3,1,4)$	2	1
$(0,2,1,2,3,4)$	0	4
$(0,2,1,3,2,4)$	2	1
$(0,2,1,3,4,4)$	2	2

To see that these cases exhaust the possibilities, first suppose a pole colour is repeated. Its two faces are either opposite, or adjacent; in the latter case the other pole is opposite one repeated face or adjacent to both. If a ring colour is repeated, the two pole faces must be adjacent, since otherwise W belongs to $\mathcal{H}_{01}$. Fix them at x^+ and y^+. Among the remaining faces x^-, y^-, z^+, z^-, an opposite repeated pair is z^+, z^-. An adjacent repeated pair is either x^-, y^-, or one of x^-, y^- together with one of z^+, z^-. These are the final three rows.

Each entry follows by reading the eight corners of the displayed cube. The projections of the nine family varieties themselves are $(6,0), (0,6)$, and $(4,4)$, as follows directly from their ring descriptions. □

Lemma 17 (Eight cubes with a common missing pair)**.** *Eight cubes from $\mathcal{H}_{\alpha\beta}$, together with any three additional five-colour cubes, have a corner solution whose target belongs to $\mathcal{H}_{\alpha\beta}$.*

Proof. Let O, P, Q count the opposite-repeat, positive adjacent-repeat, and negative adjacent-repeat copies among the chosen eight. Thus $O + P + Q = 8$; C^+ and C^- are as in Lemma 16.

If $O + \min(P, Q) \geq 4$, the 2×2 Hall argument gives an opposite-repeat corner solution from these eight copies alone. Explicitly, choose O_r whose opposite-repeat multiplicity is maximal. Each other opposite-repeat multiplicity is at most $O/2$: it is no larger than the chosen multiplicity, and their sum is at most O. Among these eight family sources, the two positions in a cell have identical neighbourhoods. Any deficient set can therefore be enlarged to a union of whole cells without enlarging its neighbourhood. Each cell has at least $O/2 + \min(P, Q) \geq 2$ neighbours, each row has at least $8 - O/2 \geq 4$, and each column has at least $O + \min(P, Q) \geq 4$. Every other nonempty union of cells has all eight copies as neighbours.

Otherwise exchange the signs so that $O + Q \leq 3$. Put $\kappa = O + Q = 8 - P$, so $P \geq 5$ and $0 \leq \kappa \leq 3$. Some positive variety A_r^+ occurs at least twice. If its eight corners have a matching among all eleven copies, we are done. Otherwise, choose a deficient Hall set D of its corners.

The two copies of A_r^+ fit every corner, so $|D| \geq 3$ and D contains a tricoloured corner. All P positive copies are therefore neighbours of D. Let R be its other neighbours and put $U = C^+ \setminus D$, $u = |U|$. Since $|D| + u \leq 8$,

$$P + |R| < |D| \leq 8 - u, \qquad |R| + u \leq \kappa - 1.$$

Remove the P positive copies and the copies in R. Every surviving copy has all its positive corners in U, and $u \leq 2$.

Every family opposite-repeat variety has four corners in C^+, so no such chosen copy remains. At least

$$\kappa - |R| \geq u + 1$$

of the chosen nonpositive copies remain, and all of them are negative adjacent-repeat copies. Each fits every corner in C^-. The remaining pool has at least

$$\kappa + 3 - |R| \geq 4 + u$$

copies.

Lemma 16 gives the following implications for every source:

$\lvert K(W) \cap C^+ \rvert \leq u$	$u = 0$	$u = 1$	$u = 2$
$\lvert K(W) \cap C^- \rvert \geq$	4	2	1.

Indeed, the family varieties have projection sizes $(6,0), (0,6), (4,4)$, and every variety outside the family has one of $(4,0), (2,1), (1,2), (2,2), (0,4)$.

Partition C^- into three pairs, indexed by the unordered pairs of ring colours. The negative half of each family opposite-repeat target is the union of two of these pairs. We will match one such four-position half using the remaining pool, then use four positive copies for its positive half.

Let j be the number of negative-family copies in the remaining pool; thus $j \geq u + 1$. If $j \geq 4$, the conclusion is immediate. If $j = 3$, another remaining copy fits at least one negative corner; choose a target whose negative half contains it and fill the other three positions with negative-family copies.

If $j = 2$, then $u \leq 1$. Two other remaining copies each fit at least two negative corners, so they can be assigned distinct ones. Any two negative corners lie in the union of two of the three pairs. Choose the corresponding opposite-repeat target and fill its other two negative positions with the two negative-family copies.

Finally, if $j = 1$, then $u = 0$, and there are at least three other copies, each fitting at least four of the six negative corners. Choose one of these copies W. Some union of two of the three pairs contains at least three of its negative corners. In that four-position half, every other copy fits at least two positions. Hence W and any two other copies can be matched to three distinct positions: two copies have at least two neighbours, and all three together have at least three. The one negative-family copy fills the fourth position.

In every case we obtain the negative half of an opposite-repeat target in $\mathcal{H}_{\alpha\beta}$. Four of the $P \geq 5$ positive copies fill its positive half. The positive copies used here are disjoint from the surviving pool. This proves the lemma. □

7.2 A uniform lower bound

Lemma 18 (A three-variety obstruction)**.** *Let $a, b, c \geq 2$ and put $\tau = a + b + c - 6$. In face order $(+x, -x, +y, -y, +z, -z)$, define*

$$U = (0, 1, 0, 2, 3, 4), \qquad V = (0, 1, 1, 2, 3, 4), \qquad W = (0, 3, 1, 4, 2, 3).$$

The collection

$$U + (\tau + 5)V + (\tau + 6)W$$

of $2\tau + 12$ cubes has no $a \times b \times c$ frame using all five colours. In particular, $U + 7V + 8W$ fails for both a $2 \times 3 \times 3$ frame and a $2 \times 2 \times 4$ box.

Proof. The corner types are

U	$003, 004, 013, 024, 032, 041, 123, 142$
V	$113, 114, 013, 024, 032, 041, 123, 142$
W	$133, 334, 012, 024, 031, 043, 132, 234.$

Triples denote cyclic classes. The only two-colour types in their union are $003, 004$, $113, 114$, and $133, 334$. An adjacent-repeat target must therefore repeat colour 0, 1, or 3. Its two repeated-colour corners force the colours on the opposite faces at the ends of its monochromatic edge to be, respectively, $\{3, 4\}$, $\{3, 4\}$, or $\{1, 4\}$. The two orders of the remaining colours give U, V, W, respectively, and their mirrors. The mirrors of U and V require 023, which is absent from the union; the mirror of W requires 021, also absent.

An opposite-repeat target requires both cyclic orientations of four distinct unordered tri-coloured triples. The union above contains both orientations only for 013 and 123. Thus no opposite-repeat target is possible. The only corner-supported targets are U, V, W.

The corners 003 and 004 of target U each require a U copy, but only one is available.

For target V, seven of its corner types are absent from W; the only shared type is 024. In addition, exactly one geometric edge in each coordinate direction has a type absent from W: these are $14, 14, 11$. Consequently, in any orientation of an $a \times b \times c$ frame, $7 + \tau$ positions require a U or V source. There are only $\tau + 6$ such copies.

For target W, seven corner types are absent from both U and V. Again, one geometric edge in each direction has a private type, now $34, 33, 34$. Hence $7 + \tau$ positions require a W source, whereas only $\tau + 6$ are available.

Every possible target therefore fails (Figure 3). □

Proposition 19. *For every $a, b, c \geq 2$,*

$$F_5(a, b, c) \geq \max\{15,\ 4(a + b + c) - 16,\ 2(a + b + c) + 1\}.$$

Proof. The three terms follow from the fourteen-cube corner obstruction (Lemma 5), the number of frame positions, and Lemma 18, respectively. □

7.3 Choosing a thin target

Lemma 20 (Choosing a target when one side has length two)**.** *Let $2 \leq b \leq c$ and suppose $N \geq 15$ five-colour sources are available. There is a corner-solvable target and an orientation for a $2 \times b \times c$ frame in which every noncorner edge position has a bichromatic requirement possessed by at least $\lceil N/2 \rceil$ sources. An adjacent-repeat target's monochromatic edge is parallel to a side of length two. If $b = 2$ and the target is opposite-repeat, its repeated-colour axis can also be placed along c.*

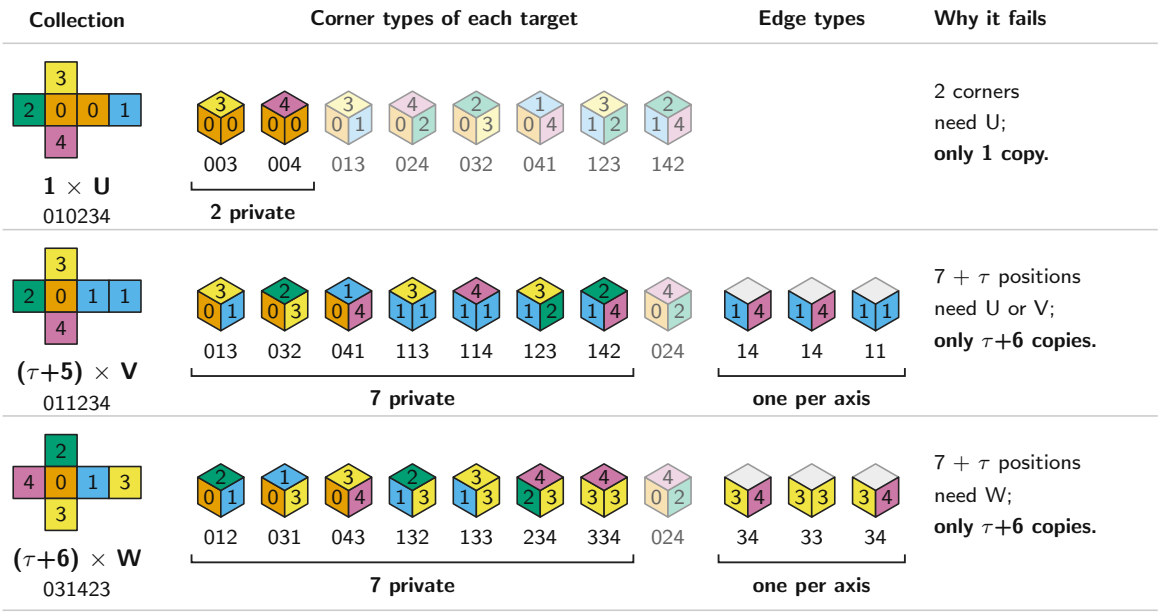


Figure 3: The uniform obstruction $U + (\tau + 5)V + (\tau + 6)W$, where $\tau = a + b + c - 6$. Each net has $+x$ at its centre, and corner types are read counterclockwise from outside. Only U, V and W can be targets. Their pools are U alone, U or V, and W alone; faded corner types also occur on a cube outside the pool. For V and W, one edge in each direction has a private type, and these edges have τ noncorner positions in total.

Proof. Put $h = \lceil N/2 \rceil$ and call a bichromatic pair rare if fewer than h sources possess it. Its missing family has at least $N - h + 1 = \lfloor N/2 \rfloor + 1 > N/2$ members, hence at least eight. The missing families of any two rare pairs intersect, so Lemma 15 gives at most two rare pairs, which are disjoint.

If no pair is rare, the fifteen-cube theorem supplies a corner target. Put its monochromatic edge, if any, parallel to a side of length two, where it contributes no noncorner positions. All actual edge requirements are then bichromatic and have at least h neighbours. For an opposite rod target, put the repeated-colour axis along c.

Otherwise choose a rare pair $\alpha\beta$. Apply Lemma 17 to eight sources from $\mathcal{H}_{\alpha\beta}$ and any three others. The resulting corner solution has a target in that family. Put its singleton $\alpha\beta$ axis along a side of length two. Every actual edge requirement joins a pole colour to a ring colour, so it avoids $\alpha\beta$ and the other possible rare pair, whose colours are both ring colours. The possible monochromatic edge is parallel to the short axis. In an opposite rod target the repeated-colour axis is perpendicular to the singleton axis, so it can simultaneously be put along c. □

7.4 The smallest thin box

Lemma 21 (Compatibility for a thin frame)**.** *Choose an orientation of a five-colour target for a $2 \times 2 \times 3$ frame such that its four noncorner edge positions are bichromatic. Every source cube fits at least four of the twelve positions.*

Proof. If either source or target is opposite-repeat, Lemma 12 gives two shared corners. The four parallel bichromatic edges have either four distinct types or two types occurring twice each. In the latter case both contain the target's repeated colour. The source's missing pairs are disjoint, so it fits at least two of these edges, giving four compatible positions.

If both varieties are adjacent-repeat, the four parallel edge types are distinct and transverse to the monochromatic edge. The source fits either all four, or at least three and a corner by Lemma 13. Again it fits four positions. □

Theorem 22 (The $2 \times 2 \times 3$ box)**.** *We have*

$$F_5(2, 2, 3) = 15.$$

In particular, any fifteen five-colour cubes contain twelve that form a $2\times2\times3$ box with monochromatic exterior faces using all five colours.

Proof. The fourteen-cube corner obstruction (Lemma 5) gives the lower bound: the eight corner cubes of any box would themselves give a corner solution.

For the upper bound, apply Lemma 20 to fifteen sources. The target has a corner matching, and each of its four noncorner edge positions has a bichromatic requirement with at least eight neighbours.

For this target and orientation, every source fits at least four of the twelve positions, by Lemma 21. Thus each source misses at most eight positions.

Apply Lemma 11 with the corners as P_0, $L = 12$, $N = 15$ and $d = 8$. It supplies a matching of all twelve positions to distinct source copies. The final matching may rearrange the preliminary corner placements. □

7.5 Sharp thresholds for $2 \times 3 \times c$ frames

Lemma 23 (Compatibility for $2\times3\times c$)**.** *Let $M = c-2 \geq 2$ and put $d = 2M+6$. Orient a five-colour target for a $2\times3\times c$ frame so that its monochromatic edge, if any, is parallel to the side of length two. An adjacent-repeat source misses at most d positions, and an opposite-repeat source misses at most $d+1$ positions. If a source misses $d+1$ positions, it is the unique opposite-repeat variety lacking the two bichromatic types appearing among those missed edge positions.*

Proof. The nonzero edge weights, counting noncorner positions, are $1, M$. By Lemma 12, for an opposite-repeat source, at most six corners are incompatible. Its two missing bichromatic pairs are disjoint, so at most three geometric edges are incompatible, with at most two parallel to either nonzero direction. If their counts are k_1, k_M, the defect is at most

$$6 + k_1 + Mk_M \leq 6 + 3 + 2(M - 1) = 2M + 7 = d + 1.$$

Equality requires six incompatible corners, two incompatible long edges, and one incompatible edge of weight 1. These three geometric edges display exactly two bichromatic types: there are at least two because any target type occurs at most twice, and at most two because the source lacks only two types. They are disjoint and determine the opposite-repeat source variety uniquely.

An adjacent-repeat source lacks only one bichromatic type. Against an opposite-repeat target, at most six corners and edge positions of total weight $2M$ are incompatible, giving defect at most d. Against an adjacent-repeat target, an incompatible edge lies transverse to the monochromatic edge, so Lemma 13 gives a shared corner. The absent type contributes weight at most $M + 1$: a doubled type has one edge in each nonzero direction. The defect is therefore at most

$$7 + (M + 1) = M + 8 \leq 2M + 6 = d.$$

If no actual edge is incompatible, the defect is at most eight, which is also at most d. □

Lemma 24 (An opposite-repeat target along the short axis)**.** *Let $c \geq 3$. For a $2 \times 3 \times c$ frame, orient an opposite-repeat target with its repeated-colour axis along the side of length two. Every source misses at most $2(c - 2) + 6$ positions.*

Proof. Every actual edge type contains the target's repeated colour. By Lemma 12, a source can lack at most one such bichromatic type, because its missing pairs are disjoint. The absent type accounts for at most two parallel geometric edges, of total weight at most $2(c-2)$. At least two corners are shared, so at most six further positions are missed. □

Theorem 25 (The remaining $2 \times 3 \times c$ frames)**.**

$$F_5(2,3,3) = 17, \qquad F_5(2,3,c) = 4c+4 \quad (c \geq 4).$$

Proof. First let $c = 3$ and take seventeen sources. Lemma 20 gives a corner target with its possible monochromatic edge along the side of length two and at least nine neighbours for each actual edge position.

Every source misses at most nine of the sixteen frame positions. Indeed, if source or target is opposite-repeat, at most six corners and three actual geometric edges are incompatible. If both are adjacent-repeat, at most two actual geometric edges are incompatible; whenever one is incompatible, Lemma 13 supplies a shared corner. If no edge is incompatible, at most eight positions are missed.

The chosen target has a corner matching, and every edge requirement has at least nine neighbours. Lemma 11, with the corners as the matched subset, completes the frame. This proves $F_5(2,3,3) \leq 17$; the collection $U + 7V + 8W$ in Lemma 18 gives equality.

Now let $c \geq 4$, put $M = c - 2 \geq 2$, and set

$$d = 2M + 6, \qquad N = 4M + 12 = 2d.$$

There are $N = 4c + 4$ frame positions. Apply Lemma 20 to any N sources. The target has a corner matching, its possible monochromatic edge lies along the side of length two, and every actual edge requirement has at least $d = N/2$ neighbours. Hall's inequality holds for corner-only sets, and for any set of at most d positions containing an edge. By Lemma 23, sets of more than $d + 1$ positions neighbour every source.

Suppose the frame has no matching. A deficient Hall set D must therefore have exactly $d + 1$ positions. Every nonneighbour misses all of those positions, so Lemma 23 says that all nonneighbours have one opposite-repeat variety V, determined by the two absent bichromatic types. Since D has at most d neighbours among $N = 2d$ copies, the inventory contains at least d copies of V.

Retarget the frame to V with its repeated-colour axis along the side of length two. By Lemma 24, every source misses at most d positions. The d copies of V fit every position, so the universal-copy consequence of Lemma 11 supplies a frame.

Thus N sources always suffice. The number of frame positions supplies the matching lower bound. □

7.6 Sharp thresholds for $2 \times 2 \times c$ boxes

Put $M = c - 2$. The frame of a $2 \times 2 \times c$ box consists of eight corners and four parallel geometric edges, each with M positions. Its size is $L = 4M + 8 = 4c$. We call an orientation of an adjacent-repeat target *transverse* if its monochromatic edge is not parallel to the long axis. For an opposite-repeat target, its *axial* orientation has its repeated-colour axis along the long direction.

Lemma 26 (Directional compatibility)**.** *Let $M \geq 2$.*

1. *For a transverse adjacent-repeat target, every adjacent-repeat source misses at most $M + 7$ positions. Every opposite-repeat source misses at most $2M + 6$ positions. An opposite-repeat source missing more than $M + 6$ positions must be one of two varieties; their missing long-edge types are complementary pairs.*

2. *For an axial opposite-repeat target V, every source misses at most $d = 2M + 4 = L/2$ positions. If $M \geq 3$, a source different from V can miss d positions only if it is one of the other two opposite-repeat varieties with the same repeated colour. These two varieties have complementary missing long-edge types.*

Proof. For (i), write the target in face order $(+x, -x, +y, -y, +z, -z)$ as

$$(r, \alpha, r, \beta, \gamma, \delta),$$

and take the long direction to be x. Its four long-edge types are

$$r\gamma, \quad r\delta, \quad \beta\gamma, \quad \beta\delta.$$

An adjacent-repeat source lacks at most one of these. If it lacks one, Lemma 13 gives a shared corner, so its defect is at most $M + 7$. If it lacks none, its defect is at most eight, also bounded by $M + 7$.

By Lemma 12, an opposite-repeat source shares at least two corners and lacks at most two long-edge types, giving defect at most $2M + 6$. To lack two of the four displayed types, its two missing pairs must be one of the disjoint matchings

$$\{r\gamma, \beta\delta\} \qquad \text{or} \qquad \{r\delta, \beta\gamma\}.$$

Each matching determines exactly one opposite-repeat variety, with repeated colour α. The two varieties have complementary missing long-edge types. Every other opposite-repeat source lacks at most one long-edge type, and hence has defect at most $M + 6$.

For (ii), let the repeated colour of V be r, and put its singleton opposite pairs on the two transverse axes. The four long-edge types are the four edges of the resulting $K_{2,2}$ on the singleton colours. The other two opposite-repeat varieties with repeated colour r each share four corners and two long-edge types with V. Their compatible position count is $4 + 2M = d$, and their missing long-edge pairs are complementary.

Every adjacent-repeat source lacks at most one long-edge type. An opposite-repeat source with repeated colour different from r also lacks at most one: one of its two missing pairs contains r, whereas no long-edge type does. By Lemma 12, these sources share at least two corners with V, so they fit at least $2 + 3M \geq d$ positions. If $M \geq 3$, this last inequality is strict. The target variety V itself fits every position. This proves both assertions. □

Lemma 27 (Concentrated opposite-repeat copies)**.** *Let $M \geq 3$ and $d = 2M + 4$. A collection of $2d = 4M + 8$ cubes containing at least $d - 1$ copies of one opposite-repeat variety forms a $2 \times 2 \times (M + 2)$ box.*

Proof. Let that variety be V with multiplicity v. Use its axial target orientation. Every source misses at most d positions. If $v \geq d$, the universal-copy consequence of Lemma 11 applies, since $N = L = 2d$.

Suppose $v = d - 1$. The same reasoning handles all position sets except sets D of size exactly d. If such a set fails Hall, no non-V copy neighbours it. Those copies must each miss d positions, so Lemma 26(ii) restricts them to the other two opposite-repeat varieties with the same repeated colour. Since $d > 8$, D contains a long-edge position. The two candidate varieties have complementary missing long-edge types, so at most one can be absent from the neighbourhood of D. All $2d - v = d + 1$ non-V copies are therefore one variety W. Using W as the axial target gives the previous case, with at least d universal copies, and completes the box. □

Theorem 28 (Thin rods)**.**

$$F_5(2, 2, 4) \leq 17, \qquad F_5(2, 2, c) = 4c \quad (c \geq 5).$$

Proof. First let $c = 4$, so $M = 2$ and $L = 16$, and take seventeen source copies. Lemma 20 gives a corner target, transverse if adjacent-repeat and axial if opposite-repeat, with at least nine sources for every long-edge type. If the target is opposite-repeat, every source misses at most eight positions by Lemma 26. Lemma 11, with the corners as the matched subset, completes the frame.

For an adjacent-repeat target, adjacent-repeat sources miss at most nine positions and opposite-repeat sources at most ten. A failing Hall set D cannot consist only of corners. Since it contains an edge position, it has at least nine neighbours; and if $|D| > 10$, it neighbours every source. Therefore a failure would have $|D| = 10$ and exactly nine neighbours. Its eight non-neighbouring source copies must all be opposite-repeat. Each must miss two long-edge types, since otherwise its defect is at most $M + 6 = 8$. The two candidate varieties in Lemma 26(i) have complementary missing long-edge types. As D contains an edge position, all eight copies must be the same opposite-repeat variety V.

For the axial target V, every source misses at most eight positions. Its eight universal copies and $N = 17 \geq L = 16$ satisfy the universal-copy consequence of Lemma 11, proving the upper bound.

Now let $M = c - 2 \geq 3$, put $d = 2M + 4$, and take exactly $N = 2d = L$ cubes. Lemma 20 gives a transverse or axial corner target as appropriate, with at least $d = N/2$ sources for each long-edge type. An opposite-repeat target is completed directly by Hall, since every source misses at most d positions.

For an adjacent-repeat target, adjacent-repeat sources miss at most $M + 7 \leq d$ positions; opposite-repeat sources miss at most $2M + 6 = d + 2$. Any failing Hall set D therefore has

$$d + 1 \leq |D| \leq d + 2.$$

It has at most $d + 1$ neighbours, leaving at least $d - 1$ non-neighbouring source copies. All are opposite-repeat. An opposite-repeat source missing only one long-edge type has defect at most $M + 6 < d + 1$, so all these copies are among the two varieties in Lemma 26(i). Since D contains an edge, all must be one variety V. Lemma 27 then supplies a box from the original collection. Thus every collection of $L = 4c$ cubes works. The number of positions gives the matching lower bound. □

Proposition 29 (Complete rod threshold)**.** *Together with Theorem 3, Theorem 22 and the sixteen-cube obstruction of Lemma 18, Theorem 28 gives*

$$F_5(2, 2, c) = \max\{15,\ 4c,\ 2c + 9\} \qquad (c \geq 2).$$

Equivalently, the values are $15, 15, 17$ *for* $c = 2, 3, 4$*, respectively, and* $4c$ *for every* $c \geq 5$*.*

Proof. The cases $c = 2, 3$ are the established corner and smallest thin-box thresholds. For $c = 4$, Lemma 18 and the seventeen-cube upper bound give equality. For $c \geq 5$, the number of positions gives $F_5(2, 2, c) = 4c$, as proved above. □

7.7 Boxes whose two shortest sides sum to at least six

Order the side lengths as $2 \leq a \leq b \leq c$ and put

$$m = a - 2, \qquad t = b - 2, \qquad M = c - 2, \qquad \tau = m + t + M.$$

The frame has $L = 8 + 4\tau$ positions. In this section assume $m + t \geq 2$.

Lemma 30 (Weighted compatibility)**.** *Let* $2 \leq a \leq b \leq c$ *with* $a + b \geq 6$*, so that* $m + t \geq 2$ *and* $M \geq 1$*. Every source misses at most* $d = \tau + M + 6$ *positions of every target frame.*

Proof. By Lemma 12, if the target is opposite-repeat, at most three geometric edges are incompatible, with at most two in any direction. Their total weight is at most $2M + t \leq \tau + M$. At most six corners are incompatible, so the claimed bound follows.

For an opposite-repeat source, take the adjacent-repeat target

$$(r, \alpha, r, \beta, \gamma, \delta),$$

whose monochromatic edge is parallel to z. A doubled bichromatic type, $r\gamma$ or $r\delta$, occupies one edge in each of the x, y directions. If such a type is missing, it and the monochromatic edge contribute one edge in each direction; the other disjoint missing type contributes at most one further edge. The missed weight is at most $\tau + M$. If neither missing type is doubled, two missing single types cannot have the same direction: the single types in each direction share a colour. Adding the monochromatic edge therefore gives at most three edges, with at most two parallel, again of weight at most $2M + t \le \tau + M$. There are at most six incompatible corners.

For two adjacent-repeat varieties, at most three geometric edges are incompatible. If there are at most two, the total defect is at most

$$2M + 8 \le \tau + M + 6,$$

because $m + t \ge 2$. If there are three, they are the monochromatic edge and a doubled missing bichromatic type, one edge in each direction. Lemma 13 gives a shared corner. The defect is at most $\tau + 7 \le \tau + M + 6$, since $M \ge 1$. □

Theorem 31. *If $2 \le a \le b \le c$ and $a + b \ge 6$, then*

$$F_5(a, b, c) = 4(a + b + c) - 16.$$

Proof. Take $N = L = 8 + 4\tau$ source copies and put $d = \tau + M + 6$. Call a bichromatic pair rare if at most $d - 1$ copies possess it. Its missing family has at least

$$N - d + 1 = 3\tau - M + 3$$

copies, exceeding $N/2$ by $m + t - 1 \ge 1$. Lemma 15 again gives at most two disjoint rare pairs.

By Lemma 11, a target avoiding all rare pairs has a full frame matching whenever its corners and monochromatic edge positions have a matching: every other position has at least d neighbours, and Lemma 30 bounds each source's defect by d.

No rare pair. For each colour reserve up to m sources having that colour on adjacent faces, taking all such sources if fewer exist. The reservations are disjoint. At least $N - 5m \ge 7m + 8 \ge 15$ sources remain when $m \ge 1$; when $m = 0$, none are reserved and $N \ge 24$. The corner theorem supplies a target on the unreserved sources. Orient its possible monochromatic edge along the shortest direction. If the target repeats colour r adjacently, a compatible source with adjacent r-faces survived the reservation, so the r-reserve has its full size m. It fills the monochromatic edge. The Hall observation completes the frame.

One rare pair $\alpha\beta$. For each of the three ring colours, reserve up to m sources in the whole collection having that colour on adjacent faces, again taking all such sources if fewer exist. At most $3m$ family copies are removed, so $\mathcal{H}_{\alpha\beta}$ retains at least

$$(3\tau - M + 3) - 3m = 3t + 2M + 3 \ge 8$$

copies, and at least eleven total sources remain unreserved. Apply Lemma 17 to eight unreserved family copies and any three further unreserved sources. The resulting corner target lies in $\mathcal{H}_{\alpha\beta}$. Orient its $\alpha\beta$ axis along the shortest direction. Any monochromatic edge is parallel to this axis and has its full reserve of m copies, by the same survivor argument. The target avoids the only rare pair, so the Hall observation applies.

Two rare pairs e, f. Their common missing variety V is the unique opposite-repeat variety with e, f as singleton opposite pairs. Its multiplicity satisfies

$$v \ge N - 2(d - 1) = 2(m + t) - 2 \ge 2.$$

If $v \ge 8$, use eight copies for the corners of V. If $4 \le v \le 7$, the family $\mathcal{H}_e$ has at least eleven copies and hence at least four not of variety V. Each shares at least four corners with V, by the

ring array. Place four such copies at distinct corners and fill the other four with V. In either case, V has no monochromatic edge and avoids both rare pairs; the Hall observation applies.

The final exceptional case: $v = 2$ or 3. Since $v \geq 2(m+t)-2$ and $v \leq 3$, we have $m+t = 2$, hence $N = 2d$. The ordering $m \leq t$ also gives $m \leq 1$. If n_e, n_f count the sources possessing the rare pairs, then

$$n_e, n_f \leq d-1, \qquad v \geq N - n_e - n_f.$$

Consequently,

$$n_e, n_f \geq d-2, \qquad n_f + v \geq d+1.$$

For each ring colour relative to e, reserve up to m sources repeating that colour adjacently, taking all if fewer exist. As in the one-rare-pair case, at least eight $\mathcal{H}_e$-sources and eleven sources in total remain unreserved. Apply Lemma 17 to eight unreserved family sources and three other unreserved sources. Orient the resulting target $T \in \mathcal{H}_e$ with its e axis shortest. Its corners have a matching on the unreserved sources, and any monochromatic edge has its full reserve of m copies by the same survivor argument.

Within this family, V shares at least four target corners and can miss at most two geometric edges, both parallel to the e axis. For a different opposite-repeat target these are its two f-edges; for an adjacent-repeat target they are its f-edge and monochromatic edge. Thus V misses at most $2m+4 \leq 6$ actual positions.

The only Hall sets not covered as before contain bichromatic edge positions of type f and no non-rare bichromatic edge. Such a set D has at least n_f neighbours. If $|D| > n_f \geq d-2 \geq 8$, every copy of V is also a neighbour, since it misses at most six positions. These are distinct from the n_f copies possessing f, so

$$|\Gamma(D)| \geq n_f + v \geq d+1.$$

This handles $|D| \leq d$; larger sets neighbour every source. Hall's theorem completes the frame in the final case.

The number of frame positions gives the lower bound, proving equality. All preliminary placements certify Hall inequalities; the final matching may rearrange them. □

7.8 Completion of the classification

Proof of Theorem 1. Order the sides as $2 \leq a \leq b \leq c$. When $a+b \geq 6$, Theorem 31 gives $F_5 = L$. The remaining possibilities are exactly $(a,b) = (2,2)$ and $(2,3)$, settled by Proposition 29 and Theorem 25. Together with the corner theorem at $(2,2,2)$, these give $F_5 = 15$ for sums six and seven, $F_5 = 17$ for sum eight, and $F_5 = L$ for every larger sum. Proposition 19 proves sharpness and yields the stated maximum formula.

For the complete box, take $\max\{abc, F_5(a,b,c)\}$ sources and first choose a frame. Every source has every colour, so any unused source can fill any face-interior position after a suitable proper rotation. All fully interior positions are unrestricted. There are enough unused sources to fill the remaining $abc - L$ positions. Conversely, any complete box supplies a frame and uses abc sources. Hence

$$G_5(a,b,c) = \max\{abc, F_5(a,b,c)\}.$$

Putting $x = a-2$, $y = b-2$, and $z = c-2$ gives

$$abc - L = xyz + 2(xy + xz + yz) \geq 0,$$

which simplifies this identity to the claimed formula for G_5.

If $(a,b) = (2,2)$, this formula is $\max\{4c, 15, 2c+9\}$: it exceeds the volume only for $c = 2,3,4$. In every other sorted case, $b \geq 3$; then $abc \geq 18 \geq 15$ and

$$abc - 2(a+b+c) - 1 \geq 2bc - 2b - 2c - 5 \geq 1.$$

The first inequality follows by increasing a from two, and the second is minimised at $b = c = 3$. There are no further exceptions. □

8 Constructing an assembly

The threshold theorem guarantees some target colouring, which need not be specified in advance. It can be found by a bounded number of capacitated matching problems, even when the side lengths are large. Berkove, Eickemeyer and Schulman [2, Section 7] formulate frame assembly as bipartite matching between individual cubes and positions. Here we compress equal source varieties and geometric requirements into a network of constant size.

Proposition 32 (Compressed construction)**.** *Given an explicit list of q sources and $a, b, c \geq 2$, one can find a frame or a complete-box assembly, or report that none exists, in $O(q)$ time. This bound holds for every explicit input, including unsuccessful ones. It uses the word-RAM model with the input sizes, dimensions and output indices fitting in a word. Write $L = 4(a+b+c)-16$ for the frame size. For sources given as 75 binary multiplicities, feasibility and a compressed assembly have polynomial bit complexity; expanding the placements costs the output size.*

Proof. Reject immediately if $q < L$, or if a complete box is requested and $q < abc$. Use $L = 8+4((a-2)+(b-2)+(c-2))$ and capped nonnegative arithmetic for these tests, recording whether a demand exceeds q. Oversized demands need not be represented in a word. Classify each source among the 75 varieties by testing its 24 proper rotations, recording both its variety and a rotation to the representative. This takes $O(q)$ time and gives 75 multiplicities and lists of actual copies.

Test each target variety with each distinct assignment of the three side lengths to its coordinate axes. There are at most six assignments, hence at most 450 configurations. When exactly two side lengths agree, three assignments suffice, giving 225 configurations; when all three agree, only 75 are needed. Every physical target is a proper rotation of one of the 75 representatives, and every proper rotation permutes the three coordinate axes. Bringing the target to its representative therefore permutes the side-length assignment. Sign changes only reverse positions along an edge; they introduce no additional demand pattern. This proves that the listed configurations cover every assembly.

For each configuration, use a flow network with 75 source-variety nodes and at most twenty requirement nodes: eight corners of demand one and twelve geometric edges whose demands are their assigned side lengths minus two. Omit zero demands. Add an arc from the network source to each variety with capacity its multiplicity, and an arc from each requirement to the sink with capacity its demand. Connect a variety to a requirement exactly when it fits, with capacity q. There are at most 97 vertices, including source and sink. Identical requirements may be merged.

An integral flow of value L saturates every requirement and gives the number of copies of each variety to place there. Conversely, any frame induces such a flow. A shortest-augmenting-path max-flow algorithm has a number of arithmetic operations bounded in terms of the network size, independently of the capacities [4]. Here the network size and the number of configurations are constants. Thus all flow tests take $O(1)$ word operations, or polynomial time in the bit lengths for binary input. If every test fails, no frame exists.

For binary input, retain the flow counts and the chosen rotations as the compressed frame assembly. For a complete box, also record counts of unused varieties and their rotations for each face-interior region and the interior; consecutive positions in a fixed coordinate order specify their placements.

Expand a successful flow into L distinct source copies and positions. A precomputed proper rotation for each compatible corner or edge, composed with the recorded source rotation, gives each placement's orientation. This takes $O(L)$ time. A complete box exists exactly when a frame exists and $q \geq abc$: fill face interiors with unused copies rotated to expose the required single colour, and fill the interior arbitrarily. Writing these placements takes $O(abc)$ time. After the initial rejection test, the number of placements is at most q, so both unsuccessful and successful inputs take $O(q)$ time. □

9 Support and extremal corner obstructions

Let $M(k)$ be the least number of cubes guaranteeing a corner solution when the collection contains at most k varieties. We determine these thresholds and classify all fourteen-cube obstructions into 56 symmetry classes. Write $K(X)$ for the set of the eight distinct cyclic corner types of X.

Directed corners. Write $V = \{0, 1, 2, 3, 4\}$ and encode a tricoloured cyclic triple (i, j, k) by the complementary ordered pair ab for which (a, b, i, j, k) is an even permutation of V. Cyclically rotating the triple preserves ab; reversing it gives ba. For example, $013 \mapsto 42$ and $031 \mapsto 24$. Write $b(r, p)$ for the bicoloured corner rrp. The tricoloured corners of an adjacent-repeat variety form a cut

$$C(S) = S \times (V \setminus S), \qquad |S| \in \{2, 3\}.$$

Its smaller side is a pair P, its repeated colour is $r \notin P$, and its other corners are $b(r, p)$ for $p \in P$. The pair, repeated colour and sign determine a unique adjacent-repeat variety. Call $C(P)$ positive and $C(V \setminus P)$ negative. For example, $(0, 1, 0, 2, 3, 4)$ has cut $C(34)$ and bicoloured corners $b(0, 3), b(0, 4)$. Colour permutations give all adjacent varieties; odd permutations also reverse the arcs. A string such as 34 in $C(34)$ denotes $\{3, 4\}$. An opposite-repeat variety with singleton opposite pairs P, Q has arcs $(P \times Q) \cup (Q \times P)$, a bidirected $K_{2,2}$ omitting its repeated colour.

The identity

$$C(S) \cap C(T) = (S \cap T) \times \big(V \setminus (S \cup T)\big)$$

gives the following tricoloured intersection sizes for smaller pairs P, Q:

$\lvert P \cap Q \rvert$	2	1	0
same sign	6	2	0
opposite signs	0	1	4

There are additionally $|P \cap Q|$ shared bicoloured corners when the repeated colours agree, and none otherwise. In particular, a singleton full intersection is tricoloured and joins opposite signs with different repeated colours. By Lemma 12, a pair involving an opposite-repeat variety shares at least two corners; two opposite varieties sharing exactly two corners share one bidirected edge.

Lemma 33. *Four copies of A, together with any four sources individually corner-disjoint from A, can build an opposite-repeat target.*

Proof. The anchor and all four other sources are adjacent-repeat. Reflect the entire instance if necessary so that A has cut $C(P)$, $|P| = 2$, and put $R = V \setminus P$. Each disjoint source has cut $C(R)$ or $C(Q)$ for a pair $Q \subset R$. Choose Q maximizing frequency among all three pair-cut classes, including those of frequency zero. Four A copies fill $P \times Q$ of the opposite target with singleton pairs P, Q. On $Q \times P$, sources of classes $C(Q), C(R)$ are unrestricted; each other pair class fits one two-position row. Neither row-only class occurs more than twice, by maximality of the chosen class. Place them in their rows and fill the remaining positions with the unrestricted sources. □

Consequently, eleven sources containing seven copies of one variety solve: one of the other four sources sharing a corner completes the seven-copy target; otherwise all four are disjoint and the lemma applies.

Theorem 34 (Support thresholds)**.** *For every integer $k \geq 1$,*

$$M(k) = \begin{cases} 8 & k = 1, \\ 13 & k = 2, \\ 15 & k \geq 3. \end{cases}$$

Three is therefore the minimum number of varieties in a fourteen-cube obstruction.

Proof. The one-variety case is immediate. Thirteen sources in two varieties include seven of one variety, so the preceding consequence gives the upper bound. For the lower bound take six copies each of the original B, C from Lemma 5. Their corner union supports only B, C: removing A from that lemma's three-variety union cannot introduce another target, and A has five corners absent from both. The sole shared corner is 042, so either remaining target requires seven copies of its own variety. Thus $6B + 6C$ fails. For $k \geq 3$, Theorem 3 and Lemma 5 give the matching bounds. □

Theorem 35 (Multiplicity bound)**.** *Every fourteen-cube obstruction has maximum multiplicity at most five.*

Proof. The seven-copy case follows above. Suppose there are exactly six copies of A and eight other sources. Two others fitting distinct A corners complete A; four disjoint others invoke Lemma 33. In a failure, therefore, at most three others are disjoint and at least five meet A, all in the same singleton corner k.

The singleton intersection implies that A is adjacent-repeat, since an opposite-repeat source shares at least two corners with every variety. Normalize the cut of A to $C(P)$ with $P = \{p, p'\}$, and put $R = V \setminus P$. The intersection table makes $k = pq$ for some $q \in R$ and forces every meeting source to have cut

$$C(T), \qquad T = \{p\} \cup (R \setminus \{q\}).$$

There are at most three varieties with this common cut. Reserve two identical B copies for its bicoloured corners, set aside any three further common-cut copies, and reserve one A copy for k. A disjoint source's intersection with $C(T)$ is

source cut	intersection with $C(T)$
$C(R)$	$(R \setminus \{q\}) \times \{p'\}$
$C(R \setminus \{q\})$	$(R \setminus \{q\}) \times \{p', q\}$
$C(\{q, r\}),\ r \in R \setminus \{q\}$	$\{rp'\}$

and always avoids k. Exactly three of the eight non-A copies remain. Each meets $C(T) \setminus \{k\}$: a common-cut copy fits all five positions, and the displayed intersections for disjoint sources are nonempty. If two remaining copies fit distinct corners there, use them and fill the other three positions with the set-aside common-cut copies. Together with the two reserved B copies and the A at k, this completes B.

Otherwise the three nonempty compatibility sets are one common singleton. Indeed, a set containing two corners can be paired at distinct corners with any other nonempty set; two different singleton sets also give such a pair. None of the three remaining copies therefore has the common cut. They are all disjoint from A, and the table identifies their shared singleton as rp'. They hence have cut $C(Q)$ with $Q = \{q, r\}$. Retarget to the opposite variety with singleton pairs P, Q: four A copies fill $P \times Q$; three disjoint sources and one B at rp' fill $Q \times P$. Every case gives an assembly, independently of the fifteen-cube theorem. □

For classification, equivalence permits common colour relabelling and simultaneous reflection of the entire collection. Reflection is not an additional move for an individual source. In directed notation this is arbitrary vertex relabelling and independent global arc reversal.

The generalized family. Let A be the opposite variety with singleton pairs $\{1, 2\}, \{3, 4\}$. For $r \in \{2, 3, 4\}$, let B_r have cut $C(01)$ and repeat r; for $s \in \{1, 2, 4\}$, let C_s have cut $C(124)$ and repeat s. In the notation of Lemma 5, $B_2 = B$ and $C_1 = C$.

Corollary 36. *With the same A, B_r, C_s, every collection*

$$4A + \sum_{r \in \{2,3,4\}} x_r B_r + \sum_{s \in \{1,2,4\}} y_s C_s, \qquad \sum_r x_r = \sum_s y_s = 5,$$

where the counts are nonnegative integers, is an obstruction if and only if $x_2 y_2 = x_4 y_4 = 0$.

Proof. When $x_2y_2 = x_4y_4 = 0$, the two groups have disjoint repeated-colour supports. Their tricoloured union is contained in that of A, $C(01)$ and $C(124)$, which misses the arcs $01, 21, 30, 34$. Its bidirected edges are $02, 04, 13, 14, 23, 24$. No four-cycle passes through 0, since its neighbours $2, 4$ have no second common neighbour; on the other four vertices the only four-cycle is A. Thus A is the only covered opposite-repeat target. For an adjacent target, the bicoloured corners determine an active B_r or C_s, up to reversing its cut. The reversed cuts require absent arcs $21, 01$, respectively. Each remaining target has the following positions private to its indicated source group:

target	private positions	demand	supply
A	$24, 31, 32, 41, 42$	5	4
B_r	$02, 03, 04, 12$ and both bicoloured corners	6	5
C_s	$10, 20, 40, 43$ and both bicoloured corners	6	5

All targets therefore fail. Missing varieties within either group can only remove suppliers. Conversely, suppose both groups contain repeat $r \in \{2, 4\}$. Build B_r: reserve one B_r at $b(r, 1)$ and one C_r at $b(r, 0)$; the other four B-group copies fill $02, 03, 04, 12$, and two A copies fill $13, 14$. These are eight distinct sources, giving a solution. Taking $y_1 = 5$ gives a 21-member subfamily. The general family also contains, for example, $4A + 2B_2 + 3B_3 + 2C_1 + 3C_4$, whose maximum multiplicity is four. □

9.1 The complete fourteen-cube classification

The preceding family accounts for most, but not all, extremal obstructions. There are two further forms, both using six varieties. Throughout this subsection equivalence means common colour relabelling and optional simultaneous reflection of the collection. A labelled collection keeps the five colour names fixed; identical physical copies remain indistinguishable.

An all-adjacent family. Choose a colour o, an ordered partition P, Q of $V \setminus \{o\}$ into pairs, and a derangement ρ of those four colours. Take one copy of each repeat-o variety with cuts $C(P)$ and $C(P \cup \{o\})$. For each $v \in P$, take three copies with cut $C(\{v, o\})$ and repeat $\rho(v)$; for each $v \in Q$, take three copies with cut $C(V \setminus \{v, o\})$ and repeat $\rho(v)$. The condition $\rho(v) \neq v$ makes every variety well defined. These collections have multiplicities $(3, 3, 3, 3, 1, 1)$.

Their tricoloured union omits exactly $Q \times P$. Its bidirected graph is the two triangles on $P \cup \{o\}$ and $Q \cup \{o\}$, sharing only o. It has no four-cycle, so no opposite-repeat target is covered. For any repeat other than o, the bicoloured corners fix the smaller pair of its unique three-copy variety; the reverse cut requires a missing arc. For repeat o, a smaller pair crossing P, Q also requires a missing arc in either orientation. Thus precisely the six source varieties are covered.

For $v \in P$, with w the other member of P, the three-copy target's private arcs are vw, ow. For $v \in Q$, with w the other member of Q, they are wv, wo. Its two bicoloured corners are also private, since ρ is bijective. Each three-copy variety therefore faces four private positions, and each singleton faces its two private bicoloured positions. Every target fails.

The colour o is recovered from the two singleton varieties, and the missing cut recovers P, Q up to reversal. The eight-element partition stabilizer acts on ρ by conjugation. For $o = 4$, $P = \{0, 3\}$ and $Q = \{1, 2\}$, its four orbits are represented by

type	ρ	labelled orbit size
within-pair transpositions	(0 3)(1 2)	30
cross-pair transpositions	(0 1)(3 2)	60
alternating four-cycle	(0 1 3 2)	60
nonalternating four-cycle	(0 3 1 2)	120

The normalized orbit sizes are $1, 2, 2, 4$, accounting for the nine derangements. There are $5\binom{4}{2} = 30$ ordered templates, giving 270 labelled collections in these four classes.

A mixed exception. The remaining form is the following single collection, in the face order $(+x, -x, +y, -y, +z, -z)$:

variety	face tuple	copies	private positions
X_0	$(0,1,0,2,3,4)$	5	6
X_1	$(0,4,1,2,3,1)$	1	2
X_2	$(0,4,1,2,3,2)$	1	2
X_3	$(0,4,1,3,3,2)$	1	2
O	$(0,4,1,2,3,3)$	3	4
Y	$(0,4,1,4,3,2)$	3	4

Only O is opposite-repeat. The arc union misses $04, 21, 23, 34$. Its bidirected edges are

$$01, 02, 03, 13, 14, 24.$$

The only four-cycle is $0{-}1{-}4{-}2{-}0$, corresponding to O. For each repeated colour, exactly two bicoloured corners occur. They fix the listed adjacent variety up to reversal, and every reverse requires a missing arc. These six varieties are therefore exactly the covered targets. The private tricoloured positions are $30, 31, 32, 40$ for X_0, none for X_1, X_2, X_3, $10, 14, 20, 24$ for O, and $12, 13$ for Y; every adjacent target also has two private bicoloured positions. Each target needs one more copy than is available. The collection fails and its labelled orbit has size 240.

Theorem 37 (Fourteen-cube obstructions)**.** *Every fourteen-cube obstruction is equivalent to a member of exactly one of the following forms:*

1. *the failing collections of Corollary 36;*
2. *the all-adjacent derangement family above;*
3. *the mixed exception above.*

There are respectively $51, 4, 1$ *symmetry classes, hence 56 in total. Their distribution by number of varieties is*

varieties	3	4	5	6	*total*
classes	5	20	26	5	56
labelled collections	840	4800	5280	510	11430.

Proof. Failure and orbit counts. The preceding arguments prove failure of all three forms. In the first form, there are 91 distributions satisfying the disjoint-repeat condition. The only residual nonidentity symmetry sends

$$(x_2, x_3, x_4; y_1, y_2, y_4) \longmapsto (y_4, y_1, y_2; x_3, x_4, x_2).$$

It fixes 11 distributions: $y_4 = x_2$, $y_1 = x_3$, $y_2 = x_4$ and $x_2x_4 = 0$. To refine the count by support, let n count normalized distributions and f those fixed by this involution:

varieties	n	f	classes $(n+f)/2$	labelled $120n$
3	7	3	5	840
4	40	0	20	4800
5	44	8	26	5280
total	91	11	51	10920

With one active repeat in each group there are $3 \cdot 3 - 2 = 7$ allowed pairs. Support sizes $(1,2)$ and $(2,1)$ in the two groups each give $5 \cdot 4 = 20$ distributions. Sizes $(1,3)$ and $(3,1)$ each give $\binom{4}{2} = 6$, while $(2,2)$ gives $2 \cdot 4^2 = 32$. The fixed-point equations give three distributions of support three and eight of support five. A fixed distribution has stabilizer of order two in the

240-element symmetry group; a nonfixed one has trivial stabilizer. Thus the labelled count in each row is $120f+240(n-f)/2 = 120n$. The other two forms contribute $270+240 = 510$ labelled collections, all of support six. The three forms cannot be equivalent, since their opposite-repeat totals are respectively four, zero and three.

Completeness by certificates. The construction of Proposition 8 applies unchanged to fourteen sources: its anchor bounds are independent of the total, and a distinct all-adjacent collection still has the eleven sources needed by Lemma 7. Theorem 35, independently of the fifteen-cube threshold, removes the eight roots with anchor multiplicity six or seven. The remaining 349 roots use total fourteen. Exactly seven roots contain the 81 explicit terminal count vectors. For 329 empty roots, Lemma 9 at $N = 14$ reduces the problem to exclusion: 133 use strictly smaller projected totals between 9 and 13, while 196 use $\lambda = b$, retaining total fourteen. For the other thirteen empty roots, zero-budget projection gives a vector $y \le x$ of total between eight and eleven. Complete coverage certificates place y in a list of 710 smaller cores. For each core c, an exclusion certificate rules out every fourteen-copy vector between c and the original upper bounds, with the original Hall budget. Additional copies are allowed in every coordinate, not only in the deleted budget set. The seven nonempty roots are enumerated. In addition to the exclusion, refinement and splitting rules, the enumeration checker accepts an exact singleton box only when its vector occurs in the terminal list. Thus every admissible vector in every root belongs to that list. Each terminal is carried to one of the 56 representatives by an explicit common colour/reflection symmetry. Separate finite checks verify the Hall deficiencies, the full symmetry group and pairwise inequivalence. The Lean endpoint establishes equivalence to exactly one of 56 concrete representatives. The geometric arguments above and a separate normal-form and orbit audit identify their union with the three stated forms and give the displayed labelled and support counts. □

In particular, any fourteen-cube collection using at least seven varieties solves. The all-adjacent obstructions are exactly the derangement family.

The threshold by deletion. The classification also gives a short proof of Theorem 3, independent of the fifteen-cube exclusion archive. Suppose an unsolvable fifteen-cube pool existed. Every deletion would be a classified fourteen-cube obstruction. If all sources were adjacent-repeat, every deletion would have profile $(3, 3, 3, 3, 1, 1)$. A variety occurring three times in one deletion had multiplicity three or four in the original pool. Deleting from that variety in the first case, or from another variety in the second, leaves multiplicity two or four, both impossible.

Otherwise there is exactly one opposite-repeat variety A: two distinct ones could both be retained by a deletion. There are adjacent sources as well, since every classified obstruction has at most four opposite copies. Deleting an adjacent copy shows that the multiplicity of A is three or four; deleting an A shows that one less is also three or four. Hence it is four. Deleting an A gives the mixed exception, so the adjacent multiplicities in the original pool are $(5, 3, 1, 1, 1)$. Delete from the five-copy variety. The resulting obstruction has four opposite copies and six varieties, whereas the first form has at most five varieties. This final contradiction proves the fifteen-cube upper bound.

Theorem 38 (Three-variety obstructions)**.** *Up to this equivalence, the fourteen-cube obstructions using exactly three varieties are precisely five collections* $4A + 5B_r + 5C_s$*. The varieties are those of Corollary 36; Class I is the original illustrated obstruction. Representatives are*

class	I	II	III	IV	V
r	2	2	3	3	4
s	1	4	1	2	2

Proof. The other two forms in Theorem 37 have six varieties. A three-variety member of the first form must concentrate each five-copy group on one variety, giving $4A + 5B_r + 5C_s$ with

$r \neq s$. Corollary 36 proves failure for all seven such pairs. A symmetry between normalized collections fixes the unique four-copy variety A, hence fixes 0 and preserves its two unordered pairs. Without arc reversal it fixes both cuts, hence every vertex. With reversal it exchanges the cuts and is $\sigma = (1\ 3)(2\ 4)$. It sends (r, s) to $(\sigma(s), \sigma(r))$, pairing $(2, 1)$ with $(3, 4)$ and $(3, 2)$ with $(4, 1)$, while fixing $(2, 4), (3, 1), (4, 2)$. This gives exactly the five inequivalent representatives displayed. □

This displayed proof uses the complete classification. The formal endpoint instead has an independent proof without interval certificates (Appendix A.3); the repository also retains the elementary necessity argument in `paper/three-variety-necessity.tex`.

10 Universal collections

Universal sets for four-colour cubes were studied in [2, Section 6]. A collection is *universal* if, for every five-colour target variety, eight of its copies can form a corner solution of that variety. The chosen copies and their placements may depend on the target. The palette is fixed, and only proper rotations are allowed, as before.

Theorem 39. *The minimum size of a universal collection is* 15*, attained by fifteen distinct adjacent-repeat varieties. Requiring only the sixty adjacent-repeat targets already forces this minimum.*

Proof. For each colour r, universality requires all four bicoloured corner types $b(r, p)$, $p \neq r$. Only adjacent-repeat sources repeating r can supply them, and each such source supplies exactly two. If there were at most two such copies, covering all four types would force exactly two copies with disjoint pairs of types. A target whose bicoloured pair equals either pair would then require two copies from a pool of one. Thus at least three adjacent-repeat sources are needed for each of the five repeated colours, giving the lower bound 15. Any universal collection of size fifteen has exactly three adjacent-repeat sources of each repeated colour and no opposite-repeat sources.

For the upper bound, use the directed-corner notation of Section 9 and read the colours modulo 5. For every r and $p \notin \{r, r+1\}$, take the positive adjacent-repeat variety with repeated colour r and smaller pair $\{r+1, p\}$. These are fifteen distinct varieties. For each r, their bicoloured pairs form a three-edge star on the other four colours, so any two required bicoloured corners can be assigned to distinct sources.

Represent each source by its smaller pair in an auxiliary multigraph on the five colours. Pairs consecutive in the colour cycle occur once and the other pairs twice. Every vertex therefore has degree six, and every pair has multiplicity at most two. A source supplies the tricoloured corner uv exactly when its pair contains u but not v. Hence one arc has at least four neighbouring sources. Two distinct arcs have at least six: if their tails agree, their neighbourhood union contains all six incident sources; otherwise each has at least four neighbours and their intersection has at most two.

The tricoloured corners of any target have at most four tails and at most three arcs per tail. A set of at least five such corners therefore contains two arcs with the same tail and one with a different tail. Writing u for the common tail and $w \neq u$ for the third tail, a source common to the six-source star and the third neighbourhood must have smaller pair $\{u, w\}$. Their intersection therefore has size at most two, and their neighbourhood union has size at least $6 + 4 - 2 = 8$. Thus a set of k tricoloured target corners has at least

k	1	2–4	5–8
neighbouring sources	4	6	8

sources. For an opposite-repeat target these bounds imply every Hall inequality. For an adjacent-repeat target, a Hall set with $k \geq 1$ tricoloured corners has at most $k + 2$ positions, and $k \leq 6$;

the same bounds again suffice. Sets containing only bicoloured corners were handled by the star construction. Hall's theorem completes every target. □

Both bounds are independent of the certified fifteen-cube corner theorem.

Corollary 40. *Every minimum universal collection consists of fifteen distinct adjacent-repeat varieties. For each repeated colour, their three smaller pairs form a spanning tree on the other four colours.*

Proof. There are exactly three adjacent-repeat sources of each repeated colour r. Their smaller pairs must cover the other four colours. If two pairs coincide, the third must be disjoint from them; a target requiring the two bicoloured corners of that third pair would have only one supplier. Thus the three pairs are distinct and cover four vertices, so they form a spanning tree. In particular, the sources are distinct varieties. The construction above chooses a star for each of these trees. □

11 Formal verification

The development uses Lean 4 and mathlib [7, 8], both at `v4.35.0`, with the physical model and palette-independent geometry of `colour-cubes-core` [6]. It connects arbitrary physical cubes to the catalogue, Hall deficiencies, the complete root cover and the certified exclusions. The cuboid deductions apply the verified corner theorem. The following table distinguishes the proof scopes; "comparison" means the separate Mathlib-only statement interface, accepted by both NanoDa and Lean's standalone kernel. The certificate column includes dependencies: the cuboid deductions use the certified corner theorem, and the symbolic classification uses the certified exhaustion by concrete representatives.

Result	Lean	Uses interval certificates	Comparison
Corner, cuboid and support thresholds	Yes	Yes	Yes
Multiplicity bound, three-variety classification and obstruction family	Yes	No	Yes
Fourteen-cube classification: 56 concrete classes	Yes	Yes	Yes
Symbolic $51 + 4 + 1$ forms and labelled counts	No	Yes	No
Universal minimum and cyclic construction	Yes	No	Yes
Minimum-universal tree structure (Corollary 40)	Yes	No	No
Construction complexity and Domino observation	No	No	No

In namespace `FiveColour.Puzzle`, the unconditional endpoints `box_frame_threshold` and `complete_box_threshold` are exported by `Solution.lean`. They use the physical box definitions and proper-rotation bridges. The sixteen-claim comparison interface describes physical copies, proper rotations and exposed faces without catalogue indices. The symbolic forms and labelled counts are proved here and checked by the separate normal-form audit. The deletion deduction is a manuscript argument; the exported threshold retains its separate certificate proof.

Appendix A gives the soundness and audit details. Appendix A.4 pins the proof artifact and gives separate commands for full verification and external comparison, together with the scope of the retained execution records.

12 Remaining questions

Does $N_5 = 15$ admit a short elementary proof, replacing the finite certificates used here? Boxes with a side of length one also fall outside the model, since a source there exposes opposite faces; their thresholds remain open.

The *Domino condition*, requiring touching interior faces to agree [2, Section 8], changes the guarantee fundamentally. For any fixed $a, b, c \geq 2$, arbitrarily many copies of one adjacent-repeat variety cannot assemble a Domino frame or box. To see this, write that variety as $D = (r, p, r, q, s, t)$, with the five colours distinct. The corner encoding in Section 9 determines a variety from its eight distinct corner types. An identical-source pool can therefore reproduce only D, up to a common proper rotation: containment of its eight corner types forces equality. Along the edge common to the p- and q-faces, every source must align its unique p and q faces with the same two perpendicular directions. This fixes its proper orientation. Consecutive copies then meet with colour s against colour t, which differ. The edge contains at least two cubes and is present in both a frame and a complete box. Thus there is no finite threshold for arbitrary multisets under the Domino condition. This observation concerns a different model and is not included in the Lean interface; its universal-collection questions are left for separate investigation.

AI disclosure

GPT-6-Astra (OpenAI), used through Codex, developed the arguments, computational search, verification programs, Lean formalisation and manuscript under the author's direction. Claude Opus 5.5 (Anthropic), used through Claude Code, contributed illustrations, the migration to the shared `colour-cubes-core` library, and review and editing of the manuscript. The author initiated and directed the research and takes responsibility for the content. The inherited geometry has its own provenance in the pinned dependency. Automated checks and AI review are not independent human peer review.

A Verification and reproduction details

A.1 Finite certificates

The repeated-variety cases have the following counts, totalling 244.

Anchor / maximum multiplicity	2	3	4	5	6	7
Adjacent-repeat	72	55	31	14	4	1
Opposite-repeat	26	21	11	6	2	1
Total	98	76	42	20	6	2

Of the 279 remaining roots, 277 admit an exclusion certificate with $\lambda = 0$. The other two, original cases 68 and 69, use $\lambda = 1$ and $G = 12$. Their $\lambda = 0$, $G = 11$ problems actually have solution-free collections, supplied and checked by physical matching in the projected manifest. The projected totals range from 9 to 15. The 101 roots whose original budget was zero retain their original problem statements.

The projected archive contains 298,167 instructions. Every certificate header is checked against its exported root case. Eighty negative controls are rejected, including all 77 possible direct closing claims against a separate $G = 11$ fixture, not a $G = 15$ root problem. This fixture is a known solution-free eleven-cube collection: it satisfies its total, fixed anchor count, coordinate bounds and zero root budget, and fails all 75 targets by physical matching. It is `data/q5-distinct-known-control.txt`; its regeneration and witness checks are in `scripts/check_five_colour_distinct.py`.

A.2 Lean implementation

Lean proves completeness and uniqueness of the 75 varieties and checks all 45,000 corner-compatibility bits using the shared corner–rotation theorem. The lower bound uses 75 defi-

cient corner sets: 72 singletons and three sets of sizes 5, 6 and 6. Its theorem is `FiveColour.Physical.lower_bound` in `FiveColour/LowerBound.lean`. In the formal proof, this replaces the elementary target classification in Lemma 5; catalogue completeness and rotation transport extend the result to every physical target.

The upper-bound formalisation follows the Hall reduction, symmetry transport, anchor selection and shared-corner argument. Finite checks verify capacities, cover representatives and the reduction from 357 roots to 279. The deletion step is `FiveColour.Physical.root_projection` in `FiveColour/RootProjection.lean`. The shared `ProjectionBridge.lean` module identifies the full admissibility predicate: coordinate bounds, total, budget and all Hall-cut families.

For efficient replay, Lean proves soundness of a packed implementation of the interval checker. For multiplicities $0 \le x_i \le 7$, index the varieties by $0, \dots, 74$ and put

$$P(x) = \sum_{i=0}^{74} x_i B^i, \qquad W(S) = \sum_{i \in S} B^{74-i}, \qquad B = 1024.$$

Then

$$x(S) = \left\lfloor \frac{P(x)W(S)}{B^{74}} \right\rfloor \bmod B.$$

Every coefficient in the product is at most $75 \cdot 7 < 1024$, so no carry can affect this coefficient extraction. Lean proves the identity, decoding, coordinate updates and equivalence with the integer inequalities, and checks all 19,125 neighbourhood weights. The evaluator in `FiveColour/PackedFast.lean` computes digit places directly as powers; Lean proves it equal to the original evaluator on every certificate. The helpers in `FiveColour/PackedComputations.lean` prove that containment in the packed all-seven digit mask establishes bounded digits and exact reconstruction. A second helper converts a neighbourhood mask to its reversed base-1024 weight by consuming successive low bits. These helpers discharge the original validity and calibration propositions.

For larger certificate cases, the generated checker specialises the cut families to the initial root. It discards alternatives that fail (4), removes fixed coordinates (subtracting their mass from the right-hand side), and merges equal remaining neighbourhoods by keeping the largest right-hand side. An alternative is also discarded if it implies a weaker retained alternative. Each original feasible cut has a checked witness pointing to a weaker replacement. Concretely, a cut $x(C) \le b_C$ may be replaced by $x(D) \le b_D$ when

$$D \subseteq C, \qquad b_C + \ell(D) \le b_D + \ell(C).$$

Indeed $x(C \setminus D) \ge \ell(C \setminus D)$ for vectors in the root. Replacement weights are calibrated in the kernel. The theorem `FiveColour.Packed.excludes_of_covered` in `FiveColour/PackedPruning.lean` transfers the resulting exclusion back to the original families at the root; it does not assume that intermediate split boxes are contained in that root. Small cases retain the original families to avoid the additional validation cost.

The 279 certificates are split into 2,926 chunks of at most 128 instructions. Composition requires exact boundary-vector agreement and a proof for every referenced box; the exporter is untrusted. All chunks and their final assembly have passed kernel checking. The case mapping and certificate hashes are in `data/q5-projected-root-manifest.json`. The generated module `FiveColour/ProjectedChunks/Threshold.lean` proves the unconditional theorems `threshold15` and `sharpThreshold15`. Both lie in namespace `FiveColour.Packed.ProjectedChunks`.

The cuboid proof in Lean follows Lemma 17. The cube exports, such as `frame20`, specialise the cuboid theorems. The equivalences in `FiveColour/CubeBoxBridge.lean` identify the cube and equal-side cuboid position models, preserving exposed faces and physical placements. Cube completion likewise reuses rectangular completion, whose generic construction is

adapted from [5]; no theorem of that paper is used. The auxiliary eleven-cube missing-pair lemma remains in `FiveColour/RingEleven.lean`, with its multiplicity argument proved without enumerating collections.

The small-support proofs in `FiveColour/SmallSupport.lean` check pair witnesses directly, without reducing them to five colour orbits.

The full cuboid classification is formalised in `FiveColour/BoxThresholds.lean`. In namespace `FiveColour.Physical`, `box_frame_threshold_iff` and `box_threshold_iff` give necessary and sufficient pool sizes for arbitrary side lengths at least two, assuming `Threshold15`. `Solution.lean` discharges this assumption and exports the unconditional `FiveColour.Puzzle` theorems `box_frame_threshold` and `complete_box_threshold`.

The rectangular model and completion are in `FiveColour/BoxModel.lean`; compatibility is connected to proper rotations, and axis assignments are realised by proper rotations. `FiveColour/EightPlusThree.lean` proves the physical prescribed-family lemma. Its proof uses a failed Hall set to obtain the small cover directly, as in Lemma 17; signed projection and small matching patterns are checked in the kernel. The weighted bounds and upper deductions are in `FiveColour/WideFrames.lean`, `FiveColour/RodFrames.lean` and `FiveColour/StripFrames.lean`. The thin-box proofs formalise both concentration and switching targets, including the equality cases of the affine geometric bounds. Directional profiles are checked for the two canonical target varieties; proved colour and rotation transport covers all targets, including their corner, edge and axis labels. The small half-mask matching step uses greedy selection after single-mask checks. `FiveColour/BoxLowerBound.lean` proves the uniform three-variety obstruction for every shape and orientation. A separate C++ audit reconstructs the geometry and checks these profiles.

A.3 Extremal obstructions

The multiplicity bound, three-variety classification and generalized obstruction family have separate physical Lean proofs, independent of `Threshold15` and the interval replay.

`ExtremalMultiplicity.lean` uses the anchor lemma and explicit corner allocations for small grouped inventories. The allocation theorem reserves distinct physical copies, and kernel checks validate both the assignments and the filtered weak-composition lists. Four disjoint sources are handled first by a separate lemma, leaving 2,702 six-copy inventories and 35 four-disjoint inventories. Their geometry uses 300 shared disjoint-group masks and 24 optional singleton references; other masks reuse the incidence data. The main-text proof instead uses the directed-cut intersection calculation and a uniform reservation argument.

For the three-variety classification, the multiplicity bound forces $(4, 5, 5)$. Normalizing the four-copy variety to one of the two anchor types leaves $2\binom{74}{2} = 5,402$ candidates. The coverage/private-position criterion solves 5,143 at one of their own source targets, and the existing four-plus-four lemma solves another 199 with a disjoint pair. Kernel checks prove that the residual list covers both anchor kinds: it has 60 records, comprising four positive target witnesses and 56 symmetry witnesses. A shared uniform Hall argument proves that all five representatives fail. Completeness of the sixteen-element stabilizer of the opposite anchor is checked on physical faces; this establishes pairwise inequivalence under every allowed global symmetry. `ExtremalClassification.lean` joins these results for arbitrary physical collections.

`ExtremalFamilyTheorems.lean` proves the generalized family as an equivalence: failure holds exactly when the two blocks have disjoint repeated-colour supports. Four enlarged support patterns require only 300 Hall witnesses; two uniform allocations prove the converse. The 21-member family and the five representatives reuse this proof. The face tuples and orbit sizes are in `data/five-colour-extremal.json`; a separate Python audit checks all 202,575 labelled $(4, 5, 5)$ collections, recovering 840 failures in five orbits of sizes $120, 120, 120, 240, 240$. It also checks all 441 normalized family distributions (91 failures and 350 solutions). These finite formal proofs supplement the elementary main-text arguments; no interval certificates are added or changed.

Complete classification. The separate 349-root proof in Section 9.1 uses 713,099 interval instructions: 228,264 for 133 strictly projected exclusions, 446,427 for 196 identity projections, 14,012 for thirteen core covers and extension exclusions, and 24,396 for seven terminal enumerations. `CoreExtension.lean` retains the coordinatewise inequality from budget projection and composes complete core coverage with exclusions under all original upper bounds. The 710 smaller cores cannot extend to failures; they are distinct from the 81 classification terminals. `EnumerationCoverage.lean` assembles all 349 roots. `Classification56.lean` connects coverage, representative failures, symmetry transport and inequivalence to physical copies; `ClassificationSubmission.lean` transfers the result to the independent comparison model. The endpoint and its shared bridges are audited in `ClassificationAudit.lean`. A separate physical audit checks the three normal forms and the union of their 11,430 labelled inventories.

Universal collections. The universal lower bound counts three sources for each repeated colour. Small kernel-checked geometry relates the bicoloured types to physical sources; the upper bound checks 75 explicit matchings and transports them to arbitrary target orientations. The minimum-collection structure in Corollary 40 is proved in `UniversalStructure.lean`: the physical supplier argument reduces to a kernel-checked lemma about three pairs on four vertices, followed by the graph-theoretic tree criterion. These proofs do not use the interval certificates.

A.4 Reproduction

The proof artifact is the repository at Git commit

`c4bd4efc20472049b7492589d8c86b573b32e0a0`.

The pinned source tree contains the mathematical sources and certificate inputs used here. The verification extracts accompanying this manuscript identify the frozen working-tree snapshot checked before that commit. The threshold archive, `q5-projected-certificates.tar.gz`, has SHA-256

`7df56cc4b252e460c9e3a762b068010b1a894405ae105218f6cea545e9b80467`.

The classification also uses three archives, whose exact hashes are retained in the verification manifests:

```
data/five-colour-classification-certificates.tar.gz
data/five-colour-classification-projections.tar.gz
data/five-colour-classification-cores.tar.gz
```

The Lean proof sources and certificate inputs in this manuscript's repository are byte-identical to those in the pinned commit.

With the pinned toolchain installed, check out that revision and run:

```
git clone https://github.com/jasondavies/five-colour-cubes
cd five-colour-cubes
git checkout c4bd4efc20472049b7492589d8c86b573b32e0a0
lake exe cache get
make check-classification-data WORKERS=2
make classification-replay WORKERS=2
make verify-full WORKERS=2
make comparator
```

The classification commands check the native data and generated sources, then build all 349 root proofs and audit the physical endpoint. `verify-full` checks the threshold audits, all 279 threshold certificate proofs, unconditional extensions and their axiom reports; building `Solution.lean` also includes the complete classification. `comparator` separately compares all

sixteen statements and rechecks the exported dependency closure with both external kernels. The latter requires `bwrap` and the pinned toolchain's `leanexport`, `leanchecker` and `nanoda_bin` executables. The pinned README and `FiveColour/README.md` give prerequisites and tuning. The shorter `make check` omits the complete certificate replay. For verification without Lean, run `make check-code WORKERS=2` and `make check-classification-data WORKERS=2`. These check the native certificates, independent audits and generated sources; they do not replace kernel checking.

A fresh checkout must build the proof after obtaining the dependency cache. Cached reruns validate Lake dependency traces; the external comparison checks the complete export afresh. Audit declarations are listed in `Audit.lean`, `ExtensionsAudit.lean`, `ClassificationAudit.lean` and the final threshold replay audit. Only `propext`, `Classical.choice` and `Quot.sound` occur; no `native_decide` is used.

The verification record checks every installed dependency's revision and tracked bytes against its pinned Git tree before and after the run. It also records source hashes, tool versions, compiler flags and concurrency settings. Ignored build caches are permitted; modified dependency sources are rejected.

All sixteen statements, shared definitions and permitted axioms passed local comparison on 9 November 2026, and both NanoDa and Lean's standalone kernel accepted the complete solution export. The run took 10 hours 32 minutes 48 seconds, using validated Lake artifacts and excluding a fresh project build. The preceding classification and full-project runs used validated caches; the external kernels checked the export afresh. These are observed execution times, not a controlled performance comparison. No complete cold-build time, aggregate peak memory or disk requirement has been measured. The retained resource report covers the outer comparison driver, not all sandboxed subprocesses. The public provenance extract `verification/optimized-comparator.json` identifies the measured source snapshot by the SHA-256 digest of its manifest of file hashes and modes, rather than a Git commit:

`940980304874bf067ea978c4a2251381be6cb0fea2cf6ffa0fca7b97303c3f2c`.

It also records dependencies, tool identities and settings. Public extracts omit local paths and process metadata while preserving the original records' checksums. The same directory retains comparison output, resource measurements and the preceding successful verification extracts, including `verification/classification.json` and its eight axiom reports. The source manifest matches the Lean proof sources and certificate inputs in this manuscript's repository.

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