hexagon:2611.00233v3

Prime factors of fusion-category dimensions: weak group-theoreticity, rank and decomposition

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Abstract

Let D be the Frobenius–Perron dimension of a weakly integral fusion category and $Q(D)$ its square-full part. For a divisor-minimal braided obstruction to weak group-theoreticity, we bound the conductors of character fields by formal codegrees and relate residue blocks to primary twists through local Weyl operators. For primes $p < q < r < s$ these restrictions give weak group-theoreticity whenever $Q(D)$ divides $p^5q^ar^b$ or $p^3q^3r^as^b$, with a, b arbitrary. Fixing the cofactor of a prime power gives a finite test even when its exponent is unrestricted.

A braided category is weakly group-theoretical for $Q(D) < 29\,160\,000$ and solvable for odd $Q(D) < 6\,087\,156\,075$. The corresponding bounds without a braiding are $D < 5400$ and odd $D < 121\,275$. Combining the dimension families with an exhaustive degree search gives rank at least 32 for a non-pointed simple integral modular category, at least 33 in nonsquare dimension, 57 in odd dimension and 75 in odd nonsquare dimension. We also obtain rank bounds without simplicity and prime-divisor criteria for a braided product decomposition.

1 Introduction

Let $\mathcal{C}$ be a fusion category over $\mathbb{C}$ and let $D = \mathrm{FPdim}(\mathcal{C})$. It is *weakly integral* if D is an integer, and *weakly group-theoretical* (WGT) if it is Morita equivalent to a nilpotent fusion category. Etingof, Nikshych and Ostrik asked whether every weakly integral fusion category is WGT [32, Section 11, Question 2]. The question is equivalent to its braided version. The Drinfeld center has dimension D^2, and weak group-theoreticity descends from the center to $\mathcal{C}$ [32, Proposition 4.1].

Prime exponents and dimension. Call a prime *repeated* if its exponent in D is at least two, and write $Q(D) = \prod_{v_p(D)\geq 2} p^{v_p(D)}$ for the square-full part. Natale proved weak group-theoreticity for nondegenerate braided categories with at most two repeated primes, and obtained weak group-theoreticity for $D < 1800$ and solvability for odd $D < 33075$ [42, Theorems 7.4, 8.1 and 8.2]. Dong extended these results to arbitrary braided categories and obtained weak group-theoreticity for $D < 120$ without braiding [22, Theorems 1.1–1.2 and Corollaries 3.2–3.3]. We bound selected exponents and allow the others to vary independently. Table 3 indexes these families. Table 6 gives their precise hypotheses. A factor d appended to a displayed prime factorization denotes a coprime square-free integer. We call N *covered* if every weakly integral braided category of dimension dividing N is WGT.

Theorem 1.1 (Two unrestricted prime powers)**.** *For primes $p < q < r < s$ and arbitrary nonnegative integers a, b, the following hold.*

(i) *Every braided category of dimension dividing $p^3q^3r^as^bd$ is WGT.*

(ii) *Every fusion category of dimension dividing pqr^as^b is solvable.*

With three repeated primes, $p^5q^ar^bd$ is covered for $p < q < r$ (Theorem 8.3). For any number of primes, fixing the complementary factor m reduces the question for divisors of $\ell^e m$, with $\ell \nmid m$ prime and $e \geq 0$, to finitely many dimensions (Corollary 16.7). Comparing the exponent bounds gives Table 1. In the braided solvability bound, $Q(D)$ is odd precisely when $4 \nmid D$.

Table 1: Dimension bounds for weakly integral fusion categories (Theorem 17.1).

Braiding	Dimension hypothesis	Conclusion
Braided	$Q(D) < 29\,160\,000$	WGT
Braided	$Q(D)$ odd, $Q(D) < 6\,087\,156\,075$	Solvable
Arbitrary	$D < 5400$	WGT
Arbitrary	D odd, $D < 121\,275$	Solvable

Rank. How many simple objects would a non-pointed simple integral modular category need? Simplicity forces its pointed subcategory to be trivial, so its unit is its only invertible simple. Such a category cannot be WGT (Corollary 17.3). If its degrees are d_i, modularity gives $d_i^2 \mid D$ and $\sum_i d_i^2 = D$. At fixed rank, these identities yield a finite search. Dimension-family exclusions prune this search before the more expensive character and local twist tests. Table 2 records the resulting bounds alongside those for categories with weaker hypotheses.

Table 2: Lower bounds on rank. Each column specifies the parity of the Frobenius–Perron dimension.

Category	Any parity	Odd
Non-pointed simple integral modular	32	57
with nonsquare dimension	33	75
Weakly integral braided, non-WGT	9	41
Weakly integral, non-WGT, without a braiding assumption	7	13

The first two rows follow from Theorem 22.2. The arbitrary-center bounds use a different argument (Theorem 23.1). Indeed, a central de-equivariantization can *increase* rank. For an odd prime $p \geq 5$, de-equivariantizing $\mathrm{Rep}(D_{2p})$ by the quotient $D_{2p} \to C_2$ gives $\mathrm{Rep}(C_p)$, of rank p, while the original category has rank $(p+3)/2$. Under the distinct hypothesis of minimality among divisors of D, a slightly degenerate obstruction has full rank at least 34 (Theorem 22.1). Its character table is indexed by pairs of simples interchanged by the transparent fermion. For strictly weakly integral non-WGT categories the lower bounds are 8 without braiding and 11 with braiding.

Perfection alone also gives the bounds 57 and 75 for nontrivial odd-dimensional braided categories, with the latter bound under the additional nonsquare hypothesis (Corollary 23.5). Alekseyev, Bruns, Palcoux and Petrov excluded nontrivial perfect integral modular categories through rank 13, and through rank 24 in odd dimension [2, Theorems 1.5 and 12.2]. Dong and Palcoux raised the odd-dimensional lower bound to 27 [24, Corollary 1.3]. The Anyon Atlas constructs modular data numerically through rank 22 [5, Section 3]. The degree enumeration in this paper uses the rank as numerical input and does not require a prior classification list. The general search excludes every rank through 31, including square dimension.

Boundary dimensions and decomposition. The first square-full dimension left by our braided comparison is $29\,160\,000 = 2^6 3^6 5^4 = 5400^2$. The slightly degenerate case is excluded throughout the family with exponent caps $6, 6, 4$. At this first dimension a modular obstruction has rank between 101 and 46 774 (Theorem 29.1). Without braiding, a simple perfect non-WGT category of dimension 5400 has rank between 11 and 462 if its fusion ring is commutative, and between 13 and 490 otherwise (Theorem 30.1). At the odd dimension $5^6 7^6 23^4$, combining the forced twist orders with their multiplicities gives rank at least 553 for any non-WGT braided category (Corollary 27.13). A different application concerns the factorization of integral modular categories. For composite square-free L, Section 31 characterizes the existence of a prime integral WGT modular category of dimension L^2 by a signed divisibility graph. For $L = 5 \cdot 7 \cdot 29$ the two possible edges require opposite signs at 29, since $5 \mid 29 + 1$ and $7 \mid 29 - 1$. Every integral modular category of this dimension therefore splits.

How the proofs fit together. The categorical starting points are the reduction theorems of Etingof–Nikshych–Ostrik and Natale, the congruence representation and the generalized indicators of Ng–Schauenburg. Our arithmetic input is the conductor bound for a character row (Theorem 2.9), coupled with block orthogonality and the integrality of complementary Weyl columns (Sections 3 and 6). For four repeated primes the isolated-singleton theorem fixes local multiplicities and degrees. For an arbitrary number of primes the fixed-cofactor theorem reduces an unbounded exponent to finitely many tests (Sections 10 and 16). The numerical bounds follow by a comparison of the uncovered exponent patterns (Theorem 17.1).

For rank, start with the degree equation in Section 18, then read the twist and character restrictions in Sections 19–21 before the exhaustive searches in Section 22. General non-WGT categories require a separate descent (Section 23). The refinements for three primes and the restrictions at the first open dimensions have their own proofs in Sections 24–30. Section 31 derives decomposition criteria from gaugings.

Two self-contained companions isolate the numerical conclusions: the 29-page dimension proof gives the bounds 5400^2 and 5400 without computation. The 17-page rank proof gives $r \geq 32$ using the same degree search and character tests. Here the dimension families provide additional pruning. The verification guide locates the calculations in each proof.

Index of the families

Table 3 lists the generating families. Products in the first five rows describe the square-full part of a braided dimension. Capital exponents are free wherever the cited result imposes no bound. Table 6, after the prime-family proofs, gives the precise order, residue and exponent conditions, including the distinction between a categorical dimension and a reduced table dimension.

Table 3: Index of infinite families and their conclusions.

Prime support	Generating forms or operation	Where the conditions enter
Two or three repeated primes	$p^A q^B$; $p^A q^B r^C$ with one small exponent, bounded triples, quartic and quintic strips, or a sextic at a smaller prime	Sections 4 and 7–8; refinements in Section 25
Four repeated primes	$P^A(uvw)^3$, $(uv)^3 P^A Q^B$, $P^a Q^B u^3 v^2$ and $P^A Q^B (uv)^2$, including height, residue and dyadic tests	Sections 9–14
Five repeated primes	$p^4(uvw)^3 t^2$, $(puvwt)^3$ and the exact minimal-obstruction restriction at three specified valuations	Section 15
Lifts and more primes	Raising designated squares to cubes; adjoining r^B to a covered N; $P^A R^B \prod t_i^{c_i}$; $p^A m$ at fixed cofactor	Sections 11, 15 and 16
Recognition and boundaries	$p^a q^b r^4$ with a principal-block degree condition; minimal obstructions with three exponents bounded by $6, 6, 4$	Theorem 8.39; Section 25.8
Without braiding	Covered D^2, ordered $uvp^A q^B$, $p^A q^B r$ and square-free D on at most five primes	Theorems 1.1 and 4.16; Corollary 15.4
Rank with prescribed dimension	$D = qM^2$ by parity, square class, prime support or largest degree; $qU^2 p^{2e}$; two isolated primes; a prime denominator; three primes with $q = 10$	Sections 19–22; Corollary 9.4
Decomposition and solvability	Prime components, signed graphs at L, L^2 (L square-free), modular and slightly degenerate solvability tests	Section 31; Proposition 17.2

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2 Reduction to a simple character table

Use the positive spherical structure of a weakly integral category over $\mathbb{C}$, so $D = \mathrm{FPdim}(\mathcal{C})$ and $d_X = \mathrm{FPdim}(X)$ are its categorical dimensions [30, Propositions 8.23 and 8.24]. Write $\mathcal{C}_{\mathrm{pt}}, \mathcal{C}_{\mathrm{ad}}, \mathcal{C}', U(\mathcal{C})$ for its pointed subcategory, adjoint subcategory, Müger center and universal grading group. We call a category *perfect* if its only invertible simple is the unit, and *fusion-simple* if its only fusion subcategories are trivial or the whole category. The corresponding ring conditions concern invertible basis elements and based subrings.

Lemma 2.1 (Minimal counterexample)**.** *Suppose every braided category of dimension a proper divisor of D is WGT. A counterexample $\mathcal{C}$ at D is integral, equals $\mathcal{C}_{\mathrm{ad}}$, and has one of two forms:*

(i) *$\mathcal{C}$ is modular, perfect and fusion-simple;*

(ii) *$\mathcal{C}' = \mathcal{C}_{\mathrm{pt}} = \mathrm{sVec}$, and the only fusion subcategories are $\mathrm{Vec}, \mathrm{sVec}, \mathcal{C}$. The ring modulo the transparent fermion is perfect and simple.*

Proof. A nontrivial universal grading reduces the question to its neutral component. A proper nontrivial nondegenerate subcategory splits off with its centralizer. A nontrivial Tannakian $\mathrm{Rep}(G)$ reduces to the neutral de-equivariantization, of dimension $D/(|G||H|)$ for grading support $H \leq G$. In each case the smaller dimensions divide D and WGT passes back by extension, equivariantization or tensor product [30, Propositions 8.15 and 8.20], [32, Proposition 4.1], [26, Theorems 3.13(i) and 4.44, Proposition 4.26], [42, Proposition 4.1]. Thus $\mathcal{C} = \mathcal{C}_{\mathrm{ad}}$ is integral by the dimension grading [36, Theorem 3.10], and its symmetric center is Vec or sVec. Every proper subcategory is integral and WGT, hence pointed in the absence of Tannakian subcategories [43, Theorem 3.1]. It thus lies in $\mathcal{C}_{\mathrm{pt}}$, which is trivial in the modular case by $(\mathcal{C}_{\mathrm{ad}})' = \mathcal{C}_{\mathrm{pt}}$, and is sVec in the slightly degenerate case by [42, Lemma 7.1]. In the latter case pull back based subrings from the fermion quotient. □

We use *ordinary modular table* for the character table of an integral modular category, and *positive reduced table* for the table of an integral slightly degenerate category after identifying its transparent fermion with the unit. A *minimal table* is one of the two categorical tables in Lemma 2.1, with the simplicity and perfectness conditions stated there. In particular, its twists, traces and indicators come from the category. Statements formulated for based rings or symmetric integral character data alone use the algebraic hypotheses specified in those statements.

Put $\Delta = D$ in case (i) and $\Delta = D/2$ in case (ii). In (ii) use the fusion ring $R = K_0(\mathcal{C})/(f - \mathbf{1})$ for the transparent fermion f. The twist gives $fX \not\cong X, X^*$. Duality-compatible orbit representatives containing $\mathbf{1}$ give structure constants $N_{XY}^Z + N_{XY}^{fZ}$ and unit coefficient δ_{Y,X^*}. These two rings are instances of the following definition.

Definition 2.2 (Symmetric integral character data)**.** Let R be a commutative fusion ring with basis $[X]$, duality involution and unit 0. Its trace τ is the coefficient of the unit; $\tau([X][Y]^*) = \delta_{XY}$ makes $a \mapsto \tau(aa^*)$ positive definite when $*$ is extended conjugate-linearly to the complex algebra. Fix a complete bijection from basis labels Y to the irreducible characters χ_Y and require $\chi_0(X) = d_X \in \mathbb{Z}_{>0}$. Put $\Delta = \sum_X d_X^2$. We require that all $\chi_Y(X)$ lie in the integer ring of a common cyclotomic field, that $S_{XY} = d_Y \chi_Y(X)$ is symmetric, and that

$$\begin{aligned} S_{\mathbf{1}X} = d_X, \quad S^\dagger S = \Delta I, \quad S_{X,Y^*} = \overline{S_{XY}}, \\ \chi_Y(X) = S_{XY}/d_Y, \quad f_Y = \|\chi_Y\|^2 = \Delta/d_Y^2 \in \mathbb{Z}. \end{aligned} \tag{2.1}$$

The involution and orthogonality above give the coefficient trace

$$\tau(B) = \sum_Y \chi_Y(B)/f_Y. \tag{2.2}$$

Galois conjugation acts on the complete set of characters. Since it commutes with duality and fixes the rational f_Y, the induced permutation π_σ preserves d_Y. Symmetry therefore gives the unsigned covariance

$$\sigma S_{XW} = S_{\pi_\sigma X,W} = S_{X,\pi_\sigma W}. \tag{2.3}$$

We use $I_s = \{X : d_X = s\}$ and $n_s = |I_s|$. A twist, a congruence representation, and a categorical indicator are additional inputs in the rows of Table 4 where they occur. For the modular table these identities are standard [29, Sections 8.13–8.14]. For the reduced table, the normalized columns give all characters by [54, Proposition 3.3] (see [7, Proposition 2.7] for the unitary case). The ratio Δ/d_Y^2 is an algebraic integer by [54, Corollary 3.4], hence an integer here.

Table 4: Hypotheses used in passing from character data to dimension and rank bounds.

Input	Consequence and use
Fusion ring with cyclotomic character values	A rational codegree f bounds the different and gives conductor dividing $2f$ (Theorem 2.7, Corollary 2.8).
Such a ring with symmetric integral character data	Unsigned Galois covariance in both indices and character orthogonality sharpen the conductor bound to f (Theorem 2.9).
Integral braided category	The LCM energy estimate of Proposition 2.3 includes the contribution of the symmetric center; the modular case uses the estimate of Dong and Palcoux [24].
Modular category or category with transparent fermion	Rotations and congruence give principal-block orthogonality (Theorem 3.2); at two the positive reduced table uses squared twists.
Integral modular category	The congruence action connects character fields to joint twist components (Theorem 24.8). At the first open dimension, its orbit and phase restrictions enter the rank bound (Theorem 29.1).

Proposition 2.3 (LCM energy and prime companions)**.** *Let $\mathcal{C}$ be integral braided, with its positive ribbon structure, and let S be its unnormalized matrix. For X simple set $h_X = \sum_{T\in\mathrm{Irr}(\mathcal{C}')} N_{XX^*}^T d_T$. Then*

$$\sum_{Y:S_{XY}\neq 0} \mathrm{lcm}(d_X, d_Y)^2 \le D h_X.$$

If X centralizes $\mathcal{C}_{\mathrm{pt}}$, is not transparent, and $p \mid d_X$ while $p \nmid \mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}})$, there is a noninvertible Y with $p \nmid d_Y$, $S_{XY} \neq 0$, and $\mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}})d_X^2 + \mathrm{lcm}(d_X, d_Y)^2 \le D h_X$.

Proof. The Hopf-link characters $a_Y(X) = S_{XY}/d_Y$ are ring characters [29, Proposition 8.13.11]. Bézout applied to $a_Y(X)$ and $a_X(Y)$ shows $S_{XY}/\mathrm{lcm}(d_X, d_Y)$ is integral, as in the modular criterion of Dong–Palcoux [24, Lemma 2.7 and the proof of Theorem 1.1]. The regular element $R = \sum_Y d_Y[Y]$ satisfies $[T]R = d_T R$. Accordingly $a_T(R)$ is D when $a_T = d$ and zero otherwise. In the positive structure $a_T = d$ means precisely $T \in \mathcal{C}'$ by the double-braiding criterion [29, Proposition 8.20.5(i)]. Symmetry and $a_Y(X^*) = \overline{a_Y(X)}$ give

$$\sum_Y |S_{XY}|^2 = \sum_Y d_Y^2 a_Y(XX^*) = \sum_T N_{XX^*}^T d_T a_T(R) = D h_X.$$

Average this rational equality over a cyclotomic splitting field. Every nonzero integral normalized entry has mean squared modulus at least one by its norm. This proves the first assertion. For the second, $a_X(R) = 0$ and the invertibles contribute $\mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}})$. Reduction at a prime above p forces a noninvertible p-free hit; retain it and all invertibles in the first inequality. □

In a modular category $h_X = 1$. In a slightly degenerate category the same holds for the full table: $gX \not\cong X$ excludes the transparent fermion from XX^*. Passage to the reduced table halves the full energy sum and the number of invertible columns.

Lemma 2.4 (Averaging over a character orbit)**.** *In ordinary or positive reduced symmetric integral data, let H be a subgroup of the cyclotomic Galois group, $e = |H \cdot Z|$ and $A_H(x) = |H|^{-1} \sum_{\sigma \in H} \sigma(x)$. Then*

$$f_Z/e = \sum_X |A_H(\chi_Z(X))|^2, \qquad 0 = \sum_X d_X A_H(\chi_Z(X)) \quad (Z \neq \mathbf{1}).$$

For the full Galois group write $K_X = \mathbb{Q}(\chi_X)$, $F_X = \mathbb{Q}(\chi_Z(X))$ and $T_X = \mathrm{Tr}_{F_X/\mathbb{Q}}(\chi_Z(X))$. Grouping columns into orbits gives

$$f_Z = \sum_{[X]} [K_Z : F_X][K_X : F_X] T_X^2, \tag{2.4}$$

$$0 = \sum_{[X]} d_X [K_X : F_X] T_X \qquad (Z \neq \mathbf{1}). \tag{2.5}$$

For any H, let A be a set of nonunits whose entries $z_X = S_{XZ}/\operatorname{lcm}(d_X, d_Z)$ are fixed by H. With $\mathsf{M}(z)$ the mean of $|\sigma z|^2$ over all conjugates, we have

$$e\left(d_Z^2 + \sum_{X \in A} \operatorname{lcm}(d_X, d_Z)^2 \mathsf{M}(z_X)\right) \leq \Delta.$$

Proof. The e conjugates of the row χ_Z are orthogonal with squared norm f_Z, and their average at X is $A_H(\chi_Z(X))$ by (2.3). Averaging preserves orthogonality to the dimension character. For the full group, the value at each of the $[K_X : \mathbb{Q}]$ conjugates of X is $T_X/[F_X : \mathbb{Q}]$. For the inequality multiply the norm identity by d_Z^2, retain the vacuum and A, then take the full Galois mean. The Galois group is abelian, so the conjugates of these entries are also fixed by H. □

Write GZ, I_pZ, $I_p^{\mathrm{tame}}Z$ for the full, inertia and chosen tame inertia orbits in the splitting field; at odd p the tame subgroup is the Teichmüller subgroup of the primary factor. Conjugates of an entry z are indexed by $\mathrm{Gal}(\mathbb{Q}(z)/\mathbb{Q})$, independently of the orbit of its row label. Thus A_H is a normalized trace, T_X an unnormalized field trace, and M a mean squared modulus. In particular, for pairwise disjoint nonvacuum sets A_i of H-fixed entries and lower bounds $c_i \leq \operatorname{lcm}(d_X, d_Z)^2 \mathsf{M}(z_X)$ for $X \in A_i$, the last inequality reads

$$|HZ|\left(d_Z^2 + \sum_i |A_i| c_i\right) \leq \Delta. \tag{2.6}$$

This applies to subsets of columns, even when several columns have the same entry. Two choices of H will recur. For one nonunit X, take H fixing $L = \mathbb{Q}(z_X)$ and A equal to the full Galois orbit of X. Then $e = [K_Z : L]$ and the sum is $[K_X : \mathbb{Q}] \operatorname{lcm}(d_X, d_Z)^2 \mathsf{M}(z_X)$. For inertia at s, all columns with K_X unramified at s have fixed entries. If Z is principal at $t \neq s$ and $t \nmid d_X$, then $\chi_Z(X) \equiv d_X$ above t gives $z_X \neq 0$, hence $\mathsf{M}(z_X) \geq 1$ by the algebraic norm bound. This applies simultaneously to any set of such columns.

Lemma 2.5 (Zeros and square-fullness)**.** *In perfect symmetric integral data, every nonunit row contains a zero and Δ is square-full. For $s = d_X, t = d_Y$, $S_{XY}/\operatorname{lcm}(s,t)$ is an algebraic integer. In a simple perfect table $S_{XY} = 0$ whenever X, Y are nonunit and $(s,t) = 1$; in particular no nonunit dimension is a prime power. If Δ is odd then duality has no nonunit fixed point.*

Proof. Apply the Bézout and Galois-mean bounds of Proposition 2.3 to either character table. If a nonunit row of dimension s had no zero, averaging its norm would give

$$\Delta/s^2 \;\geq\; \sum_Y (d_Y/\gcd(s, d_Y))^2 \;>\; \sum_Y d_Y^2/s^2 = \Delta/s^2;$$

the strict inequality holds already at $\mathbf{1}$. For ordinary modular categories the row-vanishing theorem is due to Gelaki, Naidu and Nikshych [35, Theorem 6.1]. Square-fullness in the perfect integral modular case is [14, Corollary 10.10], and the coprime-degree and prime-power obstructions in the integral modular case follow from Etingof, Nikshych and Ostrik [32, Lemma 7.1 and Corollary 7.2]. For the fusion-ring vanishing theorem see [14, Theorem 1.2]. Now $P = \prod_X XX^*$ vanishes on every non-Perron character and hence, by (2.2), $\tau(P) = \prod_X d_X^2/\Delta$ is an integer. Since $d_X^2 \mid \Delta$, no prime can occur only once in Δ.

For a non-Perron character, the labels at which $|\chi(X)| = d_X$ form a based subring: equality in the triangle inequality for a product propagates to each constituent, and the set is closed under duality. Simplicity therefore makes the inequality strict off the unit, since equality everywhere would force codegree Δ. For coprime $s, t > 1$, all conjugates of $S_{XY}/(st)$ have modulus strictly below one; its norm forces it to be zero. On the other hand a nonunit row satisfies

$$1 + \sum_{Y \neq \mathbf{1}} d_Y \chi_X(Y) = 0. \tag{2.7}$$

Modulo any prime dividing Δ, the row must meet a nonunit label of degree prime to that prime. A prime-power row cannot do so.

In odd dimension all $1/f_X$ reduce to one at two; hence (2.2) gives $\tau(B) \equiv \mathrm{Tr}(N_B) \pmod 2$ for $B \in R$. Apply this congruence to $B = X^2$ and then to $B = X$, using $\mathrm{Tr}(N_X^2) \equiv \mathrm{Tr}(N_X) \pmod 2$. It gives $\delta_{X,X^*} \equiv \delta_{X,\mathbf{1}} \pmod 2$. □

Lemma 2.6 (Small ordinary codegrees)**.** *In a nontrivial simple perfect integral modular category every formal codegree is at least five.*

Proof. Use the unnormalized modular matrix S with $S_{0Y} = d_Y$. For a nonunit X put $n = d_X$, $f = D/n^2$ and $u_Y = S_{XY}/n$. Write $H = \{Y \neq \mathbf{1} : u_Y \neq 0\}$. Orthogonality gives $\sum_{Y \in H} d_Y u_Y = -1$ and $\sum_{Y \in H} |u_Y|^2 = f - 1$. Thus the degrees in H have gcd one. The $(X, 0)$ entry of the modular relation $STS = p^+ T^{-1} S T^{-1}$, where $|p^+| = \sqrt{D}$, gives

$$\sqrt{f}\, n \leq 1 + \sum_{Y \in H} d_Y |u_Y|.$$

Symmetry and algebraic integrality of the character values give $u_Y \in (d_Y / \gcd(d_Y, n))\mathcal{O}$, where $\mathcal{O}$ denotes the algebraic integers. Averaging the squared-norm identity over the cyclotomic Galois group, each nonzero term contributes at least $(d_Y / \gcd(d_Y, n))^2$, by the arithmetic–geometric mean inequality applied to its conjugates. If $f \leq 4$, we conclude that $|H| \leq f - 1$ and every d_Y in H divides n.

Perfection and the gcd condition require at least two terms. If there are two, both degrees are proper divisors of n. Cauchy's inequality then gives $\sqrt{f}\, n \leq 1 + (n/2)\sqrt{2(f-1)}$, impossible for $f = 3, 4$ and $n \geq 2$. For three terms we have $f = 4$, and all three entries are roots of unity: each has mean square one, so all its conjugates have modulus one. At most one of the degrees can be n, by the gcd condition. If one is n, the others cannot both equal $n/2$ since $n/2 > 1$. Ordering the proper divisors therefore gives $2n \leq 1 + 11n/6$. This forces $n \leq 6$. A nonunit degree in our category has at least two distinct prime factors by Lemma 2.5; hence $n = 6$, and its only divisor admissible as a nonunit degree is 6, contradicting the gcd condition. The vacuum codegree is D and is larger than four, since such a category contains a nonunit of degree at least six. □

2.1 Fields and residue classes of characters

For an irreducible representation E of a fusion ring R, put $K = \mathbb{Q}(\chi_E(X) : X)$ and let e_E be its central primitive idempotent. The formal codegree f_E is the Schur element of τ [16, Proposition 1.15]. Thus $\dim(E) f_E = \sum_X |\chi_E(X)|^2$ and [48, Lemma 2.3]

$$\sum_X \chi_E(X^*)[X] = f_E e_E. \tag{2.8}$$

For a categorified ring K is cyclotomic in the sense of lying in a cyclotomic field [30, Theorem 8.51]; the reduced characters also lift.

Theorem 2.7 (The different divides a rational formal codegree)**.** *For an irreducible representation of a categorified fusion ring, or of a fusion ring with cyclotomic character values, a rational formal codegree f is a positive integer. The different of its character field satisfies*

$$\mathfrak{D}_K \mid f\mathcal{O}_K. \tag{2.9}$$

Only primes dividing f ramify.

Proof. For $a \in \mathfrak{D}_K^{-1}$ the element $\sum_X \mathrm{Tr}_{K/\mathbb{Q}}(a\chi_E(X^*))[X]$ belongs to R. By (2.8) it equals $f\sum_{\sigma\in\mathrm{Gal}(K/\mathbb{Q})}\sigma(a)e_{\sigma E}$, with distinct conjugate characters. Its eigenvalue af is integral. Thus $f\mathfrak{D}_K^{-1} \subseteq \mathcal{O}_K$, and $a = 1$ gives the integrality of f.

□

Corollary 2.8 (Conductor of a character field)**.** *If a cyclotomic character field K has rational formal codegree f, its conductor divides $2f$.*

Proof. Work after a finite unramified extension of $\mathbb{Q}_p$, so the ramified part E of K lies in the totally ramified cyclotomic extension L of minimal conductor p^m. Let $e = [E : F]$, where F is the unramified base, and let d_E be the different exponent at E. Theorem 2.7 gives $d_E/e \le v_p(f)$. The derivative of Φ_{p^m} gives $d_L/[L : F] = m - 1/(p - 1)$. For odd p and $e = e_0p^s > 1$, $e_0 \mid p - 1$, cyclic wild inertia gives $m = s + 1$ and L/E is tame. Thus $(d_E + 1)/e = (d_L + 1)/[L : F]$, or

$$\delta_p(e) := d_E/e = s + 1 - \frac{1 - p^{-s}}{p - 1} - \frac{1}{e}. \tag{2.10}$$

This also gives zero at $e = 1$. For $s = 0$, $e > 1$ it is positive, and for $s \ge 1$ it lies strictly between s and $s + 1$. Hence $p^m \mid f$. At two, conductor four gives normalized different one. For $m \ge 3$, minimality excludes $1 + 2^{m-1}$ from $\mathrm{Gal}(L/E)$. Every nontrivial subgroup of $\langle 5\rangle$ contains that element, so $[L : E] \le 2$. The possible quadratic generators are -1 and $2^{m-1} - 1$. For either generator σ, a primitive 2^mth root ζ satisfies $v_L(\zeta - \sigma\zeta) = 2$; this is the relative different exponent. Transitivity therefore gives $d_E/e \ge m - 1 - 2/2^{m-1} > m - 2$, whence $2^{m-1} \mid f$. □

By symmetry the field of $S_{XY}/\operatorname{lcm}(d_X, d_Y)$ lies in both character fields; its different divides both codegrees by transitivity, and the next theorem sharpens its conductor bound. In particular, the field of a character of p-power codegree lies in a p-power cyclotomic field by Kronecker–Weber and the conductor bound (for odd p, adjoining a factor of two to an odd level does not change the field). Total ramification there gives a unique prime over p in the character field [52, Lemma 3.1.13 and Theorem 7.5.5].

Theorem 2.9 (Conductor at an individual label)**.** *In symmetric integral character data (Definition 2.2), write K_Z for the field generated by the character χ_Z. Then the conductor of K_Z divides f_Z. In particular the conductor of the lcm-normalized XY crossing divides $\gcd(f_X, f_Y)$. In ordinary data, if the dyadic part of a twist has order 2^v with $v \ge 4$, then $2^{v-1} \mid f_X$ for its label X.*

Proof. The different bound (Corollary 2.8) already puts K_Z in a cyclotomic field of conductor dividing $2f_Z$. It remains to remove its last possible dyadic level. Put $m = 2^{v_2(f_Z)}$. There is nothing to check for $m = 1$. Take $L = F(\zeta)$ with F an odd cyclotomic field large enough and ζ primitive of order $2m$. Suppose the dyadic conductor is $2m$. In $G = (\mathbb{Z}/2m)^\times$ the stabilizer H of the row does not contain $e = 1 + m$, since the fixed field of e is $F(\zeta^2)$.

Write $b_Y = \chi_Z(Y) = \sum_{k=0}^{m-1} z_{kY}\zeta^k$. The coefficients belong to $\mathcal{O}_F$ by coprimality of conductors. The orthogonality of the m odd powers of ζ gives

$$z_{1Y} = m^{-1} \sum_{g \in G} \zeta^{-g} \chi_{gZ}(Y).$$

Different rows in this sum are orthogonal; the squared norm of each is f_Z. Thus

$$\sum_Y |z_{1Y}|^2 = \frac{f_Z}{m^2} \sum_{h \in H} \sum_{g \in G} \zeta^{g(h-1)} = \frac{f_Z}{m}.$$

Indeed the inner geometric sum vanishes unless $h = 1$ or $h = e$, and only 1 belongs to H. Lift e to the common splitting field, trivially on F. Its permutation of the argument labels sends z_{1Y} to $-z_{1Y}$. Every cycle on the nonzero coefficients therefore has even length and contributes to $2\mathcal{O}$ in the squared norm. This contradicts oddness of f_Z/m. Symmetry gives the crossing assertion. For the twist assertion, Theorem 3.6 puts $\mathbb{Q}(\zeta_{2^{v-1}})^+$ in K_X by the square action whenever $v \geq 4$. Its conductor is 2^{v-1}; the normalization scalar has order at most eight, hence does not affect this conclusion. □

Local projectors. Fix a prime $\mathfrak{p}$ of a splitting field over p. The full residue classes (or p-blocks) group the characters of the commutative order R with equal reductions on every basis element. Over a complete unramified coefficient ring $\mathcal{O}$ containing the residues in its residue field, the special-fiber idempotents lift. Hence for a union L of full blocks, $e_L = \sum_{X \in L} e_X$ lies in $R \otimes \mathcal{O}$, and for locally integral B,

$$\tau(e_L B) = \sum_{X \in L} \chi_X(B)/f_X \in \mathcal{O}. \tag{2.11}$$

The character transform $\Phi(B) = (\chi_z(B))_z$ sends e_z to the coordinate vector v_z and intertwines multiplication by e_L with $P_L = \sum_{z \in L} v_z v_z^*$, the coordinate projection. We keep the algebra element e_L in trace identities and the operator P_L in matrix identities.

Lemma 2.10 (Principal residues at every prime)**.** *In the data of Definition 2.2, if Z is principal at one place over p, then $\chi_Z(X) \equiv d_X$ at every place over p for every X. The congruence may be divided by a rational p-unit. In particular, a locally integral algebraic number u which vanishes at all places above p belongs to $p\mathcal{O}_{K,(p)}$ if its field K is unramified at p.*

Proof. For any σ, $\sigma\chi_Z(X) = \chi_Z(\pi_\sigma X)$ and $d_{\pi_\sigma X} = d_X$, so the congruence survives transport to each conjugate place. The last assertion follows from $p\mathcal{O}_{K,(p)} = \prod_{\mathfrak{p}|p} \mathfrak{p}\mathcal{O}_{K,(p)}$. □

Theorem 2.11 (Support in the principal residue class)**.** *In symmetric integral character data, for any label set $\mathcal{A}$ put $L = \{X : \chi_X(W) \equiv d_W \pmod{\mathfrak{p}}$ for every $W \in \mathcal{A}\}$. If $f_Z = p^a$, the support of χ_Z lies in L; it equals L when $\mathcal{A} = I_{d_Z}$ and $p \nmid d_Z$. For any Z and any Galois-stable $\mathcal{A}$, $\sum_{X \in L} |\chi_Z(X)|^2$ is an integer divisible by $p^{v_p(f_Z)}$.*

Proof. Write α for the sum. Symmetry gives $\alpha/f_Z = \tau(e_L Z Z^*) \in \mathcal{O}$. If $\mathcal{A}$ is Galois-stable, so is L at the fixed prime $\mathfrak{p}$: by (2.3), permuting X permutes the tests on W and preserves d_W. Thus $\alpha \in \mathbb{Z}$ with the asserted divisibility. Apply this to the set of *all* labels. Its principal block contains the unit, so $1 \leq \alpha \leq f_Z = p^a$ forces $\alpha = f_Z$. The support lies in this block and hence in every L. Conversely for $\mathcal{A} = I_{d_Z}$, $X \in L$ gives $\chi_X(Z) \equiv d_Z$, nonzero if $p \nmid d_Z$. □

Blocks of defect one. After a sufficiently large finite unramified base change with fraction field F, integer ring $\mathcal{O}$ and residue field k, the field factors of a block correspond to its full inertia orbits and are totally ramified. An orbit is *tame* when wild inertia fixes it pointwise. The following observation explains why a single character of small defect can determine the entire block.

Lemma 2.12 (A block with a conductor of length one)**.** *Let A be a commutative local symmetric $\mathcal{O}$-order with normalization $M = \prod_i \mathcal{O}_i$, where all residue fields are k. Suppose its perfect trace on factor i is $\mathrm{Tr}_{E_i/F}/f_i$, and put $c_i = e_i v_p(f_i) - d_i$, where e_i, d_i are its ramification index and different exponent. A factor with $c_i = 0$ is the whole block and $A = \mathcal{O}_i$. If $c_i = 1$, there are precisely two factors, both with $c = 1$, and $A = \{(x, y) \in M : \bar{x} = \bar{y}\}$.*

Proof. Since A is self-dual, the conductor $(A : M)$ is the trace dual $M^{\#} = \prod_i f_i \mathfrak{D}_i^{-1}$. A unit conductor on one factor puts its idempotent in the local ring A. If $c_i = 1$, the conductor contains its maximal ideal $\mathfrak{m}_i$ on that factor alone. Writing B for the projection to the other factors, we have $A = \mathcal{O}_i \times_k B$: common residues are necessary, and the copy of $\mathfrak{m}_i$ lifts every compatible pair. There must be another factor, since otherwise $A = \mathcal{O}_i$ has conductor exponent zero. The perfect trace identifies A with $\mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$, so A is Gorenstein. The fiber-product criterion [41, Fact 2.6 and Corollary 2.7] makes B a discrete valuation ring. The conductor of this two-factor order is the product of the two maximal ideals. □

Proposition 2.13 (The block of a defect-one character)**.** *In a local symmetric order as above, with cyclotomic characters and rational integral codegrees, suppose a character has defect $v_p(f) = 1$. Except for a single ramified quadratic branch at two, the block has two inertia orbits and is the fiber product of their valuation rings over k. In particular no unramified higher-defect character belongs to this block.*

For odd p both orbits are tame and of defect one. At $p = 3$ their lengths in symmetric integral character data are one and two. At two, if the given branch is unramified and the other has degree e, the block has rank $1 + e$ and the other defect is $1 + v_2(e)$. Consequently uniform defect one at two forces rank two; in an even-rank dyadic block of rank greater than two all defects are at least two.

Proof. At odd p the given branch is tame by Corollary 2.8, so $e_i - d_i = 1$; an unramified defect-one branch at two has the same conductor length. By Lemma 2.12 the other branch satisfies $v_p(f) = (d_E + 1)/e$, which gives defect one if it is unramified. At odd p the right side for wild degree $p^s > 1$ lies strictly between s and $s + 1$ by (2.10); hence both branches are tame of defect one. At three their lengths are one or two. The integral identity trace $\sum e_i/f_i = \sum e_i d_{X_i}^2/\Delta$ forces the sum of the lengths to vanish modulo three, since after canceling the common powers of three each unit square is one there.

At two a ramified defect-one branch satisfies $2(e_i - 1) \le d_i \le e_i$ by the first two ramification groups and the different bound. Hence $e_i = d_i = 2$ and the conductor has length zero. In the two-branch case the identity trace $1/f_1 + e/f_2$ is integral; its first summand has valuation -1, so the other does too. Since e is a power of two, even rank $1 + e$ forces $e = 1$. A defect-zero branch is unramified with conductor length zero, hence has rank one by itself. □

3 Prime blocks and primary twists

Write $s = S/\sqrt{D}$, $T = \mathrm{diag}(\theta_i)$ and $N = \mathrm{ord}(T)$ for ordinary modular data. Let v_i denote a coordinate vector, e_i a character idempotent, and 0 the unit label. A twist decomposes uniquely into primary parts $\theta_{i,p}$; we call i *p-active* if $\theta_{i,p} \neq 1$.

For a weakly integral modular category use the lift $W = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \mapsto s$, $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \mapsto t = T/\gamma$, where $\gamma^3 = p^+/\sqrt{D}$. We may take γ of two-power order, equal to one for a center. The representation factors through $\mathrm{SL}_2(\mathbb{Z}/M)$, $M = \mathrm{ord}(t)$ [21, (1-10), Theorem II(i), Proposition 6.7(ii) and Theorem 6.10]. A subscript on W, U denotes its local factor under the Chinese remainder isomorphism.

For integral slightly degenerate $\mathcal{B}$ put $\Delta = \mathrm{FPdim}(\mathcal{B})/2$ and let g be its transparent fermion. With the positive spherical structure, $\mathcal{B}$ is pseudo-unitary, so the extension theorem of Johnson-Freyd and Reutter [37, Corollary 1.3] gives a minimal modular extension of dimension 4Δ,

with matrices s, T. The Weyl matrix interchanges $v_i \mapsto v_{gi}$ and $v_i \mapsto \epsilon_i v_i$, with ϵ_i the fermion-monodromy sign. On their common $+1$ space, the vectors $(v_i + v_{gi})/\sqrt{2}$ give the normalized reduced matrix $\widehat{s}$. Table 5 gives the full normalization. Even twist powers and odd local factors preserve this space. Their Bruhat and Galois identities therefore restrict, with the same local Galois sign since the two dimension normalizations differ by a rational factor.

Lemma 3.1 (Rotation with a central insertion)**.** *Let $\mathcal{C}$ be a weakly integral modular category with its positive spherical structure. For a simple object C of its center and $k \geq 1$ put $L_k^C = \nu_{k,1}^C$. Choose a cyclotomic integer ring $\mathcal{O}$ containing the coefficients being used. The additive functional L_k^C extends $\mathcal{O}$-linearly to $\mathcal{O} \otimes K_0(\mathcal{C})$, with values in algebraic integers. We extend scalars to the splitting field when evaluating L_k^C on an idempotent.*

(i) *If p is prime, $p \mid k$, and y belongs to this algebra, then at each prime above p,*

$$L_k^C(y)^p \equiv L_{k/p}^C(y^p).$$

(ii) *Suppose $\theta_C = 1$ and A is a genuine object. Put $m = N_{p'}$, choose $b \geq \max(1, v_p(N))$ with $p^b \equiv 1 \pmod m$, and let $n \geq 2b$. Then*

$$L_m^C(A^{p^n}) \equiv L_{mp^b}^C(A^{p^{n-b}}) \pmod{p^{n-2b+1}\overline{\mathbb{Z}}_p}. \tag{3.1}$$

For $C = (i, j^*)$ under $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ write $L_k^{ij} = L_k^C$. With $A_k = sT^k s^{-1}$ one has

$$L_k^{ij}(f_z e_z) = f_z (A_k)_{iz} \overline{(A_k)_{jz}}, \tag{3.2}$$

$$L_k^{ii}(e_z) = |(A_k)_{iz}|^2. \tag{3.3}$$

Proof. The kth power of the rotation is the scalar θ_C^{-1}, so its eigenvalues are roots of unity. Its trace is additive in the object by [47, Corollary 5.5], whose coprime-index hypothesis holds for $(k, 1)$. On a genuine object, reduction of the trace to the pth power gives the trace of the pth power of the rotation. Grouping p strands identifies the latter with $L_{k/p}^C(A^{\otimes p})$ by [47, Lemmas 2.5, 2.7 and Proposition 2.8]. The Frobenius map in the commutative residue fusion algebra now gives (i) for integral virtual coefficients.

For (ii) the two sides are traces of the powers p^n, p^{n-b} of the same rotation on mp^n strands. Its eigenvalues have order dividing mp^n because $\theta_C = 1$. All its power traces lie in $\mathbb{Q}(\zeta_N)$ by [47, Proposition 5.7] and [45, Theorem 7.7], so its eigenvalues have constant multiplicity on orbits over that field. The indicated powers agree on the prime-to-p part and on roots whose p-part has order at most p^{n-b}. For a root with p-part of order $p^t > p^{n-b}$, use the subgroup acting trivially on μ_m and by units 1 mod p^b on the p-part. It fixes $\mathbb{Q}(\zeta_N)$, its orbit has size p^{t-b}, and each of the two powers is constant on that orbit, since $t \leq n$. Thus its contribution to the difference is divisible by p^{n-2b+1}.

Finally [47, Proposition 6.1] gives $L_k^{ij}([c]) = \sum_{a,b} s_{ia} s_{jb} (\theta_a/\theta_b)^k N_{ab}^c$. Contract with $f_z e_z = \sum_c \overline{\chi_z(c)}[c]$. Multiplicativity replaces N_{ab}^c by $\overline{\chi_z(a)\chi_z(b)}$ after summing on c; replacing b by b^* in the second sum gives the two conjugate matrix factors, with $s_{0z}^{-2} = f_z$. □

Theorem 3.2 (Block orthogonality)**.** *For a full p-block B of ordinary modular data,*

$$\sum_{z \in B} \overline{s_{zi}} s_{zj} = 0 \qquad (\theta_{i,p} \neq \theta_{j,p}).$$

In reduced integral data the same holds for $\widehat{s}$ at odd p, and at two whenever $\theta_{i,2}^2 \neq \theta_{j,2}^2$. In particular, in the ordinary table

$$p \nmid f_z \in \mathbb{Z}, \quad p \mid \mathrm{ord}(\theta_j) \quad \Longrightarrow \quad S_{zj} = 0. \tag{3.4}$$

Proof. Write $L_k^i = L_k^{ii}$. The insertion (i, i^*) has twist one because dual labels have the same twist. Apply Lemma 3.1(ii) with $m = N_{p'}$.

Separating blocks. In the mod-p fusion ring, residues with a given kernel and image $\mathbb{F}_{p^h}$ are Frobenius conjugate. Denote their blocks in Frobenius order by $B_0, \dots, B_{h-1}$ (all conjugates occur by local Galois symmetry). For $a \in \mathbb{F}_{p^h}^\times$, lift the Chinese-remainder element with residue a at this kernel and zero at all others to a genuine object, taking nonnegative basis coefficients. Choose $h \mid b, n$. As n tends to infinity, both powers of the object in (3.1) converge in character coordinates to $\sum_j [a]^{p^j} e_{B_j}$, where brackets denote Teichmüller lifts. Hence L_m^i and $L_{mp^b}^i$ agree on that sum. The characters $a \mapsto [a]^{p^j}$ of $\mathbb{F}_{p^h}^\times$ are distinct, so the functionals agree on each e_{B_j}. As $N \mid mp^b$, (3.3) gives

$$\sum_{z \in B} |(sT^m s^{-1})_{iz}|^2 = \mathbf{1}_{i \in B}. \tag{3.5}$$

Thus $sT^m s^{-1}$ commutes with the coordinate projector of B. Conjugating back gives the assertion because T^m distinguishes the p-primary twists. If $p \nmid f_z$, the idempotent e_z is locally integral, so $\{z\}$ is a block. Taking $i = 0$ gives (3.4).

Passage to a fermion quotient. In its minimal modular extension select all labels whose characters on the even subring have the prescribed residue of B. Their projector P is a sum of ordinary full-block projectors: equality of residues on the full ring implies equality on the even part. Since z and gz agree on the even subring, P preserves the positive fermion subspace, on which it restricts to the reduced block projector. Put $T_p = \mathrm{diag}(\theta_{i,p})$. The ordinary result gives $[P, sT_p s^{-1}] = 0$. Restrict this identity to the positive subspace at odd p, or its squared-operator version at two: s preserves that subspace and the members of a fermion pair have equal odd primary twists and equal squared dyadic twists. This gives precisely the two reduced assertions. □

Lemma 3.3 (Local modular action on residue blocks)**.** *Every full p-block subspace is invariant under the p-local modular group, in ordinary weakly integral data and also in positive reduced data at odd p. In particular W_p preserves each block.*

Proof. Its coordinate projector commutes with U_p, which is diagonal, and with $sU_p s^{-1}$ by Theorem 3.2, since U_p differs from T_p by a scalar. These elementary unipotents generate the local factor. □

3.1 When two primes never occur in the same twist

Lemma 3.4 (Twist primes seen by a row)**.** *In ordinary weakly integral modular data, if the local Weyl element W_p preserves the line of z then $p \nmid f_z$. In particular:*

1. *$p \mid f_z$ exactly when the z-row hits a p-active column;*
2. *if no twist contains both p and q, every p-active label has q-unit codegree.*

Every prime in the twist of z divides f_z.

Proof. We may assume $p \mid D$ and hence p divides the level by categorical Cauchy. Choose k divisible by its p-part and congruent to one on the complementary level. The identity $U(WUW^{-1})U = W^{-1}$ on the complementary factors identifies $sT^k s^{-1}$, up to diagonal phases on the left and right, with their inverse Weyl. If W_p preserves the line of z, the absolute value of the resulting $(0, z)$ entry is $|s_{0z}| = 1/\sqrt{f_z}$. Thus for $y = f_z e_z$, (3.3) gives $L_k^0(y) = 1$. By Lemma 3.1(i),

$$L_k^0(y)^p \equiv L_{k/p}^0(y^p) = f_z^{p-1} L_{k/p}^0(y)$$

at every prime above p. Thus $p \nmid f_z$.

Now if the row misses all active columns, then $sT_ps^{-1}v_z = v_z$. The local upper and lower unipotents both preserve the line; they generate the local group, so W_p also preserves it. On this line the two unipotents have equal scalars because they are conjugate. Comparing with $sT_ps^{-1}v_z = v_z$ gives $\theta_{z,p} = 1$. Together with (3.4) these facts give (1) and the twist assertion. For (2) the q-local group preserves every nontrivial p-twist fiber. There its upper unipotent and all its local conjugates act by scalars since the q-twist is trivial. Thus W_q preserves each coordinate line. □

Proposition 3.5 (Cauchy and vanishing at odd primes)**.** *For an integral slightly degenerate category and any odd $p \mid \Delta$, some simple has p-active twist. All such labels have zero S-entry on rows of p-unit codegree.*

Proof. If all even labels of a minimal extension were p-inactive, the p-local upper unipotent would fix the positive fermion subspace pointwise (the anomaly is dyadic). The lower unipotent would too, since s preserves this subspace. The fixed space of the local factor is invariant under the entire modular representation and contains $v_0 + v_g$, hence also $T(v_0 + v_g) = v_0 - v_g$ and sv_0. Every coordinate of sv_0 is nonzero. Thus the p-primary twist would be trivial everywhere, contrary to Cauchy's theorem in the extension [9, Theorem 3.9]. Vanishing follows from block orthogonality on the singleton block of a p-unit codegree. □

3.2 Twist orders and individual codegrees

Theorem 3.6 (Twist orders and formal codegrees)**.** *In a weakly integral modular category put $m_z = \operatorname{ord}(\theta_z)$ and $f_z = D/d_z^2$. Then*

$$(m_z)_{\mathrm{odd}} \mid f_z, \qquad m_z \mid \begin{cases} 4f_z & \text{in the integral case,} \\ 8f_z & \text{in general,} \end{cases} \qquad \operatorname{ord}(\theta_x/\theta_y) \mid 4\operatorname{lcm}(f_x, f_y), \qquad N \mid 4D.$$

The odd-part assertion holds also for an integral slightly degenerate category with reduced codegree $(D/2)/d_z^2$.

Proof. The lift $t = T/\gamma$ has $\operatorname{ord}(\gamma) \mid 8$ in the integral case and $\operatorname{ord}(\gamma) \mid 16$ in general [21, Proposition 6.7 and Theorem 6.10]. If p^k is the exact p-part of $\operatorname{ord}(t_{zz})$, each automorphism fixing K_z fixes z, so its square fixes t_{zz} [21, Theorem II]. At odd p its exponent is $\pm 1 \bmod p^k$; at two and $k \geq 4$ it is $\pm 1 \bmod 2^{k-1}$. Thus K_z contains the real subfield F at conductor p^a, with $a = k$ for odd p and $a = k - 1$ at two.

The conductor of F is p^a unless $F = \mathbb{Q}$. By Corollary 2.8, at odd p we obtain $p^k \mid f_z$; the exception $p^k = 3$ follows from Lemma 3.4. At two and $k \geq 4$ we get $2^{k-2} \mid f_z$. For an actual twist whose two-part exceeds $\operatorname{ord}(\gamma)$ this gives the same estimate. Otherwise use $2 \mid f_z$ for every nontrivial actual two-part, again by Lemma 3.4. This gives the two objectwise bounds.

For a ratio with two-part 2^k, $k \geq 4$, one of t_{xx}, t_{yy} has two-part at least 2^k, so $2^{k-2} \mid \operatorname{lcm}(f_x, f_y)$. For $1 \leq k \leq 3$ at least one actual twist has even order. Combined with the odd-part result this proves the ratio bound; taking $y = 0$ gives $N \mid 4D$ since the codegrees divide D. In a minimal modular extension each original label has codegree four times its reduced codegree, giving the last assertion. □

For modular categories of prime-power exponent, Czenky, Plavnik and Schopieray bound the exponent by the norm of the global dimension, up to a factor four at two [18, main theorem, p. 2]. Here integrality of D allows arbitrary exponent and gives an individual bound. Ising ($D = 4$, $d_z^2 = 2$, $m_z = 16$) attains the weakly integral factor eight and the global factor four. By passage to the center and [45, Theorem 5.5], the Frobenius–Schur exponent of any weakly integral category of dimension n divides $4n^2$, and its odd part divides n^2.

3.3 Mass of a primary twist sector

In an integral modular category write $\mu(A) = D^{-1}\sum_{X\in A} d_X^2$ for a set of labels A.

Lemma 3.7 (Mass and overlap of two label sets)**.** *For sets of labels A, B put $a = \mu(A)$, $b = \mu(B)$. Then*

$$D^{-1}\sum_{X\in A, Y\in B} d_X d_Y |s_{XY}| \;\geq\; ab - \sqrt{a(1-a)b(1-b)}.$$

In particular $S_{A,B} = 0$ implies $a + b \leq 1$, and hence the union of the A-row supports has mass at least a. For m labels H of common degree z and codegree $f = D/z^2$, taking $A = B = H$ gives $\sum_{X,Y\in H} |S_{XY}|/z \geq \sqrt{f}\, m(2m/f - 1)$.

Proof. The ranges of the projections $P = s^*P_A s$, $Q = TsP_B s^*T^*$ have orthonormal bases s^*v_X, Tsv_Y. Their cross inner products have moduli $|s_{XY}|$ by $sTs = \xi T^{-1}sT^{-1}$, $|\xi| = 1$. Expanding $\langle Pv_0, Qv_0\rangle$ in these bases bounds its modulus by the displayed sum. The vacuum coordinates of Pv_0, Qv_0 are a, b, equal to their squared norms. On their components orthogonal to v_0, Cauchy–Schwarz gives the claimed lower bound. □

Proposition 3.8 (Mass from a generating simple)**.** *Put $L_p = \sum_{p|\mathrm{ord}(\theta_i)} d_i^2$. If X generates $\mathcal{C}$ and $L_p > 0$, then $L_p \geq (p-1)D/(2pd_X)$. In a fusion-simple integral modular category one can take the smallest nonunit degree whenever $p \mid D$.*

Proof. Set $n = N_{p'}$, $t_i = \theta_i^n$. The indicator formula [45, Theorem 7.5] gives

$$\nu_n(X) = D^{-1}\sum_{i,j} N_{ij}^X d_i d_j\, t_i/t_j \in \mathbb{Z}:$$

it is an algebraic integer in both the p-power cyclotomic field of these phases and the prime-to-p field of the n-fold rotation. The weights sum to Dd_X. Equality $\nu_n(X) = d_X$ would force equal phases on every edge $i^* \to j$ of multiplication by X (duality preserves twists). Since X generates and X^* reverses the edges, every phase would be one. Thus $d_X - \nu_n(X) \geq 1$. The mean of a nontrivial p-power root is $-1/(p-1)$ at order p and zero otherwise. After Galois averaging, each unequal-phase pair contributes at most $p/(p-1)$ times its normalized weight to this deficit. Its first or second label is active; either choice has total weight $L_p d_X$ by reciprocity. This proves the bound. Categorical Cauchy gives $L_p > 0$ for $p \mid D$ [9, Theorem 3.9]. □

Proposition 3.9 (Odd-valuation slices)**.** *Suppose p and $v_p(D)$ are odd. Let B be a full p-block or the whole label set. Let x, y have trivial p-twists, with y fixed by p-inertia. For a complementary twist η put*

$$M_j = \sum_{\substack{i\in B,\ \theta_{i,p'}=\eta \\ \mathrm{ord}(\theta_{i,p})=p^j}} s_{xi}\overline{s_{yi}}.$$

Then $M_1 = (p-1)M_0$, also after summing over η. The same formula holds in integral positive reduced data with D replaced by Δ, provided the complementary fiber is taken up to the fermion sign. Thus on each such fiber the squared-dimension mass of order-p twists is $p-1$ times the inactive mass, and the active proportion is at least $1 - 1/p$. For the entire label set, activity at all primes in any set J of odd primes of odd valuation has proportion at least $\prod_{p\in J}(1-1/p)$, even conditional on fixed twist components outside J.

Proof. Let E project onto $B \cap \{i : \theta_{i,p'} = \eta\}$. It commutes with the local p-factor by Lemma 3.3 and commutation of distinct factors; so does $Q = sEs^{-1}$ since s normalizes that factor. Write $u = T_p, w, h_t$ for its upper unipotent, Weyl and torus. For a unit n modulo the local level p^a, the nth weighted moment of the p-twist on the fiber is

$$(Qwu^n w^{-1})_{xy} = (Qu^{-1/n}wh_{-n}u^{-1/n})_{xy} = \Big(\frac{-n}{p}\Big)(Qw)_{xy}.$$

The middle identity is Bruhat. The outer unipotents act trivially at x, y. Since inertia fixes y, the torus acts there by the Galois sign of $d_y/\sqrt{D}$ [21, Theorem II and Remark 4.5]; at odd $v_p(D)$ this is the displayed quadratic character, from $\sqrt{(-1)^{(p-1)/2}p}$. Average over n: the right side vanishes, while the left gives $M_0 - M_1/(p-1)$ by the means of p-power roots. For the reduced formula, use the restricted projector of Theorem 3.2 and the complementary twists modulo sign. The odd local group acts on the positive subspace, and its torus sign is the same since the two dimension normalizations differ by two. At $x = y = 0$ the weights are d_i^2/D or d_i^2/Δ. The active bound holds on each complementary fiber and thus on every union of them; iteration over J proves the product bound. □

Corollary 3.10 (The two masses in a defect-one block)**.** *Let B be a full p-residue block of an integral modular category, where p is odd, and assume that $v_p(f_i) = 1$ for every $i \in B$. Then B consists of one inertia orbit with trivial p-primary twist and one inertia orbit with p-primary twist of order p. If their respective lengths and dimensions are a, r and b, s, then*

$$bs^2 = (p-1)ar^2.$$

Proof. The codegree identity implies that $v_p(D)$ is odd. Proposition 2.13 gives exactly two inertia orbits; their lengths divide $p-1$ and the dimension is constant on each. Theorem 3.6 bounds all their p-primary twist orders by p. Inertia preserves these orders. Proposition 3.9, with the complementary condition omitted, says that both the trivial and the order-p masses are nonzero and have ratio $p-1$. Thus one orbit is of each kind and the displayed equation follows. □

Lemma 3.11 (The visible class at defect one)**.** *In an ordinary integral modular table, let t, s be distinct odd primes with $s \nmid t-1$. Suppose $\mathcal{U}$ is a nonempty set of s-free labels of t-defect one and $\mathcal{V}$ a nonempty set of t-defect-one labels in P_s. All these rows see only 1 and a single common square class of primitive tth roots as t-phases.*

Proof. A defect-one block has an inactive and an active orbit; the latter has one square class of primitive phases. Their lengths $a_0, a_1 \mid t-1$ and degrees d_0, d_1 satisfy $a_1 d_1^2 = (t-1)a_0 d_0^2$ (Proposition 2.13, Corollary 3.10). Thus the two degrees have equal s-valuation. In each block of a label in $\mathcal{U}$, all active representatives have s-unit degree, so every $V \in \mathcal{V}$ hits them by principality. Its row can see only phases in its own block: the local Weyl preserves the block and the complementary Weyl preserves t-phases. The active classes of every such pair of blocks therefore agree. □

4 Principal defects and fermion quotients

The conductor of a residue block bounds its principal degrees. Paired with orthogonality, this excludes a third repeated prime of exponent at most three. Two further tools will also be needed at higher exponents: comparison of the two fermion quotients at two, and positive residues of a common character value at distinct primes.

4.1 The first possible principal defect

Corollary 4.1 (Principal valuation bound)**.** *If $v_p(\Delta) \geq 2$, every ordinary or reduced principal-block row satisfies*

$$v_p(f_X) \geq 2, \qquad v_p(d_X) \leq \lfloor (v_p(\Delta) - 2)/2 \rfloor.$$

In particular the defect is at least three at odd valuation, and at valuation two or three every principal degree is p-free.

Proof. The dimension character is unramified of defect $v_p(\Delta)$. Proposition 2.13 excludes defect one; defect zero has an integral idempotent separating it from the dimension character. Now use $d_X^2 f_X = \Delta$. □

Proposition 4.2 (Prime avoidance on a principal block)**.** *Let P be a full principal p-block of ordinary integral modular data, or of positive reduced data at an odd prime. Let $E = P$; in ordinary data one may also intersect P with the fiber of trivial prime-to-p twist. Assume that the unit is the only invertible in E. Fix a label X and a prime $s \neq p$. If X is p-active, or if p and $v_p(\Delta)$ are odd and $s \nmid p-1$, then at each prime over s there is a nonunit $Y \in E$ for which $d_Y \chi_X(Y)$ has nonzero residue. In particular $s \nmid d_Y$.*

Proof. The block projector commutes with the local group (Lemma 3.3); so does the indicated complementary projector. Conjugation by the global Weyl normalizes the local group. For an active X the $X, 0$ entry of the conjugate projector is zero, whence $\sum_{Y \in E} d_Y \chi_X(Y) = 0$. For an inactive X in the second case the slice identity (Proposition 3.9) is

$$\sum_{\substack{Y \in E \\ \operatorname{ord}(\theta_{Y,p})=p}} d_Y \chi_X(Y) = (p-1) \sum_{\substack{Y \in E \\ \theta_{Y,p}=1}} d_Y \chi_X(Y).$$

In either identity the vacuum contribution is a unit at s, so a nonunit term must survive modulo each prime over s. □

In either setting of the proposition, at valuation two or three the nonunit degrees in P_p of a perfect table are p-free and have gcd one: apply prime avoidance to an active column at every other prime.

For larger defects an index inequality bounds the total ramification of a block.

Proposition 4.3 (A mixed-defect block bound)**.** *Let A be a commutative local symmetric order over an unramified p-adic integer ring, with separable cyclotomic generic algebra, rational positive integral codegrees, and a common character residue. After an unramified extension write its generic factors as E_j/K, $1 \leq j \leq t$, with totally ramified degrees $e_j = u_j p^{s_j}$, $u_j \mid p-1$. Put $a_j = v_p(f_j)$. Then*

$$\sum_{j=1}^{t} (p^{s_j} - (a_j - 1)e_j) \leq 2.$$

At two this becomes $\sum_j (2 - a_j)e_j \leq 2$.

Proof. Put $M = \prod_j \mathcal{O}_{E_j}$. The trace on E_j is $\operatorname{Tr}_{E_j/K}/f_j$, so the discriminant-index formula and self-duality of A give $2\operatorname{length}(M/A) = \sum_j (a_j e_j - d_j)$. Common residues impose $t-1$ independent conditions on M; the first two ramification groups give $d_j \geq e_j + p^{s_j} - 2$. Combining yields the bound, with $e_j = 2^{s_j}$ at two. □

4.2 The cubic exponent at two

Lemma 4.4 (An integral minimal modular extension)**.** *Let $\mathcal{B}$ be integral and slightly degenerate, with $\dim \mathcal{B} = D > 2$, pointed subcategory $\mathrm{sVec} = \langle g \rangle$, and no other proper nontrivial fusion subcategory. Every minimal modular extension has pointed subcategory sVec, adjoint subcategory $\mathcal{B}$, and precisely the proper fusion subcategories $\mathrm{Vec}, \mathrm{sVec}, \mathcal{B}$. Moreover one can choose an integral minimal modular extension. In such an extension, noninvertible simples of coprime dimensions have zero S-entry.*

Proof. By the extension theorem of Johnson-Freyd and Reutter [37, Corollary 1.3] there is a minimal modular extension $\mathcal{M}_0$, of dimension $2D$, with neutral component $\mathcal{B}$ for fermion monodromy. An odd invertible h would give a pointed subcategory of order four whose bicharacter has trivial radical: $b(g,g) = 1$, $b(g,h) = -1$, and in the cyclic case $b(h,h)^2 = -1$. Splitting it off gives $\mathcal{B} = \mathrm{sVec} \boxtimes \mathcal{N}$ with $\dim \mathcal{N} = D/2 > 1$, impossible. Hence $(\mathcal{M}_0)_{\mathrm{pt}} = \mathrm{sVec}$ and $(\mathcal{M}_0)_{\mathrm{ad}} = \mathcal{B}$. A proper subcategory crossing the grading has twice the dimension of its intersection with $\mathcal{B}$, hence dimension two or four. It is pointed or Ising. The pointed case has an odd invertible.

In the Ising case the subcategory is nondegenerate by [26, Corollary B.12] and splits off with fermion g, giving the same contradiction.

Normalize $t_X = \theta_X/\zeta$ by choosing a two-power root ζ with $\zeta^3 = p^+/\sqrt{2D}$ [21, Theorem 6.10]. At an odd prime any inertia element σ fixes t_0. The Galois image of the unit is invertible and its normalized twist is $\sigma^2(t_0) = t_0$; thus it is the unit, since $\theta_g = -1$ [21, Theorem II]. The vacuum-column ratios d_X are consequently unramified at every odd prime. Since d_X^2 is an integer, d_X is an integer or $\sqrt{2}$ times one.

In the nonintegral case the integral subcategory is exactly $\mathcal{B}$. Let $\mathcal{I}$ be Ising with fermion ψ and noninvertible σ. Condense the diagonal boson $g \boxtimes \psi$ in $\mathcal{M}_0 \boxtimes \mathcal{I}$: the local modules have dimension $(2D)4/2^2 = 2D$ and are nondegenerate [26, Propositions 4.26, 4.55(i) and 4.56(i)–(ii)]. Free modules on $X \boxtimes \mathbf{1}$, $X \in \mathcal{B}$, embed $\mathcal{B}$ braided fully faithfully since the nontrivial algebra summand contributes no morphisms by adjunction. These give all even pairs after identifying boson translates. An odd pair (Y, σ) has dimension $2n$; a free orbit gives one simple of dimension $2n$, and a fixed orbit gives two of dimension n since a cyclic stabilizer has only one-dimensional projective irreducibles. This extension is integral.

For noninvertible X in an integral extension, $\langle XX^* \rangle = \mathcal{B}$: containment in sVec gives $d_X^2 \leq 2$ since $1, g$ each occur at most once. A nonzero entry on coprime-dimensional noninvertibles would imply projective centrality [29, Proposition 8.22.5 and Lemma 9.15.1], hence membership of one in $\mathcal{B}' = \mathrm{sVec}$, impossible. □

Lemma 4.5 (The two fermion quotients)**.** *Let $\mathcal{M}$ be integral modular with fermion g and $\mathcal{B} = \langle g \rangle'$. Put*

$$R_\pm = K_0(\mathcal{B})/(g \mp 1).$$

These are symmetric integral orders with identical special fibers at two. Their blocks selected by the dimension residue have the same rank. Restrictions of modular characters have precisely the fermion orbits as fibers and belong to R_+ or R_- according as their labels are even or odd. For either quotient their codegrees are

$$f_X^\pm = \begin{cases} f_X/2 & gX = X, \\ f_X/4 & gX \neq X, \end{cases} \qquad f_X = \dim(\mathcal{M})/d_X^2. \tag{4.1}$$

A restricted character has the dimension residue if and only if the full modular character does.

Let P be the full principal two-block of $\mathcal{M}$. Then

$$\sum_{X \in P \cap \mathcal{B}} d_X^2 = \sum_{X \in P \setminus \mathcal{B}} d_X^2. \tag{4.2}$$

If $\dim \mathcal{B} = 2^{s+1}m$ with $s \in \{2, 3\}$ and m odd, the positive principal degrees are odd and its rank k satisfies $2^s \mid k$. Every full principal degree is odd or twice odd, and an even-dimensional principal label lies outside $\mathcal{B}$.

Proof. Choose duality-compatible orbit representatives in $\mathcal{B}$ ($gX \not\simeq X, X^*$ by twists). In either quotient the unit-coefficient trace satisfies $\tau(XY^*) = \delta_{XY}$, and the orders coincide modulo two, giving equal ranks of corresponding local blocks. If ϵ_Z is the fermion monodromy, then $\chi_{gX}(Z) = \epsilon_Z \chi_X(Z)$. Orthogonality with the even projector gives

$$\sum_{Z \in \mathrm{Irr}(\mathcal{B})} \chi_X(Z)\overline{\chi_Y(Z)} = \tfrac{1}{2} f_X(\delta_{XY} + \delta_{X,gY}). \tag{4.3}$$

This identifies the restriction fibers. Every character of the even subalgebra extends by semisimplicity, its value on g selects the quotient, and summing over representatives halves the displayed norm, giving (4.1). These rational codegrees are integral Casimir eigenvalues.

A restricted dimension residue extends to the full ring: each odd Y has Y^2 even and squaring is injective in the residue field. The principal labels $1, g$ have opposite dyadic twists, giving (4.2)

by block orthogonality. Corollary 4.1 at dimensions $2^s m$, $2^{s+2}m$ gives the claimed degree parities. Finally the positive block idempotent has integral trace $\sum_{Z\in P_+} d_Z^2/(2^s m)$ and each odd square equals one modulo 2^s for $s = 2, 3$. Thus $2^s \mid k$. □

Normalization in the fermion extension. Let g be the transparent fermion of $\mathcal{B}$, and $\mathcal{M}$ a minimal modular extension. On its coordinate space put $Fv_X = v_{gX}$ and $Gv_X = \epsilon_X v_X$, where ϵ_X is the monodromy sign. For an odd orbit O set $v_O = |O|^{-1/2} \sum_{Y\in O} v_Y$. A fixed odd label therefore has $v_O = v_Y$.

Table 5: Dimension, codegree and matrix conventions. In the lower panel $s = S^{\mathcal{M}}/(2\sqrt{\Delta})$.

Character data	Dimension convention	Codegree
Ordinary modular	$D = \mathrm{FPdim}(\mathcal{C})$, $s = S/\sqrt{D}$	D/d_X^2
Full modular extension	$\mathrm{FPdim}(\mathcal{M}) = 4\Delta$	$4\Delta/d_Y^2$
Positive quotient R_+	$\Delta = \mathrm{FPdim}(\mathcal{B})/2$	Δ/d_X^2
Signed order R_-	Same even source $\mathcal{B}$ of dimension 2Δ	$2\Delta/(\lvert O\rvert d_Y^2)$, $Y \in O$
Space (F, G)	**Orthonormal basis**	**Action of s**
$A = (+,+)$	$(v_X + v_{gX})/\sqrt{2}$, one per even pair	$\widehat{s}_{[X],[Z]} = 2s_{XZ}$
$B = (-,+)$	$(v_X - v_{gX})/\sqrt{2}$, same choices	$s : B \longrightarrow C$
$C = (+,-)$	v_O, one per odd orbit	$s : C \longrightarrow B$

The signed codegree uses the unit-coefficient trace on R_- in (4.1). Expanding the basis vectors with $S_{gX,Y} = \epsilon_Y S_{XY}$ gives the B, C entry $\sqrt{2|O|}\, s_{XY}$ for $Y \in O$. Opposite twists on an even pair make T exchange A and B. Equal twists on odd mates make T preserve C. Both T^2 and the odd primary twists preserve all three spaces; the dyadic observable on A is the squared dyadic twist. The integral applications below specify when the extension is integral.

Lemma 4.6 (Principal lifts at odd primes)**.** *Fix a place above an odd prime p. A positive principal pair in R_+ has a representative whose full character agrees with dimension modulo that place. Both representatives are principal if every odd degree vanishes in the residue field; otherwise exactly one is principal.*

Proof. For a lift X, write $a_Y = \overline{\chi_X(Y)}$ and $b_Y = \overline{d_Y}$ in the residue field (bars here denote reduction). The two maps agree on the even sector. For odd Y, Z, the products Y^2, YZ are even, so $a_Y^2 = b_Y^2$ and $a_Y a_Z = b_Y b_Z$. If all b_Y vanish, so do all a_Y. Otherwise fix Z with $b_Z \neq 0$. Then $a_Z/b_Z = \pm 1$ and every a_Y equals the same sign times b_Y. Passing to gX multiplies every odd evaluation by -1 and fixes the even ones, proving the assertion. □

Lemma 4.7 (Unramified signed characters from sign-only twists)**.** *Let $\mathcal{M}$ be an integral modular category with fermion g, and let $\mathcal{B} = \langle g\rangle'$. If the two-primary part of every actual twist of $\mathcal{B}$ is 1 or -1, then every character field of $K_0(\mathcal{B})$ is unramified at two. In particular this holds for the signed quotient $K_0(\mathcal{B})/(g+1)$.*

Proof. Let σ belong to inertia at two. The Galois identities [21, Theorem II] give $\sigma\chi_Y = \chi_{\widehat{\sigma}(Y)}$ and $t_{\widehat{\sigma}(Y)} = \sigma^2(t_Y)$, $t_Y = \theta_Y/\zeta$. Since the dimension character is rational, $\widehat{\sigma}(0) = 0$ and $\sigma^2(\zeta) = \zeta$. For even Y, Z the balancing formula expresses S_{YZ} in integral dimensions and actual even-sector twists, all in an odd cyclotomic field. Thus Y and $\widehat{\sigma}(Y)$ have the same restricted character and lie in one fermion orbit by Lemma 4.5. Their normalized twists agree, while Y, gY have opposite twists. Hence $\widehat{\sigma}(Y) = Y$. Symmetry of S and rationality of dimensions now give $\sigma(\chi_X(Y)) = \chi_X(Y)$ on all even Y for every X. Every character of the even subalgebra extends. □

Proposition 4.8 (Odd supports in the principal two-block)**.** *Choose an integral minimal extension $\mathcal{M}$ of a slightly degenerate minimal counterexample $\mathcal{B}$. Its positive reduced dimension is even. Two odd primes p, q of valuation two in $\dim \mathcal{B}$ cannot occur as a simple degree pq.*

If all principal two-block degrees are odd, their noninvertible supports intersect pairwise with empty total intersection. With exactly three odd primes each of their three pairs must occur as a support. When $v_2(\dim \mathcal{B}) = 3$ all principal degrees are indeed odd. Thus this two-valuation is excluded if the odd part has at most two prime divisors, or three with two valuations equal to two.

Proof. If $\Delta = \dim \mathcal{B}/2$ were odd, the positive dimension idempotent would be integral at two. Thus the full principal block would have only $1, g$ in its even sector by Lemma 4.5. Its odd mass would be two by (4.2), forcing an odd invertible, contrary to Lemma 4.4.

A label of degree pq has odd residue, hence every noninvertible principal two-block label meets $\{p, q\}$ in its support by coprime vanishing. Choose even p- and q-active columns Y, Z by reduced Cauchy. A p-bearing label has p-unit codegree and vanishes at Y; likewise at q, Z. Thus $1, g$ are their only common principal hits. Replacing Z by gZ makes the dyadic phases distinct while both contributions stay positive, contrary to Theorem 3.2.

If all principal degrees are odd, any two noninvertible labels meet by principality and coprime vanishing. They have no common prime: modulo that prime the mass identity (4.2) would read $2 = 0$, from the even invertibles $1, g$ alone. This also excludes an empty family. On two primes intersecting supports have a common prime; on three, empty total intersection forces all three pairs. The squared pair would then occur as a degree, contrary to the first assertion.

If $v_2(\dim \mathcal{B}) = 3$, Lemma 4.5 gives signed principal rank divisible by four. An even principal label would be odd-sector of degree twice odd and signed defect zero or one by (4.1), contradicting Proposition 2.13. □

4.3 Minimum positive mass and exposed faces

Fix a prime s, an integer $a \geq 1$, $L = s^a$, and ζ primitive of order L. Identify a coset of μ_s with its indices modulo L/s. Choose m coprime to s. At each j mod L permit the phases $\zeta^j \eta$ where η runs over the primitive roots of orders in a specified subset H_j of the divisors of m. These sets may vary with j and may be empty. Let μ be the Möbius function and φ Euler's totient. For $\epsilon = \pm 1$ put

$$b_j^{\epsilon} = \min\{\varphi(n) : n \in H_j,\ \mu(n) = \epsilon\}, \qquad \min \varnothing = +\infty.$$

Representations have nonnegative rational coefficients. Their mass is the sum of these coefficients.

Theorem 4.9. *For rational c_j and $x = \sum_j c_j \zeta^j$, the least mass of a representation by permitted phases is*

$$\sum_{C \in \mu_L/\mu_s} \min_{t \in \{c_j : j \in C\}} \sum_{j \in C} [b_j^+ (c_j - t)_+ + b_j^- (t - c_j)_+].$$

Here $0 \cdot \infty = 0$, and an infinite minimum means that no representation exists. Every finite minimum is attained and gives an explicit word.

Proof. Average a representation over the complementary Galois group; it fixes x, preserves mass, and acts transitively on the primitive roots of each allowed order. Their normalized trace is $\mu(n)/\varphi(n)$. If the resulting rational coefficient at j is $u > 0$, its minimum cost is $b_j^+ u$. Indeed no orbit contributes more per unit mass than $1/b_j^+$; equality comes from a minimizing orbit, since the sum over it is 1. For $u < 0$ use an orbit of sum -1 and obtain $b_j^-(-u)$. An orbit of sum zero cannot reduce the minimum. The only rational relations among the L indicated powers of ζ are the sums of the μ_s-cosets: these L/s relations have disjoint supports and $L - \varphi(L) = L/s$. Hence the possible coefficients on C are exactly $c_j - t$. The nonnegative piecewise linear cost attains any finite minimum at a breakpoint c_j, since feasibility endpoints also occur there. Different cosets are independent. □

Say that $E \subset \mu_L$ is exposed if some rational linear functional on the cyclotomic field takes value one at the indicated pure roots and strictly less than one on all other permitted phases. Require $1 \in H_j$ at every marked position.

Theorem 4.10. *E is exposed if and only if in every coset C,*

$$|E \cap C| < \sum_{j \in C \setminus E} b_j^-.$$

Proof. Average an exposing functional over the complementary Galois group. It then takes the value $t_j \mu(n)/\varphi(n)$ on the primitive order-n orbit above j. The rational values t_j satisfy precisely $\sum_{j \in C} t_j = 0$, by the coset relations and linear disjointness. At a marked root $t_j = 1$ works, since every nonidentity orbit has mean less than one. At an unmarked position with finite b_j^-, strictness requires $t_j > -b_j^-$. Summing gives necessity, including the case of a completely marked coset. Conversely choose $t_j \le 0$ on the unmarked positions with total $-|E \cap C|$. With a finite, positive total capacity, distribute this negative sum proportionally to the capacities. If an infinite capacity occurs, place it all there. For an unmarked coset take zeros throughout. The strict inequality makes every unmarked value less than one, including orbits with zero or positive mean. Composing the resulting functional on $\mathbb{Q}(\zeta)$ with normalized relative trace defines the required functional on the whole field. □

For H_j the full set of divisors of $m > 1$, $b_j^- = \ell - 1$ for ℓ the smallest prime dividing m. The criterion specializes to $|E \cap C| < s(\ell - 1)/\ell$. When $m = 1$, it says that each coset contains an unmarked root. For an arbitrary finite alphabet lacking complementary Galois invariance, choose power-basis coordinates modulo Φ_N and maximize ε subject to value one on the marked roots and value at most $1 - \varepsilon$ on every other permitted root. Exposure is equivalent to a feasible positive gap.

Lemma 4.11 (Contraction and equality for a positive cyclotomic expression)**.** *Choose a common finite level MN, $(M, N) = 1$, with one or two prime divisors of M. Write $R = \mathbb{Z}[\mu_M]$ and prescribe congruences to rational integers at every prime over specified primes of M. Give the letters $+\xi, -\xi$ ($\xi \in \mu_M$) costs $1, c$, respectively, with $c \ge 1$. Suppose every integral signed word on μ_M satisfying the congruences has cost at least g. Let an algebraic integer satisfying the same congruences in $\mathbb{Z}[\mu_{MN}]$ have a positive rational expression of mass at most g.*

1. *For $c = 1$ every complementary component belongs to $\{1, -1\}$. For $c < \ell - 1$, where ℓ is the least prime divisor of $N > 1$, every complementary component equals 1.*

2. *In either case, after absorbing permitted signs, the coefficient vector of the expression is a convex combination of integral optima on the signed configuration. With $N = 1$ this assertion holds directly. If all integral optima on its allowed columns agree, its multiplicities are therefore determined.*

3. *More generally, let a real-linear functional on the real scalar extension of the cyclotomic lattice be at most the cost on every signed column. If its value on the target equals the mass g, every positively weighted letter of an optimal expression attains equality. This conclusion applies to any actual balancing expression on the alphabet in question.*

Proof. Use $\mathbb{Z}[\mu_{MN}] = R \otimes_{\mathbb{Z}} \mathbb{Z}[\mu_N]$ by coprimality and equality of degrees. An integral functional λ on the second factor fixing 1 induces an R-linear contraction fixing the prescribed residues: for $p \mid M$ the radical of (p) at either level is generated by $1 - \zeta_{p^{v_p(M)}}$, since the prime-to-p extension is unramified. The contraction thus gives an integral target in R with the same congruences. In an integral basis of R, the columns for μ_M are unimodular by Beck–Hoşten [4, Theorems 7 and 13]; Theorem 7 reduces to square-free levels. Adjoining negative columns preserves unimodularity. Consequently every vertex of $Ax = y$, $x \ge 0$, is integral for integral y (solve in independent

columns). Positive costs make every nonempty optimal face bounded. In particular a rational signed representation of a contracted target has cost at least g.

For $c = 1$, to eliminate a particular nonsign complementary root a, use a basis in each prime-power factor obtained by omitting one root in every prime-order coset. Keep 1 and, in a factor where the component of a is not a sign, also keep that component. This is possible at two because the two positions then lie in different cosets. Taking the coefficient of 1 in the product basis kills a and takes values $0, \pm 1$ on all roots. Positive mass at a would give cost strictly below g.

For the asymmetric assertion choose independently in each u-power factor of N the functional taking value 1 at 1, -1 at one uniformly chosen nonidentity uth root and zero elsewhere. It respects every μ_u-coset relation. On a term with a nontrivial u-component the tensor product gives expected signed cost at most $c/(u-1) < 1$; the other terms cost at most one in expectation. A positive coefficient at such a term would make some contraction cost less than g. Only retained signed columns remain in either case, so the original expression belongs to their bounded optimal face. Its zero coordinates stay zero at every vertex in a convex decomposition. Finally, equality after summing a termwise functional inequality with nonnegative weights forces equality on each term with positive weight. □

4.4 Short cyclotomic sums and complementary principal labels

A cyclotomic integer has *weight at most g* if it is a nonnegative rational combination of roots of unity of total coefficient at most g.

Proposition 4.12 (Inertia forced by several positive residues)**.** *Let z be a cyclotomic integer, all of whose conjugates have modulus strictly less than a positive integer g. Suppose $z \equiv g$ at every prime above each prime in a finite set P. For every nonempty proper subset $R \subset P$ let H be the subgroup of $\mathrm{Gal}(\mathbb{Q}(z)/\mathbb{Q})$ generated by the inertia groups at $s \in R$. Then*

$$\prod_{s \in R} s < |H| g.$$

In particular any $s \geq g$ in a pair of prescribed positive residues ramifies. If $z \neq 0$ vanishes at every prime above s, then $s \geq g$ forces ramification at s.

Proof. Put $e = |H|$, $M = \prod_{s \in R} s$ and $w = \sum_{\sigma \in H} \sigma z$. Its field is unramified at the primes in R, and $w \equiv eg$ at every prime above P. Therefore $(w - eg)/M$ is integral. If $M \geq eg$, every conjugate of $1 + (w - eg)/M$ has modulus strictly less than one, forcing $w = eg - M$. A prime of $P \setminus R$ would divide M. In the last assertion every prime of $\mathbb{Q}(z)$ above s divides (z). If s were unramified, their product would be (s), so z/s would be a nonzero algebraic integer of house less than one. Lemma 2.10 supplies the all-places hypothesis when the residue comes from a principal row. □

Theorem 4.13 (Short cyclotomic sums with one or two positive residues)**.** *Let g be a positive integer and let the cyclotomic integer z have weight at most g. Suppose $z \equiv g$ at every prime over a prime $r > g$. If $r > 2g$, then z is an unsigned integral sum of exactly g roots of r-power order, unique at any common r-power level.*

Suppose also that $q < r$ is prime and $z \equiv g$ at every prime over q. At every $r > g$ one then has

$$z = h + q \sum_{i=1}^{m} \xi_i, \qquad h + qm = g,$$

with $h, m \geq 0$ and the ξ_i nonidentity r-power roots, except possibly when $r = g + 1$ and $q \mid g - 1$. In that exceptional case another possibility is

$$z = -\sum_{t \in \mathbb{F}_r^\times} \eta_t \zeta_r^t,$$

where the q-power roots η_t consist of one copy of 1 and $(g-1)/q$ complete cosets of μ_q, with repetitions allowed. In particular $z = g$ if $g < q$, and $|z| = g$ if $g = q$. If $|z| < g$ and $q \le g < 2q$, the field of z contains $\mathbb{Q}(\zeta_r)$ in the first case and at least $\mathbb{Q}(\zeta_r)^+$ in the exceptional case; in either case the subgroup of tame units of the r-power factor has orbit size $r-1$ on z.

Proof. At level s^e the integral relations among powers of ζ_{s^e} are constant on each coset of μ_s, by Φ_{s^e}. This is the prime-power case of the Rédei–de Bruijn–Schoenberg theorem [39, Theorem 5.1]. We use this also modulo another prime. In particular an unsigned s-word of length $< s$ is unique, since in each coset its least coefficient is zero.

First suppose z is a signed word of length at most g in roots of order supported on the specified primes. At $r > 2g$ alone the signed augmentation is g modulo r and lies in $[-g, g]$, forcing g positive signs. For two primes erase the r-parts, obtaining w. Its field is unramified at r, so $(w-g)/r$ is integral. All conjugates of $1 + (w-g)/r$ lie in the unit disk, meeting its boundary only at 1; the norm therefore gives $w = g$ or $g - r$.

Erase instead the q-parts, retaining signed multiplicities c_i in the r-power roots. The erased sum is g modulo q in this unramified ring. Thus $c_i - g\delta_{i0}$ is constant modulo q on each μ_r-coset. If every coset has a vacant position after marking index zero, these constants vanish. The augmentation, also the residue of w above q, is then g modulo q; this excludes $w = g - r$. Equality $w = g$ forces every erased summand to equal 1, hence the original word is unsigned and r-primary. Each nonidentity multiplicity is divisible by q (for $q = 2$ signed and unsigned multiplicities have the same parity).

The only other possibility uses all $g = r - 1$ positions, once each, on the nonidentity rth roots. Their signs are $-g$ modulo q. This excludes $q = 2$ and a positive sign, and gives $q \mid g - 1$. As q is odd, the negative summands cannot give $w = g$. Hence $w = -1$ and $\sum_t \eta_t = 1$. The μ_q-coset relations show that, after removing one copy of 1, the multiplicities are nonnegative and constant on each coset.

For the reduction to signed words choose a common cyclotomic level containing the original phases and all retained primes. Apply Lemma 4.11 with both signs of cost one. Its integral contraction preserves the prescribed residues at all places over the retained primes. The alternatives just proved have minimum signed weight g: for one prime the augmentation forces it; for two primes a vacant slot forces g positive signs, while the exceptional full coset already uses $r - 1 = g$ slots. For any complementary component other than a sign choose the contraction which annihilates it. Its positive occurrence in the original expression would leave weight strictly below g, a contradiction. Thus the complementary components are signs, and unimodularity on the retained alphabet applies to z itself.

In the unsigned expression, tame r-units stabilizing z permute the m nonidentity roots freely, counted after division by q. For $m > 0$ their stabilizer order divides m. In the exception, independence of the nonidentity rth roots over the q-power field makes η_t constant on the free orbits of a tame stabilizer. Its order divides $\sum_t \eta_t = 1$ and is therefore one. At $q \le g < 2q$, strict inequality $|z| < g$ gives $m = 1$ in the unsigned case; in the exception $g = q + 1$, $r = q + 2$. Any automorphism fixing the exceptional sum permutes its one or two positions carrying coefficient 1, hence acts on ζ_r by 1 or -1. This proves the field assertions. □

Lemma 4.14 (Complementary principal degrees at three primes)**.** *Let a perfect fusion-simple ordinary or positive reduced table have three dimension primes t, u, v. In the reduced case assume u odd. The principal block P_u contains v-free nonunits. Every such degree is $t^i u^j$ with $i, j \ge 1$ and $2j + 2 \le v_u(\Delta)$. Thus at a quartic u both complementary degrees ut^i, uv^k occur in P_u, each of defect two.*

When $u < v$ are odd, the t-parts of all v-free nonunits in P_u and all u-free nonunits in P_v are at least u.

Proof. Prime avoidance (Proposition 4.2) for a u-active column produces a v-free nonunit. It cannot have prime-power degree, and Corollary 4.1 bounds its u-height. At $v_u(\Delta) = 4$ interchange

t, v. For $u < v$ odd pair opposite labels X, Y of degrees $t^i u^j, t^k v^l$. The lcm-normalized entry has weight $g = t^{\min(i,k)}$ by balancing and strict house below g. Principality gives residue g over both u, v, on all conjugates by Galois permutation of labels and symmetry. Theorem 4.13 therefore gives $g \geq u$. □

Corollary 4.15 (The two-adic height in a reduced table)**.** *For a slightly degenerate minimal counterexample of dimension $2^a q^b r^c$ with $q < r$ odd, one has $2^{\lfloor (a-1)/2 \rfloor} \geq q$. In particular $a \geq 5$.*

Proof. The positive table has two-valuation $a - 1$. Square-divisibility bounds all two-parts by $2^{\lfloor (a-1)/2 \rfloor}$, and Lemma 4.14 bounds some two-part below by q. Sources losing a repeated prime are covered by Dong. □

4.5 The uniform theorem and separation criteria

Theorem 4.16 (Three primes with a small exponent)**.** *Let p, q, r be distinct primes, $a, b, c \geq 0$, and let d be square-free and coprime to pqr. Every braided fusion category of dimension $p^a q^b r^c d$ is WGT if $c \leq 3$. Consequently every fusion category of dimension $p^a q^b r$ is WGT without a braiding hypothesis.*

Proof. A divisor-minimal failure has a perfect simple ordinary or reduced table of square-full dimension $\Delta = D$ or $D/2$ (Lemma 2.1); all three primes must remain by Dong's two-prime theorem. In ordinary data every prime valuation is at least four by Lemma 4.14. In reduced data that lemma applies at every odd prime, and Corollary 4.15 treats two. For an unbraided category pass to its center of dimension $p^{2a} q^{2b} r^2$ and use Morita invariance and closure under subcategories [32, Proposition 4.1]. □

Lemma 4.17 (Escape from a proper set of labels)**.** *In a simple fusion ring let A be a proper dual-stable subset of the basis containing the unit and $X \in A$ a nonunit. Some product of a member of A by X or X^* has a constituent outside A.*

Proof. Otherwise every word in X, X^* is supported in A, by induction from the unit. The constituents of these words span the based subring generated by X, which is the whole ring. □

Proposition 4.18 (The gcd of the prime-free degrees)**.** *Let R be a simple integral commutative based fusion ring having both r-divisible degrees and r-free nonunit degrees, for a prime r. If the latter have minimum m, maximum M and gcd g, then $mM \geq rg$. In particular $M \geq r$ if they form a divisibility chain.*

Proof. Suppose $mM < rg$ and take $d_U = m$. For each r-free $Y \neq U^*$, the r-divisible part of UY has dimension divisible by rg: all the other terms and the total degree are multiples of g. It is therefore absent (also when $Y = \mathbf{1}$). If absent from UU^* as well, multiplication by U, U^* preserves the span of the r-free labels, contrary to Lemma 4.17. Thus UU^* contains some r-divisible B with multiplicity h. Reciprocity makes hU^* the entire r-free part of U^*B, so $r \mid hm$ and $r \mid h$. Consequently $m^2 \geq 1 + hd_B \geq 1 + r^2$, whereas $mM < rg$ gives $m < r$ since $g \leq M$. For a divisibility chain $g = m$. □

Proposition 4.19 (Separation criterion)**.** *Fix distinct primes p, q, r, exponents $a, b \geq 2$, and $c \geq 0$. Put*

$$M = p^{\lfloor a/2 \rfloor} q^{\lfloor b/2 \rfloor}, \qquad h = \max(p, q).$$

Let u and v be the least positive residues satisfying $rpu \equiv -1 \bmod q$ and $rqv \equiv -1 \bmod p$. Then

$$(M/h)^2 - 1 < r(pu + qv) \tag{4.4}$$

implies WGT at $p^a q^b r^c d$ for every square-free d coprime to pqr. The residue-independent inequality $r > M^2/(hpq)$ also suffices. In both cases c is unrestricted. In (4.4), if p, q, r are all odd, one may instead use half the left side and the residues defined with $2rp, 2rq$.

Proof. In a minimal table all three primes remain. The principal block at the odd prime h has an r-free nonunit X by prime avoidance applied to an active column. Its degree s divides M/h by Corollary 4.1 and square-divisibility. Every r-free nonunit degree is a multiple of pq dividing M, and an r-divisible degree occurs since $\Delta \mid \prod_X d_X^2$. Proposition 4.18 gives $rpq \le sM \le M^2/h$.

The nonunit terms of XX^* have degrees divisible by pq, pr or qr. Thus $s^2-1 = pqA+rpB+rqC$ with nonnegative coefficients. As $pq \mid s$, reduction modulo q, p gives $B \ge u$, $C \ge v$. This contradicts (4.4). In an odd table pair dual constituents before the same reduction to obtain the variant. □

Positive defect and active columns. The perfect double $\mathcal{Z}(\mathrm{Vec}_{A_5\times A_5})$ illustrates the role of defect zero in (3.4). Its invertibles are trivial since A_5 is centerless and perfect. Take the pure charge Z of degree $4 \cdot 3$ (standard and rotation representations), and flux Y of an involution in the second factor with centralizer character -1 on that involution. Then $d_Y = 15$, $\theta_Y = -1$ and $S_{ZY} = 4 \cdot 15 \cdot (-1) = -60$ by the charge–flux formula [40, Section 3.1, p. 4189], since a half-turn has rotation trace -1.

4.6 Common divisors and the primes of a minimal table

Balancing expresses $\chi_Y(X)$ as a positive rational sum of roots of unity of total weight d_X. In the reduced table use the actual simple lifts in the even ribbon category. Simple degrees are integral, so Galois symmetry gives $\chi_Y(X^\sigma) = \sigma\chi_Y(X)$ at the same row Y. Thus Galois permutes the based ring multiplicatively and preserves degrees and duals. In an odd ordinary table the twists, and hence all balancing phases, have odd order by the categorical Cauchy theorem [9, Theorem 3.9].

Lemma 4.20 (Common constituents and difference sets)**.** *Fix a nonunit row Y of a minimal table. Say a set of admissible phases is* exposed *if a rational linear functional on the common cyclotomic field takes value one on it and strictly less than one on the other phases. If the characters of UU^* and VV^* at Y have full-weight positive expressions supported on exposed sets E, F, for nonunits U, V, their adjoints share a nonunit whose balancing phases at Y belong to $E \cap F$.*

In particular, suppose X is a nonunit and $\chi_Y(X)$ is a scalar root of unity times a positive rational sum of s-power roots of total weight d_X, for an odd prime s. Let A be their exponent set at a common s-power level and $T = A - A$. If each μ_s-coset contains fewer than $s/2$ positions of T, then $T \cap uT$ contains a nonzero exponent for every s-adic unit u. In an odd ordinary table, fewer than $2s/3$ positions per coset suffice.

Proof. Commutativity and rigidity give $\langle UU^*, VV^*\rangle = \langle UV, UV\rangle$. If only the unit were common, it would occur once in each adjoint, so UV would be simple. This also implies simplicity for a product of chosen lifts in the positive quotient. Its double braiding would be scalar, contrary to strictness at U, V. Take a common nonunit and use its same balancing expression in both decompositions. Each exposing functional attains the full weight, so every phase in that expression lies in both sets.

Theorem 4.10 exposes the primary difference set under the stated coset bounds, using normalized complementary trace (whose minimum on odd roots is $-1/2$). The full expressions for XX^* and $X^u(X^u)^*$ have supports in T, uT, since the scalar cancels. If zero were their only intersection, the common nonunit would have character equal to its degree at Y, again contrary to strictness. □

The complementary phase at a failed exposure. If the criterion in Theorem 4.10 fails on a coset with k marked positions, let v count the unmarked positions with $b_j^- < h$ for an integer

$h \geq 2$. Since every capacity is at least one,

$$v \;\geq\; \max\left\{0, \left\lceil \frac{h(s-k)-k}{h-1} \right\rceil\right\}. \tag{4.5}$$

Each counted position admits an order n with $\mu(n) = -1$ and $\varphi(n) < h$. These possible orders are finite and explicit: $n \leq 2\varphi(n)^2$ by multiplication of the prime-power factors. If the alphabets are obtained by closing the *actual* phases of a chosen expression under complementary Galois conjugation, every such order at that position occurs in the original expression. Adding the marked pure phases for the comparison creates no negative order at an unmarked position.

Proposition 4.21. *Suppose $\chi_Y(U)$ at a fixed nonunit row has a full-mass positive expression, up to a scalar root, on primary support $A \subset \mu_L$. Set $D = A - A$ in additive exponents. If $D \cap uD = \{0\}$ for a unit multiplier u, then at least one of the two expanded adjoints of U, U^u has a coset failing Theorem 4.10, relative to its own marked set D, uD. On that coset (4.5) holds.*

Proof. Take actual simple-constituent balancing expressions in both expanded adjoints at Y. Independently in each expression close the allowed complementary orders at each primary position as above; extend the primary level if needed. New cosets with no marked root impose no restriction. If both marked sets were exposed, positivity and equality of total mass would put every constituent's phases in its respective marked set. By Lemma 4.20 the adjoints share a nonunit, also for the chosen lifts in the positive quotient. Using the same expression for that constituent on the fixed row puts all its phases in $D \cap uD = \{0\}$, contrary to strictness. □

Lemma 4.22 (Descent on the number of differences)**.** *Fix a nonunit row Y in a minimal table. Suppose the character of a nonunit X at Y is, up to a scalar root of unity, a positive s-primary word of total weight d_X on at most $m+1$ positions. Set $e = m(m+1)/2$ and*

$$g(h) = \min\{k \geq 1 : k(k+1) \geq h\}, \qquad C(e) = 1 + \max_{1 \leq h \leq e} \frac{h(h-1)}{g(h)}.$$

If $1 + 2e < s/2$ (or $1 + 2e < 2s/3$ in odd ordinary data), then $(s-1)/2 \leq C(e)$.

Proof. Keep the row Y fixed and among all such words of nonunit labels at Y, with no size restriction, choose one minimizing the number h of opposite pairs of nonzero differences. This is a positive integer by strictness, and $h \leq e$. Let U be the chosen label and D its difference set. Work at a common s-power level. For each class u of tame units modulo sign put $k_u = |(D \cap uD) \setminus \{0\}|/2$. Both adjoint spectra are exposed since $|D| = 2h + 1 \leq 2e + 1$. Lemma 4.20 supplies a common nonunit with a full word on $D \cap uD = \{0, \pm a_1, \ldots, \pm a_{k_u}\}$. Its nonzero differences, modulo sign, lie among the $a_i, 2a_i$ and the $a_i \pm a_j$ for $i < j$. There are at most $k_u(k_u + 1)$ such pairs. Minimality gives $k_u \geq g(h)$.

Tame units modulo sign act freely on opposite pairs of nonzero exponents. Each ordered pair of such pairs in D therefore contributes to at most one k_u. Thus $\sum_u k_u \leq h^2$. Since $k_1 = h$, subtraction gives $((s-1)/2 - 1)g(h) \leq h(h-1)$. □

Theorem 4.23 (A prime bound from a common divisor)**.** *Let X, Y be nonunit labels in an ordinary or positive reduced minimal table. If Y belongs to the full principal block at an odd prime p and $p \nmid d_X$, then*

$$g = \gcd(d_X, d_Y) \geq 2, \qquad p \leq b(g), \qquad b(g) = \max\{5, (g-1)g^2\}.$$

Consequently, if x is any nonunit degree and L is its largest full prime-power factor, every prime dividing Δ either divides x or is at most $b(x/L)$.

Proof. The algebraic integer $z = S_{XY}/\operatorname{lcm}(d_X, d_Y)$ has weight at most g by balancing. Since $z = (g/d_X)\chi_Y(X)$ and Y is principal, $z \equiv g$ above p. This holds at every prime above p by Lemma 2.10, since d_X/g is a p-unit here. Since $p \nmid g$, the entry is nonzero, so $g \geq 2$ by Lemma 2.5. Assume $p > b(g)$, so $p > 2g$. Theorem 4.13 writes z as a sum of g positive p-power roots. Rescaling by d_X/g gives a positive word for $\chi_Y(X)$ of full weight d_X. A single position contradicts strictness. If $g = 2$, the two positions have difference $t \neq 0$ of p-power order, and $\{0, \pm t\}$ meets its double only at zero. Since $3 < p/2$, this contradicts Lemma 4.20.

For $g \geq 3$ put $j = g - 1$ and $E = j(j+1)/2$. In Lemma 4.22, write $k(v) = \min\{k \geq 1 : k(k+1) \geq v\}$. For $1 \leq v \leq E$ we have $(v-1)/k(v) \leq j$: use $v \leq k(v)(k(v)+1)$ when $k(v) < j$ and $v \leq E$ otherwise. Hence

$$2C(E) + 1 \leq 3 + 2Ej \leq j(j+1)^2, \qquad 2(1+2E) \leq j(j+1)^2.$$

Both bounds are strictly below p, contradicting Lemma 4.22.

For the last assertion only odd $p \mid \Delta$ with $p \nmid x$ need be considered. Write $L = \ell^{v_\ell(x)}$. At a p-active column, which exists by Cauchy's theorem (Proposition 3.5 in reduced data), Proposition 4.2 supplies a nonunit $Y \in P_p$ with $\ell \nmid d_Y$. Then $g \mid x/L$, and b is nondecreasing. □

Corollary 4.24 (Degrees six and ten)**.** *If a simple in an ordinary or positive reduced minimal table has degree* 6 *or* 10*, the prime divisors of* Δ *are exactly* $2, 3, 5$*.*

Proof. For either degree x one has $x/L = 2$. The theorem restricts the primes to $2, 3, 5$. At least three must occur: otherwise every nonunit degree would contain the same prime by Lemma 2.5 and $d_Y^2 \mid \Delta$, contradicting row orthogonality (2.7) modulo that prime. □

4.7 Two principal blocks at the same witness

Lemma 4.25. *In an ordinary integral modular table let* p, q *be distinct dimension primes such that* q *divides every* p*-active degree (in particular, this holds for separated primes). Every* p*-active label* J *has a nonunit* q*-free hit* $X \in P_p \cap P_q$ *of twist one. More precisely, in the full principal* p*-block with pure* p*-twist, the sum of* $d_X\chi_J(X)$ *over the* q*-free nonunits of the principal* q*-block is* -1 *modulo a prime above* q*.*

Proof. Write E for P_p with pure p-twist. Every q-free Z is p-inactive by hypothesis. Orthogonality of J, Z on E (Proposition 4.2) multiplied by d_Z gives, modulo q,

$$d_Z + \sum_{\substack{X \in E,\ X \neq 0 \\ q \nmid d_X}} d_X \chi_J(X) \overline{\chi_X(Z)} = 0.$$

For a q-bearing X, the omitted summand was zero since $q \mid \chi_Z(X)$ by lcm divisibility. The displayed identity now also holds for q-bearing Z, because each retained X is q-free. Thus this is a relation on the whole fusion-ring basis. The distinct ring characters in the residue field are linearly independent (equivalently apply the Chinese remainder theorem after extending scalars to the residue field). The coefficient of the dimension character must vanish, proving the congruence. Complex conjugation causes no change to this principal block, since $Z \mapsto Z^*$ permutes the basis and preserves dimensions. A q-free member of E is p-inactive, so its twist is one. □

Lemma 4.26 (Lifting a positive singleton at two)**.** *Let* g *be a fermion in an integral modular category* $\mathcal{M}$ *with its positive spherical structure and even sector* $\mathcal{B} = \langle g \rangle'$*. Suppose the pair* $[X] = \{X, gX\}$ *is alone in its block of* $R_+ = K_0(\mathcal{B})/(g-1)$ *at two. The full two-block over this residue consists of* X, gX, Y, gY *with* Y *odd,* $gY \neq Y$ *and* $d_Y = d_X$*. If* $S_{XV} \neq 0$ *and* V *is odd, then* $\theta_{V,2} = \theta_{Y,2}$*.*

Proof. In characteristic two the residue on $K_0(\mathcal{B})$ determines the residue on each odd V, since V^2 is even and squaring in the residue field is injective. Thus there is a single full block over this residue. The identical special fibers of R_+ and R_- have corresponding lifted block summands of equal rank (Lemma 4.5). The signed block therefore has one character, or equivalently one odd fermion orbit. Apply Theorem 3.2 on the full block to the columns $1, g$, whose dyadic twists are opposite. Since $S_{gZ} = \pm d_Z$ according to the sector, the odd squared mass equals the even mass $2d_X^2$. A fixed label would have $d_Y^2 = 2d_X^2$, impossible for integer degrees. The odd orbit is consequently free of degree d_X. In $s^{-1} = W_{2'}^{-1} W_2^{-1}$ the dyadic Weyl preserves the block and the complementary Weyl preserves sector and actual dyadic twist. The two odd mates have equal actual twists by balancing, which proves the support assertion. □

5 Ramification and higher defects

The defect of X at p is $v_p(f_X)$. At a quartic prime the complementary labels of Lemma 4.14 have defect two; at a quintic prime they have defect three. We describe the norm spent by their characters on a fixed residue and the degrees supporting nonzero traces at a prime-square codegree.

5.1 Ramification and residue energy

For a cyclotomic integer x let $\mathsf{M}(x)$ be the mean of $|\sigma x|^2$ over its Galois conjugates and put $E(x) = [\mathbb{Q}(x) : \mathbb{Q}]\mathsf{M}(x)$. At a fixed positive degree Galois permutes the columns of a row. We bound the squared norm spent by one orbit of entries, using their common residue and ramification at p.

Theorem 5.1 (Energy at a prescribed prime residue)**.** *If a nonzero cyclotomic integer x vanishes at all primes over p, then $E(x) \geq 2p$. For odd p write the ramification index of $\mathbb{Q}(x)$ as $e = e_0 p^s$, $e_0 \mid p-1$. Then*

$$\mathsf{M}(x) \geq W_p(e) := \begin{cases} p^2 & e = 1, \\ 2p/e_0 & s = 0,\ e_0 > 1, \\ 1 + (p-1)/e_0 & s > 0. \end{cases}$$

In particular $E(x) \geq p^2$ if $e = 1$ and $E(x) \geq p^{s+1}$ if $s > 0$.

For odd p, in a p-power cyclotomic field a prescribed integer residue $0 \leq k < p$ gives $\mathsf{M}(x) \geq k(p-k)/(p-1)$.

Proof. For $p = 2$, vanishing at a single prime already gives $|N(x)| \geq 2$, whence $E(x) \geq h\, 2^{2/h} \geq 4$ for $h = [\mathbb{Q}(x) : \mathbb{Q}] \geq 1$. Now take p odd. When $e = 1$, x/p is integral and its nonzero norm gives $\mathsf{M}(x) \geq p^2$.

In the tame case take $x \in F(\zeta_p)$ with F cyclotomic of conductor prime to p. In an integral expression $x = \sum_{j \bmod p} a_j \zeta_p^j$, the sum of the a_j vanishes at every prime of F over p, hence lies in $p\mathcal{O}_F$. Subtract its average from every coefficient. This gives a unique expression of coefficient sum zero, with $\mathsf{M}(x) = \frac{p}{p-1} \sum_j \mathsf{M}(a_j)$. Inertia fixing x has order $g = (p-1)/e_0$ and its nonzero-index orbits have length g. Each nonzero orbit of coefficients costs at least g by the norm. If there is just one such orbit with common value a, then $a_0 = -ga$, so the cost is at least $g + g^2 \geq 2g$. This proves the tame bound.

For $s > 0$ the inertia conductor is p^{s+1}. With $\zeta = \zeta_{p^{s+1}}$ and F as above, the relative integral basis over $F(\zeta_p)$ gives a unique expansion $x = \sum_{j=0}^{p^s-1} \zeta^j \gamma_j$. The terms are orthogonal for the relative trace, hence $\mathsf{M}(x) = \sum_j \mathsf{M}(\gamma_j)$. Inertia fixing x has order $g = (p-1)/e_0$; it acts freely on nonzero indices and preserves coefficient means, since the coefficients are conjugate up to pth roots of unity. Some nonzero-index orbit must occur by wild ramification. Two such orbits cost at least $2g \geq g + 1$, and one together with $\gamma_0 \neq 0$ costs $g + 1$. For a single orbit with $\gamma_0 = 0$, all

its coefficients have the same residue at each prime over p. Their residues sum to zero, so each vanishes there since $p \nmid g$. The preceding tame estimate (or the unramified estimate) gives mean at least $2p/(p-1)$ for each. Their sum is at least $g+1$. Finally $[\mathbb{Q}(x):\mathbb{Q}] \geq e$ gives the energy bounds.

For the prescribed residue in $\mathbb{Q}(\zeta_p)$ choose integer coefficients c_j with $\sum c_j = k$. The same quadratic form gives $\mathsf{M}(x) = (p\sum c_j^2 - k^2)/(p-1) \geq k(p-k)/(p-1)$. At higher level the orthogonal expansion above adds the means of its coefficients. Their least residues sum to k modulo p, and $u(p-u)+v(p-v) \geq w(p-w)$ for $w \equiv u+v \bmod p$ in $[0,p)$. Iteration gives the claim. □

The bounds $2p$ and p^{s+1} are sharp. For any subgroup $H \leq \mathbb{F}_p^\times$ of index $e > 1$ and distinct cosets aH, bH, $\sum_{j\in aH} \zeta_p^j - \sum_{j\in bH} \zeta_p^j$ vanishes above p. Its coefficient sum is zero, its stabilizer is H, and its energy is $2p$. For every $s \geq 1$ take the Teichmüller subgroup $H \subset (\mathbb{Z}/p^{s+1}\mathbb{Z})^\times$ of order $p-1$. The integer $1 + \sum_{j\in H} \zeta_{p^{s+1}}^j$ vanishes above p and has mean p, by the distinct orthogonal indices modulo p^s. Its nonconstant word has length $p-1$ and misses a slot on every μ_p-coset, so its stabilizer is precisely H. Its energy is p^{s+1}.

Corollary 5.2 (Conductor and row budget)**.** *In a symmetric integral table fix an odd prime p and put $c = v_p(\Delta)$ and $t_X = v_p(f_X) = c - 2v_p(d_X)$. A row field has zero p-conductor exponent when $t_X = 0$ and conductor exponent at most t_X otherwise. The same holds for an entry field with t_X replaced by $t = \min(t_z, t_X)$. When $t \geq 1$ its wild inertia has order at most p^{t-1}.*

Suppose the entries $\chi_z(X)$ on p-bearing degrees vanish modulo p (at every prime above p). This holds if the block of z contains a p-unit degree. It also holds if $p \mid d_z$ and each nonzero entry on such a degree has $X \in P_p$ with $v_p(d_X) \geq v_p(d_z)$. For each p-bearing degree d let $u_d = (d/\gcd(d,d_z))_{p'}$. Partition the nonzero entries $\chi_z(X)/u_d$ at degree d into Galois orbits O counted with multiplicity. Write $h_O = |O|$, $e_O = e_{0,O}p^{s_O}$ for the degree and ramification index of the entry field. Then

$$\sum_{d:\, p\mid d} \sum_{O \text{ at } d} u_d^2 h_O W_p(e_O) \;\leq\; f_z - 1. \tag{5.1}$$

Proof. Corollary 2.8 also shows that a codegree-four row takes values in $\mathbb{Q}(\zeta_8)$. The stated conductor bounds follow from the same result, applied to either row since their normalized entry lies in both fields. The bound from the different is attained at conductor p^t: the full cyclotomic field has normalized different $t - 1/(p-1)$.

On a p-bearing degree d, every character of p-unit degree vanishes above p by lcm integrality. The common residue of its block gives the same vanishing for χ_z. In the second case use symmetry: $\chi_z(X) = (d_X/d_z)\chi_X(z)$, with $\chi_X(z) \equiv d_z$ and d_X/d_z integral at p. In both cases Galois covariance gives vanishing on all conjugates. Division by the integral lcm scale u_d, prime to p, preserves both integrality and vanishing. Sum the energy bounds over the orbits and reserve the unit's squared contribution one. □

If all the scales under consideration are at least u, a nonzero orbit costs at least p^2u^2 when unramified, $2pu^2$ when nontrivially tame, and $p^{s+1}u^2$ at wild depth s.

Corollary 5.3 (Small budgets and forced hits)**.** *For a row satisfying this vanishing condition, if $f_z = p^2m$, $p \nmid m$, at most $m-1$ nonzero orbits on p-bearing degrees are unramified or wild in total; the wild depth is at most one. For $m = 1$ all such orbits are nontrivially tame and number at most $(p-1)/2$.*

For any prime p, suppose there are just three dimension primes p, q, r, every nonunit degree involves at least two, and the nonunit row z has zero residue on its supported p-bearing degrees at every prime above p. Then

$$f_z \geq [K_z : \mathbb{Q}] + 2p^2.$$

Proof. For the first assertion divide the budget p^2m-1 by p^2 or by $2p$ and use the conductor bound at $t_z=2$. For the second apply (2.5) modulo q and r. Its vacuum term is one, so each reduction supplies an orbit of nonzero trace on a nonunit degree prime to the respective prime. The two orbits are distinct and both degrees are p-bearing. Each trace is a nonzero integer divisible by p. Their contributions to (2.4) together with the vacuum give the asserted bound. □

5.2 Degrees at a prime-square codegree

Proposition 5.4 (Degrees supporting a nonzero trace)**.** *Let $Z\neq\mathbf{1}$ have formal codegree p^2 in a perfect ordinary or positive reduced integral table. The nonunit degrees at which χ_Z has nonzero absolute trace all divide*

$$M=d_Z/\gcd(p,d_Z)$$

and have greatest common divisor one. If the table is fusion-simple, then M has at least three distinct prime divisors; when it has exactly three, each pair is the prime support of some such degree. In particular at a quartic prime of a three-prime minimal table there is no row of codegree p^2, for any p.

Proof. By (2.4) a nonzero trace T_X at $X\neq\mathbf{1}$ is prime to p: a nonzero multiple of p would exhaust the norm before counting the unit. Now $d_X^2\mid\Delta=p^2d_Z^2$, so $d_X/\gcd(d_X,d_Z)$ is one or p. In the latter case lcm integrality makes $\chi_Z(X)$, and hence its trace, a multiple of p. Thus $d_X\mid d_Z$. Moreover $X\in P_p$ by Theorem 2.11. If $v_p(d_X)=v_p(d_Z)>0$, then $\chi_Z(X)=(d_X/d_Z)\chi_X(Z)$ vanishes above p by principality. Lemma 2.10 gives this vanishing for every conjugate, hence $p\mid T_X$. Hence $d_X\mid M$. The dimension identity (2.5) expresses -1 as an integral combination of the indicated degrees.

In the fusion-simple case no nonunit has prime-power degree. For each prime dividing M some indicated degree omits it; this proves both assertions about prime supports. Finally $v_p(\Delta)=4$ gives $v_p(d_Z)=1$, so with only two other dimension primes M cannot have three. □

6 Local Weyl columns at separated primes

For the vacuum estimates let $\mathcal{C}$ be an integral modular category of square dimension $D=n^2$ with its positive spherical structure. Write $s=S/n$ and use the lift $t=T/\gamma$ of Section 3, with $\gamma^3=\xi=p^+/n$ and γ of two-power order. For a set J of primes of the level write $W_J=\prod_{p\in J}W_p$, with γ_J and $\theta_{Z,J}$ the J-parts of the corresponding roots. Put $M_m(Z)=\sum_Y d_Y\chi_Z(Y)\theta_Y^m$.

Theorem 6.1 (The degree-normalized vacuum column)**.** *For $m\geq1$ and all Z, $|M_m(Z)|^2/D$ is an algebraic integer. The vector $w=W_Jv_0=W_J^{-1}v_0$ is rational and*

$$k_{J,Z}=nw_Z/d_Z\in\mathbb{Z}.$$

If m is one on the primary levels in J and zero on the others, then $M_m(Z)=n\gamma_J^3\theta_{Z,J}^{-1}k_{J,Z}$. Every prime $p\notin J$ dividing f_Z divides $k_{J,Z}$.

Proof. In the congruence lift, $\sigma_a(\rho(g))=\rho(A_agA_a^{-1})$ for $A_a=\mathrm{diag}(a,1)$ by [21, Theorem II and Remark 4.5]. The torus permutations have positive signs and fix the vacuum, since its entries d_i/n are positive rational. Conjugating W_J introduces a torus on its right; the torus W_J^2 also fixes the vacuum. This gives both assertions about w.

In (3.2) take $i=j=0$. Since $\theta_{Y^*}=\theta_Y$, we have $(sT^ms^{-1})_{0Z}=(sT^ms^{-1})_{Z0}=d_ZM_m(Z)/D$. Consequently $L_m^{00}(f_Ze_Z)=|M_m(Z)|^2/D$ is an algebraic integer. For the indicated CRT exponent, the local identity $U(WUW^{-1})U=W^{-1}$ in each selected factor gives

$$(sT^ms^{-1}v_0)_Z=\gamma_J^3\theta_{Z,J}^{-1}w_Z.$$

This proves the formula for $M_m(Z)$. The rational number $k_{J,Z}$ has integral square and is therefore an integer. If $p \notin J$ choose the same CRT exponent divisible by p. Lemma 3.1(i) applied to $y = f_Z e_Z$ gives $L^{00}_m(y)^p \equiv f_Z^{p-1} L^{00}_{m/p}(y)$ above p. Here $L^{00}_m(y) = k^2_{J,Z}$, proving the divisibility. □

Lemma 6.2 (Vacuum entry and local support)**.** *For the CRT exponent of Theorem 6.1,*

$$n^{-2} \sum_i d_i^2 \theta_i^m = \xi^m (W_J)_{00}, \qquad nw \in \mathbb{Z}^{\mathrm{Irr}(\mathcal{C})}.$$

If $p \mid n$ lies outside J, then $p \mid nw_i$ for each i; thus w_i is p-integral when $v_p(n) = 1$. For $J = \{q\}$ the support is contained in the principal q-block.

Proof. Set $Z = 0$ in the theorem and use $\gamma_J^3 = \xi^m$. At any label $nw_i = d_i k_{J,i}$; if $p \nmid d_i$, then $p \mid f_i$ and the theorem supplies the factor p. The principal q-block subspace is invariant under W_q by Lemma 3.3 and contains v_0. □

Proposition 6.3 (Weyl columns on separated prime sets)**.** *Partition the primes in the modular level of an integral modular category of dimension n^2 into sets J such that each simple has twist supported in at most one set. Write γ_J for the J-part of the anomaly normalization γ, n_J for the J-part of n, and $V_J = \gamma_J^{-3} W_J$. Then*

$$(V_I - 1)(V_J - 1) = 0 \quad (I \neq J), \qquad s = \xi\Big(1 + \sum_J (V_J - 1)\Big), \qquad n_J W_J v_0 \in \mathbb{Z}^{\mathrm{Irr}(\mathcal{C})}. \tag{6.1}$$

If $p \in J$ and $v_p(d_X) < v_p(n)$, then $v_p((n_J W_J v_0)_X) = v_p(d_X)$. If every other prime of J occurs to full height in d_X, then $X \in P_p$.

A character field ramifies at primes of at most one set J. More generally, in any integral modular category a character field cannot ramify at two distinct odd dimension primes that never occur together in a twist.

Proof. In factor J let U_J denote the represented upper unipotent. Its normalized twist $\gamma_J U_J$ has the actual J-phases. Let E_J be the span of the translates under the J-factor of the nontrivial phase spaces. On each of these spaces the complementary upper unipotents are scalar; their generating conjugates have the same scalars since the entire complementary group preserves the space. The same holds on its translates, by commutation. Hence on E_J each W_I, $I \neq J$, is the scalar γ_I^3 by $U(WUW^{-1})U = W^{-1}$. On $E_J^{\perp}$ the J-upper unipotent is scalar γ_J^{-1}, so the same reasoning gives $W_J = \gamma_J^3$. This proves the zero products and the formula for $s = \prod_J W_J$. In particular on E_J one has $s = (\xi/\gamma_J^3) W_J$.

Choose m selecting J as in Lemma 6.2. Up to diagonal phases and duality, the vacuum coordinates of W_J are the entries of $sT^m s^{-1} v_0$. Subtracting the untwisted coordinate leaves a sum of $(\theta_Y^m - 1)\chi_Y(X)/f_Y$. Only J-active labels occur; by Lemma 3.4 their codegrees are supported in J. Thus $W_J v_0$ is integral away from J, while $nW_J v_0$ is integral by Lemma 6.2. In the split vacuum column all terms except the J-term are p-integral, proving the valuation assertion by comparison with d_X/n. Full height at $q \in J \setminus \{p\}$ gives $q \nmid f_X$; the proof of Lemma 3.4 shows that W_q preserves the X-line. The nonzero X-entry of $W_J v_0$ therefore gives a nonzero entry of $W_p v_0$ there, proving principality.

For the ramification assertion take two distinct primes in different sets, or any two separated odd primes. The same moving-space argument gives $(P - I)(Q - I) = 0$ for their inertia torus permutations: the complementary torus acts as a scalar on an active phase space, hence as the identity there because it permutes its coordinate vectors. Here the upper unipotent at the odd prime acts trivially on the complement of its moving space, and so does its torus. The torus signs are positive in square dimension. In arbitrary integral dimension both separated odd primes have even valuation by Lemma 3.4 applied to the opposite active sectors, so their local Galois actions also fix $\sqrt{D}$ and have positive sign. If both permutations moved the label X, the diagonal entry at X of $(P - I)(Q - I)$ would be positive. □

Proposition 6.4 (Simultaneous singleton identities)**.** *Call t* isolated *if every t-active label has pure t-power twist, and a* singleton *if $v_t(n) = 1$. Let T be a nonempty set of odd isolated primes, each occurring once in n. Write $m_t = n/t$, $\epsilon = \xi$, $j_t = t(1 - (W_t)_{00})$ and $a_t(X) = \epsilon t(W_t)_{X0}$ for $X \neq 0$. Then $\epsilon \in \{1, -1\}$, $j_t > 0$ (and $j_t < t$ if $m_t > 1$), $m_t j_t \equiv -\epsilon \pmod t$, and $\sum_{t\in T} j_t/t \leq 1 + 1/n$. Every t-active label has degree m_t, and their number is at least $(t-1)j_t$. One has*

$$tj_t = \sum_{Y \ t\text{-active}} (1 - \theta_{Y,t}). \tag{6.2}$$

For odd n, each j_t is even. Each a_t is integral, supported exactly on the nonunit t-free degrees, which belong to P_t. The column identities are

$$\begin{aligned} a_t(X) &\equiv d_X/m_t \pmod t, & \sum_{X\neq 0} d_X a_t(X) &= (n+\epsilon)j_t, \\ \sum_{X\neq 0} a_t(X)^2 &= 2tj_t - j_t^2, & \sum_{X\neq 0} a_u(X)a_v(X) &= -j_u j_v \quad (u \neq v). \end{aligned} \tag{6.3}$$

In particular, after the active labels are removed the residual squared norm is at most $B_t = j_t(t + 1 - j_t)$.

Proof. Use the singleton clusters $\{t\}$, $t \in T$, and their complement H. A t-active label Y has $f_Y = t^2$, by complementarity and $v_t(f_Y) = 2 - 2v_t(d_Y) > 0$. The equality $s = \xi W_t$ on E_t gives $(W_t)_{Y0} = 1/(t\xi)$; rationality gives the common sign. The vacuum Gauss formula gives (6.2), so $j_t > 0$. The mean of a nontrivial t-power root is at least $-1/(t-1)$, proving the count. For odd n the active labels come in distinct equal-twist dual pairs, and t is odd.

Write $z_t = (W_t - I)v_0$. By (6.1), $s = W_H + \epsilon \sum_{t\in T}(W_t - I)$, with $W_H = I$ when H is empty and $\epsilon = 1$ in that case. Unitarity at the vacuum gives the deficit-sum bound. Multiply the split vacuum column by n and reduce modulo t to get both congruences. In particular $j_t \neq t$, so $j_t < t$ when $m_t > 1$. The scaled column is integral and its vacuum entry is nonzero, so every nonunit coordinate has absolute value $< t$. This proves the support and principality assertions. Finally $z_u \perp z_v$ for $u \neq v$, $\|z_t\|^2 = 2j_t/t$, and $\langle z_t, sv_0\rangle = \epsilon\langle z_t, W_t v_0\rangle = \epsilon j_t/t$. Subtract the vacuum in these three identities to obtain (6.3); each active coordinate has square one. □

Lemma 6.5 (An isolated prime with unit vacuum deficit)**.** *Let $\mathcal{C}$ be integral modular of dimension n^2 with its positive spherical structure. Suppose $v \geq 5$ is an odd prime with $v_v(n) = 1$, and every v-active simple has v-primary twist (its prime-to-v twist is one). Put $m = n/v$ and suppose $j_v = v(1 - (W_v)_{00}) = 1$. There is exactly one active label Y_u of twist ζ_v^u for each $u \in \mathbb{F}_v^\times$, and each has degree m. With $\xi = p^+/n = \pm 1$ one has*

$$\chi_{Y_u}(Y_t) = \xi \sum_{k\in\mathbb{F}_v^\times} \zeta_v^{-u/k-kt}.$$

All active labels are self-dual with field $\mathbb{Q}(\zeta_v)^+$, and all inactive label fields are unramified at v. The active rows agree on the inactive labels X, with integral values $b_X \equiv d_X \bmod v$. Omitting the vacuum from the inactive labels,

$$\sum_X b_X^2 = v, \qquad \sum_X d_X b_X = -1 - \xi m. \tag{6.4}$$

Proof. Propositions 6.3 and 6.4 give $\sum_{Y \text{ active}}(1 - \theta_Y) = v$, with $s = \xi W_v$ on the active span. All active labels have degree m and are v-principal.

A square-Galois orbit of primitive v^eth phases, $e \geq 2$, has zero phase sum and size $v^{e-1}(v - 1)/2 > v$, which the real part of this identity precludes. At level v the cyclotomic relation forces constant multiplicity on the nontrivial phases, hence multiplicity one. Since γ has two-power order, the v-part of $\operatorname{ord}(T/\gamma)$ is exactly v.

The $Y_u, 0$ entry of the local Bruhat identity $WU^kW^{-1} = U^{-1/k}Wh_{-k}U^{-1/k}$ for $k \neq 0$ gives $\sum_{t\neq 0}(\zeta_v^{kt} - 1)\chi_{Y_u}(Y_t) = \xi v\zeta_v^{-u/k}$: the torus fixes the vacuum. Summing over $k \neq 0$ gives $\sum_t \chi_{Y_u}(Y_t) = \xi$; Fourier inversion gives the displayed entries. On the active coordinates their Gram matrix has diagonal $v(v-1)-1$ and off-diagonal $-v-1$ by Parseval. The full rows have Gram matrix $v^2 I$, so their inactive restrictions coincide, with squared norm $v+1$. Galois permutes the active rows, making their common inactive entries rational integers. Principality gives the congruences. Removing the unit from the norm and from orthogonality against the dimension character gives (6.4). Duality preserves each phase and so fixes its unique label; square twist covariance gives precisely the indicated real field in $\mathbb{Q}(\zeta_v)$. On an inactive X, the vector $v_X - b_X v_0$ and its inverse Weyl image vanish on the active sector. Both local unipotents, and hence the local torus, fix it. The torus fixes v_0 as well, so X is inertia-fixed. □

Lemma 6.6 (Column norm on isolated row sets)**.** *In the setting of Proposition 6.4, let X be any nonunit label of degree d and $[d]_t$ its centered residue at t. For $k = |[d]_t|$ put $E_t = j_t k(t-k)/t^2$. Then $\sum_{t\in T} E_t < 1$. At an unramified t one may replace E_t by $(t-1)j_t[d]_t^2/t^2$. If also $t \geq 5$, every t-active Z has $\chi_Z(X) = [d]_t$. For such an inactive X,*

$$(W_t - I)v_X = [d]_t(W_t - I)v_0, \qquad a_t(X) = -\epsilon j_t[d]_t.$$

Any set L of nonunit inactive t-unramified labels therefore satisfies $1 + \sum_{X\in L}[d_X]_t^2 \leq \lfloor (t+1)/j_t \rfloor$. In particular, for every isolated odd singleton, $j_t \mid t-1$ or $j_t \mid t+1$.

Proof. Each active Z is t-principal, of codegree t^2. Its value on X lies in a t-power cyclotomic field and reduces to d at t. Theorem 5.1 and the active count give at least E_t in the Galois mean of the normalized X-column norm. Distinct active sectors are disjoint and the vacuum also contributes. Apply the estimate at the active degree $m_t = n/t$, with $k = |[m_t]_t|$. Then $0 < j_t k < 2t^2/(t+1) < 2t-1$ and $j_t k \equiv \pm 1 \pmod t$, so $j_t k$ is 1, $t-1$ or $t+1$.

If K_X is unramified at t, symmetry gives rational integers congruent to d, hence the sharper contribution. Distinct integers congruent modulo t have squares summing to at least $(t^2+1)/2$. Each occurs on an active orbit of at least $(t-1)/2$ labels by twist covariance, exceeding the norm budget for $t \geq 5$. The one value c obeys $(t-1)j_t c^2 < t^2$ and hence is $[d]_t$. If X is inactive, $v_X - cv_0$ and its inverse Weyl image both vanish on the active sector, so the upper and lower unipotents fix it. The vacuum entry gives the formula for a_t since $W_t^{-1}v_0 = W_t v_0$. On the active rows, columns $0, L$ now form a rank-one matrix with constant row $t^{-1}(1, ([d_X]_t)_{X\in L})$. Its operator norm is at most one. The active count and integrality of $1 + \sum[d_X]_t^2$ give the stated bound. □

Lemma 6.7 (Prime supports at an isolated prime)**.** *Let t be isolated in an ordinary minimal table of square dimension. Suppose a nonunit of codegree t^2 has t-valuation at most one in its degree. The proper nonunit t-free degrees intersect pairwise in prime support, have gcd one, and include at least three distinct degrees. Here proper means smaller than the t-free part of n.*

In particular, every isolated singleton t satisfies $t \geq 5$. Writing $m = n/t$, its nonunit t-free labels satisfy

$$\sum_X \left(a_t(X)^2 - \frac{d_X}{m} a_t(X) \right) = j_t(t - j_t) - \frac{\epsilon j_t}{m}.$$

Proof. Proposition 5.4 gives t-free nonunit degrees of gcd one. They divide the t-free part of n, so deleting that full degree preserves the gcd. All t-free labels are in P_t by Proposition 6.3. Two such nonunits hit one another by principality, so share a prime. Pairwise intersection with empty total intersection requires at least three distinct proper degrees and at least three distinct primes in them.

At a singleton a t-active row exists and has degree $m = n/t$ and codegree t^2 by complementarity. The active row hits the vacuum, the full degree and all three proper degrees by principality.

Each costs at least one in its Galois-mean squared norm, excluding $t = 2$. For odd t subtract the dimension trace divided by m from the squared norm in (6.3). Each displayed summand is at least $1 - d_X/m \geq 0$. If $t = 3$, then $m \geq 2 \cdot 5 \cdot 7 = 70$ and the three distinct indices $m/d_X > 1$ are prime to three. Their reciprocals sum to at most $1/2 + 1/4 + 1/5$, so the three summands contribute at least $41/20 > 2 + 2/m$, contradicting the identity. □

Lemma 6.8 (An isolated quartic prime)**.** *Let q be an isolated prime in an ordinary minimal table of dimension n^2, where $n = q^2N$ and $q \nmid N$. If at least two simple degrees (counted with multiplicity) have q-valuation one, then at least $q(q-1)$ active labels have degree N, and some active J has degree qN. The row of each such J vanishes on all q-bearing degrees, and necessarily $q \geq 5$.*

Proof. Write A_0, A_1 for the active labels of degrees N, qN. These are all the possibilities by Lemma 3.4. Put $m = |A_0|$, $\delta_i = \sum_{Z \in A_i}(1 - \theta_{Z,q})$ and $b = q^2 W_q v_0$. Proposition 6.3 makes b integral, $v_q(b_X) = v_q(d_X)$ for $v_q(d_X) < 2$, and all height-one labels q-principal. The vacuum Gauss formula gives, for a root of unity λ,

$$\lambda b_0 = q^2 - \delta_0/q^2 - \delta_1.$$

For such a label X, every $Z \in A_0$ has $S_{XZ} \neq 0$ by principality, and $S_{XZ}/(qN)$ is integral by lcm divisibility. Their Galois-mean contribution to the normalized row norm is at least m/q^2 besides the vacuum; thus $m < q^2$. Since b_0 is a q-unit and δ_1 vanishes above q, the Gauss formula gives $m > 0$. All conjugates of the algebraic integer $1 - \delta_0/q^2$ are strict convex averages of 1 and nontrivial roots of unity, hence have modulus less than one. Therefore $\delta_0 = q^2$. The mean of $1 - \theta$ for a nontrivial q-power root is at most $q/(q-1)$, so $m \geq q(q-1)$. If A_1 were empty, $|b_0| = q^2 - 1$; with the two height-one coordinates this costs at least $(q^2-1)^2 + 2q^2 > \|b\|^2 = q^4$.

Finally let $J \in A_1$. Its row spends at least $m \geq q(q-1)$ on A_0 by principality. For height-one X, $\chi_J(X) = (d_X/d_J)\chi_X(J)$ vanishes above q, since $X \in P_q$ and d_X/d_J is a q-unit. This holds on all conjugates. A nonzero orbit would cost at least $2q$ by Theorem 5.1, exceeding the remaining norm $f_J - m \leq q$. Height two is excluded by defect-zero vanishing at the active column J. By Lemma 6.7 this row also hits at least three distinct proper q-free degrees in P_q. Together with A_0 and the vacuum they cost at least $q(q-1) + 4$, forcing $q \geq 5$. □

6.1 A denominator bound for local Weyl columns

Fix a set J of prime divisors of the level, and write $T_J = \mathrm{diag}(\theta_{z,J})$, $U_J = \prod_{q \in J} U_q$, so $T_J = \gamma_J U_J$. A J-active label has $\theta_{z,J} \neq 1$.

Theorem 6.9 (Denominators of a complementary Weyl product)**.** *For $p \notin J$ put $a = v_p(f_z)$. At every place above p, normalized by $v(p) = 1$, a nonzero entry w_{iz} of $W_J^{\pm 1}$ satisfies*

$$v(w_{iz}) \geq \begin{cases} 0 & a = 0, \\ -\lfloor (a-1)/2 \rfloor & a \geq 1. \end{cases}$$

If $v_p(f_z) \leq 2$ for every J-active label, both matrices are integral at all places above p.

Proof. Take $m > 0$ equal to one on the primary levels in J and zero on the others, also requiring $p \mid m$. Put $A = sT^m s^{-1}$ and $w = W_J^{-1}$. Since the selected factors commute, multiplying their Bruhat identities gives $U_J(W_J U_J W_J^{-1})U_J = W_J^{-1}$. Thus A is w up to left and right diagonal roots of unity. Moreover, Galois conjugation by $A_k = \mathrm{diag}(k, 1)$ as in Theorem 6.1 fixes w whenever k is one on all the selected levels. Its entries therefore belong to the cyclotomic field of conductor dividing the product of these levels, unramified at p. For $k = -1$ the same conjugation gives $\bar{w} = w^{-1} = Hw$, with $H = W_J^2$ the product of the tori at -1. It is an involutive permutation matrix in this normalization.

At $j = Hi$, $\overline{w_{jz}} = w_{iz}$. Equation (3.2) makes $f_z A_{iz}\overline{A_{jz}}$ integral. If $p \mid f_z$, Lemma 3.1(i) and $(f_z e_z)^p = f_z^{p-1}(f_z e_z)$ give strictly positive valuation. Hence $a + 2v(w_{iz}) \geq 0$, strictly if $a > 0$. Unramifiedness makes $v(w_{iz})$ an integer. This gives the asserted floor bound and also applies to $W_J = Hw$.

For the last assertion take R to be the intersection of the valuation rings above p in a common cyclotomic field. The columns $u_z = W_J v_z$ for J-active z and their adjoints lie over R, since conjugation permutes these places. Unitarity gives

$$W_J T_J W_J^{-1} = I + \sum_{z \ J\text{-active}} (\theta_{z,J} - 1) u_z u_z^*.$$

This is integral over R entrywise. Replacing T_J by U_J changes a unit scalar. The product Bruhat identity now gives W_J^{-1} over R, and its adjoint gives W_J. □

Lemma 6.10 (The tetrahedral Weyl matrix)**.** *Let u be a 3-cycle and w a nonidentity element of the normal Klein four-group K in A_4. A unitary representation of A_4 of dimension at most four is a sum of characters of A_4/K, or consists of one tetrahedral summand T and at most one such character. In the latter case T has one line at each u-eigenvalue. In a suitable orthonormal basis of these lines the matrix of w on T is $2J/3 - I$, where J has all entries one. On any character summand w acts by one. If T occurs and the u-fixed space is two-dimensional with orthonormal basis v, v', and $x = \|P_T v\|^2$, then*

$$|\langle v', wv\rangle|^2 = \tfrac{16}{9} x(1 - x).$$

Proof. The three nontrivial characters of K are cycled by u. Choose a unit vector b of the unique nontrivial K-character taking the value $+1$ on w, if this character occurs. Then $b, ub, u^2 b$ lie in mutually orthogonal character spaces and span an invariant copy of T. Its orthogonal complement has dimension at most one. If no such character occurs, K acts trivially throughout. On the displayed basis w has signs $(1, -1, -1)$. For $\omega = \exp(2\pi i/3)$, the vectors $3^{-1/2} \sum_{j=0}^{2} \omega^{-kj} u^j b$ diagonalize u. The projector onto $\mathbb{C}b$ has matrix $J/3$ in that basis, giving the formula for w. On the two-dimensional fixed space the compression of w is $I - 4P_T/3$, where P_T restricts there to a rank-one projection. Its off-diagonal entry gives the asserted square. □

6.2 Local Weyl coefficients and twist masses

The following applications use the notation and degree-normalized integers of Theorem 6.1.

Proposition 6.11. *Let p, q be separated twist primes in a perfect fusion-simple integral modular category of square dimension. At a nonunit Z suppose $a = p^{v_p(n/d_Z)} > 1$, $b = q^{v_q(n/d_Z)} > 1$, and all other primes of n occur to full height in d_Z. Let L_p, L_q bound the degrees of p-active and q-active labels, respectively. Put $f = f_Z$. Then*

$$2L_p(f - 1) \geq nb^2, \qquad 2L_q(f - 1) \geq na^2.$$

Proof. Separation forces full q-height on p-active degrees and full p-height on q-active degrees. The deficiencies at both primes, together with full height elsewhere, therefore give $\theta_Z = 1$. Each other active prime misses the row Z by defect-zero vanishing. Write $b_Y = \chi_Z(Y)$ and put

$$H_t = \sum_Y d_Y b_Y (\theta_{Y,t} - 1) \quad (t = p, q).$$

By $\sum d_Y b_Y = 0$, Theorem 6.1 gives $H_t = n\gamma_t^3 k_{t,Z}$. Every nonzero hit with nontrivial twist has only one of the two components, hence $H_p + H_q = n\xi$ by the vacuum modular relation, where $|\xi| = 1$ and ξ is a root of unity. At p-active labels, $q^{v_q(n)} \mid d_Y$ and $b \mid b_Y$ by lcm integrality. Thus $b \mid k_{p,Z}$; similarly $a \mid k_{q,Z}$. Neither integer vanishes: the other would then have absolute value

one, violating its divisibility. On the p-active labels, a union of Galois orbits, the integral entries b_Y/b satisfy $\sum |b_Y/b| \le \sum |b_Y/b|^2$. Indeed every nonzero orbit has mean square at least one by the absolute norm, followed by Cauchy–Schwarz. Consequently

$$nb \le |H_p| \le 2L_p \sum_{Y \ p\text{-active}} |b_Y| \le 2L_p(f-1)/b,$$

with one unit of squared norm reserved for the vacuum. Interchanging p, q proves the other inequality. □

A two-variable integer test. The magnitude bounds can be combined with the phase equation. Under the hypotheses of Proposition 6.11, there are nonzero integers $u \in b\mathbb{Z}$ and $v \in a\mathbb{Z}$ such that

$$u + v = 1, \qquad \frac{n}{2}\left(\frac{b|u|}{L_p} + \frac{a|v|}{L_q}\right) \le f - 1.$$

In particular $\xi = \pm 1$ in this situation. Indeed if both primes are odd, $\gamma_p = \gamma_q = 1$ so the phase equation is $k_{p,Z} + k_{q,Z} = \xi$, an integer of modulus one. If $p = 2$, it is $\xi k_{p,Z} + k_{q,Z} = \xi$; as $k_{q,Z} \neq 0$ both sides force ξ real. The other case follows on interchanging the primes. Absorb these signs into u, v to obtain $u + v = 1$. The estimate for H_p in the proof gives a squared row norm of at least $nb|u|/(2L_p)$ on the p-active sector, and likewise for q. The two sectors are disjoint and omit the vacuum. This proves the test. For fixed a, b the minimum on the left is elementary: solve $u \equiv 0 \pmod b$, $u \equiv 1 \pmod a$, and take the solution on each side of $[0, 1]$ nearest to it. Thus the test applies to prime decomposition forms without choosing individual simple labels. The same argument bounds the total number of deficient rows. Fix the primes and n in Proposition 6.11, but let Z range over all nonunit labels with the two indicated deficiencies and full height elsewhere. Since $k_{p,Z}$ is a nonzero multiple of $q^{v_q(n/d_Z)}$, the unit norm of $W_p v_0$ gives

$$\sum_Z p^{2v_p(d_Z)} < p^{2v_p(n)}, \qquad \sum_Z q^{2v_q(d_Z)} < q^{2v_q(n)}.$$

For strictness, equality would put $W_p v_0$ entirely on those twist-one labels. This vector and v_0 then have the same eigenvalue for the local upper unipotent U_p. Since $W_p^2 v_0 = v_0$ and $W_p^{-1} v_0 = W_p v_0$, the local identity $W_p U_p W_p^{-1} = U_p^{-1} W_p^{-1} U_p^{-1}$ applied to v_0 forces $W_p v_0$ proportional to v_0, contradicting its nonunit support. The strict bound for q follows in the same way.

Squared mass at a prescribed twist. Write $D = n^2$, $R = \operatorname{rank} \mathcal{C}$, $\tau_j = \sum_X d_X^2 \theta_X^j$, and $M_t = \sum_{\theta_X = t} d_X^2$. The following argument uses the genuine second indicator of the category. It applies to an integral modular category with its positive spherical structure.

Proposition 6.12. *Put $B = \sum_X d_X \nu_2(X)$. Then B is a nonnegative integer, $B \le \sum_{X = X^*} d_X \le n\sqrt{R}$, and for every unit complex number t,*

$$9M_t \le 3D + 4\operatorname{Re}(\bar{t}\tau_1) + 2\operatorname{Re}(\bar{t}^2\tau_2) \le 3D + 4n + 2n\sqrt{B}.$$

More generally, if the twist exponent is prime to all integers $1, \dots, k-1$, then $M_t \le (D + (k-1)n)/k$.

Proof. Multiply the Ng–Schauenburg formula for $\nu_2(X)$ by d_X, sum on X, and use $\sum_X N_{UV}^X d_X = d_U d_V$. This gives $DB = |\tau_2|^2$. The indicator bounds follow from its values 0 off the self-duals and ± 1 on them. Sum $d_X^2 |1 + w + w^2|^2$ with $w = \bar{t}\theta_X$. Its value at $w = 1$ is nine and every summand is nonnegative. Expanding proves the assertion since $|\tau_1| = n$. In the last assertion use $|1 + w + \cdots + w^{k-1}|^2$ in the same way. For each indicated j an automorphism takes τ_1 to τ_j, and τ_1 is n times a root of unity. □

Degree sums for a prime-power codegree. Let a row Z have formal codegree a power of p and $p \mid d_Z$. All its nonzero entries lie in P_p (Theorem 2.11). For every degree d with $v_p(d) \geq v_p(d_Z)$,

$$\sum_{d_Y = d} \chi_Z(Y) \in p\mathbb{Z}.$$

Indeed in P_p the algebraic integer $\chi_Y(Z)$ is zero modulo the prime above p; by symmetry the same holds for $\chi_Z(Y) = (d/d_Z)\chi_Y(Z)$. The sum over a degree is rational by Galois covariance. Applying any rational linear functional of value one at one to all the entries leaves this degree sum unchanged.

Lemma 6.13. *In ordinary integral modular data of dimension n^2, fix a nonunit degree z, set $f = (n/z)^2$, and suppose $p \nmid f$. All rows of degree z are unramified at p. For any nonunit Y, let e be its p-inertia orbit size and $c = d_Y / \gcd(z, d_Y)$. Then*

$$\#\{Z : d_Z = z,\ S_{ZY} \neq 0\} < \frac{f}{ec^2}.$$

Proof. Apply (2.6) to row Y, $H = I_p$, and the nonvacuum hit columns Z of degree z. Their fields are unramified at p by Corollary 2.8, so their entries are fixed by I_p. Each nonzero normalized entry has mean square at least one. Its cost is therefore at least $\operatorname{lcm}(z, d_Y)^2 = z^2c^2$; the vacuum contributes $d_Y^2 > 0$. Dividing D by ez^2c^2 gives the strict bound. The hit set is a union of full Galois orbits since (2.3) and the rational zero test preserve it. □

7 Three ordered prime powers

Write $p < q < r$ for the three repeated primes. The two larger principal blocks impose a lower bound on the p-part of certain degrees. At a quartic prime the same bound separates twist sectors; at the largest possible p-height it forces both companion exponents to be even.

7.1 Quartic primes below the largest prime

Lemma 7.1 (Separation at a quartic prime)**.** *Let an ordinary perfect fusion-simple integral modular table have dimension $t^a u^4 v^b$, $u < v$, with three repeated primes. No twist order is divisible by uv. The same holds in a reduced minimal table for odd u, v.*

Proof. Lemma 4.14 gives $X \in P_u$ of degree ut^i. Every nonunit t-free $Y \in P_v$ has degree $u^h v^j$, $h = 1, 2$. If $h = 1$, $S_{XY} / \operatorname{lcm}(d_X, d_Y)$ has weight u, strict house below u and residue u at both u, v by principality (at u both are zero). This contradicts Theorem 4.13, also at $u = 2$. Thus all such Y have u-unit codegree and vanish on u-active columns. For a joint u, v-active column, reduction of its orthogonality on P_v modulo t would leave only the vacuum. At odd u, v the same argument uses positive balancing and reduced block orthogonality. □

Lemma 7.2 (An isolated quartic in a three-prime table)**.** *A quartic prime h cannot be isolated in an ordinary three-prime minimal table.*

Proof. By Lemma 3.4 isolation makes the dimension a square. The two height-one labels of Lemma 4.14 then give an active label of codegree h^2 by Lemma 6.8, contrary to Proposition 5.4 at the three primes. □

Theorem 7.3 (A quartic below a larger prime)**.** *Let $p < q < r$. Every braided category of dimension dividing $p^a q^b r^c d$, with d coprime and square-free, is WGT when* $\min(a, b) \leq 4$.

Proof. By descent and Theorem 4.16, only an exact quartic h below a larger prime k remains. An ordinary table with h least is already impossible: Lemma 7.1 separates h from both larger primes, contrary to Lemma 7.2. Corollary 4.15 disposes of a reduced table at $h = 2$. Thus suppose h, k are odd, and denote the third prime by t. In a reduced table $t = 2$ by Proposition 4.8.

We work in the ordinary table or its integral minimal extension. In the latter, separation of h, k in the even sector extends to all labels. Indeed the joint active space is invariant under the dyadic Weyl. This operator interchanges the fermion permutation F and the monodromy sign G. A space supported on odd labels contains no G-fixed vector, but contains the F-fixed vector $v_X + v_{gX}$ for each of its fermion pairs. Thus h, k are separated by Lemma 7.1 and no character field ramifies at both (Proposition 6.3).

There is a joint t, h-active column J: in the ordinary case otherwise h is isolated; in the extension choose an even h-active pair, whose opposite twists allow a dyadically active member. Choose $A \in P_h$ of degree hk^j and $B, C \in P_k$ respectively of degrees $h^v k^l, t^w k^m$, with all powers positive, by Lemma 4.14 and Proposition 4.2. Choose full principal representatives in the extension by Lemma 4.6. Orthogonality on P_t modulo k now gives a k-free noninvertible hit V on J: the invertibles contribute one, or two in the extension since J is even. Defect-zero vanishing bounds its h-height by one. It hits both A and C by principality, so coprime vanishing gives $d_V = t^i h$, $i > 0$.

The two normalized entries $z_Y = S_{VY}/\operatorname{lcm}(d_V, d_Y)$, $Y = A, B$, belong to K_V and have strict house less than h. At A one has $t^i z_A = \chi_A(V) \equiv d_V \equiv 0$ at every place above h: Galois covariance permutes V through labels of the same degree (Lemma 2.10), and t is an h-unit. Yet $z_A \neq 0$ since $V \in P_t$ and $t \nmid d_A$. At B the principalities of V, B give residue h at both t and k. The zero-residue and two-residue assertions of Proposition 4.12 force ramification of K_V at h and at $k > h$, respectively, contradicting separation. □

7.2 Twist phases at maximal height

We first record the common indicator computation for the odd and dyadic phase arguments. Fix a prime p, write W_o for the complementary Weyl, $w = W_o v_0$, and suppose a Galois-stable label set J satisfies $|w_j|^2 = (h/D) d_j^2 \mathbf{1}_J(j)$ for an integer h. Choose m zero on the p-level and one on the complement. The complementary upper–lower–upper identity gives $|(sT^m s^{-1})_{0j}| = |w_j|$. Thus

$$b_X = \nu_m(X) = \frac{h}{D} \sum_{j \in J} d_j^2 \chi_j(X), \qquad b = \sum_X b_X [X^*].$$

These indicators are algebraic integers by (3.3) and rational by Galois symmetry. The b_X are therefore integers, constant on character orbits by symmetry of S. Orthogonality gives $\chi_j(b) = h\mathbf{1}_J(j)$, so $b^2 = hb$. Also $b = b^*$, $b_0 = \|w\|^2 = 1$ and $\|b\|^2 = h$. In each application $0 \in J$, giving $\dim b = h$.

Lemma 7.4 (An odd-order phase cannot be missing)**.** *Let an integral perfect modular category have odd dimension D, with $v_p(D)$ odd. If all p-primary twists have order at most p, every pth root of unity occurs.*

Proof. Write U, W for the p-local upper unipotent and Weyl, P for the projection onto $\ker(U - I)$. Then $Pw = w$, $Ww = sv_0$, and the local torus acts on w by the Legendre character as in Proposition 3.9. For $1 \leq t < p$, Bruhat gives $PWU^t W^{-1} w = (\frac{-t}{p}) PWw$. If ζ_p^k is missing, its Fourier projection vanishes. Conjugating by W and applying P gives $w = -GPsv_0$ with $G = \sum_{t \neq 0} (\frac{-t}{p}) \zeta_p^{-kt}$, $|G| = \sqrt{p}$. Apply the indicator computation with J the p-inactive set and $h = p$.

The coefficient vector $psPs^{-1}v_0$ of b is fixed by the local lower unipotent and transforms under the torus by the Legendre character. It therefore has an active coordinate Y, since invariance under both unipotents would give trivial torus action. Twist covariance gives $\mathbb{Q}(\zeta_p)^+ \subseteq K_Y$ and

$Y \neq Y^*$ in odd dimension makes K_Y nonreal. Its orbit has at least $p-1$ labels. Their equal nonzero coefficients exhaust the nonvacuum norm $p-1$, so $p = \dim b = 1 \pm (p-1)d_Y$, contrary to perfectness. □

Lemma 7.5 (Dyadic twists seen by a defect-one row)**.** *In an integral modular category with $v_2(D)$ odd, a label X with $v_2(f_X) = 1$ has nonzero entries only at dyadic twists $1, \eta_X$ for a single primitive fourth root η_X. Both twists occur on labels of degree d_X in its full two-block.*

Proof. By Proposition 2.13 the full two-block B has at most two inertia orbits, and if X ramifies its orbit is the entire block.

Let E project onto B, and put $Q = sEs^{-1}$. Both commute with the dyadic group by block invariance. Since $s^{-1}\mathbb{C}^B = W_o^{-1}\mathbb{C}^B$ has a nonzero vacuum coordinate and W_o preserves dyadic twists, B contains a trivial phase. To detect order four, work modulo the dyadic part 2^a of $\operatorname{ord}(T/\gamma)$. This level is divisible by eight: $s_{00} = 1/\sqrt{D}$ belongs to the level field, and the square-free part of D is even, so the quadratic discriminant has two-part eight. Write $U(1) = \lambda T_2$, with $\lambda = \gamma^{-1}$ and $\lambda^8 = 1$ by integrality [21, Proposition 6.7(ii)]. For odd k put $l = 1 + 4k$ and $M(k) = \sum_{j \in B} |s_{0j}|^2 \theta_{j,2}^{-2k}$. The unit $l \equiv 5 \pmod 8$, extended by one on the odd part of the level, fixes the vacuum label and negates $\sqrt{D}$. Thus $H_l v_0 = -v_0$. As $L(b) = sU(-b)s^{-1}$, $(QL(2k))_{00} = \lambda^{-2k} M(k)$. Taking the vacuum entry after Q in the matrix identity $U(2)L(2k) = L(2k/l)H_l U(2/l)$ now gives

$$\lambda^{2-2k} M(k) = -\lambda^{(2-2k)/l} M(k/l). \tag{7.1}$$

All divisions here are by units modulo 2^a. The exponents agree modulo eight since their difference is $8k(1-k)/l$. The map $k \mapsto k/(1+4k)$ permutes the odd residues, with inverse $h \mapsto h/(1-4h)$, so the mean of $M(k)$ is zero. The mean of θ^{-2k} over odd k is 1 at orders one and two, -1 at order four, and zero at higher dyadic orders. Thus the squared masses at orders one and two together and at order four are equal.

Inertia preserves the dyadic order and fixes each fourth root under the square action. Thus there are exactly two orbits, with phases $1, \eta_X$, and X is unramified by Proposition 2.13. On B both $U(4), L(4)$ are scalar; $U(1)L(4) = L(4/5)H_5 U(1/5)$ makes H_5 scalar as well. The units 1 mod 4 consequently fix B pointwise. One orbit is a singleton and the other has at most two members. Equal squared masses and integral degrees force both to be singletons of degree d_X. The support assertion follows from $s^{-1}\mathbb{C}^B = W_o^{-1}\mathbb{C}^B$. □

Lemma 7.6 (Two dyadic twists force an invertible)**.** *In an integral modular category with $v_2(D)$ odd, if all dyadic twists are 1 or a single primitive fourth root η, then there is a nontrivial invertible.*

Proof. Put $Z = T_2^2$, $F = sZs^{-1}$ and $P = (I+Z)/2$. As above H_5 is scalar, hence $-I$ by its vacuum entry. Using $U(8) = L(8) = I$, the identity $U(2)L(2) = L(2/5)H_5 U(2/5)$ gives $ZF = -FZ$, hence $PFP = 0$. With $a = (1+\eta)/2$, $c = (1-\eta)/2$, one has $T_2 = aI + cZ$. Up to scalar $W_2^{-1} = T_2(aI + cF)T_2$, so PW_2P acts on P by a scalar of modulus $1/\sqrt{2}$. Since $Pw = w$ and $W_2 w = sv_0$, the indicator computation applies on the trivial dyadic fiber with $h = 2$. The integral vector b has norm squared two, unit coefficient one and dimension two; its remaining entry is a positive invertible. □

Lemma 7.7 (A uniform odd-sector spin)**.** *Let $\mathcal{M}$ be an integral minimal modular extension as in Lemma 4.4. Its even dyadic twists cannot all be signs if its odd dyadic twists have a common value δ.*

Proof. Let F, G be the fermion permutation and sector sign. Write $A = (F = G = 1)$, $B = (F = -1, G = 1)$, $C = (F = 1, G = -1)$. The dyadic Weyl W interchanges F, G; W_o commutes with both. The dyadic upper unipotent $U = \lambda T_2$ exchanges A, B, squares to λ^2 there,

and equals $c = \lambda\delta$ on C. Writing $J = W|_A$, evaluate $WUW = U^{-1}WU^{-1}$ on Ux, $x \in A$. Since $WUx \in C$ and W^2 is central, $cUJ^2x = \lambda^{-2}UJx$; thus J is scalar. Project $sv_0 = Ww$ to A. The even pairs are symmetrized and w is supported on their representatives E of dyadic twist one; hence $|w_j|^2 = 4d_j^2/D$ there.

Use the indicator computation with $h = 4, J = E$, Galois-stable by twist covariance. The coefficients of b on the even basis equal those of $1+g$: the characters of each pair j, gj agree there, and the sum over the full even sector is the projection onto $\chi_j(g) = 1$. Therefore $v = b - 1 - g$ has odd support, $gv = v$, $v^2 = 2(1+g)$ and $\dim v = 2$. Its squared norm is two. A free pair gives an odd invertible; otherwise the odd simples have degree at least two, so $v = Y - Z$ with two fermion-fixed simples. Now vY has norm squared four and coefficients one at $1, g$ by reciprocity. Its remaining part is $\pm(A' + gA')$ for an even noninvertible A'. By reciprocity vA' hits Y; it is orthogonal to v and has norm squared two. Thus $vA' = Y + Z$ by positive dimension, whence $d_Y, d_Z = d_{A'} \pm 1$. Coprime vanishing in Lemma 4.4 gives $\chi_{A'}(Y) = \chi_{A'}(Z) = 0$, contradicting $\chi_{A'}(v)^2 = 4$. □

Lemma 7.8 (Twist activity in an invariant fiber)**.** *Let s be odd, with $v_s(\Delta)$ odd in ordinary or positive reduced integral modular data. Every nonempty coordinate subspace invariant under the s-local group contains an s-active label. This applies to each fiber of a complementary twist, in either fermion sector of an integral minimal extension, and to each squared dyadic twist fiber in the positive quotient.*

Proof. On an inactive subspace the s-local upper unipotent is the identity, as are its conjugates there. They generate the local group, yet the torus of a nonsquare unit at s acts by minus a permutation since its Galois automorphism negates $\sqrt{\Delta}$. Complementary twists and the fermion grading commute with the odd local group. □

Lemma 7.9 (Propagation from visible phases)**.** *Let a be a prime and Φ a set of primary phases at a different prime. For each a-active column in a specified sector suppose there is a nonzero entry on a witness row, all of whose hits in that sector have phase in Φ. Every fiber in the sector outside Φ is then a-inactive. If its coordinate span is invariant under the full a-local group of an ordinary modular table, every label on it has a-unit codegree. The same conclusion holds on a positive even fiber for odd a in an integral minimal extension. In the latter assertion the fiber is defined on pairs, using the squared dyadic twist if necessary.*

Proof. An active column outside Φ would contradict its witness. On such a fiber $T_a = I$, so the normalized upper unipotent acts by a scalar, possibly nontrivial at two. By invariance every local conjugate is scalar there; the upper and lower elementary subgroups generate the local group over $\mathbb{Z}/a^e$. Thus W_a preserves each coordinate line, and Lemma 3.4 gives the ordinary assertion. For a positive even fiber at odd a the action can first be restricted to A. If W_a preserves the symmetrized line of Y, gY, it also preserves both full lines: it commutes with the dyadic upper unipotent, whose eigenvalues on those two summands are opposite. Apply the ordinary assertion in the extension and divide the full codegree by four. The invariance required here holds for any fiber of a complementary local twist operator, intersected, at odd a, with a fermion sector. □

Lemma 7.10 (Tame orbits of a short positive word)**.** *Let s be an odd prime and let z be an unsigned word of $g < s$ roots of s-power order, of strict house below g. Its tame inertia orbit has size at least $(s-1)/\max\{h : h \mid s-1,\ h \le g\}$. The same lower bound holds for any character label whose field contains z.*

Proof. The coefficient vector at a common s-power level is unique among positive words of length g: any difference is constant on every μ_s-coset, and a nonzero constant would require s letters on one side. Strictness gives a nonidentity root. Its orbit under the stabilizer has size equal to the stabilizer order, since every nonidentity tame unit is nontrivial modulo s. This orbit fits inside the g letters. □

7.3 The two companion exponents

Theorem 7.11 (A prime-power codegree at maximal least-prime height)**.** *Let an ordinary minimal table have dimension $p^a q^b r^c$, $p < q < r$, $k = \lfloor a/2 \rfloor \geq 2$ and $p^{k-1} < q$. Some p-active label Z has $f_Z = p^{a-2j}$ and $d_Z = p^j\sqrt{q^b r^c}$. In particular b, c are even. If p is odd, then*

$$(p-1)(r-1) < H_p(r)p^{a-2}, \qquad H_x(r) = \max\{e : e \mid r-1,\ e \leq x\}.$$

Proof. For $s = q, r$ each s-active column hits a complementary label of P_s with trivial prime-to-s twist, by prime avoidance on the trivial complementary fiber (Proposition 4.2). Call these labels witnesses. By Lemma 4.14 every witness has p-height exactly k.

We find a nontrivial p-phase missed by all witness rows. If $a = 2k$ the witnesses have p-unit codegree, so any active phase works. If $a = 2k+1$, they have defect one. For odd p, apply Lemma 3.11 with $t = p$, $s = q$, taking the q-free witnesses of P_r and the witnesses in P_q. Since $q > p$, their visible p-phases are contained in 1 and one common primitive square class. A deeper phase, or the other square class by Lemma 7.4, is missed. At $p = 2$, Lemma 7.5 gives 1 and one primitive fourth root with a representative of the witness degree at both phases. Opposite witnesses hit these representatives by principality, so the nontrivial phase agrees. Lemma 7.6 and perfectness supply a phase outside the two allowed. Apply Lemma 7.9 to this nonempty p-fiber, first at q and then at r. Its labels have pure p-power codegree.

Now suppose p is odd. The orbit $\mathcal{O}$ of Z has at least $p-1$ elements, since K_Z contains $\mathbb{Q}(\zeta_p)^+$ by twist covariance and is nonreal by oddness. If $r \leq 2p^{k-1}$ the bound follows from $H_p(r) \geq 2$. Otherwise reduce the orthogonality of the Z-column to the vacuum on P_p at r. It gives a hit X of degree $p^h q^i$, $1 \leq h \leq k-1$ by principality. Pair it with a $p^k r^l$ witness in P_r. The normalized entry has residue p^h at $r > 2p^h$ and strict house below p^h. Theorem 4.13 and Lemma 7.10 give an r-inertia orbit of X of size $e \geq (r-1)/H_{p^h}(r)$.

Now K_Z is unramified at r by its codegree. At the nonzero entry S_{ZX} all e inertia conjugates of X therefore agree on Z. The field-energy bound (Lemma 2.4) and $|\mathcal{O}| \geq p-1$ give

$$e(p-1)\operatorname{lcm}(d_Z, d_X)^2 < \Delta, \qquad (p-1)(r-1) < H_{p^h}(r)p^{a-2h}.$$

Here $d_Z = p^j\sqrt{q^b r^c}$ and $d_X = p^h q^i$, so the first inequality gives $e(p-1) < p^{a-2\max(h,j)}$. For $h = 1$ the second inequality is the assertion; for $h \geq 2$ its right-hand side is at most $p^{a-h} \leq p^{a-2}$. □

Lemma 7.12 (Propagation to odd fermion orbits)**.** *In an integral minimal modular extension let p be odd. Let $\mathcal{X}$ consist of full principal representatives at p of even positive witnesses. Suppose each witness row sees at most k distinct actual dyadic phases on odd columns. If p-unit odd objects realize a set Φ of k such phases, every odd hit of every witness has phase in Φ. If each odd p-active column meets one of these rows and $v_p(\dim \mathcal{M})$ is odd, then Φ is the set of all odd-sector dyadic phases.*

Proof. Lemma 4.6 supplies the representatives. Their principalities force a nonzero entry at every p-unit odd object, so every row sees all k phases. An occupied odd-sector dyadic fiber contains a p-active column by Lemma 7.8, proving the last assertion. For a positive singleton the bound $k = 1$ follows from Lemma 4.26. Under the two hypotheses of Lemma 7.18, the positive defect-one case has $k = 2$: the dyadic Weyl preserves the full block, with its two odd orbits, and the complementary Weyl preserves the actual dyadic phase and the sector. □

Corollary 7.13 (Complementary singleton witnesses)**.** *In an integral minimal extension as in Lemma 4.4, let p, s be distinct odd dimension primes. In the positive table suppose the s-free nonunits of P_p and the p-free nonunits of P_s are nonempty sets of dyadic singleton blocks. Every p- or s-active positive pair has squared dyadic twist one, and all p- or s-active odd columns have one common actual dyadic phase. Consequently both prime valuations of Δ are even.*

Proof. On a p-active positive column, orthogonality on P_p modulo s forces an s-free nonunit hit. Singleton block orthogonality gives squared twist one on that column. Interchange p, s for the other assertion. A singleton witness has an odd mate of unchanged degree and sees only its mate's actual phase on odd columns (Lemma 4.26). Each full principal lift from one side hits the odd mate of every opposite witness by its unit degree. Hence the phases of all these mates agree. Evaluation at g makes the full principal p-block even, and the opposite odd p-unit excludes g. Its orthogonality modulo s therefore gives a witness on each odd p-active column; the same holds at s. If either valuation were odd, Lemma 7.8 would put an active column in every squared even and every actual odd fiber, contrary to Lemma 7.7. □

Proposition 7.14 (A fermion core with an odd companion valuation)**.** *Let $2 < q < r$, $k \geq 2$ and $2^{k-1} < q$. A slightly degenerate divisor-minimal counterexample of dimension $2^{2k+1}q^b r^c$ has both b, c even.*

Proof. The complementary witnesses have height k by Lemma 4.14 and square divisibility. Their odd positive codegrees give singleton dyadic blocks; apply Corollary 7.13. □

Theorem 7.15 (Lifting the companion exponents)**.** *Let $p < q < r$ and $M = p^A q^B r^C d$, with d square-free and coprime to pqr. Put $k = \lfloor A/2 \rfloor$. If $q > p^k$, every braided category of dimension dividing M is WGT. If $q > p^{k-1}$, the same conclusion holds as soon as it holds on every divisor of*

$$p^A q^{2\lfloor B/2 \rfloor} r^{2\lfloor C/2 \rfloor} d.$$

Under this second gap one may take $B = 5$ and arbitrary C. One may also take arbitrary B, C for odd p and $A \geq 4$ when $(p-1)(r-1) \geq H_p(r)p^{A-2}$. Without a gap hypothesis the caps $A = B = 5$ always suffice.

More precisely, if $p^a q^b r^c$ is the square-full part of a divisor-minimal counterexample, then $q \leq p^{\lfloor a/2 \rfloor}$, and $q > p^{\lfloor a/2 \rfloor - 1}$ forces both b, c even.

Proof. Take a least counterexample among the divisors of M. Its three exponents a, b, c satisfy $a, b \geq 5$, $c \geq 4$ by Theorems 4.16 and 7.3. With $l = \lfloor a/2 \rfloor$ one has $q \leq p^l$ by Lemma 4.14, proving the first assertion. We claim that $q > p^{l-1}$ forces b, c even. In the ordinary table this is Theorem 7.11. In a reduced table $p = 2$ by Proposition 4.8; an even a contradicts Corollary 4.15, and for odd a apply Proposition 7.14.

Now $l \leq k$: a gap at $k-1$ makes both actual exponents even, so the rounded caps suffice. For $B = 5$ the rounded q-exponent is four, covered by Theorem 7.3. When p is odd, the table is ordinary by Proposition 4.8, and Theorem 7.11 forces $(p-1)(r-1) < H_p(r)p^{a-2}$ with $a \leq A$. Finally at $A = B = 5$ we have $k = 2$ and $q > p$ automatically, so every uncovered choice has $A + B \geq 11$. □

7.4 The largest quartic prime

At $v_q(\Delta) = 5$, reducing unit orthogonality on P_q at the omitted prime forces degrees qp^i and qr^j. More precisely, for any X of degree $q^j r^b$ with $j, b > 0$, some $Y \in P_q$ of degree qp^a satisfies, with $z = S_{XY}/\operatorname{lcm}(d_X, d_Y)$,

$$r \nmid [K_Y : \mathbb{Q}(z)] \operatorname{Tr}_{\mathbb{Q}(z)/\mathbb{Q}}(z). \tag{7.2}$$

Indeed use block orthogonality if X is q-active and the odd-valuation slice of Proposition 4.2 otherwise. The respective vacuum coefficients $1, -(q-1)$ are units at r. Some r-free orbit in P_q thus contributes nontrivially modulo r; the principal height bound gives degree qp^a. Its contribution before the slice coefficient is $qp^{2a}[K_Y : \mathbb{Q}(z)] \operatorname{Tr}(z)$ by (2.3). For $j = 1$, $r^b z = \chi_Y(X) \equiv 0$ at every prime over q, so the trace is a nonzero multiple of q and $\mathbb{Q}(z)$ ramifies there (otherwise z/q is integral of house less than one). If $r \nmid f_X$, then $\mathbb{Q}(z) \subseteq K_X$ is unramified at r and (7.2) excludes wild r-ramification in K_Y. Finally, when $v_r(\Delta) = 4$, a p-free nonunit in P_r has degree $q^2 r$: degree qr paired with $qp^a \in P_q$ violates Theorem 4.13 at $g = q$.

Proposition 7.16 (Complementary fields at a quintic middle prime)**.** *Let $k \geq 3$ and consider $D \mid p^{2k}q^5r^c d$, $p < q < r$, with d coprime and square-free. Suppose $p^{k-2} < q$ and put $g = p^{k-1}$. In a divisor-minimal counterexample the source exponents at p, q are exactly $2k, 5$, respectively, and $q \leq g$. In its table some $X \in P_q$ has $d_X = gq$; every q-free nonunit $Y \in P_r$ has $g \mid d_Y$. If $q \leq g < 2q$ and $r > g$, the two fields contain $\mathbb{Q}(\zeta_r)$, except that when $r = g + 1 = q + 2$ the guaranteed common subfield is $\mathbb{Q}(\zeta_r)^+$.*

Proof. Theorem 7.15 forces $b = 5$ and excludes every $a \leq 2k - 1$. In the reduced table $p = 2$ and p-heights are at most $k - 1$. In the ordinary table a p-active fiber contains a q-active column by Lemma 7.8. Its r-free hit on P_q has p-height at most $k - 1$ by defect-zero vanishing. In either case Lemma 4.14 and $q > p^{k-2}$ force height $k - 1$ there and at least $k - 1$ on every opposite label. The principal q-height bound gives $d_X = gq$. Apply Theorem 4.13 to each opposite pairing with weight g; for general $r > g$ the proof gives tame orbit size at least $(r - 1)/\lfloor g/q \rfloor$. □

At $2k = 6$ the exceptional equality $p^2 = q + 1$, $r = q + 2$ forces $(p, q, r) = (2, 3, 5)$. In the exceptional word, if 1 occurs once, its position forces each stabilizer to fix ζ_5. Otherwise the coefficients are $1, 1, \omega, \bar{\omega}$ for a primitive cube root ω. Unless the two positions of 1 are opposite modulo five, every stabilizer fixes ζ_5; if opposite, its only nonidentity element is complex conjugation. Thus the common entry field contains $\mathbb{Q}(\zeta_5)$ or equals $\mathbb{Q}(\zeta_{15})^+$, and always contains $\mathbb{Q}(\sqrt{5})$.

The forced degree $p^{k-1}q$ also satisfies the tensor-square inequality of Proposition 4.19. For $p = 2$ and $k = 3$, Theorem 26.2 covers all odd pairs except $(3, 5)$, with both companion exponents unrestricted.

A related descent observation holds at any odd prime r. Every WGT braided category of dimension divisible by r^3 contains a nontrivial Tannakian subcategory. Indeed, a category without one would equal its core. By [43, Theorem 1.1] it would then be pointed or a product of a pointed category and an Ising category. The pointed r-primary part would have order at least r^3. Its quadratic form has a nonzero isotropic element: a cyclic subgroup of order at least r^2 has an isotropic subgroup of order r, and in the elementary abelian case a quadratic form over $\mathbb{F}_r$ in at least three variables has a nonzero zero. This gives the required Tannakian subcategory. In particular this applies to the unordered $(6, 5, 4)$ family by Theorem 8.1 whenever an odd cube survives in the dimension. For arbitrary companion exponents in $p^A q^B r^4 d$, a Tannakian $\mathrm{Rep}(G)$ with $r \mid |G|$ suffices for WGT by descent: the neutral de-equivariantization has r-valuation at most three, so Theorem 4.16 applies.

7.5 Dyadic blocks of a fermion centralizer

Let $\mathcal{M}$ be an integral modular category of dimension $D = 4\Delta$ with its positive spherical structure, a fermion g and even sector $\mathcal{B} = \langle g \rangle'$. Use the spaces A, B, C and the signed order of Lemma 4.5 and its normalization table. Write $K = W_2|_A$, $W_o = W_{2'}$, $U = \lambda T_2$, $\lambda^8 = 1$, and $\epsilon_Y = \theta_{Y,2}^2$ on an even pair. The dyadic Weyl interchanges B, C, while U interchanges A, B and preserves C.

Lemma 7.17 (Positive defect one and the dyadic level)**.** *Suppose $v_2(D)$ is odd, and let E be a positive two-block containing a character of dyadic defect one. Its squared twists are $+1, -1$, with equal sums of squared degrees at the two signs. Its rows vanish on pairs of every other squared twist. If the actual dyadic twists of* all *even labels have order dividing four, the full dyadic level divides eight, and E consists of two labels of equal degree, both unramified at two. Under this twist bound, each full character whose field is contained in a dyadic cyclotomic field takes values in $\mathbb{Z}[i]$ on even objects.*

Proof. The residue on the even ring determines the full residue by squaring. The projector onto the lifted block commutes with the dyadic group by Lemma 3.3. Its restriction to A projects onto E and commutes with K. Thus $\widehat{s}^{-1}\mathbb{C}^E = W_o^{-1}\mathbb{C}^E$ has nonzero vacuum projection, and

E contains squared phase $+1$. Apply the Bruhat mass identity (7.1) from Lemma 7.5 on A. The operators $U(2), L(2j), H_{1+4j}$ (j odd), the projector and its $\widehat{s}$-conjugate all preserve A. Here $L(t) = WU(-t)W^{-1}$. The torus negates the symmetrized vacuum ($v_2(D)$ is odd and both vacuum labels have rational characters). Its weights on E are d_Y^2/Δ. Averaging $\theta_{Y,2}^{-2j}$ over odd j gives $+1, -1, 0$ at squared twist $+1, -1$, or higher order, respectively. Thus the two signed masses are equal and nonzero. By Proposition 2.13, E has at most two inertia orbits. The two signs already occupy distinct orbits, so no higher order occurs. Since W_o preserves squared dyadic twist, the equality of subspaces above gives the support assertion.

Now assume the global bound on the even twists. Balancing puts even-even S-entries in $\mathbb{Q}(i, \zeta_m)$ for some odd m. A dyadic Galois unit $l \equiv 1 \pmod 4$, trivial on the odd factor, fixes the restricted character of every even label. Each even fermion pair is consequently fixed as a set by Lemma 4.5. The square action on actual twists fixes each representative's twist, and opposite twists distinguish the representatives. Symmetry of S and rationality of degrees now fix the values of every character on even objects. In particular an odd fermion orbit is fixed as a set. Its members have equal twists, so the unit $l = 5$ gives $\theta_{Y,2}^{24} = 1$ also on odd labels. The normalization scalar cancels in the square action since the vacuum is fixed. All actual dyadic twists have order dividing eight, as does U since $\lambda^8 = 1$. For a purely dyadic character the fixed field of the units 1 mod 4 lies in $\mathbb{Q}(i)$, giving the asserted integrality. All positive characters have dyadic ramification index at most two. The two-branch defect-one block has an unramified branch by Proposition 2.13. The other has length one or two, and length two would equate an integer square to twice another integer square by the mass equality. Thus each sign has one label, of the same degree. □

Lemma 7.18 (Signed blocks at dyadic level dividing eight)**.** *Suppose $v_2(D)$ is odd and all even actual dyadic twists have order dividing four. Then U^4 is scalar on C. If a positive two-block contains a defect-one character of degree d, the full block over it contains two odd fermion orbits, both free. Their actual dyadic twists have ratio $\pm i$ and their degrees u, v satisfy*

$$u^2 + v^2 = 2d^2, \qquad v_2(u) = v_2(v) = v_2(d).$$

If the entire odd sector has just two dyadic twists with ratio $\pm i$, then R_+ has a nontrivial invertible basis element.

Proof. The level bound just proved uses only the bound on even actual twists. Put $Z = T_2^2|_A$, $F = KZK^{-1}$. These are involutions. The torus H_5 fixes the even labels as a permutation and negates $\sqrt{D}$, hence $H_5 = -I$ on A. Modulo eight the identity $U(2)L(2) = L(2/5)H_5U(2/5)$ gives $ZF = -FZ$. Since $H_3 = H_5H_{-1} = -K^2$ on A, it commutes with both. The factorization

$$(WU)^{-1}U^2(WU) = U(-1)L(-2)U(1) = L(-2/3)H_3U(2/3) = L(2)H_3U(6)$$

gives the exact restriction

$$(WU)^{-1}U^2(WU)|_A = \lambda^4 FH_3Z. \tag{7.3}$$

Its square is scalar since $H_3^2 = I$, and WU maps A onto C.

Over a positive defect-one block, Lemma 7.17 gives two labels of degree d, both unramified at two. Their full block contains four even labels, and H_3 acts scalarly on the two-dimensional positive space. Since WU preserves the full block, its odd symmetric space also has dimension two. The invertible operator FZ anticommutes with Z, so its two eigenvalues are opposites. By (7.3) the squares of the two odd twists differ by a sign.

In the signed order R_- with its unit-coefficient trace, the two corresponding characters have equal dyadic defect if inertia interchanges them. If inertia fixes both, their local order over the completed maximal unramified base $\mathcal{O}$ is a suborder of $\mathcal{O}^2$ containing the diagonal. The ideal of coordinate differences is $2^h\mathcal{O}$ for some $h \geq 1$. Thus the order consists of pairs congruent modulo 2^h, with conductor 2^h on each branch. By self-duality the conductor on each unramified branch

is its codegree ideal (Lemma 2.12). Equal codegree valuations in (4.1) give the same fixed/free type by parity, then $v_2(u) = v_2(v)$. Column orthogonality of $1, g$ on the full block equates its odd mass to $4d^2$. If both labels were fixed, $u^2 + v^2 = 4d^2$ would have odd two-valuation on the left and even on the right. Thus both orbits are free and $u^2 + v^2 = 2d^2$, which also gives $v_2(u) = v_2(d)$.

If the only odd spins are δ_1, δ_2 with ratio $\pm i$, on C one has $U = aI + bU^2$, where, writing $x = \lambda\delta_1$, $y = \lambda\delta_2$, one has $a = xy/(x+y)$, $b = 1/(x+y)$ and $|a| = |b| = 1/\sqrt{2}$. Restrict $WUW = U^{-1}WU^{-1}$ to UA, and multiply by U^{-1} on the left. Since $WUA = C$ and W^2 is central, on A this gives

$$U^{-2}K = aK^2 + b\,U^{-1}L(-2)UK^2.$$

Here $U^{-2} = \lambda^{-2}Z$, and (7.3) gives $U^{-1}L(-2)U = cFH_3Z$ with $|c| = 1$. Since $H_3 = H_5H_{-1} = -K^2$ and $K^4 = I$, it follows that $K = a'ZK^2 + b'F$ with $|a'| = |b'| = 1/\sqrt{2}$. For the projection $P = (1+Z)/2$ this yields $PKP = a'PK^2P$.

Put $w = W_o v_{[1]}$ in A. We have $Pw = w$, while K^2w is a unit scalar multiple of w by the torus action on the vacuum and commutation of local factors. Since $\widehat{s}v_{[1]} = Kw$, $|w_{[j]}|^2 = 2d_j^2/\Delta$ on $\epsilon_j = 1$ and vanishes otherwise. Lift back to the full vacuum $w^{\mathrm{full}} = W_o v_0$. This is supported on the even labels with actual dyadic twist one: W_o preserves monodromy sector and commutes with T_2. On every even pair the reduced coefficient is $w_j^{\mathrm{full}} + w_{gj}^{\mathrm{full}}$ because W_o commutes with the fermion permutation. Exactly one representative of each positive-sign pair can contribute, so its squared weight is again $2d_j^2/\Delta$.

Take m zero on the dyadic level and one on the odd complementary level. The indicator formula at the beginning of Subsection 7.2 gives, for even Y,

$$\nu_m(Y) = \sum_{j \in \mathrm{Irr}(\mathcal{M})} |w_j^{\mathrm{full}}|^2 \chi_j(Y) = \frac{1}{\Delta}\sum_{[j]} (1+\epsilon_j) d_j^2 \chi_{[j]}([Y]).$$

This is the coefficient at $[Y]^*$ of the reduced element $1 + J$, where $J = \sum_{[j]} \epsilon_j e_{[j]}$. It is an algebraic integer by the cyclotomic integrality of indicators [46, Theorem 5.1] and rational by Galois covariance of the sign. Thus J has integral coefficients. It is self-dual with $J^2 = 1$, so has squared based-ring norm one, and its dimension is $\epsilon_1 = 1$. This gives a positive invertible basis element in R_+. Both signs occur by anticommutation, so it differs from the unit. □

8 Three repeated primes with a quartic or quintic exponent

Theorem 8.1 (Three bounded exponents)**.** *Let p, q, r be any distinct primes, and let d be square-free and coprime to pqr. Every weakly integral braided fusion category of dimension dividing $p^7q^5r^4d$ is WGT.*

The exponents in this theorem may be assigned to the primes in any order. Throughout this section, we write $p < q < r$ unless stated otherwise. A quartic below a larger prime is already covered by Theorem 7.3. Our focus is a quartic at the largest prime, and d always denotes a coprime square-free factor.

For $m \geq 1$ set $e = m(m+1)/2$. Define

$$g(h) = \min\{k \geq 1 : k(k+1) \geq h\}, \qquad C(e) = 1 + \max_{1 \leq h \leq e} \frac{h(h-1)}{g(h)}.$$

We say the odd prime s satisfies the *support bound* at m if

$$1 + 2e < 2s/3, \qquad (s-1)/2 > C(e). \tag{8.1}$$

Theorem 8.2. *Let $p < q < r$ be odd primes and set $m = \lfloor r/q \rfloor$. If $m + 1 < p$ and r satisfies (8.1), every braided fusion category of dimension dividing $p^aq^br^4d$ is WGT for all $a, b \geq 0$.*

In particular, this holds for all odd primes $p < q < r < 2q$, and for $5 \le p < q < r < 3q$. Indeed $C(1) = 1$ and $C(3) = 4$; in the additional interval $2q < r < 3q$, $q \ge 7$ gives $r \ge 17$. For $p \ge 5$ and $q \ge 11$ the range extends to $r < 4q$, since $C(6) = 16$ and $r > 3q$ suffices for the two tests at $m = 3$. More generally, for each p, the whole interval $q < r < (p-1)q$ is covered once q is sufficiently large in terms of p. A quintic at the least prime requires no size condition on the other primes.

Theorem 8.3 (A quintic and two unrestricted exponents)**.** *For any primes $p < q < r$ and any $b, c \ge 0$, every weakly integral braided fusion category of dimension dividing $p^5 q^b r^c d$ is WGT.*

Section 8.5 proves this theorem. For a middle quintic and odd primes, Theorem 8.20 handles $p^A q^5 r^4 d$ whenever $A \ge 6$ and $q > p^{\lfloor A/2 \rfloor - 2}$. This holds automatically for $A = 6, 7$. The even case combines the ordinary and fermion reductions in Sections 8.6–8.8. The unrestricted quartic problem remains open.

In a divisor-minimal obstruction all three repeated primes survive and the square-free factor disappears. A third exponent at most three is already covered (Lemmas 2.1, 2.5, Proposition 4.8 and Theorem 4.16). Thus $v_r(\Delta) = 4$ in the quartic problem, including the positive table, whose dimension is half the categorical dimension. A q-free or p-free nonunit in P_r then has degree rp^i or rq^j with $i, j > 0$, by the principal height bound and prime avoidance (Lemma 4.14). We use *house* for the largest absolute value among all conjugates.

8.1 Character values and tensor products

Lemma 8.4 (An odd word of prime weight)**.** *Let $t < r$ be odd primes. Suppose the cyclotomic integer z is a nonnegative rational combination of odd-order roots of total weight r, has strict house below r, and has respective residues $r, 0$ over all primes at t, r. Then for a t-power root η,*

$$z = \eta \left(h + t \sum_{i=1}^{m} \xi_i \right), \qquad h + tm = r,$$

with $h \ge 1$, $m \ge 1$ and nonidentity r-power roots ξ_i. In fact every root in the original positive expression has this same t-part, and the multiplicities of its distinct r-parts are uniquely determined after averaging coincident roots.

Proof. On signed words in t, r-primary roots impose cost one on $+$ and $3/2$ on $-$. Every integral word with the prescribed residues has cost at least r. Indeed, a word of length at most $r-1$ gives, on erasing r-parts, a signed t-primary sum w divisible by r in the unramified integer ring. Its strict house bound $< r$ gives $w = 0$. Erase instead t-parts; the signed coefficients $c_i - r\delta_{i0}$ are constant modulo t on each μ_r-coset. If every coset has a vacant position after marking the identity, all constants vanish, and the signed augmentation is both r and 0 modulo t. Therefore all $r-1$ nonidentity positions in the identity coset occur. They occur just once, with a common sign e since t is odd. Then $t \mid r-1$ by $w = 0$, and $e \equiv -r \equiv -1 \pmod{t}$. This word costs $3(r-1)/2 > r$.

Apply Lemma 4.11 at this asymmetric cost with retained primes t, r. The lower bound just proved uses only the two residues, so it applies to every contracted integral target. A nontrivial spectator component at an odd prime u has expected cost at most $3/(2(u-1)) < 1$. Thus every positive summand is supported at t, r.

It remains to minimize on the unsigned two-prime roots alone. Again the configuration is unimodular and an integral minimum exists. Each optimum has length r. Erasing r-parts now gives a sum w of r unsigned t-power roots, divisible by r. It is nonzero since a vanishing unsigned sum has length divisible by t. The integral norm makes w/r a root of unity. Equality in the triangle inequality makes all the t-parts the same root η. The remaining unsigned r-word has augmentation r and coefficients $c_i - r\delta_{i0}$ constant mod t on each μ_r-coset. A full nonidentity

coset makes the word zero. If all $r-1$ nonidentity positions of μ_r are occupied, the extra term cannot be outside the coset or duplicate one position, by these congruences. The complete coset again gives the prohibited zero at t. All constants therefore vanish. Strict house excludes the word supported only at one. This proves the formula. Every optimal vertex has the same η, since the ratio of two such roots lies in $\mathbb{Q}(\zeta_{t^a}) \cap \mathbb{Q}(\zeta_{r^b}) = \mathbb{Q}$. The remaining coefficients are unique: every nonzero coset has a vacant position, and a reduced unsigned r-word has a unique coefficient vector. The original expression is a convex combination of those optimal vertices. □

Lemma 8.5 (A single residue in odd order)**.** *Let s be prime and let $z = \sum_i c_i \xi_i$ be a cyclotomic integer with $c_i \in \mathbb{Q}_{\geq 0}$ and $\sum_i c_i = g \in \mathbb{Z}_{>0}$. Discard the zero coefficients; all assertions about terms concern $c_i > 0$. Suppose every prime dividing the orders of the remaining roots is at least $\ell \geq 3$, and $z \equiv g$ at all primes above $s > g$ in the common field. If $(s-g)(\ell-1) > g$, all terms of this expression have s-power order, and z has an integral unsigned s-word of length g.*

Proof. Choose a cost c with $\max(1, g/(s-g)) < c < \ell - 1$ for negative roots; positive roots cost one. Apply the asymmetric contraction in Lemma 4.11, retaining s. The residue g is preserved and a nontrivial spectator has expected cost at most $c/(\ell-1) < 1$. An integral signed s-word of cost at most g has augmentation $g \bmod s$. A negative augmentation hence costs at least $c(s-g) > g$, while augmentation g forces exactly g positive terms. Integral optima exist for the signed one-prime configuration by unimodularity. Thus any loss of cost contradicts its minimum, proving the assertion on the original terms and the integral word. □

8.2 A prime-weight crossing with one zero residue

Write μ_{s^∞} for the roots of s-power order. Let a word of weight r mean a positive rational sum of roots of unity whose coefficients sum to r. All residue conditions below are at every prime over the indicated rational prime.

Proposition 8.6. *Let $q < r$ be odd primes with $q \nmid r-1$. Suppose that a cyclotomic integer z has weight r, has strict house less than r, and satisfies $z \equiv r$ above q and $z \equiv 0$ above r. Then for some $\eta \in \mu_{q^\infty}$ one has one of the following expressions.*

1. *$z = \eta(h + q\sum_{j=1}^{m} \xi_j)$, where $h + qm = r$, $h, m \geq 1$, and $\xi_j \in \mu_{r^\infty} \setminus \{1\}$.*
2. *$r \equiv 2 \pmod{q}$ and $z = \eta(1 - \sum_{t=1}^{r-1} \eta_t \zeta_r^t)$. Here the multiset of the η_t consists of one copy of 1 and $(r-2)/q$ complete cosets of μ_q in μ_{q^∞}.*

Every weight-r expression has exactly the phases and the masses in the indicated word, after combining coincident phases.

Proof. First work with a signed integral word of length at most r in products of q- and r-power roots. Erase the r-parts to obtain w. It is divisible by r in the q-cyclotomic integers, since this field is unramified at r. If $w \neq 0$, Kronecker's theorem makes w/r a root of unity, since its house is at most one. Equality in the triangle inequality then gives exactly r terms with one common signed q-part. If $w = 0$, their signed augmentation is zero modulo q.

Let c_t be the signed coefficients obtained by erasing the q-parts, indexed at a common r-power level in additive notation. On each μ_r-coset, the numbers $c_t - r\delta_{t0}$ have a common residue modulo q, by the cyclotomic coset relations. A coset with a vacant position after marking zero has common residue zero. Any entirely occupied coset other than the identity coset would need all r letters; the empty nonzero positions of the identity coset would then contradict the residue at zero. If all common residues vanish, the augmentation is r modulo q, so $w \neq 0$, the uniform sign is positive, and each nonidentity multiplicity is divisible by q. This gives the first possibility; strictness excludes $m = 0$.

It remains to have occupied all $r-1$ nonidentity positions of the identity coset. In the case of a common signed q-part no such word can occur. With the remaining letter at zero the erased

r-word is a multiple of the complete cycle and has residue zero at q. At any already occupied position its coefficient would differ by one in absolute value from the others, against the coset relation. Finally a letter on a new coset violates the zero coefficient congruence there. Thus $w = 0$. Put $s = c_0$. The signed augmentation modulo q is $r + r(s - r)$, since the sole possible nonzero coset residue is $s - r$. Consequently $s \equiv r - 1 \pmod{q}$. There is room for at most one letter at zero. Since $q \nmid r - 1$ and $r \not\equiv 0 \pmod{q}$, we obtain $s = 1$ and $r \equiv 2 \pmod{q}$. The other signs are all negative. After division by the q-part η of the positive letter, $w = 0$ reads $\sum_t \eta_t = 1$. Subtracting the copy at 1, the integral coefficients must be constant on each μ_q-coset. These constants are nonnegative, since $q > 1$. This is the second possibility. In particular no signed integral word of length below r has the required residues.

This lower bound required only the residues: strict house entered solely to exclude $m = 0$ in the first alternative. Lemma 4.11 with both signed costs one therefore removes every spectator except a sign and gives an integral optimal word on q, r.

In the first possibility there are at most $1 + \lfloor (r - 1)/q \rfloor < r/2$ distinct primary positions. They are exposed by Lemma 4.20 and independent by the coset relations. For the second case divide by η. On the coset containing μ_r, give the functional at $t = 0$ the normalized trace T on the complementary field and at $t \neq 0$ the functional $y \mapsto T(y/(-\eta_t))$. Their sum is zero because $\sum \eta_t = 1$. Put zero on other cosets. Its maximum one is attained precisely at the r indicated roots in any common conductor. They are linearly independent over $\mathbb{Q}$: after using $1 = -\sum_{t \neq 0} \zeta_r^t$, a rational dependence would require the same rational number to be a rational multiple of every η_t, including a nonreal one. Exposure and independence fix the masses in every positive expression. □

8.3 Proof for arbitrary exponents

There are two possibilities for a putative minimal table, even without a size condition on the primes. The second uses the *canonical positive word* in a single prime-power field: in each prime-order coset subtract the least coefficient. We use additive notation for its positions, with zero the identity.

Proposition 8.7 (The unrestricted odd case)**.** *Take $p < q < r$ odd and $v_r(\Delta) = 4$, with arbitrary valuations at p, q. A minimal table has at least one of the following normalized entries $S_{XY}/\operatorname{lcm}(d_X, d_Y)$, of common weight r.*

1. *A p-free nonunit of P_q has r-height one. An entry at this label has the form $\eta(h + q\sum_{j=1}^{n} \xi_j)$ with $h + qn = r$, $n > 0$, η of q-power order and each ξ_j of r-power order.*

2. *All p-free nonunits of P_q have r-height two. The q- and r-active labels are disjoint, and a normalized entry belongs to a p-power cyclotomic field. Its canonical positive coefficient vector (c_i) has length $L \leq r$ with $L \equiv r \pmod{p}$. On every μ_p-coset the entries $c_i - r\delta_{i0}$ are constant modulo q.*

Proof of Proposition 8.7 and Theorem 8.2. Suppose a minimal counterexample exists. Its table is ordinary of odd dimension. Choose $X \in P_r$ of degree rp^i. Every p-free nonunit $Y \in P_q$ has degree $q^j r^v$ with $v = 1$ or 2. Suppose $v = 1$. The normalized entry $z = S_{XY}/\operatorname{lcm}(d_X, d_Y)$ has positive odd-order weight r and strict house less than r. Its residues are r at both r and q: use $\chi_X(Y) = q^j z$ at r, and $\chi_Y(X) = p^i z$ at q. Lemma 8.4, with $t = q$, gives at most $m + 1$ positions of r-power order up to translation. This proves the first alternative when such a label exists. Under the hypotheses of Theorem 8.2, the support bound holds at r. Lemma 4.22, applied after rescaling by $\operatorname{lcm}(d_X, d_Y)/d_Y$, gives a contradiction.

We may therefore assume every such Y has $v = 2$. Its formal codegree $f_Y = \Delta/d_Y^2$ is an r-unit, so its row vanishes on r-active columns (Theorem 3.2). If a column were both q- and r-active, orthogonality to the vacuum on P_q, reduced modulo p, would give a p-free nonunit hit

(Proposition 4.2). This proves that q- and r-activity are disjoint. By Proposition 6.3 no character field then ramifies at both q and r.

If r were also separated from p, it would be an isolated quartic prime, excluded in three-prime ordinary data by Lemma 7.2. Fix a joint p, r-active column J. Orthogonality on P_p on the trivial prime-to-p twist fiber, reduced modulo q, gives a q-free nonunit $V \in P_p$ hitting J. It has degree $p^e r$: both primes divide the degree by the exclusion of prime powers, and r^2 is impossible at a hit of J by defect-zero vanishing.

Take $A \in P_r$ of degree rq^i and a p-free nonunit $B \in P_q$, hence $d_B = r^2 q^j$. At V, A the normalized entry has weight r and is nonzero by $V \in P_p$ since $p \nmid d_A$. Its residue at each prime above r is zero by $A \in P_r$, since $\chi_A(V) = p^e S_{VA}/\operatorname{lcm}(d_V, d_A)$. If this value were unramified at r, division of the normalized entry by r would give a nonzero algebraic integer of house less than one. Thus $K_V = \mathbb{Q}(\chi_V)$ ramifies at r and is unramified at q. The field K_B is unramified at r by its codegree (Theorem 2.7).

Now form $z' = S_{VB}/\operatorname{lcm}(d_V, d_B)$. It lies in $K_V \cap K_B$, so is unramified at both q and r. It belongs to a field of p-power conductor: the entry fields are cyclotomic with no other conductor primes, and the inertia groups of the other two factors fix z'. It has weight r, strict house less than r and residue r at both p and q by the principalities of V, B.

Average its balancing expression over the q, r-cyclotomic factor. The negative mean of a complementary root has magnitude at most $1/(q-1)$. Replace a negative coefficient on a p-power root by positive coefficients on the other $p-1$ positions of its μ_p-coset. This gives a positive rational p-word of weight at most r. Subtract the minimum coefficient on each coset. The remaining coefficients c_i are nonnegative integers, since z' also has an integral expression and two such vectors differ by a constant on each coset. Write $L = \sum c_i \le r$. Reduction at p gives $L \equiv r \pmod p$. Reduction at q makes $c_i - r\delta_{i0}$ constant modulo q on each μ_p-coset.

This proves the second alternative, without any size hypothesis. To finish the WGT theorem assume (8.1). Suppose zero is the only vacant position in the identity coset and put $\rho = r - mq$. Each of its $p-1$ other coefficients is $q - \rho$ modulo q, and elsewhere all coefficients are multiples of q. Thus $L = (p-1)(q-\rho) + qn$ with $0 \le n \le m$. Reduction at p gives $n \equiv m+1 \pmod p$, impossible since $m+1 < p$ by hypothesis. There is therefore a vacant nonidentity position on the identity coset as well. All $c_i - r\delta_{i0}$ now vanish modulo q. Consequently $L \equiv r \pmod{pq}$; since $r < (m+1)q < pq$ we have $L = r$.

The equality $L = r$ also identifies the prime support of the *actual* double-braiding phases of V, B. Any physical term with nontrivial prime-to-p part would have lost weight in passing to the positive p-word: a positive normalized trace on a nontrivial complementary root is strictly less than one, and a negative trace costs at most $(p-1)/(q-1) < 1$ after its conversion to positive p-roots. This would give $L < r$. Thus all phases have p-power order. This deduction holds for every p-free nonunit $B \in P_q$.

By the choice of the trivial complementary fiber, V has p-primary twist. Both V and B are r-inactive, the latter by its r-unit codegree. Since all the phases of their double braiding are p-primary, every constituent of VB is r-inactive. Write $v = [VB]$ in the fusion basis, and $s = S/\sqrt{\Delta}$. The r-local upper unipotent fixes v. The coordinates of its Fourier transform are

$$(sv)_T = d_T \chi_T(V)\chi_T(B)/\sqrt{\Delta}.$$

They vanish at all r-active T because B has r-unit codegree. Thus the conjugate lower unipotent also fixes v. The scalar normalizing the twist has trivial r-part, as in Section 3. These two unipotents generate the local modular group, so its torus fixes v. Since $v_r(\Delta) = 4$, r-inertia fixes $\sqrt{\Delta}$; the corresponding torus operators are the unsigned Galois permutations. As B is unramified at r, this proves

$$BV^\sigma = BV \qquad (\sigma \in I_r). \tag{8.2}$$

Take any p-free nonunit $A \in P_r$, of degree rq^i. The earlier ramification argument applies to *each* such A: $\chi_A(V)$ is moved by some $\sigma \in I_r$. Evaluating (8.2) at A therefore forces $S_{AB} = 0$

for every p-free nonunit $B \in P_q$. Now A is q-inactive: q, r are separated, and its r-codegree is divisible by r (Lemma 3.4). Choose any q-active column J. Block orthogonality on P_q yields

$$0 = \sum_{W \in P_q} \chi_A(W)\overline{\chi_J(W)}.$$

At p-free nonunits the first factor vanishes by the preceding argument. At p-bearing W it is divisible by p by lcm integrality, since $p \nmid d_A$. The unit contributes one. Reduction modulo a prime above p gives the contradiction. □

The first ratio outside the length argument. Suppose p, q, r are odd, $m = p - 1$, and r satisfies (8.1). Then every q-active simple of a minimal table has degree divisible by pr^2. To see the additional factor p, retain V, A above and put $c = r - pq$. At a p-free nonunit $B \in P_q$, either $L = r$, in which case $S_{AB} = 0$ by (8.2), or the canonical word has a vacancy only at zero in the identity coset. Indeed in the other case $L \equiv r \pmod{pq}$ and $r < pq$. In the exceptional case $n \equiv m + 1 = 0 \pmod{p}$ and $0 \le n \le p - 1$, so $n = 0$. The sum of the nonidentity pth roots is -1, and therefore $z' = -(q - \rho) = c$. At least one such rational entry must occur, since a full zero rectangle was just excluded.

For each p-free $A \in P_r$ the normalized entry at A, V has weight r and residues r at p, r. Lemma 8.4 shows that its actual phases have trivial q-part. Both labels are q-inactive, hence every constituent of AV is q-inactive. For a q-active J, apply block orthogonality to the fusion vector $r[AV] - cd_V[A]$ against J on P_q. By symmetry and multiplicativity, after dividing by the common degree factors this gives

$$0 = \sum_{W \in P_q} \frac{\chi_A(W)}{d_W}(r\chi_V(W) - cd_W)\overline{\chi_J(W)}.$$

The p-free nonunit terms vanish by the two alternatives at B. If $p \nmid d_J$, then at p-bearing W the first factor is p-integral since it equals $\chi_W(A)/d_A$, the bracket has strictly positive p-valuation by $V \in P_p$, and the last factor is in $p\mathcal{O}$. This is incompatible with the vacuum term pq of valuation one. The r^2 factor follows from separation.

8.3.1 Common constituents and intersections of difference sets

Proposition 8.8 (A finite test at one row)**.** *Fix a nonunit row in either type of minimal table. Suppose that some nonunit has, up to a root of unity, a full-weight positive word on at most k positions of odd prime-power order s^a. Put $E = \binom{k}{2}$, $H = (s-1)/2$, and use $\ell = 2$ in either table or $\ell = 3$ in odd ordinary dimension. Assume $\ell(1 + 2E) < (\ell - 1)s$. Let L be the least $1 \le h \le E$ for which $(H - 1)g(h) \le h(h - 1)$, where $g(h) = \min\{j : j(j + 1) \ge h\}$. If L does not exist, there is no such word. Otherwise, with $b = \binom{k-1}{2}$, the inequality*

$$\max\{E^2,\, sb^2 + (E - b)^2\} < sHg(L)$$

forces the initial support into a translate of $\mathbb{F}_s$. In this case it suffices to examine k-subsets of $\mathbb{F}_s$ containing $0, 1$. For each such set A, start with the difference set $A - A$. If D, F are any two symmetric difference sets already obtained, each with $\ell|D|, \ell|F| < (\ell - 1)s$, then for every unit u one may adjoin $T - T$, where $T = D \cap uF$. Discard the initial set if any resulting difference set has fewer than L nonzero pairs modulo sign.

Proof. The minimization in Lemma 4.22 runs over all full words on this row, with no bound on their lengths. Its minimum number of opposite nonzero differences is therefore at least L. If the initial positions meet several cosets of μ_s, then at most b opposite differences have order s. Write x for their number. At order s^j the units act transitively on $s^{j-1}H$ opposite pairs. Thus the mean number of opposite nonzero differences common to the initial spectrum and a unit multiple

is at most $x^2/H + (E-x)^2/(sH)$. Convexity bounds this by the larger of its values at $0, b$. On the other hand every such intersection contains at least $g(L)$ pairs, by the common-constituent argument of Lemma 4.22. This proves the conductor assertion. After translation and scaling one may pad even a smaller true support to a k-set containing $0, 1$. Strictness rules out a one-point support.

Inductively each retained D contains the adjoint phases of some nonunit at this row. Apply Lemma 4.20 to that nonunit and a Galois translate of the one for F. When both sets are exposed the common constituent has phases in T, so its adjoint has phases in $T - T$. This proves the induction even for padded supports, and the minimum L justifies rejection. □

Corollary 8.9. *For odd primes $7 \le p < q < r$ and arbitrary $a, b \ge 0$, all divisors of $p^a q^b r^4 d$ are WGT dimensions if $r < 5q$, or if $q \ge 17$ and $r < 6q$. More generally, for fixed $M \ge 2$ the entire strip $M < p < q < r < Mq$ is covered when q is sufficiently large in terms of M.*

Proof. For the smaller endpoints, the support enumeration gives the following table. The commands for all listed primes are recorded under Code, data and certificates. In each row k is the number of positions and the second column lists the primes above the indicated cutoff where the scalar inequality alone still permits a word. The other primes above the cutoff are excluded by Lemma 4.22.

k	s	cutoff	surviving supports at listed primes
2	none	3	0
3	none	7	0
4	$29, 31$	23	0
5	$47, 53, 59, 61$	43	0
6	$83, 89, 97, 101, 103, 107$	79	0

For the $\binom{s-2}{k-2}$ normalized seeds use the closure rule of Proposition 8.8, storing exposed sets up to dilation. Its proof also rejects T immediately if it has fewer than $g(L)$ opposite pairs. Use $\ell = 3$ for $k \le 5$ and $\ell = 2$ for $k = 6$. A work limit counts the seed as surviving.

In Proposition 8.7 the first alternative has $k = 1 + \lfloor r/q \rfloor$ positions, while $k < p$ excludes the second as in Theorem 8.2. For $k = 4, 5$ the inequalities $r > (k-1)q$ with $q \ge 11$ give the corresponding prime cutoffs; for $k = 6$ use $q \ge 17$. For general M, choose T_k large enough that Lemma 4.22 excludes k positions at primes $s > T_k$; such a choice exists since k is fixed. It suffices that $q \ge \max_{2 \le k \le M} \lceil T_k/(k-1) \rceil$. □

8.4 Small exponents and even dimensions

Theorem 8.10 (A bounded exponent and a large companion prime)**.** *Let t, u, r be distinct primes with r odd, and $c \ge 4$. Put $w = t^{\lfloor (c-2)/2 \rfloor}$ and $e = w(w-1)/2$. If*

$$1 + 2e < r/2, \qquad (r-1)/2 > C(e),$$

then every braided fusion category of dimension dividing $t^c u^A r^B d$ is WGT for arbitrary A, B and square-free d coprime to tur. For three odd primes one may replace $r/2$ by $2r/3$.

Proof. We work in a minimal table with all three primes repeated. Except when $t = 2$ and the table is reduced, take an r-free nonunit of P_t and a u-free nonunit of P_r. Their degrees have shapes $t^i u^j$ and $t^h r^k$ and their gcd g is at most w by the principal valuation bound at t. Their normalized entry has residue g at r and weight g. The stated inequalities imply $r > 2w$ (for odd data $w \ge 3$). Thus the single-residue theorem gives an unsigned r-word on at most w positions, contrary to Lemma 4.22.

In the reduced dyadic case exchange u, r if necessary to make r the larger, preserving the inequalities since w is fixed. Use complementary witnesses in P_u and P_r; their common 2-part

satisfies $g \le 2^{\lfloor (c-1)/2 \rfloor} \le 2w$. It gives both positive residues g at u, r. Here the general phase inequality implies $r > 2w+1$ since $w \ge 2$. Hence Theorem 4.13 gives the unsigned r-word with at most $1 + \lfloor g/u \rfloor \le 1 + \lfloor 2w/3 \rfloor \le w$ positions. The same lemma finishes. □

For $t = 3$ and $c = 5$ this gives all $r \ge 17$ in either parity, and all $r \ge 11$ for three odd primes: $w = 3$, $e = 3$ and $C(3) = 4$.

The first separation in the proof above also gives a parity reduction when the second separation is unavailable. We record a version that includes a dyadic least prime.

Proposition 8.11. *Let $p < q < r$ be primes, $q \ge 5$, $m = \lfloor r/q \rfloor$ and $e = m(m+1)/2$. A minimal counterexample at quartic r has even q-valuation under either of the following hypotheses:*

$$\begin{aligned} p > 2: &\quad 1 + 2e < 2r/3, \quad (r-1)/2 > C(e), \\ p = 2: &\quad q \nmid (r-1)(r-2), \quad 1 + 2e < r/2, \quad (r-1)/2 > C(e). \end{aligned}$$

In particular both tests hold when $q + 2 < r < 2q$. In the applicable range the WGT assertion for all divisors of $p^a q^b r^4 d$ follows from the one with b rounded down to an even integer. Every divisor of $p^a q^5 r^4 d$ is therefore covered.

Proof. If a column were both q- and r-active, it would be hit by a p-free nonunit $Y \in P_q$ by principal orthogonality modulo p. Its degree is $q^j r$: a second factor of r would give defect zero and forbid the hit. Choose a q-free nonunit $X \in P_r$, of degree $p^i r$ by the principal height bound at a quartic. The normalized crossing has weight r and residues $r, 0$ at q, r respectively. In odd data Lemma 8.4 gives a translated positive r-word on at most $m+1$ positions. For $p = 2$ Proposition 8.6 gives the same conclusion because the signed alternative requires $q \mid r-2$. Scaling to the character gives a full word, excluded in both cases by Lemma 4.22. Thus q and r are separated. At odd q-valuation Lemma 7.8 would produce q-activity on a nonempty r-active fiber. Descent gives the rounding assertion, and Theorem 7.3 covers middle cap four. In the stated interval $m = 1$, $C(1) = 1$, and $r \ge 11$. □

There is also a finite-largest-prime test valid in either parity with no bound on the middle exponent. For a cap a put $G = p^{\lfloor a/2 \rfloor}$. If $q > G$, all divisors of $p^a q^b r^4 d$ are covered (Theorem 7.15). Otherwise, with $m_0 = \lfloor G/q \rfloor$, $e_0 = m_0(m_0+1)/2$, they are covered whenever

$$r > G + 1, \qquad 1 + 2e_0 < r/2, \qquad (r-1)/2 > C(e_0).$$

Indeed take complementary witnesses of degrees $p^i q^j$ in P_q and $p^t r$ in P_r. Their common weight $g = p^{\min(i,t)}$ is at most G. The two-residue theorem (Theorem 4.13) writes the normalized entry as a positive r-word on at most $1 + \lfloor g/q \rfloor$ positions when $r > g + 1$. Lemma 4.22 gives the contradiction. In odd dimension the same test may interchange p and q, using the ordinary principal block at p.

Corollary 8.12 (A middle-prime test independent of the largest prime)**.** *Let $p < q$ be odd primes, let $G = p^{\lfloor A/2 \rfloor}$, and put $M = \lfloor G/q \rfloor$. If $M + 1 < p$ and $q \ge (M+1)^2$, then every braided category of dimension dividing $p^A q^B r^4 d$ is WGT for* every *prime $r > q$, with B arbitrary.*

Proof. If $M = 0$, use Theorem 7.15. For $M > 0$, the preceding large-prime test and Theorem 8.2 both use r-words on $1 + m$ positions for $m = \lfloor \min(r, G)/q \rfloor$: if $r < G$, the latter theorem applies; if $r > G$, the former applies since the odd numbers r, G differ by at least two. Thus $1 \le m \le M$ and $r > mq$. For $m = 1$ the bound at r is immediate as $r \ge 7$. For $m \ge 2$ set $e = m(m+1)/2$. At $h \le e$ one has $(h-1)/g(h) \le m$: if $g(h) < m$ this follows from $h \le g(h)(g(h)+1)$, and otherwise use $h \le e$. Hence $2C(e) + 1 \le 3 + 2em \le m(m+1)^2$. The exposure condition $2(1 + 2e) \le m(m+1)^2$ also holds. Now $r > mq \ge m(m+1)^2$ verifies both inequalities. The equality $r = G$ is impossible for the distinct primes. □

Proposition 8.13 (Transport of deep quartic twists)**.** *Let an ordinary or positive reduced minimal three-prime table at $p < q < r$ have $v_r(\Delta) = 4$. If the r-part of some twist has order greater than r^2, then for every complementary witness $X \in P_r$ of degree $p^i r$ and every occupied nonidentity coset of μ_{r^2} in the r-primary twists, that coset contains a label Z with $2\gcd(p^i, d_Z) \geq r$. The same holds upon exchanging p, q. In particular the r-primary level divides r^2 if some witness has $2p^i < r$ or $2q^j < r$.*

Proof. Fix a nonidentity coset. Each of its labels Z is r-free by Theorem 3.6 (the odd twist part divides the ordinary or reduced codegree). Put $g = \gcd(d_X, d_Z)$ and suppose $2g < r$. Principality gives $S_{XZ}/\operatorname{lcm}(d_X, d_Z) \equiv g$ above r, since d_Z/g is a unit there. The one-residue short-word theorem (Theorem 4.13) provides an unsigned r-primary word of length g, and the functional in Lemma 4.20 identifies its roots with the actual monodromy phases. Wild inertia over the r^2-cyclotomic level fixes the sum: the row field of X has conductor exponent at most two because its r-defect is two (Corollary 2.8). Uniqueness of the unsigned word and $g < r$ force this inertia to fix each root, since any nontrivial orbit has length at least r. Thus all phases lie in μ_{r^2}, as does the r-twist of X by twist–codegree divisibility. Every constituent of XZ remains in the original coset by balancing. The odd primary twist is independent of the representative of a fermion pair, so this also holds on the positive quotient. If $2g < r$ throughout the coset, its coordinate span is invariant under multiplication by X and by X^* (another witness of the same degree). It is then invariant under every constituent of their tensor words, by positivity. These generate the entire based ring by simplicity, whereas Z^*Z contains the unit, which lies outside the coset. This is impossible. □

8.5 A quintic valuation

Here $S_0 = S/\sqrt{\Delta}$, $D = \Delta$ for an ordinary table, and $U_v = T_v$ at an odd prime because the normalization anomaly is two-primary.

Lemma 8.14 (Gauss sums at a quadratic class)**.** *Let $D = p^{2k+1}n^2$, $k \geq 1$, $p \nmid n$, be the dimension of an ordinary integral modular category, with p odd. Fix one of the two square classes of primitive pth roots. Suppose every label whose p-part of the twist is neither 1 nor in that class has degree $p^i n$ for some $i \geq 0$. Let L_i count the degree-$p^i n$ labels at orders p^a with $a \geq 2$, and let M_i count such labels at one fixed primitive root of the other class. Then*

$$p^{k+1} \;\Big|\; \sum_i p^{2i}(L_i + pM_i).$$

Each L_i is divisible by p. In odd dimension L_i and M_i are even.

Proof. Square Galois covariance preserves degrees. A square-unit orbit of primitive roots of order p^a, $a \geq 2$, is a union of full μ_p-cosets, so its sum is zero and its size is divisible by p. At order p, multiplicity is constant within each square class. In odd dimension every nonunit has a distinct dual of the same twist (Lemma 2.5).

Write $L = \sum_i p^{2i} L_i$, $M = \sum_i p^{2i} M_i$ and put $A = (p^{2k+1} - L)/p$. By the odd-valuation slice (Proposition 3.9), the squared-dimension mass at twist one is $n^2 A$, and the mass at order p is $(p-1)n^2 A$. In units of n^2 the mass per primitive phase is therefore M on the other square class and $2A - M$ on the chosen class. The sums of the roots in these classes add to -1 and differ by a quadratic Gauss sum of squared modulus p. Thus for $T = \sum_X d_X^2 \theta_{X,p}$ we have $|T|^2 = pn^4(A - M)^2$. The regular-object indicator at an exponent selecting the p-primary twist is $|T|^2/D$, a rational algebraic integer by [45, Theorem 7.5] summed over the simples with their dimension multiplicities. It follows that $p^k \mid A - M$, as asserted. □

Corollary 8.15 (Exceptional labels and principal inertia)**.** *In Lemma 8.14, suppose either that all exceptional labels have degree n, or that $k \leq 2$. Then $p^{k+1} \mid L_0 + pM_0$. If $L_0 + M_0 > 0$, the*

number N of degree-n exceptional labels satisfies $2N \geq c(p-1)p^k$, where $c = 2$ in odd dimension and $c = 1$ otherwise. For $U \in P_p$ of p-height j, an inertia orbit on U of length e at a prime $s \neq p$ then satisfies

$$c(p-1)ep^{2j} < 2p^{k+1}.$$

Proof. Higher strata in the congruence vanish under either hypothesis: $p \mid L_i$ gives $p^3 \mid p^{2i}(L_i + pM_i)$ for $i \geq 1$. Thus a nonzero $L_0 + pM_0$ is at least cp^{k+1}, by parity in odd dimension. Its contribution to N is $L_0 + (p-1)M_0/2 \geq (p-1)(L_0 + pM_0)/(2p)$. Each of these labels has codegree p^{2k+1}, so its field is unramified at s. Lemma 2.4 gives $eN(p^j n)^2 < D$. □

Lemma 8.16 (A trace on the trivial twist fiber)**.** *Let an ordinary odd minimal table at three primes have dimension $D = t^5 n^2$, with t prime and $t \nmid n$. Suppose the t-primary twist level is t and one quadratic class of primitive tth roots occurs only on rows of codegree t^3 and field $\mathbb{Q}(\zeta_t)$. If that class occurs, then for each prime $s \mid n$ at least one s-free nonunit in P_t is fixed by s-inertia.*

Proof. Fix a label Z in that class and an outer prime $s \mid n$. Its degree is tn. The two complementary local groups act by scalars on each twist fiber of this class. Indeed the fibers are preserved by each complementary group, their labels have complementary unit codegrees, and the upper unipotents and their conjugates generate these groups. Thus the t-local Weyl W agrees with S_0 up to a root of unity on each such coordinate vector. The same holds for the torus translates of Z.

Put $U = U_t$. For $a \in \mathbb{F}_t$ set

$$M_a = \sum_V d_V \overline{\chi_Z(V)} \theta_{V,t}^a = \frac{D}{d_Z} (S_0 U^a S_0^{-1})_{0Z}.$$

Orthogonality gives $M_0 = 0$. For $a \neq 0$ use the local Bruhat identity

$$WU^a W^{-1} = U^{-1/a} W H_{t,-a} U^{-1/a}.$$

It gives M_a equal to a root of unity times $\sqrt{D}$. Indeed the diagonal factors contribute phases, and the vacuum entry of W at the torus translate has modulus $d_Z/\sqrt{D}$ and agrees with that of S_0 up to a root of unity. Since $\sqrt{D} = t^2 n\sqrt{t}$ and $s \mid n$, summing over a gives

$$\sum_{\theta_{V,t}=1} d_V \overline{\chi_Z(V)} \equiv 0 \pmod{s\overline{\mathbb{Z}}}.$$

This set of labels is Galois stable. Its vacuum contribution is one. Grouping by Galois orbit therefore supplies an s-free nonunit X with trivial t-twist and nonzero absolute trace of $x = \chi_Z(X)$. Its row meets Z, so $X \in P_t$ by the prime-power codegree support theorem (Theorem 2.11).

Suppose every such s-free nonunit is moved by s-inertia. Let A be the span of the s-local translates of the s-active coordinates. It is t-locally invariant. The chosen quadratic class is absent from A, since it is absent on s-active coordinates and the s-group preserves their t-phases. On $A^\perp$ the s-upper unipotent and all its conjugates are the identity. Let P project onto the U-fixed space. For a nonzero torus eigenvector $v \in PA$ with eigencharacter λ, the Fourier projector at a missing phase ζ_t^c vanishes on $W^{-1}v$. The same Bruhat identity gives

$$0 = \sum_{a \bmod t} \zeta_t^{-ca} PWU^a W^{-1} v = v + \left(\sum_{a \neq 0} \zeta_t^{-ca} \lambda(-a) \right) PWv.$$

Consequently $PWv \neq 0$, and the parenthesized sum has the same nonzero value for every c in that class. Replacing c by cu^2 multiplies it by $\lambda(u^2)^{-1}$. Thus every square torus acts trivially on PA. On $PA^\perp$ the s-group acts trivially, so on the entire t-inactive space

$$(H_{t,u^2} - 1)(H_{s,b} - 1) = 0.$$

Both tori act by unsigned Galois permutations since their automorphisms fix $\sqrt{D}$. Take b moving X. If H_{t,u^2} also moved X, the coefficient at X in the mixed difference of v_X would be positive. Hence all square t-units fix X. The entry field $F = \mathbb{Q}(x) \subseteq K_X \cap K_Z$ has degree at most two, since $K_Z = \mathbb{Q}(\zeta_t)$. Also $[K_X : F] \geq 2$, since s-inertia fixes K_Z and moves X.

Write u' for the other outer prime. The s-free nonunit X has degree $t(u')^i$: the principal valuation bound gives t-height at most one and a nonunit prime-power degree is impossible. The t-valuation of d_Z is also one. Since $X \in P_t$, symmetry gives $x = (d_X/d_Z)\chi_X(Z) \equiv 0$ above t. Its nonzero absolute field trace is an integer multiple of t. In the trace-norm identity (Lemma 2.4) this orbit contributes

$$[K_Z : F][K_X : F] \operatorname{Tr}_{F/\mathbb{Q}}(x)^2 \;\geq\; (t-1)t^2.$$

Reducing the ordinary dimension–trace identity at u' supplies a second, distinct orbit, of degree ts^j, with a nonzero trace also divisible by t. It contributes at least t^2. Together with the positive vacuum term this exceeds $t^3 = f_Z$. □

Proposition 8.17 (An odd quintic with two complementary witnesses)**.** *Let an ordinary odd minimal table have three dimension primes t, u, s, where $s > \max(t, u)$ and $v_t(D) = 5$. Suppose the s-free nonunits of P_u and the u-free nonunits of P_s all have t-height two. Some s-free nonunit of P_t is fixed by s-inertia.*

Proof. Suppose all s-free nonunits of P_t are moved. Call the two sets in the hypotheses the witnesses. They exist by prime avoidance. Each witness has t-defect one. Lemma 3.11 gives a single visible quadratic class for both sets. Each s-active or u-active label hits a corresponding witness by principal-block orthogonality modulo the omitted prime (Proposition 4.2). All their t-phases are therefore 1 or in the chosen class. There is a label outside these phases: either a deeper twist occurs or Lemma 7.4 gives the opposite primitive class. Lemma 7.9 applied at s, u gives every label Z of such a fiber pure t-power codegree. It follows that $D = t^5 n^2$ and the possible codegrees t, t^3, t^5 have degrees $t^2 n, tn, n$, respectively. Each exceptional Galois orbit contains at least $t - 1$ labels: square covariance gives at least $(t-1)/2$ distinct phases, and both duals occur at each phase in odd dimension.

The row of Z is supported on P_t. The dimension–trace equation reduced at each outer prime gives separate nonunit orbits of degrees tu^i and ts^j with nonzero entry trace. Each contributes at least one to the trace norm, while the vacuum contributes $[K_Z : \mathbb{Q}] \geq t - 1$. Hence $f_Z \neq t$. If an exceptional label had degree n, take an s-free nonunit $U \in P_t$. It has degree tu^i and inertia orbit length $e \geq 2$ by assumption. Corollary 8.15, with $k = 2$, $j = 1$ and $c = 2$, would give $(t-1)e < t$. Every exceptional row therefore has codegree t^3.

To determine its field, take the space A spanned by the s-local translates of the s-active labels as in Lemma 8.16. On A the only t-phases have order at most t. Thus $U_t(t)$ and its conjugates are trivial there. The upper–lower factorization

$$U_t(1)L_t(b) = L_t(b/(1+b))H_{t,1+b}U_t(1/(1+b))$$

for the lower unipotent $L_t(b)$ at $b = tj$ gives $H_{t,1+tj} = 1$ there: $U_t(1/(1+tj))$ agrees with $U_t(1)$ and the lower parameters are multiples of t. On $A^\perp$ the s-group is trivial. The mixed-difference argument of Lemma 8.16 shows that every label moved by s-inertia is fixed by wild t-inertia; both tori again fix $\sqrt{D}$. Apply this to the s-free trace-contributing orbit above. Its entry field F is tame at t, and its nonzero trace is divisible by t since $d_Z = tn$. If K_Z were wild at t, then $t \mid [K_Z : F]$ and this orbit alone would cost t^3, before the vacuum term. The codegree makes K_Z a field of t-power conductor (Theorem 2.7 and Corollary 2.8). Tameness now gives $K_Z \subseteq \mathbb{Q}(\zeta_t)$, with equality by its degree bound. The local Galois unit $1 + t$ fixes Z, and its square fixes the twist of Z. Hence that twist has t-part of order at most t. This holds on every exceptional row, and the other rows already have t-phase of order at most t. The global t-primary level is thus t, and Lemma 8.16 gives the required contradiction. □

Corollary 8.18 (A least quintic in odd dimension)**.** *For odd primes $p < q < r$ the conclusion of Theorem 8.3 holds for all $b, c \geq 0$.*

Proof. In a divisor-minimal failure all three primes survive and the least valuation is five by Theorem 7.3. Opposite r-free nonunits of P_q and q-free nonunits of P_r have common p-part at least q by Lemma 4.14, and at most p^2 by square divisibility. Hence every such witness has p-height two. An r-free nonunit of P_p has degree pq^i. Pair it with a q-free witness of P_r. The normalized entry has weight p and residue p at r. All twist primes are at least p, and $(r-p)(p-1) > p$. Lemma 8.5 gives an unsigned r-word of length $p < r-1$. Lemma 7.10 thus moves every r-free nonunit of P_p under r-inertia, contradicting Proposition 8.17. □

8.5.1 A quintic at the middle prime in odd dimension

Let $p < q < r$ be odd primes. There are two ways to control a middle quintic. The first uses short words at q, independently of the two other exponents. The second uses the exponent at p to shorten a word at r.

Proposition 8.19 (Short words at the middle prime)**.** *Put $j = \lfloor q/p \rfloor$. Suppose*

$$1 + j(j+1) < 2q/3.$$

Then every divisor of $p^A q^5 r^4 d$ gives WGT for every A. If $(r-q)(p-1) > q$, the two exponents A and C may both be arbitrary in $p^A q^5 r^C d$.

Proof. In a minimal table with three primes the middle valuation is exactly five by Theorem 7.3. First a p-free nonunit $A \in P_r$ has q-height two: height one would give weight q with a complementary $qp^i \in P_q$, with residue q at both q, r. By Theorem 4.13 the house would be q, contradicting strictness.

Suppose now that an r-free nonunit $U \in P_p$ has degree $p^i q$. Compare two character values on this one row. Choose $A \in P_r$ as above, of degree $q^2 r^l$, and a p-free $B \in P_q$ of degree qr^k. Each crossing has common weight q. At U, A the residues at p, r are both q, so Theorem 4.13 gives an unsigned r-word on at most $1 + j$ positions; its signed alternative would require $r = q + 1$. At U, B the residues at p, q are $q, 0$. Lemma 8.4 gives, up to a scalar, an unsigned q-word on at most $1 + j$ positions. Rescaling each normalized crossing gives the corresponding character value of full weight d_A or d_B.

The adjoints have full words on r-power and q-power roots, respectively, each on at most $1 + j(j+1) < 2q/3$ positions. Both supports are exposed and meet only at 1, contradicting Lemma 4.20. Thus every r-free nonunit of P_p has q-height two.

For any r-free nonunit $X \in P_q$, $d_X = qp^i$. Pair it with such a p-free witness of P_r. Their common weight is q and the residue at r is q. If $(r-q)(p-1) > q$, Lemma 8.5 gives an unsigned r-word of length $q < r-1$. By Lemma 7.10, r-inertia moves every such X, contradicting Proposition 8.17 with $t = q$, $s = r$. For $C = 4$ the complementary range has $r < 2q$ and is covered by Theorem 8.2. □

Theorem 8.20 (A least-prime exponent bound for a middle quintic)**.** *For $p < q < r$ odd and $A \geq 6$, suppose $q > p^{\lfloor A/2 \rfloor - 2}$. Every weakly integral braided fusion category of dimension dividing $p^A q^5 r^4 d$ is WGT. In particular $A = 6$ and $A = 7$ work for all odd prime triples.*

Proof. A divisor-minimal failure has table of dimension $p^\alpha q^5 r^4$, by the smaller-quartic and cubic-exponent theorems (Theorems 7.3 and 4.16). Theorem 8.3 gives $h = \lfloor \alpha/2 \rfloor \geq 3$. Consider opposite nonunits of P_q, P_r, respectively r-free and q-free. Their common p-part is at least q by Lemma 4.14. Since $q > p^{h-2}$, their individual p-heights are $h-1$ or h.

They cannot all have height h. At $\alpha = 2h$ all these witnesses would have p-unit codegree. But a p-active fiber contains a q-active column by the odd q-valuation, and orthogonality on

P_q modulo r gives a witness hitting it. At $\alpha = 2h+1$ every witness would have p-defect one. As in Theorem 7.11, its two orbit degrees agree: their squared ratio is a_0 or $2a_0$ with $a_0 \mid p-1$, and the primes of the degrees are all at least p. The opposite witnesses thus see the same primitive quadratic class, as in Proposition 8.17. The missing-phase argument gives a row of pure p-codegree, which would force the q-valuation to be even.

Consequently there exists an opposite pair of common weight $G = p^{h-1}$. Write $M = \lfloor G/q \rfloor$, so $1 \le M < p$. For $r > G$ the two-residue theorem gives an r-word on at most $1+M$ positions: $r = G+1$ is impossible for two odd numbers. For $r < G$ the odd q-valuation gives a column active at both q and r; the first step of the proof of Proposition 8.7 then gives an r-word on at most $1 + \lfloor r/q \rfloor$ positions. Explicitly, that column hits a p-free member of P_q of r-height one, and one pairs it with a q-free member of P_r of r-height one. Lemma 8.4 applies. Thus in either range we have a word on at most $1+m$ positions, with $m = \lfloor \min(r, G)/q \rfloor \le M$.

If $q \ge (M+1)^2$, then $r > mq \ge m(m+1)^2$. For $m \ge 2$ the estimates of Corollary 8.12 give both support inequalities at r, and for $m = 1$ they hold since $r \ge 7$. This contradicts Lemma 4.22. In particular $h \ge 4$ is excluded, since then $q > p^2 \ge (M+1)^2$.

It remains that $h = 3$ and $q < (M+1)^2$. Since $q \ge 5$ we must have $M \ge 2$, so $q < p^2/2$. For $j = \lfloor q/p \rfloor$ this gives $j \le (p-1)/2$. We claim $1 + j(j+1) < 2q/3$. For $j = 1$ use $q \ge 5$. For $j \ge 2$ we have $p \ge 5$ and $2p - 3(j+1) \ge (p-3)/2 \ge 1$. Consequently $3 + 3j(j+1) < 2(pj+1) \le 2q$. Proposition 8.19 now gives the contradiction. In particular its unrestricted-A conclusion holds throughout $2q < p^2$. □

8.5.2 The ordinary table at dyadic valuation five

Lemma 8.21 (The mass outside two dyadic phases)**.** *Let an integral modular category have dimension $D = 2^{2k+1}n^2$, with n odd and $k \ge 1$. Fix a primitive fourth root i. If every label with two-primary twist outside $\{1, i\}$ has dimension n and such a label exists, there are at least 2^k such labels.*

Proof. For each exponent m, the regular-object indicator $D^{-1}|\sum_X d_X^2 \theta_X^m|^2$ is an algebraic integer. Suppose the largest dyadic twist order is $2^s \ge 8$, and let L count labels at that order. Choose m equal to 2^{s-1} on the dyadic level and zero on its odd part. The Gauss sum is $n^2(2^{2k+1} - 2L)$. Its indicator is rational, so $2^k \mid L$.

It remains to consider twists of dyadic order dividing four. The vacuum calculation in the proof of Lemma 7.5, on the whole label set, equates the squared-dimension mass at orders one and two together with the mass at order four. Each therefore equals $D/2$. Let u, v count the exceptional labels at $-1, -i$, respectively. Choosing m one on the dyadic level and zero elsewhere gives the Gauss sum $n^2((2^{2k} - 2u) + i(2^{2k} - 2v))$. Its integral indicator gives $2^{2k-1} \mid u^2 + v^2$. Hence u, v are divisible by 2^{k-1}, and their quotients have the same parity: this follows from $v_2(x^2 + y^2) = 2\min(v_2(x), v_2(y))$ when the valuations differ, and $2v_2(x) + 1$ when they agree. Thus $2^k \mid u + v$ and $u + v \ge 2^k$ whenever an exceptional phase occurs. □

Lemma 8.22 (The weight-two crossing)**.** *In a minimal table let $q < r$ be odd primes, with $U \in P_2$ of degree $2q^i$ and $V \in P_r$ of degree $4r^j$, $i, j > 0$. Suppose N labels have odd degree n divisible by q^i and pure dyadic codegree f. Then the r-inertia orbit size e of U satisfies $e \ge (r-1)/2$ and $4eN < f$.*

Proof. The crossing of U, V has weight and residue two at $r \ge 5$, so Theorem 4.13 and Lemma 7.10 give the orbit bound. The N labels are unramified at r; principality and odd degree make their crossings with U nonzero. These distinct fixed entries cost at least $(2n)^2$ each, besides the vacuum, in Lemma 2.4. Hence $eN(2n)^2 < \Delta = fn^2$ when $N > 0$. □

Proposition 8.23 (An ordinary quintic at two)**.** *There is no ordinary minimal table of dimension $2^5 q^b r^c$ for odd primes $q < r$ and any b, c.*

Proof. Apply the proof of Theorem 7.11 with least prime two and height two. It gives a fourth root i for which the set of dyadic phases outside $\{1, i\}$ is nonempty, and every label there has pure two-power codegree. Thus b, c are even. Put $n = q^{b/2}r^{c/2}$. Every such Z has degree $2^j n$ and codegree 2^{5-2j}, $0 \le j \le 2$. Its row is supported on the principal two-block by Theorem 2.11. Reduce the dimension-trace equation (Lemma 2.4) at q and at r. The vacuum forces two distinct nonunit trace orbits in this block, of degrees $2q^h$ and $2r^l$: a prime-power degree is impossible, and an even principal degree has two-height at most one. Each orbit costs at least one in the trace norm, already excluding codegree two. At codegree eight one has $j = 1$. For a representative U of either orbit, $\chi_U(Z)$ vanishes above two by principality. By symmetry the same holds for $x = \chi_Z(U)$ since d_U/d_Z is a two-adic unit. Principality holds at all primes above two by Galois stability. The trace of x from its entry field therefore belongs to $2\mathbb{Z}$. The two nonzero orbit traces now cost at least $4 + 4$ besides the vacuum by Lemma 2.4. Consequently all exceptional labels have degree n and codegree 32.

The r-free degree $2q^h$ just obtained and a q-free degree $4r^l$ in P_r satisfy Lemma 8.22. Thus $4eN < 32$ and $e \ge 2$, giving $N \le 3$ contrary to Lemma 8.21. □

8.5.3 The fermion quotient at dyadic valuation five

Lemma 8.24 (Sign twists and the principal two-block)**.** *Let an integral slightly degenerate category be the even sector of an integral modular category of dimension $D = 4\Delta$. If the dyadic twist of every even label is a sign, then $v_2(\Delta)$ is even. If also $v_2(\Delta) \ge 3$, no member of the positive principal two-block has dyadic defect two.*

Proof. Use the fermion spaces A, B, C of Lemma 4.5, with dyadic unipotent $U = \lambda T_2$ and Weyl W. Dyadic inertia fixes the even labels individually by Lemma 4.7. The torus W^2 therefore acts on A, B by the common Galois sign ϵ of $\sqrt{D}$. It acts by the same sign on $C = WB$. Let $J = W|_A$ and $P = WU : A \to C$. This is a bijection, since U exchanges A, B. Applying $WUW = U^{-1}WU^{-1}$ to Ux, using $U^2 = \lambda^2$ on A, B, gives $UPx = \epsilon\lambda^{-2}PJx$. Since $J^2 = \epsilon$, the dyadic phases of odd labels differ only by a sign: each odd fermion orbit has a vector in C of its dyadic phase. Denote the two possible odd-sector dyadic phases by $\pm\delta$. The even contributions to the positive Gauss sum cancel in fermion pairs. Hence $p^+/\delta = (\xi/\delta)\sqrt{D}$ lies in an odd-conductor cyclotomic field K, with ξ the multiplicative central charge. Squaring shows that $(\xi/\delta)^2 \in K$, so the odd-order part of ξ/δ belongs to K and its dyadic part has order at most four. Thus $\sqrt{D} \in K(i)$. Were $v_2(D)$ odd, the quadratic discriminant of $\mathbb{Q}(\sqrt{D})$ would be divisible by eight, impossible in this cyclotomic field of conductor dividing four times an odd integer. We conclude that $v_2(D)$ and $v_2(\Delta) = v_2(D) - 2$ are even.

Choose k zero on the dyadic level and two on the odd level. In the modular indicator sum for even V, the two fusion inputs belong to the same fermion sector, so the dyadic twist ratio is a sign and $\nu_k(V) = \nu_2(V)$. For $b = \sum_Y \nu_k(Y)[Y^*]$, (3.3) gives $\chi_x(b) = f_x|(sT^k s^{-1})_{0x}|^2$. Odd-local Bruhat at the unit parameter -2 identifies this absolute entry with that of the vacuum column w of the odd Weyl, up to a torus permutation. For clarity, let d_o be the product of odd primes t with $v_t(D)$ odd and let $d_o^* = \prod_t (-1)^{(t-1)/2} t$ over the same primes. The Galois identity used in Lemma 6.2 conjugates the odd Weyl by $\mathrm{diag}(a, 1)$, introducing on its right the odd torus. On the vacuum this torus has the sign of the corresponding odd inertia action on $\sqrt{D}$, namely $\sigma_a(\sqrt{d_o^*})/\sqrt{d_o^*}$. Thus $w_i \in \sqrt{d_o^*}\,\mathbb{Q}$ and $|w_i|^2$ is d_o times a rational square. Since $v_2(D)$ is even, $d_o f_x = d_o D/d_x^2$ is a rational square. Also w has even support and trivial actual dyadic twist, by commutation with the grading and the dyadic unipotent. It follows that $gb = b$ and that for even x at most one of $\chi_x(b), \chi_{gx}(b)$ is nonzero. Consequently $W^+ = \sum_{[V]} \nu_2(V)[V^*]$ in the positive quotient has eigenvalues $w'_x = (\chi_x(b) + \chi_{gx}(b))/4$ that are integer squares. Here the average first selects the even terms, then divides out the two representatives of each pair; $gb = b$ gives $\nu_2(gV) = \nu_2(V)$.

Modulo two, $(W^+)^2$ equals the Casimir $\sum_{[V]}[V][V^*]$. Nonselfdual pairs cancel, and any

self-dual pair has a self-dual representative by the opposite twists of gV and V^*. Thus $4 \mid w'_x$ whenever f_x^+ is even. If x were principal of defect two, sign-only balancing makes the positive fields unramified at two. Hence $v(V) = (\chi_x(V) - d_V)/2$ is locally integral. Orthogonality gives $\sum_{[V]} v(V)v(V^*) = (f_x^+ + \Delta)/4 \equiv 1 \bmod 2$. After cancellation of nonselfdual pairs, Frobenius in the residue field gives $\sum_{[V]} \nu_2(V)v(V^*) \neq 0 \bmod 2$. But this sum is $(w'_x - w'_0)/2$, divisible by two. □

Lemma 8.25 (A squared dyadic twist fiber)**.** *On the positive space let $K = W_2|_A$. If the odd Weyls preserve the line of Z, then $Z \in P_2$ and the Z-column of K agrees up to phase with that of $\widehat{s}$. If a power of the squared dyadic twist is $R = I + (u-1)P$, where P is a coordinate projection and u a root of unity, then $(2 - u - u^{-1}) \operatorname{Tr}(PKPK^{-1})$ is an algebraic integer.*

Proof. The dyadic Weyl preserves each positive two-block by the full lift in Lemma 4.5. The product factorization of $\widehat{s}$ gives the column assertion; its vacuum entry forces $Z \in P_2$. The scalar lifts cancel in $RKR^{-1}K^{-1}$, an element of a finite modular image. Expanding its integral trace and subtracting $\dim A + (u + u^{-1} - 2) \operatorname{rank} P$ gives the claim. □

Proposition 8.26 (The reduced quintic at two)**.** *There is no slightly degenerate divisor-minimal counterexample of dimension $2^5 q^b r^c$ with $2 < q < r$ and b, c arbitrary.*

Proof. Write Δ for the positive quotient dimension and $D = 4\Delta$ for the dimension of the integral minimal modular extension from Lemma 4.4. Let g be the transparent fermion of the even sector. By Theorem 7.15, b, c are even. Write $\Delta = 16n^2$ with n odd. The complementary r-free nonunits of P_q and q-free nonunits of P_r all have two-height two: comparison across the two blocks gives two-part at least q, while square divisibility gives at most four. Thus $q = 3$ and the respective degrees are $4q^i, 4r^j$.

Call a pair deep if its squared dyadic twist is nontrivial. Every complementary witness has odd positive codegree, hence a singleton two-block. By Corollary 7.13 deep pairs are inactive at both q, r. Lemma 7.9 gives them pure dyadic positive codegree. It is 16: codegree one would exhaust the dimension and four is excluded at the quartic valuation by Proposition 5.4. Thus $d_Z = n$.

If a deep pair exists, positive principal two-block orthogonality modulo r gives an r-free nonunit hit of degree $2q^i$ by the principal valuation bound. A complementary witness has degree $4r^j$. Lemma 8.22 now gives $4eN < 16$, $e \geq 2$, hence a unique deep pair Z. The odd Weyls preserve its line. Apply Lemma 8.25 with P projecting onto Z and $u = \theta_{Z,2}^2$. It makes Z principal, hence $\chi_Z(Z)$ a dyadic unit, and $(2 - u - u^{-1})|K_{ZZ}|^2$ integral. But $|K_{ZZ}|^2 = |\chi_Z(Z)|^2/16$ has valuation -4, while $v_2(2 - u - u^{-1}) \leq 2$. This excludes all deep pairs.

All even dyadic twists are therefore signs. Lemma 8.24 excludes defect two in the positive principal two-block. Since $v_2(\Delta) = 4$, the principal valuation bound now makes every degree of that block odd.

By Corollary 7.13, the odd mates of the complementary witnesses and all odd q- or r-active columns have a common phase δ. Lemma 7.7 gives an odd label Y of another phase. Its fiber is inactive at q, r, so Y has pure dyadic full codegree by Lemma 7.9.

Take an odd mate T of a complementary r-free witness, so $r \nmid d_T$ and its phase is δ. The restrictions of T, Y to the even ring give distinct signed characters. Orthogonality over even fermion-pair representatives, reduced modulo r, supplies a nonunit r-free even pair V hit by Y: the vacuum contributes one and $r \mid \chi_T(V)$ at every r-bearing degree by lcm integrality. Since f_Y is a power of two, Theorem 2.11 puts V in the full principal two-block, hence its pair is in the positive principal block. Its degree is odd by the preceding paragraph, so it is a power of q, impossible in the perfect simple quotient. □

Conclusion of Theorem 8.3. For odd p use Corollary 8.18. For $p = 2$ Propositions 8.23 and 8.26 exclude the ordinary and reduced tables. Smaller least-prime valuations already give WGT by Theorem 7.3, and the two-prime theorem of Dong [22, Theorem 3.1].

8.6 A quintic at three in even dimension

Lemma 8.27 (Weight three at two and three)**.** *In an ordinary or positive reduced minimal table with dimension primes $2, 3, r$, $r > 3$, let $\gcd(d_V, d_B) = 3$. Set $z = S_{VB}/\operatorname{lcm}(d_V, d_B)$. If $z \equiv 1$ at every prime above two and $z \equiv 0$ above three, then $z = \lambda\sqrt{-3}$ for a dyadic root of unity λ.*

Proof. A signed integral word of length at most three represents z: at a common level $2^a 3^b r^c$ the root lattice on the odd factors has a totally unimodular root configuration by [4, Theorems 3 and 13]. A basis of the 2-power lattice consists of one member of each opposite pair, so tensoring gives disjoint signed copies of the same unimodular matrix. Integral minimum length follows.

Erase the 3-parts of the word. The remaining sum w is divisible by three in its unramified ring and has house at most three. If $w = 0$ the only possibility is a cancelling pair (a vanishing triangle would require the prime three in the complement). Thus $z = \lambda(\alpha - \beta)$ with λ prime-to-three and α, β of 3-power order. If $w \neq 0$, Kronecker and equality in the triangle inequality give $z = \lambda(\alpha + \beta + \gamma)$ with the same conventions. In either case λ has trivial r-part: invert its r-part while fixing the other factors and reduce both values modulo two. They are both 1 and the reduction is faithful on the odd-order roots.

Erase the dyadic factor modulo two. In the pair case the vector of $\alpha, \beta, 1$ modulo two is a union of complete μ_3-cosets by the cyclotomic relations. Its odd cardinality is at most three, so it is the identity coset and $\{\alpha, \beta\} = \{\omega, \omega^{-1}\}$ for a primitive cube root ω. In the triple case the vector of $\alpha, \beta, \gamma, 1$ has even cardinality at most four and is a union of complete triples. Thus it is empty and the word is $\lambda(1 + 2\xi)$ for a 3-power root ξ. Strict house excludes $\xi = 1$. At order at least nine, the adjoint support $1, \xi, \xi^{-1}$ has one mark in each occupied μ_3-coset. Its dilation by the 3-adic unit four meets it only at one. Lemma 4.20 after scaling to the character value at B gives a contradiction. Thus $\xi = \omega^{\pm 1}$ and in both branches $z = \pm\lambda(\omega - \omega^{-1})$. □

Proposition 8.28 (Transfer at two and three)**.** *In an ordinary or positive reduced minimal three-prime table at $2, 3, r$, a row $U \in P_2$ of degree $2^i 3$ forces every odd-degree label to have the same 3-primary twist as U. No such row exists when $v_3(\Delta)$ is odd. The same implications hold in a full integral modular table if there is a subset $E \subseteq P_3$ with the following properties. It is invariant under ternary inversion ι, contains the vacuum as its only invertible, and supports restricted orthogonality at unequal ternary phases and the odd-valuation mass identity. At each odd-degree nonunit of E, the crossing with U is negated by ι and is nonzero.*

Proof. Let ι invert the 3-power roots and fix the other primary factors, and put $H = \chi_U + \chi_{U^\iota}$. The two row labels have the same 3-twist by Galois square covariance. Each odd nonunit $B \in E = P_3$ in the minimal tables has degree $3^h r^j$ with $h, j > 0$. The normalized crossing has weight three and residues $1, 0$ at $2, 3$, respectively, by principality. Lemma 8.27 therefore gives the required negation and nonvanishing. In all cases $H(B) = 0$ at each such B, and $B^\iota \neq B$. On every even B both terms in H vanish modulo any prime above two by principality. If an odd-degree J has different 3-twist, block orthogonality gives $\sum_{B \in E} \chi_J(B)\overline{H(B)} = 0$. The vacuum contributes two. On every even B, lcm divisibility makes $\chi_J(B)$ divisible by two, so all other nonzero contributions have dyadic valuation greater than one. This proves the transfer; in particular every odd-degree label is 3-inactive by the vacuum. Now the odd labels of E number one modulo two: all except the vacuum are paired by ι. At odd $v_3(\Delta)$ the principal-block mass identity of Prop. 3.9 reads

$$\sum_{B \in E,\ \operatorname{ord}(\theta_{B,3}) = 3} d_B^2 = 2 \sum_{B \in E,\ \theta_{B,3} = 1} d_B^2.$$

The left is zero modulo four whereas the right is two. □

Lemma 8.29 (Three-primary phases at odd valuation)**.** *If an ordinary or positive reduced minimal table has odd $v_3(\Delta)$, then some 3-twist lies outside $\{1,\eta\}$ for each primitive cube root η.*

Proof. If all phases are in $\{1,\eta\}$, the first paragraph of the proof of Lemma 7.4 and its immediately preceding indicator construction (with $p=h=3$) give an integral self-adjoint virtual element b with $b^2=3b$, $b_1=1$, $\dim b=3$. In the ordinary table this uses the nontrivial local Legendre sign from $v_3(\Delta)$ odd, and applies also in even dimension; odd dimension in that lemma is used subsequently for the nonreal character orbit. For clarity, the missing Fourier coefficient gives $w=-GPsv_0$ on the inactive phase projector, $|G|^2=3$. The indicator construction gives $\chi_j(b)=3\mathbf{1}_{\theta_{j,3}=1}$, rational integral coefficients and the stated normalization. For a positive reduced table we obtain the same element as follows. In the integral minimal modular extension let g be the fermion and A the positive space spanned by $(v_Y+v_{gY})/\sqrt{2}$ for even fermion pairs. Write W_o for the product of the local Weyls away from three and $w=W_ov_0$. It interchanges the fermion permutation and its monodromy sign. Therefore the coordinates of $W_o(v_0+v_g)/\sqrt{2}$ in the pair basis are $2w_Y$, with $w_Y=w_{gY}$ for even Y. For m zero on the ternary level and one on its complement, the complementary Weyl identity and the ordinary indicator formula give, for even V,

$$\nu_m(V)+\nu_m(gV)=\sum_{[Y]}|2w_Y|^2\,\widehat{\chi}_{[Y]}([V]).$$

Indeed the odd-sector contributions cancel and each even pair contributes twice the same character. If all ternary phases in the positive table lie in $\{1,\eta\}$, the proof of Lemma 7.4 applies its ternary Bruhat identity on A and gives $|2w_Y|^2=3d_Y^2/\Delta$ at inactive pairs and zero elsewhere. The displayed sum is rational by Galois invariance and is an algebraic integer as a sum of indicators. It therefore defines the same integral virtual element in the positive ring, with norm squared three and dimension three.

Let $M=N_b$; $M^2=3M$ and M is symmetric integral. At a diagonal $a=1$ or 2 the off-diagonal squared row norm is $a(3-a)=2$, so the row has two neighbors with signs. Rows at $0,3$ are isolated. The unit component is a cycle of length three or four: a longer cycle has nonneighbors with a unique length-two path, contradicting the matrix equation. A triangle has trace three (the trace is a multiple of three and the unit diagonal is one), hence rank one and all three diagonals one. Its positive dimension eigenvector forces equal dimensions, contradicting perfectness. On a four-cycle let A oppose the vacuum. Adjacent diagonals sum to three, so the two neighbors Y,Z have diagonal two. Their dimension equations give $d_Y,d_Z\in\{d_A-1,d_A+1\}$. Coprime vanishing gives $\chi_A(Y)=\chi_A(Z)=0$, impossible because $b=1\pm Y\pm Z$ and the only eigenvalues of b are zero and three. □

Theorem 8.30 (A pure ternary codegree)**.** *In an ordinary minimal table of dimension $2^a3^5r^4$ with $r>3$, the exponent a is even. For some primitive cube root η, the set of labels with 3-primary twist outside $\{1,\eta\}$ is nonempty, and every such label has formal codegree 27 or 243.*

Proof. By prime avoidance, P_2 contains r-free nonunits. Their degrees have positive 2- and 3-heights, and hence 3-height exactly two by the transfer proposition. The 2-free nonunits of P_r likewise have degree $9r$ (Section 7.4).

All these witnesses have ternary defect one. By Lemma 3.11 their rows see only 1 and a common primitive cube root η. By principal orthogonality modulo the omitted prime, each 2-active or r-active column hits one of the witnesses. Their 3-phases therefore lie in $\{1,\eta\}$. The preceding lemma gives a nonempty 3-phase fiber outside the pair. Lemma 7.9 at $2,r$ gives every row Z on it 2- and r-unit codegree, so a is even, and $f_Z=3^{5-2j}$ for $0\le j\le 2$. The value three is impossible: orthogonality to the vacuum reduced at each of $2,3,r$ requires three distinct nonunit hits, since a hit omitting two primes would have prime-power degree. The Galois-averaged squared norm of each nonzero algebraic integer is at least one, so the vacuum and these three hits already exceed three. (Equivalently average the entire row norm over the splitting field.) □

Lemma 8.31 (The trace restriction at codegree 27)**.** *Suppose there is an ordinary minimal table of dimension $2^a3^5r^4$ and a label Z of pure codegree 27, so a is even and $d_Z = 2^{a/2}3r^2$. There is a unique Galois orbit of r-free nonunits with a nonzero absolute trace on this row. For its representative U write $d_U = 2^i3$. Its field equals K_Z, is of 3-power conductor, and*

$$9 \cdot 2^i \equiv \pm 1 \pmod{r}.$$

There is likewise exactly one odd nonunit orbit contributing nonzero trace, of degree $3r^j$, $j = 1, 2$, with field K_Z.

Proof. Every hit lies in P_3 by Theorem 2.11. The integral trace–dimension relation of Lemma 2.4 at Z reduced modulo r and modulo two gives a separate trace-contributing r-free and odd nonunit orbit. Every such label U has exactly 3-height one by the principal valuation bound and absence of prime-power nonunit degrees. Then $\chi_U(Z)$ vanishes above three by principality, and $\chi_Z(U) = (d_U/d_Z)\chi_U(Z)$ does too since the ratio is a 3-unit. This holds at every prime above three, so its absolute field trace is an integer multiple of three (reduce the sum of all conjugates). For $F = \mathbb{Q}(\chi_Z(U))$ the exact contribution to the orbit trace norm is $[K_U : F][K_Z : F] \operatorname{Tr}_{F/\mathbb{Q}}(\chi_Z(U))^2$ by Lemma 2.4. A nonzero contribution is at least nine. Three such orbits are impossible by the positive vacuum term, and either relative field index exceeding one would likewise cost at least $18 + 9 + 1 > 27$. Thus for each of these two orbits the entry field equals both row fields, and the corresponding absolute trace is ± 3. The dimension trace identity modulo r now reads $1 \pm 3d_U = 0 \bmod r$ for the unique r-free orbit. □

Proposition 8.32 (The odd witness at the ternary quintic)**.** *Let an ordinary minimal table have dimension $2^a3^5r^4$, with $r > 3$. If $r \geq 11$, no such table exists, for any a. If $r = 7$, all exceptional ternary phases of Theorem 8.30 have codegree 243; every r-free nonunit of P_3 has tame r-inertia orbit of size exactly two, and every r-active label has even degree.*

Proof. Take any r-free nonunit $A \in P_3$ and any odd nonunit $Y \in P_r$. Their degrees have the forms $3 \cdot 2^i$ and $9r$. Their normalized crossing z has weight three and strict house less than three, and $z \equiv 3$ above r by r-principality. If K_A is r-unramified and $r \geq 7$, the algebraic integer $(z - 3)/r$ has house less than one. This would give $z = 3$, which is impossible. In particular the r-free trace witness of Lemma 8.31 cannot exist, so every exceptional label has codegree 243.

For $r \geq 7$ the single-residue theorem gives an unsigned r-word of length three. Lemma 7.10 gives orbit length at least $(r - 1)/3$. By Corollary 8.15, applied with $j = 1$ and $k = 2$, the length is strictly less than three. This excludes every $r \geq 11$.

At seven every such A has full and tame inertia orbit size two. The word at any odd nonunit $Y \in P_7$ therefore consists of a single order-three orbit of roots; wild inertia fixes the word, and its orbits on the three terms are singletons, so all terms have order seven. Thus z is one of the two quadratic periods, whose sum is -1. Let A' be the conjugate under inversion at seven. Every label in P_3 is 7-inactive: an exceptional row of codegree 243 has degree prime to three, hence hits all of P_3 by principality and only 7-inactive labels by its 7-unit codegree. Thus A, A' are 7-inactive. The function $H = 3(\chi_A + \chi_{A'}) + \dim$ equals 7 at the vacuum and vanishes on every odd nonunit $Y \in P_7$ since $\chi_A(Y) + \chi_{A'}(Y) = -d_Y/3$. Orthogonality on P_7 to any 7-active row J now precludes d_J odd: at even Y the factor $\chi_J(Y)$ would be even by lcm integrality, leaving the odd vacuum term. □

Theorem 8.33 (A sextic dyadic and a quintic companion)**.** *Let $2 < q < r$ be primes. Every weakly integral braided fusion category of dimension dividing $2^6q^5r^4d$ is WGT, for any square-free d coprime to $2qr$.*

Proof. In a divisor-minimal counterexample, Proposition 7.16 forces the first two source exponents to be six and five and $q \leq 2^2$, so $q = 3$. The r-valuation is four by the reduction in Section 8.

Proposition 8.37 excludes a slightly degenerate obstruction at ternary valuation five. It suffices to exclude an ordinary table of dimension $2^6 3^5 r^4$ with $r < 11$, by the preceding proposition.

There is a row $U \in P_3$ of degree 12, by the same Proposition 7.16. Every 3-free nonunit of P_r has degree $4r$ or $8r$. Paired with any degree-12 row of P_3, it gives a normalized entry of weight four, of strict house less than four and of residue four at both three and r. The two-residue theorem (Theorem 4.13) gives a tame r-inertia orbit of size $r-1$ on the entry, also in its signed alternative. In particular every degree-12 principal row is ramified at r, and the orbit of U has size at least $r-1$.

A codegree-27 row would, by Lemma 8.31, require an r-unramified, r-free row of P_3 of degree $3 \cdot 2^i$, with $9 \cdot 2^i \equiv \pm 1 \pmod r$. Its dyadic part is at least four by Lemma 4.14 applied against a 3-free nonunit of P_r, and at most eight by square divisibility. Degree 12 has just been excluded for this row. Degree 24 would require $r \mid 71 \cdot 73$, impossible as both numbers are primes larger than 11. Thus Theorem 8.30 supplies a nonempty set of exceptional phases, all of degree $n = 8r^2$ and codegree 243. Apply Corollary 8.15 with $k = 2$, $j = 1$ and the row U. It would give $9(r-1) \le 27$, whereas $r \ge 5$. □

8.7 An even-dimensional reduction by the fermion

Let the divisor-minimal slightly degenerate category $\mathcal{B}$ have dimension $2^{2k} q^b r^c$, with $2 < q < r$, $k \ge 3$, $2^{k-2} < q$ and b odd. Use the positive quotient R and integral minimal extension of Lemma 4.5. Complementary witnesses are the r-free nonunits of P_q and the q-free nonunits of P_r in R.

Proposition 8.34. *All even-sector dyadic twists have order dividing four, and all twists of $\mathcal{M}$ have dyadic order dividing eight. Put $\epsilon_Y = \theta_{Y,2}^2 \in \{1, -1\}$ on fermion pairs. Each complementary witness X has a unique mate X' in its reduced two-block. They have equal degrees and opposite squared dyadic twists.*

Proof. Complementary witnesses exist on both sides. For opposite witnesses their dyadic gcd is at least q by Lemma 4.14. Since $2^{k-2} < q$ and $d_X^2 \mid \Delta$, every witness has dyadic height $k-1$, hence positive dyadic defect one.

Lemma 7.17 confines all witness rows to the two squared phases $+1, -1$, since $v_2(4\Delta)$ is odd. Every nonempty squared dyadic twist fiber contains a q-active label by oddness of b. It has a nonzero entry on a complementary witness in P_q by principal-block orthogonality modulo r. Thus all actual even dyadic twists have order dividing four. The second assertion of the lemma now bounds the full level and gives the equal-degree mate. □

Theorem 8.35 (Odd middle valuation in a reduced table)**.** *Under the hypotheses of the preceding proposition there is no slightly degenerate minimal counterexample. In particular this excludes every reduced obstruction with source dimension $2^6 3^b r^c$, b odd.*

Proof. All complementary witnesses have positive defect one and the even twists satisfy the order bound of Lemma 7.18, by Proposition 8.34. Over each witness of degree d there are two free odd orbits whose actual dyadic twists have ratio $\pm i$ and whose degrees u, v satisfy $u^2 + v^2 = 2d^2$ and $v_2(u) = v_2(v) = k-1$.

Choose a complementary witness of P_r, so $d = 2^{k-1} r^j$. The odd parts of u, v involve only q, r by square divisibility. After canceling 2^{k-1} and the common r-power r^t, their squares sum to $2r^{2(j-t)}$. If $t < j$, neither remaining integer is divisible by r, as the other would also have to be divisible by r. They are therefore powers of q, at least one equal to one. This gives $1 + q^{2e} = 2r^{2(j-t)}$, where $q < r$ forces $e \ge 2$. But q^e is then a nontrivial perfect power among the first coordinates of solutions of $x^2 - 2y^2 = -1$. This is excluded by [17, Theorem 6.1]: for $(1+\sqrt{2})^n = A_n + B_n\sqrt{2}$, the associated Pell sequence A_n has no perfect power greater than one. The positive solutions used here have odd n and first coordinate A_n. Thus $t = j$ and both odd

mates have degree d. At $q = 3$ just reduce $u^2 + v^2 = 2d^2$ modulo three to see that both degrees are q-units. Thus in every case odd q-unit objects realize two distinct dyadic phases. In the full principal q-block, evaluation at the fermion forces even sector, and evaluation of the fermion at an odd q-unit excludes the fermion itself. On each odd q-active column, block orthogonality modulo r therefore gives an r-free nonunit witness. Lemma 7.12 with $p = q$ and $k = 2$ shows that these are all the odd-sector dyadic phases.

Lemma 7.18 now gives a nontrivial invertible in the perfect positive quotient, a contradiction. □

Proposition 8.36 (Forcing the top dyadic height)**.** *Suppose $2 < q < r$, b is odd, and a slightly degenerate minimal counterexample has source dimension $2^{2k+1}q^b r^c$, with $k \geq 2$. Put $G = 2^{k-1}$. If $G < 2q$ and $r > G + 1$, such a counterexample cannot occur.*

Proof. In the positive table, pair an r-free nonunit of P_q with a q-free nonunit of P_r. Such witnesses exist by prime avoidance, and the common weight g is a power of two between q and 2^k by Lemma 4.14 and square divisibility. If either witness has dyadic height less than k, then $q \leq g \leq G < 2q$ and $r > g + 1$. The positive residues at q, r and strict house force the normalized entry to be $g - q + q\xi$ with ξ a nonidentity r-primary root by Theorem 4.13. This two-point word has three differences, and the difference-set argument (Lemma 4.20) excludes it for $r \geq 7$: its support is exposed since $3 < r/2$, and a tame dilate moves the unique nonzero opposite pair. Here $r \geq 7$ holds whenever $g \geq q$ is possible, since $g \geq 4$ and $r > g + 1$.

Thus all complementary witnesses have maximal height k and singleton positive two-blocks. Corollary 7.13 contradicts odd b. □

8.8 A quintic valuation at three in the fermion extension

Proposition 8.37. *A slightly degenerate minimal obstruction cannot have dimension $2^a 3^5 r^c$ for any prime $r > 3$ and positive exponents a, c.*

Proof. Use the integral minimal extension $\mathcal{M}$ of $\mathcal{B}$ with fermion g; a missing repeated prime is covered by the two-prime reduction. Its only invertibles are $1, g$. For a noninvertible X, the constituents of XX^* generate $\mathcal{B}$, whose centralizer is $\langle g \rangle$. Thus strict house and coprime vanishing hold for all full noninvertibles (Lemma 4.4). None can have prime-power degree t^i: for $s \in \{3, r\} \setminus \{t\}$ a t-free nonunit of positive P_s exists by prime avoidance, and its full principal lift (Lemma 4.6) would have a nonzero coprime crossing.

We first exclude $U \in P_2(\mathcal{M})$ of degree $2^i 3$, $i > 0$. Put

$$E = P_3(\mathcal{M}) \cap \{B : \theta_{B,3'} = 1\},$$

where $3'$ denotes the entire prime-to-three factor. The only invertible in E is the vacuum, since $\theta_g = -1$. Evaluation at g makes every label of $P_3(\mathcal{M})$ even. The projector onto E commutes with the local ternary modular group, as do its two factors. Block orthogonality at unequal ternary phases and the odd-valuation mass identity hold on E by Lemma 3.3, Proposition 3.9 and the proof of Proposition 4.2.

An odd-degree nonunit $B \in E$ has degree $3r^j$ with $j > 0$, by the exclusion of prime powers and the principal height bound at valuation five. The number $z = S_{UB}/\operatorname{lcm}(d_U, d_B)$ has weight three, residue one above two by $U \in P_2$, and residue zero above three by $B \in P_3$. The weight-three argument of Lemma 8.27 gives $z = \lambda\sqrt{-3}$ with λ a dyadic root of unity. To justify its use in the full extension, the cyclotomic calculation requires only the three level primes and strict house. Its remaining difference-set step uses the adjoints of B and its Galois translate, evaluated at U. Both fusion labels are even, and tensoring by g switches the sign of their twists, so neither BB^* nor $B^\sigma(B^\sigma)^*$ contains g. The vacuum is their only possible invertible constituent. All noninvertible constituents satisfy the strictness just proved. These are the two uses of simplicity in Lemma 4.20.

Ternary inversion preserves E by square covariance and negates these nonzero crossings. Proposition 8.28 on E now excludes U, since $v_3(\dim \mathcal{M}) = 5$.

Every r-free noninvertible $U \in P_2(\mathcal{M})$ consequently has $v_3(d_U) = 2$. Indeed its degree has both primes $2, 3$ and $d_U^2 \mid \dim \mathcal{M}$. We need a full r-principal even label V of odd degree with $v_3(d_V) = 2$. Take a 2-free positive nonunit from P_r and lift it as above. If its degree were $3r^j$, compare it in the positive table with an r-free nonunit of P_3, necessarily of degree $3 \cdot 2^k$. The normalized crossing has weight 3 and residue 3 above both 3 and r. Indeed if R is the P_3 witness the entry is $\chi_R(V)/r^j$, whose denominator is a unit at three. The equality case of Theorem 4.13 gives modulus three, contrary to strictness.

Both V and the r-free noninvertibles of $P_2(\mathcal{M})$ have ternary defect one. Lemma 3.11 supplies a common visible primitive phase η.

In every even pair J, gJ at least one representative has nontrivial dyadic twist. For it, orthogonality to the vacuum on the full P_2 yields an r-free noninvertible hit, since modulo r the invertibles $1, g$ together contribute two. Its row sees only $1, \eta$ at three. The two representatives have the same ternary twist, so every positive label has ternary phase 1 or η. This contradicts Lemma 8.29 in the positive table. □

Proof of Theorem 8.1. If the quartic is below a larger repeated prime, Theorem 7.3 applies. For a largest quartic and a least quintic use Theorem 8.3. The remaining ordering has a middle quintic. In odd dimension Theorem 8.20 applies at exponent seven. For a dyadic least prime and $q \geq 5$, use Theorem 7.15 with $2^{\lfloor 7/2 \rfloor - 1} = 4 < q$: rounding down the middle cap leaves a quartic below the largest prime, covered by Theorem 7.3. For $q = 3$, Theorem 8.33 covers dyadic exponents at most six. At exponent seven Theorem 8.30 excludes the ordinary table by dyadic parity, and Proposition 8.37 excludes the fermion case. Each reduction allows divisors and the coprime square-free factor. □

8.9 A principal-block criterion for individual categories

Fix a place over an odd prime r in a common splitting field. Write P_r for the principal character block there, using the positive quotient in the slightly degenerate case.

Proposition 8.38 (Principal blocks under gauging of order prime to r)**.** *Let $\mathcal{C} = \mathcal{E}^G$ be an integral modular or slightly degenerate gauging of $\mathcal{D} = \mathcal{E}_e$, with $r \nmid |G|$ and $\mathrm{Rep}(G) \cap \mathcal{C}' = \mathrm{Vec}$. Every member of $P_r(\mathcal{C})$ has neutral flux. The G-orbits supporting at least one member of this block are exactly the orbits in $P_r(\mathcal{D})$. In particular one block contains a degree divisible by r if and only if the other does.*

Proof. The grading support is a normal subgroup $H \leq G$. Charges in $\mathrm{Rep}(G/H)$ centralize every homogeneous object, so $\mathrm{Rep}(G) \cap \mathcal{C}' = \mathrm{Vec}$ forces $H = G$. The equivariant braiding on a homogeneous object of grade g acts on a pure charge π by its equivariant structure at g [26, Section 4.4.4 and proof of Proposition 4.56(i)]. Thus evaluation of a charge at that flux gives its ordinary character at g, up to inversion. Since the character table of the r'-group G is invertible at r, a principal simple has neutral flux. For a neutral simple Z over U and neutral equivariant object W we have $\chi_Z(W) = \chi_U(FW)$, where F forgets equivariance. The terms in the underlying orbit give equal normalized traces because FW is invariant. Every simple in $\mathcal{E}$ occurs under an equivariant simple, whose degree is its degree times an r-unit [13, Corollary 2.13]. In fact its orbit size divides $|G|$, as does the stabilizer order, and the multiplier class on a stabilizer I is killed by $|I|$ by restriction–corestriction. The exact sequence for the $|I|$th power on $\mathbb{C}^\times$ therefore represents the class by a cocycle in $\mu_{|I|}$. The resulting central extension of I has order prime to r. Its ordinary irreducible degrees, including the projective degree in question, divide that order. The integral block idempotent of $P_r(\mathcal{D})$ is G-invariant and lifts to a neutral element of the local Grothendieck algebra by $|G|^{-1}\,\mathrm{Ind}$. Evaluation shows that a principal Z lies over $P_r(\mathcal{D})$.

Conversely, note first that every homogeneous simple has integer degree: an equivariant lift has integer degree and its ratio to the original degree is an integer, so the latter is a rational algebraic integer. Put $K = \{g : \mathcal{E}_g$ contains a simple of r-unit degree$\}$. Products, duals and conjugation make K a normal subgroup. For a pure charge π, its row at a simple of flux g is $d_W \operatorname{tr}_\pi(g)/\dim \pi$ up to inversion. Thus π is principal exactly when its character is congruent to $\dim \pi$ on K. Taking scalar products with nontrivial irreducibles of K shows their multiplicity in $\pi|_K$ would be divisible by r. By Clifford theory any occurring multiplicity divides $\dim \pi$, an r-unit. Hence the principal charges are exactly $\operatorname{Irr}(G/K)$.

Let F_0 forget equivariance and then project to the neutral grade; it is an integral linear map. Fix any neutral lift Z' above U. For the primitive character idempotents e_Z, orthogonality gives

$$\chi_U(F_0(e_Z)) = \frac{d_Z}{|G| d_{Z'}} \sum_{\pi \in \operatorname{Irr}(G)} \dim(\pi) N^Z_{\pi Z'}.$$

Indeed the left side is $f_Z^{-1} \sum_{W \text{ neutral}} \overline{\chi_Z(W)} \chi_{Z'}(W)$. The indicator of neutral flux at W is the regular charge character $|G|^{-1} \sum_\pi \dim(\pi) \chi_W(\pi)$. Use multiplicativity of χ_W and symmetry of S. The displayed sum is zero unless Z lies above the orbit of U. For $c = F_0(\sum_{Z \in P_r(\mathcal{C})} e_Z)$, evaluating at the unit gives $\chi_{\mathbf{1}}(c) = |G|^{-1} \sum_{\pi \in \operatorname{Irr}(G/K)} (\dim \pi)^2 = |G/K|/|G|$, an r-unit. If $U \in P_r(\mathcal{D})$ its value on the r-integral element c is congruent to that same unit, proving some lift principal. In the slightly degenerate case take the quotient by the transparent fermion in both rings. The fermion belongs to the neutral sector and its forgetful image is the transparent fermion of $\mathcal{D}$, so F_0 descends to the based quotients. Averaged induction commutes with this quotient. All sums and orthogonality identities above are now on fermion pairs. □

For passage from an integral modular category to the centralizer of a fermion, each positive principal pair has a full principal lift at r by Lemma 4.6.

Theorem 8.39 (Recognition on the original principal block)**.** *Let $\mathcal{C}$ be integral modular or slightly degenerate, of dimension $p^a q^b r^4$, with distinct p, q, r and r odd. If all degrees in $P_r(\mathcal{C})$ are prime to r, then $\mathcal{C}$ is weakly group-theoretical. The same conclusion holds with dimension dividing this number.*

Proof. Induct on dimension, using the known result at r-valuation at most three. A Tannakian $\operatorname{Rep}(G)$ with $r \mid |G|$ suffices by de-equivariantization, since its neutral dimension has lower r-valuation and weak group-theoreticity passes back. If $r \nmid |G|$, Proposition 8.38 applies directly. The smaller neutral component is integral, since a lift of each simple has dimension an integral multiple of its degree; its center is trivial or the image of the same transparent fermion. We may therefore assume that there is no nontrivial Tannakian subcategory.

A nontrivial proper nondegenerate subcategory splits off. The principal blocks of a tensor product are exactly the products of the principal blocks, so induction applies to both factors. If $\mathcal{C}$ is modular and has a fermion, pass by the preceding even-sector argument to $\langle f \rangle'$ and use the faithful two-grading. An ordinary category still remaining has no proper nontrivial subcategory: its center would be either trivial, Tannakian, or contain a fermion, by the classification of symmetric fusion categories. The pointed case is immediate; otherwise it is perfect and fusion-simple.

If $\mathcal{C}$ is slightly degenerate, any proper subcategory strictly between sVec and $\mathcal{C}$ is itself nondegenerate over sVec, again by the absence of Tannakian subcategories. It splits off with its centralizer in the tensor product relative to sVec: the usual centralizer dimension formula gives the factorization. The positive based ring is the product of the two positive based rings (the simples of the relative product are the simultaneous fermion orbits of pairs). Induction applies to these smaller factors; weak group-theoreticity passes through their product and condensation of the diagonal boson. With no such intermediate subcategory, the positive table is simple, and either pointed or perfect. In each remaining perfect fusion-simple table at the three dimension primes, Lemma 4.14 forces an r-bearing degree in P_r, contrary to hypothesis.

□

8.10 Equality in the multiplier criterion

Let $W(r, m+1)$ have the meaning of Lemma 4.22: a nonunit at a nonunit row of an odd ordinary minimal table cannot have a full positive r-primary word on at most $m+1$ positions, up to a common root of unity. Put $E = m(m+1)/2$, $H = (r-1)/2$ for an odd prime r, and assume $1 + 2E < 2r/3$. Failure of W requires an integer $h \le E$, $h > 0$, satisfying

$$h(h-1) \ge (H-1)k, \qquad k = \min\{j \ge 1 : j(j+1) \ge h\}.$$

Equality imposes two more restrictions.

Proposition 8.40. *At equality $h(h-1) = (H-1)k$, the following hold.*

1. *If $k \ge 2$, then $h < k(k+1)$.*
2. *Write $n = h - k$. For every $b > 1$ dividing H and every prime $\ell \nmid b$ for which -1 is a power of ℓ modulo b, $v_\ell(n)$ is even.*

Proof. Use the minimizing word and its h opposite difference pairs A from Lemma 4.22. The tame multiplier group modulo sign is cyclic of order H, acts freely on the pairs, and satisfies $|A \cap uA| \ge k$ for $u \ne 1$. Equality in the count forces every nonidentity overlap to equal k and all pairs to lie in one free orbit: pairs from different orbits would contribute to no overlap. Identify the orbit with the group G. Then A is a cyclic (H, h, k) difference set.

Suppose $h = k(k+1)$ and fix $u \ne 1$. Let $B = A \cap uA$. The common word on $\{0\} \cup \{\pm b : b \in B\}$ has at least h differences. The upper bound is exactly $k(k+1) = h$, with the possible differences given by $b, 2b$ and $b \pm b'$ for different members of B. Thus all these pairs are distinct and are realized; write A' for the resulting h-pair mask. It too is minimizing and lies in a single free orbit. Since it contains B, that orbit is the original one. Comparing the two words yields $c_g = |A' \cap gA| \ge k$ for every $g \in G$. The difference-set equations and double counting give

$$\sum_g (c_g - k) = h - k, \qquad \sum_g (c_g - k)^2 = (h-k)^2.$$

Indeed $\sum_g c_g = h^2$ and $\sum_g c_g(c_g - 1) = kh(h-1)$. All excesses are nonnegative, so one equals $h - k$. In particular $A' = gA$ for some g.

Both B and $2B$ lie in A'. Multiplication by two on the original free orbit is consequently one tame multiplier v: at one pair it agrees up to sign modulo its order with a tame unit, and every pair on this orbit has the same order. Here $v \ne 1$ since $r > 3$. Therefore $B \subseteq A' \cap v^{-1}A'$ and equality follows from the common cardinality k. Every intersection of two distinct translates of A is consequently a translate of $A \cap v^{-1}A$.

The H translates of A are the blocks of a symmetric design with H points and H blocks, and any two points lie in exactly k blocks. Let t be the stabilizer size of a block intersection and s the number of blocks containing that intersection. All intersections form one orbit of length H/t. Two blocks containing the intersection meet exactly in it. Counting ordered pairs of distinct blocks gives $s(s-1) = t(H-1)$. But two points of the intersection give $s \le k$, while $H - 1 = h(h-1)/k > k(k-1)$. This proves the first assertion.

For the second, choose a character χ of G of order b and put $\alpha = \sum_{a \in A} \chi(a)$. The difference-set equations give $\alpha\overline{\alpha} = n$. In $\mathbb{Q}(\zeta_b)$, ℓ is unramified and the decomposition group of each prime over ℓ contains the Frobenius ℓ. Thus it contains complex conjugation. At that prime the valuations of $\alpha, \overline{\alpha}$ agree and the valuation of n equals $v_\ell(n)$. □

9 Two squared primes and their local columns

Theorem 9.1 (Heights at the two higher primes)**.** *Let $p < q$ be the two primes with exponent caps A, B and let u, v be two other primes. Put $h = \lfloor A/2 \rfloor$, $k = \lfloor B/2 \rfloor$. If a divisor of $p^A q^B (uv)^2 d$*

supports a braided counterexample to WGT, where d is coprime and square-free, then

$$p < \min(u, v), \qquad h, k \geq 2, \qquad h + k \geq 6, \qquad h \geq 3 \text{ if } p \text{ is odd.}$$

In particular $p^7 q^5 (uv)^2 d$ is covered in either order of the two higher primes. When $p < q$ are odd, $p^5 q^B (uv)^2 d$ is covered for every B.

The numbers h, k bound the exponents of p, q in a simple degree. In an odd minimal root a twist involving both p and q forces its two heights to lie in the rectangle $\mathcal{H}(h, k) = \{(i, j) \in \mathbb{Z}^2 : 0 < i < h,\ 0 < j < k\}$. For $h, k \geq 2$ the inequality $h + k \geq 6$ says that this rectangle has at least three lattice points. For even roots, the local columns lead to the finite trace calculation in Theorem 9.16. Section 10 excludes an odd least prime of height two with any companion height. The only remaining root needed for the theorem is the odd root $p^3 q^2 uv$, treated in Section 12. Section 17 then obtains the four dimension cutoffs by comparing the remaining profiles.

Work in a minimal table as in Lemma 2.1. We determine which primes are isolated, then apply the simultaneous column identities of Proposition 6.4. We write P_t for the principal t-block and $[x]_t$ for the centered residue at an odd prime.

Lemma 9.2 (Three pair degrees)**.** *In an ordinary table let a, b, c be distinct primes of valuation two in D, and suppose the degrees ab, ac, bc all occur. On the union of the a-, b- and c-active labels the ℓ-primary twist is one for every other prime ℓ.*

Proof. A nonunit in P_ℓ must hit ab, ac, bc by principality, hence has at least two of their three primes by coprime vanishing. A b-active and a c-active column therefore share only the vacuum hit on P_ℓ, by (3.4). Their ℓ-primary phases agree by block orthogonality. All three sectors are nonempty by Cauchy, so cycling the pair gives a common phase. It is one: the complementary Weyl W' keeps v_0 in the vacuum U_ℓ-eigenspace. Its image has a nonzero b-active projection since W_ℓ preserves that sector and $W_\ell W' v_0 = s v_0$ has full support. □

Proposition 9.3 (Isolation at two squared primes)**.** *Let an ordinary minimal table have dimension $p^A q^B u^2 v^2$, $A, B \geq 2$, with four distinct primes. Both u, v are isolated and at least five, and the dimension is a square n^2. At $t \in \{u, v\}$ all active labels have degree n/t. Writing $w = uv/t$, both a degree $p^i w$ and a degree $q^j w$ occur, with $i, j > 0$. Moreover every p-free nonunit of P_q is divisible by uv, and such a label exists. Every q-active label has degree divisible by puv, and conversely every p-active label has degree divisible by quv.*

Proof. A t-active column hits only t-free degrees. Reducing its orthogonality to the vacuum modulo p or q, and excluding nonunit prime-power degrees, gives the two asserted degrees. A p-free nonunit of P_q must hit both $p^i u$ and $p^{i'} v$, hence is divisible by uv. Orthogonality on P_q of a q-active column modulo p supplies such a label. If the column were also t-active, the required hit would have t-unit codegree, a contradiction. This separates u, v from p, q. A column active at both u, v has all nonunit hits divisible by pq, contrary to vacuum orthogonality modulo p.

Complementary localization now gives codegree t^2 on the t-active labels; indeed its only prime is t and its positive valuation is $2 - 2v_t(d)$. This gives even A, B and degree n/t. Lemma 6.7 gives $t \geq 5$.

Compare a q-active column Z with a u-active column on P_q. Their q-phases differ. Any nonunit common hit is u-free by defect zero and hence divisible by p, by the preceding description of p-free nonunits in P_q. If $p \nmid d_Z$, lcm divisibility makes every nonunit contribution vanish modulo p, contradicting the vacuum term one. Complementary localization gives $uv \mid d_Z$. Interchange p, q for the other assertion. □

If three positions are squared, applying the proposition to each pair isolates all four primes and gives all three pair degrees.

Corollary 9.4 (Rank from two isolated primes)**.** *In an ordinary minimal table of dimension* $(p^h q^k uv)^2$ *with four distinct primes and isolated singletons* u, v*, the rank satisfies*

$$R \geq 7 + (u-1)j_u + (v-1)j_v.$$

If the dimension is odd, then $R \geq 13 + uj_u + vj_v$*. Here* $h, k \geq 2$ *and* j_u, j_v *are the two vacuum deficits.*

For a bound depending also on the heights put $x = p^h$*,* $y = q^k$*,* $n = xyuv$*,* $c = q\lceil 4/q \rceil$*. Then*

$$R \geq L + 2 - \max\{16/x^2 + 16/y^2,\ 16/(xy)^2 + 16/c^2\}, \tag{9.1}$$

where

$$\begin{aligned} L = 17 - 16/n^2 + (u-1)(1-16/u^2) + (v-1)(1-16/v^2) \\ + 4 - 16(x^{-2} + y^{-2})(u^{-2} + v^{-2}) + 1 - 16/(uv)^2. \end{aligned}$$

Proof. Put $n = p^h q^k uv$. By Proposition 6.4 the u- and v-active sets have sizes at least $(u-1)j_u, (v-1)j_v$, and their degrees are $n/u, n/v$. The mixed inner product (6.3) gives a nonunit whose degree omits both u and v. Proposition 9.3 gives four further degrees with prime supports $\{p,u\}, \{q,u\}, \{p,v\}, \{q,v\}$ and a p-free degree divisible by uv. These six distinct nonunit degrees and the unit lie outside the two active sets. In odd dimension, Theorem 10.3 supplies uj_u, vj_v labels of the two large degrees. Each of the six other degrees occurs on a distinct dual pair by Lemma 2.5.

For the height-dependent estimate let $\delta_Z = 1 - 16(d_Z/n)^2$. Each δ_Z is nonnegative by the small-codegree bound and Theorem 20.8, and $R = 16 + \sum_Z \delta_Z$. The vacuum, the two active sectors, the four degrees supported on one higher prime and one singleton, and the common nonunit omitting u, v contribute at least L after adding the initial 16. There are also a p-free and a q-free nonunit divisible by uv, by Proposition 9.3. If distinct, they contribute at least $2 - 16/x^2 - 16/y^2$. If equal, their degree is uv. A q-active label has degree divisible by puv, and its denominator is a multiple of q by Theorem 3.6. Its denominator is at least four, hence at least c. Together these two labels contribute at least $2 - 16/(xy)^2 - 16/c^2$. □

Lemma 9.5 (Two squared primes in the reduced table)**.** *A slightly degenerate minimal counterexample cannot have dimension* $2^a q^b u^2 v^2$*, with* q, u, v *distinct odd primes,* $a \geq 1$*,* $b \geq 2$*.*

Proof. In the reduced table choose a u-active column by Proposition 3.5. Its nonunit hits are u-free; orthogonality modulo two gives an odd one, of degree $q^i v$, $i > 0$. Likewise $q^j u$ occurs. Lift both to the integral minimal extension with fermion g. A noninvertible q-free label of its principal two-block hits both lifts, hence has $uv \mid d_X$ by coprime vanishing. Take a u-active even label Y, choosing between Y, gY to make its two-primary phase nontrivial. On the full principal two-block its vacuum orthogonality modulo q has no noninvertible contribution: a q-free one has u-unit codegree. The two invertibles contribute one each, contradicting $q \nmid 2$. □

For four surviving repeated primes, this excludes the reduced case with two specified squared source positions: a squared two would leave two-valuation one in the reduced dimension, and a square-free two leaves an odd reduced dimension (Lemma 2.5 and Proposition 4.8).

9.1 The two singleton columns

For the rest of this section, a *common* label is a nonunit whose degree omits both specified singletons. For four primes write $n = Cuv$, $C = p^h q^k$. We call the degrees $p^i w, q^j w$ of Proposition 9.3 the *axes at* w.

Lemma 9.6 (Inertia from principal residues)**.** *Let* $z = S_{XY}/\operatorname{lcm}(d_X, d_Y)$ *be a normalized entry on two nonunits,* $g = \gcd(d_X, d_Y)$*,* m *the least dimension prime, and* $b = m/(m-1)$*. Suppose* $z \equiv g$ *at every prime above* t*, as occurs for* $Y \in P_t$ *and* $t \nmid d_X$*. If* e *is the ramification index*

of $\mathbb{Q}(z)$ at t, then $t \le beg$. More generally $st \le beg$ when $z \equiv g$ also above a distinct prime s unramified in $\mathbb{Q}(z)$. The t-inertia orbit of each label has length at least

$$L(t,g) = \begin{cases} (t-1)/\max\{e : e \mid t-1,\ e \le g\}, & t > 2g, \\ 1, & t \le 2g. \end{cases}$$

Proof. Take $s = 1$ for the first assertion. The sum w of the inertia translates of z has $(w-eg)/(st)$ integral: $(z-g)/s$ is integral, and its inertia trace vanishes at every prime over t in the unramified fixed field. If $st > eg$, strict house $|z| < g$ at every embedding gives $|1+(w-eg)/(st)| < 1$ at every embedding, so $w = eg - st$. The balancing expression for z has positive rational weights summing to g. By Cauchy its roots have only dimension primes in their orders, so their Ramanujan means are at least $-1/(m-1)$. Thus $eg - st \ge -eg/(m-1)$. For $st \le eg$ the bound is immediate.

For $t > 2g$ Theorem 4.13 gives an unsigned t-word of length g. Both character fields contain z; apply Lemma 7.10. □

For a four-prime root with $h, k \ge 1$, write I for the nonunit labels of degrees prime to uv. Each such degree is $d = p^i q^j$, $i, j > 0$, and its field is unramified at one of u, v by Proposition 6.3. At that endpoint t, the opposite axes are t-principal, and their gcds with d are at most p^i, q^j. Lemma 9.6 yields the first bound below. The unramified estimate in Lemma 6.6 gives the second:

$$t \le \frac{m}{m-1} \min(p^i, q^j), \qquad (t-1) j_t [d]_t^2 < t^2. \tag{9.2}$$

We use a_t, j_t, B_t as in Proposition 6.4.

Proposition 9.7 (A congruence obstruction on the two repeated prime powers)**.** *Let $n = Crs$, $C = xy$, with $x = p^A, y = q^B$ positive powers of two distinct primes and r, s two further odd primes. Suppose r, s are isolated in an ordinary perfect fusion-simple modular table of dimension n^2. If*

$$C \equiv \pm 1 \pmod{rs}, \qquad H := (1-x^{-1})^2 + (1-y^{-1})^2 > 1 + C^{-1},$$

no such table exists. For a center and $C \equiv 1 \pmod{rs}$ the weaker hypothesis $H \ge 1$ suffices. In particular the obstruction applies whenever $A = B = 2$ and $C \equiv \pm 1 \pmod{rs}$.

Proof. Put $m = sj_r + rj_s$ and let ρ be the least positive residue of ϵC^{-1} modulo rs. By the two-column identities, $m \equiv -\rho \pmod{rs}$ and $1 < m \le rs + 1/C$, so $m = rs - \rho$. The complementary Weyl column $W_C v_0$ has vacuum entry $1/n + \epsilon m/(rs)$. Its coordinate on an axis $p^i s$ has modulus $r^{-1}|p^i/C - a_r| \ge (1 - 1/y)/r$, since a_r is a nonzero integer. Summing over the four distinct axes gives

$$H(r^2 + s^2) \le (\rho - \epsilon/C)(2rs - \rho + \epsilon/C).$$

If $C \equiv \pm 1$, then $m > 1$ forces $\rho = 1$ and the right side is strictly below $2rs(1 + C^{-1})$ (below $2rs$ for a center, where $\epsilon = 1$). Use $r^2 + s^2 \ge 2rs$. For $A = B = 2$, $H \ge (1 - 1/4)^2 + (1 - 1/9)^2 > 1 + 1/36 \ge 1 + 1/C$. □

Proposition 9.8 (Bounds for the other three primes)**.** *Under the hypotheses and notation of the two-column proposition, let $u < v$, $L = \min(p^h, q^k)$ and $b = m/(m-1)$. Then*

$$u \le bL, \qquad v \le bLB_u, \qquad \max(p,q) \le b \min(vB_u, uB_v), \qquad \min(\max(p,q), v) \le bu. \tag{9.3}$$

In particular the least prime is one of p, q. For the family with $h, k \le 2$, specifying any one of its four primes leaves only finitely many choices of the other three in an ordinary minimal table.

Proof. The mixed identity makes I nonempty; (9.2) gives $u \le bL$. The comparison of a common label at v with the axis of smaller prime-power part has gcd at most L and inertia length at most $|I| \le B_u$. Lemma 9.6 gives $v \le bLB_u$.

Compare a $p^i v$ axis with a p-free label in P_q; by Proposition 9.3 the latter has degree $q^a uv$, $a \geq 0$. The gcd is v, and every q-inertia translate of the axis spends a unit of B_u (its degree differs from n/u). This gives $q \leq bvB_u$ and, by both interchanges, the third bound. An axis $p^i u$ compares both with $q^j u \in P_v$ and with $q^a uv \in P_q$. Both gcds equal u, so if $q, v > bu$ its field would ramify at v and q, contrary to isolation. This gives the last bound by symmetry. If $u = m$, every other prime is at least $u + 2 > bu$, impossible.

A prescribed repeated prime with bounded exponent bounds L, then u, v and the other prime by (9.3). A prescribed v bounds u and both other primes. For fixed u, either $v \leq 2u$ or both repeated primes are at most $2u$; bounded exponents then bound L and v. □

Lemma 9.9 (Multiplicity at an isolated odd singleton)**.** *Suppose an ordinary minimal table has odd root n and an isolated prime $t \geq 5$ of root-valuation one. If H distinct proper nonunit t-free degrees occur, then*

$$H \min\{2j_t^2,\, 2^{v_2(t-1)}\} \;\leq\; B_t = j_t(t + 1 - j_t).$$

In particular, no isolated singleton of an odd minimal square root can be five.

Proof. An unramified label and its distinct dual cost at least $2j_t^2$ by Lemma 6.6. A ramified label has a pure t-power field by isolation and Proposition 6.4. Its field is nonreal since the dual is distinct. Its stabilizer in the cyclic cyclotomic Galois group omits the unique involution, so its orbit size is a multiple of $2^{v_2(t-1)}$. Every translate costs at least one in squared norm. Sum these bounds on the H proper degrees. At $t = 5$ Lemma 6.7 gives $H \geq 3$. Both even deficits $j_t = 2, 4$ would require twelve units, exceeding $B_t = 8$. □

Lemma 9.10 (A unique complementary degree)**.** *Let $u, v \geq 5$ be distinct isolated odd singletons in an ordinary minimal table of square root n. Suppose $D = n/(uv)$ is its only nonunit degree prime to uv, and the signed coordinates of these labels are centered residues at u, v. If $u < v$ then $D \equiv \pm 1 \pmod{v}$. If its multiplicity M exceeds one, then $M = j_u$, $v = uj_v \pm 1$, and all common labels are unramified at v. For odd n the centering and multiplicity conditions are automatic.*

Proof. For the moment name the endpoints t, w without ordering. The signed entries are the centered inverses $A = [w^{-1}]_t$, $B = [t^{-1}]_w$. The integer $wA + tB$ is one modulo tw and has absolute value at most $tw - (t + w)/2$, so $wA + tB = 1$. The mixed column identity gives $M|AB| = j_t j_w$. If $M = 1$ the unique label is Galois-fixed; an active row at each endpoint gives $1 + [D]_t[D]_w = 0$ by orthogonality and Lemma 6.6.

Assume $M > 1$. At most one endpoint ramifies at each label. At an unramified endpoint its deficit divides its signed coordinate; both $j_t \mid A$ and $j_w \mid B$ would force $M \leq 1$. Thus one endpoint, say w, is unramified at every common label. A t-active row Z and a w-active row now give $1 + [D]_w \sum_{d_X = D} \chi_Z(X) = 0$. Integrality gives $[D]_w = \pm 1$, hence $|B| = j_w$ and $j_t = M|A|$. Reducing at t gives $MD \equiv \pm 1 \pmod{t}$.

Since $M \mid j_t$ and $j_t \mid t - 1$ or $t + 1$ by Lemma 6.6, $M \mid t \pm 1$ and the centered residue of D has magnitude at least $(t - 1)/M$. If $|A| \geq 2$, the active rows at t charge at least

$$|A|(t - 1)(t - (t - 1)/M)/t^2 \;\geq\; 1 - 1/t^2$$

to the D-column norm. At w, $|[D]_w| = 1$ charges at least $j_w(w - 1)/w^2$. This is greater than $1/t^2$ since $tj_w = |1 - wA| \geq 2w - 1$. The two sets of active rows are disjoint.

Thus $|A| = 1$, $M = j_t$, and $w = tj_w \pm 1$. Parity gives $j_w \geq 2$ even and $w > t$. For odd n, a common label and its distinct dual share their signed coordinate by rationality; its square is at most $B_t/2$, which forces centering at each endpoint. □

Corollary 9.11 (Three omitted primes)**.** *Let $u < v < w$ be isolated odd singletons at least five. If for both $t = u, v$ the only common nonunit degree at t, w is $n/(tw)$, and both signed coordinates are centered, the table cannot exist.*

Proof. Put $C = n/(uvw)$. The preceding lemma gives $Cu, Cv \equiv \pm 1 \bmod w$, hence $u \equiv \pm v \bmod w$, impossible for distinct odd primes $u < v < w$. □

Proposition 9.12 (Full valuations under isolation)**.** *Let $n = p^h q^k uv$, $h, k \geq 1$, and suppose all four primes are isolated. Every occurring degree $p^i t$ or $q^j t$ at $t = u, v$ has $i = h$ or $j = k$, respectively. If uv occurs, then all common degrees are $C = p^h q^k$. If n is odd and $k = 2$, the degree uv occurs.*

Proof. By Proposition 6.3, full isolation puts each label in the principal block of every prime at which its degree is deficient.

An axis $p^i t$ with $i < h$ would have to hit the coprime opposite axis $q^j w$. A deficient common degree would likewise have to hit uv if it occurs.

For n odd and $k = 2$, suppose uv were absent. A p-free nonunit of P_q is divisible by uv by Proposition 9.3. Defect zero excludes q^2, leaving quv. Together with its distinct dual it gives, by Lemma 6.8, a q-active row vanishing on every q-bearing degree. Its vacuum orthogonality modulo p would require a nonunit hit of degree uv. □

Proposition 9.13 (One unrestricted prime and three squares)**.** *For any four distinct primes ℓ, a, b, c, every weakly integral braided fusion category whose square-full part divides $\ell^A (abc)^2$ is weakly group-theoretical.*

Proof. Descend over source divisors. Losing a repeated prime leaves Theorem 4.16 or Dong's theorem. With all four retained, the reduced case was excluded above. In the ordinary table Proposition 9.3 leaves $n = \ell^h abc$ with four isolated primes, a, b, c odd, and degrees ab, bc, ca present. For the pair b, c, the preceding proposition applies with higher powers ℓ^h, a: the degree bc occurs, so the only common degree is $n/(bc)$, and it occurs by (6.3). The same holds for the other pairs. The other proper coordinates force centering since $B_t \leq (t+1)^2/4$. Apply Corollary 9.11. □

9.2 Trace and norm tests for two higher powers

The two Weyl columns have prescribed norms and scalar product. Keeping the two axis sets separate gives congruences in addition to the norm bounds.

Lemma 9.14 (Trace and Gram constraints on degrees)**.** *For a root $n = p^h q^k uv$ as above, let m_d be the multiplicity of each degree d and $A_t(d) = [d/(n/t)]_t$ for $t = u, v$. Put $w = uv/t$. For $b = p, q$ set $D_t(b) = \{b^i w : 1 \leq i \leq v_b(n)\}$, and put $G = \{p^i q^l : 1 \leq i \leq h, 1 \leq l \leq k\}$. The two axis sets at t each contain an occurring degree. One has*

$$\sum_{d \in D_t(b)} m_d d A_t(d) \equiv \epsilon j_t \pmod{pq/b},$$

$$\sum_{d \in D_t(p) \cup D_t(q) \cup G} m_d A_t(d)^2 \leq B_t, \qquad \sum_{x \in G} m_x A_u(x) A_v(x) = -j_u j_v. \tag{9.4}$$

An occurring degree has $j_t k(t-k) < t^2$, $k = |[d]_t|$. An occurring $x = p^i q^l \in G$ has an unramified endpoint t satisfying (9.2) and $A_t(x) = -\epsilon j_t [x]_t$; if $m_x = 1$ both endpoints do.

For finer multiplicity bounds, write $\mu = 2$ for odd n, $\mu = 1$ otherwise. An occurring axis $d = b^i w$ has $m_d \geq \max(\mu, L(t, w), L(pq/b, w))$. An occurring common degree $x = p^i q^l$ has $m_x \geq \max(\mu, L(u, g), L(v, g))$ at $g = \min(p^i, q^l)$. Any degree occurring once has unramified field.

Proof. The two axes and a common degree all occur by Proposition 9.3 and the mixed identity (6.3). Each shares the norm $B_t \leq (t+1)^2/4$ with the other two, so its signed entry must be the centered residue $A_t(d)$. The pq/b-free nonunit degrees in the t-column are precisely the occurring members of $D_t(b)$. Reduce the dimension trace in (6.3) at pq/b; the other two assertions in

(9.4) are the single and mixed column norms. Lemma 6.6 gives the energy and unramified tests. Finally an axis $b^i w$ has gcd w with both the opposite t-free axis and a b-free label of $P_{pq/b}$ as supplied by Proposition 9.3. A common label compares at each endpoint with an axis of gcd at most g. Apply Lemma 9.6 and in odd dimension count distinct duals. □

The finite feasibility problem. For each prime and height window fix the vacuum data $(u, v, j_u, j_v, \epsilon, C \bmod uv)$ with $C = n/(uv)$ satisfying (6.3). Unknowns are nonnegative integral multiplicities of the two axis types at each endpoint and of common degrees. The complete test on these variables is (9.4) with the occurrence, multiplicity and ramification restrictions above. A degree appearing in both common lists has a single multiplicity. A calculation may relax constraints or project away variables: finding no solution to the relaxation excludes the original window. To merge candidate degrees, retain every statistic actually used (trace residues, squared costs at the relevant endpoints and the signed mixed product); each positive aggregate is at least the minimum of the positive multiplicity floors of its constituents. Identical statistics give the same contributions to every retained constraint. For a nonempty axis one may keep the smallest cost for each trace residue since only upper bounds on that cost remain. These operations never exclude a genuine table. In Section 13.1 the periodic residue-and-capped-power state additionally covers unbounded heights; a bounded exponent window alone gives no such conclusion.

Lemma 9.15 (Two trace divisors)**.** *For $h, k \geq 2$ in the preceding setting take $u < v$ and $U \geq 2$ such that $U \geq \lfloor (u+1)/j_u \rfloor - 1$. One can always take $U = u$, and for odd n one can take $U = (u-1)/2$ since j_u is even and $u \geq 5$. For a finite set of positive integers E prime to u, define the test $\mathcal{T}_{u,U}(P, a, E)$ by*

$$P \mid 1 + \sum_{d \in E} m_d \, d[d]_u, \qquad \sum_{d \in E} m_d [d]_u^2 \leq U, \qquad m_d \in \{0\} \cup \{L(P,a), L(P,a)+1, \ldots\}. \tag{9.5}$$

Here the m_d are nonnegative integers to be chosen. Then

$$\begin{aligned} q > 2v \quad &\Longrightarrow \quad \mathcal{T}_{u,U}(q, v, \{p^i v : 1 \leq i \leq h\}), \\ v > 2p^h \quad &\Longrightarrow \quad \mathcal{T}_{u,U}(v, p^h, \{p^i q^j : 1 \leq i \leq h,\ 1 \leq j \leq k,\ u \leq 2\min(p^i, q^j)\}). \end{aligned}$$

In particular $u \leq 2p^h$, $v \leq p^h U + 1$, $q \leq vU + 1$, and $q \leq 2u$ when $v > 2u$.

Proof. Both tests are the signed dimension trace at u, reduced at the omitted prime P. Only its P-free nonunit terms survive. For $P = q > 2v$ these are the axes $p^i v$: comparison with a q-principal p-free label of degree $q^j uv$ has gcd v. For $P = v > 2p^h$ these are the common degrees; comparison with a v-free p-axis has gcd at most p^h. In either case positive multiplicity forces P-ramification by Lemma 9.6, hence u-unramifiedness by isolation. Thus $a_u = -\epsilon j_u [d]_u$, and in the second case $u \leq 2\min(p^i, q^j)$ by (9.2). Cancel $j_u < u < P$ in the trace. The inertia-orbit bound and the unramified rank bound of Lemma 6.6 give (9.5). Some multiplicity is positive, whence $L(P,a) \leq U$ and $P \leq aU + 1$ (also if $P \leq 2a$). The other bounds follow from (9.2) and (9.3). □

Theorem 9.16 (Total height at two higher primes)**.** *In an ordinary minimal table of dimension $(2^h q^k uv)^2$, where q, u, v are distinct odd primes and $h, k \geq 2$, either $h + k > 6$ or $h = k = 3$.*

Proof. Order $u < v$, put $C = 2^h q^k$, $n = Cuv$, and use Lemma 9.14. At $t = u, v$ call d admissible when $j_t |[d]_t| (t - |[d]_t|) < t^2$. Write $F_t(d)$ if $A_t(d) = -\epsilon j_t [d]_t$ and $(t-1) j_t [d]_t^2 < t^2$. The primes and signs to compare come from Lemma 9.15:

$$5 \leq u \leq 2^{h+1}, \quad u < v \leq 2^h U + 1, \quad 3 \leq q \leq vU + 1, \qquad U = \max(2, \lfloor (u+1)/j_u \rfloor - 1),$$

with $q \leq 2u$ if $v > 2u$. For $t = u, v$ use $1 \leq j_t < t$, $j_t \mid t-1$ or $j_t \mid t+1$, $C(uv/t) j_t \equiv -\epsilon \pmod t$, and $0 < n - \epsilon - C(v j_u + u j_v) < n$.

For $a = 2, q$ and $w = uv/t$ choose an admissible axis $d = a^i w$. When present it costs at least $H_t(d)A_t(d)^2$, where

$$H_t(d) = \max\{L(t, w), L(2q/a, w), 1 + \mathbf{1}_{\neg F_t(d)}\}.$$

Write $c_t(a)$ for the smallest such cost (infinite for an empty axis). The negative mixed sum requires some common degree $x = 2^i q^l$ admissible at both ends with $A_u(x)A_v(x) < 0$. Put $g = \min(2^i, q^l)$ and count the endpoints with $t \le 2g$ and $F_t(x)$. This count f is positive. The multiplicity is at least $M = \max\{L(u, g), L(v, g), 3 - f\}$, hence at each endpoint

$$MA_t(x)^2 + c_t(2) + c_t(q) \le B_t.$$

The table gives all surviving choices. The vacuum counts are 12, 14, 92, 18 and 365 at heights $(2, 2)$, $(2, 3)$, $(3, 2)$, $(2, 4)$ and $(4, 2)$, respectively. To obtain the rows, run $1 \le i \le h$, $1 \le l \le k$ in the last inequality for each vacuum class. At the indicated t the 2-axis has budget $K = B_t - MA_t(x)^2 - c_t(q)$. The final column contains all its nonempty trace residues modulo q within K.

h	k	ϵq	u	v	j_u	j_v	(i, l)	t	K	residues
3	2	-13	7	11	2	3	$(3, 2)$	11	10	$9, 12$
4	2	3	5	11	1	2	$(4, 2)$	11	8	1
4	2	11	5	7	2	1	$(2, 1), (2, 2)$	5	2	$4, 10$
4	2	11	7	13	2	2	$(4, 2)$	13	12	$4, 5, 10$
4	2	13	11	17	1	3	$(4, 2)$	11	6	$2, 4, 5, 7, 10, 12$
4	2	-41	13	17	3	2	$(4, 2)$	13	16	15
4	2	23	13	17	3	2	$(4, 2)$	13	15	$11, 20, 21$

Indeed each positive multiplicity m_d costs $m_d A_t(d)^2$ and contributes $m_d d A_t(d)$ to the trace. Adjoining one admissible $d = 2^i w$ at a time therefore adds these pairs for $m_d = 0$ or $H_t(d) \le m_d \le K/A_t(d)^2$, discarding total costs above K. For instance, in the first row only two copies of 14 or one of 56 fit. In table order the required residues $\epsilon j_t \bmod q$ are $10, 2, 2, 2, 1, 38, 3$; each is absent. The enumeration produces these seven terminal rows. Their trace residues can also be checked independently. □

For other heights, the general trace and Gram calculation uses the same additions to impose all of (9.4). At each axis retain the cheapest nonempty choice in the required trace class and subtract its cost. For each common degree add $m_x(A_u(x)^2, A_v(x)^2, A_u(x)A_v(x))$ to the two costs and the mixed sum, retaining costs within both remaining budgets. The final mixed sum must equal $-j_u j_v$. At an odd least prime p use even multiplicities and $t \le (p/(p-1))g$ in the unramified test.

10 Mixed twists and intermediate heights at four primes

Let the ordinary minimal table have odd root $n = Cuv$, where $C = p^h q^k$, $h, k \ge 2$ and $u < v$ are isolated singletons. A common degree omits both u, v; an axis at $w = u, v$ has degree $p^i w$ or $q^l w$. We use the signed columns a_t, deficits j_t, principal blocks P_t, and centered residues $[x]_t$ of Section 9. Put $b = m/(m-1)$ for the least dimension prime m. In what follows the crucial distinction is whether a label's character field ramifies at a singleton.

10.1 The local level and its two moving orbits

Here the local results hold for *any* isolated singleton t in an odd ordinary minimal root, independently of the other exponents. By Lemmas 6.7 and 9.9 one has $t \ge 7$. Put $j = j_t$ and write W for its Weyl operator. On the nontrivial t-local space $s = S/n = \epsilon W$ by Proposition 6.3. Each active label has codegree t^2. The field of every active or t-ramified label is pure at t and nonreal,

and local $-I$ exchanges its label with its dual. Since the anomaly is two-primary, the t-part of the level is the largest order of a t-twist.

Lemma 10.1. *The local level is t. Every active row field is $\mathbb{Q}(\zeta_t)$ and there are exactly j free Galois orbits of active labels.*

Proof. For an active label of maximal twist order t^a, square Galois covariance puts $\mathbb{Q}(\zeta_{t^a})^+$ in its character field. This field is nonreal and contained in $\mathbb{Q}(\zeta_{t^2})$ by Corollary 2.8 and $f = t^2$; thus $a \le 2$. If $a = 2$, the field is the full $\mathbb{Q}(\zeta_{t^2})$ of degree $h = t(t-1)$. The vacuum leaves just $t^2 - h = t$ in the trace budget of Lemma 2.4. A wild value orbit has trace divisible by t by total ramification, so a nonzero trace costs at least t^2. A tame value orbit has degree dividing $t-1$, so a nonzero trace costs at least $h/(t-1) = t$. Consequently at most one nonunit orbit can have nonzero trace. The traced dimension equation would give 1 as a multiple of its nonunit degree, a contradiction.

At level t purity puts every active field in $\mathbb{Q}(\zeta_t)$. It contains the real subfield and is nonreal, hence equals $\mathbb{Q}(\zeta_t)$. The rational twist sum forces equal multiplicity on the nontrivial phases; (6.2) gives j labels per phase and thus j full orbits. □

Lemma 10.2 (The trace interval for a singleton column)**.** *In a centered singleton column, in either parity, put $m = n/t$, $j = j_t$. The multiplicity F of the full degree m satisfies*

$$tj - j(1 - j/t) + \frac{(t+1)\epsilon j}{mt} \le F \le tj + j(1 - j/t) + \frac{(t-1)\epsilon j}{mt}.$$

Thus $F > (t-1)j$ if $jm > t+1$, and $F = t$ if $j = 1$ and $m > t+1$.

Proof. For a proper t-free nonunit degree d put $I = m/d$ and $A = [I^{-1}]_t$. Since $I > 1$ and $IA \equiv 1 \pmod t$, positive IA is at least $t+1$, and negative IA is at most $1-t$. Hence

$$-A^2/(t-1) \le A/I \le A^2/(t+1).$$

Removing the F full labels leaves trace $(t + \epsilon/m)j - F$ and norm $2tj - j^2 - F$ by Proposition 6.4. Summing the two bounds over the remaining labels gives the interval. □

Theorem 10.3 (The two ramified orbits at an isolated singleton)**.** *Let t be an isolated singleton in an odd ordinary minimal root n, and write $m = n/t$, $j = j_t$. Exactly tj labels have degree m. The inactive t-ramified labels form two orbits of size j, one of degree m and one of a proper degree $d_* < m$. Their character field is the degree-j subfield of $\mathbb{Q}(\zeta_t)$, with j even and $(t-1)/j$ odd. The evaluations of a fixed active character on either orbit are the conjugates of an integer plus a signed Gaussian period, of trace $-\epsilon a_t(X)$ with $a_t(X) \equiv \pm 1 \pmod j$.*

Put $A_* = a_t(d_*)$ and sum over the unramified proper t-free nonunit labels in dual pairs. Then

$$t - j = A_*^2 + 2j\sum [d]_t^2, \qquad A_* d_* - 2\epsilon \sum d[d]_t = \epsilon, \qquad A_*^2 \equiv 1 \pmod{2j}.$$

If k_t distinct proper t-free nonunit degrees occur, then $(t-1)/j \ge 2k_t - 1 \ge 5$. In particular the odd part of $t-1$ is at least five. Any two distinct such degrees x, y satisfy $t \le b\gcd(x,y)$, where $b = \ell/(\ell-1)$ for the least dimension prime ℓ. For a label Y of degree d_* one has $S_{XY} \in \mathbb{Q}$ whenever $d_X \notin \{d_*, m\}$.

Proof. The local representation. Let R count the ramified inactive labels, P the average over the upper unipotents, and H the diagonal torus of $\mathrm{SL}_2(\mathbb{F}_t)$. If $Px = x$ and the H-average of x vanishes, then $PW^{-1}PWPx = x/t$. Indeed the zero parameter gives x/t and Bruhat decomposition expresses the sum over the nonzero parameters as a full torus sum between P and W, which commute with and normalize H, respectively. On the negative eigenspace of $-I$ the dimension is $D = ((t-1)j + R)/2$ and the upper-fixed rank is $b_0 = R/2$ (conjugate pairs contribute one each). For the unipotent averages P_L attached to the $t+1$ lines, the identity and

transitivity on ordered distinct lines give $\mathrm{Tr}(P_L P_{L'}) = b_0/t$ for $L \neq L'$. Thus the sum of these projections has trace $(t+1)b_0$ and squared trace norm $2(t+1)b_0$. Cauchy–Schwarz yields $R \leq 2j$.

The period lattice. For a ramified inactive orbit O let $e = |O|$, $K = K_X$ for $X \in O$ and $x_0 = e^{-1}\sum_{Y\in O} v_Y$. The nonreal field K has odd index in $\mathbb{Q}(\zeta_t)$ and e is even. For the full tame degree $e = t-1$ we use only $e \geq j$ (the deficit is at most $t-1$); the exclusion of that case follows below from the representation and rank bounds. For the period estimate assume the index exceeds one, so $e \leq (t-1)/3$. Choose representatives Z_i of the j active orbits and form $\alpha_i = \chi_{Z_i}(X) \in \mathcal{O}_K$ and $L_i = \mathrm{Tr}_{K/\mathbb{Q}}\,\alpha_i$. The projection identity applied to $x = v_X - x_0$ gives $\|(I-P)sx\|^2 = (1-1/t)(1-1/e)$. Its coordinates on active orbit i are the conjugates of $(\alpha_i - L_i/e)/t$, equally often. Thus, writing Var for the Galois-mean squared distance from the mean,

$$\sum_i \mathrm{Var}(\alpha_i) = t(1-1/e), \qquad \sum_i L_i^2 < et^2/(t-1).$$

For the strict bound apply Lemma 2.4 to the X-row. On the $t-1$ labels in each active orbit, $\chi_X(Z_i) = (\sqrt{f_X}/t)\alpha_i$. Their traced cost is $(t-1)f_X L_i^2/(et^2)$, besides the vacuum. Since $e \leq (t-1)/3$, the sum of the L_i^2 is $< t^2/2$, while the L_i are all ed_X modulo t by principality. Hence they have the same value L.

Let B fix K in $\mathbb{F}_t^\times$. The periods $\eta_a = \sum_{b\in aB}\zeta_t^b$ form an integral basis of K of trace -1, by grouping the permuted basis $(\zeta_t^b)_{b\neq 0}$. Write $\alpha_i = \sum_a c_{ia}\eta_a$. Each row sums to $-L$ and Fourier orthogonality gives

$$\sum_{i,a}(c_{ia} + L/e)^2 = e-1, \qquad \text{since } \mathrm{Var}(\alpha_i) = (t/e)\sum_a (c_{ia}+L/e)^2.$$

If $e \mid L$, each centered row is integral of sum zero and even squared norm, contradicting the odd right side. Otherwise, for $r \equiv -L \bmod e$, $0 < r < e$, each row has squared norm at least $r(e-r)/e \geq (e-1)/e$, with the minimum attained when its coefficients differ by at most one. Thus $j \leq e$ in this case as well. There are at most two ramified inactive orbits. At least one of the three distinct proper degrees of Lemma 6.7 therefore supplies an unramified dual pair; the rank budget gives $3j \leq t+1$. This excludes full degree $e = t-1$ when $t > 5$ by $R \leq 2j$, and $t = 5$ is already excluded by Lemma 9.9.

Parity in the vacuum column. The vector $x_0 - (L/e)v_0$ and its inverse Weyl image vanish on active coordinates, so it is locally invariant. Its vacuum coordinate gives $ea_t(X) = -\epsilon jL$. Also $\sum c_{ia}^2 = e - 1 + jL^2/e \equiv -jL \equiv 0 \pmod 2$. Hence jL^2/e is an odd integer. Since $|jL^2/e| = |La_t(X)|$, both factors are odd and $\nu_2(j) = \nu_2(e) = \nu_2(t-1)$.

Each active signed coordinate is one by Proposition 6.4; the $(t-1)j$ active labels thus leave exactly $j(t+1-j)$ in the inactive squared norm.

Dividing by j, unramified terms $j[d_X]_t^2$ are even and each ramified orbit contributes the odd integer $ea_t(X)^2/j = jL^2/e$. Since $t+1-j$ is even, the number of ramified orbits is zero or two. In the latter case $R \leq 2j$ and $e \geq j$ force $e = j$ on both. Equality in the row minima forces $r = 1$ or $e-1$ and one coefficient in each row to differ from the others by ± 1. The periods sum to -1, so the evaluations are integers plus signed periods, with trace $L = -\epsilon a_t(X)$ congruent to $\pm 1 \bmod j$.

The full and proper degrees. Lemma 6.7 supplies at least three distinct proper t-free nonunit degrees. None is active and at most two can ramify, by the orbit count above. Choose a label X of an unramified one. Its normalized diagonal entry is unramified at t and has positive residue d_X there because $X \in P_t$. Lemma 9.6 gives $t \leq bd_X \leq bm/3 \leq m/2$, using the odd index $m/d_X > 1$. Thus $jm > t+1$. Lemma 10.2 supplies a full label beyond the $(t-1)j$ active ones. An inactive full label has codegree t^2 and nonreal character field unramified away from t, so it ramifies at t. The orbit count therefore gives two inactive ramified orbits of size j. If both were full, the lower trace bound would give $j(1-\epsilon/m) \leq j(t-1-j)/(t-1)$, again impossible. Thus exactly one is full. Subtracting the tj full labels from the two identities and using $a_t(d) = -\epsilon j[d]_t$ on

unramified dual pairs gives the displayed residual identities. Their norm and the oddness of $(t-1)/j$ give the congruence and the bound on k_t. Since $k_t \geq 3$, the odd index $(t-1)/j$ is at least five. For distinct proper degrees x, y, at least one of the labels is unramified at t. Their normalized entry has the positive principal residue $\gcd(x, y)$ at t; Lemma 9.6 gives the bound. All labels of degree d_* belong to the ramified orbit: their common coordinate is odd while an unramified coordinate is divisible by the even j. Finally K_Y is pure at t; if $d_X \notin \{d_*, m\}$ its character field is unramified at t (by defect zero when $t \mid d_X$). The two fields intersect in $\mathbb{Q}$, giving the rational entry. □

Proposition 10.4 (Two residues at an isolated prime)**.** *In an odd ordinary minimal table let s be an isolated singleton. Put $b = m/(m-1)$ for the least dimension prime m. Suppose X, Y are s-free, s-inactive nonunits whose normalized entry z has positive residue $g = \gcd(d_X, d_Y)$ at every prime over a second prime $a \neq s$. Then*

$$\min(a, s) < g, \qquad a \leq bg\frac{s-1}{2s}.$$

In particular, for $n = p^h q^k uv$ with u, v isolated singletons, if a common degree D belongs to P_q, every axis $p^i w$ $(w = u, v)$ satisfies $q < \frac{b}{2} p^{\min(i, v_p(D))}$, independently of the other higher exponent.

Proof. Principality gives the second positive residue, at s. Purity implies that z is unramified at a or s, so Proposition 4.12 gives $\min(a, s) < g$. By Lemma 9.6, $as \leq beg$ whenever one prime is unramified and e is the inertia degree at the other. If z ramifies at s, its field is pure there and $e \leq j_s \leq (s+1)/3 \leq (s-1)/2$ by Theorem 10.3. If it ramifies at a, at least e translates of X are s-unramified and cost at least e in the rank budget, so $e \leq (s+1)/j_s - 1 \leq (s-1)/2$. If neither prime ramifies take $e = 1$. The last assertion uses $s = uv/w$ and the q-principality of D.

□

Corollary 10.5 (Parity, ramified degrees and singleton orders)**.** *At an isolated singleton t in an odd ordinary minimal root, a proper t-free degree has odd signed coordinate exactly when its labels ramify at t. One has $v_2(j_t) = v_2(t-1)$ and $(t-1)/j_t \geq 2z+1$ if z different proper degrees are unramified there. In particular the odd part of $t-1$ is at least five. At two isolated singletons some common degree consists of one ramified orbit at one endpoint and has odd centered residue at the other. In the four-odd-prime family $p^A q^B (uv)^2 d$ all divisors are covered if u or v has the form $2^a + 1$ or $3 \cdot 2^a + 1$, with d coprime and square-free.*

Proof. By Theorem 10.3, the unique proper ramified orbit has size j_t and odd signed coordinate; the unramified coordinate $-\epsilon j_t [d]_t$ is even. Its degree therefore differs from all z unramified degrees. The index, parity and multiplicity bounds follow from that theorem. In the mixed Gram sum divided by $j_u j_v$, dual pairs unramified at both ends contribute even integers, whereas a common degree ramified at t contributes $\pm a_t(d)[d]_{uv/t}$. The sum is -1, giving the claimed orbit. For the family assertion a lost repeated prime or a third singleton is already covered, and an odd reduced minimal table is excluded. Proposition 9.3 would therefore isolate u, v, contradicting the odd-index bound. □

Proposition 10.6 (Degrees supporting mixed twists)**.** *In an odd ordinary minimal table of root $n = p^h q^k uv$ with $h, k \geq 2$, every p, q-mixed label Z has*

$$d_Z = p^i q^l uv, \qquad 1 \leq i < h, \quad 1 \leq l < k.$$

For any such mixed column, at each $t = u, v$ and $s = p, q$ there is a nonunit t-free hit in P_s. Both higher valuations of the hit are deficient, and the higher prime other than s divides its degree.

Proof. The lower bounds follow from Proposition 9.3; any active prime divides the formal codegree, giving the strict upper height bounds. For the final assertion reduce the orthogonality

of the mixed column with the vacuum on P_s modulo t. The hit omits t, cannot have full height at either active prime by defect zero, and cannot omit the other higher prime by the same two-axis argument. □

Lemma 10.7 (A single common degree)**.** *Let D be the only common nonunit degree at the two isolated singletons of an odd four-prime root. Its labels form one ramified orbit at one endpoint t, and are unramified at $w = uv/t$. Moreover $a_t(D) = \pm 1$, $[D]_w = \pm 1$, $j_t D \equiv \pm 1 \pmod t$, and $j_w(1 + j_t) \le w + 1$.*

Proof. Write M for the multiplicity. Since $M|a_u a_v| = j_u j_v$ and $M \ge 2$, at some endpoint t we have $j_t \nmid a_t$. Every D-label there ramifies, so M is a multiple of j_t by Theorem 10.3. At the other endpoint $|a_w| = j_w|[D]_w|$. Cancelling in the mixed identity gives $(M/j_t)|a_t[D]_w| = 1$, proving $M = j_t$ and both unit assertions. The vacuum congruence gives $j_t D \equiv \pm 1 \bmod t$, and the unramified rank budget gives the final inequality.

□

10.2 Fusion transport and bounds at arbitrary heights

Return to $m = \min(p, q, u, v)$, $b = m/(m-1)$. For distinct unramified, inactive t-free nonunit degrees x, y each has a distinct dual pair; hence Lemma 6.6 gives

$$|j_t[x]_t[y]_t| < t/4, \qquad -\epsilon j_t[x]_t[y]_t \equiv xy/(Cw) \pmod t, \quad w = uv/t. \tag{10.1}$$

The integer on the left of the congruence is even. In particular $xy/(Cw)$ cannot be an odd integer k in $(0, 3t/4]$.

Lemma 10.8 (A forbidden square residue)**.** *At a singleton $t \ge 7$ with nonzero inactive ramification, no t-free inactive nonunit Z satisfies $j_t^2 d_Z \equiv \pm 1 \pmod t$.*

Proof. Let $j = j_t$ and $H = (t-1)/j$. The two-orbit theorem and $3j \le t + 1$ give H odd, $H \ge 3$, j even. First suppose Z is unramified at t. Choose the sign of $c = \pm[d_Z]_t$ giving $c \equiv H^2 \pmod t$. The dual-pair rank budget gives $2c^2 \le (t+1)/j - 1 \le H$, hence $|c| \le (H-1)/2$. Write $H^2 - c = kt$. Since $t \ge 2H + 1$ and $|c| \le (H-1)/2$, we have $0 \le k \le (H-1)/2$ (the quotient is strictly below $(H+1)/2$). Reduction modulo H gives $k + c \equiv 0$; the bounds force $k = -c$. Thus $H = kj$, contrary to parity. If Z ramifies, its signed vacuum coefficient must instead have absolute value H: it is centered and satisfies $a_t(Z) \equiv -\epsilon j_t d_Z \equiv \pm j_t^{-1} \pmod t$. The other ramified orbit and an unramified dual pair cost at least $j + 2j^2$ in the same budget. Such a pair exists among the three compulsory distinct proper degrees, of which at most two have ramified orbits. Thus $H^2 \le jH + 1 - 3j$, forcing $H < j - 1$, whereas the signed period trace gives $H \equiv \pm 1 \pmod j$ and $H \ge 3$. □

Lemma 10.9 (Transport from an inactive singleton orbit)**.** *Let $t \ge 5$ be one of the two isolated singletons of an odd ordinary minimal table with root $p^h q^k uv$, $h, k \ge 2$. Suppose X, Y are t-inactive, K_X is pure at t, and S_{XY} is moved by t-inertia. For any t-unramified label Z, put $c = [d_Z]_t$, allowing $c = 0$. Then*

$$\chi_Y(Z) \ne c \quad \Longrightarrow \quad d_X d_Z \ge (t-1)n/t.$$

Proof. On every active row T one has $\chi_T(Z) = c$ by the unramified-column formula, including $c = 0$ by defect zero. If XZ has no active constituent, then $F = XZ - cX$ and $s^{-1}F$ vanish on active coordinates, the latter since $\chi_T(F) = 0$. Thus both elementary unipotents fix F, hence so does the local torus. Its unsigned permutations act on the ring since $\chi_T(\sigma X) = \sigma\chi_T(X)$ and characters separate elements. As $\sigma Z = Z$, evaluating $\sigma F = F$ at Y gives $(\chi_Y(\sigma X) - \chi_Y(X))(\chi_Y(Z) - c) = 0$. The first factor is nonzero for some σ.

If $\chi_Y(Z) \neq c$, there is therefore an active constituent. Since $S_{XZ} \in K_X \cap K_Z = \mathbb{Q}$, balancing XZ puts the sum of its active phases, counted with multiplicity, in the complementary cyclotomic field and thus in $\mathbb{Q}$. The nontrivial tth roots form a permuted basis, so all $t-1$ occur. Each active constituent has degree n/t. □

Proposition 10.10 (The two squared primes at arbitrary heights)**.** *Let $n = Cuv$, $C = p^h q^k$, be an odd ordinary minimal root with $h, k \geq 2$, $u < v$, $b = m/(m-1)$ for the least prime m, and $J = \max(p^{h-1}, q^{k-1})$. Some common degree differs from C, and*

$$v \leq b \min(u, J).$$

In fact $v \leq b \min(p^i, q^l)$ for each occurring common degree $p^i q^l$. A strict reverse of the displayed inequality excludes counterexamples at divisors of $p^A q^B (uv)^2 d$ for four odd primes, $h = \lfloor A/2 \rfloor, k = \lfloor B/2 \rfloor \geq 2$, and coprime square-free d.

Proof. The compulsory axes $p^i u, q^l u$ at v have gcd u. Their degrees are distinct, proper and v-free, so Theorem 10.3 gives $v \leq bu$. Comparing a common degree with each of these axes gives $v \leq b \min(p^i, q^l)$ for its own exponents i, l. If C alone occurred, Lemma 9.10 would give $v = uj_v \pm 1 > bu$, since $j_v \geq 2$, $u \geq 5$ and $b \leq 3/2$. Some common degree therefore has a prime-power factor at most J.

Under divisor descent, an odd reduced table is excluded by Proposition 4.8. Loss of a repeated prime leaves the three-prime small-exponent case, and a higher root height below two is covered by Proposition 9.13. Otherwise the singletons persist and J only decreases. □

10.3 Isolation of a quartic prime

Theorem 10.11 (Full isolation at four odd primes)**.** *Let $n = p^h q^k uv$ be an odd ordinary minimal root, $h, k \geq 1$. If all four primes are isolated then uv is absent and neither h nor k equals two.*

Proof. Proposition 9.13 excludes height one. With $h, k \geq 2$, full isolation and the occurrence of uv would make $p^h q^k$ the sole common degree by Proposition 9.12, contrary to Proposition 10.10. Either height equal to two forces uv by the same isolation proposition. □

10.4 The two higher prime powers in increasing order

Close singletons rule out the full common degree, independently of the two higher exponents. This elementary estimate has two applications: it treats a quartic valuation at the least prime with arbitrary companion height, and later it applies to the odd sextic root.

Theorem 10.12 (The full common degree is absent)**.** *Let $n = Cuv$ be an odd ordinary minimal root with C the product of two prime powers of exponents at least two. Let p be the least of the four primes and $b = p/(p-1)$. If $u \leq bp^2$ and $u < v \leq bu$, the full degree C is absent.*

Proof. For $p = 3$, $u \leq 13.5$ leaves only $u = 11$ by Corollary 10.5 and Lemma 9.9, and no admissible $v \leq 3u/2$ remains. Let $p \geq 5$ and $\delta = v - u$. Since $\delta \leq u/(p-1) \leq p^3/(p-1)^2 < p+3$, the even gap satisfies $\delta \leq p+1$.

If C occurs, its signed coordinates are the centered inverses $a_u = [v^{-1}]_u$, $a_v = [u^{-1}]_v$; $va_u + ua_v = 1$. At the endpoint t with odd coordinate the C-labels ramify. One orbit costs $j_t a_t^2$ in B_t, so $|a_t| < \sqrt{t} \leq bp < p+2$. Also $\delta a_t \equiv \pm 1 \bmod t$. Since δa_t is even and $t \geq (p-1)\delta$, $|a_t| \geq (t-1)/\delta > p-2$. Oddness gives $|a_t| = p$. Hence $t > p^2$, $\delta = p+1$, and $t = p(p+1) \pm 1$: here $t - 1 \leq p\delta < 2t - 1$. If $t = v$, then $u = p^2$ or $p^2 - 2$, contrary to primality and $u \geq p^2 - 1$. The choice $t = u = p^2 + p - 1$ gives composite $v = p(p+2)$. The remaining inverse pair is

$$(u, v) = (p^2 + p + 1, (p+1)^2 + 1), \qquad (a_u, a_v) = (-p, p+1).$$

The C-orbit is v-unramified, so $j_v \mid p+1$. Since $v - 1 = (p+1)^2$, the two-orbit theorem excludes all inactive v-ramification, contrary to the compulsory inactive full orbit at v in Theorem 10.3. □

Theorem 10.13 (A quartic exponent at the smaller higher prime)**.** *Let $q < p$ be odd primes, and let u, v be two further distinct primes. For every $A \geq 0$ and every square-free d coprime to $pquv$, all braided fusion categories of dimension dividing $q^5 p^A (uv)^2 d$ are WGT.*

Proof. After divisor descent it suffices to exclude an ordinary odd root $n = Cuv$, $C = q^2 p^h$, $h \geq 2$, in a minimal table. Indeed, losing a repeated prime leaves at most two or three with an exponent at most two. In a reduced table the dimension either is odd or has two-valuation one at a dyadic singleton, contrary to Proposition 4.8 and Lemma 2.5. Proposition 9.3 makes the ordinary higher valuations even and the singletons odd; Proposition 9.13 excludes a root height below two. Order $u < v$. By (9.3) the least prime is q, and $b = q/(q-1)$.

At an unramified endpoint t, a common degree x satisfies $t \leq bq^{v_q(x)}$ by (9.2). Since $t > bq$, write $x = q^2 p^i$. For $i < h$ the full q-height and deficient p-height put x in P_p by Proposition 6.3. Then degree uv is absent by coprime vanishing. A p-free nonunit of P_q contains uv by Proposition 9.3; its q-height is below two by defect zero, leaving degree quv. The comparison of x with quv has gcd q and positive residue at $p > bq$, hence ramifies there. Every proper common label is therefore unramified at both singletons. The mixed Gram identity has a ramified common label by Corollary 10.5, so C occurs.

Proposition 10.10 gives $u \leq v \leq bq^2$ and $v \leq bu$, so Theorem 10.12 excludes C. □

Proposition 10.14 (An unrestricted power at the least odd prime)**.** *Let $n = p^h q^2 uv$ be an odd ordinary minimal root, for distinct primes with p least, $u < v$, $h \geq 3$. Set $P = p^{h-1}$ and $b = p/(p-1)$. Then*

$$u \leq \begin{cases} (1 + b^2/2)P, & p \geq 5, \\ (9P + 2)/4, & p = 3. \end{cases}$$

Consequently the strict reverse bound suffices for WGT for the braided source family $p^A q^B (uv)^2 d$, $h = \lfloor A/2 \rfloor$, $4 \leq B \leq 5$, with any coprime square-free decoration d.

Proof. Suppose the asserted bound fails, so $u > b^2 P$. By (9.2) every common degree is $L = p^h q$ or $C = p^h q^2$. Proposition 10.10 forces L to occur. It is q-principal since the p-valuation is full and the vacuum denominator has a q-part. Hence uv is absent. By (9.2), $q \geq u/b > bP$. A q-free nonunit in P_p has degree $p^i uv$, $1 \leq i < h$ (compare both q-axes); such a label exists by reducing principal-block orthogonality at q. Pairing with L forces q-inertia of size at least $q/(bP) > 1$ at every L-label. They are therefore unramified at both singletons. At each t their comparison with both opposite axes gives $t \leq bq$ and full p-axis height since $t > bP$. The mixed column from Theorem 10.11 has, by Proposition 10.6, a t-free hit in P_q with deficient heights. Both common degrees and all p-axes have full p-height, so the hit has degree $Y = p^l qw$, $1 \leq l < h$, $w = uv/t$. Comparing it with $p^h t$ at q gives q-inertia of size at least $q/(bP)$ here too. The disjoint unramified orbits of L, Y then give $j_t |[L]_t [Y]_t| < bPt/(2q) \leq b^2 P/2$. But $LY/(Cw) = p^l$ is odd, so the even representative in (10.1) has modulus at least $t - p^l \geq u - P > b^2 P/2$ (also for $p = 3$). The bounds increase with h; smaller roots with $h' \leq 2$ or $k' \leq 1$ descend by Theorem 10.13 or Proposition 9.13. □

Corollary 10.15 (Reduction to an odd sextic root)**.** *Let $p < q$. A divisor-minimal counterexample to WGT among the divisors of $p^A q^B (uv)^2 d$ would have an ordinary table of square root $n = p^i q^j uv$, with $p < \min(u, v)$ and $i, j \geq 2$. If p is odd, then $i \geq 3$. If $i + j < 6$, the only possibility is an odd root $p^3 q^2 uv$ with p least.*

Proof. All four repeated primes must survive: loss of one leaves at most two, or three with an exponent at most two, covered by Theorem 4.16 and Dong's theorem. In a reduced table a dyadic singleton would leave valuation one, contrary to Lemma 2.5; an odd reduced dimension is also impossible by Proposition 4.8. These observations exclude two outside the higher positions, and Lemma 9.5 excludes two in a higher position. In the ordinary table Proposition 9.3 makes the higher valuations even. A root height below two is covered by Proposition 9.13. Lemma 6.7

excludes a dyadic singleton, so an even root has $p = 2$ and $i + j \geq 6$ by the dyadic enumeration in Theorem 9.16. In an odd root the least prime cannot be a singleton by (9.3); Theorem 10.13 excludes height two there, at every companion height. Now $i \geq 3$ and $j \geq 2$ leave only $(3, 2)$ below total height six. □

11 Lifting the smaller prime exponents to cubes

11.1 Odd cubic primes and complementary supports

Lemma 11.1 (A degree at an odd cubic prime)**.** *In an ordinary or positive reduced minimal table suppose that an odd prime p has $v_p(\Delta) = 3$. If a nonunit degree has prime support $\{p, q\}$, then $q \mid p - 1$. The same conclusion holds throughout an integral minimal modular extension of a slightly degenerate counterexample, including its odd sector.*

Proof. Write X for the indicated simple. Every label $Z \in P_p$ has p-free degree by Corollary 4.1. For a noninvertible Z either $q \mid d_Z$ or $S_{XZ} = 0$ by coprime vanishing (Lemmas 2.5 and 4.4). Thus $d_Z \chi_X(Z)$ vanishes above q in either case. Apply Proposition 4.2; in the extension intersect P_p with the trivial prime-to-p twist fiber to remove the extra invertible fermion. The restriction preserves the vacuum and the p-local action, and $v_p(4\Delta) = v_p(\Delta)$. □

Proposition 11.2 (Small exponents at four repeated primes)**.** *At four repeated primes a divisor-minimal counterexample cannot have both a square and a cube among its prime powers. If u is an odd cubed prime and another prime v is cubed, the product of the two remaining primes divides $u - 1$.*

Proof. Write p, q for the cube and square. We seek a column J active at both primes. In the ordinary table q-activity exists by categorical Cauchy. Separation would give a q-active label with p-unit codegree (Lemma 3.4), impossible at odd p-valuation of the dimension. In the reduced table with p, q odd, a q-active fiber contains a p-active label by Lemma 7.8. A squared two would leave valuation one after reduction. For a cubed two pass to the integral minimal extension: choose an even q-active label, then choose from its fermion pair a dyadically active J.

In every case each noninvertible degree on P_p omits p, by Corollary 4.1, or by Proposition 4.8 for the extension at two. A hit on J also omits q by defect-zero vanishing. Consequently the degrees of the proper hits are divisible by both remaining primes. Their contributions to $\sum_{X \in P_p} d_X \chi_J(X) = 0$ cannot cancel the unit modulo an odd one of those primes. In the extension the two invertibles contribute two since J is even, giving the same contradiction. This proof also applies to a square and cube in ordinary perfect fusion-simple integral modular data and, at odd p, q, in positive reduced data.

For two cubes u, v let p, q denote the other primes. If v is odd or the table is ordinary, prime avoidance for a v-active column on P_v, first at q and then at p, gives two v-free degrees with supports $\{u, p\}$ and $\{u, q\}$. If $v = 2$ in a reduced counterexample, the same two supports occur in the full extension by Proposition 4.8. Lemma 11.1 now gives $pq \mid u - 1$ in both cases. The prime-avoidance proof applies equally at a second prime of valuation two whenever that prime is ordinary or odd in the reduced table. □

Lemma 11.3 (Divisor descent for designated squares)**.** *Consider a covered family with designated exponent caps two, a coprime square-free decoration, and fixed caps at all other positions. Raise selected designated caps to three. Assume all divisors with a missing repeated prime are already covered. Any divisor-minimal obstruction to the enlarged family must retain all repeated primes, have actual valuations two or three at the designated positions, and have at least one exact cube at a raised position. The possible assignments are consequently determined by ruling out the remaining square/cube patterns in the minimal obstruction.*

Proof. A surviving repeated prime has valuation at least two. If all its designated valuations are two, the obstruction divides the original covered cap, including its square-free decoration. Throughout the pattern tests these are valuations of the category: the positive fermion table subtracts one from the dyadic valuation only; odd valuations are unchanged. The reduction in Lemma 2.1 treats any square-free factors in the same way for both families. □

Theorem 11.4 (Lifting squared primes to cubes)**.** *Let* Π *be a set of four primes, let* $a_p \geq 0$ *be integers, and fix* $S \subseteq \Pi$ *of size at least two. Put*

$$M = d \prod_{p \in \Pi \setminus S} p^{a_p} \prod_{s \in S} s^2, \qquad d \text{ square-free and coprime to } \prod_{p \in \Pi} p.$$

Suppose every braided category of dimension dividing M *is WGT. Then this remains true after raising any proper subset of the designated squares to cubes. All of them may be raised if* $|S| \geq 3$*, or if some odd* $s \in S$ *satisfies* $\prod_{p \in \Pi \setminus S} p \nmid s - 1$*. In particular every divisor of* $\ell^H (abc)^3 d$ *is covered for arbitrary* $H \geq 0$*.*

Proof. After loss of a repeated prime, at most two remain, or three with one exponent at most three. By Lemma 11.3 at least one designated cube must survive. Proposition 11.2 excludes any remaining square, so all of S must consist of exact cubes. If $|S| \geq 3$, choose two odd members $u < w$ and a third member v. The proposition at u, v would give $w \mid u - 1$. Thus $|S| = 2$ and the same proposition makes the product outside S divide $s - 1$ for each odd $s \in S$. For the final assertion use Proposition 9.13 as the starting family. □

The same lift applies to the two squares of Theorem 9.1, in particular for higher caps $(7, 5)$.

12 Four odd primes at root heights three and two

Here $n = p^3 q^2 uv$ is the odd root left by Corollary 10.15, with p least and $u < v$ the isolated singletons. Put $b = p/(p-1)$, $\delta = v - u$ and

$$A = p^2 q, \qquad B = p^2 q^2, \qquad L = p^3 q, \qquad C = p^3 q^2.$$

A nonunit degree is common if it omits both u and v. The general singleton bound $v \leq bu$ leaves just one pair when $p = 3$. For $p \geq 5$ the full degree C is excluded by combining the close-singleton obstruction of Section 10 with a bound on u. Lemma 12.4 organizes the remaining field and trace arguments by the four possible common degrees.

As $v - u \geq 2$, one has $u \geq 2p - 1 > bp$. The bound in Proposition 10.10 therefore restricts common degrees to A, B, L, C. A common label with full p-valuation and deficient q-valuation is q-principal, and symmetrically with p, q exchanged, by Proposition 6.3 for the cluster $\{p, q\}$. Proposition 9.3 supplies a p-free nonunit in P_q whose degree contains uv; since its q-height is deficient, that degree is quv whenever uv is absent. Theorem 10.11 supplies a mixed p, q-active row Z.

We use one trace test throughout. Let T be active at $t \in \{u, v\}$ and let $w = uv/t$. Suppose all proper common hits of Z, T, on the whole table or on P_q, have degree d. Then

$$[d]_t = \pm 1 \quad \text{if unramified at } t, \qquad a_t(d) = \pm 1 \quad \text{if ramified at } t.$$

Indeed ramification status is constant on a proper degree by Corollary 10.5. In the first case $\chi_T(X) = [d]_t$ on every hit. In the second, K_X is pure at t and K_Z is unramified there, as purity would contradict the square Galois action on its nontrivial q-twist ($q > 3$). Thus $\chi_Z(X)$ is integral rational and constant on the orbit. The orbit sum of $\chi_T(X)$ is $-\epsilon a_t(d)$ by Theorem 10.3, and the principal block is Galois stable. In both cases orthogonality makes the indicated integer divide the vacuum term one.

By Proposition 10.10, $v \leq bu$ and at least one of A, B, L occurs.

Lemma 12.1 (Axis heights forced by a ramified common degree)**.** *If a label of degree B or L ramifies at a singleton t, every p-axis has full height. For L the q-axes are full as well, no degree pqs, $s = u, v$, occurs, and $qw \le bp^3$, $w = uv/t$.*

Proof. The gap bound gives $u \ge (p-1)\delta + 1 \ge 2p - 1 > bp$; also $q > bp$. Write D for the ramified label. It is principal at $a = p$ for $D = B$ and at $a = q$ for $D = L$. For a label X of degree ds, $s = t, w$, consider its normalized pairing z with D. For the axis and witness degrees considered below, ds differs from both d_D and $Cw = n/t$. Thus z is rational by Theorem 10.3. Now let $s' = uv/s$. If $D = B$ and $d = p^i$, $i \le 2$, then $p \mid z$ by p-principality and $z/p \equiv p^{i-1} \pmod{s'}$, impossible since $s' > bp$ by the strict house and balancing mean. If $D = L$ and $d = p^i$ $(i = 1, 2)$, both q and s' divide $p^i - z$, giving $qs' \le bp^i$. For $d = p^i q$ $(i = 0, 1)$, $q \mid z$ and $z/q \equiv p^i \pmod{s'}$ give $s' \le bp^i$. Again $q, s' > bp$ exclude these possibilities. In particular the rational pairing of L with the compulsory $p^3 t$ gives $qw \le bp^3$. □

Corollary 12.2 (The least prime three)**.** *An odd ordinary minimal root $27q^2uv$ is impossible.*

Proof. Propositions 10.14 and 10.10 give $u \le 20$, $u < v \le 3u/2$, and an odd part at least five in both $u - 1$ and $v - 1$. The only possible pair is $(u, v) = (19, 23)$. At an unramified endpoint (9.2) forces the 3-part of every common degree to exceed nine, leaving only L, C. If L ramified at either endpoint, Lemma 12.1 would give $qw \le 81/2$, impossible for $q \ge 5$ and $w \ge 19$. The mixed parity constraint therefore forces C to occur. Its coordinate at 19 is $[23^{-1}]_{19} = 5$, hence odd and ramified. The residual norm of Theorem 10.3 would give $25 \le 19 - j_{19}$. □

12.1 Common degrees when the least prime is at least five

For the remainder let $p \ge 5$. We first exclude C and then use the ramification of a proper common degree to eliminate the remaining patterns. A mixed row on P_q supplies the same two possible witness heights in both steps.

Lemma 12.3. *If $n = p^3q^2uv$ with $p \ge 5$ least, then $u \le bp^2$.*

Proof. Assume $u > bp^2$. The only possible common degrees are L and C by (9.2), and L occurs by Proposition 10.10. Comparison of L at an unramified endpoint gives $q \ge u/b > p^2$. An L-label ramified at t would give $qw \le bp^3$ by Lemma 12.1, contrary to $q > p^2$ and $w > bp$. Thus L is unramified at both singletons; then $v \le bq$ by the q-axis comparison. The mixed Gram identity and even multiplicity exclude L alone, so C occurs. The centered inverse coordinates $A_u = [v^{-1}]_u$ and $A_v = [u^{-1}]_v$ on C satisfy $vA_u + uA_v = 1$. Hence C is ramified at the odd coefficient endpoint t, and unramified at the even endpoint w. If $q \le bp^2$, its orbit cost gives $A_t^2 \le t + 1 - j_t \le t - 1$, and $\delta A_t \equiv \pm 1 \pmod{t}$ with δ even implies $\delta \ge \sqrt{t-1} > p$. Yet $q \le p^2 + p + 1$ and $q \le bp^2 < u$, so $v - q \le q/(p-1) \le p + 2 + 3/(p-1) < p + 3$. Both $v - q$ and $u - q$ are positive and even, giving $\delta \le p - 1$.

Suppose $q > bp^2$. The comparison of L with the p-axes forces all of them to have full height since $t > bp^2$. On P_q a mixed column, reduced at t, therefore needs a hit $Y_l = p^l qs$ with $1 \le l \le 2$, $s = uv/t$. Comparing it at q with $p^3 t$ forces both to be q-ramified, hence unramified at t. The even quantity in (10.1) for L, Y_l is congruent to p^l. As $t > p^2$, this requires $p^l > 3t/4$, forcing $l = 2$ and $v < 4p^2/3$. Now $\delta \le u/(p-1) < 2p$ so $p \nmid \delta$. At the even endpoint w, the identity $pY_2 = sL$ shows that $p[Y_2]_w \pm \delta[L]_w$ is a multiple of w. It is nonzero because $0 < |[L]_w| < \sqrt{w}/2 < p$. Write $c = [L]_w$, $d = [Y_2]_w$, and $a = [s^{-1}]_w$, where $s = uv/w$. The two unramified dual pairs and at least two full-degree labels give $2j_w(c^2 + d^2) + 2a^2/j_w \le w + 1$ by the unramified column norm. Since $|a| \ge (w-1)/\delta$ by parity, Cauchy–Schwarz and the arithmetic–geometric mean inequality give

$$p - 1 \le w/\delta \le \sqrt{p^2 + \delta^2}\,\sqrt{c^2 + d^2}/\delta \le \frac{\sqrt{p^2 + \delta^2}(w+1)}{4(w-1)} < \frac{7\sqrt{5}}{24}p < p - 1.$$

Here $w > p^2 \ge 25$. This excludes the second alternative. □

Lemma 12.4 (The odd sextic root)**.** *No ordinary minimal table has root p^3q^2uv with $p \geq 5$ least.*

Proof. The possible common degrees are exhausted by the following cases. The principal memberships listed in the middle column are the ones forced by the complementary full height in the p, q cluster.

degree	membership	use in the elimination
$C = p^3q^2$	full at both	excluded by the gap bound
$A = p^2q$	neither height full	cannot occur alone; forces B
$B = p^2q^2$	P_p	cannot occur alone; must supply the ramified orbit
$L = p^3q$	P_q	if A occurs, unramified at both ends

Removing the full degree. These bounds give $\delta \leq u/(p-1) \leq p^3/(p-1)^2 < p+3$. Thus $\delta \leq p+1$, $u \geq (p-1)\delta + 1 \geq 2p-1$, and Theorem 10.12 excludes C. We determine which of A, B, L must carry the ramification required by the mixed Gram identity. We shall use the following consequence of the gap bounds. For $s = u, v$ and $\delta z \equiv \pm p \pmod{s}$ one has

$$|z| \geq \frac{s-p}{\delta} > \max\left\{\frac{\sqrt{s}}{2}, \frac{s+1}{\delta+3}\right\}. \tag{12.1}$$

The first inequality follows from parity. For the square-root bound $(s-p)/\sqrt{s}$ increases with s. At $s_0 = (p-1)\delta + 1 \leq p^2$ one has $2(s_0 - p)/\delta = 2(p-1)(1-1/\delta)$, greater than p for $\delta \geq 4$ and equal to $p - 1 > \sqrt{2p-1}$ for $\delta = 2$. The second bound says $3s > (p+1)\delta + 3p$. Using $s \geq s_0$ gives this for $\delta \geq 4$; at $\delta = 2$ equality could occur only for $p = 5, s = 9$, which is composite.

The sole-A case. Put $F = C/A = pq$. Suppose A occurs alone. Use the two endpoints and signed coordinates from Lemma 10.7. The coordinate congruence $a_t \equiv A/(Cw) = 1/(Fw) \pmod{t}$ now gives $F \equiv \pm\delta^{-1} \pmod{t}$. Since $A = pF$ we obtain $pj_t \equiv \pm\delta \pmod{t}$. At w one has $C = A^2/p \equiv 1/p$ by $[A]_w = \pm 1$. The vacuum congruence $j_w Ct \equiv \pm 1 \pmod{w}$ therefore gives $\delta j_w \equiv \pm p \pmod{w}$. Since $pj_t \geq 2p > \delta$, the first congruence and parity give $pj_t \geq 2t - \delta > p\delta$, so $j_t \geq \delta + 2$. At w, (12.1) gives $j_w(\delta+3) > w+1$, contrary to $j_w(1+j_t) \leq w+1$. Finally, if uv occurred, full-height principality would exclude both p^2q^2 and p^3q and force A. The exclusion of the full common degree C would then make A the sole common degree, as just ruled out.

When A occurs. We first record a restriction common to $D = B, L$. If an A- or D-label ramifies at a singleton t, with partner w, then its normalized pairing $z = S_{A,D}/D$ with any label of the other degree is rational. Indeed the degrees are distinct, neither is full at t, and the rational-entry assertion of Theorem 10.3 applies.

For $D = L$ a rational pairing is itself impossible. The integer z is divisible by q since $L \in P_q$ and $\chi_L(A) = z$. Both labels are principal and units at u, v, so $z/q \equiv p^2 \pmod{uv}$. The house and balancing mean give $-p^2/(p-1) \leq z/q < p^2$, forcing $uv \leq bp^2$, whereas $u, v \geq p+2$. Hence, if L occurs, no A- or L-label can ramify at either singleton.

Some common label is singleton-ramified by the divided mixed Gram identity and even orbit sizes. The full degree C is absent. If B were absent, the preceding paragraph would exclude coexistence with L, and A as the sole common degree was already ruled out. Thus B occurs. In the presence of L, it carries the compulsory ramification; in its absence at least one of A, B does. Choose this ramified label and a label of the other degree. Their pairing $z = S_{A,B}/B$ is rational by the first paragraph. Since $B \in P_p$, $p \mid z$. The singleton congruences give $z/p \equiv pq \pmod{uv}$, and the same strict house and mean bounds yield $uv \leq bpq$. Also $v - u \geq 2$ and $v \leq bu$ give $u \geq 2p - 1 > b^2p$, so $q \geq uv/(bp) > bv$.

Finally $B \in P_p$ excludes the coprime degree uv; the complementary p-free witness in P_q has degree quv. Comparing it at q with $p^i s$ for $s = u, v$ gives q-inertia at least $q/(bs) > 1$ on the corner. Isolation of either singleton then excludes ramification there.

For the rest of the proof we will repeatedly use two consequences at an endpoint s, with $s' = uv/s$. Once the p-axes are full, all proper common hits of Z and an s-active row on P_q have degrees

$$A, \qquad Y_1 = pqs', \qquad Y_2 = p^2qs'.$$

Indeed the two heights are deficient, p-free nonunits carry uv, and a hit omits s. A Y_1-hit is s-unramified: its comparison in P_q with p^3s has gcd p, and $q > bp$ forces q-ramification. Also if a common label of degree x is s-unramified, an s-unramified Y_l is excluded whenever $xY_l/(Cs')$ is an odd integer at most $3s/4$, by (10.1). For $x = A$ these quotients are $1, p$; for $x = L$ they are p, p^2. Throughout $p < 3s/4$.

The sole-B case. Under the sole-degree hypothesis, Lemma 10.7 provides endpoints t, w with $a_t(B) = \pm 1$, $[B]_w = \pm 1$, $j_tB \equiv \pm 1 \pmod t$ and $j_w(1 + j_t) \le w + 1$. As $C = pB$, the first congruence gives $pw \equiv \pm 1 \pmod t$. All p-axes have full height by Lemma 12.1. Since B is the sole common degree, the proper P_q-hits at t are Y_1 (unramified) or Y_2.

If a Y_2-hit were unramified at t, its dual-pair budget would give $0 < j_t[Y_2]_t^2 < t/2$. This even integer is congruent to $j_t(pq)^2 \equiv \pm 1 \pmod t$ since $pw \equiv \pm 1$, impossible.

Nor can Y_2 ramify there: B is already the unique proper ramified degree. Thus only Y_1 is hit, and the trace test gives $|[Y_1]_t| = |[q]_t| = 1$. As $p\delta \equiv \pm 1$ and $j_t(pq)^2 \equiv \pm 1 \pmod t$, we get $j_t \equiv \pm\delta^2 \pmod t$. Here $j_t < t/2$ and $\delta^2 < 3t/2$ since $\delta \le p + 1$, $t > (p - 1)\delta$, and $p \ge 5$. Their sum and absolute difference are even and less than $2t$, forcing $j_t = \delta^2$.

Finally $p\delta \equiv \pm 1 \pmod t$ and $0 < p\delta \le v < 2t - 1$, so $t = p\delta + \sigma$ and $w = t + \eta\delta$, with signs σ, η. Thus $pt \equiv h := \delta + \eta\sigma \pmod w$. From $C = pB$, $[B]_w = \pm 1$ and the vacuum congruence, $j_wh \equiv \pm 1 \pmod w$. Thus $j_w(\delta + 1) \ge j_wh \ge w - 1$ by parity. But $j_w(1 + \delta^2) \le w + 1$ by the sole-degree budget, impossible since $j_w(\delta^2 - \delta) \ge 4$.

When A is absent. Suppose A is absent. Then L occurs by the exclusion of a sole B. If L is unramified at both singletons, mixed Gram parity forces B to ramify at one. Thus in either case Lemma 12.1 gives full p-axes. At a singleton s the proper P_q-hits all have degree $Y_2 = p^2qs'$. Indeed if L ramifies anywhere the lemma excludes Y_1 outright. Otherwise a Y_1-hit is unramified and its quotient with L is p, also excluded. Write $Y_s = p^2qs'$.

Whenever L is unramified at s, so is its whole degree. If Y_s were unramified there, the trace test would give $[Y_s]_s = \pm 1$. But $pY_s = s'L$ would give $\delta[L]_s \equiv \pm p \pmod s$, contradicting (12.1) and $|[L]_s| < \sqrt{s}/2$. Hence Y_s ramifies there, with $a_s(Y_s) = \pm 1$ by the same test; equivalently $pq \equiv \pm 1 \pmod s$.

If L is unramified at both endpoints, Y_s is the unique proper ramified degree at each. This contradicts the ramification of a common degree. Thus L ramifies at t and Y_w at $w = uv/t$. If L alone is common, Lemma 10.7 gives $[L]_w = \pm 1$. As $L = p^2(pq)$ and $pq \equiv \pm 1 \pmod w$, one has $p^2 \equiv q^2 \equiv -1 \pmod w$: the positive sign would give $p \equiv \pm 1$, impossible for $1 < p < w - 1$. The compulsory axis p^3t then has signed coordinate $[q^{-2}]_w = -1$ and must ramify at w, where the distinct degree Y_w already ramifies.

Therefore B occurs. At t all Y_t-hits are unramified since L is the proper ramified degree. The trace gives $Y_t \equiv \pm 1 \pmod t$. Since $Cw = pqY_t$, the vacuum yields $j_t^2B \equiv 1 \pmod t$, contrary to Lemma 10.8.

The final ramification budget. We now have both A, B and $q \ge uv/(bp) > bv$ by the two cases for A. All p-axes are unramified at both singletons and have full height. Indeed if B ramifies at either singleton this follows from Lemma 12.1; otherwise apply (10.1) to B, p^ir at $s = uv/r$ for $r = u, v$. At $i = 1, 2$ the quotient $p^{i-1} \le p < 3s/4$ is forbidden.

We claim that A is unramified at both endpoints. Suppose instead it ramifies at t. All A-labels are then pure at t, unramified at $w = uv/t$, and L is absent by the A, L comparison above. None can belong to P_q: its rational normalized pairing with p^3t would give two positive residues p^2 at q, w, hence $qw \le bp^2$, contrary to $q > bv$. The P_q-witnesses are now Y_1, Y_2 at w. Since A is unramified there, the quotient test excludes all unramified Y_l, including every Y_1. Only ramified Y_2 remain.

Their trace criterion gives $a_w(Y_2) = \pm 1$, or $pq \equiv \pm 1 \pmod{w}$. Hence $|[A]_w| \geq p-1$ as $w \geq 2p-1$. But its unramified dual-pair budget gives $|[A]_w| < \sqrt{w}/2 \leq bp/2 < p-1$, a contradiction.

Thus A and any L are unramified at both singletons. By mixed Gram parity B ramifies at some t. At t the proper P_q-hits are A or ramified Y_2 by the same quotient test and the exclusion of Y_1.

Since B is the unique proper ramified degree at t, no ramified Y_2 occurs. The remaining hits are unramified of degree A, so $[A]_t = \pm 1$ by the trace criterion.

Since $C = A^2/p$, the vacuum gives $\delta j_t \equiv \pm p \pmod{t}$ and therefore $j_t(\delta+3) > t+1$ by (12.1). The p^3w-axis has at least δ labels, all t-unramified. By the q-principal comparison with quv its multiplicity is at least $(q-1)/w$ for $q > 2w$; for $q \leq 2w$ this bound follows from duality. Thus its even multiplicity is at least $t/(bp) - 1/w > \delta - 2$: use $t > (p-1)\delta$ if $\delta \leq p-1$, and $t \geq p^2+2$ by primality if $\delta = p+1$. The vacuum, this axis and an A dual pair give $j_t(\delta+3) \leq t+1$ in the unramified budget, a contradiction. □

Proof of Theorem 9.1. The root heights i, j in Corollary 10.15 satisfy $i \leq h$, $j \leq k$. Its even case uses the dyadic enumeration of Theorem 9.16. The corollary gives each asserted bound except possibly $h+k \geq 6$ at an odd root p^3q^2uv. Such a root is excluded by Corollary 12.2 at $p = 3$ and by Lemma 12.4 at $p \geq 5$. □

13 Four repeated primes with an unrestricted exponent

Theorem 13.1. *Let P, Q, u, v be distinct primes with Q greater than at least one of P, u, v. For every nonnegative integer A, the integer $P^5Q^Au^3v^2$ is covered.*

Write a four-prime ordinary minimal root as P^hQ^kuv, with $u < v$ the isolated singletons. Here P, Q, u, v are distinct primes and $h, k \geq 2$. Proposition 9.3 supplies degrees $P^au, Q^bu, P^{a'}v, Q^{b'}v$, with positive powers. The same proposition gives $Q \mid d_J$ for every P-active label J. Consequently for $k = 2$ Theorem 6.9 gives Q-integrality of W_Pv_0. In particular $P^hW_Pv_0$ is integral: the singleton denominators are already removed by Lemma 6.2. This holds for arbitrary h and arbitrary prime positions.

13.1 An arbitrary exponent at the larger of two higher primes

Fix distinct primes P, Q, u, v with $P < Q$. We prove coverage of $P^5Q^Au^3v^2e$ for arbitrary A and coprime square-free e. Theorem 11.4 raises either square of the two-square base, and both if $PQ \nmid s-1$ for some odd $s \in \{u, v\}$. The odd base is Theorem 10.13. For $P = 2$, a lost repeated prime leaves a three-prime family with a square, or fewer. With four retained, Lemma 9.5 excludes the reduced table. An ordinary root has dyadic height at most two; height one is covered by Proposition 9.13. It remains to exclude $n = 4q^kuv$, $q = Q$, $u < v$ and $k \geq 1$.

Use the trace and Gram feasibility problem of Section 9.2 with $h = 2$. The window bounds u, q, v; $q^k \bmod uv$ determines the signed coordinates. For all positive i the signatures of q^i needed for the two axis tests and the common Gram equation are

$$(q^i \bmod 2uv,\ \min(q^i, 4)).$$

Starting at $i = 1$, multiply the first component by q modulo $2uv$ and cap the second component after multiplication. Stopping at the first repeated pair is exhaustive by determinism. We permit all these signatures at any k, relaxing the condition $i \leq k$. The cap is exact for $\min(2^a, q^i)$, $a = 1, 2$. Merge equal axis and common signatures by the rule following (9.4).

At each endpoint t, put $m_t = 4q^kuv/t$. Since $v \leq 4u+1$ and $4q^ku > v+1$, the full-multiplicity bound of Lemma 10.2 reserves a full label beyond the $(t-1)j_t$ full labels already reserved. Its

signed coordinate is one. The residual squared budget is $j_t(t+1-j_t)-1$ and the degree trace is $(j_t-1)m_t+\epsilon j_t$. Further full labels remain allowed.

Minimize each axis cost at its trace residue and impose the common norm and mixed constraints of (9.4) with these residual budgets. The recurrence gives 279 vacuum residue rows, 53 after both axis budgets, and two after the common Gram sum. Both rows have $q=3$, $u=5$, $v=7$, $j_u=1$, $\epsilon=-1$. Their $(j_v, 3^k \bmod 35)$ values are $(2,17)$ and $(1,27)$.

One calculation rules out both rows. At five the remaining squared norm is four and the degree trace is -1. Both axis types and a common degree must still occur, so there are exactly four remaining labels, each of signed coordinate $[d_X]_5=\pm 1$. A 3-free one can only have degree 14: the alternative 28 has $|[28]_5|=2$. Its signed trace contribution is -14. At least one such axis is compulsory, and the trace modulo three requires a second one. There are exactly two since the other axis and a common degree are also compulsory. The other axis has degree $7\cdot 3^b$ with b odd because $|[7\cdot 3^b]_5|=1$. Every inactive label is unramified at this unit-deficit singleton. A common degree 2^i3^a must therefore satisfy $5\le 2\cdot 2^i$ by the opposite axis, hence $i=2$. Its residue has magnitude one precisely for even $a\ge 2$. The trace would give

$$\pm 4\cdot 3^a \ \pm 7\cdot 3^b = 27.$$

As the exponents have opposite parity, the 3-valuation gives $\min(a,b)=3$. Hence $b=3$, $a\ge 4$, and $4\cdot 3^{a-3}$ would have to be 6 or 8. This is impossible.

For any specified dyadic root height $H\ge 2$, replace four by 2^H in the cap and window and allow all powers 2^i with $1\le i\le H$. Again $2^H q^k u > v+1$ forces the extra full label. The algorithm thus tests all opposite heights at once for each input H.

If $Q<P$ and Q exceeds a singleton, then the least of the four repeated primes is a singleton, which is excluded by Corollary 10.15. The same divisor descent and cube lift apply. This proves Theorem 13.1 in its stated generality.

The two dyadic exponent boxes. By the four-prime descent used above, the dyadic boxes $2^9q^5(uv)^2e$ and $2^5q^9(uv)^2e$ reduce to ordinary roots of higher heights at most $(4,2)$ or $(2,4)$. The trace calculation in Theorem 9.16 excludes every such root. A dyadic singleton is excluded by Corollary 10.15. Thus for $(8,4,2,2)$ the remaining roots are odd, of form P^4Q^2uv with P least and P,Q mixed (Theorems 10.11, 10.13 and Section 12). The complementary-prime integrality above applies to these roots.

Corollary 13.2 (A dyadic family with two unrestricted exponents)**.** *Let q,w,t be distinct odd primes. Suppose $t\equiv 1 \pmod 8$, every odd prime divisor of $t+1$ is $1 \pmod 4$, and $\left(\frac{q}{t}\right)=1$, $\left(\frac{w}{t}\right)=-1$. Then both $2^Aq^Bw^3t^2$ and $2^Aq^Bw^2t^3$ are covered for every $A,B\ge 0$. If $2q\nmid w-1$ or $2q\nmid t-1$, then $2^Aq^B(wt)^3$ is covered.*

Proof. In the base family with two squares, a minimal obstruction would have ordinary root 2^hq^kwt with isolated singleton t (Propositions 9.3 and 9.5). By Proposition 6.4 and Lemma 6.6, its deficit j divides $t-1$ or $t+1$ and satisfies $2^hq^kwj\equiv\pm 1 \pmod t$. Both 2 and -1 are squares modulo t. For any odd prime ℓ dividing $t-1$ or $t+1$, quadratic reciprocity gives $\left(\frac{\ell}{t}\right)=\left(\frac{t}{\ell}\right)=1$, using $\ell\equiv 1 \pmod 4$ in the second case. Thus j is a square modulo t, while the congruence forces it to have the same quadratic character as w. This excludes the two-square base. Apply Theorem 11.4 to its two designated squares. □

14 Residue conditions at two isolated primes

We use the exact full multiplicity and residual trace and norm of Theorem 10.3. Write $m=n/t$, $j=j_t$, $[\cdot]_t$ for centered residue, and ϵ for the common anomaly. Let k_t count the distinct proper nonunit t-free degrees.

Consider an ordinary odd four-prime minimal root $n = P^h Q^k uv$ with $h, k \geq 2$ and isolated $u < v$.

Corollary 10.5 gives the singleton parity, ramification and odd-index restrictions. Its family assertion covers the two-square base whenever the odd part of $u-1$ or $v-1$ is at most three. Theorem 11.4 then permits either square to be cubed, and both under its divisibility hypothesis. At $(A, B) = (8, 4)$ the even-core result also allows arbitrary prime assignments when either specified odd singleton satisfies this condition.

The residual norm also restricts the pair of singletons. Write $a(t) = 2^{\nu_2(t-1)}$ and $o(t) = (t-1)/a(t)$. At some endpoint t there is a common proper degree ramified there (10.5). These labels are unramified at the other endpoint w by separation of inertia. They contribute at least j_t unramified labels there. The two compulsory axis degrees at w are distinct from the common degree and each other, and at least one supplies an additional unramified dual pair. The norm formula therefore gives

$$H_w = (w-1)/j_w \geq j_t + 3.$$

Since H_t is an odd divisor of $t-1$ and $a(t) \mid j_t$, a necessary condition on the two primes is

$$o(v) \geq a(u) + 3 \quad \text{or} \quad o(u) \geq a(v) + 3.$$

Furthermore $t - 1 = j_t H_t \leq (o(w) - 3)o(t)$, hence

$$u \leq o(u)o(v) - 3\min\{o(u), o(v)\} + 1, \qquad v \leq 3u/2.$$

Thus prescribed odd parts give only finitely many possible pairs, independently of both higher primes and exponents. In particular $o(u) = o(v) = 5$ forces $u \leq 11$ and then $u = 11$. The interval $11 < v \leq 33/2$ contains no prime $1 + 5 \cdot 2^k$ $(k \geq 1)$. More generally if both odd parts are at most seven, the single-end obstruction leaves only $5, 7$. If $o(u) = o(v) = 7$, the bound gives $u \leq 29$ and $v \leq 43$. There is just one prime with odd part seven in this range, namely 29. If the odd parts are five and seven, $u \leq 21$ and $v \leq 31$. The only possible pair is $11, 29$, and it violates the closeness bound. Thus both odd parts at most seven are impossible.

14.1 A residue criterion for the two higher exponents

Fix distinct odd primes P, Q, u, v with u, v the designated squares. For a divisor-minimal obstruction to $P^A Q^B (uv)^2 e$, loss of a repeated prime is covered by Theorem 4.16 or Dong's theorem; the reduced table and a third root height one are excluded by Propositions 4.8, 9.3 and 9.13. Thus its ordinary root is $P^h Q^k uv$, $h, k \geq 2$. At $t = u, v$ set $w = uv/t$ and $H_t = (t-1)/j_t$. These are odd indices at least five, with $3j_t \leq t + 1$ and $j_u + 3 \leq H_v$ or $j_v + 3 \leq H_u$, as proved above. The vacuum congruence reads

$$P^h Q^k w \equiv \epsilon H_t \pmod{t}.$$

In $G = \mathbb{F}_u^\times \times \mathbb{F}_v^\times$ let E be the set of pairs $(\epsilon H_t / w)_t$ for all such indices and both common signs. Every surviving exponent pair satisfies $P^h Q^k \in E$; the same necessity holds after any group homomorphism. We use its quadratic characters to constrain both parities and cyclic quotients to eliminate an unrestricted exponent.

Quadratic characters. Write χ_t for the Legendre symbol modulo t. For $t \equiv 1 \pmod 4$ and any odd divisor H of $t-1$, quadratic reciprocity gives $\chi_t(H) = (t/H) = 1$, where the right side is the Jacobi symbol. At such an endpoint every residual four-prime root therefore satisfies

$$\chi_t(P)^h \chi_t(Q)^k = \chi_t(w). \tag{*}$$

The same character argument at a ramified endpoint also applies with any number of other primes.

Lemma 14.1. *In an odd ordinary minimal table of dimension n^2, at most one isolated singleton $t \equiv 1 \pmod 4$ satisfies $\chi_t(n/t) = -1$. The other prime factors of n may have any heights.*

Proof. Among any two isolated singletons, a common inactive orbit ramifies at one endpoint t (Corollary 10.5). There $H_t = (t-1)/j_t$ is odd by Theorem 10.3, and the vacuum congruence gives $n/t \equiv \epsilon H_t \pmod t$. If $t \equiv 1 \pmod 4$, both factors on the right have character one. □

For each endpoint congruent to one modulo four, equation $(*)$ is linear over $\mathbb{F}_2$ in the parities of h, k. If these equations are inconsistent, every box $(A, B, 3, 2)$ is covered, for arbitrary A, B, by descent and the four-prime cube lift (Theorem 11.4). The cubed companion can be chosen freely. In particular it suffices that $\chi_t(P) = \chi_t(Q) = 1$ and $\chi_t(w) = -1$. The fixed primes $P = 13$, $Q = 17$, $t = 101$, $w = 103$ illustrate the test since $35^2 \equiv 13$, $44^2 \equiv 17 \pmod{101}$, and $103 \equiv 2$ is a quadratic nonresidue modulo 101.

If $\chi_t(P) = 1$, $\chi_t(Q) = -1$ and $\chi_t(w) = (-1)^\delta$, $\delta \in \{0, 1\}$, the possible source exponents at Q are $2k \equiv 2\delta \pmod 4$. Hence for $K = 4b + 2\delta$, $b \geq 0$, one has the fixed-prime lift

$$B(A, K, 2, 2) \quad \Longrightarrow \quad B(A, K + 3, 3, 2).$$

Indeed, the largest allowed Q-valuation at most $K + 3$ is at most K; first apply this with both squares intact, then apply the cube lift. The output $3, 2$ can again be permuted. More generally the admissible parities of both higher positions can be used together. For each admissible pair (α, β), round A down to the largest nonnegative integer $\leq A$ congruent to $2\alpha \pmod 4$, and do the same to B using 2β. A pair with no such corner is discarded. Coverage of each resulting corner with the two squares implies coverage of $(A, B, 3, 2)$.

At root heights $(4, 2)$ both higher powers are squares, so $\chi_t(w) = -1$ alone excludes that root. At every lower corner of the source $(8, 4, 2, 2)$, the higher root heights either sum to at most five or one is at most one, unless they are exactly $(4, 2)$. Those smaller cases are covered by Sections 9 and 12. Thus this quadratic nonresidue criterion already covers the complete divisor box $(9, 5, 3, 2)$ for every such assignment.

The same argument also handles two endpoints $u, v \equiv 3 \pmod 4$. For $m_t = (t-1)/j_t$ one has $\chi_t(j_t) = (-1)^{(m_t+1)/2}$ by the reciprocity sign. Putting $C = P^h Q^k$, multiplication of the two vacuum characters cancels the common sign $-\epsilon$, and reciprocity at u, v gives

$$\chi_u(C)\chi_v(C) = -\chi_u(j_u)\chi_v(j_v).$$

If all prime factors of $(u-1)(v-1)/4$ are one modulo four, then m_u, m_v are both one modulo four. Thus the right side is -1, excluding any square C, in particular root heights $(4, 2)$. If in addition $\chi_u(P) = \chi_v(P)$ and $\chi_u(Q) = \chi_v(Q)$, every pair of heights is excluded. For example take $(P, Q, u, v) = (13, 17, 443, 467)$. Here $(u-1)/2 = 13 \cdot 17$ and $(v-1)/2 = 233$. Modulo 13 the two singletons give $1, -1$, and modulo 17 they give $1, 8$. All four residues are squares, so reciprocity gives the two character equalities and the all-height conclusion applies. For either odd residue class, the image of E under (χ_u, χ_v) tests the four possible parity pairs with the same sign at both endpoints. The corner rule then applies to every choice of exponent caps.

Cyclic quotients and jumps of a higher exponent. Project G, E to either endpoint or retain both.

In $G/\langle P\rangle$, let d be the order of the image of Q and set

$$R = \{i \in \mathbb{Z}/d\mathbb{Z} : Q^i\langle P\rangle \text{ meets } E\}.$$

Then every surviving k satisfies $k \bmod d \in R$, simultaneously for all h. Conversely, those are exactly the $k \geq 2$ allowed by this residue test because the powers with $h \geq 2$ exhaust $\langle P\rangle$.

Suppose all divisors of $P^A Q^B (uv)^2 e$ are covered, where e is any coprime square-free factor. Set $k_0 = \min\{k \geq 2 : 2k > B,\ k \bmod d \in R\}$, with $k_0 = \infty$ if R is empty. Then $P^A Q^{2k_0-1} u^3 v^2 e$

and all its divisors are covered, with ∞ denoting an unrestricted cap. On the two-square box the higher valuation of a divisor-minimal obstruction is exactly $2k$. Theorem 11.4 then lifts either singleton to a cube. In particular one can start with $B = 0$ and arbitrary A.

At one endpoint put $G_t = \mathbb{F}_t^\times/\langle -1, P\rangle$. If the odd part of $t-1$ is a prime $\ell \geq 5$, then $H_t = \ell$. Thus $\bar{Q}^k = \bar{\ell}/\bar{w}$ in G_t, which has no solutions or one class $k = c \pmod d$, where $d = \mathrm{ord}_{G_t}(\bar{Q})$. In the latter case every $k_0 \geq 2$ in that class gives the lift

$$B(A, 2k_0, 2, 2) \implies B(A, 2k_0 + 2d - 1, 3, 2)$$

for the same prime assignments.

For example, let $u = 113$, $v \equiv 14$, $P \equiv 1$, and $Q \equiv 3$ modulo 113. The only allowed H_u is 7. The element 3 has order 112, with $3^{44} = 56$ and $3^{100} = 57$ modulo 113. Thus $14 \cdot 3^k \equiv \pm 7$ is equivalent to $k \equiv 44 \pmod{56}$. Consequently the box $(A, 87, 3, 2)$ is covered for every A, and for every $b \geq 0$ one obtains the transfer

$$(A, 88 + 112b, 2, 2) \implies (A, 199 + 112b, 3, 2).$$

The primes $(P, Q, u, v) = (227, 3, 113, 127)$ realize this assignment, with Q least and $127/113 < 3/2$.

The two-endpoint test can be sharper. If $u = 449$, $v = 463$ and P, Q have the residues of $23, 13$ modulo uv, the calculation in the finite quotient gives $d = 168$ and $R = \{42, 46, 126, 130\}$, proving $(A, 83, 3, 2)$ for arbitrary A. If Q instead has the residues of 67, then R is empty. Already at 449 the only allowed H is 7 and $23^{112}, 67^{112}, 7^{112}, 463^{112}$ are $1, 1, 1, -1$ respectively. Also $(-1)^{112} = 1$. Thus the two higher exponents are both unrestricted in this family. The set R can be computed from discrete logarithms at the selected endpoints and elementary linear congruences. This eliminates h by the generalized Chinese remainder theorem and never enumerates source dimensions or any finite range of the two heights.

Arbitrarily prescribed heights. Fix an odd prime Q, an integer $L \geq 2$ and a distinct odd prime t. Fix the residue of P modulo t, of multiplicative order m. For all but at most $2m(L-1)\tau(t-1)$ nonzero residues of the other singleton w modulo t, the box $(A, 2L+1, 3, 2)$ is covered for every A, for distinct odd primes P, Q, t, w. Here τ counts positive divisors and the cube may be put at either singleton. Indeed, for $2 \leq k \leq L$ the vacuum congruence restricts w to the residues $\epsilon H P^{-h} Q^{-k}$, with $H \mid t-1$ odd, $H \geq 5$, and h taken modulo m. This proves the count. The divisor descent above gives the asserted box.

The residue criterion excludes prime assignments even within the ratio bound of Proposition 10.10. Fix Q, L and let $t > Q$ tend to infinity through primes. Take $P \equiv 1 \pmod t$, so that $m = 1$. The prime number theorem gives $\sim t/((Q-1)\log t)$ primes w in $(t, Qt/(Q-1)]$, all in distinct classes modulo t. At most $2(L-1)\tau(t-1) \leq 4(L-1)\sqrt{t}$ are exceptional. The proportion covered by this test therefore tends to one in this interval. One may choose any prime $P > w$ in the class 1 modulo t by Dirichlet's theorem. Then Q is the least prime and $w \leq Qt/(Q-1)$ as required. Other necessary conditions may also exclude some or all of these assignments.

14.2 Finite windows with one unbounded height

The odd least-prime window. For the feasibility problem following (9.4) at odd least prime P and root $P^h Q^2 uv$, all three primes Q, u, v are free. For $h \geq 3$ use the window of Propositions 9.8 and 10.14, Lemma 9.15. Its first limit is

$$u \leq \begin{cases} (9P^{h-1} + 2)/4, & P = 3, \\ (1 + P^2/(2(P-1)^2))P^{h-1}, & P \geq 5. \end{cases}$$

Put $U = \max(2, \lfloor (u+1)/j_u \rfloor - 1)$; then $v \leq P^h U + 1$ and $Q \leq vU + 1$. Remove $5, 7, 13$ as isolated singleton positions by Lemma 9.9 and Corollary 10.5.

Proposition 10.10 gives $v \le Pu/(P-1)$. The deficits are even, with $3j_t \le t+1$ and $(t-1)/j_t$ odd (Theorem 10.3); each proper degree occurs in dual pairs. At height $h = 4$ the two budgets and the sign of one mixed contribution suffice. For given u, v, j_u, j_v, ϵ, solve

$$Q^2 \equiv -\epsilon(P^4 v j_u)^{-1} \pmod{u}, \qquad Q^2 \equiv -\epsilon(P^4 u j_v)^{-1} \pmod{v}.$$

Combine root pairs by CRT and retain primes in the window. At each t reserve twice the smallest admissible squared cost on each of the two axes (odd duality). Since the mixed sum is $-j_u j_v < 0$, some admissible common degree y must have $A_u(y)A_v(y) < 0$, fit both residual budgets with multiplicity two, and pass the unramified-endpoint test in Lemma 9.14. The enumeration implements these necessary tests and gives

P	3	5	7	11	13	17	19
vacuum classes	14	59	181	1338	2706	8065	12977
positive axis budgets	0	3	0	1	0	1	1
negative common term	0	0	0	0	0	0	0

Thus the other primes are unrestricted at each of these least octic primes. The test can be applied at any other odd P and any root height $h \ge 3$ in $P^h Q^2 uv$, replacing fourth powers by hth powers and using the corresponding proven window.

A general prime-size bound and the dyadic case. Let P, Q, u, v be distinct primes, $H \ge 2$ and $A \ge 0$. Set $b = P/(P-1)$ and $c = 1$ for $P = 2$, $c = 2$ for odd P. The bounds on the two axes imply that

$$Q > 1 + \frac{b^3}{c} P^{2H} \quad \Longrightarrow \quad P^{2H+1} Q^A u^3 v^2 e \text{ is covered.}$$

For the two-square base, Corollary 10.15 gives a putative ordinary minimal root $P^h Q^k uv$, with P least and $2 \le h \le H$, $k \ge 2$. After ordering $u < v$, Proposition 9.8 gives $u \le bP^h$ and $\min(Q, v) \le bu$. Lemma 9.15 gives $Q \le vU + 1$, where U can be taken at most u/c. If $Q > bu$, then $v \le bu$ and hence $Q \le bu^2/c + 1$. If $Q \le bu$ this bound holds as well. Substitution gives the contradiction, and Theorem 11.4, raises either of the two squared positions to a cube.

For $H = 3, 4$, consider all ordinary minimal roots $2^H q^k uv$, with q, u, v distinct odd primes and $k \ge 2$. Run the periodic vacuum, two axis trace and signed Gram tests of Section 13, with cap 2^H and all positive powers of q. Record just the q values among the surviving classes, denoting their set by S_H. The finite recurrence gives

H	vacuum	axes	Gram	$\lvert S_H \rvert$	$\max S_H$
3	3967	582	98	9	37
4	87793	5704	1638	39	283

Here $S_3 = \{3, 5, 7, 11, 13, 17, 19, 31, 37\}$ is contained in S_4. Two independent calculations give the same sets. The first enumerates residues by multiplication until repetition. The second reconstructs the vacuum residue by CRT, checks membership in the cyclic group generated by q, and retains minimal costs for each partial Gram sum. Allowing exponent signatures for all positive exponents includes every signature with actual exponent at most k. Thus an omitted q rules out all k and all prime assignments to u, v.

Consequently $2^{2H+1} q^A u^3 v^2 e$ is covered for $q \notin S_H$, arbitrary A and coprime square-free e. Apply the two-square descent of Section 13.1: heights one and two are already covered, and $S_3 \subseteq S_4$ covers height three when $H = 4$. The periodic test excludes the remaining height, and Theorem 11.4 raises either square.

Unbounded opposite height at three. There is an odd version of the same projection with no bound on the opposite height. Consider an odd ordinary minimal root P^hQ^kuv, P least, $u < v$ isolated, $h, k \geq 2$. Put $b = P/(P-1)$. Use $u \leq bP^h$, $v \leq bu$ and $Q \leq v \max(2, \lfloor (u+1)/j_u \rfloor - 1) + 1$ (Propositions 9.8, 10.10 and Lemma 9.15). Remove singleton values $5, 7, 13$ (Lemma 9.9 and Corollary 10.5). Apply Theorem 10.3 and Corollary 10.5. At t the proper-degree budget is $j_t(t - j_t)$, multiplicities are even, and an odd coordinate occupies the unique ramified orbit of size j_t. Branch on a ramified common endpoint, where both axes are unramified. Retain the constraints $j_t \mid t - 1$, $(t-1)/j_t$ odd, j_t even, $3j_t \leq t + 1$.

The finite calculation in `odd_periodic.py` uses all positive powers of Q, represented by $(Q^i \bmod m, \min(Q^i, P^h))$ until repetition. The moduli are uv for the vacuum and the common degree tests, and Pt for a Q-axis at t (signed coordinate modulo t, trace modulo P). At the P-axis the exact powers $P, \ldots, P^h$ suffice. Apply the axis minimization of (9.4) with these signatures, the inertia floors and the preceding ramification conditions. For the remaining common degrees keep the minimum v-cost at fixed u-cost, signed Gram sum and flag for the ramified orbit. Each equal-signature merge retains all genuine solutions by the feasibility specification. The program first runs the relaxed test, then tries the two choices of ramified endpoint for each survivor. Its counts are

(P, h)	window	vacuum and P-axes	both axes	mixed	remaining Q
$(3, 3)$	4376	9	4	2	11
$(3, 4)$	284978	23	20	11	$5, 7, 11, 19, 29$

The two remaining sign classes at height three both have $(u, v, j_u, j_v) = (23, 31, 2, 2)$.

A second congruence removes every remaining class. At the endpoint t carrying a proper ramified common orbit, Theorem 10.3 gives the exact trace $\sum A_t(D)D = \epsilon j_t$ and squared cost $\sum A_t(D)^2 = j_t(t - j_t)$ on the proper labels, counted with multiplicity. Both identities may be halved by duality.

For $m = Q^s$ the trace calculation retains all degrees $D = P^iQ^aw^\delta$, $0 \leq i \leq h$, $a \geq 0$, $\delta = 0, 1$, having at least two distinct prime factors, as required by the prime-power exclusion of Lemma 2.5. The signatures use residues modulo mtw (or the least common multiple) and the same height cap. An odd coordinate is allowed only at a common degree. For a common degree the two signed coordinates must be admissible and their gcd bound must permit unramifiedness at one endpoint (Lemma 9.14). Choose t by Corollary 10.5, so that its proper ramified common orbit also has odd centered degree at w. The row cost of a dual pair is $A_t(D)^2$ and its trace increment $DA_t(D)$ mod m. The recurrence uses each signature zero or more times within the exact cost $j_t(t - j_t)/2$, with three flags recording the two mandatory axes and the ramified common orbit. The latter must occur exactly once, contributing $j_t/2$ pairs of just one signature. An accepted state requires all three flags and residue $\epsilon j_t/2$ mod m. For each tested modulus both endpoints are allowed. Retesting surviving rows gives the following counts:

h	mixed	Q^2	Q^3	Q^4
3	2	0	0	0
4	11	5	1	0

The single class after Q^3 is $(Q, u, v, j_u, j_v, \epsilon) = (7, 103, 149, 2, 4, -1)$. Thus neither height admits an odd ordinary minimal root with $P = 3$, for any other odd prime Q or any $k \geq 2$. The height-two case with $P = 3 < Q$ is covered by Theorem 10.13. By the same reduced-case exclusion, one-high-prime descent and cubic lift as in the dyadic projection, every divisor of $3^9Q^Au^3v^2e$ is covered for every odd prime $Q > 3$, arbitrary $A \geq 0$, any further distinct primes u, v, and coprime square-free e. An even singleton cannot yield an ordinary minimal obstruction by Corollary 10.15.

Consequently an ordinary obstruction to the unordered $(8, 4, 2, 2)$ family must be odd, with the octic prime least and at least 23. The four dimension thresholds do not require the finite windows of this section.

15 Five repeated primes and local Weyl columns

Theorem 15.1. *For arbitrary distinct primes p, a, b, c, d, the integer $p^4(abcd)^2$ is covered. Any three of the four squared exponents can be raised to three.*

The square of a product of five primes gives the starting case. Its proof also yields the unbraided solvability assertion of Theorem 1.1. We then pass to the higher exponents by studying the supports of principal witnesses.

Lemma 15.2 (Intersecting degree supports)**.** *A family of pairwise-intersecting two-element sets is contained in a star or in a triangle. In the graph of two-position degrees, disjoint edges are impossible whenever their two labels are principal at a common prime omitted by both.*

Proof. Given edges ab, ac, either all contain a, or there is an edge bc. In the latter case any edge meeting all three lies in the triangle. The cases with at most one edge are immediate. In the stated application disjoint edges give coprime degrees, hence zero crossing; common principality at an omitted prime gives a nonzero residue. □

15.1 The square of a product of five primes

Theorem 15.3. *Let n be the product of five distinct primes. Every weakly integral braided fusion category of dimension dividing $n^2 d$, with d square-free and coprime to n, is WGT.*

Proof. At a least counterexample among its divisors, losing a repeated prime is covered by Theorems 4.16 and 11.4. The reduced case has either two-valuation one or odd reduced dimension, excluded by Lemma 2.5 and Proposition 4.8, respectively. Thus the table is ordinary with square root n.

Every isolated singleton is at least five by Lemma 6.7.

Suppose first some twist contains two primes a, b. Put $C = ab$ and let $u < v < w$ be the others. Hits on that column omit a, b by defect-zero vanishing. Reducing its vacuum orthogonality on P_a and P_b at each of u, v, w forces the three pair degrees uv, uw, vw in both blocks. Lemma 9.2 separates u, v, w from a, b. A twist meeting two of u, v, w would share only the vacuum hit with the mixed column. Thus the last three primes are isolated, and are odd. For any two of them s, t with partner x, a nonunit degree prime to st must contain both a, b: otherwise it must hit degree st in the corresponding principal block, contrary to coprime vanishing. It cannot be C, since C and st would be coprime x-free degrees in P_x. Thus its only possible value is Cx, and the mixed column norm ensures it occurs. At s the distinct proper degrees tx, Ct, Cx share the norm $j_s(s+1-j_s) \le (s+1)^2/4$, so each signed coordinate is centered. Corollary 9.11 now rules out this triple of isolated primes.

If all primes are isolated they are odd. Each proper degree occurs with its distinct dual, so again its signed coordinates are centered. The two-prime degrees form an intersecting graph by Lemma 15.2: two disjoint edges omit the fifth prime, where both are principal. If $\{s, t\}$ meets every edge, the only common nonunit degree is $n/(st)$ by the mixed norm identity. It occurs at least twice by duality. No vertex w has two distinct smaller partners of this kind, by Corollary 9.11. This excludes a graph supported on at most three vertices: every pair in a containing triple meets every edge. Otherwise Lemma 15.2 gives a star with at least three leaves. For its center h every pair $\{h, t\}$ meets every edge, so at most one prime is smaller than h. Choose larger leaves c, d and a third leaf b. A label of degree hb is unramified at one of c, d, say c, by Proposition 6.3. Pair it with a label of degree hd. Both degrees omit c and the fifth prime, hence both labels are principal at each by Proposition 6.4. Their normalized entry has strict house below h and residue h at both primes, contradicting Proposition 4.12 since $c > h$. □

Corollary 15.4 (Solvability without a braiding assumption)**.** *Every fusion category of square-free dimension with at most five prime divisors is solvable; if integral it is group-theoretical. For four*

distinct primes with $\min(u,v) < \min(p,q)$*, every fusion category of dimension dividing* uvp^aq^b *is solvable, with arbitrary* $a, b \geq 0$*. In particular this includes* $2pq\ell^h$ *for any distinct odd primes* p, q, ℓ*.*

Proof. Pass to the Drinfeld center, whose dimension is the square of the given dimension. The five-square theorem applies in the first case. In the second, Corollary 10.15 excludes every minimal obstruction, since the least prime would have to be p or q. WGT descends from the center by Morita invariance and closure under subcategories [32, Proposition 4.1]. In both cases $4 \nmid \mathrm{FPdim}(\mathcal{C})$. To prove solvability, and for later use at the odd cutoffs, recall the group-extension argument. A WGT category is Morita equivalent to a nilpotent category built by extensions; the group orders multiply to its dimension. If 4 does not divide the dimension, each group has odd order or order twice odd. In the latter case the sign of the regular permutation representation gives an odd-order kernel of index two. The odd-order theorem [33, p. 775] and [32, Proposition 4.5(i)] now give solvability. In integral square-free dimension use also [32, Proposition 9.1]. □

Proof of Theorem 1.1. For $p < q < r < s$, Corollary 10.15 covers all divisors of $p^2q^2r^as^bd$: the least prime of an obstruction would have to be r. Theorem 11.4 raises both smaller powers to cubes, since q is odd and $rs \nmid q - 1$. The assertion without braiding follows from Corollary 15.4 with p, q as the two singletons. □

15.2 Prime support of principal witnesses

Work in a perfect fusion-simple ordinary integral modular table with dimension Δ and prime support P. A squared prime has valuation two in Δ. A witness (s, A) means that the principal s-block P_s contains a nonunit whose degree is supported in $A \subseteq P$. If $|A| < 2$, no such witness exists. If $(s, A), (t, B)$ are witnesses with $A \cap B = \varnothing$ and at least one of $s \notin B$, $t \notin A$ holds, their entry vanishes by coprimality and is nonzero by principality (Lemma 2.5). For increasingly ordered odd primes, a two-element $A = \{u, v\}$ with $u < v$ and $v_u(\Delta) = 3$ is also impossible, since $v \mid u - 1$ by Lemma 11.1.

Lemma 15.5. *Call two primes mixed if one column is active at both. Every prime of odd valuation in* Δ *mixes with every other prime. The following rules produce witnesses.*

1. *Let* J *be* s*-active and let* $q \neq s$*. In a witness* (s, A) *one can omit* q *and every squared prime at which* J *is active. One can also omit* s *if* $v_s(\Delta) \leq 3$*.*

2. *Let* J *be* s*-active, and* $w \neq s$ *a squared prime. There is a witness on* P_s *omitting* w*, every squared prime active at* J*, and one prescribed prime* $q \nmid d_J$*. One may instead take* $q = s$ *whenever* $v_s(d_J) \leq 1$*, in particular if* $v_s(\Delta) \leq 4$*.*

Proof. Categorical Cauchy gives an active column at each prime. If no column were active at both s, u, every u-active column would have s-unit codegree (Lemma 3.4). This is impossible if $v_s(\Delta)$ is odd since $f_J = \Delta/d_J^2$. For the first rule apply prime avoidance on P_s at q (Proposition 4.2). A hit on J has degree prime to every squared active prime by defect-zero vanishing (Theorem 3.2). The extra omission follows from Corollary 4.1.

For the second rule, choose a w-active, s-inactive column K as follows. The product of the complementary Weyl elements sends the vacuum into the trivial s-twist fiber. Its projection to the w-active sector is nonzero, since the s-local Weyl preserves that sector and completes the product to the global Weyl, whose vacuum column has full support. The J, K columns are orthogonal on P_s, with vacuum contribution one. If $q \mid d_X$ and $q \nmid d_J$, then $\chi_J(X) = (d_X/d_J)\chi_X(J)$ vanishes modulo a prime above q. For $q = s$ this still holds if $v_s(d_J) = 1$: the quotient is locally integral and $\chi_X(J) \equiv d_J \equiv 0$ by principality. Thus there is a q-free common nonunit hit. Defect-zero vanishing omits w and all squared primes active at J. Finally $s \mid f_J$ by Lemma 3.4, so $2v_s(d_J) \leq v_s(\Delta) - 1$.

□

Allowing more than two active primes on one column. The first rule of Lemma 15.5 applies to every squared prime simultaneously active on the chosen column. In an ordinary minimal table with $k \geq 3$ dimension primes, choose $k-2$ primes of which one, s, has valuation two or three and all the others have valuation two. No column can be active at all $k-2$ primes. Otherwise apply prime avoidance on P_s at one of the two complementary primes. The required nonunit omits that prime, s by the principal valuation bound, and each other chosen prime by defect-zero vanishing. Its degree is supported at most at the last complementary prime, contrary to Lemma 2.5. Both complementary valuations are arbitrary.

15.3 A cube and two squares among five primes

Theorem 15.6 (Two squares beside a cube)**.** *A divisor-minimal braided obstruction with exactly five repeated primes cannot have one valuation three and two other valuations two. The two remaining valuations may be arbitrary. Moreover, for five distinct primes $p_1, \dots, p_5$, every weakly integral braided fusion category of dimension dividing*

$$(p_1p_2p_3p_4p_5)^3 d$$

is weakly group-theoretical, where d is square-free and coprime to the five primes.

Proof. Denote the cubed prime by p, the squared primes by u, v, and the others by a, b. In the ordinary case p mixes with u, v by Lemma 15.5. Its first rule, applied to P_p with additional omitted prime b or a, gives respective witnesses $(p, \{v, a\})$ and $(p, \{u, b\})$. Their disjoint supports contradict principality.

For a positive reduced table, a squared prime two would leave two-valuation one in the reduced dimension, contradicting its square-fullness. If p is odd, then u, v are odd and the same prime avoidance and defect-zero arguments apply on the positive table. The joint columns exist because a nonempty u- or v-active fiber contains a p-active pair at odd p-valuation (Lemma 7.8). The argument also permits $a = 2$ or $b = 2$.

If $p = 2$, pass to the integral minimal modular extension with fermion g. Its full principal two-block has only odd degrees. Its noninvertible degree supports intersect pairwise and have empty total intersection (Proposition 4.8), so none is a singleton. At each $w = u, v$ take an even w-active label by reduced Cauchy and choose its fermion mate if necessary to make its actual dyadic twist nontrivial. In the full block orthogonality the two invertibles $1, g$ contribute two because the column is even. Reducing modulo the odd primes b, a in turn therefore gives the same disjoint supports as above. Coprime vanishing holds in the full extension by Lemma 4.4, contrary again to two-principality.

For the divisor assertion, a drop to four or fewer repeated primes is covered by Theorems 4.16 and 11.4 and the two-prime theorem. If all five survive and at least two are squared, either a cube occurs or the five-squares theorem (Theorem 15.3) applies. If precisely one square remains, there are four cubes and one square w. Choose three distinct odd cubed primes p, u, v with $u < v$. A joint p, w-active column exists as above (a squared two in a reduced source is already impossible). Orthogonality on P_p modulo the fifth prime forces a degree of prime support $\{u, v\}$. By Lemma 11.1, the odd cubic valuation at u requires $v \mid u - 1$, a contradiction.

It remains possible to cube the fifth prime as well. Suppose every valuation is three and write h for the least prime. No degree can have exact support $\{a, b\}$ for two odd primes $a < b$ among the five, by Lemma 11.1. Fix distinct primes $p, s \neq h$. Every s-free nonunit in P_p has degree divisible by h: it omits p, and omitting h as well would leave at most the two remaining odd primes. The same holds with p, s exchanged.

In an ordinary table all these witnesses have h-defect one. If h is odd, since it is least, their two h-block orbits have equal degrees; only 1 and one square class of primitive hth roots are visible on each row, as in Theorem 7.11. At $h = 2$ the same conclusion uses one primitive fourth root in place of the class (Lemma 7.5). A witness in P_s hits all the s-free active mates of the

witnesses in P_p, so both witness sets share one allowed class. Prime avoidance now places every p- or s-active label in the allowed h-phases. There is a phase outside this list by Lemma 7.4 or 7.6, using odd dimension for odd h. On its fiber Lemma 7.9 gives p-unit codegree, incompatible with $v_p(\Delta) = 3$.

In the reduced case $h = 2$ has positive valuation two. The complementary witnesses for any two odd primes p, s have dyadic height one and singleton positive two-blocks. Corollary 7.13 excludes their odd valuations. □

15.3.1 Four squares beside an unrestricted power

Theorem 15.7 (Five-prime lifting)**.** *Fix five distinct primes p, a, b, c, d, a nonnegative integer H, and a square-free integer e coprime to them. Suppose every weakly integral braided category of dimension dividing $p^H(abcd)^2e$ is WGT. Then the same holds for every divisor of*

$$p^H(abc)^3d^2e.$$

Any one of the four companions may be assigned to the squared position. All four may be cubed if their least member h is less than p and at least one of the other three, say v, satisfies $p \nmid v - 1$.

Proof. Theorems 4.16 and 11.4 cover the loss of a repeated prime since at most one unrestricted exponent remains. By Lemma 11.3 it remains to test patterns containing a cube. If there is a cube and at least two squares among the designated positions, Theorem 15.6 applies. If there are three cubes and one square d, choose two odd cubed primes $u < v$ and call the third t. On P_t, use a joint t, d-active column and reduce vacuum orthogonality modulo p. As in the proof of Theorem 15.6, every hit omits t, d, so the reduction forces a nonunit with support exactly $\{u, v\}$. If $t = 2$ in the fermion case use the full two-block: the two invertibles contribute two and p is odd. A squared two is impossible in a reduced source. In every case Lemma 11.1 would give $v \mid u - 1$. This proves the first assertion.

For the last assertion it suffices to exclude four exact cubes. Denote by q, s the two cubed primes other than h, v. An s-free nonunit of P_q omits q. If it also omitted h, it would have support $\{v, p\}$; since v is an odd cubic prime, Lemma 11.1 would force $p \mid v - 1$. Thus every such nonunit is h-bearing, as is every q-free nonunit of P_s. These are precisely the witness properties used in the all-cubed part of Theorem 15.6: h is the least prime of the minimal table, h, q, s have odd valuation three, and the opposite witnesses have h-height one. In the ordinary case their defect-one blocks confine the h-twists on the q-active columns; a missing phase then forces q-unit codegree despite cubic valuation. In the reduced case $h = 2$, and their positive singleton blocks force a common dyadic phase on the odd sector by the same full-extension argument. Neither argument uses the valuation at p. □

15.4 The slightly degenerate case of one quartic and four squares

If the quartic prime is odd, a squared two would give $v_2(\Delta) = 1$, contrary to square-fullness. If two is absent or occurs in the coprime square-free factor, the source or its positive quotient would have odd dimension, excluded by Proposition 4.8. Thus it suffices to consider a quartic at two.

Proposition 15.8 (A quartic two and four odd squares)**.** *There is no reduced minimal obstruction of source dimension 2^4n^2 for n a product of four distinct odd primes.*

Proof. Write $\Delta = 8n^2$, let P be the positive principal two-block, and choose the integral minimal modular extension. By Lemmas 4.4–4.5 and Proposition 4.8 the degrees in P are odd, its rank is divisible by eight, and no two distinct primes of n can occur as their exact product in a simple degree. The positive quotient is perfect and simple, hence has no nonunit prime-power degrees. Each nonunit of P therefore has degree n or n/a for some prime $a \mid n$.

Call an even pair deep when its squared dyadic twist is nontrivial. If a deep pair were also a-active for $a \mid n$, its row would vanish on every a-bearing degree by defect-zero vanishing. On P its row is orthogonal to the vacuum by reduced dyadic block orthogonality. Reducing there modulo a different $b \mid n$ leaves only the unit: no nonunit in P omits both a, b. Thus no odd prime is active on a deep pair.

A deep pair exists: otherwise the even dyadic twists are signs, contrary to Lemma 8.24 at odd $v_2(\Delta)$. Each deep squared-twist fiber is inactive at every odd prime. By the positive-fiber assertion of Lemma 7.9, its labels have pure dyadic positive codegree. It is 2 or 8. A codegree of two leaves room for at most one nonunit hit besides the vacuum by the mean-square bound. The dimension orthogonality equation then implies a nonunit degree divides one. Hence every deep pair Z has $d_Z = n$, $f_Z = 8$. Its row hits every label of P by principality and oddness of n. Since the row norm is eight and $|P|$ a positive multiple of eight, $|P| = 8$; every entry there has Galois mean squared modulus one and is a root of unity, and the row vanishes elsewhere. This last assertion uses the arithmetic–geometric mean inequality and Kronecker's theorem for each nonzero algebraic integer.

The odd Weyls preserve each deep line, so Lemma 8.25 puts its label in P. Dimension orthogonality on a deep row modulo each $a \mid n$ requires a label of degree n/a. With these four degrees and the vacuum there are at most three deep labels. A power of the squared twist is $I - 2E$, where E projects onto the m pairs of maximal squared twist order, $1 \le m \le 3$. Their K-columns agree with the reduced columns up to phase and have squared modulus $1/8$ on these m coordinates. The lemma gives $4\,\mathrm{Tr}(EKEK^{-1}) = m^2/2 \in \mathbb{Z}$, so $m = 2$.

On each degree n/a occurring once in P, both maximal-twist characters take the same value, 1 or -1. Indeed P is Galois stable, so its unique label of that degree has rational character field. The sign is determined by reduction of the dimension equation modulo a. At least three of the four degrees n/a occur just once because P has size eight and contains the unit and the two maximal-twist labels. If all four occur once, the two rows agree on at least five of their eight unit-modulus coordinates, violating orthogonality. Otherwise they agree on four coordinates including the unit, and hence they are opposite on each of the other four. In particular their sums on the two labels at the repeated degree n/a are negatives of one another. Both sums are congruent to $-(n/a)^{-1}$ modulo a by dimension orthogonality. Since a is odd, this is impossible. □

15.5 One quartic and four squares in the ordinary case

Write $n = p^2abcd$ and $\Delta = n^2$, and let $\beta = m/(m-1)$ for the least prime $m \mid n$. Two primes *mix* if some twist is active at both; a prime with no partner is *isolated*. In this section a *singleton* is a prime of valuation one in n.

We recall three facts about an isolated singleton t from Section 6. Each t-free label lies in P_t. Its signed coordinate in the local Weyl column is a nonzero integer off the vacuum, and the sum of their squares over *proper* degrees (smaller than n/t) is at most $B_t = j_t(t+1-j_t) \le (t+1)^2/4$, for a positive integer j_t. On proper t-free labels unramified at t we have the stronger bound

$$1 + \sum_X [d_X]_t^2 \le (t+1)/j_t,$$

where $[d]_t$ denotes the centered residue. There are at least three distinct proper t-free degrees. Consequently all signed coordinates at these degrees are centered: an uncentered coordinate would exhaust B_t by itself. An isolated singleton is at least five. For two isolated singletons $u < v$, if the only nonunit degree omitting both is $n/(uv)$, then $v \mid n/(uv) \pm 1$. For three such primes $u < v < w$, this unique-degree property cannot hold at both pairs u, w and v, w (Lemma 9.10 and Corollary 9.11). Their mixed column norm always supplies at least one common degree.

Lemma 15.9. *Suppose X has degree Ax, where x is a singleton and A is a power of p. For each singleton $q \ne x$ there is a normalized crossing of X with common weight at most A and*

positive residue equal to that weight at q. Thus $q > \beta A$ forces K_X to ramify at q. If $y \neq q, x$ is an isolated singleton and K_X is unramified there, then $q \leq \beta A y$.

Proof. Choose an x-free nonunit of P_q by prime avoidance. It hits X by q-principality. If e is the inertia index at q of the normalized entry field, Lemma 9.6 gives $q \leq \beta e \gcd(d_X, d_Y) \leq \beta e A$. All inertia translates of X are proper y-free labels. If unramified at y, there are at most y of them by the displayed bound. Their number is at least e because the entry lies in K_X. □

15.5.1 Which primes can mix

Proposition 15.10 (The unique possible mixed pair)**.** *In an ordinary minimal table with square root p^2abcd, at most one pair of distinct primes can mix. If p, a mix, the other three primes $b < c < d$ are isolated and all their pairwise products occur in P_p. For any two of them with third prime x, the only possible common nonunit degrees are px and p^2ax, and at least one of pb, pc occurs. The quartic prime cannot be least when all five primes are odd.*

If two squared primes u, v mix, write c, d for the remaining squared primes. Then p, c, d are isolated. The sole nonunit degree omitting both c, d is p^2uv. At each $t \in \{c, d\}$, writing $w = cd/t$, the sole p-free nonunit degree in P_t is uvw. Both p^2c, p^2d occur in each of P_u, P_v.

Proof. If p, a mix, the second rule of Lemma 15.5 supplies a witness on P_p omitting p, a and any specified one of b, c, d. Prime-power degrees are impossible, so all three pair degrees bc, bd, cd occur on P_p.

A second partner would force a disjoint pair of these p-free degrees on P_p, contrary to principality and coprime vanishing. In the one-partner case the other three primes are isolated. Indeed, an occurring degree uv at two squared primes forces every nonunit of P_t, $t \notin \{u, v\}$, to carry u or v, by t-principality. A joint t, u-active column would yield a v-free nonunit hit on P_t by prime avoidance, whereas each hit omits u. The same holds for v. Applying this to bc, bd, cd proves isolation. For any two of the three isolated primes with third member x, a common nonunit degree can only be px or p^2ax. Indeed on omitting the two endpoints a p-free degree cannot hit their coprime product in P_p. A degree $p^i a$ misses x and would have to hit the same product by x-principality. A degree p^2x belongs to P_a by full p-height in the cluster $\{p, a\}$, and a degree pax belongs to P_p by full a-height. Thus both encounter the same coprime product. This exhausts the possible degrees. If px occurred for neither $x = b$ nor $x = c$, the pairs c, d and b, d would have unique common degrees $n/(cd), n/(bd)$. At each endpoint another proper degree (the product of the other two isolated primes) ensures centering, since one uncentered coordinate exhausts the residual budget. This contradicts Corollary 9.11.

Choose one such px. At each of the other three companion primes $q > \beta p$ its field ramifies by Lemma 15.9. These primes belong to distinct twist clusters, so at most one can ramify. If p were least and odd, each companion would exceed $\beta p = p^2/(p-1)$, a contradiction.

Now suppose u, v are squared and mix. The preceding case forbids either from mixing also with p. On the mixed column J every hit omits u, v by defect-zero vanishing. Vacuum orthogonality on P_u reduced successively at p, c, d forces supports cd, pc, pd in P_u, since the lesser supports have prime-power degree. The same holds on P_v. For any prime $t \notin \{c, d\}$ every nonunit of P_t hits the t-free degree cd, so contains c or d. A t, c-mixed column would require a d-free P_t-hit, by vacuum orthogonality, whereas all its hits omit c. The same argument separates d from t. Finally a c, d-mixed column would share only the vacuum hit with J. Thus c, d are isolated and at least five (Lemma 6.7).

The mixed norm at the isolated columns c, d gives a common nonunit omitting both by Proposition 6.4. Any such X must contain u, v: otherwise it would have to hit a coprime cd from the corresponding principal block. If its p-height were less than two, the complementary cluster $\{p, u, v\}$ puts X in P_p by full height at u, v (Proposition 6.3), again impossible at the p-free degree cd. Thus $D = p^2uv$ is the unique common degree.

The prime p is isolated by the first case and the isolation of c, d. A witness of degree $p^i c$ on P_u or P_v cannot have $i = 1$: it would then belong to P_p and have to hit a coprime p-free nonunit of P_c. The same argument at d supplies the two squared axes in the assertion. Finally, a p-free nonunit of P_t must meet both $p^2 t$ witnesses in the opposite principal blocks whenever it omits u or v. Thus it contains both u, v, and it must also meet $p^2 w$ by t-principality. This forces uvw. Prime avoidance gives its existence. □

15.5.2 A quartic at two

Proposition 15.11 (Separation among four odd squares)**.** *Let an ordinary minimal table have dimension $16M^2$, with M the product of four distinct odd primes. No two odd primes occur in the same twist. No degree has the form $2a$ for $a \mid M$ prime. A degree $4a$ can occur only if at most one of the other prime factors s of M satisfies $s \geq 11$ or mixes with two and satisfies $s \geq 5$.*

Proof. Suppose a column is active at odd primes $a < b$, and write $c < d$ for the other two. Every hit on this column omits a, b by defect-zero vanishing. On each of P_a, P_b, reducing vacuum orthogonality successively at $2, c, d$ forces all three supports $cd, 2d, 2c$, since a nonunit cannot have prime-power degree. Cross a label of support $2c$ in P_a with one of support $2d$ in P_b. Their lcm-normalized entry z has strict house below its weight $g \in \{2, 4\}$ and has residue g at both a, b. The short-word theorem (Theorem 4.13) forces $a = 3$ and $g = 4$. If $b \geq 7$ it gives $z = 1 + 3\xi$ for a nonidentity b-power root. Its three difference positions $1, \xi, \xi^{-1}$ are exposed since $3 < b/2$, and a tame dilate intersects them only at one. The difference-set lemma (Lemma 4.20) excludes this. Hence $b = 5$ and K_{4c} ramifies at five by the two-residue inertia bound (Proposition 4.12). In particular all pairs of odd primes at least five have already been separated.

Here $d \geq 11$. Choose a c-free nonunit of P_d by prime avoidance. Its crossing with the label of degree $4c$ has common weight $h \leq 4$ and residue h above d. If it were unramified there, $(z' - h)/d$ would be an algebraic integer of strict house less than $2h/d < 1$, contradicting strictness. Thus K_{4c} also ramifies at d. This contradicts separated inertia (Proposition 6.3) at $5, d$ and proves separation.

Apply the same argument to a putative label of degree $2a$: each $s \neq a$ dividing M with $s \geq 5$ gives an a-free witness in P_s, now of common weight at most two. There are at least two such s, forcing simultaneous ramification. For $4a$ the ordinary prime-avoidance argument applies at each $s \geq 11$. When s mixes with two, use a joint active column for this avoidance; every hit on it has two-height at most one, so $s \geq 5$ already suffices. □

Corollary 15.12. *In an ordinary minimal table of dimension $16(abcd)^2$, every prime is isolated.*

Proof. The four odd primes are pairwise separated by Proposition 15.11. If two had a partner, the one-partner proof would force a degree $2x$ for one of the other three singletons. The same proposition rules this out. □

15.5.3 All five primes isolated

If two were a singleton it could not be isolated. Thus either $p = 2$ or n is odd. We first show that a degree cannot contain p to exponent exactly one in either case.

Lemma 15.13 (No intermediate height at an isolated odd quartic)**.** *In an ordinary minimal table of odd dimension n^2, suppose h is isolated and $v_h(n) = 2$. Then every simple degree has h-valuation zero or two.*

Proof. Any label of h-valuation one and its distinct dual give two such labels. The Gauss calculation in the proof of Lemma 6.8 then gives

$$\sum_{\substack{X \ h\text{-active} \\ d_X = n/h^2}} (1 - \theta_X) = h^2.$$

The summation set is a union of dual pairs with equal twists, so its sum is twice an algebraic integer. This is impossible for odd h. □

The intermediate height at two. Put $M = abcd$. We first exclude all degrees of two-valuation one. Two such labels are already impossible by Lemma 6.8. Suppose there is just one, X. The proof of that lemma up to the Gauss identity still gives $\delta_0 = 4$ and $2 \le |A_0| < 4$, where A_0 consists of the dyadically active labels of odd degree M and $\delta_0 = \sum_{Z \in A_0}(1 - \theta_Z)$. Indeed the upper bound uses only X and the mean square bound on its crossings with A_0; the lower bound uses only the Gauss identity. If a label J of degree $2M$ were dyadically active, the final paragraph of that proof applies without a second height-one label: its row vanishes on all even degrees and hits the vacuum, A_0, and three distinct proper odd degrees, impossible at codegree four. All active labels therefore have degree M. The Gauss identity gives $|(4W_2v_0)_0| = 3$. The integral vector $4W_2v_0$ has squared norm 16. The unique height-one label costs at least four, the vacuum nine, and each odd nonunit contributes at least one, by Proposition 6.3. Besides A_0 there is room for at most one proper odd nonunit label Y. On the row of an active Z all degree multiples of four vanish. Reducing dimension orthogonality at each prime of M thus forces that prime to be missing in X or Y, since all remaining nonunit degrees on the row equal M. If Y is absent then X has pure dyadic degree; if present then $\gcd(d_X, d_Y) = 1$, impossible because $X \in P_2$ and d_Y is odd.

In both cases the degrees are now subproducts of the five pairwise coprime integers $H = p^2, a, b, c, d$. Join two positions if their product occurs as a degree. Two disjoint edges would be coprime rows on the principal block of the fifth prime, where principality forces them to hit each other. If two singleton vertices s, t meet every edge, the sole possible nonunit degree omitting both is $n/(st)$, and it occurs by the mixed column norm. By the three-singleton result recalled above, no singleton vertex has two smaller singleton partners with this property.

At least one degree Hx occurs. Otherwise all edges are on the four singleton vertices. If supported on at most three of them, every pair in a containing triple would meet every edge. Otherwise Lemma 15.2 gives a star on four vertices. Its center h has two larger leaves c, d; let b be the remaining leaf. A label of degree hb is unramified at one of c, d, say c, by separated inertia. Its crossing with a label of degree hd has weight h, is strictly smaller in house, and has positive residues at c and p by principality. Since $c > h$, Proposition 4.12 forces ramification at c, a contradiction.

For $p = 2$ there cannot be two edges $4a, 4b$. Their weight-four crossing would ramify at both other singleton primes, each at least five, by the same two-residue result. As these primes are isolated this is impossible. Thus $4a$ is the unique edge on four and every edge meets a. At most one other odd prime is smaller than a by the three-singleton result. On the other hand Proposition 15.11 allows at most one of the other three to exceed seven, so the other two are $5, 7$, both smaller than a. This excludes the dyadic quartic.

15.6 Three isolated primes in odd dimension

For the rest of the odd five-prime argument put $H = p^2$ and $\beta = m/(m-1)$, where m is the least prime. The following calculation allows an arbitrary complementary factor G.

Lemma 15.14 (Centered Weyl column for a separated set of odd primes)**.** *In an odd ordinary minimal table of square root $n = HN$, let $H > 1$ be the full part of n supported on a nonempty set of odd dimension primes which is separated from its complement: every twist is supported on one side or the other. Put $J = H(1 - (W_H)_{00})$ and $a_H(X) = \epsilon H(W_H)_{X0}$ for nonunits, with $\epsilon = \xi$. Then J is a positive even integer and*

$$JN \equiv -\epsilon \pmod{H}, \qquad a_H(X) = [d_X/N]_H.$$

In particular its nonunit support is exactly the degrees deficient at some prime of H. No assumption on the internal prime powers or internal separation in H is needed.

Proof. The split Weyl column is integral after scaling each local column by its part of n (Proposition 6.3); reduction of the vacuum and nonvacuum entries modulo H gives the displayed congruences before centering. Every H-active label Z has degree Nk_Z with $k_Z \mid H$ by complementarity; on its active subspace $s = \epsilon W_H$, so $a_H(Z) = k_Z$. Here $k_Z < H$ since $f_Z = 1$ is impossible in a nontrivial table, and thus $k_Z \le H/3$. Choose an even exponent projecting the modular level to its odd H-part. The vacuum Gauss identity of Lemma 6.2 gives

$$HJ = \sum_{Z \ H\text{-active}} k_Z^2(1 - \theta_Z).$$

Since H is odd, equal-twist dual pairs show J is even. Its positivity follows by taking real parts (a nonidentity phase exists by Cauchy). Let $m \ge 3$ be the least prime of H and $\beta = m/(m-1)$. The Galois mean of each $1 - \theta_Z$ is at most β by the Ramanujan mean formula. Subtracting the vacuum and the active coordinates from the norm H^2 leaves at most

$$2HJ - J^2 - HJ/\beta \le (2 - 1/\beta)^2 H^2/4.$$

Distinct duals have the same signed coordinate (the Weyl column is rational and invariant under duality). Each inactive nonunit coordinate therefore has magnitude at most $(2-1/\beta)H/(2\sqrt{2}) < H/2$, since $\beta \le 3/2$. Together with the active bound, this proves centering. □

Proposition 15.15 (Three isolated prime positions with full complementary degrees)**.** *Let $n = QGcd$ be an odd ordinary minimal square root, where $Q = p^2$, p, c, d are distinct isolated primes, $c < d$ are singletons, $G > 1$, and $\gcd(G, pcd) = 1$. For each pair of the three positions Q, c, d, suppose that every nonunit degree prime to both equals the full complementary divisor: QG, Gd, Gc for the pairs $(c, d), (Q, c), (Q, d)$, respectively. Then degree Qc cannot occur.*

Proof. For $H = Q, c, d$ let $z_H = (W_H - I)v_0$. Its vacuum coordinate is nonzero and its other coordinates are supported precisely on the degrees deficient at H, by Lemma 15.14. For $H = Q$ this means precisely the p-free degrees, by Lemma 15.13. The vectors for distinct positions are orthogonal by (6.1), so every pair has a common nonunit coordinate to cancel the vacuum product. The pairs (c, d) and (Q, d) therefore supply the degrees QG and Gc, respectively.

At c, d the only common nonunit degree is $QG = n/(cd)$. Since n is odd, Lemma 9.10 gives $d = cj_d \pm 1$, with j_d even and positive by Theorem 10.3. Hence $d \ge 2c - 1 > 3c/2$. If Qc occurred, Qc and Gc would be distinct proper d-free nonunit degrees, with greatest common divisor c. The same theorem gives $d \le \ell c/(\ell - 1) \le 3c/2$ for the least dimension prime ℓ, a contradiction. □

Corollary 15.16 (Odd roots with five repeated primes)**.** *In an odd ordinary minimal root $n = p^2uvcd$ at five distinct primes, no two squared primes mix. Under full isolation there cannot be exactly two degrees of the form p^2x with x a singleton.*

Proof. If u, v mix, put $G = uv$ and order the isolated singletons $c < d$. The mixed-square structure supplies p^2c, p^2d, Gc, Gd and the sole common degree p^2G at c, d. For $\{t, w\} = \{c, d\}$ every p-free t-free nonunit is t-principal by isolation and has degree Gw by the same structure. This is precisely the excluded configuration.

In full isolation the possible degrees are subproducts of the five coprime factors p^2, u, v, c, d (no intermediate p-height). Write p^2c, p^2d for the two stated degrees, with $c < d$. Any two-factor degree on the ordinary primes must meet both of these edges, so only cd is possible. For each pair of positions in p^2, c, d, a nonunit degree omitting the pair is therefore the full product of the other three factors: one factor is forbidden by the exclusion of prime-power degrees, and a two-factor degree would be a further p^2-edge or an ordinary edge other than cd. The same proposition again contradicts the degree p^2c. □

15.7 Full isolation in odd dimension

Proposition 15.17 (Two axes at an isolated quartic)**.** *Let an ordinary minimal table of odd square root $n = p^2abcd$ have all five primes isolated. Put $H = p^2$. There are at least two distinct companion primes s, t for which Hs, Ht occur as simple degrees. Consequently at least two of the four companions are strictly smaller than $\beta p^2/2$, where $\beta = m/(m-1)$ and m is the least prime dividing n.*

Proof. By the no-intermediate-height lemma, all degrees are subproducts of H, a, b, c, d. The graph whose edges are occurring two-factor degrees has pairwise intersecting edges. The argument in Section 15.5.3 shows that an H-edge occurs. Suppose it is uniquely Hs. Every edge then meets s. For each other companion w the only nonunit degree omitting s, w is $n/(sw)$, and it occurs. The unique-complementary-degree lemma applies at all these pairs.

Corollary 9.11 permits at most one companion smaller than s. For each of the at least two companions $w > s$, Lemma 9.10 gives signed s-coordinate ± 1 at $n/(sw)$. This odd coordinate forces ramification at s. These are distinct proper degrees, whereas Theorem 10.3 allows only one proper ramified orbit. This proves the first assertion.

Finally choose any two H-edges. Their normalized crossing has weight H and positive residues at the two omitted singleton primes. The two-residue estimate at an isolated prime, applied in both orders, puts each omitted prime below $\beta H/2$. □

Proposition 15.18 (Crossings of two quartic axes)**.** *In an odd, fully isolated minimal root p^2abcd, let Hx, Hy be two occurring degrees, where $H = p^2$. For the complementary companions z, w one has $zw \le \beta H$. In particular p cannot be least.*

Proof. The crossing has weight H and positive residue H at z, w. If nonrational, its field ramifies at a prime t common to the two character fields. By their codegrees $t \in \{z, w\}$. Both degrees would belong to the sole proper t-ramified orbit of Theorem 10.3, impossible since they differ. The crossing is therefore an integer $H - kzw$, $k \ge 1$, and its balancing mean is at least $-H/(m-1)$, where m is the least prime (Lemma 9.6). This yields the inequality. If p is least, $zw \ge (p+2)^2 > p^3/(p-1) = \beta H$, whereas Proposition 15.17 supplies two axes. □

Proposition 15.19 (No remaining quartic axes in odd dimension)**.** *An odd ordinary minimal table of root p^2abcd cannot have all five primes isolated and at least three occurring axes p^2t.*

Proof. Put $H = p^2$ and $N = abcd$. No ordinary pair can be an edge: it would miss one of the at least three H-edges. For each companion t, prime avoidance on P_t forces a p-free nonunit degree $E_t = N/t$, since no proper subproduct of the other three companions can occur. This is the only nonunit degree omitting both p, t.

Proposition 15.18 excludes p least. Order the companions $a < b < c < d$, so a is least, $a \ge 11$ and $\beta = a/(a-1)$. For every axis Hx and companion $w \ne x$, the degrees Hx, E_w are distinct and proper at w, with gcd x. Theorem 10.3 gives $w \le \beta x$. In particular Ha cannot occur since $b \ge a + 2 > \beta a$. The three axes are Hb, Hc, Hd and $d \le \beta b$; from $c \ge b + 2$ it follows that $b \ge 2a - 1$.

Fix a companion t. Its unique proper ramified orbit is given by Theorem 10.3. An E_t-label has codegree $(Ht)^2$, so its nonreal character field ramifies at p or t, and at only one by isolation. Its signed coordinate $B_t = [H^{-1}]_t$ has odd parity exactly in the latter case (Corollary 10.5); the choice is common to all its labels.

If Ht occurs, its nonreal field ramifies at another companion u, by its codegree. Were E_t ramified at t, the unique proper ramified degrees at t, u would be E_t, Ht respectively. The former contains u and the latter t, leaving no common ramified orbit at either endpoint, contrary to Corollary 10.5. Consequently all E_b, E_c, E_d ramify at p.

Choose a p-active row Z and a t-active row T. Their only possible common nonunit hits have degree E_t, by defect-zero vanishing and full heights. The rows are orthogonal with vacuum

contribution one. If E_t ramifies at p, the constant evaluation on T is $[E_t]_t$; it therefore divides one and $|B_t| = j_t$. If E_t ramifies at t, its labels form one orbit. The evaluations on Z are constant integers, since the two character fields are pure at distinct primes; the evaluations on T sum to $-\epsilon B_t$ by Theorem 10.3. Thus $|B_t| = 1$. Since $HB_t \equiv 1 \pmod t$, we have $Hj_t \equiv \pm 1 \pmod t$ in the first case and $Hj_t \equiv \pm j_t \pmod t$ in the second.

The fully separated Weyl sum on the vacuum, with the trivial auxiliary two-primary factor, gives

$$H = J + \sum_{t \in \{a,b,c,d\}} Hj_t/t + \epsilon/N, \qquad J = H(1 - (W_p)_{00}) \in \mathbb{Z}$$

(Lemma 15.14). If E_a also ramifies at p, an integer equals $\pm 1/a \pm 1/b \pm 1/c \pm 1/d + \epsilon/N$. This has magnitude less than one, so it vanishes. Multiplication by the odd N gives four odd terms and the extra odd ϵ, an impossibility. If E_a ramifies at a, the integer instead equals $\pm j_a/a \pm 1/b \pm 1/c \pm 1/d + \epsilon/N$. It again has magnitude less than one, since $3j_a \le a+1$, $a \ge 11$ and $b \ge 2a-1$. But it is nonzero: $j_a/a \ge 2/a > 3/(2a-1) + 1/N$. This exhausts the possibilities. □

15.8 The remaining odd configuration

Proposition 15.20 (The three isolated primes beside a mixed quartic)**.** *Let the odd ordinary minimal table have root p^2abcd, with a mixing with p and $b < c < d$ isolated. Then a is the least of all five primes. Put $\beta = a/(a-1)$. All three low degrees pb, pc, pd occur, and $2a - 1 \le b < c < d \le \beta b$.*

Proof. Put $\beta = m/(m-1)$ for the least prime m. By Proposition 15.10, the three pair products bc, bd, cd occur, as does at least one of pb, pc. Whenever px occurs, take any other singleton t and let y be the third. At t the degrees px, xy are distinct and proper with gcd x; hence $t \le \beta x$ by Theorem 10.3. Conversely, if px is absent, the only common nonunit degree of the other two singletons $r < t$ is $n/(rt)$. Lemma 9.10 gives $t = rj_t \pm 1 \ge 2r - 1$.

If pb were absent, pc would occur and give $d \le \beta c$, whereas $d \ge 2c - 1 > \beta c$. Thus pb occurs and $d \le \beta b$. Absence of either other low degree would force $d \ge 2b - 1 > \beta b$. The least prime cannot be b, since $d \ge b+2$, or p, by Proposition 15.10; hence it is a. Finally $b+2 \le d \le ab/(a-1)$ and parity give $b \ge 2a - 1$.

□

After loss of a repeated prime or a drop of the quartic valuation use Theorems 11.4, 4.16 and 15.6. The reduced core was excluded in Section 15.4. For the ordinary root p^2abcd the preceding witness arguments leave the following cases.

support or parity	obstruction or remaining argument
$p = 2$	degree edges at the dyadic quartic (Section 15.5.3)
odd, two squared primes mix	three isolated corners (Corollary 15.16)
odd, no two primes mix	quartic axes (Propositions 15.17–15.19)
odd, the quartic mixes	three-prime cycle below
2 squared	the dyadic partner cases of Section 15.10

15.9 The odd five-prime root

Suppose all five primes are odd. Permute the companion primes so that $a < b < c < d$. By Corollary 15.16 and Propositions 15.17–15.19 and 15.20, p mixes with a, a is the least of the five primes, and the remaining three primes are isolated. Mixing means that the order of some simple twist is divisible by both primes. Put $H = p^2a$ and $\beta = a/(a-1)$. We use the constraints

$$11 \le b < c < d, \qquad d \le \beta b.$$

For an isolated pair r, t with third prime x, the only possible common nonunit degrees (those prime to rt) are px and $Hx = n/(rt)$. We call these the low and high degrees of the pair. All six degrees bc, bd, cd, pb, pc, pd occur (Propositions 15.10 and 15.20).

For $t \in \{b, c, d\}$ put $m_t = n/t$. The signed Weyl coordinate of a nonunit t-free degree h is $A_t(h) = [h/m_t]_t$, with $m_t j_t \equiv -\epsilon \pmod t$. At proper degrees an odd coordinate detects ramification; an unramified label has $A_t(h) = -\epsilon j_t [h]_t$. We use the trace and norm of Proposition 6.4, in particular the mixed sum $-j_r j_t$ away from the vacuum and the bound $j_t(1 + \sum_{X \in L} [d_X]_t^2) \le t + 1$ for any set L of nonunit inactive t-unramified labels.

15.9.1 The three-prime cycle

Theorem 10.3 gives one inactive full orbit and one proper ramified orbit at each t, both of size j_t, and $\sum_{\text{proper}} A_t(d_X) d_X = \epsilon j_t$. Each pair needs a ramified orbit at an endpoint by Corollary 10.5. Its common degrees px, Hx, indexed by the third vertex x, are proper, and different pairs have disjoint degree sets. Orient each pair from its ramified endpoint: the three arrows form a directed cycle. All other proper degrees at t are unramified there, in particular the product of its two neighbors.

For a high degree on $\{u, v\}$ the two signed coordinates are the centered inverses $[v^{-1}]_u$ and $[u^{-1}]_v$. Indeed, the Bezout identity $v[v^{-1}]_u + u[u^{-1}]_v = 1$ follows by reducing modulo uv and using centering. Consequently exactly one coordinate is odd, and its endpoint ramifies. Thus whenever the high degree occurs, it accounts for the unique proper ramified orbit on its edge. If the low degree also occurs there, all of its labels are unramified at both endpoints. At t the compulsory degree rs, where r, s are the other vertices, is also unramified: its degree differs from both eligible degrees on each incident edge and only one proper orbit ramifies.

When the high degree is absent from an edge $u \to v$, only the low degree px contributes to its mixed Gram equation. Its multiplicity is j_u, its ramified coordinate has magnitude one, and $|[px]_v| = 1$; indeed the product of these last two integers has magnitude one after dividing the mixed equation by $j_u j_v$. In particular

$$pav \equiv \pm 1 \pmod u, \qquad px \equiv \pm 1 \pmod v.$$

Two consecutive low-only arrows $u \to v \to z$ would therefore give $pz \equiv \pm 1$ and $paz \equiv \pm 1$ modulo v. This is impossible since $1 < a < v - 1$. Thus at least two edges carry a high degree.

All three high degrees are forced. On a high edge $u < v$, the odd coordinate z at its ramified endpoint t satisfies $z^2 \le t + 1 - j_t \le t - 1$ by the singleton norm, hence $z^2 \le t - 2$ by parity. The even integer $(v - u)z$ is ± 1 modulo t, so

$$(v - u)^2 \ge (t - 1)^2/(t - 2) > t \ge b.$$

The window $d - b \le b/(a - 1)$ now gives $b > (a - 1)^2$.

If $t \to r$ is low-only, the incoming edge $s \to t$ carries a high degree. Its low degree pr is unramified at t. The outgoing congruence $par \equiv \pm 1 \pmod t$ gives $a|[pr]_t| \ge t - 1$, since the only integer multiple of a congruent to ± 1 with magnitude less than $t - 1$ would be ± 1. Its dual-pair budget gives $2j_t[pr]_t^2 \le t + 1 - j_t \le t - 1$. Thus $t \le 1 + a^2/(2j_t) \le 1 + a^2/4$. But $t \ge b > (a - 1)^2 > 1 + a^2/4$ for $a \ge 3$. It follows that every edge carries a high degree.

The cyclic divisibility contradiction. At $s \to t \to r$ let

$$A = [s^{-1}]_t, \qquad B = [r^{-1}]_t, \qquad w = [rs]_t.$$

The incoming high degree Hr has even coordinate A at t and multiplicity j_s; the outgoing high degree Hs has odd coordinate B at t and multiplicity j_t.

The degree rs contributes at least $2j_t^2w^2$ to the residual norm at t. Pairing this cost first with the incoming high cost j_sA^2 and then with the outgoing high cost j_tB^2 gives

$$|Aw| \le \frac{t+1-j_t}{2\sqrt{2j_s}} < \frac{t}{4}, \qquad |Bw| \le \frac{t+1-j_t}{2\sqrt{2j_t}} < \frac{t}{4}.$$

Here $j_s, j_t \ge 2$. The vertex window gives $2/3 \le r/t, s/t \le 3/2$, strictly inside $(1/4, 7/4)$. For $x = r, s$ the only representative of x modulo t that can have magnitude less than $t/4$ is $x - t$: all $x + kt$ with $k \ge 0$ exceed $t/4$, and all those with $k \le -2$ are less than $-t/4$. As $Aw \equiv r$ and $Bw \equiv s$ modulo t, this gives $Aw = r - t$ and $Bw = s - t$. As A is even, B odd, and $w \ne 0$,

$$\nu_2(r - t) > \nu_2(s - t).$$

Applied at the three vertices of the directed cycle this gives a strict cyclic inequality. The odd minimal table is impossible.

15.10 A squared prime two

The remaining root has a squared two in the dimension and an odd quartic prime p. Two primes *mix* if the order of some simple twist is divisible by both. There is one mixing pair and it contains two. Its partner is p or another squared prime u (Proposition 15.10 gives at most one pair, and Lemma 6.7 excludes an isolated squared two). Let P_ℓ be the principal residue block at ℓ. In a perfect fusion-simple table nonunit labels of coprime degrees have zero S-entry, while a label in P_ℓ hits every label whose degree is prime to ℓ (Sections 2–4).

The odd denominator. Suppose first that two mixes with p. Write $n = 2p^2bcd$ and order the three isolated primes as $b < c < d$. For an isolated pair the only possible common nonunit degrees are px and $2p^2x$, where x is the third isolated prime. Each product of two isolated primes occurs, and at least one of pb, pc occurs (Proposition 15.10 and its proof). Fix such a low degree pt, and let r, s be the other isolated primes. If a dyadically active J had $p \nmid d_J$, the second rule of Lemma 15.5, with principal prime two, $w = t$ and $q = p$, would give a nonunit on P_2 of degree supported on r, s. Its degree must be their product, coprime to the odd degree pt it must hit by principality. Thus $v_p(f_J) = 4 - 2v_p(d_J) \le 2$ on the dyadically active sector.

If two mixes with a squared prime u, the quartic prime p is isolated, so every dyadically active label has p-unit codegree by Lemma 3.4.

In both cases Theorem 6.9 makes W_2v_0 p-integral. Its other odd denominator primes have height one in n and are removed by Lemma 6.2. The same lemma gives rationality and nW_2v_0 integral. Thus

$$x = 2\epsilon W_2v_0 \quad \text{is integral}, \qquad \|x\|^2 = 4,$$

where $\epsilon = \xi = \pm 1$ by the presence of an odd isolated singleton (Proposition 6.4). Choose the anomaly lift $\gamma = \epsilon$.

The four positions. The vector x lies in P_2 and hence has odd degrees on its support. The complementary Weyl factor commutes with T_2 and carries W_2v_0 to sv_0. Thus the mass $\sum_{i:\theta_{i,2}=\eta} d_i^2/n^2$ of each dyadic phase equals $\sum_{i:\theta_{i,2}=\eta} x_i^2/4$. In particular x cannot be supported at a single position: the vacuum and a dyadically active label (by Cauchy) have different phases and both have positive mass. Hence its four nonzero coordinates are signed units. Let K be the number of active positions. By the vacuum Gauss identity (Lemma 6.2), with an odd exponent selecting the dyadic part,

$$4 - K + \sum_{i=1}^{K} \eta_i = 2x_0.$$

Here $\eta_i \ne 1$ are the phases on those positions.

At any odd isolated singleton t separated from two, $(\epsilon W_2 - I)v_0$ and $(W_t - I)v_0$ are orthogonal. Indeed ϵW_2 fixes the t-moving subspace and W_t fixes its orthogonal complement, by the proof of Proposition 6.3. Their vacuum coordinates are $x_0/2 - 1$ and $-j_t/t$, both nonzero. The singleton column therefore forces a nonunit t-free position of x. This position is dyadically inactive, since all dyadically active labels have full t-height by separation.

When two mixes with p, no position can serve two different isolated primes. Of the only possible common nonunit degrees $px, 2p^2x$, the second is even and the first is coprime to the occurring odd degree given by the product of the opposite pair. Neither can lie in P_2. Thus three distinct nonvacuum positions are inactive. An active position must also occur, so $K = 1$ and $x_0 = 0$. The Gauss identity would say $3 + \eta_1 = 0$, which is impossible.

When two mixes with u, the isolated singletons c, d again require two distinct positions, since their only common nonunit degree is $2up^2$. Both positions have p-divisible degrees: for $t = c, d$ and $w = cd/t$, every p-free t-free nonunit has degree $2uw$ (Proposition 15.10 and its proof). Every active position also has degree divisible by p, by separation from p. If $x_0 = 0$, the Gauss identity and $K \leq 2$ give $K = 2$ with both phases -1. If $x_0 \neq 0$, exactly one active position remains, and $3 + \eta_1 = 2x_0$ gives $x_0 = 1$, $\eta_1 = -1$. All nonvacuum positions have now been accounted for, and their degrees are multiples of p. The phase-mass identity implies $T_2^2 = I$ on the entire table.

Put $Q = s(I - T_2)s^{-1}/2$. As $U_2 = \epsilon T_2$, the local level divides two. The Weyl in $\mathrm{SL}_2(\mathbb{F}_2)$ is an involution, and the unipotent identity of Theorem 6.9 gives

$$\epsilon W_2 = T_2(sT_2s^{-1})T_2, \qquad 4Qv_0 = 2v_0 - T_2x.$$

The vacuum coefficient of $4Qv_0$ is K by the phase masses. Its dimension is zero, since the normalized dimension row is v_0^*s, s^2 fixes v_0, and $T_2v_0 = v_0$. Reducing the dimension modulo p gives $p \mid K$, impossible for $K = 1, 2$. This proves the assertion for four squares in Theorem 15.1. The three cubic companions follow by Theorem 15.7.

16 Extending the support by new prime powers

Throughout, Δ is the dimension of the ordinary or positive reduced table attached to a divisor-minimal obstruction (Lemma 2.1).

16.1 Two unrestricted exponents

Lemma 16.1. *Let r, p, t divide Δ, with r odd, $p \neq r$ and $t \neq r$. In the reduced case assume also that t is odd. The primes p, t may coincide. There exist nonunit labels $X \in P_t$ and $Y \in P_r$ with $r \nmid d_X$ and $p \nmid d_Y$. Every such pair satisfies*

$$r \leq B(g) := \max\{5, (g-1)g^2\}, \qquad g = \gcd(d_X, d_Y) \geq 2.$$

In particular its common degree factor is prime to both p and r. At t, its height is at most $\lfloor v_t(\Delta)/2 \rfloor - 1$.

Proof. Categorical Cauchy supplies active columns at t, r, using its odd-prime form on the positive table (Section 3). Apply prime avoidance (Proposition 4.2) first on P_t at r and then on P_r at p. The valuation assertion follows from Corollary 4.1. No principal block at two is used in the reduced case, even when $p = 2$.

For $g = \gcd(d_X, d_Y)$ the principal crossing at r gives $r \leq \max\{5, (g-1)g^2\}$ by Theorem 4.23, in the ordinary and reduced cases alike. The case $g = 1$ is impossible: coprimality forces the entry to vanish, whereas $Y \in P_r$ and $r \nmid d_X$ give $\chi_Y(X) \equiv d_X \not\equiv 0 \pmod{\mathfrak{r}}$.

□

Proposition 16.2. *For every $k \geq 0$ and integers $C_1, \dots, C_k \geq 2$, the procedure below produces sufficient prime inequalities for every divisor of*

$$P^A R^B \prod_{i=1}^{k} t_i^{C_i} e$$

to be a braided weakly group-theoretical dimension. Here all the displayed primes are distinct, $t_1 < \cdots < t_k$, R is odd, $A, B \geq 0$, and e is coprime and square-free. Any of the C_i may be indeterminates. The inequalities are independent of A, B, and admit prime specializations for every choice of the C_i.

Proof. Enumerate the subsets S of displayed primes retained as repeated primes in a hypothetical divisor-minimal source. If $|S| \leq 2$, use Dong's theorem. For $|S| = 3, 4, 5$ terminate when respectively at most $2, 1, 0$ of the surviving caps can exceed three, by Theorems 4.16 and 11.4 and Theorem 15.6. Count P, R and every symbolic cap as possibly exceeding three. For every other support consider an ordinary table and all possible reduced tables. A reduced table requires a surviving repeated dyadic prime, either P or t_1, whose source cap permits a valuation at least three. This exhausts the possibilities by square-fullness of Δ and evenness of the reduced dimension (Proposition 4.8).

On such a branch take $r = R$ if it belongs to S and otherwise take the largest surviving companion. This r is odd: a nonterminal branch has at least three surviving primes, and P is the only available noncompanion if R is absent. Choose $p \in S \setminus \{r\}$, taking $p = P$ when P survives. Choose $t \in S \setminus \{r\}$ odd in the reduced case. Such a t exists since at most one surviving prime is two. In Lemma 16.1 the gcd is divisible by neither r nor p. For every other surviving companion use the bound $h_i = \lfloor C_i/2 \rfloor$ on its height, subtracting one at t if it is such a companion. For a numerical cap at the reduced dyadic prime one can use $\lfloor (C_i - 1)/2 \rfloor$ instead. The product W of the resulting prime powers bounds the gcd. If $W = 1$ the branch is impossible. Otherwise record $r > B(W)$; one may choose the smallest convenient monomial among the allowed pairs (p, t). Every factor of W is a companion preceding r in the order $t_1, \dots, t_k, R$. Thus the hypotheses can be satisfied successively by choosing primes.

The certificate records the terminal theorem or the pair (p, t) for every branch. A hypothesis $r > B(V)$ covers a recorded W whenever $V \geq W$ throughout the prescribed order and heights. For monomials V, W, write V/W as a product of powers of t_1 and of the ratios t_i/t_{i-1}. The exponents in this expression are the successive suffix sums of the exponent differences. Nonnegative suffix sums suffice; for affine exponents with independent $h_i \geq 1$ this is checked by checking their slopes and their values at $h_i = 1$. Replaying these checks for every branch proves the divisor assertion by Lemma 16.1 and the minimal-counterexample reduction, without substituting individual primes. □

Corollary 16.3. *Let P, R, u, v, w be distinct primes, with $u < v < w$ and R odd. If $R > (uv - 1)(uv)^2$, every weakly integral braided fusion category whose dimension divides*

$$P^A R^B (uvw)^3 e$$

is weakly group-theoretical, for arbitrary $A, B \geq 0$ and any square-free e coprime to the five primes.

Proof. Apply Proposition 16.2 with three companion caps equal to three. On any nonterminal branch both P, R and at least two companions survive: otherwise the terminal theorem for at most four primes applies. In either table take $r = R$, $p = P$ and t the largest surviving companion, necessarily odd. Its valuation is at most three, so the certificate removes its whole height; each other companion has height at most one. Hence $W \leq uv$ and $B(W) \leq B(uv) = (uv-1)(uv)^2$. □

16.2 One new prime of arbitrary height

Proposition 16.4. *Let $p_1 < \cdots < p_k < r$ be primes, where $k \geq 2$, and let $a_i \geq 2$. Suppose $N = \prod_{i=1}^{k} p_i^{a_i}$ is covered, and put $T = \prod_{i=1}^{k} p_i^{\lfloor a_i/2 \rfloor}$. If*

$$r > \frac{T^2}{p_1 p_k},$$

then Nr^b is covered for every $b \geq 0$. For $k = 2$ the weaker inequality $r > T^2/(p_1 p_2^2)$ suffices.

Proof. Fix b, e and take a counterexample of least dimension D among their allowed divisors. Its ordinary or positive reduced table has dimension $\Delta = D$ or $D/2$ and a perfect, simple, integral based ring. Moreover Δ is square-full, $d_Y^2 \mid \Delta$ for each label Y, and $\Delta \mid \prod_Y d_Y^2$ by Lemmas 2.1 and 2.5. If $v_r(D) \leq 1$, include the possible factor r in the square-free decoration of the covered prefix. Thus $r^2 \mid \Delta$, since r is odd. Dong's two-prime theorem applies unless at least two prefix primes are repeated in D. All repeated prefix primes remain in Δ, where valuation one is impossible. Their largest, h, is odd. Put $T' = \prod_i p_i^{\lfloor v_{p_i}(\Delta)/2 \rfloor}$.

Choose an h-active column by categorical Cauchy, using (Proposition 3.5) in the reduced case. Prime avoidance produces an r-free nonunit X in the principal h-block, so $d_X \leq T'/h$ by Corollary 4.1 and Proposition 4.2. Every r-free degree is at most T'. Some degree is divisible by r since $\Delta \mid \prod_Y d_Y^2$. Each such degree is at least rp_1, since nonunit degrees cannot be prime powers (Lemma 2.5). The bound $T'^2/h \leq T^2/p_k$ holds also when $h < p_k$, since then $T' \leq T/p_k$. Under the displayed inequality, therefore, $d_X T' < rp_1$. For every r-free label Y, every constituent of XY is r-free by its degree. The same holds for X^*Y, contradicting support escape (Lemma 4.17) for the proper set of r-free labels.

For $k = 2$ both prefix primes survive. Each r-free nonunit degree is divisible by $p_1 p_2$. The gcd bound of Proposition 4.18 gives $rp_1p_2 \leq d_X T' \leq T^2/p_2$. This recovers precisely the residue-independent clause of Proposition 4.19. □

Starting from Dong's theorem on two repeated primes, repeat Proposition 16.4 along a chain of increasing primes. The new exponent is unrestricted at every step. It may enter the thresholds at subsequent steps. For each finite list of exponent values the resulting inequalities admit infinitely many prime tuples by successive choice of the primes. Any previously covered prefix can shorten the chain.

16.3 Bounding a prime exponent

Lemma 16.5 (Factorial spectral projector)**.** *In a table with unit-coefficient trace and dimension Δ, let B have coefficients integral at a chosen place over p in the distinguished basis. Suppose $\chi_0(B) = m \in \mathbb{Z}_{>0}$ and every other eigenvalue belongs to $\mathbb{Z} \cap (-m, m)$. Then $v_p(\Delta) \leq v_p((2m-1)!)$.*

Proof. The polynomial $\prod_{j=1-m}^{m-1}(B - j)$ equals $(2m-1)!e_0$, by evaluation at every character. Its unit coefficient is $(2m-1)!/\Delta$ and is integral at the place by the integral structure constants. □

Theorem 16.6. *In a minimal table let p divide Δ and write $\Delta = p^a m$ with $a = v_p(\Delta) \geq 1$. Put*

$$u(m) = \prod_{\ell \mid m} \ell^{\lfloor v_\ell(m)/2 \rfloor}, \qquad c = 2^{\mathbf{1}_{3 \mid m}}, \qquad t = (p-1)c.$$

There is a nonunit X in the full principal p-block with $p \nmid d_X [K_X : \mathbb{Q}]$. Every such X satisfies

$$a \leq v_p((2td_X - 1)!) < 2c\, d_X \leq 2c\, u(m).$$

The bound $a \leq v_p((2td_X - 1)!)$ also holds for a specified nonunit X whose field is tame at odd p, or unramified at $p = 2$. If K_X is unramified at odd p, one may use $t = c$ in the factorial.

Proof. The unit coefficient of the p-local principal-block idempotent is $\Delta^{-1}\sum_{X\in P_p} d_X^2$. Thus $p^a \mid \sum_{X\in P_p} d_X^2$, so this sum vanishes modulo p. The block is Galois-stable and the unit orbit contributes one. Some other orbit has degree and size prime to p. Its field is tame at odd p, and unramified at two if $p = 2$, since the wild inertia or the entire dyadic inertia, respectively, is a p-group. Also $d_X \mid u(m)$.

Fix a primitive ℓth root η_ℓ for every $\ell \geq 5$ dividing m. On the ℓ-power roots prescribe a linear functional L_ℓ taking 1 at 1, -1 at η_ℓ, and zero elsewhere. These values sum to zero on each coset of μ_ℓ, exactly the cyclotomic relations at a prime power. Form their tensor product with the normalized trace on the dyadic factor, $(p-1)$ times the normalized trace on the p-factor for odd p, and twice the normalized trace on a complementary three-factor. Include the dyadic factor just once if $p = 2$. Call the result F. Then $F(1) = t$, and F is integer-valued on cyclotomic integers: check on roots, using the normalized traces $1, -1/(\ell-1), 0$ at orders $1, \ell, \ell^k$ with $k > 1$. For unramified odd p, use the prime-to-p conductor of K_X and extend by normalized trace at p to evaluate balancing words. Integrality on K_X is preserved and $t = c$.

On K_X the functional is a sum of embeddings with coefficients integral at p. Indeed, take a cyclotomic field containing K_X and the marks, with only the tame p-level, omitting the dyadic level at $p = 2$. Such a field exists by the local inertia description and the conductor bound in Corollary 2.8. On this field L_ℓ is normalized trace after multiplication by $(\ell-1)(1-\eta_\ell^{-1})/\ell$. The factor $\ell-1$ cancels the p-part of the degree of the ℓ-primary cyclotomic field. At a complementary three the factor 2 does this for $p = 2$. The remaining local degrees are p-units. One may evaluate F on any higher balancing level by normalized relative trace; the smaller field suffices for the coefficients of the embeddings. Galois covariance therefore gives a p-locally integral linear combination B of the $[X^\sigma]$ with eigenvalues $b_Y = F(\chi_Y(X))$.

We prove $|b_Y| < td_X$ for $Y \neq 0$. On roots $|F| \leq t$. A root attaining t is a product of a subset of the $-\eta_\ell$; a root attaining $-t$ is the negative of such a product. Indeed all unmarked odd coordinates must be trivial, and the signs of the marked coordinates determine the dyadic sign. Equality at Y forces every phase of a positive balancing expression to attain the same endpoint. Its product with its conjugate gives a full-mass word for XX^* supported on

$$T = \left\{ \prod_\ell v_\ell : \ v_\ell \in \{1, -\eta_\ell, -\eta_\ell^{-1}\} \right\}.$$

There is a rational functional taking 1 exactly on T and strictly smaller values on every other root. At ℓ give 1 the value 1, give $\eta_\ell^{\pm 1}$ the value -1, and give the other ℓth roots the value $1/(\ell-3)$. Give higher-order roots the value zero. The coset sums are zero. Multiply these functionals and the normalized traces at all remaining factors. Since $\ell \geq 5$, the only possible coordinates of absolute value one are the three marked positions and the dyadic signs. The sign making the product equal to one is precisely the sign in T.

Take a Galois multiplier σ squaring each η_ℓ and fixing the other primary factors. Then $T \cap \sigma T = \{1\}$, since $\{\pm 1\} \cap \{\pm 2\}$ is empty modulo every $\ell \geq 5$. By Lemma 4.20 the two adjoints have a common nonunit V, including for the chosen lifts in the positive quotient. Their exposing functionals put every phase of the same balancing expression for $\chi_Y(V)$ in $T \cap \sigma T$, contradicting strictness.

Apply Lemma 16.5 to B with $m = td_X$. Legendre's formula gives the stated strict upper bound. □

16.4 A finite reduction for a fixed complementary factor

Set $L(m) = 2 \cdot 2^{\mathbf{1}_{3\mid m}} u(m)$. Let $w(m)$ be $u(m)$ divided by its largest full prime-power factor, with $w(m) = 1$ if $u(m) = 1$, and put

$$B(m) = \max\{5, (w(m)-1)w(m)^2\}.$$

Corollary 16.7 (Fixed complementary factor)**.** *If a minimal table has dimension $p^a m$ with $a > 0$ and $p \nmid m$, then $p \le B(m)$ and $a < L(m)$. Consequently, for a fixed m and any prime $p \nmid m$, to prove weak group-theoreticity for all braided fusion categories of dimensions $p^j t$ with $t \mid m$ and $j \ge 0$, it suffices to check the divisors of m and the dimensions*

$$p^j t, \qquad t \mid m, \quad p \le B(m), \quad 1 \le j < L(m) + \mathbf{1}_{p=2}.$$

This is also a finite reduction when p varies.

Proof. The exponent bound is Theorem 16.6. Choose the p-free nonunit X supplied by that theorem. For $p \ge 7$, Theorem 4.23 gives $p \le b(d_X/L)$ with L the largest full prime-power factor of d_X. Since $d_X \mid u(m)$, deleting its largest primary factor leaves $d_X/L \le w(m)$. Indeed, deleting instead the prime component corresponding to the largest factor of $u(m)$ can only increase that quotient. Thus $p \le B(m)$, also immediate for $p \le 5$.

Now choose an obstruction of least dimension in the divisor family and write $p^j t$ for its dimension. The case $j = 0$ is among the divisors of m. Otherwise the part m' of its table dimension prime to p divides m. Both B and L are nondecreasing under divisibility. If p occurs in the table, the two bounds give $p \le B(m)$ and table exponent at most $L(m) - 1$. Passing to the category adds one only for a positive fermion quotient at $p = 2$. If p is absent from the table then necessarily $p = 2$ and $j = 1$, also in the displayed range. This proves the finite reduction. □

The normalized trace projector. The unmarked normalized trace gives another form of the projector in Theorem 16.6. If X is a specified nonunit, $h = [K_X : \mathbb{Q}]$, and H clears the normalized traces of its cyclotomic integers with H/h integral at p, take $B = (H/h) \sum_{X' \in \mathrm{Gal}\cdot X} [X']$. Its eigenvalues are $H\tau(\chi_Y(X))$, rational integers of absolute value strictly less than Hd_X at nonunits. The same projector therefore gives $a \le v_p((2Hd_X - 1)!)$. One admissible trace denominator is the product of $\ell - 1$ over primes of a conductor containing K_X, by the normalized trace of a primary root.

16.5 Catalogue with hypotheses

Table 6 lists the dimension families proved here. In its WGT rows for braided categories, a product N means that every weakly integral braided fusion category with $Q(D) \mid N$ is WGT. Thus all divisors and all coprime square-free factors are included. This convention follows from the square-fullness of the minimal table and the evenness of a positive reduced table (Lemma 2.5 and Proposition 4.8). A WGT category with $4 \nmid D$ is solvable (Corollary 15.4). Letters at different prime positions denote distinct primes. Capital exponents A, B, C may vary independently unless a bound is given. Prime order matters only where indicated.

In the three-prime rows, $W(s, k)$ excludes a crossing of full positive weight whose s-primary phases occupy at most k positions, after a common rotation, at nonunit labels of an odd ordinary minimal table. The precise test and its arithmetic criteria are in Section 24, (8.1), and Propositions 8.40 and 25.14. Each row cites the result specifying the conditions on its free primes and exponents.

Table 6: Dimension families, grouped by the hypotheses on their prime divisors.

Dimension family	Conditions and references
Uniform braided families	
$p^A q^B$	At most two repeated primes [22, Theorem 3.1].
$p^A q^B r^C$	For $p < q < r$, it suffices that $A \le 5$, $B \le 4$, or $C \le 3$. The unordered caps $(7, 5, 4)$ also suffice (Theorems 8.3, 7.3, 4.16 and 8.1).

Table 6 continued

Dimension family	Conditions and references
$P^A(uvw)^3$; $(uv)^3P^AQ^B$	The first holds in every order. The second holds for $u < v < P < Q$ (Theorems 11.4 and 1.1).
$P^aQ^Bu^3v^2$	For $a = 5$, it suffices that $Q > \min(P, u, v)$. The caps $(a, B) = (7, 5)$ hold in every order. The caps $(9, 5)$ also hold unless P is the least of four odd primes and $P \geq 23$ (Theorems 13.1, 9.1 and 11.4, Section 14.2).
$p^4(uvw)^3t^2$; $(puvwt)^3$	Both five-prime families hold in every order (Theorems 15.1 and 15.6).
Three repeated primes with size, residue or word conditions	
$t^au^bs^C$	Put $L = t^{\lfloor a/2\rfloor}u^{\lfloor b/2\rfloor}$. The size test is $stu\max(t, u) > L^2$; the residue and support alternatives are Proposition 4.19 and Theorem 8.10.
$p^Aq^Br^C$, $p < q < r$	With $h = \lfloor A/2\rfloor$, $q > p^h$ suffices. For $q > p^{h-1}$ the two companion caps may be rounded down to even; see Theorem 7.15 for the odd-prime refinement.
$p^Aq^Br^4$, $p < q < r$	At odd p, $k = 1 + \lfloor r/q\rfloor < p$ and $W(r, k)$ suffice (Theorem 8.2). For the explicit strips and bounded-companion alternatives see Proposition 8.7, Corollary 8.9 and Proposition 25.15, Proposition 8.11 and Corollary 8.12.
$p^Aq^5r^C$, $p < q < r$ odd	For $j = \lfloor q/p\rfloor$, $1 + j(j + 1) < 2q/3$ gives $C = 4$. If also $(r - q)(p - 1) > q$, then C is free. For $A \geq 6$, $q > p^{\lfloor A/2\rfloor-2}$ is another sufficient condition at $C = 4$ (Proposition 8.19 and Theorem 8.20).
$p^{2h}q^Br^4$; $p^{2h}q^{2k}r^4$; $p^{2h}q^Br^C$	For odd $p < q < r$, use the top-height tests at $h, k \geq 2$ (Theorem 25.1 through Proposition 25.3). For $h \geq 3$, $r > p^{h-1}$ and $W(r, p^{h-2})$ suffice for unrestricted B, C (Theorem 25.6).
$P^6Q^Br^C$	At $2 = P < Q < r$ only $(Q, r) = (3, 5)$ remains (Theorem 26.2). At odd $P, Q < r$, use $r > P^2$ and $W(r, 1 + \lfloor P^2/Q\rfloor)$ (Theorem 25.9). For $P < Q$, the quartic test uses $W(r, 1 + \lfloor\min(r, P^2)/Q\rfloor)$; the remaining unrestricted tests below P^2 are in Theorem 25.9, Corollary 25.10 and Section 25.4.
Four and five repeated primes with additional conditions	
$P^AQ^B(uv)^2$	Order $P < Q$ and $u < v$. It suffices to exclude ordinary roots P^hQ^kuv with $P < u$, $u \geq 5$, $3 \leq h \leq \lfloor A/2\rfloor$, $2 \leq k \leq \lfloor B/2\rfloor$, and $h + k \geq 6$. For $P = 2$, equality requires $h = k = 3$. Simultaneous trace, norm and size tests appear in Sections 9, 10 and 13.
$P^AQ^B(uv)^2$ at odd primes	The residue pairs modulo (u, v) and the quotient by $\langle P\rangle$ constrain k for arbitrary h (Section 14.1). Finite height windows, odd and dyadic, are in Section 14.2.
$2^Aq^Bw^3t^2$	For $t \equiv 1 \pmod 8$, suppose every odd prime divisor of $t + 1$ is $1 \pmod 4$ and $(q/t) = 1$, $(w/t) = -1$ in Legendre symbols. The caps at w, t may be interchanged, and both may be three if $2q \nmid w - 1$ or $2q \nmid t - 1$ (Corollary 13.2).
Raising squares to cubes	In a covered four-prime product with a designated set S of at least two squares, any proper subset may be cubed. All may be cubed when $\lvert S\rvert \geq 3$ or some odd $t \in S$ satisfies $\prod_{z\notin S} z \nmid t - 1$, where z ranges over the other displayed primes. In a covered $p^H(uvwt)^2$, any three squares may be cubed. All four may be cubed if their least prime is less than p and $p \nmid z - 1$ for one of the other three (Theorems 11.4 and 15.7).
Arbitrary support and finite reductions	
$P^AR^B\prod_i t_i^{c_i}$	For odd R and $c_i \geq 2$, test each surviving support by Proposition 16.2. For $P^AR^B(uvw)^3$, $u < v < w$, the inequality $R > (uv - 1)(uv)^2$ suffices (Corollary 16.3).
Nr^B with covered N	Write $N = \prod_{i=1}^k p_i^{a_i}$ with $a_i \geq 2$, $p_1 < \cdots < p_k < r$, $k \geq 2$, and set $T = \prod_i p_i^{\lfloor a_i/2\rfloor}$. It suffices that $rp_1p_k > T^2$, or that $k = 2$ and $rp_1p_2^2 > T^2$ (Proposition 16.4).
p^Am, fixed m	For $p \nmid m$ a finite list suffices even with p variable, including the additional dyadic exponent for a reduced table (Corollary 16.7).

Table 6 continued

Dimension family	Conditions and references
Conclusions without a braiding assumption	
D^2 in a braided family	Apply the corresponding condition to $Q(D^2) = D^2$. This gives WGT for every source of dimension D ([32, Proposition 4.1]).
$D \mid uvp^Aq^B$; D square-free	The first family is solvable when $\min(u, v) < \min(p, q)$. Square-free D on at most five primes gives solvability, and gives group-theoreticity in the integral case (Corollary 15.4). Without an ordering condition, $D \mid p^Aq^Br$ gives WGT (Theorem 4.16).
Rank families at the specified dimension	
Here R is the rank of a non-pointed simple integral modular category and $D = qM^2$ with q square-free, except where a minimal obstruction is specified. Table 2 gives the uniform bounds.	
$q = 1$	Put $B = M/d_{\max}$. The bounds for $B \geq 5$, $4 \nmid M$, and $B \geq 6$ are 36, 39, and 46, respectively (Theorem 22.2).
$q > 1$	Let a be the least prime of M. For prime q, $R > qa^2$ since formal codegree q is impossible (Lemma 20.1). For composite q, $R > q + (a^2 - 1)(1 + \sum_{p \mid q,\, p \text{ odd}} (p-1)/2)$ (Corollary 19.10, including $q = 15$).
Prescribed prime support	For odd $s \mid M$, $s \nmid q$, collected in S, $R \geq B(q)(1 + \sum_{s \in S}(s-1))$ with $B(q)$ as in Section 19.2. The joint-orbit, denominator and simultaneous-deficit bounds are Propositions 19.3, 19.8, 22.3 and 19.9.
$D = qU^2p^{2e}$	For fixed q, U and odd $p \nmid qU$, (19.1) is uniform in $e \geq 1$. For $q = 3$ the ten-object refinement is Corollary 19.14.
$q = 10$, three primes	For $10M^2$ on three primes, $R \geq 36$ in the ordinary case. At *reduced* dimension $10M^2$ a minimal obstruction has full rank at least 68 (Proposition 22.4).
Two isolated primes	For the ordinary minimal root P^hQ^kuv of the four-prime family above, $R \geq 7 + (u-1)j_u + (v-1)j_v$, where j_u, j_v are its vacuum deficits. At odd dimension $R \geq 13 + uj_u + vj_v$ (Corollary 9.4). Estimate (9.1) also uses the sizes of the two higher prime powers.
$D = M^2$, a prime denominator	A unique $M/d = p$ requires $v_p(M) = 1$ and at least three prime factors of M/p (Proposition 20.4). The further trace bounds are in Section 21.3.
Minimal obstructions and decomposition	
Five primes; three primes	A minimal obstruction on five repeated primes cannot have exact valuations $3, 2, 2$ at three positions, whatever the other two (Theorem 15.6). Under the unordered caps $(6, 6, 4)$, a minimal obstruction must be ordinary with exact valuations $6, 6, 4$ and the quartic prime largest (Section 25.8). The only possible even dimension of such a minimal obstruction is $2^63^65^4$. At $5^67^623^4$ every non-WGT braided category has rank at least 553 (Corollary 27.13).
$D \mid p^aq^br^4$	An integral modular or slightly degenerate category is WGT if the degrees in its principal block at odd r are prime to r (Theorem 8.39).
Prime supports of braided factors	For integral WGT modular categories use the components determined by $s \mid \ell^{2j} - 1$, $1 \leq j \leq \lfloor v_\ell(D)/2 \rfloor$. Theorem 31.3 also treats integral categories with all repeated primes in one component and categories with pointed core.
$D = L$ or L^2	For composite square-free L, Theorem 31.5 gives the connected-graph existence test for a tensor-prime integral WGT category at L (positive signs) and a prime integral WGT modular category at L^2 (one sign per target).
Modular or slightly degenerate dimensions	With $e = 1$ or 2, respectively, let U be the square root of the largest square divisor of D/e. If $U < 60$ the category is solvable. For a WGT category it suffices that no order of a nonabelian finite simple group divides U (Proposition 17.2).

17 The four dimension bounds

Theorem 17.1 (Uniform dimension bounds)**.** *The four bounds in Table 1 hold. Every strictly weakly integral fusion category of dimension less than* 10800 *is weakly group-theoretical.*

Proof. We compare the prime exponents of a divisor-minimal braided obstruction with those allowed by the family theorems. With a braiding, the square-full part of its dimension divides $Q(D)$. Without a braiding, the Drinfeld center has dimension N^2, so a valuation e in a minimal obstruction requires valuation at least $\lceil e/2 \rceil$ in N. At most two repeated primes are covered by Dong's theorem [22, Theorem 3.1]. The following lower bounds use only the families needed for this comparison.

# repeated primes	$Q(D)$	$Q(D)$ odd	N	N odd
3	$2^6 3^6 5^4$	$3^6 5^6 7^4$	5400	165375
4	6300^2	121275^2	6300	121275
5	$2^3\, 2310^2$	$3^3\, 15015^2$	9240	135135
≥ 6	30030^2	255255^2	30030	255255

At three primes $p < q < r$, the exponents must be at least $(6, 5, 4)$ in that order, by Theorems 4.16, 7.3 and 8.3. The unordered box $(7, 5, 4)$ is covered by Theorem 8.1. Thus the braided comparison costs at least $p^6 q^5 r^4 \min(p^2, q, r)$; for the center use the ceilings of half the three original lower exponents. Both costs increase with the primes.

At four primes write the exponents in decreasing order as a, b, c, d. Theorem 11.4 gives $b \geq 4$. If $c \geq 4$ or $c = d = 3$, the respective lower profiles are $(4, 4, 4, 2)$ and $(4, 4, 3, 3)$. Their minimum is attained by assigning the larger exponents to the smaller primes. Otherwise $d = 2$ and $c \leq 3$. Theorems 9.1 and 11.4 give $\lfloor a/2 \rfloor + \lfloor b/2 \rfloor \geq 6$. This leaves $(6, 6, 2, 2)$ or $(8, 4, 2, 2)$ as lower profiles. For odd primes the least of the four products is $(3^2 5^2 7^2 \cdot 11)^2 = 121275^2$. For even support the only profile with minimum below $(2^2 3^2 5^2 \cdot 7)^2 = 6300^2$ is $(8, 4, 2, 2)$. When two is in a higher position the dyadic $(9, 5)$ box at the two higher primes and the lift of one square to a cube cover all $a \leq 9, b \leq 5$. The two placements follow from the dyadic enumeration in Theorem 9.16 and the lift in the paragraph The two dyadic exponent boxes. The next possible exponents have $a \geq 10$ or $b \geq 6$. When two is in a lower position the product is already at least $(2 \cdot 3^4 5^2 \cdot 7)^2$. Taking square roots after rounding odd valuations upwards gives the same lower limits for N.

At five primes let P be their product and $p < q$ the two least primes. Theorems 15.6 and 15.1 cover the profiles $(3, 3, 3, 3, 3)$ and $(4, 3, 3, 3, 2)$ in every order. Hence one valuation is at least five, two are at least four, or a quartic is accompanied by four valuations at least three. All three alternatives cost at least $p^3 P^2$. For centers we may test the even exponent caps in N^2 directly, since the covered families include every divisor. A prime then occurs to height at least three in N, or two primes occur to height at least two, so $N \geq p^2 P$. Six primes give the last row immediately.

Taking column minima gives 29160000, 6087156075, 5400 and 121275. A Python check reproduces each entry of the table by enumerating exponent vectors in order of increasing product and, for the two center columns, testing the doubled exponents. WGT descends from the center, and $4 \nmid D$ when $Q(D)$ is odd, so the solvability argument of Corollary 15.4 applies to both odd columns. For a strictly weakly integral category the canonical dimension grading has integral neutral component of dimension at most $N/2$ [36, Theorem 3.10]. The bound 5400 applies to this component, and WGT passes to its extension. □

The first exponent profiles left by this comparison are $2^6 3^6 5^4$ and $3^5(5 \cdot 7 \cdot 11 \cdot 13)^2$ in the braided columns. The boundary dimensions for centers are $2^3 3^3 5^2$ and $3^2 5^2 7^2 \cdot 11$.

17.1 Solvability

For a WGT category of dimension D, the group-extension argument of Corollary 15.4 proves solvability if no order of a nonabelian finite simple group divides D. Braiding gives a smaller integer to test. For $m \geq 1$ put $R(m) = \prod_p p^{\lfloor v_p(m)/2 \rfloor}$, the square root of its largest square divisor.

Proposition 17.2 (Solvability from the square divisors)**.** *Let $e = 1$ in a nondegenerate weakly integral braided category of dimension D and $e = 2$ in a slightly degenerate one. If the category*

is WGT, it is solvable whenever no order of a nonabelian finite simple group divides $R(D/e)$. The condition $R(D/e) < 60$ suffices even without the WGT hypothesis.

Proof. For $e = 1$ the WGT assertion is [43, Corollary 5.2]: the integers u with $u^2 \mid D$ are exactly the divisors of $R(D)$, and a nonsolvable finite group has a nonabelian simple composition factor. For either value of e, each Tannakian $\mathrm{Rep}(G)$ satisfies $e|G|^2 \mid D$ by the centralizer formula: its centralizer contains the product of $\mathrm{Rep}(G)$ with the symmetric center. Thus $|G| \mid R(D/e)$. Under the first hypothesis every such G is solvable, and [43, Theorem 5.1] applies. Under the second, use $Q(m) \le R(m)^3$ and $Q(2m) \le 4Q(m)$ to obtain $Q(D) \le 4 \cdot 59^3 < 29160000$. Theorem 17.1 first gives WGT; the smallest nonabelian simple group has order 60. □

In particular $Q(D) < 3600e$ suffices for solvability. Indeed the dimension bound first gives WGT. A nonsolvable Tannakian group would have order at least 60 divisible by four, so $e|G|^2$ would itself be square-full and divide $Q(D)$. Likewise $16e \nmid D$ suffices once WGT is known. The center of Vec_{A_5} and its product with sVec attain the respective thresholds and contain $\mathrm{Rep}(A_5)$. Together with the classification of Xu and Dong [53, p. 292, abstract], the dimension bound also gives the following small-range classification: every nontrivial perfect weakly integral modular category of dimension < 14400 is a twisted center $\mathcal{Z}(\mathrm{Vec}^{\omega}_{A_5})$, of dimension 3600.

For an odd-dimensional modular category put $\mathcal{E} = \mathcal{C}_{\mathrm{ad}} \cap \mathcal{C}_{\mathrm{pt}} = \mathrm{Rep}(H)$, $g = \mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}})$ and $e_0 = \mathrm{FPdim}(\mathcal{E})$. The symmetric center of $\mathcal{C}_{\mathrm{ad}}$ is $\mathcal{E}$, so the modular quotient $(\mathcal{C}_{\mathrm{ad}})_H$ has dimension $D/(ge_0)$ [26, Proposition 4.26 and Corollary 4.31(i)]. Solvability passes in both directions [32, Proposition 4.5(i), (iv)]. Iteration ends at a perfect category, which is trivial precisely in the solvable case since every nontrivial solvable category has a nontrivial invertible object.

17.2 Dimension of a simple perfect category

Corollary 17.3 (Dimension in arbitrary rank)**.** *Every nontrivial simple perfect integral modular category has dimension at least* 29160000*, and at least* 6087156075 *in odd dimension. Without a braiding assumption every non-WGT weakly integral fusion category has dimension at least* 5400*, and at least* 121275 *in odd dimension. The odd lower bound also holds for every nontrivial perfect weakly integral fusion category.*

Proof. A nontrivial simple modular category has no nontrivial Tannakian subcategory: simplicity would force that subcategory to be the whole category, whereas a nontrivial symmetric category is degenerate. An integral WGT braided category with no nontrivial Tannakian subcategory is pointed by [43, Theorem 3.1]. Perfection rules out a nontrivial pointed category. The first two assertions now follow from Theorem 17.1 and $Q(D) \le D$. The same theorem gives the unbraided bounds, since solvability implies weak group-theoreticity and a nontrivial solvable category has a nontrivial invertible object [32, Proposition 4.5(iv)]. In the unbraided statement the non-WGT assumption matters even for simple perfect categories: $\mathrm{Rep}(A_5)$ is such a category of dimension 60 and rank five. □

18 Enumerating degree multisets from formal codegrees

To prove the modular and reduced bounds of Table 2 and Theorem 22.1, we first enumerate possible multisets of degrees. The dimension families apply in both cases. The transparent fermion pairs the simples in the slightly degenerate case, so the rank and dimension of the positive table are half the corresponding categorical quantities. Simplicity and perfection of this table follow from divisor-minimality (Lemma 2.1).

18.1 Common degree constraints

For the ordinary table put $\Delta = D$. For the positive fermion quotient put $\Delta = D/2$ and take as basis the fermion orbits in $K_0(\mathcal{C})/(g-1)$, with multiplication coefficients $N_{XY}^Z + N_{XY}^{gZ}$. In either case the dimension vector (d_i) has unique unit at 0, and $S_{0i} = d_i$, $S^*S = \Delta I$. The formal codegree of the character $\chi_X(i) = S_{Xi}/d_X$ is $f_X = \Delta/d_X^2 \in \mathbb{Z}$. In the ordinary case Corollary 17.3 and its proof show that the category is non-WGT. The table arguments below use only simplicity, perfection and modularity in this case; divisor-minimality supplies simplicity and perfection for the fermion quotient. Every dimension-family test is applied to D, including in the reduced case.

Let $H_X = \{i \neq 0 : S_{Xi} \neq 0\}$ for $X \neq 0$, and put $x = d_X$. The vacuum and row-norm identities give

$$\sum_{i \in H_X} d_i \chi_X(i) = -1, \tag{18.1}$$

$$x^2 + \sum_{i \in H_X} \operatorname{lcm}(x, d_i)^2 \leq \Delta. \tag{18.2}$$

Indeed, the nonzero $S_{Xi}/\operatorname{lcm}(x, d_i)$ are algebraic integers by Lemma 2.5. The arithmetic-geometric mean inequality applied to all conjugates shows that the mean squared modulus of each is at least one. The same lemma gives coprime-degree vanishing, the occurrence of a zero in every nonunit row, and at least two prime factors in every nonunit degree. Consequently the least possible nonunit degree is 6, or 15 when Δ is odd.

Lemma 18.1 (Two prime-free duality orbits)**.** *For each prime p, at least two distinct nonunit duality orbits have degree prime to p.*

Proof. By (18.1), every nonunit row meets a nonunit of degree prime to p. If there were only one such orbit, each row would meet every member, since character values on dual arguments are conjugate. Symmetry of S and its nonzero vacuum entries would give a nonunit row having no zero, contrary to Lemma 2.5. □

Write $\Delta = qM^2$ with q square-free. Square divisibility $d_i^2 \mid \Delta$ gives $d_i \mid M$. Attach a denominator $b_i = M/d_i$ to each nonunit. Take $w = 1$ when each label is counted separately. For odd Δ, Lemma 2.5 pairs each nonunit with its distinct dual, so we can take $w = 2$ and count one denominator per pair. In increasing order with repetitions, the denominators satisfy

$$q = \frac{1}{M^2} + w \sum_j \frac{1}{b_j^2}, \qquad f_j = q b_j^2, \qquad q b_1^2 < r. \tag{18.3}$$

Here and in each computation r is the number of labels in the table. The last inequality follows by comparing every term of $\sum_i 1/f_i = 1$ with $1/f_{\min}$, strictly at the vacuum. Thus (q, b_1) ranges over an explicit finite set determined by the rank.

For a sorted prefix, let L be its lcm (initially 1), $E = L^2(q - w\sum_{\text{prefix}} b_j^{-2})$, and h the number of remaining nonunit positions or pairs. For a lower bound a on their degrees, the next denominator satisfies

$$b \geq b_{\text{prev}}, \qquad wL^2 < Eb^2 \leq (wh + a^{-2})L^2. \tag{18.4}$$

The vacuum makes the lower bound strict and $M \geq ab$ bounds its contribution above. With $g = \gcd(b, L)$, $k = b/g$, appending b replaces (L, E) by $(kL, k^2E - w(L/g)^2)$. Testing integer squares or using integer square roots gives both endpoints exactly. Lemma 18.1 implies that the lcm of the nonunit denominators remains M after deleting any one nonunit or a whole dual pair, since an orbit prime to p supplies the full p-part of M.

Proposition 18.2 (Terminal equations)**.** *With one position left, $M = L$ and $E = 1 + wd^2$ for a divisor d of L. With two positions left put $M = kL$ and denote their degrees by $d \leq e$. Then*

$$d, e \mid L, \qquad (k, de) = 1, \qquad Ek^2 = 1 + w(d^2 + e^2). \tag{18.5}$$

These restrictions enumerate every completion when supplemented by the denominator order and the checks in (18.3).

Proof. Every earlier degree is a multiple of k. For each $p \mid k$ the two required p-free orbits must both meet the two remaining positions. Thus $p \nmid de$, whence $d, e \mid L$. The residual mass gives (18.5). With one position left no such prime can occur. □

For $w = 1$, (18.5) implies $(10, k) = 1$ by reducing modulo four or five and using $(k, de) = 1$. For this individual-label recursion there are two ways to enumerate the last two degrees. One groups divisors $e \mid L$ by their square modulo E and checks the square in (18.5). The other takes the next denominator $b = M/e$ from (18.4) with $h = 2$, recovers $e = L/(b, L)$ and $k = b/(b, L)$, and tests $Ek^2 - e^2 - 1$ for the square of a divisor of L. For the second parametrization, coprimality of e and k gives $(b, L) = L/e$, so every allowed completion occurs, including $d = e$.

The prefix also bounds the prime support. For an odd prime $p \mid M$ with $p \nmid q$, the two square classes of p-primary twists require two eligible packets of $(p-1)/2$ labels of equal degree, or a larger packet from a higher twist order. Here a label is eligible when $p \mid qb^2$ by Theorem 3.6. At odd dimension, distinct duals have the same actual twist, and a packet costs that many dual pairs. Thus a new odd prime $p \nmid qL$ can enter M only if $p \le h + 1$. Known eligible closed runs of lengths m_j supply at most $\sum_j \lfloor m_j/((p-1)/2) \rfloor$ packets. Only the last run can grow. If just two packets fit, equality of squared masses in the two classes forces their degrees to agree. The twist counts follow from Propositions 19.1 and 19.3.

With two positions left, the only possible new odd prime outside qL is 3. For two individual labels ($w = 1$), $(k, 10) = 1$ fixes the valuations at 2 and 5. Moreover, for each $p \mid k$ both remaining degrees divide $L/p^{v_p(L)}$, so (18.5) gives $Ek^2 \le 1 + 2(L/p^{v_p(L)})^2$. This bounds $v_p(k)$ before the degrees are chosen. If the possible supports and valuations all lie in covered divisor families, the whole prefix is excluded.

With three individual labels of degrees d_1, d_2, d_3 left, write $M = kL$. If k is even, reducing $Ek^2 = 1 + d_1^2 + d_2^2 + d_3^2$ modulo four shows that all three degrees are odd. Reduction modulo eight now gives $v_2(k) = 1$ and E odd. With four positions left any new odd prime outside qL is at most five. In the reduced search the two-adic valuation used for a family is $v_2(2qL^2k^2)$.

At a completed list the row of degree x can hit a nonunit of degree y only if $(x, y) > 1$ and $x^2 + \mathrm{lcm}(x, y)^2 \le \Delta$. By (18.1) the gcd of all such candidates must be one. The ordinary searches apply the character, local-block, twist-mass and dimension-family tests after this degree check or earlier on a whole prefix when all completions violate them. A failed necessary condition discards the corresponding list. For the fermion quotient we use only identities established for that table, including its odd-primary twist identities.

Proposition 18.3 (Completeness of the degree search)**.** *Fix the table rank r, choose the ordinary or positive reduced table and choose the counting weight w of* (18.3)*. Take the finitely many square-free q and positive B with $qB^2 < r$. Prescribe $b_1 = B$ and start with the empty prefix, $L = 1$, $E = q$ and $h = (r-1)/w$. Require h to be integral. Keep the prefix and its last denominator together with L, E, h. Append each integer in* (18.4) *and close one or two remaining positions by Proposition 18.2. Test integrality of all degrees, the sorted order, and the original dimension equation. Every category in the specified class gives exactly one sorted list in this search. A prefix may be discarded whenever a necessary condition in that same table is violated by every completion.*

Proof. The square-free decomposition of the table dimension determines q, M uniquely. Sorting the nonunit denominators determines the branch, including repeated entries. The update retains the defining residual mass E/L^2. It bounds the next integer above at each state with $E > 0$, and each update decreases h. The terminal equations include the original pair of degrees for every completion; at the one-position boundary the same assertion uses $M = L$. Conversely the explicit final tests recover the dimension identity in the original units. A complete denominator list

determines M, since its lcm is M by Lemma 18.1. Grouping the last two divisors in a different order, or dispatching disjoint prefix subtrees, therefore leaves the enumeration exhaustive. The terminal arithmetic conditions are necessary; a retained list still requires the character and twist tests. □

19 Counting labels by prime components of the twist

For an odd prime p put $h_p = (p-1)/2$. A label with p-primary twist of exact order p^j has $p^{j-1}h_p$ distinct square Galois translates of its phase, all at the same degree and with unchanged complementary twist. We call such a collection of labels a packet. Theorem 3.6 requires $p^j \mid f_X$. In odd dimension each phase occurs in dual pairs. We first determine what happens within a fiber of the complementary twist, and then combine the degree and multiplicity requirements at several primes. For a set T of primes, a complementary fiber F fixes the specified twist components outside T; omitted components are summed over. On a nonempty fiber put $m(F) = \sum_{i\in F} d_i^2$ and $g(F) = \gcd_{i\in F} d_i$. In the positive fermion quotient the components are those of the twist modulo the fermion sign. Each odd component is well defined. The following estimates use the same fibers: at an odd prime of even valuation they equate primitive masses, at an odd prime of odd valuation they compare active and inactive masses, and at two they bound cardinality when $2 \mid q$. The extra two-label test in Lemma 19.6 uses the full ordinary action.

19.1 Local fibers and simultaneous prime bounds

Proposition 19.1 (Even valuation)**.** *Let p be an odd prime with $v_p(\Delta) > 0$ even. In either table, intersect a full p-block, or the whole label set, with prescribed complementary twist components. In this intersection the squared-degree masses at the $p-1$ primitive pth roots are equal. If any phase of exact order p^j occurs, $j \geq 1$, at least $p-1$ phases of that order occur there. In particular the whole table contains at least $p-1$ active labels.*

Proof. Take the vacuum at both entries of the local Bruhat calculation in Proposition 3.9. The block and complementary-phase projectors commute with the local factor. At even valuation the local torus fixes the vacuum vector with positive sign because p is unramified in $\mathbb{Q}(\sqrt{\Delta})$. The odd local unipotent acts by the actual primary twists. Hence $\sum d_i^2 \theta_{i,p}^u$ over the intersection is independent of the unit u modulo the local level. These are all conjugates of the first moment, so the moment is rational. Its normalized trace to $\mathbb{Q}(\zeta_p)$ kills the roots of higher p-power order. The unique rational relation among the pth roots then equates the masses at the nontrivial ones. The proof on fermion pairs uses the positive subspace of Section 3; the vacuum sign is unchanged. When $j \geq 2$, a single square orbit has $p^{j-1}h_p \geq p-1$ phases. For the whole table, categorical Cauchy or its reduced form in Section 3 supplies an active label. In ordinary data a related single-cycle obstruction appears in [6, Lemma 4.7 and Theorem 4.8]. □

Lemma 19.2. *Let t be an odd prime with $v_t(\Delta)$ odd. For any nonempty complementary fiber F for $\{t\}$, put $g = \gcd\{d_i : i \in F\}$ and $h_t = (t-1)/2$. Then*

$$t \mid \sum_{i\in F} (d_i/g)^2.$$

At least $h_t + 1$ distinct t-primary phases occur in F, and F contains at least $h_t + 2$ labels.

Proof. Apply Proposition 3.9 with both test labels equal to the vacuum and with the whole label set as its block. Sum over all unspecified complementary components. The squared mass at exact t-order t in F is $t-1$ times the t-inactive squared mass. At any degree and exact t-primary order t^j, inertia at t preserves the degree, the complementary components and the order. Its square action realizes all square-unit translates, of which there are $t^{j-1}h_t$. Multiplicities are

equal within each such set of phases. Thus for $j \geq 2$ the number of these labels at each fixed degree is a multiple of t. Dividing squared masses by g^2 gives the displayed divisibility.

If a higher-order phase occurs it gives at least th_t distinct phases and $th_t \geq h_t + 2$. Otherwise the positive squared masses and the slice identity require both an inactive label and an orbit of at least h_t distinct primitive phases. If precisely $h_t + 1$ labels occur, they consist of one inactive label of degree b and one label at each phase of a square orbit, all of a common degree a. The slice identity reads $h_t a^2 = (t-1)b^2$, or $a^2 = 2b^2$, impossible for positive integers. This proves both assertions. □

For example, the congruence alone excludes a singleton fiber. At $t \equiv 3 \pmod 4$ it also excludes any two-label fiber, since at least one of the two degrees divided by g is prime to t and -1 is not a square modulo t.

For a set T of odd primes of odd valuation put $\Gamma(\varnothing) = 1$, and, for $t_0 = \min T$ if T is nonempty, put

$$\Gamma(T) = (h_{t_0} + 2) \prod_{t \in T \setminus \{t_0\}} (h_t + 1).$$

Every nonempty complementary fiber for T contains at least $\Gamma(T)$ labels. Indeed refine it successively by the t-components for $t \neq t_0$. Each nonempty intermediate fiber has at least $h_t + 1$ nonempty children by Lemma 19.2. At the last prime each terminal parent has at least $h_{t_0} + 2$ labels. The chosen order maximizes this bound because $(h+2)/(h+1)$ decreases with h.

If Δ is odd, a complementary fiber that contains no vacuum consists of pairs of distinct duals of equal degree and twist, as in Lemma 2.5. In this case the same argument counts pairs: the Galois action commutes with duality and each phase in a square orbit still occurs at the same degree, while all masses in the slice identity can be divided by two. Such a fiber therefore contains at least $2\Gamma(T)$ labels. The fiber containing the vacuum is only charged $\Gamma(T)$ labels below.

Proposition 19.3. *Let u be any odd prime divisor of Δ, $T = \{t : t \mid q,\ t \text{ odd},\ t \neq u\}$, and $\delta = 2$ for odd Δ, otherwise $\delta = 1$. Set $s_u = u - 1$ if $u \nmid q$ and $s_u = (u-1)/2$ if $u \mid q$. There are at least $\delta s_u \Gamma(T)$ u-active labels, and*

$$\mathrm{rk}_{\mathrm{table}} \geq (1 + \delta s_u)\Gamma(T), \qquad \sum_{\substack{d>1 \\ u \mid \Delta/d^2}} \left\lfloor \frac{m_d}{\delta(u-1)/2} \right\rfloor \geq \frac{s_u}{(u-1)/2}\,\Gamma(T).$$

Proof. Categorical Cauchy supplies a nontrivial u-primary phase. If higher u-order occurs, square covariance supplies at least $u(u-1)/2$ distinct nontrivial phases. If only order u occurs, at odd valuation there are at least $(u-1)/2$ such phases. At even valuation every nontrivial u-phase has the same squared mass by Proposition 19.1, hence all $u - 1$ occur. Each resulting nontrivial u-fiber contains no vacuum, so its cardinality is at least $\delta\Gamma(T)$. The trivial u-fiber, containing the vacuum, contributes at least $\Gamma(T)$. At any fixed degree the u-active multiplicity is divisible by $\delta(u-1)/2$ by square covariance (and dual pairing in odd dimension). Its codegree is divisible by u by Theorem 3.6. Dividing the active count into these degree packets proves the claimed bound. □

Lemma 19.4 (Dyadic torus on a complementary fiber)**.** *If $2 \mid q$, every nonempty fiber of any collection of odd-primary components has at least two labels in ordinary or positive reduced data.*

Proof. Work in the ordinary modular representation, using a minimal modular extension of dimension 4Δ in the reduced case, with the congruence lift described in Section 3. Let $W, U(x), L(x), H_a$ denote the Weyl, upper and lower unipotents and diagonal torus in the dyadic local factor, with $H_a = \mathrm{diag}(a, a^{-1})$. The fiber space is preserved by W and $U(2)$, since the odd twist operators belong to commuting local factors. In the reduced case these two operators also preserve the positive fermion subspace: even powers of the global twist preserve it, and the dyadic Weyl is

the global Weyl times the inverses of the odd local Weyls, all of which preserve it (Section 3). The even local unipotents are powers of $U(2)$.

For a one-dimensional character λ of the subgroup generated by $W, U(2)$, Weyl conjugacy gives $\lambda(L(x))\lambda(U(x)) = 1$ for every even x. In the local coefficient ring $\mathbb{Z}/2^a\mathbb{Z}$ the matrix identity

$$U(2)L(2) = L(2/5)H_5U(2/5)$$

therefore gives $\lambda(H_5) = 1$. On the other hand, the local H_5 is the signed Galois permutation with pure dyadic multiplier five ([21, Theorem II]). This automorphism negates $\sqrt{2}$, fixes i, and fixes all odd-conductor roots. It therefore negates $\sqrt{\Delta}$ for every even square class q. Every ordinary label here is integral, and in the extension the two representatives of each reduced label are integral. Applying the Galois formula to the positive vacuum coordinates $d_i/\sqrt{\Delta}$ (or $d_i/(2\sqrt{\Delta})$ in the extension) shows that H_5 acts as minus a permutation on the specified ordinary basis, respectively on the basis of positive pair sums. For the latter, invariance of the positive subspace, already shown for the displayed product representing H_5, ensures that a pair sum maps to a pair sum with this common minus sign. On a singleton fiber it thus acts by -1, a contradiction. □

Refine any fixed odd-primary fiber successively by every $t \in T$ as in Lemma 19.2. This produces at least $\prod_{t\in T}(h_t + 1)$ distinct tuples of odd phases. Each tuple has at least two labels by Lemma 19.4. Consequently for $2 \mid q$ one may use

$$\Gamma_2(T) = 2\prod_{t\in T}(h_t + 1)$$

in both bounds of Proposition 19.3. At table rank at most 30 this leaves only $u = 3$ as an additional odd prime for $q = 10$ or 14, and no such prime for $q = 22$ or 26. Since the dimension of a nontrivial simple perfect integral modular category contains at least three primes, for every odd prime $p \geq 5$ the square class $q = 2p$ forces rank at least $3(p + 1)$: any third prime u has even valuation and $u \geq 3$. In particular the square classes 22 and 26 have rank at least 36 and 42 respectively, independently of the exponents.

Lemma 19.5 (Two labels in a local invariant subspace)**.** *Fix the linear congruence lift for an integral modular category. Let p be a prime and suppose that the span of two coordinate vectors v_X, v_Y is invariant under the p-local congruence factor. If its local Weyl element W_p does not preserve $\mathbb{C}v_X$, then $p = 2$, $d_X = d_Y$, and the ratio of their actual p-primary twists is a primitive fourth root.*

Proof. The local upper unipotent U_p acts diagonally with ratio ω of p-power order: the anomaly normalization cancels in the ratio. Its two eigenvalues are distinct, since otherwise its conjugate lower unipotent is scalar too, and the two generate the local group. The two-dimensional unitary action is irreducible. Indeed, an invariant line would be a U_p eigenline, whose orthogonal complement is a second invariant eigenline, contrary to the hypothesis on W_p. The group element $W_p^2 = -I$ is central. After multiplying W_p by a scalar of absolute value one, its restriction is a non-scalar Hermitian involution of the form

$$S' = \begin{pmatrix} a & b \\ \bar{b} & -a \end{pmatrix}, \qquad a \in \mathbb{R}, \quad b \neq 0, \quad a^2 + |b|^2 = 1.$$

The relation $(W_pU_p)^3 = W_p^2$ implies that $S'U_p$ has projective order three: projective order one would make S' diagonal. Its two eigenvalues therefore have ratio a primitive cube root, and their trace has absolute value one. Taking the trace in the displayed basis gives

$$a^2|1 - \omega|^2 = 1.$$

In particular $a \neq 0$.

Write $s = W_{p'}W_p$ with the complementary Weyl element commuting with U_p and thus preserving every U_p-eigenspace. If the vacuum eigenvalue of U_p were different from both eigenvalues on our subspace, the vacuum entries of both s-columns would vanish. It thus agrees with exactly one. On its eigenspace, the two columns are multiples of the same vector $W_{p'}v_X$ or $W_{p'}v_Y$. Their strictly positive vacuum entries give $d_Y/d_X = |a|/|b|$ or $|b|/|a|$. Consequently a^2 is rational. Thus $\omega + \omega^{-1}$ is a rational algebraic integer of absolute value at most two. A nontrivial prime-power root of unity with this property has order $2, 3$ or 4. At these orders a^2 is respectively $1/4, 1/3, 1/2$. The first two yield irrational vacuum ratios, and the last gives $d_X = d_Y$ as claimed. □

Lemma 19.6 (Dyadic tests in an even square class)**.** *Let $D = qM^2$ in an ordinary non-pointed fusion-simple integral modular category, with q square-free and $2 \mid q$. A two-label odd-primary twist fiber in a union of full dyadic blocks has equal degrees. If $p \mid M$ is odd and $p \nmid q$, and m_d counts nonunit labels of degree d with $p \mid D/d^2$, then for $h = (p-1)/2$ one has*

$$\sum_d \left(\lfloor m_d/h \rfloor + \lfloor m_d/(2h) \rfloor\right) \geq 6.$$

An odd-length run at odd denominator needs a partner orbit at even denominator c of length $4^{v_2(c)}$.

Proof. The span of such a fiber is invariant under the dyadic congruence factor by Lemma 3.3, and $v_2(f_i) = 1 + 2v_2(b_i) > 0$. Lemma 3.4 excludes an invariant coordinate line, so Lemma 19.5 applies.

At an odd prime $p \mid D$ with $p \nmid q$, put $h = (p-1)/2$ and let m_d count labels of each eligible nonunit degree ($p \mid D/d^2$). Put $P = \sum_d \lfloor m_d/h \rfloor$, $H = \sum_d \lfloor m_d/(2h) \rfloor$. We claim $P + H \geq 6$. A phase of order p^2 or higher gives at least ph distinct phases by square Galois covariance. Each nonempty fiber contains at least two labels, by invariance under the dyadic action and Lemma 3.4. Translation by square units therefore gives $P \geq 2p$. Otherwise both classes of nontrivial p-phases occur by Proposition 19.1. In each class the fiber is either of size at least three, supplying three h-packets, or has two equal degrees, supplying two h-packets and one unit of H. The two classes are disjoint.

At odd b, the dyadic defect is one and the character field is unramified at two by Theorem 2.9. Its block contains one other inertia orbit by Proposition 2.13. If that orbit is unramified, the two labels have equal degree by the two-label argument applied to the whole block. Otherwise the other orbit has even denominator c and length $4^{v_2(c)}$ by the same proposition. An odd-length run at odd denominator needs at least one such partner orbit. A run of n equal even denominators c supplies at most $\lfloor n/4^{v_2(c)} \rfloor$. Both this bound and $P + H \geq 6$ can be applied to a prefix by allowing the last run to grow and reserving the unused slots for any missing packets. □

Proposition 19.7 (Degrees at the minimum fiber size)**.** *In the setting of Lemma 19.2, if a complementary fiber has exactly $h_t + 2$ labels, its degree multiset is $\{r, r, (2r)^{[h_t]}\}$ for some positive integer r, or, only at $t = 3$, three copies of r.*

Proof. The fiber space is invariant under the odd local factor $G = \mathrm{SL}_2(\mathbb{Z}/t^a\mathbb{Z})$, since its complementary primary twist operators commute with this factor. Use the ordinary congruence lift; in the reduced case restrict the odd factor to the positive fermion space as in Section 3. The determinant character of this G-representation cannot take the value -1: reduction from $\mathrm{SL}_2(\mathbb{Z})$ is onto, its abelianization is generated by the upper unipotent, and the image of this generator in G has odd order. Let v generate the local units. Its quadratic character is negative. The torus H_v acts on the fiber as minus a permutation, since its Galois sign on every integral vacuum coordinate $d_i/\sqrt{\Delta}$ is $(v/t) = -1$. In the reduced lift the two ambient representatives are integral, the normalization has the extra rational factor two, and their positive sums give the same signed action. Thus the number of cycles of this permutation must be even, by $\det(-P) = (-1)^{\#\mathrm{cycles}(P)}$.

Higher order is too large unless $t = 3$ and the three labels form a single orbit of primitive ninth phases. Its single cycle is impossible. For $t \geq 5$ there are exactly two inactive labels and h_t active ones. The active set is one cycle under v^2 acting on twists, so the two inactive labels must be exchanged by H_v. They have the same degree r. The slice identity gives active degree $2r$. The same argument applies at three with two inactive labels; with one inactive and two active, the active labels must be exchanged, and the slice identity gives three equal degrees. □

For instance take an even-dimensional table, u an odd divisor of positive even valuation, and another odd prime t of odd valuation. If exactly $2(h_t + 2)$ degree packets of size h_u are available at u, all must be used. No higher u-order can occur: it would give at least $uh_u > 2h_u$ distinct phases, each costing at least $h_t + 2$ labels. Each nontrivial u-phase therefore has the minimum number $h_t + 2$ of labels. Degrees are constant as multisets over each of the two u-square classes by inertia; the two class masses agree. The first pattern of Proposition 19.7 has mass $2tr^2$ and the exceptional ternary one has mass $3r^2$. When $t = 3$ these masses cannot coincide across the two different patterns, since their ratio would be twice a rational square. Hence all nontrivial u-phases have the same multiset of degrees. The available packets themselves, counted once per h_u labels, must number four at r and $2h_t$ at $2r$, or six at r for the equal-degree ternary alternative.

19.2 Simultaneous bounds on the missing denominators

We now combine several even-valuation primes. Work with ordinary integral modular data of dimension $D = qM^2$, where q is square-free and the nonunit denominators $b_i = M/d_i$ are integers. Put $T = \{t : t \mid q,\ t \text{ odd}\}$, $a_t = (t + 1)/2$, and $P = \prod_{t \in T} a_t$, with empty product one. Define

$$B(q) = \max\Big(\{2^{\mathbf{1}_{2 \mid q}} P\} \cup \{(a_t + 1)P/a_t : t \in T\}\Big).$$

Every nonempty fiber obtained by prescribing odd-primary twist components outside T contains at least $B(q)$ labels. Indeed, refine by the phases at $t \in T$. Each successive refinement has at least a_t children by Lemma 19.2. Leaving t to the last step gives at least $a_t + 1$ labels in each of its parent fibers. Alternatively, after all refinements, each terminal fiber contains at least two labels if $2 \mid q$, by Lemma 19.4.

Proposition 19.8. *Let S be any set of odd prime divisors of D of even valuation. At least $B(q) \sum_{p \in S}(p - 1)$ labels have twist order divisible by some $p \in S$.*

Proof. Project the twists to the components at S and partition their values by the vector of exact orders. In a nonempty stratum let $A \subseteq S$ consist of the active coordinates. With all other components fixed, an active p-coordinate takes at least $p - 1$ values of the prescribed order by Proposition 19.1. This remains true if only the other S-coordinates are fixed: choose a full complementary twist occurring over them. Successively projecting away coordinates now bounds the number of distinct tuples in the stratum below by $\prod_{p \in A}(p - 1)$. For nonempty A this product is at least $\sum_{p \in A}(p - 1)$, since each factor is at least two. Every $p \in S$ occurs in at least one active set by categorical Cauchy. Hence there are at least $\sum_{p \in S}(p - 1)$ distinct nontrivial projected tuples, each supporting at least $B(q)$ labels by the fiber estimate. These tuples have disjoint fibers. □

For a nonempty prefix let h be the unfilled slots, L its lcm, b_* its last denominator and m_b its multiplicities. At an odd $p \mid L$ with $p \nmid q$, only runs with $p \mid b$ are eligible. Put $u_p = (p - 1)/2$. Each closed run contributes whole u_p-packets, whereas every label already seen in the open run may count. The prefix capacity at p and the total deficit are

$$C_p = u_p \sum_{b:\, p \mid b} \lfloor m_b/u_p \rfloor + \mathbf{1}_{p \mid b_*}(m_{b_*} \bmod u_p), \qquad H = \sum_{\substack{p \mid L \\ p \nmid 2q}} \max\{0, B(q)(p - 1) - C_p\}.$$

Apply Proposition 19.8 to the set of primes with positive deficit. Its prefix union has at most the sum of their capacities, even if different primes use the same label. At most h further labels occur in the suffix, so $H \le h$. An unseen odd $\ell \mid M$, $\ell \nmid qL$, has capacity zero. Adjoining it gives $H + B(q)(\ell - 1) \le h$. This cutoff leaves two and all primes in qL available (`optimized/nonsquare/joint_capacity.hpp`).

The trivial S-fiber adds $B(q)$ labels to the nontrivial ones, whence $r \ge B(q)(1+\sum_{p\in S}(p-1))$. At $q = 6$ adding five and seven gives $4(1+4+6) = 44$; at $q = 15$ adding seven gives $9(1+6) = 63$, with unrestricted positive even valuations.

19.3 Multiplicity at primes of defect one

Here D denotes the ordinary modular dimension unless specified otherwise. The defect of a label at p is $v_p(f_X)$.

Proposition 19.9 (Simultaneous defect-one primes)**.** *In a nontrivial ordinary or positive reduced simple perfect table, let $x > 1$ have multiplicity m and codegree $f = \Delta/x^2$. For $J = \{p \text{ odd prime} : v_p(f) = 1\}$ one has*

$$m \le f - 1 - \sum_{p\in J} \frac{p-1}{2}.$$

Proof. At each $p \in J$ the p-part of the twist of a degree-x label is trivial or has order p, by Theorem 3.6. For $S \subseteq J$, write m_S for the number of such labels with order p at every $p \in S$ and inactive at the other primes of J. Then

$$m_S \le C_S = \frac{f}{\prod_{p\in J} p} \prod_{p\in S} (p-1).$$

In fact $v_p(\Delta) = 2v_p(x)+1$ is odd. In a fixed prime-to-p twist fiber, let M_j be the squared mass of the labels of p-primary order p^j. By Proposition 3.9, $M_1 = (p-1)M_0$. Thus M_0 and M_1 are at most $1/p$ and $(p-1)/p$ times the fiber mass, respectively, even in the presence of higher orders. Conditions already imposed at other primes give a union of such fibers. Imposing the conditions successively multiplies their upper bounds. Division by x^2 gives the displayed inequality.

All C_S are integers and their sum is f. In the empty pattern the vacuum gives strict inequality. For $S = \{p\}$ take the full principal p-block and the fiber where the complementary twist is trivial (up to fermion sign in the reduced table). The vacuum contributes to M_0, hence the same blockwise relation gives an order-p label in this fiber. It lies outside the degree-x class: square-fullness gives $v_p(\Delta) \ge 3$, and principal labels have defect at least two by Corollary 4.1. This label lies in the singleton pattern, so $m_{\{p\}} < C_{\{p\}}$. At fixed degree the local quadratic Galois action permutes the primitive p-twists in each square class transitively and preserves inactivity at the other primes. Thus $m_{\{p\}}$ is a multiple of $(p-1)/2$, as is $C_{\{p\}}$. Each singleton saves at least $(p-1)/2$, and the empty pattern saves one. □

Corollary 19.10 (A rank bound for a square class)**.** *Suppose $D = qM^2$ with $q > 1$ square-free and composite, and let a be the least prime divisor of M. With no bounds on the exponents in M,*

$$\operatorname{rk}(\mathcal{C}) > q + (a^2-1)\left(1 + \sum_{\substack{p\mid q \\ p \text{ odd}}} \frac{p-1}{2}\right).$$

For $D = 15M^2$, the following direct count gives rank at least 33, and at least 55 when M is odd. In the latter case the enumeration of Theorem 22.2 strengthens the bound to 75.

Proof. Each degree divides M. If m degrees equal M, all the others are at most M/a, and the unit is strictly smaller: the table is square-full and at least three distinct primes divide M. Thus $q < m + (\mathrm{rk}(\mathcal{C}) - m)/a^2$. Apply Proposition 19.9 with $f = q$ when $m > 0$. For $m = 0$ the same upper bound on m holds since $1 + \sum_{p|q,\ p \text{ odd}} (p-1)/2 \le q$.

For $q = 15$ this gives $r > 27$ for even M. For odd M it gives $r > 47$, and $r \equiv D \equiv 7 \pmod 8$ raises the lower bound to 55.

Take M even and $r \le 32$; then $30 \mid M$. At least three nonunit degrees are five-free: two are required by Lemma 18.1, and two nonzero squares cannot sum to -1 modulo five. They are at most $M/5$. Write b for the multiplicity at $M/2$. The following normalized square-mass maxima use $1/M^2 \le 1/900$. Nonunit tails outside the maxima and halves cost at most $1/9$ each.

$$\begin{array}{r|l} m \le 10 & 10 + 18/4 + 3/25 + 1/900 < 15 \\ m = 11,\ b \le 14 & 11 + 14/4 + 3/9 + 3/25 + 1/900 < 15. \end{array}$$

The simultaneous defect-one bound gives $m \le 11$; $m = 11$ forces $b \le 15$ by the vacuum. Thus $(m, b) = (11, 15)$, leaving mass $1/4$. At $r \le 31$ its upper bound is $1/9 + 3/25 + 1/900 < 1/4$. At $r = 32$ there are five nonunit tails. If $4 \mid M$, at least three are odd by reduction modulo four. Choose three odd and three five-free tails. Their overlap has size $j = 1, 2, 3$ and they have denominators at least $4, 5, 20$ in the odd-only, five-free-only and common classes. The other $j - 1$ tails cost at most $1/9$ each. But

$$\max_{1 \le j \le 3} \left(j/400 + (3-j)/16 + (3-j)/25 + (j-1)/9 \right) + 1/900 < 1/4.$$

If $2 \parallel M$, the five tail squares sum to $(M/2)^2 - 1 \equiv 0 \pmod 8$. With o odd tails this gives $o + 4(5 - o) \equiv 0 \pmod 8$, hence $o = 4$. Their denominators are 2 mod 4 and larger than two, giving the final contradiction $4/36 + 1/9 + 1/900 < 1/4$.

□

If q is prime, degree M is impossible by Lemma 20.1. The same comparison, with $m = 0$, therefore gives $r > qa^2$.

Testing several defect-one degrees together. For an odd prime $p \mid q$ put $w = (p-1)/2$. Select any set of *closed* runs of nonunit denominators b_j prime to p, with multiplicities m_j. A closed run is one whose multiplicity can no longer grow. There must be integers $0 \le t_j \le m_j$ divisible by w such that

$$\sum_j \frac{m_j - t_j}{b_j^2} < \frac{q}{p}, \qquad \sum_j \frac{t_j}{b_j^2} < \frac{(p-1)q}{p}.$$

Indeed t_j counts the labels of exact p-order in that run. No higher order is possible there. The slice identity of Proposition 3.9 gives the two total mass upper bounds. Both inequalities are strict on the selected runs: the principal block lies outside all of them by Corollary 4.1, and contains the vacuum and positive exact-p mass by the same slice identity. Scaling by $\mathrm{lcm}(b_j)^2$ gives a bounded integer sum with two strict endpoints. The program `code/defect_mass.py` can apply it to a prefix or a complete profile. It uses at most the six smallest eligible runs, so discarding any further runs is conservative.

Partition of the defect-one degrees.

Proposition 19.11. *In either integral categorical table, let p be an odd prime with $v_p(\Delta)$ odd. The multiset of degrees d with $v_p(\Delta/d^2) = 1$ partitions into blocks of the following two forms, with exponents denoting multiplicities:*

$$\{r^{[k^2]}, (kr)^{[p-1]}\} \quad (k^2 \mid p-1), \qquad \{r^{[2k^2]}, (2kr)^{[(p-1)/2]}\} \quad (2k^2 \mid p-1).$$

For $p = 3$ these become three copies of r, or two copies of r and one of $2r$.

Proof. By Proposition 2.13 a block containing a defect-one character consists of precisely two tame inertia orbits of defect one. The proof of Corollary 3.10 applies in both tables: the odd-primary twist order divides the codegree and the two masses have ratio $1 : (p-1)$ by Proposition 3.9. Thus one orbit has trivial primary twist and the other order p, with respective lengths $a, b \mid p-1$ and degrees r, s satisfying $bs^2 = (p-1)ar^2$. The square Galois action forces $(p-1)/2 \mid b$. If $b = p-1$, then $a = (s/r)^2$ is an integer square. If $b = (p-1)/2$, then $2a = (s/r)^2$ is an even integer square. This gives the two lists. □

Corollary 19.12 (Integer feasibility for defect-one degrees)**.** *Let m_d count the labels at each defect-one degree of Proposition 19.11. Form the finite list $\mathcal{T}$ of its two allowed block multiplicity vectors whose support is among these degrees and whose entries do not exceed m_d. The required integer condition is*

$$m_d = \sum_{T \in \mathcal{T}} n_T T_d \quad \text{for every such } d, \qquad n_T \in \mathbb{Z}_{\geq 0}.$$

For $p = 3$ the two column types are $3[d]$ and $2[d] + [2d]$. At an incomplete prefix enforce the equalities on closed degrees and allow unused slots at compatible unprocessed degrees.

Proof. Count the blocks of each permitted type. Conversely any integral solution partitions the multiplicity multiset into these types. This converse concerns only the multiset; the local action and the rest of the table impose further conditions. Projecting away unprocessed degrees only relaxes a necessary condition. □

The complement of the defect-one blocks has its own necessary signed-mass test when $v_p(D) \geq 3$. For a nonunit degree d occurring m_d times there, let t_d, u_d count its labels of primary twist order p and of order at least p^2, respectively. In ordinary even dimension they are multiples of $h = (p-1)/2$ and ph, respectively. In odd dimension the multiples double because duality pairs labels of the same twist. Proposition 3.9 on the whole set, less the same identity on the defect-one blocks, gives

$$\sum_d d^2 [t_d - (p-1)(m_d - t_d - u_d)] = p - 1.$$

The right hand side is the omitted vacuum. On the principal block alone the same target must be attained by $\sum d^2[t_d - (p-1)i_d]$, where $i_d, t_d \geq 0$, $i_d + t_d \leq m_d$. Indeed Corollary 4.1 removes all defect-one labels from this block, which is Galois- and duality-stable. The order-p counts have the same divisibility, and the inactive counts are even in odd dimension. The implementation in `code/odd_blocks.py` checks both integer conditions by suffix bounds and gcds, and the block partition by exact multiset subtraction. Each matching is a necessary condition.

Exact prefix test for a family of odd primes. Write $D = qM^2$ with q square-free and put $z = M/d$ for a simple degree d. Let $p \mid q$ be an odd prime such that $8 \nmid p-1$ and the odd part of $p-1$ is square-free. An individual label has p-defect one precisely when $p \nmid z$. Set $w = (p-1)/2$. The counts at these denominators must be a nonnegative sum of the two patterns

$$p \text{ labels at } z, \qquad w \text{ labels at } z \text{ and two at } 2z.$$

In the first block of Proposition 19.11, $k = 1$ or 2. In the second block $k = 1$. The case $k = 2$ in the first list gives two copies of the second pattern, which proves the assertion. The system of Corollary 19.12 reduces to a recurrence on doubling chains. Process each completed run at z in increasing order. Subtract from its multiplicity the counts already required by the run at $z/2$, leaving v. Negative v fails. Set y equal to the least nonnegative residue of $-2v$ modulo p. One requires $wy \leq v$, and reserves $2y$ slots at $2z$. Indeed $v = px + wy$ and $w^{-1} \equiv -2 \pmod{p}$. Choosing $y + kp$ in place of y can be reversed by replacing kp mixed tiles by kw homogeneous

tiles at z and $2k$ at $2z$. Thus the least feasible choice suffices. A pending denominator skipped by the search, or more pending slots than remain, rules out that prefix. The primes through 31 to which the test applies are $3, 5, 7, 11, 13, 23, 29, 31$. When several primes are tested, a label can meet requirements at more than one prime. At a given denominator the maximum of their reservations therefore gives a valid joint lower bound. Summing these maxima over different denominators bounds the number of remaining slots.

Principal witnesses on the degree list. For distinct dimension primes r, p with r odd, and a dimension prime $t \neq r$ (odd in a reduced table), Lemma 16.1 and Corollary 4.1 require nonunit degrees x, y with $r \nmid x$, $t^2 \mid \Delta/x^2$, $p \nmid y$ and $r^2 \mid \Delta/y^2$. Their gcd g satisfies $g \geq 2$ and $r \leq \max\{5, (g-1)g^2\}$. The test permits $p = t$ and $x = y$. It is implemented in `code/spectrum_crossing.py`.

19.4 Mass and multiplicity of active degrees

For an odd prime p of positive even valuation in the table dimension, Proposition 19.1 equates the squared masses of the two square classes at exact order p in any full block and complementary fiber, including when a higher-order phase occurs. Set $h = (p-1)/2$, doubled when the table dimension is odd. At degree d of multiplicity m, let $n = \lfloor m/h \rfloor$. If $p \nmid f_d$ there are no active labels. Otherwise let k be the signed difference of the two numbers of exact-order-p packets at this degree. Necessarily $|k| \leq n$ and $\sum_d k_d d^2 = 0$. For fixed k, at most $h(n-\epsilon)$ labels are active, where $\epsilon = 1$ if $n - |k|$ is odd and either $p^2 \nmid f_d$ or $n - |k| < p$, and $\epsilon = 0$ otherwise. Indeed opposite-sign packets cost two slots of size h, and a packet of higher order costs a multiple of ph. Maximizing the resulting upper bound on active mass subject to the one integer equality yields a direct test, implemented in `code/class_mass.py`. The same condition applies separately to the principal p-block and its complement. For a full principal block candidate P, local integrality of the block idempotent at the vacuum gives $p^{v_p(D)} \mid \sum_{Y \in P} d_Y^2$. Necessarily $p^2 \mid f_Y$ for all its labels. Each row of degree prime to p meets every column in P; these necessary conditions allow a preliminary enumeration of P.

For a prefix whose runs are closed and with fewer than $2h$ unfilled slots, a closed eligible run of size at least $2h$ permits an internal balance. Otherwise each eligible closed denominator supplies at most one packet. Put L equal to their lcm and take subset sums of $(L/b)^2$ over the available packets. With fewer than h slots remaining, a collision between different subsets is necessary. With at least h slots, one additional packet may instead complete a difference of two subset sums to $(L/c)^2$ for a future denominator c. This forces $c \mid L$, since $(L/c)^2$ is an integer. Only divisors of L at least the next permitted denominator and divisible by p need examination. Here $p \nmid q$ and the support prime p is already known. Higher primary orders would require ph slots in a single run. This proves the prefix test implemented in `code/class_mass.py` without anticipating the final value of M.

The mass forced by a generating simple. Let $D = qM^2$ with q square-free in the ordinary simple perfect integral modular case. Write m_d for the number of simples of degree d, g for the least nonunit degree, and $f_d = D/d^2$. In particular every occurring d divides M.

Proposition 19.13. *For every prime $p \mid D$, let $h = 1$ if $p = 2$, and let $h = (p-1)/2$ for odd p, doubled when D is odd. Then*

$$2pg \sum_{d>1:\, p \mid f_d} h \lfloor m_d/h \rfloor d^2 \;\geq\; (p-1)D.$$

Proof. Let L_p be the total squared degree of the labels whose twist is p-active. Categorical Cauchy supplies such a label. A smallest nonunit simple generates the category, so Proposition 3.8 gives $2pgL_p \geq (p-1)D$. Every active label satisfies $p \mid f_d$ by Lemma 3.4. For odd p, fix a

degree and an exact primary twist order p^j, $j \geq 1$. The Galois square action on primitive roots has orbits of size $p^{j-1}(p-1)/2$ with constant multiplicity. In odd dimension every label in a nontrivial twist fiber pairs with a distinct dual of the same degree and twist. Thus the active count at each degree is a multiple of h. The vacuum is inactive. □

Here is a bound involving only a denominator threshold and a count. Take an odd $p \mid M$, $p \nmid q$. Suppose all nonunit degrees with $p \mid M/d$ have $M/d \geq B$. Suppose also that K labels are known to be p-inactive. The number of p-active labels is a multiple of h, each of degree at most M/B. Therefore

$$R \;\geq\; K + h\left\lceil \frac{q(p-1)B^2}{2pgh} \right\rceil. \tag{19.1}$$

Even before the type is known, $B = p$, $K = 1$ and $g \leq U = M/p^{v_p(M)}$ are valid. Indeed $p \mid D$, so reduction of $\sum d_i^2 = D$ modulo p forces a nonunit p-free degree, which divides U. In particular $R \geq 1 + h\lceil qp(p-1)/(2Uh)\rceil$ at $D = qU^2p^{2e}$ for every $e \geq 1$ and $p \nmid qU$. This form bounds complete ranges of exponents at a fixed cofactor.

Corollary 19.14. *Let $D = 3M^2$ and suppose ten labels have degree $M/2$. For every $p \geq 5$ dividing M, equation (19.1) holds with $K = 13$. In particular, throughout $D = 2^{2a}3^{2b+1}5^{2c}$, $a, c \geq 1$, if every nonunit 5-eligible denominator M/d is at least B, then*

$$R \;\geq\; 13 + 2\left\lceil \frac{3B^2}{5g} \right\rceil.$$

Here 5-eligible means $5 \mid M/d$.

Proof. Degree M is impossible by the ordinary small-codegree bound. In the three-defect-one partition (Proposition 19.11), a copy of $M/2$ belongs either to an equal triple or to a block containing two copies of $M/4$ and one of $M/2$. Ten is not a multiple of three; the latter block must occur. The resulting twelve labels have codegrees 12 or 48, so their twists are p-inactive for $p \geq 5$. The vacuum is a thirteenth such label. In the exceptional numerical case $M/4 = 1$ the mixed block would require two distinct invertibles, contrary to perfection. □

A divisor from the regular indicator. Suppose that $D = qM^2$ is the ordinary table dimension, p is an odd prime dividing M but not q, and there is no twist of p-primary order at least p^2. Let C be the squared-degree mass on any one nontrivial p-primary phase. It is positive and independent of the chosen phase by Proposition 19.1. Then

$$(qM/p) \mid C.$$

Indeed, take the higher indicator of the regular object at the prime-to-p part k of the twist exponent. The ordinary indicator formula [45, Theorem 7.5] and $\sum_X d_X N_{ij}^X = d_i d_j$ give the algebraic integer

$$\sum_X d_X \nu_k(X) = D^{-1}\left|\sum_i d_i^2 \theta_i^k\right|^2 = (D - pC)^2/D.$$

An integer z with $D \mid z^2$ is divisible by qM. Since $qM \mid D$, the claim follows. The common phase mass is thereby constrained for every square-free kernel.

In particular consider the family $D = qU^2p^{2e}$, $e \geq 1$ and $p \nmid qU$. Put $\delta = 2$ for odd D and 1 otherwise, $h = \delta(p-1)/2$. A degree $d > 1$ can carry a nontrivial phase only if $p \mid f_d$ and $m_d \geq h$. If all such degrees with $p^2 \mid f_d$ have multiplicity less than ph, the higher-order hypothesis follows from the inertia orbit bound. If just one eligible degree d remains, the number of copies it requires is at least

$$\delta(p-1)\frac{qUp^{e-1}}{\gcd(qUp^{e-1}, \delta d^2)}.$$

Indeed $C = k'\delta d^2$ with integer $k' > 0$, the same for all $p-1$ phases. With any number of eligible degrees, C is a multiple of both qM/p and $\delta \gcd_d(d^2)$. The total eligible mass, with multiplicities rounded down to multiples of h, must reach $p-1$ times the lcm of these integers. The implementation is `code/regular_indicator.py`.

20 Character rows of small codegree

Let X be a nonunit in a nontrivial fusion-simple perfect integral modular category. Write $n = d_X$, $u_Y = S_{XY}/n$, $f = D/n^2$ and $c_Y = \operatorname{mean} |u_Y|^2$, where the mean is the normalized cyclotomic trace. The $(X, 0)$ modular relation, its Galois conjugates and Cauchy–Schwarz give

$$\sqrt{f} \le \frac{1}{n} \sum_Y d_Y \sqrt{c_Y}, \qquad \sum_Y c_Y = f. \tag{20.1}$$

On a nonzero entry $c_Y \ge (d_Y / \gcd(n, d_Y))^2$ by lcm integrality. We use $d_Y \ge 6$ at nonunits (Lemma 2.5) and $\sqrt{D} \ge 5400$ (Corollary 17.3). When $D = M^2$, write $b_Y = M/d_Y$ for the denominator, including $b_0 = M$.

20.1 Trace and norm restrictions

Lemma 20.1 (Prime codegrees)**.** *No prime is a formal codegree of a nontrivial perfect integral modular category.*

Proof. By Lemma 2.5, D is square-full. If $D = pn^2$ and $f_X = p$, square-fullness implies $p \mid n$. All degrees divide n and principal support and principal defect (Theorem 2.11 and Corollary 4.1) force the supported degrees to divide n/p. Now (20.1) gives $\sqrt{p} \le p^{-1} \sum c_Y = 1$. □

Lemma 20.2 (Integral trace vectors with a prescribed orbit)**.** *Let d_i be the degrees of a nontrivial simple perfect table with integral degrees, symmetric S and ordinary or positive reduced orthogonality. Write $D = \sum_i d_i^2$ for its table dimension and m_y for the multiplicity of the degree y. Fix a nonunit label X of degree x and put $f = D/x^2$. There is an integer $1 \le h \le \min(m_x, f-1)$ and a family of integers z_i satisfying*

$$z_0 = h, \qquad \sum_{i \ne 0} d_i z_i = -h, \qquad \sum_{i \ne 0} z_i^2 = h(f-h).$$

For $i \ne 0$ with $\gcd(x, d_i) = 1$ one has $z_i = 0$. For every i,

$$\frac{d_i}{\gcd(x, d_i)} \frac{h}{\gcd(h, \operatorname{lcm}(1, \dots, m_{d_i}))} \mid z_i.$$

Among the m_x labels of degree x, at least h have a common coordinate z_i. In the ordinary case, if $h = 1$ one also has $1 + \sum_{i \ne 0} d_i |z_i| \ge \lceil \sqrt{D} \rceil$.

Proof. Let $K = \mathbb{Q}(\chi_X(i) : i)$, with $\chi_X(i) = S_{Xi}/x$. This is an abelian extension, and its $h = [K : \mathbb{Q}]$ embeddings give h distinct character rows. They have common degree x and are mutually orthogonal with squared norm f. Set $z_i = \operatorname{Tr}_{K/\mathbb{Q}} \chi_X(i)$. These ordinary integers are the coordinates of the sum of the h rows. Its squared norm is hf. It is orthogonal to the Perron character, and its vacuum coordinate is h. These facts give the equations and imply $h < f$. Galois conjugation permutes the arguments of each normalized row because S is symmetric and the degrees are rational and Galois-invariant. The trace is therefore constant on each orbit of labels, including that of X. Simplicity gives the asserted vanishing at coprime nonunit degrees. Put $s = d_i / \gcd(x, d_i)$ and $a = \chi_X(i)/s$. By Bezout, $a = S_{Xi} / \operatorname{lcm}(x, d_i)$ is an algebraic integer. The field $\mathbb{Q}(a)$ is contained in the two row fields of X and i by symmetry. Its degree e divides h and also divides the degree of the row field of i, which is at most m_{d_i}. Hence e

divides $\gcd(h, \operatorname{lcm}(1, \dots, m_{d_i}))$, and $z_i = s(h/e)\operatorname{Tr}_{\mathbb{Q}(a)/\mathbb{Q}}(a)$ has the claimed divisor. For an ordinary rational character, $h = 1$ gives $z_i = \chi_X(i)$. The $(X, 0)$ entry of $STS = p^+T^{-1}ST^{-1}$ with $|p^+| = \sqrt{D}$ gives $\sqrt{D} = |\sum_i d_i\theta_i z_i|$, and the triangle inequality concludes. □

Partitioning the labels at a fixed degree. At a degree x of multiplicity m, compute the set of admissible lengths h from the trace-vector lemma. The m labels form a disjoint union of Galois orbits, so m must be a sum, with repetition allowed, of these lengths. At prime-power codegree $f = p^a$ we use three optional restrictions. Only positions in the principal p-block can support any of the conjugate rows, so $p^2 \mid f_Y$ at such a position. If $p \mid x$, at a supported column of degree y every $\chi_Y(X')$ for a conjugate X' reduces to x modulo the chosen prime above p. Their sum $(x/y)z_Y$ is a rational algebraic integer divisible by p. This requires $py/\gcd(x, py) \mid z_Y$, also at the coordinates of degree x. Finally when $f = p^2$ a nonzero trace at a nonunit requires $y \mid x/\gcd(x, p)$ by Proposition 5.4. A repeated coordinate costs its multiplicity times its square in the trace budget, as before. The shared orbit test applies these restrictions. The scoped searches also use `code/trace_rows.py` and `code/orbit_screen.py`.

Proposition 20.3. *No formal codegree of a nontrivial simple perfect integral modular category equals six.*

Proof. For a row of degree n and codegree f, let H be its nonunit support. The sum of its entries over all labels of each degree is integral, since the Galois action on columns permutes the summands. If $f = 6$, the dimension bound $6n^2 = D \geq 29\,160\,000$ of Corollary 17.3 gives $n \geq 2205$. Each nonunit dimension has at least two distinct prime divisors, by coprime vanishing and row orthogonality. The character conductor divides f by Theorem 2.9, so $u_Y \in \mathbb{Z}[\zeta_3]$ and $c_Y = |u_Y|^2$ is an Eisenstein norm. The possible positive norms up to five are $1, 3, 4$, and $\sum_{Y\in H} c_Y = 5$. Thus the nonunit norms are $(4, 1)$, $(3, 1, 1)$ or five ones.

In the first case the two supported degrees b, c are coprime. Their gcds with n are coprime divisors of n, and the product of $a_Y = d_Y/\gcd(d_Y, n)$ over the two hits is at most two, since $a_Y^2 \leq c_Y$. Hence $b/n \leq 2/c$ and $c/n \leq 2/b$. As $b, c \geq 6$, the right side of (20.1) is at most $1/n + 1$. In the second case all three degrees divide n. At most one equals n, by their gcd. If there is none, their squared ratios to n sum to at most $3/4$. If one equals n, the other two are coprime divisors, so that each ratio is at most $1/6$. In either case Cauchy–Schwarz bounds the right side by $1/n + \sqrt{5(1 + 2/36)} < \sqrt{6}$.

In the remaining case the five entries are sixth roots of unity and all five supported degrees divide n. Write $k = n/d_Y$ on these entries and m for the number with $k = 1$. Their gcd property says that the lcm of these five k is n. The principal 3-block contains the vacuum, and the squared row norm on it is a positive multiple of three, by Theorem 2.11. Corollary 4.1 shows that each supported degree in this block has $3 \mid k$. Thus at least two of the five denominators k are divisible by three. We need

$$\sqrt{6} \leq 1/n + \sum_{Y\in H} 1/k.$$

If $m = 0$ and at most three denominators are two, the sum is at most $3/2 + 2/3$. If at least four are two, the degree at the remaining denominator divides two by the lcm property, which is impossible. If $m = 1$ and at most one of the four proper denominators is two, the only way to meet the inequality is $(2, 3, 3, 3)$: otherwise the sum including m is at most $1 + 1/2 + 2/3 + 1/4$. This would give $n = 6$. If at least two denominators are two, the remaining two are multiples of three. If either is three, the degree at the other divides two by the lcm property. Otherwise the sum including m is at most $2 + 2/6$.

If $m = 3$, the two proper degrees b, c are distinct and coprime, hence $b/n, c/n \leq 1/6$. The integral sum of the three entries at degree n must be zero by row orthogonality, since the other terms have total absolute value at most $1 + b + c < n$. A real row is impossible, as three signs cannot sum to zero. A nonreal row is orthogonal to its complex conjugate row, so $\sum_Y u_Y^2 = 0$.

The six nonzero squares must contain two copies of each cube root of unity. But the vacuum and each of the two proper degrees contribute a rational root of unity, hence three squares equal one. Four or more full degrees are impossible by the gcd property and perfection.

Suppose finally $m = 2$. If the three proper degrees repeat, write them as a, a, b; the gcd property forces a, b coprime and distinct. We have $(2a + b)/n \le 2/b + 1/a$. If either is six the other is at least 35. Otherwise both are at least ten. These bounds give $2+(2a+b)/n+1/n < \sqrt{6}$. Hence the three proper degrees are distinct. Their entries are rational roots of unity, by the degree-sum integrality above. The conjugate-row argument forces the entire row to be real. The sum at the full degree is $-2, 0$ or 2, and the three proper degrees total at most $3n/2$. Thus this sum is zero. Ordering the proper denominators as $k_1 < k_2 < k_3$, the remaining signed row equation says

$$|1/k_1 - 1/k_2 - 1/k_3| = 1/n.$$

Indeed all the degrees are at least six, and only the largest degree can balance the other two to within one. The mass bound gives $2/k_1 + 2/n \ge \sqrt{6} - 2$, so $k_1 \le 4$. Next $2/k_2 \ge 1/k_1 - 1/n$, so $k_2 \le 8$. Finally $1/k_3 \ge 1/(k_1k_2) - 1/n \ge 1/32 - 1/2205 > 1/33$. We obtain $n = \operatorname{lcm}(k_1, k_2, k_3) \le 4 \cdot 8 \cdot 32 < 2205$, a contradiction. □

Proposition 20.4 (A rational row of prime-square codegree)**.** *In an ordinary fusion-simple perfect integral modular category, if the row of a nonunit simple Z is rational and has codegree p^2 for a prime p, then $p \nmid d_Z$. In particular, if $D = M^2$ and the denominator $M/d = p$ occurs exactly once, then $v_p(M) = 1$ and M/p has at least three distinct prime divisors.*

Proof. Write $z = d_Z$ and $u_Y = \chi_Z(Y) \in \mathbb{Z}$. A nonzero u_Y has nonzero trace. Thus if $p \mid z$, every nonunit degree hit by this row divides z/p, by Proposition 5.4. The vacuum modular relation and the integer row norm give

$$p \le \frac{1}{z} + \frac{1}{p}\sum_{Y\neq 0} |u_Y| \le \frac{1}{z} + \frac{1}{p}\sum_{Y\neq 0} u_Y^2 = \frac{1}{z} + \frac{p^2-1}{p} < p.$$

Here $z > p$ by exclusion of prime-power degrees. The contradiction proves the first statement. A singleton degree gives a rational row by Galois covariance. The three-prime requirement then follows from Proposition 5.4 with $z/\gcd(z, p) = z$. □

20.2 A saturated prime-power degree class

Here R denotes the rank. Since the vacuum contributes positive mass, the multiplicity at a nonunit degree of codegree f is strictly less than f.

Proposition 20.5. *Let n be the largest nonunit degree, of codegree $f \in \{8, 16, 25, 27\}$. If $f(R - f) \ge 25$ at rank R, its multiplicity is at most $f - 2$.*

Proof. Suppose the multiplicity is $f - 1$ and let A be the sum of twists of those labels. The twists have prime-power order supported at the prime dividing f and belong to $K = \mathbb{Q}(\zeta_{4f})$ by Theorem 3.6. Reduction at that prime gives $A \equiv -1$. The remaining squared degrees sum to n^2. Every conjugate of the Gauss sum has absolute value $\sqrt{D}$, so $|\sigma A| \le 1 + f/\sqrt{D}$. The normalized trace of $A\bar{A}$ lies in $[K : \mathbb{Q}]^{-1}\mathbb{Z}$, and $[K : \mathbb{Q}] \le 40$ for the four listed codegrees. Its nonzero norm puts the mean at least one, and $(1 + 27/5400)^2 < 1 + 1/40$ makes it exactly one. Equality in arithmetic–geometric mean makes A a root of unity. Put $\eta = -A$. Multiplying the Gauss sum by $\bar{A}$ and taking real parts and normalized trace gives $\sum_{Y\notin I} d_Y^2(1 - \operatorname{mean}(\theta_Y/\eta)) \le \sqrt{D}$, where I is the largest degree class. A nontrivial root of order k has mean $\mu(k)/\varphi(k) \le 1/2$. Choose an odd dimension prime ℓ outside $4f$; there are at least three dimension primes. The ℓ-active squared mass is at most $2\sqrt{D}$, since η is ℓ-inactive. Proposition 3.8 bounds the least nonunit degree g below by $(\ell - 1)\sqrt{D}/(4\ell)$. Now $n^2 \ge 1 + (R - f)g^2$ is impossible when $f(R - f) \ge 25$

and $\ell \geq 5$, or when $f(R-f) \geq 36$ and $\ell = 3$. We can take $\ell \geq 5$ except possibly at $f = 25$. The sole additional case is $f = 25, R = 26$, where the vacuum and one nonunit Y would give $n^2 = 1 + d_Y^2$. □

20.3 Orbits and multiplicity at codegree eight

Lemma 20.6 (Common projection of orthogonal rows)**.** *Let $u_1, \dots, u_k$ be mutually orthogonal vectors of squared norm f. If an orthogonal projection P satisfies $Pu_i = v$ for every i, then $k\|v\|^2 \leq f$.*

Proof. $k^2\|v\|^2 = \|P\sum_i u_i\|^2 \leq \|\sum_i u_i\|^2 = kf$. □

Theorem 20.7 (Ordinary rows of codegree eight)**.** *Let x be a degree of formal codegree eight in a nontrivial fusion-simple perfect integral modular category. It occurs at most four times. If x is even, its multiplicity is even and there are three nonunit labels with degrees $u \leq v \leq w$ dividing $x/2$ and satisfying $|u + v - w| = 1$. If x is a largest degree, at least three odd primes divide x and it occurs at most twice, exactly twice when even.*

Proof. Rational rows at even degree. Suppose the row at X is rational, and put $z_Y = \chi_X(Y) \in \mathbb{Z}$. For $f_X = 8$ the principal-support theorem places every hit in the principal dyadic block. Since x is even, the nonzero integer $\chi_Y(X) = (x/d_Y)z_Y$ is even at every hit. Put $a = x/2$. Orthogonality gives $\sum_{Y\neq 0} z_Y^2 = 7$ and $\sum_{Y\neq 0} d_Y z_Y = -1$. If $|z_Y| = 1$ then $d_Y \mid a$, and if $|z_Y| = 2$ then $d_Y \mid x$. There are either seven entries of absolute value one, or three such entries and an entry of absolute value two. In the latter case the degree at the double entry cannot be x: the three single entries could cancel at most $3x/2$, whereas $2x - 1 > 3x/2$. This degree is therefore at most $x/2$, giving $\sqrt{8}\,x \leq 1 + 5x/2$, impossible for $x \geq 6$.

In the former case all seven degrees divide a and are at least six, so $a \geq 6$. The positive and negative degree sums differ by one. The smaller is at least $2\sqrt{2}\,a - 1 > 5a/2$, by the modular relation. There must be three labels of one sign and four of the other. All three degrees in the smaller group by cardinality must equal a, since a proper divisor of a is at most $a/2$. The four degrees in the other group sum to $3a \pm 1$. Exactly two must equal a: one or fewer gives at most $5a/2$, and three or more gives at least $3a + 6$. The two proper divisors $u, v \leq a/2$ left over therefore sum to $a - 1$. They cannot both be $a/2$, so one is at most $a/3$, forcing $a \leq 6$. This is impossible since u, v are proper divisors of a of size at least six.

Thus no row of this even degree is rational. All row fields at codegree eight have conductor dividing 8 by the conductor bound of Theorem 2.9. Their Galois orbit lengths are powers of two, hence all are even. The multiplicity of x is their sum.

The three trace coordinates. For a row at X of degree x write $h = [K_X : \mathbb{Q}]$ and $a_Y = h^{-1}\operatorname{Tr}_{K_X/\mathbb{Q}}\chi_X(Y)$. The conductor divides 8. Normalized traces of the integers of $\mathbb{Q}(\zeta_8)$ are integral since powers of ζ_8 have normalized traces 0 or ± 1. Since $h < 8$, the preceding rational-row argument gives $h = 2$ or 4. For $h = 4$, the norm identity $\sum_{Y\neq 0} a_Y^2 = 8/h - 1 = 1$ leaves one entry ± 1, contrary to $\sum_{Y\neq 0} d_Y a_Y = -1$. Thus $h = 2$, and there are exactly three nonunit entries ± 1. Each of their degrees divides x: the normalized trace of $S_{XY}/\operatorname{lcm}(x, d_Y)$ is integral, and multiplying by $d_Y/\gcd(x, d_Y)$ gives a_Y. In an imaginary quadratic field the orbit is X, X^* with equal twists. Average their two ordinary modular identities to obtain $\sqrt{8}\,x \leq 1 + \sum_{a_Y\neq 0,\, Y\neq 0} d_Y$. At least one of the three degrees is a proper divisor of x by their signed sum, so the right hand side is at most $1 + 5x/2$, a contradiction. The real quadratic subfield of $\mathbb{Q}(\zeta_8)$ is $\mathbb{Q}(\sqrt{2})$. In this field write $S_{XY}/\operatorname{lcm}(x, d_Y) = A + B\sqrt{2}$ with $A, B \in \mathbb{Z}$. If $v_2(d_Y) \geq v_2(x)$, the factor $x/\gcd(x, d_Y)$ is odd. Multiplication by it gives $\chi_Y(X)$, which reduces to zero above two at every hit by principality. Thus A is even. Then $a_Y = (d_Y/\gcd(x, d_Y))A$ cannot be ± 1. Each of the three degrees therefore divides $x/2$, and their signed sum gives the stated equation.

The uniform bound of four. Let I consist of the m labels of degree x. We use the mass inequality of Lemma 3.7:

$$\sum_{X,Y\in I} |S_{XY}|/x \;\geq\; \sqrt{8}\, m(2m/8-1).$$

Suppose $m \geq 5$. Some I-row meets I. Every I-row is supported in the principal dyadic block P, where every degree divides x by square divisibility and the principal defect bound Theorem 2.11 and Corollary 4.1.

If x is odd, principality and symmetry give support exactly P for every I-row. A hit within I then gives $I \subset P$. By the row norm $|P| \leq 8$. Apart from the vacuum and I, it contains at least two distinct proper degrees, since a single nonunit divisor of x would divide every nonunit support degree, contradicting (18.1). Thus $m = 5$, and the two proper degrees y, z are distinct coprime divisors of x. All eight entries on the support are roots of unity by equality in the Galois mean norm bound. The entries at y, z are signs, since each occurs once in the support, and the sum over I is integral by Galois symmetry. As $1 + y + z < x$ (use $y, z \geq 6$ and $yz \mid x$), orthogonality forces $1 \pm y \pm z = 0$, impossible for odd y, z.

If x is even, the parity just proved gives $m \geq 6$. By the three-coordinate argument above, each row in I has entries $a_Y + b_Y\sqrt{2}$, $a_Y, b_Y \in \mathbb{Z}$. The rational parts occur just at the unit and three divisors of $x/2$, all with value ± 1. The averaged norm gives $\sum b_Y^2 = 2$. The two nonzero irrational coefficients therefore occur at conjugate columns of the same degree and have opposite signs. If an I-row meets I, both irrational entries occur there and have absolute value $\sqrt{2}$. The row itself belongs to P by symmetry and thus hits every odd degree. Since $4 \mid D$, the number of odd degrees (counted with multiplicity) is a positive multiple of four. Consequently the four rational coordinates are precisely the four odd labels. The signs there are fixed across all I-rows meeting I: their vacuum value is 1 and the signed sum of the three nonunit odd degrees must be -1, since the two irrational entries cancel. Only the largest degree can oppose the other two; a tie for largest would leave a signed sum of absolute value at least the smallest degree, which is at least six. Thus the signs are unique, including when the smaller degrees coincide. The normalized rows $\chi_X/\sqrt{8}$ are orthonormal. Each row meeting I has squared projection $1/2$ on the unit vector given by the four fixed signs divided by two. Lemma 20.6 permits at most two such rows. Their total contribution on $I \times I$ is at most $4\sqrt{2}$, below the displayed lower bound for $m \geq 6$.

Even largest degree. Assume its multiplicity is at least three. The uniform bound argument shows that at most two rows of degree x meet a column of that degree. Choose one that misses them all. On its nonunit support y is a proper divisor of x. Write c for the Galois mean of $|\chi_X(Y)|^2$, so $c \geq 1$ on the support and the sum of the nonunit c's is seven. If $y \leq x/4$, then $y\sqrt{c} \leq xc/4$. If $y = x/3$, its rational part is zero by the three-coordinate calculation, since y does not divide $x/2$. Thus $\chi_X(Y) = \pm\sqrt{2}$ and the same inequality holds. For $y = x/2$ use $2\sqrt{c} \leq c+1$. With k supported labels of degree $x/2$, the modular inequality (20.1) yields $\sqrt{8} \leq 1/x + (7+k)/4$. The dimension bound gives $x > 13$, so $k \geq 5$. At least two other nonunit hits are required: any single mixed-prime degree $y \mid x$ has $\gcd(y, x/2) > 1$, whereas the nonunit support degrees have gcd one. Thus at least seven nonunit positions share norm seven. All have Galois mean squared norm one, so are roots of unity. This is impossible in the real character field $\mathbb{Q}(\sqrt{2})$ of the chosen row, since its real roots of unity are rational signs. Finally the parity just proved gives even multiplicity.

Prime divisors of a largest degree. At codegree eight every supported degree divides x. If x is odd, their nonunit gcd is one and each has at least two prime divisors, so x needs three odd primes. If x is even, the preceding proof in fact excludes any x-row missing all columns of degree x (the contradiction after choosing such a row used only its vanishing there). Its three odd nonunit hits therefore exist by the uniform bound argument, and their signed sum is -1. The same gcd argument applies to these three odd divisors of x.

Odd largest degree. It remains to take x odd. A row of degree x cannot vanish on the set I

of all positions of that degree: all its supported degrees divide x, and if they were proper then (20.1) would give $\sqrt{8} \le 1/x + 7/3$. Hence $I \subset P$ by the principal-support argument above. Its common support P consists precisely of the odd-degree labels. Suppose $m = |I|$ is three or four; the larger values have been excluded. The unit, I, and at least two proper nonunit positions lie in P, and $|P| \le 8$. Modulo four the dimension shows $4 \mid |P|$. Hence exactly eight entries occur and all are roots of unity of two-power order. Their sum A over I is an integer congruent to m modulo two.

For $m = 4$, there are three proper odd nonunit positions. The estimate $|xA| \le 1 + 3x/3 < 2x$ forces $A = 0$. The signs at the three proper positions are fixed by their signed degree sum -1. Indeed each singleton degree carries a rational root of unity; if two degrees coincide their two roots have integral sum in $\{-2, 0, 2\}$, because a sum ± 1 would require a third root in their ratio. A zero pair sum is impossible already at the remaining nonunit degree. If all three degrees coincided, their common degree would divide one. The four rows consequently agree on four signed entries including the vacuum, contradicting Lemma 20.6 since $4 \cdot 4 > 8$.

For $m = 3$, the four proper nonunit degrees y_i have sum at least $x - 1$ since A is odd. Their gcd is one as they divide x. The integers $z_i = x/y_i$ in increasing order are odd, at least three, have lcm x, and satisfy $\sum_i 1/z_i \ge 1 - 1/x$. Here $x > 105$ by the dimension bound. The first two denominators must be three since $1/3 + 3/5 < 1 - 1/105$. Three copies of three are impossible: the remaining mixed-prime divisor y of x has a prime in common with $x/3$. Thus $z_3 = 5$, since $2/3 + 2/7 < 1 - 1/105$, and $z_4 \le 7$, since $2/3 + 1/5 + 1/9 < 1 - 1/105$. The lcm is at most 105, a contradiction. □

20.4 The principal ternary block at codegree nine

Theorem 20.8 (Exclusion of codegree nine)**.** *A nontrivial fusion-simple perfect integral modular category has no label of formal codegree nine.*

Proof. Write $D = 9n^2$, $I = \{X : d_X = n\}$, $m = |I| > 0$, and let P be the principal 3-block. Every I-row is supported in P, whose degrees divide n, by Theorem 2.11 and Corollary 4.1. Here $n \ge 1800$ by Corollary 17.3.

Suppose first that $3 \mid n$. For $X \in I$ put $u_Y = \chi_X(Y)$. At a nonzero entry with $v_3(d_Y) = v_3(n)$, symmetry and principality give $u_Y = (d_Y/n)\chi_Y(X) \equiv 0$ above three. Its field lies in $\mathbb{Q}(\zeta_9)$ by Theorem 2.9. A wildly ramified entry would cost at least nine in the row norm over its conjugate columns by Theorem 5.1; the vacuum already costs one. Hence $u_Y \in \mathbb{Q}(\zeta_3)$. Its trace is zero by Proposition 5.4, since $d_Y \nmid n/3$. The two terms for Y, Y^* cancel in the twisted dimension sum, since u_Y is purely imaginary and duality preserves twists. This holds at every embedding. Each remaining nonunit degree on the support divides $n/3$. Its mean squared entry c_Y is at least one when nonzero. The vacuum modular relation gives $3 \le 1/n + \sum_{Y \ne 0} c_Y/3 = 1/n + 8/3$, a contradiction.

We now have $3 \nmid n$. For any $X \in I$ and $Y \in P$, $\chi_Y(X) \equiv n \ne 0$ above three, so the X-row has support exactly P. Thus $|P| \le 9$. We claim P meets I. Otherwise all nonunit degrees in P are proper divisors of n with gcd one. At most seven of the eight available positions have degree $n/2$; all other proper degrees are at most $n/4$, since $3 \nmid n$. The mass of P is consequently at most $1 + 7(n/2)^2 + (n/4)^2 < 2n^2$, whereas Lemma 3.7 makes it at least mn^2. It remains to exclude $m = 1$ with $P \cap I = \varnothing$. Its unique row is rational, so the nonunit squared norm eight allows eight signs, one entry of absolute value two and four signs, or two entries of absolute value two. The last option is impossible: the two nonunit degrees would have gcd one by orthogonality, but every two in P have nonzero crossing by principality and hence share a prime by Lemma 2.5. In the middle case all five proper degrees must equal $n/2$ to meet the modular bound $3n - 1 \le \sum_{Y \ne 0} d_Y |u_Y|$: the first smaller degree loses at least $n/4$. This contradicts their gcd. In the first case at least four degrees must equal $n/2$, since $3n/2 + 5n/4 < 3n - 1$. Their signs agree by $b_Y u_Y \equiv n \pmod 3$, where $b_Y = n/d_Y$. The other at most four degrees, each at most $n/4$, cannot cancel them to

within one.

An I-label in P sees every 3-free degree by principality. Conversely all degrees in P divide n. Thus P is the set of 3-free labels and its twists are pure powers of three, by Lemma 3.4 applied to an I-row. Every 3-active label has 3-free degree since $v_3(D) = 2$, so three is an isolated singleton. This contradicts Lemma 6.7, which excludes an isolated singleton at three. Its proof uses only coprime vanishing, the principal-block congruences and the singleton column identities, all available for the present fusion-simple perfect table. □

20.5 Integral traces in the principal dyadic block

Assume $D = M^2$, let $n = M/a$ be the largest simple degree, and suppose $a > 1$ is a power of two. Write $f = a^2$ and m for the multiplicity of n. Theorem 2.11 and Corollary 4.1 put the support of each degree-n row in the principal dyadic block P, where $4 \mid f_Y$. Thus a hit of degree $d = M/b$ requires $2 \mid b$. Its nonunit support has gcd one by orthogonality to the vacuum. Consequently the lcm of the even nonunit denominators is M. If $2m > f$, the degree-n square submatrix contains a nonzero entry by Lemma 3.7. We can then choose its row X in P, by symmetry and the support theorem. Otherwise choose any row X of degree n.

For each nonunit degree $d = M/b$ with even b, put

$$t = d/\gcd(n,d) = a/\gcd(a,b), \qquad \alpha_Y = \chi_X(Y)/t = S_{XY}/\operatorname{lcm}(n,d) \quad (d_Y = d).$$

Lcm integrality and the character-field conductor bound (Corollary 2.8) place α_Y in the integers of a cyclotomic field $K = \mathbb{Q}(\zeta_{2^j})$. Enlarge K to a common such field. The standard integral basis $1, \zeta, \dots, \zeta^{[K:\mathbb{Q}]-1}$ is orthonormal for normalized trace of $u\bar{v}$. In particular the normalized trace of α_Y is its integer constant coefficient k_Y, and the mean C_Y of $|\alpha_Y|^2$ is the sum of the squares of its integer coefficients. Reduction at the unique prime above two sends ζ to 1.

There are two circumstances prescribing the residue $\delta \in \{0,1\}$ of every α_Y at a given degree. If n is even and $n/\gcd(n,d)$ is odd, then $\delta = 0$. In fact $\alpha_Y = 0$ off P, whereas on P symmetry gives $(n/\gcd(n,d))\alpha_Y = \chi_Y(X) \equiv n \equiv 0$. If the selected X is known to be in P and t is odd, then $\delta = d \bmod 2$ because $t\alpha_Y = \chi_X(Y) \equiv d$. In all other cases the residue may be left unspecified.

Let s be the multiplicity of d, $T = \sum_{d_Y=d} k_Y$, $B = \sum_{d_Y=d} C_Y$. The algebraic integer $\sum_{d_Y=d} \alpha_Y$ is rational by the Galois action on columns, so it equals T. If δ is prescribed then

$$T \equiv B \equiv s\delta \pmod 2, \qquad C_Y \geq F(k_Y), \qquad F(k) = k^2 + \mathbf{1}_{k \not\equiv \delta} \pmod 2.$$

Indeed the nonconstant coefficients must include an odd one when the constant coefficient has the wrong parity. Without prescribed residue put $F(k) = k^2$ and omit both congruences. Write $|T| = sj + r$ with $0 \leq r < s$. Discrete convexity and evenness of F give $B \geq (s-r)F(j) + rF(j+1)$.

The search chooses integers (T_d, B_d) with these group bounds and

$$\sum_d dtT_d = -1, \qquad \sum_d t^2 B_d = f - 1.$$

The first identity averages orthogonality to the vacuum. The second averages the row norm, whose vacuum contribution is one. A final inequality comes from the ordinary modular relation. At every Galois conjugate the absolute value of the twisted dimension sum of the row is $\sqrt{D} = M$. The triangle inequality and averaging therefore require

$$M - 1 \;\leq\; \sum_d dt \,\min\{\sqrt{sB_d},\, B_d/\sqrt{\eta_d}\}, \qquad \eta_d = \begin{cases} 2, & \delta = 0 \text{ prescribed}, \\ 1, & \text{otherwise}. \end{cases}$$

Here each nonzero α_Y costs at least η_d in averaged squared norm. The two bounds are Cauchy–Schwarz and $\sqrt{C_Y} \leq C_Y/\sqrt{\eta_d}$, respectively. The program rounds each upper bound upwards in units of $1/256$. It searches the at most $f - 1$ nonvacuum norm units by degree and prunes targets incongruent to the gcd of the remaining dt. The test is necessary for the existence of the selected row.

The search is implemented in `code/principal_cover.py`.

21 Square dimensions and rows of codegree sixteen

Write $D = M^2$ and $R = \mathrm{rk}(\mathcal{C})$. The squared mass of the principal dyadic block limits the number of labels of degree $M/4$. In particular, Theorem 21.4 bounds the multiplicity of $M/4$ through rank 31.

21.1 Concentration at a prime-power codegree

Proposition 21.1. *Let x be a nonunit degree of prime-power formal codegree $f = p^a$ and multiplicity $m > f/2$ in a nontrivial fusion-simple perfect integral modular category. Every label of degree prime to p belongs to the principal p-block P and has p-power twist order. Put $k = 2m - f$. At least k of the degree-x labels belong to P. If $p \mid x$ and $p \nmid y$, then a label of degree y satisfies*

$$\sum_{Z : p \nmid d_Z} \left(d_Z / \gcd(d_Z, y)\right)^2 + k\left(x / \gcd(x, y)\right)^2 \le D/y^2.$$

For each other prime $\ell \mid D$, let h have the value from Proposition 19.13, let g be the least nonunit degree and m_d the multiplicity of d. Then

$$2\ell g \sum_{\substack{p \mid d \\ \ell \mid D/d^2}} h\lfloor m_d/h \rfloor d^2 \;\ge\; (\ell - 1)D.$$

Proof. Every row at x is supported in P by Theorem 2.11. Its degree class I has mass m/f in D-normalized squared degrees, so P has mass at least m/f by Lemma 3.7. Thus $I \cap P$ has mass at least $2m/f - 1$ and contains at least k labels. Choose $X \in I \cap P$. For $p \nmid d_Y$, principality gives $\chi_X(Y) \equiv d_Y \neq 0$ at the chosen prime over p. Thus Y belongs to the support of the X-row and lies in P. For $\ell \neq p$ we have $\ell \nmid f_X$, so Theorem 3.2 forces $S_{XY} = 0$ if Y is ℓ-active. Consequently the twist of Y has p-power order. Its row has nonzero entries on all labels of degree prime to p by principality, and on each of the k chosen degree-x columns by symmetry. The sets are disjoint if $p \mid x$. Applying the Galois-mean bound to the nonzero lcm-scaled entries gives the norm inequality. Finally every ℓ-active label has degree divisible by p. Apply the packet and mass argument of Proposition 19.13 on these degrees. □

Theorem 21.2. *An odd degree x of formal codegree 16 has multiplicity at most eight in a nontrivial fusion-simple perfect integral modular category.*

Proof. Suppose $m \ge 9$. Every degree-x row is supported exactly on P: for $Y \in P$, the reduction $\chi_Y(X) \equiv x$ is nonzero, and symmetry gives $S_{XY} \neq 0$. By Proposition 21.1 all odd labels lie in P. In fact P consists entirely of odd labels. For an even degree d_Y in P, every X-row at degree x would have mean $|\chi_X(Y)|^2 \ge (d_Y / \gcd(d_Y, x))^2 \ge 4$. The normalized column at Y has norm one, whereas its m entries on those rows would contribute a mean at least $4m/16 > 1$. Thus every member of P has odd degree and divides x, since $D = 16x^2$. Any two nonunit degrees there have a common prime: principality requires their entry to be nonzero and simplicity excludes a nonzero crossing at coprime degrees. The support degrees other than one have gcd one by dimension orthogonality. Thus at least three distinct proper nonunit divisors of x occur. If $o = |P|$, the row norm gives $o \le 16$, and $4 \mid o$ since $4 \mid D$ and all degrees outside P are even. For $m \ge 9$, the vacuum, the m degree-x positions and the three proper degrees require $o \ge 13$.

Thus $o = 16$. Every entry of each X-row on P has Galois-mean squared modulus one, so is a root of unity. In particular $m \le 12$. Put $A = \sum_{Z : d_Z = x} \chi_X(Z)$. It is an integer, since the dimension class is Galois-stable and Galois action on entries can be taken on the column labels (the dimensions are rational and preserved). Each summand is a dyadic root of unity by the individual conductor bound. Its dyadic reduction is therefore one, so $A \equiv m \pmod 2$.

If $m = 12$, the three proper degrees occur once each. They are odd proper divisors at most $x/3$. Dimension orthogonality gives $|A| \le 1 + 1/x < 2$, so $A = 0$. At each singleton degree the

root entry is rational, hence a sign. The three signs are fixed across all degree-x rows by their weighted sum -1: two different choices would give a zero sum on a nonempty proper subset of the three distinct degrees, which is impossible. The rows therefore agree on the vacuum and three signs. Lemma 20.6 gives $4m \le 16$, a contradiction.

If $m = 11$, the four proper degrees have sum at least $x - 1$, since A is an odd integer. Write them as x/a_i where the a_i are odd and at least three; their least common multiple is x by the gcd-one property, and at least three are distinct. At least two a_i are three since $1/3 + 3/5 < 1 - 1/x$. There are exactly two threes by distinctness. The two remaining distinct cofactors must be five and seven: $2/3 + 2/7 < 1 - 1/x$ and $2/3 + 1/5 + 1/9 < 1 - 1/x$. All these bounds hold since the dimension threshold gives $x \ge 5400/4 = 1350$. The four cofactors have lcm 105, a contradiction.

If $m = 10$, the five proper degrees are at most $x/3$, so $|A| \le 5/3 + 1/x < 2$ and $A = 0$ by parity. On each proper degree class, the sum of its entries is an integer of absolute value at most its multiplicity and congruent to that multiplicity modulo two. It can therefore be represented by a sum of formal signs, one for each position. Together they satisfy $\sum_{j=1}^{5} \varepsilon_j y_j = -1$, where the y_j are the five proper odd degrees. Any two sign assignments solving this equation have identical sign totals on each degree class: the subset on which they differ has weighted signed sum zero and even size. If its size is four, the remaining entry would have signed degree -1. If its size is two, the two flipped degrees are equal and their signs opposite. At least three distinct degrees occur among the five, so one of them is a singleton. Its coordinate is a fixed sign in all ten rows. The rows agree also at the vacuum. Lemma 20.6 on those two coordinates gives $2m \le 16$.

If $m = 9$, the six proper degrees are at most $x/3$. The odd integer A satisfies $|A| \le 2 + 1/x$, so $A = \pm 1$. For a proper degree d write n_d for its multiplicity and B_d for its class sum. As above the B_d are realized by signs, with $1 + Ax + \sum_d dB_d = 0$. At fixed A any two sign solutions differ on an even number of positions. Flipping two just exchanges signs at equal degrees. Flipping four leaves two proper degrees and x with signed sum -1. This is impossible, since the two proper degrees share a prime dividing x. Flipping all six would give $Ax = -1$. Thus for each A all B_d are fixed across the corresponding rows.

At least three distinct B_d must be nonzero, by the same common-factor argument. If $\sum_d |B_d| \ge 4$, Cauchy–Schwarz gives $\sum_d B_d^2/n_d \ge 16/6$. Otherwise precisely three B_d are ± 1. Their three multiplicities are odd and sum to at most five, giving $\sum_d B_d^2/n_d \ge 1 + 1 + 1/3 = 7/3$. Including the vacuum, the common orthogonal projection of the rows with fixed A onto the vacuum and the constant vectors on proper degree classes has squared norm at least $10/3$. Since the rows have squared norm sixteen, Lemma 20.6 allows at most four such rows for each sign of A. This excludes nine rows. □

21.2 Fourteen labels of formal codegree sixteen

Write $s = S/M$, $n = M/4$, and let P be the principal dyadic block. If $8 \mid M$, a degree-n row vanishes at every denominator $b \equiv 2 \pmod 4$. Indeed $\alpha = 2s_{XY}$ is integral by lcm integrality, while principality gives $(b/2)\alpha = \chi_Y(X) \equiv n \equiv 0$ above two. Its e distinct conjugates occur in distinct columns and have total squared norm at least $e2^{2/e}/4 \ge 1$, before the vacuum. If n is odd and at least four denominators four occur, each of those rows has support exactly P; a nonzero such column would meet all four rows, again exhausting its norm before the vacuum. Odd denominators already vanish by the principal defect bound. Consequently, in either case the nonunit denominators divisible by four have lcm M, by row orthogonality.

Proposition 21.3 (Fourteen dyadic rows and their principal block)**.** *If the denominator four occurs exactly fourteen times, its class I is not contained in P. Moreover*

$$\sum_{\substack{4 \mid b_Y \\ b_Y \neq 4}} b_Y^{-2} > 1/16, \qquad \sum_{4 \nmid b_Y} b_Y^{-2} < 1/16.$$

Both sums include the vacuum when eligible. In particular, three denominators six are impossible.

Proof. Suppose $I \subset P$. The sums $w_Y = \sum_{X \in I} \chi_X(Y)$ are even integers. For odd $d_Y > 1$, all fourteen summands have nonzero residue, and their squared moduli have Galois mean an even integer in $[14, 16)$, hence fourteen. Thus the columns $Y, 0$ have squared norm $14/16$ on I, and orthogonality gives $|w_Y| \leq 2$. Equality would make their complementary restrictions proportional with ratio of modulus one, contrary to their unequal vacuum moduli. Hence $w_Y = 0$, contradicting $\sum_{Y \neq 0} d_Y w_Y/2 = -7$ modulo two.

The mass inequality of Lemma 3.7 gives $I \cap P \neq \varnothing$. If n were odd, a principal member would meet all of I, forcing $I \subset P$. Thus $8 \mid M$. A member outside P misses I by symmetry, so its support is among the other denominators divisible by four. The vacuum modular relation gives their mass at least $D/16$. Equality forces $M = 16$ at the vacuum coordinate, impossible. The remaining mass bound follows since I has mass $14D/16$. □

Theorem 21.4. *At rank at most* 31 *the denominator four has multiplicity at most thirteen in a nontrivial fusion-simple perfect integral modular category of square dimension.*

Proof. Sixteen copies exhaust the dimension; fifteen are excluded by Proposition 20.5, since the vacuum and six labels with nontrivial odd-primary twist give $R \geq 22$. Suppose there are fourteen. The set $E = I \setminus P$ is nonempty by the proposition. Its rows miss I and see only pure dyadic twists. Some odd-primary twist is nontrivial, so their available squared mass is strictly less than $2n^2$. The mass inequality makes E a singleton $\{Z\}$, and Galois stability makes $u_Y = \chi_Z(Y)$ integral. At a nonunit hit write $d_Y = n/h_Y$. The preceding zero gives $h_Y \geq 2$ integral; principality makes it even when u_Y is odd. The row identities give

$$\sum u_Y^2 = 15, \qquad \sum u_Y/h_Y = -1/n, \qquad \sum |u_Y|/h_Y \geq 4 - 1/n.$$

Here $n \geq 1350$ by Corollary 17.3. With at most eight hits the absolute weights are $(3, 2, 1, 1)$, $(3, 1^6)$ or $(2^3, 1^3)$. The first has total weight seven, too small. In the others expand each weight into unit fractions. Both signs have mass at least $2 - 1/n$, so one side has four terms and the other five. All four denominators on the first side are two, since any larger one loses at least $1/6$. The other five must sum to $2 \pm 1/n$. Four or five twos overshoot. With three twos the last two reciprocals equal $1/2$ or differ from it by at least $1/42$; the equality pairs are $(3, 6), (4, 4)$. With two twos the last three sum to one or at most $11/12$; with at most one two the sum is at most $1/2 + 4/3$. Thus there are at least nine hits.

These hits, the vacuum and the thirteen labels of $I \cap P$ belong to P and have pure dyadic twists. At least six further labels have nontrivial odd-primary twists, by Proposition 19.8 at two distinct odd primes. The full dyadic block of Z, intersected with the trivial odd-phase fiber, therefore has at most $31 - 9 - 1 - 13 - 6 = 2$ labels. Its span is dyadically invariant. A single label contradicts Lemma 3.4; two have equal degree by Lemma 19.5, contrary to the uniqueness of Z outside P. □

21.3 Square dimension with denominators four, five and seven

Keep $D = M^2$ and $b_Y = M/d_Y$. Two restrictions will be used repeatedly. The field of a prime-power degree class is controlled by the square Galois action on twists, and every nonzero entry in a prime-square row is supported on a pure primary twist.

Two prime-power degree classes in square dimension. Write $D = M^2$. Suppose i labels of degree $M/4$ and j of degree $M/5$ comprise their respective full degree classes. Put $t = 400 - 25i - 16j$, and let A, B be their respective sums of twists. When $1 \leq t \leq 17$ both A, B are rational integers. They satisfy

$$-i \leq A \leq i, \quad A \equiv i \pmod 2, \qquad -j/4 \leq B \leq j, \quad B \equiv j \pmod 5, \qquad |25A + 16B| < t.$$

Indeed the complement has squared mass $(t/400)M^2$. Independent embeddings of the dyadic and quintic twist fields give $|A/16 + B/25| \le t/400 + 1/M$ at every embedding. The variance of the dyadic integer A is an integer by the orthonormal power basis for dyadic fields. It is at most $256(t/400 + 1/M)^2 < 1$, because $M \ge 5400$. The quintic sum B lies in $\mathbb{Q}(\sqrt{5})$: its twists have 5-power order and their multiplicities in this degree are constant under squares of the units by odd-primary twist covariance. For two conjugates, $|B - B'| \le 50(t/400 + 1/M) < \sqrt{5}$. A nonrational integer of $\mathbb{Q}(\sqrt{5})$ has conjugate difference at least $\sqrt{5}$, so B is rational. Reduction at the local primes gives the congruences; the lower bound on B uses the normalized trace of a five-power root. Integrality now gives $|25A + 16B| \le t$.

Equality would make the two classes contribute $\pm(t/400)D$ to the Gauss sum. Taking real parts with the appropriate sign and averaging conjugates bounds the residual squared mass with twist different from the opposite sign by $2M$. Choose a dimension prime $\ell \notin \{2, 5\}$, which exists by the two-prime theorem. Both degree classes are ℓ-inactive by Lemma 3.4; thus $L_\ell \le 2M$. Proposition 3.8 implies $g \ge (\ell - 1)M/(4\ell) \ge M/6$ for the smallest nonunit degree. Every nonunit denominator M/d consequently belongs to $\{1, 2, 3, 4, 5, 6\}$. Multiplying $\sum_{d>1} m_d/(M/d)^2 = 1 - 1/M^2$ by 3600 is impossible since $0 < 3600/M^2 < 1$. This proves strictness.

For example, immediate exclusions of exact multiplicity pairs are

$$(1, 23),\ (2, 21),\ (3, 20),\ (5, 17),\ (7, 14),\ (10, 9),\ (12, 6),\ (14, 3),\ (15, 1).$$

The pairs $(12, 6)$ and $(10, 9)$ cannot be increased in either coordinate without exceeding total dimension, so their two-prefix tests apply whenever at least the displayed multiplicities occur. For $(11, 7)$ the only permitted pair is $(A, B) = (-1, 2)$; for $(13, 4)$ it is $(1, -1)$.

Trace and isolation at a prime denominator. Let p be an odd prime, $y = M/p$, $J = \{X : b_X = p\}$, and $T = \{X \in J : \theta_X = 1\}$. Set $t = |T|$ and $w_Z = \sum_{X \in T} \chi_X(Z)$. The twists in J are pure p-power twists by Lemma 3.4. Odd-primary twists transform by squares under the modular Galois permutation, so T is Galois-stable, and all w_Z are rational integers. Each J-row vanishes outside the principal p-block P (Theorem 2.11); this block is supported on $p \mid b_Z$ (Corollary 4.1). Every label hit by one of these rows has pure p-power twist, by Lemma 3.4(1) at the other primes. For $t > 0$, orthogonality and the vacuum modular relation yield

$$\sum d_Z w_Z = 0, \qquad \sum d_Z \theta_Z w_Z = tM\xi, \qquad \sum |w_Z|^2 = tp^2.$$

Here ξ is the multiplicative central charge. Since M is an integer and the middle sum lies in a p-power cyclotomic field, $\xi \in \mu_8$ forces $\xi = \pm 1$. The mean real part of a p-power root of unity lies in $[-1/(p-1), 1]$. The positive and negative parts of $\sum d_Z w_Z$ have equal mass. Averaging the middle identity and applying Cauchy–Schwarz gives

$$\sum_Z d_Z |w_Z| \ \ge\ 2(p-1)tM/p, \qquad \sum_{w_Z \ne 0} b_Z^{-2} \ \ge\ 4(p-1)^2 t/p^4. \tag{21.1}$$

If $p \mid y$, this support misses J: Proposition 5.4 gives zero absolute trace of $\chi_X(Z)$ for $X, Z \in J$, since $y \nmid y/p$. The sum over each complete Galois orbit in T is therefore zero there. If $p \nmid y$, every J-row has support exactly P by symmetry and the principal congruence Theorem 2.11. If P meets J, it contains J by this same symmetry. At such a principal row the congruence ensures a hit at every p-free degree, all of whose twists must therefore be pure p-power. Since $v_p(D) = 2$, every p-active label has p-free degree (Lemma 3.4). Thus p is isolated and Lemma 3.4(2) puts every p-active label in J. In particular, if $j = |J| < p - 1$, P cannot meet J. Indeed, in this case no label in J has order p^2 or higher by the square-orbit size $p(p-1)/2$; categorical Cauchy and equality of masses at the $p - 1$ exact primitive p-phases at even valuation would require at least $p - 1$ labels in J. These observations allow (21.1) to charge only the other denominators divisible by p whenever $j < p - 1$.

Thirteen fours and four fives. Suppose the two multiplicities are thirteen and four. Write I, J for the classes at $n = M/4$, $y = M/5$. The preceding inequalities force $(A, B) = (1, -1)$ and $\delta = 11/400$. The four twists in J must be the four primitive fifth phases, since a higher phase requires at least ten labels of that degree and the rational sum is -1. Write ζ for a primitive fifth root. Let H comprise the labels outside $I \cup J$ whose twists have five-power order (including order one). The residual deficit relative to -1, after averaging over embeddings, is at most $(\delta - 9/400)D + M = D/200 + M$. Every label in H contributes at least $3d_Y^2/4$ to this deficit. Define

$$\eta = D^{-1} \sum_{Y \in H} d_Y^2 \le \tfrac{4}{3}(1/200 + 1/M).$$

By Theorem 2.11, Lemma 3.4, Corollary 4.1, J-rows are supported on the principal five-block P, only at labels of five-power twist, and miss I because $5 \nmid 16$. If $X \in J$ were outside P, symmetry would also exclude all of J from its support. Its modular relation would then give $1 \le \sqrt{25\eta}$, a contradiction. Thus $J \subset P$.

For $X \in J$ put

$$R_X = 5\xi\theta_X^{-1} - \sum_{Z \in J} \theta_Z \chi_X(Z) = \frac{25}{M} \sum_{Z \in H} s_{XZ}\theta_Z d_Z.$$

The J character fields ramify only at five and lie in $\mathbb{Q}(\zeta)$: their degrees are at most four, hence their conductor has no wild five-part (Corollary 2.8). All twists on their support have five-power order. The full modular relation therefore places the dyadic root ξ in a five-power cyclotomic field, so $\xi = \pm 1$. Consequently $R_X \in \mathbb{Z}[\zeta]$. Unitarity, also after permuting rows by Galois, gives $\sum_{X \in J} \operatorname{mean} |R_X|^2 \le 625\eta$.

Suppose $5 \mid y$. Since $J \subset P$, every $\chi_X(Z)$ in the J-sum reduces to $y = 0$ above five. Each R_X belongs to $(1 - \zeta)\mathbb{Z}[\zeta]$. Every nonzero element of this ideal has mean squared norm at least $5/2$: using all five powers of ζ, subtract a common integer from its coefficients to give them sum zero. The mean norm is then $5/4$ times the sum of their squares, at least $5/2$. For local five-inertia σ one has $R_{\sigma X} = \sigma^2 R_X$. Indeed $\theta_{\sigma Z} = \sigma^2\theta_Z$ and $\sigma\chi_X(Z) = \chi_{\sigma X}(Z) = \chi_X(\sigma Z)$, so a change of index in the J-sum proves the assertion. Each orbit in J has at least two labels by its primitive fifth twist. Any nonzero R_X would therefore cost at least five, whereas $625\eta < 5$. All residuals vanish. Projecting $sT(d_X \mathbf{1}_{X \in J})$ onto J now exhausts its norm. Its vacuum coordinate forces $B = 0$, contradicting $B = -1$.

Suppose $5 \nmid y$. As $J \subset P$, the isolation argument above puts all five-active labels in J; every twist on H is therefore trivial.

The same deficit bound improves to $\eta \le (1/200 + 1/M)/2$. But each R_X has residue $-4y \ne 0$ at five and costs at least one, contradicting $4 \le 625\eta < 2$.

One denominator $p \ge 5$. If J consists of one label Z, Proposition 20.4 gives $p \nmid y$ and at least three prime divisors of y. The twist is trivial, since every nontrivial phase would have a square orbit of at least $(p - 1)/2 > 1$ labels. Its exact support P is disjoint from J by the preceding argument, and every nonunit in P has degree a proper divisor of y by Proposition 5.4. Equation (21.1) gives $\sum_P b_Y^{-2} \ge 4(p - 1)^2/p^4$. At $p = 5$, fourteen fours would leave only $1 - 14/16 - 1/25 < 64/625$ for P. For both $p = 5$ and 7 we can count its labels. Put $H = |P| - 1$, $W = \sum_{P \setminus \{0\}} |w_Y|$. The proper hit degrees are at most $y/2$, at least one at most $y/3$ by their gcd one. Equation (21.1) gives $2(p - 1)y \le 1 + Wy/2 - y/6$. As $y \ge 5400/p > 6$, and $W \equiv \sum_{Y \ne 0} w_Y^2 = p^2 - 1 \pmod 2$, one has $W \ge 4(p - 1) + 2$. Now $k^2 \ge 3k - 2$ for positive integers k gives $2H \ge 3W - (p^2 - 1) \ge 30$ for either prime. Choose a prime $\ell \mid y$ with $\ell \ge 7$ for $p = 5$, or $\ell \ge 5$ for $p = 7$: three distinct primes divide y, none equal to p. The ℓ-active labels of Proposition 19.1 miss P, Z and all denominator-four labels by twist purity. The four classes counted are disjoint, since P requires $p \mid b$ and misses Z. Thus $R \ge (1 + H) + 1 + m_4 + (\ell - 1)$, which gives $R \ge m_4 + 23$ at five and $R \ge m_4 + 21$ at seven.

Eleven denominators four preclude six denominators five. Suppose $D = M^2$ and eleven labels have largest degree $M/4$. Let J be the labels of degree $y = M/5$ and suppose $j = |J| \geq 6$. Their multiplicity is 6 or 7 by the dimension identity. Write I for the eleven largest labels. The remaining squared mass is δD, where

$$\delta = 1 - \tfrac{11}{16} - \tfrac{j}{25} \leq \tfrac{29}{400} < \tfrac{2}{25}.$$

Every J-row is supported on the principal five-block P (Theorem 2.11); this block misses I by Corollary 4.1. The actual twists in J have five-power order (Lemma 3.4). They have order at most five because a primitive higher phase would have a square orbit of at least ten labels at this degree. By Proposition 19.1, the squared masses at the four primitive fifth phases agree even if other labels have twist order divisible by 25.

First suppose $5 \mid y$. Equation (21.1) for all twist-one labels of J charges only the other multiples of five. Even one such label would require mass $64D/625 > \delta D$. Hence all j twists in J are primitive fifth phases.

An odd $j = 7$ is impossible since there are two elements in each square orbit. For $j = 6$ the two quadratic classes have packet counts c_+, c_- with $c_+ + c_- = 3$. They contribute respective squared masses $2c_+y^2, 2c_-y^2$, differing by at least $2y^2$. The class I has no five-active labels. The remaining mass $\delta D < 2y^2$ cannot restore the required equality.

If $5 \nmid y$, their exact support P has normalized mass at least $j/25$ by Lemma 3.7. It misses I and $\delta < j/25$, so P meets J. The isolation argument above puts all five-active labels in J.

Cauchy's theorem and the equal exact fifth-phase masses now give exactly four active labels, one at each primitive phase, because $j < 8$.

Five is an isolated singleton in the ordinary square root M. Its signed singleton identities (Proposition 6.4) give deficit one because $\sum_{\text{active}}(1 - \theta_{Z,5}) = 5$. Their nonunit integer coordinates a_Z satisfy

$$\sum a_Z^2 = 9, \qquad \sum d_Z a_Z = 5y + \epsilon, \qquad a_Z \equiv d_Z/y \pmod 5, \quad \epsilon \in \{1, -1\}.$$

Only five-free degrees have a nonzero coordinate and all such degrees divide y. At the j full labels every coordinate is one by its residue and the norm budget. The remaining budget $9 - j \leq 3$ forces all other nonzero coordinates to be signs. If $a_Z = -1$, its degree satisfies $y/d_Z \equiv -1 \pmod 5$, hence $d_Z \leq y/4$. Consequently $5y + \epsilon \geq jy - (9-j)y/4$, impossible at $j = 6, 7$ and $y = M/5 \geq 1080$. This is also the $p = 5$, deficit-one instance of the centered singleton trace bound Lemma 10.2, with centering forced by the displayed norm budget.

Mass at exact quintic multiplicities. At exact multiplicity $j = 6, 7$ of the denominator five, its other multiples must carry squared mass at least $2D/25$, independently of the first denominator. To see this, replace δ throughout by $\sum_{5 \mid b,\ b \neq 5} b^{-2}$ (including the vacuum denominator). The principal five-block is supported on $5 \mid b$ by Corollary 4.1 and only these denominators can be five-active by Lemma 3.4. All mass tests in the proof therefore hold with this smaller quantity.

More generally, at a prime $p \geq 5$ write $\eta_p = \sum_{p \mid b,\ b \neq p} b^{-2}$. If $2 \leq j < p - 1$, then $\eta_p \geq 2/p^2$. Indeed, if $p \mid y$ and $\eta_p < 2/p^2$, even one inactive label in J would require $\eta_p \geq 4(p-1)^2/p^4 > 2/p^2$ by (21.1). Higher-order twists need at least $p(p-1)/2$ labels, so J consists of one square packet of $(p-1)/2$ primitive phases. Their opposite class needs $(p-1)y^2/2 \geq 2y^2$ outside J. If $p \nmid y$, the exact support P has normalized mass at least j/p^2 by Lemma 3.7. If $\eta_p < j/p^2$, it meets J, forcing all p-activity into J by the isolation argument. Equality of the primitive masses would then require $p - 1 > j$ active labels. This proves the stronger $\eta_p \geq j/p^2$ in this case. Thus $j = 2$ at five requires $2/25$; $j = 3$ requires $3/25$, since an orbit size divides 20 and three labels must include a rational row, forcing $5 \nmid y$ by Proposition 20.4.

Proposition 21.5. *At rank at most* 34*, fourteen labels of denominator four preclude any label of denominator seven. More precisely, if both denominators occur and there are at least fourteen*

fours, a unique seven forces rank at least $m_4 + 21$; multiplicities two, four, five and six of seven are impossible at every rank; and multiplicity three forces rank at least 35.

Proof. Write $p = 7$, $j = |J|$, $m_4 = |\{X : b_X = 4\}|$, and $\eta = \sum_{7 \mid b_Z,\, b_Z \neq 7} b_Z^{-2}$, including the vacuum. The dimension equation gives $1 \le j \le 6$ and $\eta \le 1 - 14/16 - j/49$.

For $j = 1$ the preceding single-denominator count at $p = 7$ gives $R \ge m_4 + 21$.

For $2 \le j \le 5$ there are no wild septic twists in J, and its active labels occur in packets of three under square translation. At $j = 2$ all twists are trivial, so (21.1) gives $\eta \ge 288/2401 > 1-14/16-2/49$. For $j = 4, 5$ at least one label is inactive, giving $\eta \ge 144/2401 > 1-14/16-j/49$. For $j = 3$ either all are trivial, which gives $\eta \ge 432/2401 > 1 - 14/16 - 3/49$, or they form one square class. In this case the opposite class needs $3/49$ compensating mass. Suppose $R \le 34$.

Each of the three opposite primitive seventh phases has squared mass at least y^2. The degree multisets on these opposite phases agree by the local square Galois action; this assertion concerns the seventh component, with the complementary component unrestricted. Let s be their common label count. Every such label has $b = 7k'$, $k' \ge 2$, by Lemma 3.4 and the exact multiplicity of seven. It contributes at most $y^2/4$. Since the rank leaves at most $34 - 14 - 3 - 1 = 16$ other nonunit positions, $4 \le s \le 5$. For $s = 5$ at least four labels on each phase have denominator 14, since $3/4 + 2/9 < 1$. The fifth label gives strictly greater mass than y^2, so each of the three original phases also needs an additional label. This requires at least $15 + 3 > 16$ positions. Thus $s = 4$ and all twelve labels on the three opposite phases have denominator 14. Put $k = M/28$. The remaining $h \le 4$ nonunit degrees satisfy

$$1 + \sum_{i=1}^{h} d_i^2 = 2k^2,$$

because $14/16 + 3/49 + 12/196 = 1 - 1/392$.

If an odd prime $\ell \ge 5$ different from seven divides M, at least $\ell - 1$ labels are ℓ-active, all among the h remaining positions by Lemma 3.4. This forces $\ell = 5$ and $h = 4$. No wild quintic phase can occur in four labels. Equality of squared masses at the four primitive quintic phases forces all four degrees to coincide. Then $1 + 4d^2 = 2k^2$ contradicts parity. Otherwise M has support contained in $\{2, 3, 7\}$. The two-prime and ordered three-prime divisor families (Theorems 4.16, 7.3 and 8.3) dispose of all cases except $v_2(M) \ge 3$, $v_3(M) \ge 3$, $v_7(M) \ge 2$: the corresponding covered ordered exponents of D are $v_2(D) \le 5$, $v_3(D) \le 4$, or $v_7(D) \le 3$. In particular k is even. Reducing the residual equation modulo four forces exactly three odd degrees among the $h \le 4$ terms. Each such degree is divisible by both three and seven, since no nonunit degree is a prime power. The degrees of denominators $4, 7, 14$ are likewise divisible by three. There remains at most one nonunit of degree prime to three, contradicting Lemma 18.1.

Finally $j = 6$ uses $6/49 > 1/16$ on denominators prime to four, contrary to Proposition 21.3. □

21.4 Prefix witnesses and terminal compatibility

Lemma 21.6 (Prime heights on three-prime support)**.** *Suppose exactly three primes divide M in square dimension. A denominator p^a forces $p^2 \mid M$, four denominators four force $8 \mid M$, and a denominator five forces $125 \mid M$. For each $\ell^e \| M$ some nonunit denominator is divisible respectively by $\operatorname{lcm}(\ell^e, p)$, $\operatorname{lcm}(\ell^e, 4)$ and $\operatorname{lcm}(\ell^e, 25)$.*

Proof. A prime-power denominator row is supported in the principal p-block, so its hit denominators are divisible by p. Their nonunit degrees have gcd one and each contains at least two support primes. Their common multiple must therefore contain all three primes, giving $p^2 \mid M$. With four fours the zero argument in Section 21.2 makes all hit denominators divisible by four, and gives $8 \mid M$ in the same way. At a denominator five, Proposition 5.4 puts the nonzero-trace degrees among the divisors of $(M/5)/\gcd(M/5, 5)$, again with gcd one. Their common multiple

contains all three primes, so $125 \mid M$ and each trace-hit denominator is divisible by 25. A hit prime to ℓ supplies each asserted lcm. □

Reserving the remaining mass. In the square search the possible support consists of the primes in the prefix lcm L, the prime two, and odd $p \le h + 1 - H$, where h is the number of vacant nonunit positions and H is the known packet deficit of Section 19.2. If only three primes are possible, all must occur. For each, distinguish whether its full power in M equals that in L or is larger. In the latter case no prefix degree is prime to that prime. Reserve enough suffix positions for the two prime-free nonunits of Lemma 18.1, or three at $2, 5$ by the dimension equation modulo $4, 5$. At $2, 3$ their total numbers are respectively 3 mod 4 and 2 mod 3. Also reserve any missing witness from Lemma 21.6. These three sets of positions are disjoint, since every nonunit retains two support primes. Round their denominator lower bounds to allowed multiples no smaller than the last prefix member, say $a_1, \dots, a_v$. In units of L^{-2} their mass is at most $W = \sum_j \lceil L^2/a_j^2 \rceil$, and the vacuum contributes at most one. Thus, when $E > W + 1$, the next denominator satisfies

$$b^2(E - W - 1) \le (h - v)L^2.$$

Discard $v > h$ and retain the weakest bound over feasible prime heights. When dimension-family pruning is selected, the ordered three-prime exclusions (Theorems 8.3, 7.3 and 4.16) require the valuations of D to be at least $(6, 5, 4)$, giving additional lower bounds on the same reserved denominators. At the two-position closure, $(k, 10) = 1$ fixes the heights at two and five; a fixed height below a required family bound excludes every completion. With three positions left, if $k = M/L$ is even, all three remaining degrees must be odd by reduction modulo four. Hence $Ek^2 \equiv 4 \pmod 8$, forcing $v_2(k) = 1$ and E odd.

Two compatible phase fibers. For $q = 3$ and $r \le 31$, the joint fiber bound gives support $\{2, 3, p\}$, $p = 5$ or 7. An order-p^2 phase would give at least $p(p-1)/2$ nontrivial phases, each supporting three labels, besides the trivial fiber with at least three. Already at $p = 5$ this exceeds 31. Thus the nontrivial p-phases have order p and equal positive squared mass W, by Proposition 19.1. Choose a fiber in each of the two square classes of exponents modulo p. Their degree counts a_d, b_d must satisfy

$$\sum_d a_d d^2 = \sum_d b_d d^2 = W > 0, \qquad (a_d + b_d)(p-1)/2 \le m_d.$$

Only nonunit degrees with $p \mid D/d^2$ are eligible. In each fiber $3g^2 \mid W$, where g is the gcd of its degrees, by Lemma 19.2. The terminal screen enumerates one-fiber vectors, groups them by W, and tests pairs in each group.

21.5 Integral constraints on tensor products

Let H be the additive monoid generated by the nonunit degrees. Every nonunit degree a satisfies $a^2 - 1 \in H$, by the decomposition of $X \otimes X^*$. For fixed X of degree a, the undirected fusion graph is connected since X, X^* generate. An edge between nonunit degrees u, v requires both $au - v$ and $av - u$ to lie in H, allowing one unit to be subtracted from the first if $u = a$ and from the second if $v = a$. All nonunit degrees must be reachable from a in this possible-edge graph. Consecutive distinct degrees consequently have ratio at most the least nonunit degree.

Lemma 21.7 (Multiplication by a smallest nonunit)**.** *Fix X of least nonunit degree x. Let C_{ab} sum the multiplicities of the degree-b constituents of $X \otimes U$ over the m_a simples U of degree a. Then*

$$C_{ab} = C_{ba}, \qquad \sum_{b>1} bC_{ab} = xam_a - \mathbf{1}_{a=x}, \qquad 0 \le C_{ab} \le m_a m_b \lfloor \min(xa/b, xb/a, ab/x) \rfloor.$$

Proof. Rigidity and braiding give $N_{XU}^{V} = N_{XV^*}^{U^*}$, and duality preserves degrees. The unit occurs once, precisely when $U = X^*$. The three individual bounds come from taking dimensions in the three products obtained by reciprocity. Summing gives all assertions. □

The multiplication screen removes entry values inconsistent with each integer row sum, propagates the bounds to the transposed row, and also checks individual product dimensions. At degree b an individual product can use at most the aggregate bound and at most m_b times the single-pair bound. The implementation limits the propagation passes and the row sums tested; these limits only weaken the screen.

22 Rank bounds for modular categories and fermion quotients

Write r for the number of labels of the table, $\Delta = qM^2$ with q square-free, and $b_i = M/d_i$ for the denominators. In the modular case r and Δ are the rank and dimension of the category. In the positive fermion quotient both quantities are halved. We use the row and twist conditions proved in Sections 19–21 to conclude the finite searches defined by (18.3) and (18.4).

22.1 The positive fermion quotient

Theorem 22.1. *A divisor-minimal slightly degenerate non-WGT obstruction has rank at least* 34.

Proof. By Lemma 2.1 and Proposition 4.8, its positive table is simple and perfect, and Δ is even. We first describe a constraint specific to the two smallest codegrees in this search.

For a row X of prime-power codegree $f = p^a$, every hit lies in the principal p-block by Theorem 2.11. Two nonunit hits cannot have coprime degrees. Indeed, one of their degrees, say d_Y, would be prime to p, and the column of the other hit Z is principal. Then $\chi_Z(Y) \equiv d_Y$ at a prime above p, contradicting coprime vanishing. The gcd of all nonunit hit degrees is one by (18.1), so at least three different degrees occur. Each costs at least one in the mean squared row norm. Thus $f \geq 4$ in the positive table.

At $f = 4$ all three hits have mean squared norm one, hence their character values are roots of unity. At each hit degree there is only one nonzero entry. The column Galois action preserves this degree and its nonzero entry, so the value is rational and equals 1 or -1. Write $x = d_X$ and $a < b < c$ for the hit degrees. Lcm integrality makes a, b, c divisors of x. All are proper divisors, since otherwise the gcd of the other two would divide the third. The signs satisfy $|a + b - c| = 1$ and are determined by a, b, c. For even x each divides $x/2$, since $\chi_Y(X) = \pm x/d_Y$ reduces to x in the principal block at two.

Suppose x has multiplicity m. Then $m < 4$ by the dimension sum. The nonunit supports of any two such rows have inner product -1, so they meet at one or three labels. Three common labels would give identical signs and prevent orthogonality. The two rows thus meet at a unique label with opposite signs, and no label belongs to three supports. Their union has size $3m - \binom{m}{2}$.

These observations also bound whole prefixes. At codegree four, $q = 1$, $x = M/2$. If the remaining denominators are at least s, then $M \geq 6s$. The three distinct hit denominators are even and greater than two, and are multiples of four when $4 \mid L$, where L is the prefix lcm. If exactly one hit denominator b is known, the signed reciprocal equation gives $1/b \leq 2/s + 1/M$. If two are known, $|1/b - 1/c| \leq 1/s + 1/M$. Count the known entries eligible for either alternative and add the number of remaining slots. If this count is smaller than $3m - \binom{m}{2}$, some whole triple is known. Its signed reciprocal sum is $\pm 1/M$, whence $M \mid L$, and therefore $M = L$. The triple can then be checked by a signed sum of divisors of L.

Through half-rank 14, the arbitrary-precision enumeration gives eight row-feasible lists. Their categorical dimensions 2Δ are at most $2 \cdot 3240^2 < 29\,160\,000$. At half-ranks 15 and 16, the enumerator using the signed triples gives 6 and 256 lists. All six at half-rank 15 lie in the WGT

dimension families. At half-rank 16 the first exclusions are a dimension family (133), the gcd of the eligible hit degrees (104), the signed supports at codegree four (18), and codegree three (1). For every family comparison we use 2Δ. The row checker reads the lists for half-rank 15 and half-rank 16. This exhausts the reduced ranks through 16. The reduced verifier checks the stored lists; its replay option repeats the reduced recursion. The ordinary exclusion of codegree nine does not enter either calculation. □

22.2 The modular bounds

Theorem 22.2. *Let $\mathcal{C}$ be a non-pointed fusion-simple integral modular category of dimension $D = qM^2$ and rank r. Then $r \geq 32$. For $q > 1$ one has $r \geq 33$. If D is odd then $r \geq 57$, and if also $q > 1$ then $r \geq 75$. In square dimension put $B = M/d_{\max}$. Then $r \geq 36$ if $B \geq 5$, $r \geq 39$ if $4 \nmid M$, and $r \geq 46$ if $B \geq 6$.*

In odd dimension count the nonunit denominators by dual pairs. Since all degrees are odd, $r \equiv D \equiv q \pmod 8$. Let $f = qb^2$ and $s = \min_{p|f} v_2(p-1)$. Every run at b contains a multiple of 2^{s-1} pairs. Indeed the character field K has conductor dividing f. Complex conjugation acts nontrivially on K, since its label differs from its dual. On $\mathbb{Q}(\zeta_f)$, complex conjugation is a 2^{s-1}-st power, as one sees on each cyclic prime-power factor. Its root maps to an element of order divisible by 2^s in $\mathrm{Gal}(K/\mathbb{Q})$. The asserted divisibility follows, and it multiplies the odd factors supplied by the defect-one block partitions.

For the nonsquare odd search this gives short bounds on each remaining denominator. If the support has precisely three primes in increasing order, their exponents in D are at least $(6, 5, 4)$ by the dimension families. For each p at least two p-free nonunit pairs are required; at three, at least four, since $1 + 2t \equiv 0 \pmod 3$ for their number t. These sets of pairs are disjoint for three-prime support, since a nonunit degree contains at least two primes. Their denominators contain the full p-part of M. For additional primes outside q, the necessary number of eligible packets follows from the complementary-fiber bounds of Proposition 19.8. Sorting these mandatory lower bounds on the unfilled slots gives an upper bound on the available reciprocal-square mass. The vacuum term is at most $(15b_{\mathrm{last}})^{-2}$; the implementation rounds each summand upwards on a grid of mesh 2^{-64}.

Proof of Theorem 22.2. Simplicity makes the unit the unique invertible. Put $n = d_{\max}$ and $f_{\min} = D/n^2 = qB^2 < r$. In fact $f_{\min} \leq r-2$: equality $f_{\min} = r-1$ would give $\sum_{i \neq 0}(n^2 - d_i^2) = 1$, while every positive term is at least $2n - 1 > 1$. Lemma 20.1 and Proposition 20.3 give $f_{\min} \geq 8$, excluding $r \leq 9$. Theorem 20.8 removes the square root $B = 3$. For the remaining roots use Proposition 18.3 and the following necessary tests.

Test	Mathematical input
Dimension families	Table 6, Corollary 17.3; prefix bounds in Section 18 and paragraph 21.4
Prime support and multiplicity	Propositions 19.8, 19.9, 19.11, 20.7; Lemma 21.6
Characters and phase masses	Lemma 20.2, Sections 20.5–21, Proposition 19.13, paragraph 21.4
Tensor products	The monoid and graph test above; Lemma 21.7

The shared optimized verifier uses the same square and nonsquare generators at every rank. Here its `-families` option uses the dimension-family catalogue: a covered dimension is excluded before the character tests, and family height bounds strengthen the same prefix recursion. Each category supplies a unique branch. A prefix is discarded only when every completion violates a necessary condition; arithmetic beyond native width passes to the square or nonsquare arbitrary-precision recursion. The single certificate records the sources, mode, complete task partition and terminal exclusions for the whole range. Its counts of generated lists and survivors are

Rank	Square D	Nonsquare D	Survivors
9–24	0	0	0
25	1	0	0
26	3	0	0
27	9	0	0
28	65	0	0
29	655	0	0
30	14 482	1	0
31	396 110	121	0

Each rank in the first row was exhausted separately. The independent mode generates 453 041 lists. Family prefix constraints reduce this to 411 447, of which the catalogue excludes 22 964 before the character tests. Every generated list fails a necessary condition, so $r \geq 32$. For $q > 1$, the further rank-32 verification excludes all 3 081 terminal profiles and gives $r \geq 33$.

For odd D, the paired recursion visits 14 520 prefixes for $r \leq 56$ and leaves no type. With $q > 1$ it visits 128 prefixes for $r \leq 74$ and leaves no type. Both the unrestricted odd result and the odd nonsquare result are regenerated by the paired-odd verifier.

Finally let $q = 1$. Then $B^2 < r$ and $B \geq 4$ by the exclusions at small codegree and at nine. For $B \geq 5$ the complete search through $r = 35$ emits 437 lists in ranks 30 through 35. If $4 \nmid M$, odd M is already covered; for $M \equiv 2 \pmod 4$ the lcm of the prefix may never be divisible by four and the terminal root must satisfy the same congruence as M. These restrictions leave 6 456 profiles across ranks 30 through 38. With $B \geq 6$ the inequality $B^2 < r$ starts the search at 37, and it emits 80 profiles through rank 45. The three restricted enumerations, their ledgers and the degree lists give zero survivors in all three cases. □

The shared implementation accepts higher ranks. The thirteen-fours bound is used only under the hypothesis $r \leq 31$ of Theorem 21.4. At higher rank the general saturation bound allows fourteen. The codegree-eight and phase-order tests likewise check their prime-support and actual-rank hypotheses before applying the stronger restriction. The published certificate proves the range through 31.

22.3 Rank bounds at specified prime supports

The prime support can be used before any large denominator is generated. Let P be the primes dividing M. Since the table dimension is square-full, P is also the support of $\Delta = qM^2$. Put $a_p = v_p(M)$ and let w be one in even dimension and two in odd dimension. In $(\mathbb{Z}/p^2\mathbb{Z})^\times$ let H_p be generated by the squares of the other support primes. Write w_p for the smallest wj with $j \geq 2$ for which -1 is a sum of j elements of wH_p modulo p^2. If the sumset never contains -1, this support is impossible.

Proposition 22.3. *At least w_p nonunit labels have denominator divisible by p^{a_p}. For $t \geq 1$ put $I_t = \{p \in P : p^{a_p} \geq t\}$. If $|P| \geq 3$, the number of nonunit denominators at least t is at least*

$$\max\left\{ \max_{p \in I_t} w_p,\ \left\lceil \frac{\sum_{p \in I_t} w_p}{|P| - 2} \right\rceil,\ \max_{\substack{p \in P,\ p \geq t \\ p \nmid 2q}} w(p-1) \right\}.$$

An empty maximum contributes zero. Any lower bounds for the a_p may replace their exact values here.

Proof. The squared degrees prime to p, excluding the vacuum, sum to -1 modulo p^2. They occupy at least two duality orbits by Lemma 18.1; in odd dimension every orbit contributes twice the same square. Each such label contains the full p-part of M in its denominator. Conversely, its degree retains at least two support primes, so it can witness at most $|P| - 2$ of the missing primes.

This gives the first two bounds. At an odd $p \nmid q$, the two square classes of Proposition 19.1 require $w(p-1)$ labels with $p \mid b_i$. □

Proposition 22.4. *Suppose $\Delta = 10M^2$ has exactly three prime divisors. A non-pointed fusion-simple integral modular category with this dimension has rank at least* 36*. A divisor-minimal slightly degenerate obstruction with this* reduced *dimension has full rank at least* 68*.*

Proof. Write $P = \{2, 5, u\}$. Outside the three-prime families the dimension exponents in increasing prime order are at least $(6, 5, 4)$. For the ordinary table this gives $2^{a_2} \geq 8$, $5^{a_5} \geq 25$, and $u^{a_u} \geq u^3$ for $u = 3$ or u^2 otherwise. For the reduced table apply the same families to 2Δ; there the lower bound on 2^{a_2} is four. The three disjoint sets of prime-free witnesses have sizes at least $3, 3, 2$, respectively. This follows from Proposition 22.3, using modulo four that odd squares are one, and modulo five that two nonzero squares never sum to -1. In addition at least $u-1$ denominators are divisible by u, by Proposition 19.1.

Proposition 19.9 gives at most seven occurrences at $b = 1$. At five, Proposition 19.11 partitions the labels at $5 \nmid b$ into blocks of five equal degrees and blocks of two each of $d, 2d$ (a block of four each can be split into two for this test). Both propositions apply to the positive fermion quotient at odd primes.

The finite check enumerates the multiplicities of every P-smooth denominator at most 50. Only primes $u \leq r$ need be considered because of the $u-1$ required nonunits. For a fixed prefix, raise the height floors to the powers seen in its lcm. Subtract known witnesses from the three requirements, then apply them as lower bounds on the remaining sorted denominators. The u-packet count gives additional lower bounds on the last slots. Round each floor up to the least P-smooth integer above the last processed denominator and at least the floor. Counting the unused slots at these lower bounds gives an upper bound on their mass, also when the actual rank is smaller than the cap. The vacuum contributes at most the inverse square of the product of the three height floors.

At five the block condition is the integer system of Corollary 19.12, evaluated on doubling chains by the exact prefix recurrence following that corollary. A prefix fails if no choice is possible or its maximum obtainable mass is below ten. The short exact check exhausts these prefixes and leaves no u at rank cap 35 (ordinary) or 33 (reduced). The latter cap is on fermion pairs, giving the full rank stated. A separate support enumeration gives the same ordinary exclusion, with its first admissible prefix at rank 36. □

For bounds depending on two isolated primes and their heights, see Corollary 9.4 and its estimate (9.1).

23 Rank bounds for general weakly integral categories

Theorem 23.1. *A weakly integral non-WGT fusion category satisfies the following lower bounds on its rank.*

	arbitrary dimension	*odd dimension*	*strictly weakly integral*
arbitrary fusion category	7	13	8
braided fusion category	9	41	11

We first reduce the problem to integral categories, then use induction to the Drinfeld center in small ranks and count twists in the modularization in odd braided dimension.

23.1 Descent by the dimension grading

The dimension grading of a weakly integral category $\mathcal{C}$ is faithful over an elementary abelian two-group E, with neutral component the integral subcategory $\mathcal{B} = \mathcal{C}_{\mathrm{int}}$ [36, Theorem 3.10].

Since WGT passes to extensions [32, Proposition 4.1], a non-WGT $\mathcal{C}$ has non-WGT $\mathcal{B}$, which inherits any braiding. Moreover

$$\operatorname{FPdim}(\mathcal{C}) = |E| \operatorname{FPdim}(\mathcal{B}), \qquad \operatorname{rank}(\mathcal{C}) \geq \operatorname{rank}(\mathcal{B}) + |E| - 1.$$

Thus odd dimension implies integrality. With or without braiding, the least possible non-WGT rank, whenever examples exist, is the same for integral and weakly integral categories. Similarly, a rank-minimal counterexample equals its adjoint subcategory by its universal grading.

In the braided case a nonneutral component consisting of a single simple X forces $\mathcal{B}$ to be symmetric. Indeed $X^* = X$ and $X \otimes Y \simeq d_Y X$ for every simple Y of $\mathcal{B}$. Balancing gives $M_{X,Y} = \theta_Y^{-1}\,\mathrm{id}$ for the double braiding. The hexagon identities then give $M_{X\otimes X,Y} = \theta_Y^{-2}\,\mathrm{id}$. Its restriction to $\mathbf{1} \otimes Y$ is the identity, and every simple Z of $\mathcal{B}$ occurs in $X \otimes X$ with multiplicity d_Z by reciprocity. Hence Y centralizes $\mathcal{B}$. A symmetric fusion category over $\mathbb{C}$ is WGT by Deligne's theorem. It follows that a braided non-WGT category satisfies

$$\operatorname{rank}(\mathcal{C}) \geq \operatorname{rank}(\mathcal{B}) + 2(|E| - 1).$$

The two strict bounds in Theorem 23.1 now follow from the integral bounds below. If $E \neq 1$ and $\mathcal{B}$ has odd dimension, the braided bound is at least 43.

23.2 Small ranks from formal codegrees

Let $D = \operatorname{FPdim}(\mathcal{C})$. For a commutative integral fusion ring of a category, its character rows χ are orthogonal. Their squared norms f_χ are integers dividing D, and the dimension character has the largest norm D since $|\chi(X)| \leq d_X$. The coefficient of the unit in the primitive-idempotent expansion of 1 gives $\sum_\chi 1/f_\chi = 1$. For divisibility, the summands A_χ of $I(\mathbf{1})$ in the center, where I is right adjoint to the forgetful functor $F\colon \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$, have dimension D/f_χ, trivial twist and multiplicity one [49, Theorem 2.13]. Forgetting preserves dimension, so D/f_χ is an integer. The rational algebraic integer f_χ is therefore a divisor of D. In odd dimension duality pairs all nonunit simples. Indeed $\tau(b) = \sum_\chi \chi(b)/f_\chi$ for the unit-coefficient trace, so reduction at a prime above two gives $\tau(b) \equiv \operatorname{Tr}(N_b)$ for each basis element and its square. Since $\operatorname{Tr}(N_X^2) \equiv \operatorname{Tr}(N_X)^2 \equiv \operatorname{Tr}(N_X)$, we obtain $\delta_{X,X^*} = \delta_{X,\mathbf{1}}$. The trace of duality in the basis of primitive idempotents now pairs all characters except dimension by conjugation. We use commutativity in the braided case and, apart from pointed rings, through rank six and through odd-dimensional rank thirteen by [1, Theorems 1.5 and 1.8]. In the braided case each degree d divides D by modular divisibility in the center, which contains a braided copy of $\mathcal{C}$. Without braiding we apply the dimension families to the center of dimension D^2: it is Morita equivalent to $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}$, and WGT passes to subcategories [32, Proposition 4.1].

Here is the entire reciprocal enumeration. Reserve the dimension row for last, and give each preceding denominator weight $w = 1$, or $w = 2$ for conjugate pairs in odd dimension. For a remaining fraction P/Q in lowest terms and s entries still preceding D, try in nondecreasing order all integers $wQ/P < x \leq (ws+1)Q/P$, odd in the paired case. For the final x, the ratio $k = D/x$ divides Q and $D = Q(wk+1)/P$: Q divides the prefix lcm, which divides D, and $k \mid D = Q((wk+1)/P)$ with $\gcd(k, (wk+1)/P) = 1$. Test each divisor and check every earlier denominator divides D. Rank one or two gives $D = 1$ or 2. The optional three-term cutoff at $w = 1$ uses $D \leq Qx(Qx+1)/(Px-Q)$ after trying x; comparison to a fixed threshold reduces to a convex quadratic, whose maximum on an interval is attained at an endpoint.

Codegree two. Codegree two in rank > 2 is unique, hence its row is integral and has values $1, -1$ on $1, g$ and zero elsewhere by dimension orthogonality. These are the only invertibles, since each character is nonzero on every invertible. Every other character is one on g, so its permutation fixes every noninvertible: its sole eigenvalue -1 already occurs on $\{1, g\}$. Stabilizer divisibility [23, Lemma 2.2(i)] makes their degrees even and $D \equiv 2 \pmod 4$. The codegrees

$2, 3, 7$ cannot all occur [51, Corollary 2]. The enumeration `code/unbraided_small.py` covers every D^2 through rank five; at rank six these last two obstructions and the covered centers remove, respectively, $26, 232, 1310$ of 1568 tuples (`data/unbraided_small.txt`). This proves the unrestricted integral bound seven.

For the braided bound take a counterexample of least rank. Its universal grading is trivial, since WGT passes to extensions. Its invertibles are therefore transparent: their double braiding defines a grading character trivial on the adjoint category. At codegree two, g fixes a noninvertible and balancing gives $\theta_g = 1$. Thus $\mathcal{C} = \mathcal{B}^{C_2}$ with g the sign. A fixed simple of $\mathcal{B}$ has two lifts exchanged by g [13, Corollary 2.13]; only the vacuum is fixed. The other simples have half the integral degree of a free-orbit lift and hence integral degree; $\mathrm{FPdim}(\mathcal{B}) = D/2$ is odd. The involution of its commutative ring fixes just the dimension idempotent, by its trace in the simple basis. Each other pair χ, χ' restricts to one character via the orbit-sum map from $K_0(\mathcal{C})$; their distinct restrictions exhaust the rows other than $2, D$ because the image is the invariant algebra. The restricted norm is $2 + \frac{1}{2}\sum_{Y \neq 1} |\chi(Y) + \chi'(Y)|^2 = f_\chi$ by orthogonality. All intermediate codegrees thus divide $D/2$.

Using the divisor families and Theorem 17.1, `code/formal_sieve.cpp` leaves no tuple through rank seven and 52 tuples in 38 dimensions at rank eight (`data/formal_residual.txt`). At codegree two the cutoff is $2 \cdot 6\,087\,156\,075$, since the square-full part of D is odd and at most $D/2$; otherwise use $29\,160\,000$. One tuple has all codegrees distinct, so the rows are integral, whereas the Gram determinant $\prod f_i$ is nonsquare. Each other tuple has two threes and all remaining codegrees distinct. Each row of norm three has two nonunit entries, both roots of unity: dimension orthogonality rules out only one and the conjugate mean square of a nonzero algebraic integer is at least one. With two invertibles its values must be $1, 1, -1$ on the two invertibles and one degree two, impossible for orthogonal rows. With one invertible, $1 + a\alpha + b\beta = 0$, $a, b > 1$, forces consecutive degrees and opposite signs: $a = b$ would give the noninteger rational algebraic integer $-1/a$, and $|a - b| \leq 1$. Two orthogonal rows share one position of opposite sign; hence three consecutive degrees occur and the Gram determinant is square. With three invertibles the rows restrict to the nontrivial characters of C_3 and vanish elsewhere. Replacing them by their sum and their difference divided by $\sqrt{-3}$ gives a rational table, so $(\prod f_i)/3$ is square. The two nontrivial eigenvalues of a generator occur on the unit orbit, hence it fixes every other label; their degrees are divisible by three.

The two square-class requirements leave respectively two and five tuples. In the one-invertible case subtract the unit and three consecutive squares from $D = 576\,298\,800$, and test whether the remainder is four squares of nonunit divisors. In the three-invertible case, $D \equiv 3 \pmod 9$ leaves only $(3, 3, 5, 8, 121, 14600, 2649920, 351101150400)$. Its integral norm-five row is one on the three invertibles and has two more signs. Its signed dimension sum gives nonunit degrees $3a, 3(a + 1)$. The five nonunit degrees divided by three are divisors of $D/3$ with squared sum $(D - 3)/9$. Neither integer test has a solution, by the pair sums in `code/formal_check.py`. This proves the integral braided bound nine.

23.3 The odd unbraided boundary

Ranks $3, 5, 7, 9$ yield $1, 3, 13, 115$ paired tuples, all covered at D^2. At rank eleven there are 2799. We justify the additional conditions in `code/odd_unbraided.py`, whose complete counts and last eight tuples are in `data/odd_unbraided.json`. Write $f_1, \ldots, f_5$ for the paired denominators. When distinct, each pair is Galois-stable. Restrict the real parts of its characters and the dimension row to one representative per dual pair and the unit. The column metric is $\mathrm{diag}(1, 2, \ldots, 2)$ and the resulting rational Gram matrix has entries $D, f_1/2, \ldots, f_5/2$ on the diagonal. Hence $D \prod f_i$ is a square.

We give the subcategory test used at $3, 7$ and $5, 11$. Suppose the earlier rows vanish outside a subcategory $\mathcal{H}_0$ of dimension h and every other row equals dimension on it. At a single conjugate pair of prime codegree $f = 2h + 1$, write a_i for its values on the exterior pairs and $t_i = a_i + \bar{a}_i$.

The row field is imaginary quadratic because the row is nonreal and has one possible nontrivial conjugate. Orthogonality to its conjugate and its row norm give $\sum t_i^2 = 1$, $-\sum(a_i - \bar{a}_i)^2 = f$. The unique nonzero trace is -1 at a pair of degree h by dimension orthogonality. In $\mathbb{Q}(\sqrt{-m})$, m square-free, the odd trace implies $m \equiv 3 \pmod 4$; write $a_i - \bar{a}_i = k_i\sqrt{-m}$ with $k_i \equiv t_i \pmod 2$. Then $m\sum k_i^2 = f$ gives $m = f$. The pair has values $(-1 \pm \sqrt{-f})/2$ and all other exterior values vanish. By orthogonality each subsequent character takes value h on both new simples. Adjoining these simples gives a fusion subcategory $\mathcal{H}$. Indeed, the character formula for N_{UV}^W vanishes for U, V inside the enlarged set and W outside, being $d_U d_V \sum_\chi \overline{\chi(W)}/f_\chi = 0$. It also gives $UY = d_U Y$ for every exterior Y and every U inside, by evaluation on the characters. The $\mathcal{H}$-module component containing Y has only this simple, hence $\mathrm{FPdim}(\mathcal{H}) \mid d_Y^2$ [30, Proposition 8.15 and Remark 8.17].

Starting at the unit gives the pointed C_3 when $f_1 = 3$ and then a subcategory of dimension 21 at a single pair of sevens. A subsequent pair at 43 would give a subcategory with degrees $1, 1, 1, 3, 3, 21, 21$, contrary to the odd rank-seven classification [1, Theorem 1.7]. In this greedy sequence uniqueness of each pair follows from the remaining reciprocal sum. At dimensions three and twenty-one all exterior degrees are multiples of 3 and 21 respectively, giving $D \equiv 3 \pmod 9$ and $D \equiv 21 \pmod{882}$ in the latter case.

For use here and in the perfect case, consider any row χ of codegree five. If the unit were the only invertible, orthogonality to dimension would require at least two dual pairs of hits: their degrees have gcd one. Equality of the mean-square bound then gives two pairs of root-of-unity values $u, \bar{u}, v, \bar{v}$. Their traces $s = u + \bar{u}, t = v + \bar{v}$ and degrees $d, e > 1$ satisfy $s^2 + t^2 = 3$ and $ds + et = -1$ by orthogonality to $\bar{\chi}$ and to dimension. Rational traces are integers and cannot square to three in this way. Otherwise elimination gives the quadratic trace $-2d/(d^2 + e^2)$ of s, strictly between -1 and zero. Thus there are at least three invertibles. The order of the group of invertibles is odd and at most five, since the row is nonzero on the group. At order three there would be one extra dual pair of roots $a, \bar{a}$. A trivial restriction to C_3 gives $3 + a^2 + \bar{a}^2 = 0$ by orthogonality to $\bar{\chi}$, while a nontrivial restriction gives both $a + \bar{a} = 0$ by dimension orthogonality and $a^2 + \bar{a}^2 = 0$. Consequently there are five invertibles, and the Galois translates of the restriction give four codegree-five rows. They restrict to the four nontrivial characters of C_5 and vanish elsewhere. A single pair of elevens adds two objects of degree five. In rank eleven the remaining two dual pairs have degrees divisible by 55, so $(D - 55)/6050$ must be two positive squares.

We recall the indicator identity $\mathrm{Tr}(\theta^a_{I(X)}) = D\nu_a(X)$ of Ng–Schauenburg [45, Theorem 4.1]. If g is an invertible of prime order p fixing some simple under left tensor product, its pointed category admits a rank-one module category. Its associator class is therefore trivial, so $\nu_p(g) = 1$ and $\nu_1(g) = 0$. Positivity of dimensions forces every summand of $I(g)$ to have pth-root twist, and the relation at $a = 1$ forces the dimension at each of the p twists to be D/p, by the minimal polynomial of a primitive pth root. Write $r = \mathrm{rank}(\mathcal{C})$ and, for a simple central object A, put $m_X(A) = [F(A) : X]$. Frobenius reciprocity gives $[I(X) : A] = m_X(A)$ and $F(I(X)) = \bigoplus_Y Y X Y^*$. In particular when the Grothendieck ring is commutative and g invertible,

$$\langle I(g), I(g)\rangle = r, \qquad \langle I(1), I(g)\rangle = \#\{Y : gY = Y\}.$$

Lemma 23.2. *Suppose $r = 11$ and $f_1 = 3$. There are three invertibles, with generator g fixing every noninvertible, and $D \equiv 3 \pmod 9$. For u_j the least nonnegative residue of $D/f_j - 1$ modulo three, one has $\sum u_j = 8$. If*

$$\frac{D}{3} - \sum_j u_j \frac{D}{f_j} = 1$$

and the dimension character is the only character of codegree D, then $\mathcal{C}$ de-equivariantizes by a central $\mathrm{Rep}(C_3)$ to a fusion category of dimension $D/3$.

Proof. The subcategory argument with $h = 1$ gave the three invertibles, the fixed noninvertibles

and the dimension congruence. Pair the nonunit summands of $I(1)$ into dual pairs, choosing A_j in each; their degrees are D/f_j. Put $v_j = m_g(A_j) + m_{g^2}(A_j)$. Since the central category has odd dimension and duality interchanges g and g^2, and since the external dimensions are multiples of three, $v_j \equiv u_j \pmod 3$, $\sum v_j = 8$. The known summands contribute at least $\sum\lceil v_j^2/2\rceil$ to $\langle I(g), I(g)\rangle$. The two nontrivial twists each require at least one more summand. If some $v_j \geq 3$, the five nonnegative integers of sum eight give $\sum\lceil v_j^2/2\rceil \geq 10$ (minimum attained at $3,2,1,1,1$). This would exceed the available norm eleven. Thus $v_j = u_j$. At twist one the remaining mass in $I(g)$ is the displayed gap. A gap of one supplies an invertible central lift of g with twist one. Its cube lies above 1 and is the unit because $I(1)$ has no other invertible summand. This gives the asserted central Tannakian subcategory. Its embedding is fully faithful after forgetting since its generator maps to g. □

Lemma 23.3 (Schur elements of cyclic invariants)**.** *Let B be a finite-dimensional semisimple complex algebra with a cyclic C_ℓ-action, ℓ prime, preserving a symmetrizing trace τ. Write $\tau|_{\mathrm{End}(V)} = f_V^{-1}\,\mathrm{Tr}_V$, with Tr_V the ordinary matrix trace. Suppose B^{C_ℓ} is commutative, and let e be one of its primitive idempotents. On a free block orbit e is the sum of the identities of ℓ scalar blocks, so $\tau(e) = \ell/f_V$. On a fixed block it has matrix rank one and $\tau(e) = 1/f_V$.*

Proof. On a free orbit the invariant algebra projects isomorphically to any constituent matrix block. Its commutativity makes that block scalar, and invariance makes the ℓ trace weights equal. On a fixed block a generator acts by conjugation by a matrix T with T^ℓ scalar. Rescaling T makes $T^\ell = 1$. Its centralizer is a product of endomorphism algebras of its eigenspaces. Since the centralizer is commutative, each nonzero eigenspace has dimension one. The trace formula follows by evaluating the corresponding projector. □

Lemma 23.4. *The tuple $(3, 7, 55, 179, 24323; 5028658635)$ cannot occur in a non-weakly-group-theoretical rank-eleven category.*

Proof. For this tuple $\sum u_j = 8$, the twist-one deficit is one, and D occurs only once. Lemma 23.2 therefore supplies a de-equivariantization $\mathcal{E}$ of dimension $M = D/3$. The canonical action of C_3 on $\mathcal{E}$ recovers $\mathcal{C}$ by equivariantization. In the equivariantization a fixed simple contributes three simple equivariant structures (the stabilizer is cyclic); a free orbit contributes one simple. The dual action by the three invertibles of $\mathcal{C}$ permutes the three structures in the first case and fixes the resulting simple in the second. Since the three invertibles are the only nonfixed labels in $\mathcal{C}$, the unit is the only fixed label of $\mathcal{E}$. Thus a noninvertible of $\mathcal{C}$ maps to a sum of three distinct simples of equal dimension in $\mathcal{E}$.

The subcategory of dimension 21 yields seven invertibles in $\mathcal{E}$. The other three dual pairs in $\mathcal{C}$ have dimensions $21e_i$. Here $\sum e_i^2 = (D-21)/882 = 5701427$. The three e_i are distinct. The calculation in `code/odd_unbraided.py` tests $5701427 - 2t^2$ for a positive square for $1 \leq t \leq \lfloor\sqrt{(5701427-1)/2}\rfloor$ and finds none. Hence $\mathcal{E}$ has rank 25 and its 18 noninvertibles occur in three sets of six, each degree divisible by seven. Its group C_7 fixes every one under both left and right tensor product.

Put $B = K_0(\mathcal{E})\otimes\mathbb{C}$. The forgetful map F from $K_0(\mathcal{C})\otimes\mathbb{C}$ has image B^{C_3}, since the orbit sums and the unit span it. This image has dimension nine (one fixed label and eight free label orbits). As $F(g) = 1$, the two scalar blocks on which g acts nontrivially lie in the kernel; dimension now gives equality. For each remaining primitive source idempotent e_χ, its coefficients at $1, g, g^2$ all equal $1/f_\chi$ by the character-idempotent formula, and no noninvertible label contributes to the unit after forgetting. Thus $\tau_B(F(e_\chi)) = 3/f_\chi$. By Lemma 23.3 this equals $3/f_V$ on a free block orbit and $1/f_V$ on a fixed block. The latter alternative would give $f_V = f_\chi/3$, impossible for any of the eight characters of codegrees $7, 55, 179, 24323$ by algebraic integrality. The eight free orbits already account for 24 scalar summands of the 25-dimensional algebra B. The last summand is scalar and carries the dimension character. In particular B is commutative.

Apply the indicator identity and the induction norm count in $\mathcal{E}$ for each $h \in C_7 \setminus \{1\}$. For any canonical $A \subset I(1)$ let $x_A = \sum_{h \neq 1} m_h(A)$. There are 25 canonical summands, six of each of the four listed codegrees and the unit. From the eighteen fixed simples, $\sum_A x_A = 6 \cdot 18 = 108$. The congruence $x_A \equiv d_A - 1 \pmod 7$ follows from divisibility of external degrees. At the eighteen summands of codegree $55, 179, 24323$ it requires $x_A \geq 6$. Thus $x_A = 6$ there and vanishes on the rest. Summing the twist-one mass over the six nontrivial h, the deficit from the canonical objects is

$$6M/7 - 6 \sum_{A:\, f_A \in \{55,179,24323\}} d_A = 6M/7 - 6(M - 1 - 6M/7) = 6.$$

For any noncanonical summand Z contributing to that deficit, put $s_Z = \sum_{h \neq 1} m_h(Z)$. Its contribution is $s_Z d_Z$. As $m_1(Z) = 0$ we have $d_Z \equiv s_Z \pmod 7$ and $1 \leq s_Z \leq d_Z \leq 6$, whence $d_Z = s_Z$. Each d_Z divides M by modular divisibility in the center, so is odd. Thus all $d_Z = 1$ and six central invertibles of twist one lift the six nontrivial elements of C_7. Indeed, two lifts of the same element would produce a nontrivial invertible above 1, whereas the dimension character alone has codegree M. They yield a central $\mathrm{Rep}(C_7)$ with faithful forgetful restriction. All powers of a twist-one invertible have twist one. The second de-equivariantization has dimension $D/21 = 5 \cdot 11 \cdot 13 \cdot 179 \cdot 1871$ and is therefore solvable by Corollary 15.4. Weak group-theoreticity goes back up both equivariantizations. □

For clarity, the successive counts in `data/odd_unbraided.json` are as follows. Of the 2799 paired tuples, 2134 have a covered center. The stabilizer congruence removes 533, the cubic induction norm 17, the central C_3 descent 88, and the real determinant 19. The dimension-55 and dimension-21 tests remove six and one, respectively. Lemma 23.4 excludes the last. This proves the general odd bound 13. If the Grothendieck ring is noncommutative, the published bound for a nonpointed odd-dimensional integral category is 23 [1, Theorem 1.8].

Corollary 23.5. *A nontrivial perfect integral fusion category of odd dimension has rank at least* 15*. With a braiding its rank is at least* 57*, and at least* 75 *in nonsquare dimension.*

Proof. An odd-dimensional WGT category is solvable by the group-extension argument of Corollary 15.4 and the odd-order theorem. A nontrivial solvable category contains a nonunit invertible [32, Proposition 4.5(iv)]. Thus it suffices to exclude rank thirteen in the first assertion. At that rank the ring is commutative by [1, Theorem 1.8]. Perfection rules out a row of codegree three (its support would be a single dual pair of nonunit degree, contradicting orthogonality), and the norm-five calculation above excludes five. Hence the reciprocal denominators are odd and at least seven. In a commutative perfect odd-dimensional integral fusion ring arising from a category, a character χ of codegree seven has nonreal field contained in $\mathbb{Q}(\zeta_7)$ by Corollary 2.8. Its support consists of the unit and two or three dual pairs, by the preceding norm and gcd estimates. With three pairs all six nonunit values are roots in μ_{14}. Orthogonality to $\bar{\chi}$ says that the seven squares of the entries sum to zero. The cyclotomic relation forces these squares to run through μ_7. Orthogonality to the dimension character is then a relation on all seventh roots: after writing each value as a signed seventh root, the powers are distinct and the signed coefficients are its simple degrees. The vacuum coefficient is one, so every coefficient equals one, contrary to perfection.

With two pairs, all Galois conjugates of χ have the same five-label support and are mutually orthogonal, so the field has degree at most five. The only nonreal proper subfield of $\mathbb{Q}(\zeta_7)$ is $\mathbb{Q}(\sqrt{-7})$. The two nonzero squared absolute values are positive integer norms summing to three. Thus on a pair A the character is $\epsilon = \pm 1$ and on a pair B its values have norm two and sum $\delta = \pm 1$ by the norm form of that field. Their degrees satisfy $1 + 2\epsilon d_A + \delta d_B = 0$, so $d_B > d_A$. Associate the ordered pair $A \to B$ to $\{\chi, \bar{\chi}\}$. Two such pairs have distinct heads, because $\chi - \bar{\chi}$ is supported just at the head, and two differences with that same support would be proportional and orthogonal. Their tails too are distinct: at a common tail the inner product of the two character

sums $\chi + \bar{\chi}$ would be 4 ± 8. For distinct heads and tails this inner product is four from the unit and ± 4 per head–tail coincidence. Its vanishing therefore requires exactly one such coincidence. Three pairwise consecutive directed edges with distinct heads and distinct tails would form a cycle, which strict degree increase precludes. There are at most two complex-conjugate pairs of codegree seven.

At rank thirteen the recursion with six paired denominators at least seven and one final D gives 520 tuples. Of these, 17 have $D \geq 121275$ and at most two paired denominators equal to seven. Testing their squares by the exponent comparisons of Theorem 17.1 leaves the paired denominators and dimensions

$$(7, 7, 11, 11, 31, 4785; 1038345), \qquad (7, 7, 9, 11, 95, 575; 7571025).$$

In both, $v_7(D) = 1$ and no other paired denominator is divisible by seven. In $D\tau = \sum_\chi (D/f_\chi)\chi$, reduce at a prime above seven. The two codegree-seven pairs give two distinct reduced characters: the two quadratic-conjugate values agree at the unique prime of $\mathbb{Q}(\sqrt{-7})$ over seven, and the values on each pair A, B are nonzero modulo seven. The two supports differ, since their edges are consecutive and directed by strictly increasing degrees. The surviving relation contains only the dimension character and these two distinct characters, each of the latter with nonzero coefficient $2D/7$. This contradicts independence of distinct characters over the residue field, even if one is the dimension character. Hence rank thirteen is impossible.

For a perfect braided category of odd dimension every fusion subcategory is perfect. Its symmetric center is Tannakian, and a nontrivial group of odd order is solvable, hence has a nontrivial linear character. It follows that each subcategory is nondegenerate. Successive Müger decompositions express the category as a product of nontrivial simple perfect modular categories. Ranks multiply, and Theorem 22.2 gives 57 for every factor. If the total dimension is nonsquare, at least one factor has nonsquare dimension and hence rank at least 75. □

There is a consequence for simple integral categories in which no braiding exists. Their formal codegrees exclude 3 provided the category is nonpointed. Indeed, their Drinfeld centers are simple by [38, Theorem 5.1] and integral by the forgetful functor. They are nonpointed since their global dimension is a nontrivial square, whereas a simple pointed modular category has prime dimension. A codegree f of the original category gives a simple of dimension D/f in the center [49, Theorem 2.13]. Its modular codegree is f^2, so the assertion follows from Theorem 20.8.

23.4 Counting twists in odd braided dimension

Suppose that $\mathcal{C}$ is an odd-dimensional braided non-WGT category of least rank R. Its universal grading is trivial. If $\mathcal{C}$ is modular, $(\mathcal{C}_{\mathrm{ad}})' = \mathcal{C}_{\mathrm{pt}}$ makes it perfect, and Corollary 23.5 already gives $R \geq 57$. Otherwise its symmetric center is $\mathrm{Rep}(G)$ for a nontrivial group G of odd order. Write k for its number of conjugacy classes. The central de-equivariantization $\mathcal{M}$ is modular, $\mathcal{C} = \mathcal{M}^G$, and $\mathrm{FPdim}(\mathcal{C}) = |G|D$ for $D = \mathrm{FPdim}(\mathcal{M})$ [26, Section 4]. The category $\mathcal{M}$ is non-WGT since WGT passes to equivariantizations, so at least three primes divide D by [32, Theorem 1.6]. Braided equivalences preserve the canonical positive spherical structure. Thus every twist of $\mathcal{M}$ occurs in $\mathcal{C}$ under the dominant forgetful functor. Write t for the number of its distinct nontrivial twists. We have

$$R \geq k + 2t.$$

Indeed, all k central simples have twist one, and a nontrivial twist accounts for at least one pair of distinct duals in odd dimension.

A lower bound on the number of twists. Fix a set S of primes of even valuation in D. Partition the projected S-tuples of twists by the exact orders of their coordinates. In a nonempty part let $A \subset S$ be the primes where the coordinate is nontrivial. Fixing the other coordinates still

leaves at least $p-1$ choices at $p \in A$ by Proposition 19.1, first with the full complementary twist fixed and then after projection. Successively forgetting coordinates gives at least $\prod_{p\in A}(p-1)$ distinct tuples in this part. For nonempty A this is at least $\sum_{p\in A}(p-1)$. Every prime occurs by categorical Cauchy, and the vacuum contributes the trivial tuple. There are therefore at least $1+\sum_{p\in S}(p-1)$ projected twists.

At a prime p of odd valuation, every nonempty fiber over complementary twist coordinates has at least $(p+1)/2$ different p-phases. A phase of order at least p^2 supplies enough square Galois conjugates. Otherwise Proposition 3.9 at the vacuum gives positive inactive mass and order-p mass equal to $(p-1)$ times that mass. An active phase has $(p-1)/2$ square conjugates.

If D has at least two primes of odd valuation, use these two fiber bounds over the projection to any third prime q. The latter has at least $(q+1)/2$ values by categorical Cauchy, square Galois conjugation and the vacuum. There are at least $2\cdot 3\cdot 4 = 24$ twists. If precisely p has odd valuation, choose two other primes $u<v$. There are at least $(p+1)(u+v-1)/2 \geq 22$ twists (the minima for $p=3,5,\geq 7$ are respectively $22, 27, 28$). In each nonsquare case $R \geq k+42$.

It remains to consider $D=n^2$. In this case $t \geq \sum_{p|n}(p-1) \geq 12$. Simple degrees in either category are odd: in a braided integral category of dimension N they divide N, by the embedding into its Drinfeld center. The sums of squared degrees therefore give $R \equiv |G|D \equiv |G| \equiv k \pmod 8$. If t attains $\sum_{p|n}(p-1)$, each active set is a singleton and all its phases have order p. Indeed, for distinct odd primes the product of any two or more of the numbers $p-1$ exceeds their sum. A phase of order p^j, $j\geq 2$, has at least $p(p-1)/2 > p-1$ square conjugates.

We will use the following equality argument twice. Suppose all nontrivial phases are pure of prime order and only the unit of $\mathcal{M}$ has twist one. At each p, let $m_p>0$ be the common mass at each primitive pth root (Proposition 19.1). The dimension and the real Gauss sum satisfy

$$n^2 = 1+\sum_{p|n}(p-1)m_p, \qquad \tau^+ = 1-\sum_{p|n} m_p = -n.$$

Writing P for the greatest prime of n gives $n^2 \leq 1+(P-1)(n+1)$, hence $n\leq P$. This contradicts the presence of three distinct prime divisors.

If $R-k=24$, then $t=12$ and the primes are $3,5,7$. Every nontrivial twist uses exactly two simples of $\mathcal{C}$, so its twist-one simples are exactly those in $\mathrm{Rep}(G)$. Dominance and the fact that their images are multiples of the unit imply that only the unit of $\mathcal{M}$ has twist one. This is the equality argument above. Consequently $R-k\geq 32$. Since $k\geq 3$ for a nontrivial odd-order group, a counterexample of rank below 41 must have $R-k=32$ and $k\in\{3,5,7\}$.

The orbits at the boundary. No nonunit G-orbit of simples in $\mathcal{M}$ is its own dual: duality would fix a point of that odd orbit. The odd rank-seven classification [1, Theorem 1.7] applied to $\mathrm{Rep}(G)$ allows orders $3,5,7,21,39,55$. For a nonabelian group $F_{pq}=C_q\rtimes C_p$ of order pq $(p<q)$ the class number is $p+(q-1)/p$ by counting in the normal subgroup and its nontrivial cosets. Thus the choices at $k=3,5,7$ are respectively C_3, C_5, F_{21}, and C_7, F_{39}, F_{55}. All stabilizers $H\leq G$ have trivial Schur multiplier: restriction to cyclic Sylow subgroups detects every primary part of $H^2(H,\mathbb{C}^\times)$ by restriction–corestriction. A pair of dual nonunit orbits with stabilizer H therefore contributes $k(H)$ pairs of simples to $\mathcal{C}$ [13, Corollary 2.13]. Their total is 16.

At most 16 nontrivial twists occur, forcing $3,5 \mid n$. If $7\nmid n$, the only possible support is $3,5,11$ and $t=16$. Each orbit-pair then has a distinct nontrivial twist, so only the unit has twist one, and all phases are pure and tame by equality in the lower bound. The Gauss sum again excludes this case. Thus n has precisely the prime divisors $3,5,7$. At least thirteen pairs of nonunit orbits are needed: with only twelve, the same equality argument applies. In particular a stabilizer with $k(H)\geq 5$, or two orbit-pairs with $k(H)\geq 3$, would use more than the sixteen available pairs of lifts.

For each $p \mid \gcd(|G|,n)$ at least one nonunit orbit has size prime to p, since otherwise $n^2\equiv 1 \pmod p$ on summing its squared degrees by orbits. For $G=C_5, F_{55}$ take $p=5$, and for

$G = C_7, F_{21}$ take $p = 7$. A stabilizer containing the corresponding Sylow subgroup has at least five classes in each case. The only possibilities are therefore $G = C_3, F_{39}$. In each, one orbit-pair has stabilizer C_3 and the other thirteen have trivial stabilizer. Let $s = |G|/3 \in \{1, 13\}$. The degree d of a simple in the exceptional pair satisfies $1 + 2sd^2 \equiv n^2 \equiv 0 \pmod 3$, so $3 \nmid d$.

The exceptional pair has twist one. At any other twist its mass would contribute $2sd^2 \not\equiv 0 \pmod 3$; the mass at every other nontrivial phase would be divisible by three. But each nontrivial phase has a distinct phase of equal mass: vary a nontrivial primary coordinate, using Proposition 19.1 at order p and a square Galois translate at higher order. This proves the claim. At most thirteen nontrivial twists remain. A mixed phase would increase the minimum 12 by at least $(3-1)(5-1) - (3-1) - (5-1) = 2$. At $5, 7$ all phases are therefore pure and tame. At 3 there are either both cubic phases or one square orbit of three ninth roots. Every nontrivial phase contains exactly one free dual orbit-pair: in the tame case a spare pair at a nontrivial phase would force $d_1^2 + d_2^2 = e^2$ on comparing with a phase of equal mass, impossible for odd degrees modulo eight. The degree at each p is consequently constant across its phases. By Lemma 3.4(2) and Theorem 3.6, a simple at such a phase has codegree a nontrivial power of p. Since $D = n^2$, its degree is n/p^{j_p} with $j_p \geq 1$.

Put $b = 2|G| \in \{6, 78\}$. The mass per phase at p is now $b(n/p^{j_p})^2$. The full set of primitive pth roots sums to -1; a square orbit of ninth roots sums to zero. Thus the Gauss sum is real, say ϵn. Subtract it from n^2 to cancel the twist-one mass. In both cubic cases the equation is

$$1 - \epsilon/n = b\left(\frac{3}{3^{2j_3}} + \frac{5}{5^{2j_5}} + \frac{7}{7^{2j_7}}\right), \qquad \epsilon = \pm 1, \quad 105 \mid n.$$

For $b = 6$ individual terms force $j_3, j_5 \geq 2$. With $j_7 \geq 2$ the right side is below 0.29. With $j_7 = 1$ it is above 1.07 when $j_3 = 2$, and below 0.94 otherwise. For $b = 78$ the componentwise minima are $(3, 2, 2)$. There the sum exceeds 1.17. Increasing any one coordinate makes it smaller than 0.96. All these bounds are outside $[1 - 1/105, 1 + 1/105]$. This excludes the three boundary ranks and completes the proof of Theorem 23.1.

24 Cyclotomic support and local ramification

Throughout these sections a minimal table is one of the two tables in Lemma 2.1. The dimension of a positive fermionic quotient is half the dimension of its category. Write P_t for the principal block at a chosen prime above t, and call X *t-active* when the t-part of θ_X is nontrivial. The twist graph has the dimension primes as vertices and an edge t, u when a simple twist is divisible by tu. An edge need not have the same witness as another edge. We call t, u *separated* when this edge is absent. In the ordinary square-dimensional case put $n = \sqrt{D}$, $s = S/n$, $f_X = D/d_X^2$ and $K_X = \mathbb{Q}(\chi_X)$. A nonzero S-entry will be called a hit. A full word at nonunits X, Y is a positive rational combination of phases, of total mass d_X, equal to $\chi_Y(X)$ up to a common root of unity. Write $W(t, k)$ for the impossibility of such a word on at most k positions of t-power order in odd ordinary minimal data. All assertions about divisor families allow a coprime square-free factor d.

24.1 Two phases of arbitrary relative order

In this subsection exposure is taken among all roots at any common multiple of the conductors of the expressions compared.

Theorem 24.1. *If a full-weight word has exactly two distinct phases at a pair of nonunits in a minimal ordinary or positive reduced table, their ratio has order $2, 3, 4, 5$ or 6. At order five, a constituent in one of the two expanded adjoints from Proposition 4.21 has a balancing phase of exact order ten. In odd ordinary data only order three is possible.*

Proof. Write the ratio as ρ and its order as n. We prove that for $n \geq 7$ the set $D = \{1, \rho, \rho^{-1}\}$ is exposed at every common conductor. We may choose any primitive nth root for ρ by Galois symmetry. We first give a construction that handles all but six orders at once.

Split off a prime-power factor s^b of the common conductor, writing each root as αy, with α of s-power order and y of order coprime to s. At a marked primary position write y_i for the complementary component of its marked root. The three primary positions will always be distinct in this construction. Prescribe there the functional $f_i(y) = \mathrm{Tr}_{\mathrm{norm}}(y/y_i)$, normalized absolute trace on the complementary field. It takes the value 1 on a root only when $y = y_i$, and lies in $[-1, 1]$ at all roots. On each μ_s-coset of primary positions with k marks, put $-\sum f_i/(s-k)$ at each vacant position. These assignments define a rational functional by linear disjointness and the primary cyclotomic relation that the values sum to zero on each coset.

If $s \mid n$ and $s \geq 7$, then $k \leq 3 < s/2$, giving strictness off the marks. If $s^2 \mid n$ for an odd prime s, the three positions lie in three distinct cosets, so $k \leq 1 < s/2$. If $8 \mid n$, use $s = 2$. The three marked cosets are again distinct, and every complementary root has odd order and normalized trace at least $-1/2$. Indeed the trace of a primitive dth root is $\mu(d)/\varphi(d)$ by summing its conjugates. Thus the vacant position opposite a mark has value at most $1/2$.

The remaining $n \geq 7$ divide 60. At $m = \mathrm{lcm}(n, 2)$ the table gives an integer functional L in the basis $1, z, \ldots, z^{\varphi(m)-1}$, where z is a primitive mth root. Its value on D and its maximum on the other mth roots are recorded in the last two columns.

n	$(L(z^i))_{0 \leq i < \varphi(m)}$	on D	elsewhere
10	$(2, 2, 1, -1)$	2	1
12	$(2, 2, 1, 0)$	2	1
15	$(2, -1, 2, 1, -1, 1, -1, 1)$	2	1
20	$(2, 2, 1, 0, -1, -1, -1, -1)$	2	1
30	$(4, 4, 2, 3, 3, 1, 1, -1)$	4	3
60	$(4, 4, -1, -1, 1, 2, -1, 3, -1, 1, 1, 1, -1, -1, -1, -1)$	4	3

The entries can be checked by the recurrence $L(z^{j+d}) = -\sum_{i<d} a_i L(z^{j+i})$ for $\Phi_m(x) = x^d + \sum_{i<d} a_i x^i$. Divide by the value on D. Such a functional on an even level m extends to every multiple by normalized relative trace. Its absolute value on μ_m is at most one by sign symmetry. Outside μ_m, an increased old prime-power level gives trace zero, and a new odd prime gives zero or a factor of absolute value at most $1/2$.

The adjoint of the two-phase word gives a full-weight positive expression supported on D. Choose a Galois multiplier u with $u \not\equiv \pm 1 \pmod{n}$; such a unit exists for every $n \geq 7$ and lifts to the common conductor. Then $D \cap uD = \{1\}$. As in Lemma 4.20, the two adjoints share a nonunit. Exposure forces its balancing phases at Y to lie in this intersection, contrary to strictness. At order five compare difference sets $\{0, \pm d\}$ and $\{0, \pm 2d\}$. They meet only at zero. Proposition 4.21 forces failed exposure in a five-coset with three marked and two unmarked positions. Formula (4.5) with $h = 2$ gives an unmarked position admitting an order-two complementary root. Its primary part is a nonidentity fifth root, so this actual phase has order ten. In odd ordinary data all balancing phases have odd order, which excludes this case and every even relative order. □

A test from the unresolved family. At $2^6 3^6 5^4$ one surviving complementary crossing can have the normalized word $z = 1 + 3\zeta_5$. This word has strict house below four, residue four over three and five, and tame trace one at five. On the alphabet μ_{30} the pure coefficient vector for $|z|^2$ is $(10, 3, 0, 0, 3)$. The minimum in Theorem 4.9 occurs at $t = 3$ and is thirteen:

$$|z|^2 = 7 + 3(-\zeta_5^2) + 3(-\zeta_5^{-2}).$$

The required adjoint mass is sixteen. Given *any* $\omega \in \mu_{30}$, add $\omega(1 + \zeta_3 + \zeta_3^2) = 0$ to the right. This gives a positive integral expression of mass sixteen containing ω. Thus even the sharpest one-entry support bound at this alphabet retains every phase. Nevertheless Theorem 24.1 forces

the indicated order-ten event once this two-position alternative is selected: after scaling, the crossing gives a full character word. It is a disjunction about specified products. The joint restrictions on the surviving modular data are developed in Section 29.

24.2 Fusion support of the exceptional cyclotomic word

For an ordinary or positive reduced crossing of nonunits X, Y, write $\ell = \operatorname{lcm}(d_X, d_Y)$, $g = \gcd(d_X, d_Y)$ and $z = S_{XY}/\ell$. Choose actual even lifts in the reduced case. A constituent V of XY gives its monodromy phase $\theta_V/(\theta_X\theta_Y)$ mass $N_{XY}^V d_V$. The total mass is $g\ell$.

Proposition 24.2. *Suppose the signed alternative in the two-residue test of Theorem 4.13 occurs at distinct odd primes $q < r$. Thus $g = r - 1$ and $k = (r-2)/q$ is a positive integer. Its $r-1$ phases $-\eta_t\zeta_r^t$ each have mass ℓ in XY. All constituents have dyadic twist $-\theta_{X,2}\theta_{Y,2}$, and total dimension at least $(r-3-k)\ell$ comes from constituents active at both q and r. In particular the signed alternative requires a column mixing the two odd primes.*

Proof. The stated alternative writes $z = -\sum_{t=1}^{r-1}\eta_t\zeta_r^t$, where the η_t comprise 1 and k complete μ_q-cosets. The nonidentity rth roots are independent over the q-power field. Some η_t is nonreal, so conjugating the q-parts alone changes z. An unsigned r-primary optimum is therefore impossible. By the signed-word classification in the proof of Theorem 4.13 every integral optimum has the displayed form, and independence fixes each η_t. Lemma 4.11 with $c = 1$ gives these same multiplicities for the physical expression divided by ℓ. At most one phase cancels the r-part of $\theta_X\theta_Y$, and at most $k+1$ cancel its q-part, since each coset uses each position at most once. The remaining number is at least $r - 3 - k > 0$, as $q \geq 3$ and $r \geq 5$. □

The unsigned alternatives of Theorem 4.13 also determine all monodromy phases and masses. At one residue with $r > 2g$, at most $g < r/2$ positions are used. At two residues with $g < r$, write $z = h + q\sum_{j=1}^{m}\xi_j$ with $h + qm = g$. If $h = 0$, the number b of distinct positions satisfies $2b \leq 2m \leq g$. If $h > 0$ then $2b \leq 2 + 2m \leq g + 1 \leq r$. In both cases $b < r/2$, since r is odd. Theorem 4.10 therefore exposes the marked positions even with all complementary orders allowed. The coset relations give their independence. Thus each monodromy phase in the displayed integral word has mass ℓ times its multiplicity and every constituent has the same prime-to-r twist as $\theta_X\theta_Y$.

At $g = 4$, assume the residues at $3, 5$ are four and the house is strictly less than four. The signed profile puts mass ℓ at each of its four phases. The only other expression in Theorem 4.13 is $1 + 3\xi$ with ξ of five-power order. Since $\chi_Y(X) = (d_X/4)z$ has full weight, the two-phase theorem gives $\operatorname{ord}(\xi) = 5$. Their masses are $\ell, 3\ell$, respectively.

Let H be tame five-inertia, of order four, acting trivially on the prime-to-five factor at a common conductor. In both cases $\sum_{\sigma\in H}\sigma z = 1$: the four conjugates of a primitive fifth root sum to -1, and $\sum\eta_t = 1$. The proof identifies the constituent masses before applying the trace; thus it applies at arbitrarily large common conductor.

Proposition 24.3 (A single coset of wild phases)**.** *Let s, r be distinct odd primes, let ω, ζ be primitive sth and rth roots, and let θ have s-power order. Write $e(i,j) = \omega^i\zeta^j$ on $\mathbb{F}_s \times \mathbb{F}_r$. Let T be a set of at least two points with distinct first coordinates and distinct, nonzero second coordinates. Suppose that*

$$(i,j), (i',j') \in T - T,\ j = j' \quad \Longrightarrow \quad i = i'.$$

If, up to a scalar root, the normalized character $\chi_Y(U)/d_U$ at nonunit labels U, Y in a minimal ordinary or positive reduced table is a convex combination with positive rational coefficients of $\{1\} \cup \theta e(T)$, then $\theta^s = 1$.

Proof. On $\mathbb{Q}(\omega, \zeta)$, a rational linear functional is specified by its values on the s-by-r array $e(i, j)$, subject precisely to zero row and column sums, by the geometric-sum relations and the degree $(s-1)(r-1)$. Consider any set of h marked entries using each column at most once, with k_i marks in row i and $2 \max_i k_i < h$. It is exposed in the signed array. To see this, take two copies of the h marks and match them while forbidding pairs in the same row. Every allowed pair extends to a perfect matching: after deleting the pair, a source subset from two or more rows sees all remaining targets, and a single row satisfies Hall's inequality since $2k_i \le h - 1$. Average these matchings uniformly. In each used column put 1 at its mark and at every other row put the negative of the probability that the match ends in that row. All row and column sums vanish, and at least three rows contain marks, so every unmarked absolute value is strictly below 1. In unused columns put zero. Two marks in different rows and columns are also exposed: extend them in two different ways to three marks in different rows and columns. The extensions exist because s, r are distinct odd primes. Average the functionals exposing these triples.

Suppose θ has order s^t, $t \ge 2$. The ratios of the word form the union $D = C \cup H \cup H^{-1}$, where $C = e(T - T)$ and $H = \theta e(T)$. In the direct sum decomposition of $\mathbb{Q}(\theta, \zeta)$ over $\mathbb{Q}(\omega, \zeta)$ by cosets $\theta^a \mu_s$ of the s-primary parts, these sets occupy three different summands. Indeed $[\mathbb{Q}(\theta, \zeta) : \mathbb{Q}(\omega, \zeta)] = s^{t-1}$, with basis θ^a, $0 \le a < s^{t-1}$. In C only the identity has row zero. The other row sizes satisfy $k_i = k_{-i}$ by symmetry, so $2 \max k_i < |C|$. The preceding construction therefore exposes C and, using prefactors $\theta^{\pm 1}$ on their summands, H, H^{-1}. Use zero on unused summands. This exposes D at conductor $2rs^t$. Normalized relative trace extends exposure to every common conductor, as in Theorem 24.1.

Let σ fix all s-power roots and invert ζ. It preserves each of the three summands. If (i, j) belongs to both $T - T$ and its image under column negation, then $(-i, j)$ also belongs to $T - T$ by central symmetry. Uniqueness in column j gives $i = 0$, hence $j = 0$. In T the uniqueness of the row forces a common point with its image to have $j = -j$, impossible. Thus $D \cap \sigma D = \{1\}$. The adjoints of U and σU have a common nonunit constituent. By exposure its balancing expression at Y uses only the intersection, contradicting strictness. □

The same proof with $D = C$ shows that a word supported on $e(T)$ alone, up to a scalar, is impossible. In this assertion columns of T may include zero, since the argument uses only differences. For example, the proposition applies to $T = \{(j, b + j) : 0 \le j < m\}$ whenever $2 \le m \le s$, $r \ge 2m - 1$, and all its columns are nonzero. The column of a difference uniquely determines its integer step in $[-m + 1, m - 1]$. For the signed four-term crossing, take ζ to be the fifth root at the distinguished singleton. In order of the remaining exponents $2, 3, 4$, the ternary parts are $\theta, \theta\omega, \theta\omega^2$, on choosing one of the primitive cube roots as ω. Division by the singleton gives the progression $s = 3, r = 5, m = 3, b = 1$, hence $\theta^3 = 1$.

At the remaining level $\theta^3 = 1$ there are exactly two Galois orbits of signed words with the four equal weights. Indeed, the two positions with ternary part 1 are either opposites in $\mathbb{F}_5^\times$ or a pair with quotient 2 up to inversion. Multiplying the positions by a unit is transitive on each type, and ternary inversion exchanges the assignments on the other two positions. The four nonidentity fifth roots are independent over $\mathbb{Q}(\omega)$, so the stabilizer of the word is exactly the stabilizer of its array. For an opposite pair it is generated by complex conjugation, giving $\mathbb{Q}(z) = \mathbb{Q}(\zeta_{15})^+$; for a nonopposite pair it is trivial, giving $\mathbb{Q}(z) = \mathbb{Q}(\zeta_{15})$. Thus if either label is self-dual, the opposite signed profile is forced, since the unsigned value $1 + 3\xi$ is also nonreal. For both orbits the adjoint word contains $1, \xi, \xi^{-1}$ for a primitive fifth root ξ, from the repeated ternary part. The identity $1 + \xi + \xi^{-1} = -\xi^2 - \xi^{-2}$ replaces an equal positive amount at these three positions with strictly less total weight in the signed alphabet. The saved weight can be distributed uniformly over all roots of the common conductor since their sum is zero. Thus neither orbit has a proper exposed adjoint face in the full signed alphabet.

24.3 Two positive residues just above the larger prime

Write μ_{v^∞} for the group of all v-power roots of unity. In the normalized crossing of two integral simples, a positive expression of weight g is a sum of roots of unity with nonnegative rational coefficients totaling g. The weight from balancing is $g = \gcd(d_X, d_Y)$. The following result also determines the monodromy phases of the product.

Proposition 24.4. *Let $s > q \geq 5$ be odd primes, $g = s + t$, $1 \leq t < q$, and suppose every prime in the phase orders is at least $\ell \geq 3$. Let z be a cyclotomic integer with a positive expression of weight g, congruent to g at all primes above q and s. For a residue a modulo q with representative $0 \leq a_0 < q$, put $w(a) = \min(a_0, c(q - a_0))$. Suppose some real $1 < c < \min(\ell - 1, q - 1)$ satisfies*

$$\left(1 + \frac{2t}{s}\right)^2 < \frac{2(q-2)}{q-1}, \quad c(s-t) > g, \quad w(g) > t, \quad (s-1)w(a) + w(g+a) > g \quad (a \neq 0).$$

Then every positive weight-g expression for z has the same phase multiplicities. They are nonnegative integers. The q-parts of the terms consist of t copies of 1 and s copies of one q-primary root η. All remaining parts are s-primary. At every s-position x its multiplicity n_x satisfies $n_x \equiv g\, 1_{x=1} \pmod{q}$.

Proof. Give positive roots cost one and negative roots cost c. First consider an integral signed word in q- and s-primary roots with the two residues and cost at most g. Erase the q-parts and let m_x be the signed multiplicity at s-position x. Modulo q, the numbers $m_x - g1_{x=1}$ are constant on each coset of μ_s by the cyclotomic relations. The identity coset cannot have a nonzero constant a by the last cost inequality. A nonidentity coset with nonzero constant would cost at least s, besides the identity position's cost $w(g) > t$. Thus all constants vanish and the signed augmentation is g modulo q.

Erase instead the s-parts, obtaining a signed q-primary word of value v. Its residue over s gives $\alpha = (v - t)/s$ integral, as s is unramified in the q-power field. Each conjugate of α has absolute value at most $1 + 2t/s$. A nonzero cyclotomic integer in a q-power field of mean square less than $2(q-2)/(q-1)$ is a signed q-power root. Indeed, write it as an integer sum of q-power roots. Cassels's trace identities [15, (3.4) and (3.16), pp. 114–115] make different μ_q-cosets orthogonal for normalized trace and give the contribution $\sum_{i<j}(b_i - b_j)^2/(q-1)$ for a coset with coefficient vector (b_i). A nonconstant coset contributes at least one. In a single coset, two coefficient levels at distance one give numerator $k(q-k)$; below $2(q-2)$ one of their multiplicities must be one. If a pair differs by at least two, its contribution and those with the remaining $q-2$ points total at least $4 + 2(q-2)$. This proves the assertion. Also $v = t$ would give signed augmentation t modulo q, contradicting $g \equiv t + s$. Hence $v = t \pm s\eta$ for a q-power root η.

For the minus sign, prescribe value $-c$ at η and $c/(q-1)$ elsewhere in its μ_q-coset, zero on other cosets. This is a rational-linear functional with real values, bounded by the assigned costs on signed roots. Its value at $t - s\eta$ is at least $c(s-t) > g$. For the plus sign, mark 1 and η, assigning value 1 at each marked position and $-k/(q-k)$ at the other positions of a coset containing k marks. The resulting functional is strictly below cost on every signed letter other than the marked positive letters. Its value at $t + s\eta$ is g. Thus cost at most g forces an unsigned word of length exactly g with precisely the stated q-parts, by linear independence of the one or two marked roots.

Apply Lemma 4.11 with retained primes q, s and the cost $c < \ell - 1$. The integral lower bound g just proved holds for every target with the two residues. Thus every phase of the original expression has only q- and s-parts, and its coefficients are a convex combination of integral optima.

For uniqueness it suffices that the union of the s-positions of two optima have fewer than s points in each μ_s-coset. Each optimum has at most $1 + \lfloor g/q \rfloor$ positions, including 1 since $w(g) > 0$. Hence the union has size at most $1 + 2\lfloor g/q \rfloor < s$ (use $q \geq 5$, $t < q < s$). These s-roots are independent over the q-power field, so the two coefficient sums of q-roots agree separately

at each position. The union of their q-roots is contained in $\{1, \eta, \eta'\}$; these are independent if distinct since $q \geq 5$. All multiplicities therefore coincide. □

For $g = 25$, $q = 7$, $s = 23$, $t = 2$, take $c = 2$ and $\ell = 5$. The inequalities are $(27/23)^2 < 5/3$, $42 > 25$, $w(25) = 4 > 2$. For $a = 1$ one has $22 + w(26) = 26$, and for other nonzero residues $22w(a) \geq 44$.

Corollary 24.5. *At any nonunit crossing of the two-residue type just described, tame* 23*-inertia has orbit size* 22 *on the normalized entry* z*. In particular both fields of the two labels ramify at* 23*. This applies to all opposite complementary rows of the principal seven- and* 23*-blocks with common* 5*-part* 25*, for arbitrary exponents of the other degree factors.*

Proof. The 23-projection contains 1 and at most three further positions. If it is just 1, then $z = 2 + 23\eta$. If $\eta = 1$ this violates strictness; otherwise the full-mass two-position character has relative order at least seven, excluded by Theorem 24.1. By uniqueness every stabilizer in the order-22 tame group preserves the nonzero positions (in additive 23-exponents). Its order divides their number, since that group acts freely there. Only order two could survive. In that case the projection is $A = \{0, a, -a\}$. The full positive adjoint word on the same row has 23-components in $A - A = \{0, \pm a, \pm 2a\}$ and 7-components in $\{0, \pm b\}$, writing b for the exponent of η. This entire product of sets is exposed. In each primary factor mark the prescribed positions, give them value one, and use $-k/(v - k)$ elsewhere in a μ_v-coset with k marks. Each density is less than one half, so the product functional (with normalized trace on any complementary factor) exposes the prescribed set. An independent 7-unit moves the nonzero pair off itself, and a tame 23-unit moves the two nonzero opposite pairs off themselves (at most four of eleven multipliers can give a match modulo sign). The original and conjugate adjoints would then have a common nonunit whose balancing phases are all 1, contradicting strictness (Lemma 4.20). This rules out the last stabilizer. □

Proposition 24.6. *In the* $g = 25$*,* $q = 7$*,* $s = 23$ *crossing of Proposition 24.4 in an ordinary minimal table on* $5, 7, 23$*, the four* 23*-positions belong to* μ_{23} *and, in additive exponents, are affinely equivalent to* $\{0, 1, 3, 14\} \subset \mathbb{F}_{23}$*. Their multiplicities are four at zero and seven at each of the other three. At most two letters have* 7*-part different from* η*. The joint inertia orbit of the entry at* $7, 23$ *has size* $22\varphi(\operatorname{ord}\eta)$*. With* $v_{23}(23) = 1$*,*

$$v_{23}(z - 2 - 23\eta) = 1/22$$

at every prime above 23*. At the prime-weight crossing* $z = \eta(2 + 7\sum_{i=1}^{3} \xi_i)$ *of Proposition 8.7 at* $7, 23$*, the same affine classification gives* $v_{23}(z) = 1/22$*.*

Proof. Write A for the 23-projection of the weight-25 word, $D = A - A$ and $B = \{0, \pm b\}$ for the differences of the two possible 7-parts. The rectangle $B \times D$ exposes the adjoint word even when the two components are correlated. Use the product of the two coset functionals in Corollary 24.5, and the normalized trace on the remaining primary factor. Indeed the negative values in the 7- and 23-factors have magnitudes bounded by $3/4$ and $13/10$, respectively, since $|A| \leq 4$. Their product is at most $39/40 < 1$. In the remaining factor the trace is at least $-1/4$. Thus the functional attains one precisely on this rectangle. An independent tame 7-unit moves the nonzero pair in B off itself. For every 23-unit u, comparison with the conjugate rectangle consequently gives a common nonunit whose entire word on the fixed row is pure 23-primary on $D \cap uD$. Likewise comparison of the rectangle with any subsequently forced pure mask uses precisely their 23-coordinate intersection.

Every pure word on this row has at least five opposite pairs of nonzero differences. Otherwise a minimizing one with $h \leq 4$ has exposed differences and violates $10 \min\{j : j(j+1) \geq h\} \leq h(h-1)$ (Lemma 4.22). Thus each of the intersections just obtained contains at least two opposite pairs. If A meets two order-23 cosets, at most three of the at most six opposite pairs in D have order

23. With l such pairs the average overlap over all units is at most $(l^2 + (6-l)^2/23)/11 < 2$, a contradiction. Hence $A \subset \mu_{23}$ since it contains the identity. The remaining finite calculation takes place in $\mathbb{F}_{23}$. Normalize two marked positions to $0, 1$ and take the other two in increasing order. For each of the $\binom{21}{2} = 210$ supports, start with the set B of its nonzero differences modulo sign. Store one representative per dilation class. Given two stored sets B, C and $u \in \mathbb{F}_{23}^{\times}$, put $H = B \cap uC$. Reject if $|H| < 2$ or if the nonzero differences of $\{0\} \cup (\pm H)$ comprise fewer than five opposite pairs. Otherwise add these differences if the resulting support L is exposed, i.e. $3(1+2|L|) < 46$. Repeat until rejection or no new dilation class appears. The preceding rectangle argument justifies each use of the initial set, and later sets are pure. Of the 18 distinct initial dilation classes, the surviving supports among the 210 are exactly the 12 affine translates and dilates of $\{0, 1, 3, 14\}$ normalized in this way. The closure calculation with `check 23 4 -ell 3 -classes` reproduces these counts (the least conductor prime in this exposure test may be taken to be three). A starting mask with fewer than five opposite pairs is likewise impossible, since its mean overlap over nonidentity tame units is at most $4 \cdot 3/10 < 2$. Thus the same four-position proof gives precisely the indicated affine class. Fewer than four positions give at most three difference pairs. The congruences in Proposition 24.4 now fix the four multiplicities.

The full 23-unit group has trivial stabilizer of these positions: a stabilizer acts freely on the three nonzero ones and has order dividing 22. Under a joint action the 23-projection is still preserved by uniqueness, so the same argument applies. With the 23-action then trivial, the 7-component must fix η since it has multiplicity 23. This gives the orbit size.

Write $0, a_1, a_2, a_3$ for the projected positions. Their affine centroid is outside the support, since the centroid of the displayed representative is 16 modulo 23. In particular $U = 7\sum_{j=1}^{3} a_j \neq 0 \pmod{23}$. Color two letters with 7-part one and the others with 7-part η. This convention works also when $\eta = 1$. Let V be the sum of the exponents of the two colored letters. For $\pi = \zeta_{23} - 1$, the linear coefficient in $z - (2 + 23\eta)$ is $V + (U - V)\eta$ modulo π. It is nonzero: if $\eta = 1$ it equals U, and otherwise the reduction of η has 7-power order greater than one and hence lies outside $\mathbb{F}_{23}$. This proves the first valuation. For the pure prime-weight word the linear coefficient after division by η is U, and its constant term is 23. The same argument applies. □

Lemma 24.7. *Let X, Y be integral labels in a normalized minimal table of dimension D, let $L = \operatorname{lcm}(d_X, d_Y)$ and $z = S_{XY}/L$. Let G be the Galois group of the abelian field generated by z. For every character ψ of G,*

$$L\left|\sum_{u \in G} \psi(u)u(z)\right| \leq \sqrt{D}.$$

Proof. Choose a Galois lift σ_u of every $u \in G$. The labels X^{σ_u} are distinct since their entries at Y are distinct. The same holds for Y^{σ_u}. The submatrix of S/L on these rows and columns has entries $(uv)(z)$ because degrees are integral. Testing on a unit vector with coordinates $\psi(v)/\sqrt{|G|}$ gives the claimed inequality by whole-table orthogonality. □

In either crossing of Proposition 24.6, if $\eta \neq 1$ and $Q = \operatorname{ord}(\eta)$, then

$$\frac{\sqrt{D}}{L} \geq 19\cos(2\pi/11)\sqrt{23Q}.$$

Indeed $G = (\mathbb{Z}/Q)^{\times} \times \mathbb{F}_{23}^{\times}$ by the proven orbit size and conductor, also for the pure prime-weight word by the same stabilizer argument. Use a primitive character at Q and a nontrivial character χ at 23. Summing first at Q kills each term with 7-part one and gives a Gauss sum of modulus $\sqrt{Q}$ for every remaining term. Summing at 23 kills the zero position and yields a common Gauss sum of modulus $\sqrt{23}$ times $\sum_{j=1}^{3} m_j\overline{\chi(a_j)}$, where m_j counts the letters with 7-part η at a_j. Thus $\sum_j m_j \geq 19$. For each origin $c = 0, 1, 3, 14$ in the affine representative found by the four-position enumeration, use the character sending the primitive root 5 to $\exp(2\pi ik/22)$ with $k = 18, 7, 11, 9$, respectively. At the three other points shifted by $-c$, its exponents modulo 22

are respectively $\{0,2,4\},\{10,11,14\},\{11\},\{2,4,5\}$. An overall dilation rotates all three values together. Projection onto the bisector of the resulting arc gives the indicated bound. In particular if $D = 5^6 7^{2e} 23^4$ and $5^3 7^{e-1} 23 \mid L$, then $\eta = 1$ without a separation hypothesis: otherwise the upper bound is 161 while $19\cos(2\pi/11)\sqrt{23 \cdot 7} > 19\sqrt{161/2} > 161$.

24.4 Crossings of weight five at the primes three and five

Consider an ordinary minimal table with dimension primes $2, 3, 5$. For a jointly 3- and 5-active column, suppose $v_5(D) = 4$. Proposition 4.2 gives a two-free nonunit $Y \in P_3$ meeting the column with $d_Y = 3^j 5$; the height at five cannot be two by defect-zero vanishing. Take any three-free nonunit $X \in P_5$ of degree $2^i 5$. The normalized crossing $z = S_{XY}/\ell$, $\ell = \operatorname{lcm}(d_X, d_Y)$, is congruent to 5 above three and to 0 above five by principality. Apply Proposition 8.6 with $(q, r) = (3, 5)$. Its actual phase masses are either $2\ell, 3\ell$ at $\eta, \eta\xi$, for ξ primitive of order five by Theorem 24.1, or precisely ℓ at each of $\eta, -\eta\eta_t\zeta_5^t$, $t = 1, 2, 3, 4$. In particular, tame five-inertia has trace 5η in either case. In the second case there are four phases at which the dyadic monodromy equals -1 and a fifth at which it is 1.

Both witnesses may be chosen in the fibers of trivial prime-to-principal twist, by Proposition 4.2, so their actual dyadic twists are trivial. In the signed case at most one of the $r-1$ negative letters cancels the r-part of $\theta_X\theta_Y$, and at most $k+1$ cancel its q-part, where $k = (r-2)/q$. Thus at least $(r-3-k)\ell$ of the total dimension belongs to summands active at all three primes $2, q, r$. In particular, at the sextic frontier, absence of a twist of order divisible by 30 forces the first spectrum for every such crossing.

There is also an exact local valuation at the sextic frontier: at every prime above five, $v_5(z) = 1/4$, with $v_5(5) = 1$. This is immediate in the first case from $2 + 3\xi = 5 + 3(\xi - 1)$. In the second case expand in $\pi = \zeta_5 - 1$. The constant coefficient vanishes and, up to a unit, the linear coefficient is $\sum_{t=1}^{4} t\eta_t$. If $\theta\mu_3$ is the ternary coset of the three nondistinguished letters, this reduces to $a + \theta B$, $1 \le a \le 4$, $B \in \mathbb{F}_{25}$. When $\theta^3 \neq 1$ its residue is outside $\mathbb{F}_{25}$, since $3 \| 25 - 1$. Hence the coefficient is nonzero. If $\theta^3 = 1$, there are indices i, j with ternary part one and k, l with parts ω, ω^2 respectively, with ω primitive of order three. The linear coefficient is $(i + j - l) + (k - l)\omega$, nonzero at five as $k \neq l$ and $\omega \notin \mathbb{F}_5$. Tame five-inertia consequently gives at least four different conjugates of this crossing and of each witness. Indeed it fixes the ternary coefficients, and the leading residue of π is multiplied by each fourth root of unity in the residue field.

24.5 The conductor of a cyclic character subfield

Lemma 24.7a (Local ideals on a joint twist fiber)**.** *Let V have a basis (v_i) and commuting actions of $G_p = \mathrm{SL}_2(\mathbb{Z}/p^{b_p}\mathbb{Z})$ for a set S of odd primes. Write $U_p(x), L_p(x)$ for the upper and lower unipotents, and $H_{p,a} = \operatorname{diag}(a, a^{-1})$ for the diagonal torus element or its action. Suppose $U_p(1)v_i = \theta_{i,p}v_i$. For $0 \le k_p < b_p$ and units $a_p \equiv 1 \pmod{p^{k_p}}$ put*

$$E = \operatorname{span}\{v_i : \theta_{i,p}^{p^{k_p}} \neq 1 \text{ for every } p \in S\}, \qquad A = \prod_{p \in S} (H_{p,a_p} - I).$$

Then $AV \subseteq \sum_{g \in \prod_{p\in S} G_p} gE$. For $S = \varnothing$ take $E = V$ and $A = I$.

Suppose also that $G_2 = \mathrm{SL}_2(\mathbb{Z}/2^{b_2}\mathbb{Z})$ commutes with the odd factors and satisfies $U_2(1)v_i = \delta\theta_{i,2}v_i$ for a common scalar δ. For $0 \le k < b_2$, let $a \equiv 1 \pmod{2^k}$ be a unit and let P_a be a scalar multiple of $H_{2,a}$ that permutes the basis vectors. If $\theta_{i,2}^{2^k} = 1$ on all basis vectors of E, then $(P_a - I)A = 0$.

Proof. Work first in one factor $G = \mathrm{SL}_2(\mathbb{Z}/p^b\mathbb{Z})$, including $p = 2$. Modulo the normal subgroup generated by $U(p^k)$ all $U(x)$ and $L(x)$ with $x \in p^k\mathbb{Z}/p^b\mathbb{Z}$ are trivial, by taking powers and

conjugating by the Weyl element. For a unit $t = 1 + x$ the identity

$$U(x)L(1) = L(1/t)H_tU(x/t)$$

then gives $H_t = 1$ in the quotient because $1 - 1/t = x/t$. Consequently $H_t - I$ belongs to the two-sided ideal of $\mathbb{C}[G]$ generated by $U(p^k) - I$. Indeed the latter ideal contains $h - I$ for each conjugate h of $U(p^k)^{\pm 1}$, and hence for every product of such conjugates.

Set $B_p = U_p(p^{k_p}) - I$. Expand each $H_{p,a_p} - I$ as a sum of terms $c\, g_p B_p h_p$, with $g_p, h_p \in G_p$ and $c \in \mathbb{C}$. Commuting distinct factors puts every term of A in the form $c'(\prod g_p)(\prod B_p)(\prod h_p)$. The diagonal operator $\prod B_p$ has image exactly E. This proves the inclusion and also shows that G_2 preserves E whenever present.

If $E \neq 0$ and all its dyadic phases have order dividing 2^k, then $U_2(2^k)|_E = \delta^{2^k} I$. Its image in $\mathrm{PGL}(E)$ is the identity. Applying the normal-subgroup calculation there shows that $H_{2,a}|_E$, and hence $P_a|_E$, is scalar. A scalar that permutes a nonempty set of basis vectors is one. Commutation now gives $P_a = I$ on every odd translate gE. If $E = 0$ the first inclusion gives $A = 0$. □

For example, take dyadic level eight and a character of dyadic conductor eight. It is nontrivial at $a = 5$, the generator of the kernel of reduction of units modulo four. Here $k = 2$ and $5^{-1} = 5 \pmod 8$, so the displayed factorization is $U(4)L(1) = L(5)H_5U(4)$. If every dyadic phase on a nonzero E has order at most four, then $U(4) = \delta^4 I$ there and both $U(4)$ and its Weyl conjugate $L(-4) = L(4)$ act trivially in $\mathrm{PGL}(E)$. Since $L(5) = L(1)L(4)$, H_5 is projectively trivial, and its unsigned permutation fixes E pointwise. For a concrete joint realization, label the simples of $\mathcal{Z}(\mathrm{Vec}_{C_{24}})$ by $(x, y) \in (\mathbb{Z}/24)^2$, where y indexes the character sending 1 to ζ_{24}^y. The label $(1, 0)$ has trivial twist and its row contains ζ_{24}. Take $F = \mathbb{Q}(\sqrt{6})$, of conductor 24. Here E is the space of labels with nontrivial ternary twist, and $(1, 1)$ has twist ζ_{24}, of order 8 at two and 3 at three.

Theorem 24.8. *Let X be simple in an integral modular category with its positive spherical structure. Let F be a cyclic subfield of its character field K_X, with conductor f. There is a simple V such that* $\mathrm{lcm}(f, \mathrm{ord}\,\theta_X)$ *divides the order of the actual twist θ_V. At every prime p with $v_p(f) \leq v_p(\mathrm{ord}\,\theta_X)$ one may simultaneously prescribe $\theta_{V,p} = \theta_{X,p}$.*

Proof. Use the lift $U(1) \mapsto T/\gamma$ of Section 3 at its congruence level L [21, Theorem II]. The scalar γ has two-power order, so the eigenvalues of $U_p(1)$ are the actual p-primary twists for odd p, and those of $U_2(1)$ are $\gamma^{-1}\theta_{i,2}$. The tori act by Galois permutations with common sign $\epsilon_D(a) = \sigma_a(\sqrt{D})/\sqrt{D}$, as follows by applying Galois symmetry to $s_{i0} = d_i/\sqrt{D}$. Take the torus convention in which the unsigned permutation corresponds to σ_a (replacing all characters by their inverses gives the other convention).

Inflate a faithful character of $\mathrm{Gal}(F/\mathbb{Q})$ to a character ψ of $(\mathbb{Z}/L\mathbb{Z})^\times$. Its conductor is f because its kernel cuts out F. Let J consist of the primes for which $c_p = v_p(f) > v_p(\mathrm{ord}\,\theta_X)$, and put $k_p = c_p - 1$. Take V spanned by the labels with the same actual twist components as X outside J. Each local group at $p \in J$ preserves V because it commutes with the complementary upper unipotents. On the orbit of X under the product of the local tori in J, the unsigned action is the permutation action on a quotient of this abelian product. Its stabilizer fixes every $\chi_X(Y)$ and hence F. Thus ψ restricted to the product occurs on the orbit, and the represented tori have a common eigenline of character $\epsilon_D\psi$ in V.

For each $p \in J$ choose a local unit $a_p \equiv 1 \pmod{p^{k_p}}$ with $\psi(a_p) \neq 1$, extending it by one in the other factors. Such a unit exists by the local conductor of ψ. Among the odd primes in J set aside those for which $c_p = 1$ and $v_p(D)$ is odd. Call this set R, and write S for the other odd primes in J. For $p \in S$ the local quadratic sign is trivial at a_p. Indeed it is trivial on the entire local factor when $v_p(D)$ is even and on the principal units at every odd prime. Hence the eigenvalue of $A = \prod_{p \in S}(H_{p,a_p} - I)$ on the chosen line is $\prod_{p \in S}(\psi(a_p) - 1) \neq 0$.

The space E of Lemma 24.7a is therefore nonzero. Its labels satisfy $p^{c_p} \mid \mathrm{ord}\,\theta_i$ for every $p \in S$. If $2 \in J$, the unsigned permutation $P_{a_2} = \epsilon_D(a_2)H_{2,a_2}$ has eigenvalue $\psi(a_2)$ on the same

line. Thus $(P_{a_2} - I)A \neq 0$. The dyadic assertion of the lemma supplies a label of E whose dyadic twist order is at least 2^{c_2}.

Fix the actual twist components of the resulting label outside R. Proposition 3.9 gives a proportion at least $\prod_{p \in R}(1 - 1/p) > 0$ of squared-dimension mass active at every $p \in R$ in this fiber. Since $c_p = 1$ for these primes, any such label has all required orders and retains the prescribed twists outside J. □

Corollary 24.9. *If* $\operatorname{ord} \theta_X$ *is maximal under divisibility among the orders of all simple twists, then* $K_X \subseteq \mathbb{Q}(\theta_X)$.

Proof. For every cyclic subfield of K_X, the theorem and maximality force its conductor to divide $\operatorname{ord} \theta_X$, so it lies in $\mathbb{Q}(\theta_X)$. Their compositum is K_X, since the characters of the finite abelian group $\mathrm{Gal}(K_X/\mathbb{Q})$ separate its elements. □

Corollary 24.10. *If three primes ramify in the field of a single row, and at least one of their inertia images has order greater than two, some simple has twist order divisible by their product. More precisely, absence of such a twist forces the three inertia images to be the distinct subgroups of order two in a Klein four group.*

Proof. Let $G = \mathrm{Gal}(K_X/\mathbb{Q})$ and let I_p, I_q, I_r be the three nontrivial inertia images in G. By Theorem 24.8 every character of G is trivial on at least one of these subgroups if the stated twist is absent. Thus the three proper annihilators cover $G^\vee$. Recall the elementary three-subgroup argument. None of these subgroups is contained in the union of the other two: a group covered by two subgroups equals one of them. Each two of them therefore have the same intersection N as all three; for if x were common to the first two but outside the third, adding x to an element only in the third gives one in none. Modulo N, the pairwise intersections are zero. Choose a only in the first and b only in the second. For any nonzero a' in the first, both $a + b$ and $a' + b$ are in the third, whence $a - a' = 0$. By symmetry all three images have size two. Thus the quotient is a Klein four group; dualizing gives the claimed inertia images. □

A second consequence uses the actual twist of the given row. If q and r ramify in K_X and θ_X is p-active for a third prime p, some twist is active at all three. Indeed the two proper annihilators of the inertia images cannot cover the dual of $\mathrm{Gal}(K_X/\mathbb{Q})$. Choose a character outside both: its cyclic field has conductor divisible by qr, and the theorem retains the p-activity of X.

For example, in the absence of a twist of order divisible by 30, each complementary witness whose tame five-inertia orbit has length four is unramified at two or at three. This applies to the weight-five crossing of Proposition 8.6 as well as the complementary weight-four crossing of Section 24.2.

More directly, if the twist of J is active at two of the three dimension primes and its row field ramifies at the remaining prime, choose a cyclic subfield ramified there. The theorem produces a twist active at all three. Hence, at the ordinary sextic core in the absence of a twist of order divisible by 30, every jointly active row is unramified at the third prime. In particular every jointly two- and three-active column has a five-unramified field. The principal hit chosen above at two in the trivial prime-to-three twist fiber of P_3 therefore has odd degree 75 or 225, and its twist has three-power order (possibly one). This applies even when both two and three mix separately with five.

24.6 Changing the primary components of a twist

Work first in ordinary integral modular data with positive spherical structure. Write $s = S/\sqrt{D}$, v_X for the coordinate vector of a simple, and W_I for the product of the local Weyl operators at the primes of a subset I of the congruence level. We use the linear lift of Dong–Lin–Ng [21, Theorem II], with the normalization of Section 6. Thus $s = W_I W_J$ when I, J partition the level

primes. A common scalar convention for s does not affect any conclusion below. Different local groups commute.

Theorem 24.11. *Let A, B be sets of labels such that their coordinate subspaces are invariant under W_I, W_J, respectively. Then the submatrix $s_{A,B}$ has exactly $|A \cap B|$ nonzero singular values, all equal to one. In particular*

$$\sum_{X \in A, Y \in B} |S_{XY}|^2 = D\,|A \cap B|.$$

If $S_{XY} \neq 0$, any choice of the I-components of θ_Y and the complementary components of θ_X occurs as a simple twist. If $I = \{p\}$, the realizing simple can be chosen in the full p-block of X.

Proof. Write P_A, P_B for the coordinate projectors. The two Weyl operators are unitary, so the invariance gives

$$P_A s P_B = W_I P_{A \cap B} W_J.$$

For the second statement take A to prescribe the J-components of X, and B the I-components of Y. The nonzero entry forces a nonempty intersection. At a singleton I we may intersect A with the full p-block of X by Lemma 3.3. □

Proposition 24.12. *Fix a simple X, a prime $p \mid D$, its full p-block and all of its prime-to-p twist components. In this set there is a simple Z with trivial p-twist and $(W_p)_{XZ} \neq 0$. There is a simple Y in the same set with nontrivial p-twist and $(W_p)_{YX} \neq 0$ if and only if $p \mid f_X$. If X is itself p-active, the inactive Z has $p \mid f_Z$.*

Proof. Use the proof of Theorem 24.11 with Y there the tensor unit. Since $s_{X0} = d_X/\sqrt{D} \neq 0$, a term $(W_p)_{XZ}(W_{p'})_{Z0}$ is nonzero, giving the first assertion. If $p \nmid f_X$, its locally integral primitive idempotent makes its full p-block a singleton, and it is inactive by Theorem 3.2 and Lemma 3.4. Conversely suppose $W_p v_X$ has only inactive coordinates. The local upper unipotent is a scalar multiple of the actual p-twist, so both it and its conjugate by W_p^{-1} preserve the X-line. They generate the local group; hence W_p preserves the line. By Lemma 3.4 this implies $p \nmid f_X$. Every coordinate reached by W_p is in the stated block and fiber. An inactive Z with $p \nmid f_Z$ would lie in a singleton block and could not be joined by W_p to an active X. □

A simple with twist order maximal under divisibility satisfies $\mathrm{rad}(f_X) = \mathrm{rad}(\mathrm{ord}\,\theta_X)$. Indeed Proposition 24.12 would enlarge the twist order at any missing prime of f_X; the converse inclusion follows from Lemma 3.4. More generally, if no twist is divisible by $p \prod_{t \in T} t$, then every label active at all primes of T has p-unit codegree and its row vanishes on p-active labels. For square $D = n^2$, this

is precisely $v_p(d_X) = v_p(n)$.

The conductor-by-conductor bounds for inactive labels in the same fiber are proved in Proposition 24.18.

24.7 A congruence for primes missing from joint twists

Proposition 24.13. *Let p be any prime and let T be a nonempty set of other odd primes in a square-dimensional integral modular category. Prescribe some twist components outside $\{p\} \cup T$. Suppose no label in that fiber is active at every prime of $\{p\} \cup T$. If X is in the fiber and $p \mid f_X$, then for every label U and arbitrary independent local t-inertia automorphisms σ_t,*

$$\prod_{t \in T} (1 - \sigma_t)\chi_X(U) \in p\mathcal{O},$$

where $\mathcal{O}$ is the ring of integers of a common cyclotomic field. The ideal here is generated by p, including any ramification.

Proof. All indicated local groups preserve the fiber. By Lemma 24.7a at $k_t = 0$, the image of $P = \prod_{t\in T}(I - H_t)$ lies in the span of T-local translates of its simultaneous T-active coordinate space: the ideals generated by $I - U_t(1)$ contain the respective torus differences and the factors commute. The p-local modular factor preserves the joint space, on which its upper unipotent acts by the normalization scalar, hence the Weyl W_p acts by γ_p^3 there and on its translates (Proposition 6.3). Thus $W_pPv_X = \gamma_p^3 Pv_X$. Write $B = \prod_{u\neq p} W_u$ over the complementary local level factors, so $s = BW_p$. If $k = v_p(\sqrt{f_X}) \geq 1$, every entry of the X-column of B has p-valuation at least $1 - k$. This is the proof of Theorem 6.9 for the product of the complementary factors: choose an exponent divisible by the full p-level and equal to one on its complement. The Bruhat identity identifies $sT^m s^{-1}$ with B^{-1} up to diagonal phases and a scalar unit. The mixed indicator and the p-strand congruence give $2v_p(B_{iX}) + 2k > 0$, using the local torus at -1 on the other insertion as in that theorem. The entries of B are unramified at p by Galois covariance, so the valuation bound follows. It applies to every Galois conjugate of X. Multiply $sPv_X = \gamma_p^3 BPv_X$ by $\sqrt{f_X}$ and use Galois covariance of characters. □

Corollary 24.14. *Suppose there is no twist of order divisible by* 30 *in an ordinary minimal table on* $2, 3, 5$. *At every complementary* P_3, P_5 *crossing of weight four with five-free and three-free degrees, respectively, the lcm-normalized word is* $1 + 3\xi$, $\operatorname{ord}\xi = 5$. *At every crossing of an odd* $Y \in P_3$ *of degree* $3^j 5$ *with a three-free* $X \in P_5$ *of degree* $2^i 5$, *it is* $\eta(2 + 3\xi)$ *with* η *ternary. If* X *is dyadically active in the latter case,* $\eta = 1$.

Proof. Section 24.2 and Proposition 8.6 classify both words without bounds on the other exponents. In their signed alternatives the four nonidentity five-positions have coefficients $-\eta\eta_a$, $a = 1, 2, 3, 4$, with the multiset of the η_a one copy of one plus one coset of the cubic roots in the ternary roots. At weight four $\eta = 1$ and the constant coefficient $c_0 = 0$; at weight five $c_0 = \eta$. If both witnesses are dyadically inactive, all four negative packets are dyadically active. At most one cancels the five-twist of the product of the factors, and at most two cancel its three-twist. This forces a twist divisible by 30.

If the P_3 witness is dyadically active, apply the preceding congruence with $p = 3, t = 5$ in its dyadic twist fiber. Its 3-defect is positive by principality, and the opposite degree divided by the weight is a 3-unit. Thus all five-inertia conjugates of the normalized word agree modulo $3\mathcal{O}$. A tame generator permutes the four nonzero five-positions transitively and fixes the coefficient at zero. Over the ternary cyclotomic integers the only relation on these five roots modulo three is the constant vector, by linear disjointness and the monic cyclotomic basis. The four nonzero coefficients would agree modulo $3\mathcal{O}$. Distinct three-power roots differ by an element of 3-valuation at most $1/2$. This rules out the signed alternative.

If the P_5 witness is dyadically active, use $p = 5, t = 3$. The opposite degree divided by the weight is a five-unit, hence $c_0 - c_a$ is invariant modulo five under ternary inertia for every a, by the same integral-basis argument now reduced modulo five. At weight four this would fix all the distinct η_a under ternary inversion. At weight five one η_a equals one, so this first fixes 2η, whence $\eta = 1$, and then every η_a, again impossible. Reduction modulo a prime over five is injective on three-power roots. In the unsigned weight-five case the difference $(\eta - \sigma\eta)(2 + 3\xi)$ is divisible by five; the second factor has five-valuation $1/4$, forcing $\eta = 1$ as asserted. □

The congruence extends to whole residue blocks. In the absence of a joint p, t, r twist, let B be a full p-block containing a label of positive p-defect. If local t-inertia moves its residue character, every label in B is r-inactive. Indeed each member has positive p-defect, for one of defect zero is a singleton block. An r-active member would have t-invariant residue character by the singleton congruence proved above. Galois covariance in the *argument* shows that this applies to the entire t-orbit of B even at a fixed prime above p.

24.8 Character fields at twist one

Let S be ordinary symmetric integral modular data of dimension D, with $SS^* = DI$. Let G be the Galois group over $\mathbb{Q}$ of a cyclotomic field containing all the character fields $K_X = \mathbb{Q}(\chi_X)$. For a character ψ of G, use the same symbol for its descent to a quotient whenever it is trivial on the kernel.

Proposition 24.15. *Choose representatives X and Y of any two lists of distinct Galois orbits (the two lists may intersect). Form a matrix indexed by the representatives for which ψ descends to their character fields. Put $L_{XY} = \operatorname{lcm}(d_X, d_Y)$, $z_{XY} = S_{XY}/L_{XY}$ and $F_{XY} = \mathbb{Q}(z_{XY})$. The following matrix has operator norm at most $\sqrt{D}$:*

$$A_{XY} = L_{XY}\sqrt{[K_X : F_{XY}][K_Y : F_{XY}]} \sum_{u \in \mathrm{Gal}(F_{XY}/\mathbb{Q})} \psi(u)u(z_{XY}) \qquad \text{if } \psi \text{ descends to } F_{XY}.$$

The entry is zero otherwise. In particular every row and every column has squared norm at most D.

Proof. On the orbit of Y, take the unit vector whose coordinates are $\psi(v)/\sqrt{[K_Y : \mathbb{Q}]}$, with v ranging over $\mathrm{Gal}(K_Y/\mathbb{Q})$. On the orbit of X take the analogous vector with $\overline{\psi}$ in place of ψ. These are orthonormal families on each side. The matrix coefficients of S between them are the double averages of $L_{XY}\psi(u)\psi(v)(uv)(z_{XY})$. Here lifts from the two fields give the same answer by symmetry and the covariance of characters, since the degrees are rational. Summing on the fibers of restriction to F_{XY} gives zero if ψ is nontrivial on a kernel; otherwise it gives the displayed coefficient. Thus A is a compression of S between two isometries. □

Proposition 24.16. *In ordinary integral modular data with positive spherical structure let u, v, w be three distinct primes. Suppose the other local groups act by scalars. Let $X \in P_u$ have u-power twist and $Y \in P_v$ have v-power twist, allowing twist one in either case. Write G for the Galois group of the cyclotomic splitting field. Let ψ be a character of G. For $j = u, v$ suppose that inverting just the j-primary component of ψ leaves its kernel unchanged. In particular this holds for a character of prime-power conductor. If*

$$\sum_{a \in G} \psi(a)\, a(S_{XY}) \neq 0,$$

then P_u and P_v have twist-one members in a common full w-block. Their character fields both contain the cyclic field cut out by ψ.

Proof. By Proposition 24.15 the orbit vectors affording $\overline{\psi}$ at X and ψ at Y have a nonzero matrix coefficient for s. Now s is a scalar multiple of $W_u W_w W_v$. Move W_u^* onto the first vector and W_v onto the second. For completeness, when D is nonsquare the diagonal torus is $H_a = \epsilon_D(a)P_a$, with P_a the permutation matrix and $\epsilon_D(a) = a(\sqrt{D})/\sqrt{D}$ (Theorem 24.8). This quadratic character is unchanged by inversion in any subset of the local factors. Thus the conjugation rule for a Weyl and a torus also holds with P_a. A local Weyl inverts the torus on its own factor, so both new characters have the original kernel. The new row vector is supported on pure u-power twists in P_u, and the new column vector on pure v-power twists in P_v. A nonzero W_w-entry between their supports preserves the complementary twists and the full w-block (Lemma 3.3). Hence the two twists are one. The stabilizer of any nonzero coordinate of an eigenvector is in the kernel of its character, giving the two field inclusions. □

24.9 Multiplicities of quadratic inertia characters

In square-dimensional ordinary integral modular data let H_p be the group of local diagonal tori at an odd prime p, acting on the simple labels by the Galois permutations. Use the linear lift

of Section 6. For a set T of odd primes dividing the level, consider a coordinate subspace E stable under the local groups at every $p \in T$. For example, one may prescribe the complementary primary twist components, and for a single p one may also prescribe its full p-block. Write $H = \prod_{p\in T} H_p$ and let $\lambda = \prod_{p\in T}(\frac{\cdot}{p})$ be its product of quadratic characters. Retain the H-orbits in E on which λ is trivial on the stabilizer; write $m_1, \dots, m_b$ for their sizes.

Proposition 24.17. *One has $2^{|T|} \mid b$. Moreover, putting $K = \mathbb{Q}(\sqrt{p} : p \in T)$,*

$$\prod_{i=1}^{b} m_i \ \in \operatorname{disc}(K)^{\,b/[K:\mathbb{Q}]}\, N_{K/\mathbb{Q}}(K^\times)\,(\mathbb{Q}^\times)^2.$$

In particular, in every full five-block and specified prime-to-five twist fiber, the number of five-inertia orbits of tame order four is even, and so is the number of tame order two.

Proof. The rational λ-eigenspace V has the orthogonal basis of signed orbit sums on the retained orbits. Its positive form thus has Gram matrix $\operatorname{diag}(m_1, \dots, m_b)$. Write $p^* = (-1)^{(p-1)/2}p$. On V the Weyl operator W_p preserves the character, since it inverts the local torus and the latter acts by signs. For a cyclotomic automorphism σ_a the congruence representation gives $\sigma_a(W_p) = H_{a,p}W_p$, with the opposite torus convention producing the same signs. This follows by conjugation by $\operatorname{diag}(a, 1)$ in Lemma 6.2. Consequently $A_p = \sqrt{p^*}\,W_p$ has rational matrix on V. Here $W_p^2 = H_{-1,p} = (\frac{-1}{p})I$. Unitarity gives $A_p^* = A_p$ and $A_p^2 = pI$. These commuting operators make V a vector space over the stated multiquadratic field K, of degree $2^{|T|}$. The zero-dimensional case is immediate.

The rational symmetric form B is balanced over K since all A_p are self-adjoint. Nondegeneracy of the field trace defines a symmetric K-bilinear form B_K by $\operatorname{Tr}_{K/\mathbb{Q}}(uB_K(x, y)) = B(ux, y)$ for all $u \in K$. Diagonalize it over K. Taking the trace of a one-dimensional summand with coefficient c gives determinant $\operatorname{disc}(K)N_{K/\mathbb{Q}}(c)$ up to rational squares, proving the formula. For $p = 5$, the retained orbit sizes have the form $2 \cdot 5^a$ or $4 \cdot 5^a$. The prime-to-five unit of every nonzero norm from $\mathbb{Q}(\sqrt{5})$ is a square modulo five: in $x^2 - 5y^2$ the two summands have valuations of opposite parity, and -1 is a square modulo five. The discriminant is 5 and the unit part of a rational square is a square. Since two is a nonsquare, the number of tame-order-two orbits is even. Subtraction from the even dimension proves the assertion at tame order four. □

24.10 The conductor filtration inside a twist fiber

Let $D = n^2$ in ordinary integral modular data. Fix an odd prime p, one full p-block (or all blocks), and the exact prime-to-p twist components. Write E for the resulting nonempty fiber and P_E for its coordinate projector. Let Γ_E be the subgroup of all diagonal tori preserving E, and let b count its orbits on the p-inactive labels of E. Write

$$m(\alpha) = \sum_{X\in E:\, \theta_{X,p}=\alpha} d_X^2, \qquad J = \{j \geq 2 :\ m(\alpha) > 0 \text{ for some } \alpha \text{ of order } p^j\}.$$

Proposition 24.18. *All nonidentity pth roots have the same mass m_*. One has*

$$m_* \leq (p + 1)m(1), \qquad b \geq |J| + \mathbf{1}_{m_*>0} + \mathbf{1}_{m_*<(p+1)m(1)} \geq |J| + 1.$$

For the number of inactive labels, replace $|J|$ by $\sum_{j\in J} |G_j^2|$, where $G_j = (\mathbb{Z}/p^{\lfloor j/2\rfloor})^\times$. At odd p one has $|G_j^2| = \varphi(p^{\lfloor j/2\rfloor})/2$. If all inactive labels lie in one orbit and the fiber is active, its only twists at p are the pth roots, with $m_* = (p + 1)m(1)$ and total mass $p^2m(1)$. In particular $m(1) = d_Y^2$ for a singleton Y.

Proof. Write U, L, H, W for the local upper, lower, torus and Weyl operators. Put $v = nP_EW_{p'}v_0$. The upper unipotents and diagonal torus at p fix v, as does Γ_E, since conjugation across $W_{p'}$

only inverts the complementary tori and all tori fix the vacuum. The vector Wv has entries d_X on E and zero elsewhere.

At local level p^a let P_j average over the kernel of reduction modulo p^j, for $1 \le j \le a$. Its lower–diagonal–upper factorization is unique, the diagonal entries being 1 mod p^j. Averaging the factors in this order therefore gives $P_j v = \mathrm{avg}_{p^j|t}\, L(t)v$. Conjugation by W shows that WP_jv is exactly the projection of Wv onto p-twists of order dividing p^j. (Take $j = 1$ to mean the identity projection when $a = 1$.) The mutually orthogonal vectors $(P_j - P_{j-1})v$, $2 \le j \le a$, are thus nonzero precisely at $j \in J$. Each is upper-fixed and Γ_E-fixed, because its projectors commute with the local group and the other local factors. They lie in the space of invariant vectors on the inactive labels, of dimension b.

Fix such a nonzero layer $v_j = (P_j - P_{j-1})v$, on which the action factors through level p^j. Put $l = \lceil j/2 \rceil$ and $d = j - l$. The kernel of reduction modulo p^l is abelian, of elements $I + p^l A$ with $A \in \mathfrak{sl}_2(\mathbb{Z}/p^d)$. As v_j is upper- and torus-fixed, the characters in its decomposition on this kernel have the form $A \mapsto \exp(2\pi i c A_{21}/p^d)$. Some c is a unit: otherwise $L(p^{j-1})$ fixes v_j, although Wv_j has exact twist order p^j. Every square-unit multiple of this c occurs by torus invariance. Their character components are orthogonal, and are upper-fixed since upper conjugation fixes A_{21}. They give the indicated number of independent vectors on the inactive labels within this layer; different layers are orthogonal.

It remains to calculate in the quotient $G = \mathrm{SL}_2(\mathbb{F}_p)$ on $u = P_1 v$; its Weyl image gives all masses at orders one and p. Let B be its upper Borel, P_B its average, and $A = P_B W P_B$ on B-fixed vectors. Then

$$A = P_G - p^{-1}(I - P_G).$$

Indeed the two double cosets are B and BWB, of relative sizes $1 : p$. Averaging over G therefore gives $P_G = (I + pA)/(p + 1)$ on this space. Write $u = u_0 + u_1$ with $u_0 = P_G u$ and $u_1 = u - u_0$. The mass at one equals $\|Au\|^2$, since Wu is torus-fixed. For $t \in \mathbb{F}_p^\times$, the element $W^{-1}U(t)W$ belongs to BWB, so the tth moment of the masses is $\langle u, Au \rangle$. Finite Fourier inversion gives

$$m(1) = \|u_0\|^2 + p^{-2}\|u_1\|^2, \qquad m(\alpha) = (p+1)p^{-2}\|u_1\|^2 \quad (\alpha \in \mu_p \setminus \{1\}).$$

Each nonzero u_i supplies a further orthogonal invariant vector on the inactive labels. There is at least one such vector: $v \ne 0$ is upper-fixed, hence there are inactive labels, so $m(1) > 0$. □

The gap between the two masses also gives a signed integer identity at every local level:

$$m(1) - m_* = n \sum_{X \in E:\, \theta_{X,p} = 1} d_X \, (W_{p'} v_0)_X.$$

Indeed, for a unit t in the local ring, $W^{-1}U(t)W \in BWB$ for its upper triangular subgroup B. Its coefficient at v equals $\langle v, Wv \rangle$. Averaging such coefficients over the units kills roots of order at least p^2 and averages a primitive pth root to $-1/(p-1)$. The entries $n(W_{p'}v_0)_X$ are rational integers divisible by p by Lemma 6.2.

The same result holds at $p = 2$ whenever the dyadic anomaly scalar γ is 1 or -1. If $\gamma = -1$, tensor the local lift with the sign character of $\mathrm{SL}_2(\mathbb{F}_2) \cong S_3$: the normalized upper generator then becomes the actual dyadic twist, the diagonal tori are unchanged, and the Weyl changes only by a sign. In particular this applies if a nonempty exact odd-twist fiber has only dyadically inactive labels. The dyadic group preserves this fiber, so its upper acts there by γ^{-1}, and Bruhat gives $W_2 = \gamma^3 I$ there. Its square is an unsigned torus permutation, hence $\gamma^6 = 1$; the lift has dyadic γ, giving the assertion.

At $D = 2^6 3^6 5^4$ a joint three–five column whose five-twist has order 25 consequently requires two Galois-inequivalent five-inactive labels in its full five-block with the same dyadic and ternary twist components (inequivalence is under Γ_E). This strictly strengthens a count of two labels when those labels are conjugate. There is a joint three–five column at this dimension, so absence

of a twist of order divisible by 30 supplies the odd-twist fiber just used and gives the dyadic assertion on every block and complementary fiber. In particular, if $b = 1$ there and the fiber is dyadically active, its actual dyadic twists are $1, -1$ and the mass at -1 is precisely three times the mass at 1.

Proposition 24.19 (Joint conductor-profile transform)**.** *In ordinary integral modular data of square dimension $D = n^2$, assume that the dyadic anomaly is 1 or -1 whenever two occurs, with the sign correction above in the latter case. Write $M_{\boldsymbol{i}}$ for the squared-degree mass at twists of exact order $\prod_p p^{i_p}$. On an array indexed by these exponents apply the following operation in each prime coordinate, with the other indices fixed:*

$$(M_0, M_1, M_2, \ldots) \longmapsto \left(M_0 - \frac{M_1}{p^2 - 1},\, \frac{p^2 M_1}{p^2 - 1},\, M_2, \ldots \right).$$

All entries of the resulting array V are nonnegative, and the number of its positive entries is at most the number of distinct Galois orbits of twist-one labels.

Proof. Put $Q_{p,0} = P_{G_p}$, $Q_{p,1} = P_1 - P_{G_p}$, and $Q_{p,j} = P_j - P_{j-1}$ for $j \geq 2$. Then $V_{\boldsymbol{j}} = n^2 \| \prod_p Q_{p,j_p} v_0 \|^2$. To see this, compress each local twist-order projector, conjugated by its Weyl, to the space of upper-Borel fixed vectors. At order one its compression is $Q_{p,0} + p^{-2} Q_{p,1}$, at order p it is $(1 - p^{-2}) Q_{p,1}$, and at order p^j, $j \geq 2$, it is $Q_{p,j}$. These are the squared-norm formulas in the proof above, applied to an arbitrary upper-Borel fixed vector and polarized. Distinct local factors commute, so their compressed projectors can be multiplied inside the vacuum coefficient. Inverting the resulting two by two tame relation gives the stated array. Its nonzero vectors are orthogonal, fixed by every torus and every upper unipotent. They are consequently supported on twist-one labels and constant on their Galois orbits. □

In the forward relation of Proposition 24.19, a profile n' can contribute to order m only if $n' = md$ with d square-free and coprime to m. Every positive V-profile also occurs as an actual twist order. Thus every occurring m for which no such strict extension occurs gives its own positive entry. At the even core without a twist order divisible by 30, this applies to *every distinct mixed order*, even when one mixed order divides another. Hence their number is bounded by the number of Galois orbits at twist one. If all three pairs occur there are at least three such orbits, of which at least two are inequivalent nonunit orbits.

There is an unconditional form on the branch admitting an order divisible by 30. Fix $a \geq 0$ and take E to be the union of all dyadic phases of exact order 2^a. Work only with the local groups at three and five, with initial vector $nP_E W_2 v_0$. It is fixed by both upper Borels and by all tori, and its two local Weyls give the degree vector on E. Applying the proof of Proposition 24.19 only at the two odd primes gives

$$\#\{(b, c) \in \mathcal{Z}_{>0}^2 : 2^a 3^b 5^c \in \mathcal{T}\} \;\leq\; \#(\{X : \mathrm{ord}(\theta_X) = 2^a\}/\,\mathrm{Gal}).$$

Here $\mathcal{T}$ is the set of actual twist orders. Indeed each profile with both odd exponents positive already specifies every odd prime, so no other profile can contribute to it. This holds for each a, with arbitrary dyadic central charge, and with the same proof for any prescribed set of odd primes in a Galois-stable union of complementary twist fibers.

Primitive odd components at a fixed dyadic phase. The filtration can equally be applied at just three and five, fixing the dyadic twist. In the next formulas the total masses sum over both possible signs at three and all four possible primitive phases at five, as appropriate. Fix an individual nontrivial dyadic phase δ at $D = 2^6 3^6 5^4$. Write M_{ij} for the mass with $\theta_{X,2} = \delta$ and odd primary orders $3^i, 5^j$, $i, j \in \{0, 1\}$. Then

$$M_{11} \leq 8 M_{01}, \qquad M_{11} \leq 24 M_{10}, \qquad M_{10} \leq 8 M_{00}, \qquad M_{01} \leq 24 M_{00},$$

and

$$192M_{00} - 24M_{10} - 8M_{01} + M_{11} \geq 0.$$

Every one of the eight individual phase pairs (ζ_3^u, ζ_5^v), $u = 1, 2$ and $1 \leq v \leq 4$, has mass $M_{11}/8$. This follows by applying the individual balance successively, fixing the other primary twist each time.

If this mass is positive, the degree transfers and activity height bounds give the triple degrees $2^i 3^j 5$, $0 \leq i \leq 2$, $1 \leq j \leq 2$. The full quartic tame five-inertia result for jointly three- and five-active labels shows that the unique involution of tame five-inertia moves every such label. It fixes each of these eight phases, by the square action on the fifth roots, and preserves the degree. Thus each degree occurs with even multiplicity at each phase. Since every such degree is divisible by 15,

$$3600 \mid M_{11}.$$

There are at least sixteen labels across the eight phases, and at least four distinct full Galois orbits: the square classes of the two odd components are unchanged under the square Galois action. All these conclusions concern exact odd primary orders 3 and 5.

Opposite dyadic phases for arbitrary central charge. Write $v_E = \sum_{X \in E} d_X v_X$ for the degree vector restricted to a fiber. Let $p = 2$, fix the odd twist components and optionally one full two-block, and write $m(\delta)$ for the mass of an individual dyadic phase in the resulting set E. For the multiplicative central charge ξ one has

$$m(\delta) = m(-\delta) \qquad (\delta \neq \pm\xi).$$

In fact the integral lift has $\gamma^3 = \xi$, $\gamma^8 = 1$, and the actual dyadic twist operator is $\gamma U(1)$. Conjugating the upper unipotents on the vacuum shows that $L(y)v_E = \gamma^y v_E$ for the standard lower unipotents. The dyadic diagonal tori $H(a)$ fix v_E. For odd u the identity

$$U(u) = L(u^{-1})H(-u)WL(u^{-1}), \qquad W = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

therefore gives

$$\sum_{X \in E} d_X^2 \theta_{X,2}^u = \gamma^{u+2u^{-1}} v_E^* W v_E = \xi^u v_E^* W v_E.$$

The exponent calculation uses $u^{-1} \equiv u \pmod 8$ (enlarge the local modulus to include 8 if necessary). The coefficient $v_E^* W v_E$ is an integer by the local Weyl vacuum column formula. Thus $\sum_{X \in E} d_X^2 \theta_{X,2}/\xi$ is rational. In the power basis of a dyadic cyclotomic field, opposite roots cancel and one representative of each opposite pair gives a basis. Rationality forces equality of the two coefficients except in the pair $\{1, -1\}$.

If ξ is nonreal, every nonempty odd-twist fiber in a full two-block has the same positive mass at its dyadic phases 1 and -1, since it contains a dyadically inactive label by Weyl routing. Consequently any joint ternary and quintic phases, including arbitrary higher orders, then occur with dyadic phase exactly -1 in that block. At any central charge, when a nonexceptional δ admits a triple with odd primary orders 3 and 5, the opposite phase $-\delta$ contains the same mass at each of the eight individual odd-phase pairs.

Wild inertia at an inactive prime. Work in ordinary integral modular data of square dimension. At a prime p let $U(t)$, $L(t)$, $H(a)$ be the upper, lower and diagonal subgroups of the local modular representation at level p^e, and W_p its Weyl element. Torus actions are the unsigned Galois permutations. At two a common normalization scalar shifts the phase labels; we formulate the local statement for the actual representation and use its categorical consequences only at odd p.

Proposition 24.20. *Suppose v is fixed by $U(t)$ and has torus character ψ. Let p^b be the conductor of ψ^2, $b \geq 1$. The squared masses of $W_p v$ on the distinct upper-unipotent phases are constant on each coset of μ_{p^b} in μ_{p^e}. In particular the squared mass on any single phase is at most $p^{-b}\|v\|^2$. The assertion also applies to the global normalized S-image of a vector with these local properties, since the complementary Weyl is a commuting unitary.*

Proof. If $j = v_p(t) < b$, choose a unit a congruent to one modulo p^j with $\psi(a)^2 \neq 1$ (at $j = 0$ there is no restriction). Take y satisfying $ty = a^{-1} - a$. For $B = \left(\begin{smallmatrix} a & y \\ 0 & a^{-1} \end{smallmatrix}\right)$ the matrix $L(-t)BL(t)$ is upper triangular with first diagonal entry a^{-1}. Its character on v is the inverse of that of B. Unitarity gives $\langle v, L(t)v \rangle = 0$. The Fourier coefficients of the phase masses of $W_p v$ are these lower-unipotent coefficients, up to the sign of t. They vanish off $p^b\mathbb{Z}/p^e$, proving the asserted invariance by finite Fourier inversion. □

For odd p, let an inactive label have local inertia representation of conductor p^b with $b \geq 2$ on its orbit (equivalently wild inertia has order p^{b-1} there). The permutation representation of its cyclic local orbit admits a character ψ of that conductor. Squaring leaves its wild conductor unchanged. A character vector supported on this orbit supplies v in Proposition 24.20. Every occupied phase coset contains at least $p^b - 1$ nonidentity phases and at least $(p-1)p^{b-1}$ of order $\geq p^b$. If the local involution $H(-1)$ acts nontrivially on the orbit, take ψ faithful, so $\psi(-1) = -1$. The Weyl image is odd under the same involution, which preserves each phase. Each occupied phase now requires two different labels. The same doubling holds for every nonselfdual inactive label, irrespective of its local tame group: choose a character of its full Galois orbit with p-primary part faithful on wild inertia, and independently with 2-primary part nontrivial on complex conjugation. Global duality commutes with the Weyl and pairs the resulting coordinates at every fixed twist.

24.11 Ramification at half a self-dual degree

Lemma 24.21 (Weighted moments in a residue block)**.** *Let R be the local integer ring at a prime above p in a splitting field, e_B a locally integral block idempotent, and A_j elements of the based order over R. Suppose $\chi_Y(A_j) = p^{a_j}\alpha_{j,Y}$ with $a_j \in \mathbb{Z}_{\geq 0}$ and $\alpha_{j,Y} \in R$ for every $Y \in B$. For a monomial $x^{\boldsymbol{k}}$ of weight $w = \sum_j a_j k_j < v_p(\Delta)$ one has*

$$\sum_{Y \in B} d_Y^2 \prod_j \alpha_{j,Y}^{k_j} \in p^{v_p(\Delta)-w} R.$$

The sum for a polynomial over R all of whose monomials have weight below $v_p(\Delta)$ therefore vanishes modulo pR. If the unit lies in B, the nonunit contributions cannot sum to zero modulo pR when the vacuum evaluation is nonzero modulo pR.

Proof. The coefficient trace $\tau(e_B \prod_j A_j^{k_j})$ belongs to R. Its character expansion by (2.11) is p^w/Δ times the indicated sum. The polynomial assertion follows by linearity, including the constant monomial. □

Theorem 24.22. *Let X be self-dual of degree $2p$ in an ordinary or positive reduced minimal table, where $p \geq 5$ is prime, and put $a = v_p(\Delta)$. If K_X is unramified at p, then $a \leq 5$. When $a \geq 4$, some nonunit Y in the full principal p-block has $\chi_Y(X)/p$ a root of $T^2 - T - 1$ or $T^2 + T - 1$. In particular $\sqrt{5} \in K_X \cap K_Y$. Consequently every self-dual simple of degree ten in a minimal table with $v_5(\Delta) \geq 4$ ramifies at five.*

Proof. Set $a_Y = \chi_Y(X)/p$. Principality gives residues zero above p at every conjugate, and K_X is unramified there, so a_Y is an algebraic integer. It is totally real by self-duality and has strict house below two if Y is a nonunit. The roots of $T^2 - a_Y T + 1$ are algebraic integers all of whose conjugates have absolute value one. By Kronecker $a_Y = \xi + \xi^{-1}$ for a root of unity ξ. The

full-weight word $p\xi + p\xi^{-1}$ has two distinct terms. Thus $\operatorname{ord}(\xi^2) \in \{2,3,4,5,6\}$ by Theorem 24.1. The resulting values, respectively, are 0, ± 1, $\pm\sqrt{2}$, the roots of $T^2 \mp T - 1$, and $\pm\sqrt{3}$.

Let $m(z) = \sum_{Y \in P_p,\, a_Y = z} d_Y^2$. The principal block is Galois-stable, hence conjugate values have equal masses; $m(2) = 1$. Use Lemma 24.21 with $B = P_p$, $A_1 = [X]$ and weight one. For $0 \le j < a$ it gives

$$\sum_z m(z) z^j \equiv 0 \pmod{p}.$$

If neither golden quadratic occurs, $z^3 - z$ vanishes at $0, \pm 1$ and sums to zero on each of the conjugate pairs $\pm\sqrt{2}, \pm\sqrt{3}$. The vacuum contributes six. Thus $a \ge 4$ would imply $p \mid 6$. This proves the quadratic-field conclusion and rules out the case $p = 5$.

For $p \ge 7$ use $P(T) = 3T^5 - 10T^3 + 7T$. It is odd, vanishes at $0, \pm 1$, and sums to zero on the two roots of $T^2 - T - 1$ since the traces of T, T^3, T^5 there are $1, 4, 11$. The other quadratic has their negatives as roots. Therefore $\sum_z m(z)P(z) = P(2) = 30$. If $a \ge 6$ all three moments give $p \mid 30$, a contradiction. □

Theorem 24.23. *Let $d_X = 2p$ with $p \ge 5$ prime. Suppose K_X is unramified at p and at five, and $v_p(\Delta) \ge 4$. Then $v_p(\Delta) = 4$, and there is a p-free nonunit $Y \in P_p$ with*

$$\chi_Y(X) = \pm p(1 - \omega), \qquad \omega \text{ a primitive cube root.}$$

In particular $\mathbb{Q}(\sqrt{-3}) \subseteq K_X \cap K_Y$. In ordinary data all three pairs from $\{2, 3, p\}$ occur in twist orders. For $p = 5$, unramifiedness at five alone gives all these conclusions.

Proof. For $d_X = 2p$, Theorem 4.23 gives $\operatorname{supp}(\Delta) \subseteq \{2, 3, 5, p\}$ since $b(2) = 5$. For $p = 5$ the support is exactly $\{2, 3, 5\}$ by Corollary 4.24, and Lemma 4.14 gives $v_5(\Delta) \ge 4$ in either table.

The hypotheses and Corollary 2.8 give $K_X \subset L = \mathbb{Q}(\zeta_{2^u 3^v})$, where $u, v \ge 1$ may be enlarged. For $Y \in P_p$, write $\alpha_Y = \chi_Y(X)/p$; its integrality follows from its residues as in Theorem 24.22. It has a signed word of length at most two in L. Indeed the balancing expression has rational weight two; normalized relative trace into L does not increase its signed weight (on omitted prime factors use the rational normalized trace). For completeness, choose one of each pair of opposite dyadic roots for the dyadic integral basis. In each dyadic coordinate the relations on ternary powers are precisely the constant coefficient shifts on μ_3-cosets. An integral representation exists since α_Y is integral, and subtracting a median on each coset minimizes total absolute coefficient over rational shifts while retaining integrality.

Let $H = \operatorname{Gal}(L/\mathbb{Q}(\zeta_3))$ and put $\beta_Y = |H|^{-1} \sum_{s \in H} s\alpha_Y$. On signed roots this trace fixes the sixth roots and kills all others. Its denominator $|H|$ is prime to p. For a nonunit Y, $|\beta_Y| < 2$ by strictness, so β_Y is zero or an Eisenstein integer of squared modulus one or three. The latter possibilities follow directly from adding at most two sixth roots. The vacuum value is $\beta_0 = 2$.

Use the p-local orbit average $B = |H|^{-1} \sum_{s \in H} [X^s]$, whose eigenvalues on P_p are $p\beta_Y$. Apply Lemma 24.21 to B, B^* with weights $(1, 1)$. For every integral polynomial P in $z, \bar{z}$ of total degree less than $v_p(\Delta)$ it gives

$$\sum_{Y \in P_p} d_Y^2 P(\beta_Y, \overline{\beta_Y}) \equiv 0 \pmod{p}.$$

Take $P = (z + \bar{z})(z\bar{z} - 1)$. The vacuum contributes 12 and all nonunits of squared modulus zero or one contribute zero. Thus some p-free nonunit has squared modulus three and nonzero real part. Its value is $\pm(1 - \omega)$. Both terms of its signed word survived the trace, so $\alpha_Y = \beta_Y$. To bound the p-valuation, put $x = z + \bar{z}$, $t = z\bar{z}$ and take the degree-four polynomial

$$x^4 - 2x^2 t - 2t^2 - 3x^2 + 6t.$$

The nonvacuum pairs (x, t) are $(0, 0)$, $(\pm 1, 1)$, $(\pm 2, 1)$, $(0, 3)$ and $(\pm 3, 3)$; the polynomial vanishes on each. The vacuum $(4, 4)$ gives 72, so $v_p(\Delta) \le 4$.

In an ordinary table, XY has exactly two twist values of relative order six, each carrying dimension pd_Y. To see this, the two phases in the displayed word for $\chi_Y(X)$ are adjacent vertices of the regular hexagon. A rational functional on $\mathbb{Q}(\zeta_6)$ is one at these two vertices and strictly less at the others (assign the value one on the two adjacent vertices, which form a rational basis). The even-level trace argument of Theorem 24.1 extends this functional to every common conductor. Equality for the full-weight balancing expression forces the stated support; the two adjacent roots are linearly independent, so their masses agree with those in the word.

At least one fiber is two-active and at least one is three-active. If either prime is separated from p, that fiber must have full p-height (Lemma 3.4), impossible since its mass has p-height one. If two and three are separated, each constituent of a two-active fiber has full ternary height. Since $p \neq 3$, its mass pd_Y then forces full ternary height in d_Y, so K_Y is unramified at three by Corollary 2.8. This contradicts $\omega \in K_Y$. □

Corollary 24.24. *In the degree-ten case of Theorem 24.23, the principal five-block contains at least seven distinct five-free nonunits.*

Proof. In the notation of the proof, write $\beta = a + b\omega$ with $a, b \in \mathbb{Z}$, and reduce (a, b) modulo five. The nonvacuum points satisfy $a^2 - ab + b^2 < 4$. They are distinct after reduction. The six lines through the vacuum $(2, 0)$ with slopes $\infty, 0, 1, 2, 3, 4$ contain respectively $1, 3, 1, 2, 3, 3$ of them, as one sees from the three norm values $0, 1, 3$. Suppose at most six points have nonzero squared mass modulo five. They can be covered by at most three lines avoiding the vacuum. Indeed place at most two in each of three groups, using distinct slopes in a group: list the points in consecutive blocks of equal slope and deal cyclically into the groups, since each block has length at most three. A pair determines a line missing the vacuum, and a singleton permits a choice. The product of the corresponding affine forms is nonzero at the vacuum and zero on all contributing nonunits. Each coordinate is a linear combination of $\beta, \bar{\beta}$ over the five-local integers of $\mathbb{Q}(\omega)$, since $\omega - \bar{\omega}$ is a five-unit. This contradicts Lemma 24.21 with the two averaged elements of the preceding proof and degree at most three. A nonzero squared mass modulo five requires some five-free label. □

25 Two unrestricted prime powers and a sextic prime

25.1 Forcing a mixed column at the largest quartic prime

In this subsection $p < q < r$ are odd primes. Proposition 4.8 rules out a positive reduced table when these are the repeated primes.

Theorem 25.1. *Let $h \geq 2$, $U = p^{h-1}$, $G = p^h$, and $m = \lfloor G/q \rfloor$. Suppose*

$$(p-1)q > U, \qquad W(s, j+1) \text{ holds for every } 1 \leq j \leq m \text{ and every prime } s > jq.$$

For every prime $r > q$ and every $B \geq 0$, each braided fusion category of dimension dividing $p^{2h}q^B r^4 d$ is weakly group-theoretical.

Proof. In a remaining divisor-minimal table the r-valuation is four, while square divisibility bounds every p-height by h. Lemma 4.14 gives an r-free nonunit in the principal q-block and a q-free nonunit in the principal r-block. For every such pair the degrees are $p^i q^j$ and $p^t r$, and the weight of the lcm-normalized entry is $g = p^{\min(i,t)} \geq q$. Thus $m = 0$ is impossible.

If $r \leq U$, then $j = \lfloor r/q \rfloor$ satisfies $1 \leq j \leq m$, $r > jq$ and $j + 1 < p$. In Proposition 8.7, the first alternative gives a word excluded by $W(r, j+1)$. The proof of Theorem 8.2 rules out the second alternative using $j + 1 < p$; its scalar test is used only for the first alternative.

Suppose $r > U$. If any of the above pairs has $g \leq U$, then $r \geq g + 2$ since both are odd. Theorem 4.13, applied to the two principal residues, expresses the normalized entry as a positive r-word of weight g on at most $1 + \lfloor g/q \rfloor$ positions. Multiplying by $\operatorname{lcm}(d_X, d_Y)/d_Y$ gives a

character of full weight d_X. Since $r > g \geq q\lfloor g/q \rfloor$, the word hypothesis excludes this pair. Consequently every opposite pair has $g = G$. All its witnesses have p-height exactly h, forcing the dimension's p-valuation to be exactly $2h$ and their codegrees to be p-units.

Every q-active column has a nonzero entry at an r-free nonunit of the principal q-block by Proposition 4.2, applied modulo r. This row has p-unit codegree, hence vanishes on every p-active column by Theorem 3.2. Thus p and q never occur together in a twist. The same argument on the principal r-block separates p from r. By Lemma 7.2, the quartic r cannot be isolated, so some column is both q- and r-active.

At this column apply Proposition 4.2 on the principal q-block modulo p. The resulting p-free nonunit has degree $q^j r^v$ for $v \in \{1, 2\}$ by Lemma 4.14 and square divisibility. At $v = 2$ its row would vanish on this column by its r-unit codegree. Thus $v = 1$, exactly the hypothesis that gives the first word in Proposition 8.7. Its support has at most $1 + \lfloor r/q \rfloor$ positions, with no restriction on p in this construction. If $r < G$, this contradicts $W(r, 1 + \lfloor r/q \rfloor)$. If $r > G$, apply Theorem 4.13 to an opposite pair of weight G: the resulting word contradicts $W(r, 1 + m)$. Equality $r = G$ is impossible. □

Corollary 25.2. *The word hypothesis in Theorem 25.1 holds whenever $q \geq (m + 1)^2$. In particular the theorem applies whenever $h \geq 3$ and $q > p^{h-1}$. For $h \geq 6$ it applies throughout $(p - 1)q > p^{h-1}$.*

Proof. With $E = j(j+1)/2$, put $g(v) = \min\{t \geq 1 : t(t+1) \geq v\}$ and $C(E) = 1+\max_{1\leq v\leq E} v(v-1)/g(v)$ as in Lemma 4.22. At $j = 1$ the two-position theorem applies. For $j \geq 2$, the scalar estimates in the proof of Theorem 4.23 bound both $3(1 + 2E)/2$ and $2C(E) + 1$ by $j(j + 1)^2$. This is strictly less than s since $s > jq$ and $q \geq (m + 1)^2$. If $h \geq 3$ and $q > p^{h-1}$, then $m + 1 \leq p$ and $q > p^{h-1} \geq p^2$. Finally, if $h \geq 6$ and $(p - 1)q > p^{h-1}$, then $m + 1 \leq p(p - 1)$ and $q > p^{h-2} \geq p^4 > (m + 1)^2$. □

The all-r criterion in Corollary 8.12 required $m + 1 < p$. For instance at $p = 3$, $q = 11$, $h = 3$ one has $m + 1 = 3$. Theorem 25.1 covers $3^6 11^B r^4 d$ for every B and prime $r > 11$. This includes $r = 23$, where the ratio test of Theorem 8.2 also has $1 + \lfloor r/q \rfloor = 3$. Direct use of the strict scalar inequalities gives examples below $q = p^{h-1}$: the same theorem covers $5^6 23^B r^4 d$ and $7^6 47^B r^4 d$ for all B and all r above the middle prime. Indeed their respective values of m are five and seven, and for $j = 1, \ldots, 7$ the exact values of $2C(j(j + 1)/2) + 1$ are $3, 9, 33, 63, 108, 193, 1527/5$. Each required inequality follows from $s > jq$. These examples illustrate the criterion; no prime value is used in its proof.

Proposition 25.3. *Let $h, k \geq 2$, $U = p^{h-1}$, $V = q^{k-1}$, and $m = \max\{\lfloor U/q \rfloor, \lfloor V/p \rfloor\}$. For any prime $r > \max(U, V)$ for which $W(r, 1 + m)$ holds, every braided fusion category of dimension dividing $p^{2h} q^{2k} r^4 d$ is weakly group-theoretical.*

Proof. Repeat the opposite-pair argument of Theorem 25.1 first with p as the common degree prime. Every common weight $g \leq U$ would force a word on at most $1 + \lfloor U/q \rfloor$ positions, so all complementary degrees have top p-height. The codegree and prime-avoidance argument isolates p. Now use the principal blocks at p and r instead, so the common degree prime is q. Lemma 4.14 gives for every such pair a common weight $g = q^a \geq p$. At $g \leq V$ we have $r \geq g + 2$ and Theorem 4.13 gives a word on at most $1 + \lfloor V/p \rfloor$ positions. Thus all those witnesses have upper height k, forcing valuation $2k$ and q-unit codegrees. Applied on the principal blocks at p, r, prime avoidance now isolates q. No twist can then contain r and either other prime, contrary to Lemma 7.2 at the quartic r. □

25.2 An isolated prime with unramified principal witnesses

This obstruction depends only on the exponent at the least prime. It does not require a quartic at either companion.

Lemma 25.4 (Trace and mass at an isolated prime)**.** *Let p be isolated in an odd ordinary minimal table with $v_p(D) = 2h$, $h \geq 3$. Write $D = p^{2h}n^2$. Each p-active Z has degree $p^j n$ with $0 \leq j < h$, and K_Z has p-power conductor.*

1. *If $j > 0$, no $X \in P_p$ of height $h-1$ can have nonzero absolute trace T_X in the Z-row of (2.4).*
2. *If all active labels have degree n, their number N satisfies $N \geq 2p^{h-1}(p-1)$. In that case P_p has no nonunit of height $h-1$.*

In particular some active column has no hit on a principal row of height $h-1$ unramified at p.

Proof. The descriptions of d_Z and K_Z follow from isolation, Lemma 3.4 and Corollary 2.8. In (1), $x = \chi_Z(X)$ satisfies $x/p^{h-1-j} \in \mathcal{O}_{K_Z}$ by lcm divisibility. By symmetry this quotient equals $\chi_X(Z)d_X/(p^{h-1-j}d_Z)$, whose second factor is a p-unit. Principality gives zero residue above p. Since K_Z is totally ramified at p, the rational trace of the quotient from $F_X = \mathbb{Q}(x) \subseteq K_X \cap K_Z$ is divisible by p. A nonzero T_X would thus have square at least $p^{2(h-j)} = f_Z$, leaving no room for the unit.

For (2), choose a CRT exponent m equal to one on the p-level and zero on the complementary level, including the dyadic anomaly. Lemma 6.2 and Proposition 6.3 give

$$p^{-h} \sum_{Z \ p\text{-active}} (1 - \theta_{Z,p}) = p^h(1 - (W_p)_{00}) \in \mathbb{Z}.$$

The integer is positive by taking real parts and even since distinct dual labels have the same twist and p is odd. Each summand on the left before division has normalized trace at most $p/(p-1)$, giving the lower bound. Any $X \in P_p$ hits every one of these labels by principality. If $v_p(d_X) = h-1$, then $S_{XZ}/(p^{h-1}n)$ is a nonzero algebraic integer by lcm divisibility. Its Galois mean squared modulus is at least one, by the norm and the arithmetic–geometric mean inequality. Averaging the row norm and counting the unit gives $N < p^2$. Finally, for an unramified X the intersection $K_X \cap K_Z$ is rational, so every nonzero hit has nonzero trace; (1) and (2) give the last assertion. □

Theorem 25.5 (Trace budget for two companion primes)**.** *Suppose p is isolated in an odd ordinary minimal table of dimension $p^{2h}n^2$, $h \geq 3$, with n supported on exactly two primes $q < r$, both different from p. Put $\ell = \min(p, q)$. For each $s = q, r$ and $1 \leq t \leq h-2$, call t eligible at s if*

$$s > p^t, \qquad (s - p^t)(\ell - 1) > p^t.$$

At an eligible height put

$$b_s(t) = \frac{s-1}{\max_{a \mid s-1,\, a \leq p^t} a/\gcd(a,2)}, \qquad c_s(t) = (p-1)b_s(t).$$

At every other height set $b_s(t) = c_s(t) = 1$. Let

$$A_s = \min_t p^{2(t-1)}c_s(t),$$

omitting eligible t for which $W(s, p^t)$ holds, and interpreting an empty minimum as infinity. Then $A_q + A_r < p^{2h-4}$.

Proof. Suppose first that Z is p-active of degree $p^j n$, $j > 0$. For each companion s, the unit term of (2.5) is one, so reduction modulo s supplies an s-free nonunit X with nonzero trace T_X on the Z-row. It belongs to P_p by Theorem 2.11. Write $d_X = p^t u^i$, where u is the other companion and $t, i > 0$ by Lemma 4.14. Defect-zero vanishing excludes $t = h$, and Lemma 25.4 excludes $t = h-1$. If $j \leq t$, the proof of that lemma gives $p^{t+1-j} \mid T_X$: divide $\chi_Z(X)$ by p^{t-j}, then use

its zero residue and total ramification of its own field at p. If $j > t$, integrality alone suffices. In either case $d_Z^2 T_X^2 \geq p^{2(t+1)} n^2$.

At an eligible height cross X with a u-free nonunit of P_s. The gcd is some $w = p^k$ with $1 \leq k \leq t$, and the normalized entry has residue w over s. Lemma 8.5 gives a positive integral s-word of weight w, since $(s-w)(\ell-1) > w$. It is unique: a word shorter than s misses a position in every μ_s-coset, so its coefficients are the unique nonnegative coset representatives with minimum zero. Strictness gives a nonidentity term. The tame stabilizer acts freely on these terms and preserves multiplicities, hence has order $a \mid s-1$ with $a \leq w$. If a is even, its involution acts as conjugation on the pure word. The entry field is then real whereas K_X is nonreal by odd duality. Thus $[K_X : \mathbb{Q}] \geq b_s(t)$. Since $s - 1 > p^t$ (both s, p^t are odd), the entry ramifies at s. Separation gives $K_X \cap K_Z = \mathbb{Q}$, so the full orbit of X agrees on the p-active columns. The square action at p produces at least $(p-1)/2$ phases in the orbit of Z, each with a distinct dual pair, hence $[K_Z : \mathbb{Q}] \geq p - 1$. Finally $W(s, p^t)$ would forbid the word just obtained upon rescaling to full weight.

The X-orbit therefore costs at least $p^{2(t+1)} n^2 c_s(t)$ in (2.4) multiplied by d_Z^2: at eligible heights use the two field degrees over their rational intersection, and otherwise just the trace square. The q-free and r-free orbits are disjoint by degree. Counting also the unit yields $p^4 n^2 (A_q + A_r) < p^{2h} n^2$, as claimed.

If all N active labels have degree n, any s-free nonunit $X \in P_p$ hits all of them and has $1 \leq t \leq h-2$ by the same height bounds. Such an X exists by Proposition 4.2. At an eligible height its orbit agrees on these columns, and each entry is a nonzero integer multiple of $p^t n$. The row averaged over the orbit has norm squared $D/[K_X : \mathbb{Q}]$ by orthogonality. At an ineligible height take the Galois mean of the squared row norm and use the lcm divisibility as in Lemma 25.4(2). In both cases $N p^{2t} b_s(t) < p^{2h}$, counting also the unit. In particular a permitted height exists on each axis. As $p^{2t} b_s(t) \geq p^2 A_s/(p-1)$ and $N \geq 2p^{h-1}(p-1)$ by that lemma, $A_s < p^{h-1}/2$ at each companion. Their sum is less than $p^{h-1} \leq p^{2h-4}$. □

Theorem 25.6. *Let $p < q < r$ be odd primes, $h \geq 3$, $U = p^{h-1}$ and $H_t = \max\{a : a \mid r-1,\ a \leq p^t\}$. Suppose $r > U$, $W(r, 1 + \lfloor U/q \rfloor)$, and for every $1 \leq t \leq h-2$ either*

$$W(r, p^t) \qquad \text{or} \qquad (p-1)(r-1) \geq p^{2(h-t-1)} H_t.$$

Then every braided fusion category of dimension dividing $p^{2h} q^B r^C d$ is weakly group-theoretical for all $B, C \geq 0$. In particular $r > U$ and $W(r, p^{h-2})$ suffice.

Proof. In a remaining divisor-minimal table take complementary nonunits of the principal q- and r-blocks, of degrees $p^i q^b, p^k r^c$. If their common weight g is at most U, then $r \geq g + 2$ and the two residues in Theorem 4.13 give a full r-word on at most $1 + \lfloor U/q \rfloor$ positions. Thus every such pair has weight p^h and their codegrees are p-units. Prime avoidance and defect-zero vanishing isolate p and force $D = p^{2h} n^2$, as in Theorem 25.1. This argument only uses $q < r$ and $r > U$ and works in either order of p, q.

In Theorem 25.5, all heights at r are eligible since $r > U \geq p^{t+1}$. Every retained height gives $p^{2(t-1)} c_r(t) \geq p^{2(t-1)}(p-1)(r-1)/H_t \geq p^{2h-4}$, a contradiction. The specialization uses $1 + \lfloor U/q \rfloor \leq p^{h-2}$. □

Corollary 25.7. *For $h \geq 3$, the conclusion of Theorem 25.6 holds whenever*

$$r > \max\{p^{h-1},\ 33,\ (p^{h-2} - 1)^3\}.$$

In particular, $3^6 q^B r^C d$ is covered for all B, C and all odd primes $3 < q < r$ with $r \geq 11$. At $C = 4$ all primes $3 < q < r$ are covered.

Proof. The scalar test gives $W(s, j+1)$ for $s > \max(33, j^3)$ and $j \geq 1$. For $j = 1, 2, 3$ substitute directly. For $j \geq 4$, write $E = j(j+1)/2$ and use the $g(v)$ of Corollary 25.2. If $g(v) \leq j - 2$ then

$v \le g(v)(g(v)+1)$; if $g(v) \ge j-1$, then $v-1 \le E-1 \le (j-1)^2$. Thus $(v-1)/g(v) \le j-1$ and $2C(E)+1 \le 3+j(j+1)(j-1) \le j^3$. Also $3(1+2E)/2 < j^3$. Take $j = p^{h-2}-1$. At $p=h=3$ the exact test gives $W(r,3)$ for every prime $r \ge 11$, and $r > 9$ throughout that range.

The quartic assertion follows also for the smaller primes from Corollary 25.10 below. □

25.3 The trace budget at height three

For the two companions $s = q, r$ put

$$e_s = \frac{s-1}{\max_{a \mid s-1,\, a \le p} a}, \qquad e_s^+ = \frac{s-1}{\max_{a \mid s-1,\, a \le p}(a/\gcd(a,2))}.$$

Proposition 25.8 (Height-three substitutions)**.** *Suppose p is isolated in an odd ordinary minimal table of dimension p^6n^2 with n supported on precisely q, r, $q < r$. There is only one height $t = 1$ in Theorem 25.5. The following are lower bounds for its two terms.*

condition at $s \in \{q,r\}$	*lower bound for A_s*
$s > p, \quad (s-p)(\min(p,q)-1) > p$	$(p-1)e_s^+$
otherwise	1

If their sum is at least p^2, isolation is impossible. When $p < q$ it suffices that $(p-1)(e_q+e_r) \ge p^2$. More generally, for $p < q < r$, $r > p^2$ and $W(r, 1+\lfloor p^2/q \rfloor)$, every divisor of $p^6q^Br^Cd$ is a WGT dimension if $(p-1)e_r \ge p^2$.

Proof. The table substitutes $h = 3$, $t = 1$ into the trace budget (an omitted height has infinite cost). When $p < q$, both companions are eligible since $s - p \ge 2$ and $p \ge 3$, and $e_s^+ \ge e_s$. For the final implication use Theorem 25.6 at $h = 3$, where the single numerical alternative is exactly $(p-1)e_r \ge p^2$. □

Theorem 25.9. *Let p, q, r be distinct odd primes with $q < r$, and d square-free and coprime to pqr. Suppose $W(r, 1+\lfloor \min(r, p^2)/q \rfloor)$ holds. If $r > p^2$, every divisor of $p^6q^Br^Cd$ is a WGT dimension for all $B, C \ge 0$, in either order of p, q. If $p < q < r < p^2$, the conclusion holds with $C = 4$ and B arbitrary.*

Proof. For $p < q < r < p^2$ the position bound is less than p since $(p-1)q > p^2$. The first alternative of Proposition 8.7 gives the forbidden word, and the proof of Theorem 8.2 excludes the second at this bound. For $r > p^2$, a divisor-minimal obstruction has ordinary data by Proposition 4.8. The complementary-pair argument of Theorem 25.6 at $h = 3$ isolates p at valuation six, in either order of p, q. In Proposition 25.8 the companion $r > 2p$ is eligible and $e_r^+ \ge e_r \ge p+1$ since $r - 1 > p^2$. The other companion costs at least one, giving the contradiction $A_q + A_r \ge 1 + (p-1)(p+1) = p^2$. □

In particular all r are covered for the sextic–quartic conclusion if $p \ge 5$ and $q^3 \ge p^4$. Indeed, with $j = \lfloor \min(r, p^2)/q \rfloor$ we have $j^3 < r$: for $r \le p^2$ use $r^2 \le p^4 \le q^3$ and $j < r/q$, and at $r > p^2$ use $(p^2/q)^3 \le p^2 < r$. For $j \ge 4$ or $r > 33$, use the scalar obstruction in Corollary 25.7. At $j = 0, 1, 2$ strictness and the small scalar bounds suffice (at $j = 2$, $r > 2q$). At $j = 3$ and $r \le 33$ we have $r > 3q$, so the only possible q is 7 and thus $p = 5$, incompatible with $q^3 \ge p^4$.

For $p = 5, q = 7$ and $r = 29, 31$, the four-position calculation in Corollary 8.9 gives $W(r, 4)$, so both companion exponents are unrestricted. For $p = 5, q = 7, r = 23$ the word bound $W(23, 4)$ remains undecided. Nevertheless $e_7 = 2$ and $e_{23} = 11$, so Proposition 25.8 excludes isolation at five for arbitrary companion exponents whenever the five-valuation is six.

Corollary 25.10. *Let $p < q < r$ be odd primes. For each line of the following table every divisor of $p^6 q^B r^C d$ is covered, for all $B, C \geq 0$ and all coprime square-free d:*

p	q	r
3	$q > 3$	$r \geq 11$
5	$q > 5$	$r \geq 29$
7	$q > 7$	$r \geq 53$.

For $C = 4$ the conclusion holds at $p = 3, 7$ for every q, r. At $p = 5$ the sole assignment left by these tests is $(q, r) = (7, 23)$.

Proof. In the three lines $r > p^2$, and $1 + \lfloor p^2/q \rfloor$ is at most $2, 4, 5$, respectively. At two positions use Theorem 24.1. For four and five positions use the exposed-support calculations of Corollary 8.9. At five positions these give the cases $53, 59, 61$; the scalar bound applies from 67 onward. For the quartic the only smaller triple at $p = 3$ is $(3, 5, 7)$, where $r < 2q$. At $p = 5$, $r < 29$ gives $r < 3q$ except for $(q, r) = (7, 23)$. Both ratio tests follow from Theorem 8.2. When $p = 7$ and $r < 53$ we have $r < 5q$, so Corollary 8.9 applies. □

For $r > 33$ the scalar bound alone has a useful consequence for the location of any surviving odd triple:

$$q^3 < p^4, \qquad q^{3/2} < r < (p^2/q)^3.$$

Indeed with $j = \lfloor \min(r, p^2)/q \rfloor$ a failure of the test forces $r \leq j^3$. Since $j < r/q$ and $j < p^2/q$, the two strict inequalities follow. For a fixed least prime p the possible companion primes in this interval are finite even when B is unbounded.

Applying the multiplier criterion. Let $p < q < r$ be odd primes, and d coprime and square-free. Take $m = \lfloor \min(r, p^2)/q \rfloor$. Write $E = m(m+1)/2$ and $H = (r-1)/2$. Under the exposure condition $1 + 2E < 2r/3$ in Section 8.10, if its scalar inequality and Proposition 8.40 exclude every $h \leq E$, then every divisor of $p^6 q^B r^4 d$ is a WGT dimension for all $B \geq 0$, by Theorem 25.9. The exponent at r is also arbitrary when $r > p^2$. Independently, if $m = \lfloor r/q \rfloor$ works and $m + 1 < p$, the same holds for $p^A q^B r^4 d$ with both A, B arbitrary (Proposition 8.7 and Theorem 8.2).

The first part of Proposition 8.40 gives $W(577, 10)$ and $W(1783, 14)$. At $(m, r) = (9, 577), (13, 1783)$ the only scalar survivors are $(h, k) = (42, 6), (90, 9)$ respectively, with equality and $h = k(k+1)$. In fact for every integer $k \geq 2$ with $r = 2k^3 + 4k^2 + 1$ prime, the same argument applies whenever $k(k+1) \leq E$ and $E(E-1) < (H-1)(k+1)$: for each lower j the largest scalar quotient $h(h-1)/j$ at $h = j(j+1)$ is increasing in j, and beyond $k(k+1)$ the new denominator is at least $k+1$. The first example applies for either middle prime $59, 61$ and every prime $10 < p < q$ at arbitrary first and middle exponents. In the second example one may take $q = 131$ or 137 and $14 < p < q$.

The second part gives $W(180547, 64)$, with $E = 2016$. The sole scalar survivor is $h = 2016$, $k = 45$ (compare the preceding endpoint $44 \cdot 45$ by the same monotonicity). Here $H = 90273$, $h - k = 1971 = 3^3 \cdot 73$, and $3^{51} \equiv -1 \pmod{30091}$ with $30091 \mid H$. The odd valuation at three excludes the mask. The scalar inequality and the order-two quotient condition both allow these three examples: H is odd in the last two and $h - k = 36$ is a square in the first.

25.4 Two unbounded exponents below the least-prime square

For odd primes $p < q < r$ put $g = p^2$. We give a criterion for every divisor of $p^6 q^B r^C d$, with both B, C unbounded, when $r < g$. Here and below d is coprime and square-free.

Proposition 25.11 (Two positive residues at arbitrary ratio)**.** *Let $q < s$ be primes, g a positive integer with $q \nmid g$, and set $m = \lfloor g/s \rfloor$, $t = g - ms$. Assume $m \geq 1$ and $q \geq 5(m+2)$, or $m = 1$,*

$q \geq 7$ and $s < g < 2s - 2$. Let the cyclotomic integer z be congruent to g at every prime above q and s, and have a positive rational expression of mass g whose phase orders have no prime divisor below five. All such expressions have the same multiplicities, which are integers.

Every occurring phase is a product of a q-power root and an s-power root. Counting repetitions, its q-parts form t copies of 1 and s copies of each member of a multiset of m q-power roots. The multiplicity at each s-position x is $g\delta_{x,1} \pmod q$. In particular the two projections have at most $m + 1$ and $1 + \lfloor g/q \rfloor$ positions, respectively.

Proof. We may enlarge the level to a common multiple of qs and any two levels being compared. For an integral signed word in q- and s-primary roots with the prescribed residues, let P, N be its positive and negative counts and suppose $P + 2N \leq g$. Erasing q-parts gives coefficients h_x at the s-positions, and a value congruent to g modulo q in the unramified s-power field. By the integral coset relations $h_x - g\delta_{x,1}$ is constant modulo q on each μ_s-coset C. Choose the representative a_C of the constant minimizing $w(a) = \max(a, -2a)$. At all positions except possibly 1 the cost is at least $w(a_C)$, giving $\sum_C w(a_C) \leq g/(s-1) < m + 2$.

Erase the s-parts instead. The resulting q-primary value is $v = t + s\alpha$ with α integral, by the residue over s and unramifiedness. Subtracting t and dividing by s gives a signed rational expression of absolute length at most $(g + t)/s < m + 2$ for α. Its integral coefficient vectors differ by constants on each μ_q-coset. Subtracting a median per coset minimizes absolute length even among rational vectors, so α has an integral word of length at most $m + 1$. Let L be the sum of its signs. Counting the h_x modulo q, and reducing v above q, gives

$$P - N \equiv g + s \sum_C a_C \equiv t + sL \pmod q.$$

Hence $\sum_C a_C = L - m$ as integers: their absolute values are at most $m + 1$ and $2m + 1$, and $q > 3m + 2$.

Mark 1 and every q-root in the chosen word for α. On a μ_q-coset containing k marks prescribe a functional equal to 1 there and $-k/(q-k)$ on the unmarked positions. Its coset sums are zero, so these prescriptions give a rational-linear functional. There are at most $m + 2 < q/2$ marks on each coset in either range. Thus the functional has absolute value at most one and yields $t + sL \leq P + N \leq g - N$. In particular $d = m - L \geq 0$. In the range $m = 1$, $s < g < 2s - 2$, the bound $\sum_C w(a_C) \leq g/(s-1) < 2$ already rules out negative a_C. In the other range $m + 2 \leq q/5$, the lower functional bound is $-1/4$, so $t + sL \leq P + N/4 \leq g - 7N/4$. If some a_C is negative, write $a_C = -b_C$ on these cosets, put $B = \sum b_C$ and $b = \max b_C$. Then $B \geq d$ and $2b \leq m + 1$. At each of at least $s - 1$ positions of such a coset there are at least b_C negative letters, or there are at least $q - b_C$ positive letters. The number of positions where the first bound fails is at most $g/(q-b)$. Consequently

$$N \geq B(s-1) - \frac{bg}{q-b} > Bs\left(1 - \frac{1}{s} - \frac{m+1}{q-b}\right) > \frac{4}{7}Bs,$$

using $g < (m+1)s$, $b \leq B$, $1/s < 1/15$ and $(m+1)/(q-b) < 2/9$. This contradicts $N \leq 4sd/7$. Thus every $a_C \geq 0$ in both ranges. Their sum is $-d \leq 0$, so all vanish, $d = 0$ and $N = 0$. The word for α has signed sum m and length at most $m + 1$; by parity it consists of m positive letters. The functional now takes value g on v, so $P = g$ and every q-part is marked. The marks are linearly independent, since they miss a position in each μ_q-coset. This gives the stated q-multiplicities, and $a_C = 0$ gives the congruence on the s-multiplicities.

Lemma 4.11 with $c = 2$ eliminates all spectator components (their primes are at least five) and puts the physical coefficient vector in the convex hull of integral signed optima. In each optimum the nonidentity s-positions have multiplicity at least q and the identity occurs since $q \nmid g$. Two optima together use at most $1 + 2\lfloor g/q \rfloor < s$ such positions in total, since $g < (m+1)s$ and $q \geq 2(m+1) + 3$ in either range. Their union is independent over the q-power field by the coset basis and coprimality of the conductors. The sums of q-roots therefore agree at each

s-position. The union of the q-positions contains at most $1 + 2m < q$ roots and is likewise independent. Thus every joint multiplicity agrees, proving uniqueness. □

At $(q, s, g) = (7, 23, 25)$ the arbitrary-ratio bound would require $q \geq 15$. The second range applies since $23 < 25 < 44$, $7 \nmid 25$ and $25/22 < 2$. This recovers the joint multiplicities of Proposition 24.4 for alphabets with prime divisors at least five.

In odd ordinary data the difference-set argument applies simultaneously to any number of primary coordinates. Suppose a full positive word, up to a scalar, uses at most k_s positions in each coordinate s of a finite set of odd primes, with constant complementary part. Write $E_s = k_s(k_s - 1)/2$, $H_s = (s - 1)/2$ and $u(h) = \min\{j \geq 1 : j(j + 1) \geq h\}$. Such a word is impossible if, in every nonconstant coordinate, the following conditions hold with $(s, k, E, H) = (s, k_s, E_s, H_s)$:

$$2(2E + 1) < s, \qquad (H - 1)u(h) > h(h - 1) \quad (1 \leq h \leq E). \tag{25.1}$$

Indeed, at the fixed row choose a full word minimizing the sum of its numbers h_s of opposite nonzero differences, subject to $h_s \leq E_s$ and to the initially constant coordinates staying constant. Its adjoint support lies in the product of the coordinate difference sets. The coset functionals in Lemma 4.20 have absolute value strictly less than one off the marks under the first inequality. Their product, with normalized trace on the complement, therefore exposes this box. In each nonconstant coordinate the nonidentity tame multipliers modulo sign have average overlap at most $h_s(h_s - 1)/(H_s - 1)$ by freeness. Choose independently one with overlap $j_s < u(h_s)$, using the second inequality. Lemma 4.20 supplies a common nonunit whose phases belong to the intersection of the two boxes. In each such coordinate its opposite differences number at most $j_s(j_s + 1) < h_s$, as in Lemma 4.22. This contradicts minimality; a word constant everywhere already contradicts strictness.

For a rectangle of at most k positions at r and two at $q \geq 7$, the q-condition is automatic ($E_q \leq 1$). Thus only (25.1) at r, k need be checked.

Theorem 25.12. *Suppose $p \geq 5$ and $(p^2 + 2)/2 < r < p^2$. Let $k = 1 + \lfloor p^2/q \rfloor$ and*

$$e_s^+ = \frac{s - 1}{\max_{a|(s-1),\ a \leq p} a/\gcd(a, 2)}, \qquad s = q, r.$$

If (25.1) holds at $s = r, k$, and $(p - 1)(e_q^+ + e_r^+) \geq p^2$, then all divisors of $p^6 q^B r^C d$ are WGT dimensions for all B, C.

Proof. In a remaining minimal table opposite r-free and q-free nonunits of P_q and P_r have common degree factor $p^i \geq q$ (Lemma 4.14). As $i \leq 3$ and $q > p$, the sole alternative to full p-height is $i = 2$. Their normalized crossing would then have the two residues and weight $g = p^2$ of Proposition 25.11. Its r-projection has at most k positions, excluded by the mixed support obstruction. Thus every such witness has full p-height three. The opposite principal hits separate p from both q and r as in Theorem 25.9, without a hypothesis on their exponents.

Here both companion primes are eligible. Proposition 25.8 excludes this isolated sextic by the stated orbit inequality. □

Here are the small-prime substitutions into the preceding theorem. Each entry in the third column is e_r^+ in the order displayed.

p	r	e_r^+	q
5	$17, 19, 23$	$8, 6, 22$	$11 \leq q < r$
7	$29, 31$	$4, 6$	$17 \leq q < r$
7	$37, 41, 43, 47$	$12, 8, 6, 46$	$13 \leq q < r$

The difference budget E in (25.1) is at most three in the first two rows and six in the last. For $(p, r) = (7, 29)$ the values e_q^+ are $8, 6, 22$ at $q = 17, 19, 23$. At $(7, 31)$ and $(7, 43)$ one needs $e_q^+ \geq 4$; an odd stabilizer of index two would require $q \leq 15$ and $q \equiv 3 \pmod 4$, and an even stabilizer of index one would require $q - 1 \leq 7$. The remaining entries give the required sum directly.

Corollary 25.13. *For odd primes $p < q < r$ with $p \geq 5$ and $2q > p^2$, all divisors of $p^6 q^B r^C d$ are WGT dimensions for all B, C.*

Proof. Here $k \leq 2$. For $r > p^2$ apply Theorem 25.9. Otherwise $2r - 2 > p^2$ since $r \geq q + 2$, and (25.1) holds. Put $h = (p+1)/2$. For either $s = q, r$ we have $s - 1 \geq (p-1)h > p(h-1)$, so $e_s^+ \geq h$. Equality would require an effective stabilizer $a/\gcd(a, 2) \geq p - 1$ for a divisor $a \leq p$ of $s - 1$. Its only possible value is p, so equality forces $s = ph + 1$. The two distinct primes cannot both have this value; hence $e_q^+ + e_r^+ \geq 2h + 1 = p + 2$, as required. □

For $g = p^2$ and $s = r < p^2$, Proposition 25.11 gives the two projection sizes in the product obstruction (25.1). The isolation and trace argument of Theorem 25.12 applies whenever $q \geq 5(\lfloor p^2/r \rfloor + 2)$, both projection tests hold and the isolated-orbit inequality is satisfied.

The small least prime. For $p = 3$ the conclusion of Corollary 25.13 follows from Corollary 25.10 except at $(3, 5, 7)$. In any minimal obstruction in this remaining unbounded-exponent family a complementary crossing of common weight nine is necessary. Moreover, *every* such crossing has, up to the choices of primitive roots $\eta \in \mu_5$, $\xi \in \mu_7$, the unique normalized spectrum

$$z = 1 + \xi + 3\eta + 4\eta\xi.$$

Indeed the common 3-weight of a 7-free nonunit of P_5 and a 5-free nonunit of P_7 is nine or 27 by Lemma 4.14. If all pairs have weight 27, full 3-height isolates a sextic three as before. The isolated-orbit argument applies with $e_5^+ = 4$, $e_7^+ = 2$ and $2(4 + 2) > 9$: the height-one auxiliary pure words are available since $(s - 3)(3 - 1) > 3$ for $s = 5, 7$.

At weight nine use the proof of Proposition 25.11 with negative cost $7/4$, permitting contraction at the spectator prime three. The coset cost is at most $9/(7-1) < 7/4$, so each a_C is nonnegative and their sum is 0 or 1. Here $-3 \leq L - 1 \leq 1$, hence the congruence $\sum a_C \equiv L - 1 \pmod 5$ is equality. The functional on at most three marks in a five-coset takes values in $[-3/2, 1]$. Consequently $2 + 7L \leq P + 3N/2 \leq 9 - N/4$, forcing $L = 1$, $N = 0$ and all $a_C = 0$. As in the proposition the two five-parts have masses $2, 7$ and the two seven-parts have masses $4, 5$, with 1 first in each coordinate (combining the masses in a singleton projection). A singleton projection would give an excluded one- or two-phase word (Theorem 24.1). Write the nonzero exponents of the two projections as a, b, respectively.

If either relative order is wild, its adjoint projection $\{0, \pm a\}$ or $\{0, \pm b\}$ has at most one mark per prime-order coset. In the product coset functional the largest possible single negative magnitude is $3/2$ and the product of the two negative magnitudes is at most $(3/2)/4 < 1$. Tensoring with the normalized spectator trace (at least $-1/2$) exposes the adjoint rectangle. Independent units moving the nonzero pair off itself in each coordinate then contradict the common-nonunit argument. Thus both relative orders are tame. If an initial corner is missing, after scaling the exponents and choosing signs, the adjoint marks are the 3×3 square of positions $0, \pm 1$ in each coordinate with $(1, 1), (-1, -1)$ omitted. The following functional exposes these seven marks. In that square put 1 on the marks and 0 on the omitted corners. In the other four columns put $-3/4$ on row zero and $-1/2$ on rows ± 1. In each of the other two rows put $-3/2, -1, -1$ on columns $0, 1, -1$ and $7/8$ on the four remaining columns. Every row and every column sums to zero, so the array is a linear functional on the cyclotomic configuration. Extend by zero on other product cosets and tensor with the normalized spectator trace. As all its values are greater than -2, equality at one is possible only on the seven marks. The independent dilations again yield a contradiction. With all corners occupied the margins force exactly the four stated coefficients. In particular the value generates $\mathbb{Q}(\zeta_{35})$ by uniqueness of the two projections, and its tame trace at five is $4(1 + \xi) - (3 + 4\xi) = 1$. These are necessary conditions. The $(3, 5, 7)$ assignment with two unrestricted companion exponents stays open; its quartic case follows from Section 25.2.

Twist-one orbits at the least prime three. For a minimal table in the unbounded-exponent case $(3,5,7)$ of Section 25.4, the weight-nine pair may be chosen in the pure five- and seven-twist fibers. Indeed Proposition 4.2 gives each hit in those fibers; if all their complementary pairs had full three-height, the vacuum-block argument of Section 25.4 would again isolate three. Take faithful characters at five and seven of orders four and six. At $1+\xi+3\eta+4\eta\xi$ their mixed Fourier coefficient is four times a product of two nonzero Gauss sums. On $(\mathbb{Z}/35)^\times$ their product has kernel generated by simultaneous negation: the two images intersect in $\{1,-1\}$ and the characters are faithful. Inverting either factor preserves this kernel. Therefore Proposition 24.16 supplies twist-one labels in P_5, P_7 in the same full three-block with fields containing $\mathbb{Q}(\zeta_{35})^+$. Each orbit has at least 24 members, since a nonunit in odd dimension has nonreal field and this real subfield has degree 12. More precisely, use characters with local orders $(4,6),(4,3),(2,6)$. All three mixed coefficients are nonzero by the same product of Gauss sums. With fixed generators a,b of the order-four and order-six groups, their kernels on $(\mathbb{Z}/35)^\times$ are respectively $\langle(a^2,b^3)\rangle$, $\langle(1,b^3)\rangle$, and $\langle(a,b^3)\rangle$. Inverting either component preserves each kernel. The kernel intersections are pairwise trivial, so any two of the three fields generate $\mathbb{Q}(\zeta_{35})$. In each of P_5, P_7 therefore either a twist-one row contains $\mathbb{Q}(\zeta_{35})$, or at least $24+12+6=42$ nonunit twist-one labels occur in three distinct orbits. The common three-block can depend on the character.

25.5 The sextic exponent at either smaller prime

For any fixed distinct odd primes P,Q, Theorem 25.9 gives $P^6Q^BR^C$ for all sufficiently large primes R, independently of both B and C. An explicit sufficient threshold is $R>\max\{P^2,33,\lfloor P^2/Q\rfloor^3\}$ by the scalar word bound in Section 25.3 and Corollary 25.7.

A cubic scalar bound. For integer $j\ge 1$ put $E=j(j+1)/2$. For $1\le h\le E$ one has

$$\frac{h}{k(h)}\le\sqrt{h+1/4}+1/2\le(j+1)/\sqrt{2}+1/2,\qquad k(h)=\min\{v:v(v+1)\ge h\}.$$

Consequently $3+2h(h-1)/k(h)<(j+1)^3/\sqrt{2}$. To verify the strict estimate, replace $2(h-1)$ by $j(j+1)-2$. The difference between the proposed upper bound and the resulting expression is $(j+1)(j+3)/\sqrt{2}-(j^2+j)/2-2>0$. Also $3(1+2E)/2<(j+1)^3/\sqrt{2}$ for $j\ge 1$. Thus $r\ge(j+1)^3/\sqrt{2}$ forces $W(r,j+1)$ by Lemma 4.22.

A uniform floor criterion. For $5\le p<q$ put $J=\lfloor p^2/q\rfloor$. For every $r>q$, all divisors of $p^6q^Br^4d$ are WGT dimensions if

$$J\le 1\quad\text{or}\quad(J+1)^3\le\sqrt{2}\,Jq.$$

For $r>p^2$ both companion exponents can be arbitrary. Indeed put $j=\lfloor\min(r,p^2)/q\rfloor$. At $j=0,1$ use strictness and Theorem 24.1, respectively. At $2\le j\le J$, the function $(x+1)^3/x$ increases on $x\ge 1$, so $(j+1)^3\le\sqrt{2}\,jq<\sqrt{2}\,r$. The preceding cubic bound and Theorem 25.9 finish the proof.

Two convenient sufficient tests are $(J+2)^2\le\sqrt{2}\,q$ and $(p^2/q+1)^3\le\sqrt{2}\,p^2$. For the first use $(J+1)^3/J<(J+2)^2$ for $J\ge 2$; for the second use the same monotonicity at $x=p^2/q\ge J$. In particular either $\sqrt{2}\,q^3\ge(p^2+2q)^2$ or $q^3\ge\alpha p^4$ for fixed $\alpha>1/\sqrt{2}$ and sufficiently large p suffices. For $\alpha=4/5$, $p\ge 125$ suffices since $(5/4)^{1/3}+p^{-2/3}<27/25+1/25<2^{1/6}$. For $(p,q)=(31,97)$ the floor test applies although $q^3<p^4$.

25.6 A quotient of the tame multiplier group

Let $W(r,m+1)$ denote the exclusion of a full positive r-primary word on at most $m+1$ positions for a nonunit label at a nonunit row in odd ordinary minimal data. A common scalar root is allowed. The following test keeps track of where differences lie in the group of tame multipliers.

Proposition 25.14. *Let r be an odd prime, $E = m(m+1)/2$, $H = (r-1)/2$, and assume $1 + 2E < 2r/3$. If $W(r, m+1)$ fails, there is an integer $1 \le h \le E$ with*

$$k = \min\{j \ge 1 : j(j+1) \ge h\}, \qquad \delta = h(h-1) - (H-1)k \ge 0.$$

If $\delta < 2(h-1)$, the following further condition holds for every divisor b of H. There are nonnegative integers x_i indexed by $i \in \mathbb{Z}/b$ with $\sum_i x_i = h$, and integers $e_j \ge 0$ with $\sum_j e_j = \delta$, such that

$$\sum_i x_i x_{i+j} = \frac{H}{b}k + (h-k)\mathbf{1}_{j=0} + e_j.$$

In particular, when H is even, the interval $[h-k-\delta, h-k+\delta]$ contains the square of an integer congruent to h modulo two.

Proof. By Lemma 4.22, a minimizing word at the fixed row has $h \le E$ opposite difference pairs A and every nonidentity tame multiplier satisfies $|A \cap uA| \ge k$.

If the populations of A on the free tame orbits are $h_1, \dots, h_v > 0$, then summing the overlaps over $u \ne 1$ gives $\sum_a h_a(h_a - 1)$. Two or more nonempty orbits decrease this sum from $h(h-1)$ by at least $2(h-1)$. This proves $\delta \ge 0$ and, under the strict bound on δ, identifies A with an h-element subset of a single cyclic orbit. Identify that orbit with $\mathbb{Z}/H$. Under reduction modulo b let x_i count the members of A in residue class i. The left side of the displayed equation counts all ordered pairs with difference j modulo b. Each of the H/b group differences in that class occurs at least k times; difference zero in the full group occurs h times. Summing gives the asserted equations and total excess. At $b = 2$ subtraction of the odd equation from the even gives $(x_0 - x_1)^2 = h - k + e_0 - e_1$. The two populations sum to h. □

For arbitrary odd primes $p < q < r$ put $m = \lfloor r/q \rfloor$. If $m + 1 < p$ and Proposition 25.14 has no feasible h at (r, m), every divisor of $p^A q^B r^4 d$ is a WGT dimension for all $A, B \ge 0$ and every coprime square-free d. Indeed its conclusion excludes the first alternative of Proposition 8.7, and $m+1 < p$ excludes the second as in Theorem 8.2. For the sextic superfamily one can instead use $m = \lfloor \min(r, p^2)/q \rfloor$ without the hypothesis $m + 1 < p$, by Theorem 25.9; if $r > p^2$ the exponent at r may also be arbitrary.

Here are instances where the scalar inequality alone leaves a feasible h but the order-two quotient excludes it. Every column uses the same rule. The last column brackets the interval in the preceding column by consecutive squares of the required parity.

m	r	h	k	δ	$[h-k-\delta, h-k+\delta]$	bracketing squares
17	3877	153	12	12	$[129, 153]$	$11^2, 13^2$
29	17981	435	21	21	$[393, 435]$	$19^2, 21^2$
33	26449	552	23	23	$[506, 552]$	$22^2, 24^2$
34	29453	595	24	30	$[541, 601]$	$23^2, 25^2$
45	66889	1035	32	14	$[989, 1017]$	$31^2, 33^2$

To check uniqueness of h without enumerating supports, on the interval $k(k-1) < h \le k(k+1)$ the scalar quotient $h(h-1)/k$ increases. Its value at a complete right endpoint is $(k+1)(k(k+1)-1)$, increasing with k. It suffices to compare the last two positions of the last complete interval and of the terminal interval, and the endpoint of the preceding complete interval. For example the first line gives $W(3877, 18)$. It applies at middle prime 223 or 227 for every prime $18 < p < q$ and arbitrary first and middle exponents. At $p = 67$, $q = 223$, Theorem 25.9 asks for this same word obstruction, with $h = 153$ left by the scalar test.

25.7 A small intersection at the largest quartic prime

Proposition 25.15 (Seven positions and the quartic strip)**.** *At every prime $s \ge 127$, no full positive s-primary word of a nonunit at a nonunit row has at most seven positions, in*

either type of minimal table. Consequently every divisor of $p^A q^B r^4 d$ is a WGT dimension for $11 \le p < q < r < 7q$, $q \ge 19$, arbitrary $A, B \ge 0$ and coprime square-free d.

Proof. Write $H = (s-1)/2$, $g(v) = \min\{t \ge 1 : t(t+1) \ge v\}$. Among all full positive s-primary words of nonunit labels on the given row, take one minimizing the number h of nonzero opposite pairs in its difference set D. We allow overall root-of-unity factors. The initial word gives $h \le 21$. Since $43 < s/2$, Lemma 4.22 applies. Every intersection of D with a unit multiple of D contains at least $g(h)$ opposite pairs: a common nonunit of the two adjoints has its balancing support in that intersection. Counting the nonidentity tame multipliers gives

$$(H-1)g(h) \le h(h-1).$$

The maximum of $h(h-1)/g(h)$ for $1 \le h \le 21$ is 95, attained at $h = 20$. Thus $s > 193$ is impossible.

The primes from 173 *to* 193. At $s = 173$ the inequality allows $h = 19, 20$, and at $179, 181, 191, 193$ only $h = 20$. In all cases $g(h) = 4$.

Let l be the number of nonzero opposite pairs in D of exact order s. For a uniform random unit at the chosen s-power level, the expected overlap on an orbit of n opposite pairs containing b pairs of D is b^2/n. The order-s orbit has H pairs, and every higher-order orbit has at least sH. Summing gives

$$4H \le l^2 + (h-l)^2/s.$$

If $l \le 18$, the right side is at most $18^2 + 20^2/173 < 344 \le 4H$. Thus at most one pair has higher order. No nonidentity tame multiplier modulo sign can match a higher-order pair with itself, so every overlap for such a multiplier lies in $\mathbb{F}_s$. The mean nonidentity overlap there is $l(l-1)/(H-1) \le 380/85 < 5$. Some intersection therefore equals $T = \{0, \pm x_1, \dots, \pm x_4\}$. A common nonunit has support contained in T. Its differences lie in $T - T$, which has at most $4 \cdot 5 = 20$ pairs and at least h. These actual differences are exposed. Comparing their dilates as above shows that $T - T$ meets each of its dilates in at least four pairs.

The following short finite check is on differences of four residues and is independent of the dimension. Scale one pair to ± 1 and choose the other three from the $H - 1$ remaining pairs. Form the mask of $x_i, 2x_i, x_i \pm x_j$ modulo sign. Retain masks with at least nineteen pairs at $s = 173$ and twenty elsewhere, and test their intersections with every nonidentity multiple. The columns count normalized choices of T, including choices with the same mask. The loops in the residue enumeration reproduce the counts using only integer arithmetic.

s	choices	at least the required size	all overlaps at least four
173	98770	52012	0
179	109736	50376	0
181	113564	52658	0
191	134044	65108	0
193	138415	67813	0

This excludes every possible T in this range.

The primes from 127 *to* 167. Let h_0 be the least $h \le 21$ satisfying $(H-1)g(h) \le h(h-1)$. Every word on this row has at least h_0 differences modulo sign, by the minimizing argument above. Here $g(h_0) = 4$. The initial seven positions lie in one coset of μ_s. Indeed, if split across cosets, at most $\binom{6}{2} = 15$ of their nonzero opposite differences have exact order s, by summing the binomial coefficients of the coset populations. All difference spectra in this initial comparison are exposed since $43 < s/2$. Averaging the overlaps over all units as above gives $4H \le 15^2 + 21^2/s < 252 \le 4H$. Translate the initial positions to $\mathbb{F}_s$. We can now minimize over full words supported there up to translation: every common nonunit obtained from them is again of this kind.

We describe the finite test explicitly. A forced mask D means that on the fixed row some nonunit has adjoint phases in $\{0\} \cup \{\pm d : d \in D\}$, with one entry per opposite pair. Use such a mask only if $2(1 + 2|D|) < s$. The starting mask consists of all differences of a candidate support, modulo sign. For a lower bound L on every true word's difference count, any two forced masks, including a mask with itself, give for each unit u a common nonunit with phases in $T = \{0\} \cup \pm(D \cap uF)$. Consequently reject the seed if $|D \cap uF| < g(L)$ or $T - T$ has fewer than L nonzero opposite differences. Otherwise its nonzero differences form a further forced mask whenever exposed. This implication holds equally for masks that contain the true adjoint support strictly. It follows from Lemma 4.20, by exposing both adjoints and applying strictness to their common constituents.

At 127 only, first suppose that the true minimum is 17 or 18. The average nonidentity overlap of a minimizing word is $h(h-1)/62 \le 306/62 < 5$. Some intersection therefore contains exactly four opposite pairs. Enumerate all $\binom{62}{3} = 37820$ choices after scaling one pair to ± 1, and start the mask test from $T - T$ with $L = 17$. Exactly 1024 pass its comparison with its own dilates; none passes subsequent comparisons. Thus $L = 19$ is valid at 127. Elsewhere take $L = h_0$.

Pad the initial support to seven points. Among all ordered triples of distinct points in the padded support choose (a, b, c) minimizing the least positive residue of $(c-a)/(b-a)$; use a, b as the affine coordinates $0, 1$. The smallest remaining point is this minimum. Generate supports in increasing order, skipping any prefix with a smaller normalized third point on one of its triples. Also skip a prefix of length j with v existing difference pairs if $v + 21 - \binom{j}{2} < L$. Both omissions preserve at least one allowed representative of each possible support. Counts are

s	L	seven-point seeds	pass own dilates	survive all tests
127	19	7648354	181274	0
131	17	14348304	541163	0
137	17	17850526	284626	0
139	17	19150174	237257	0
149	18	23066734	6567	0
151	18	24674574	3705	0
157	19	24343632	5	0
163	19	29683165	0	0
167	19	33726047	0	0

The normalized-support enumeration gives the exact loops. It restricts forward comparisons to intersections with at most six pairs and at most 41 processed masks per seed. Reaching that limit counts as survival. Thus each zero comes from contradictions alone.

Passage to dimension families. For $r < 6q$ use Corollary 8.9. For $6q < r < 7q$ with $q \ge 19$ prime, one has $r \ge 127$ (for $q = 19$ the least prime above 114 is 127). Here $1 + \lfloor r/q \rfloor = 7 < p$. The seven-position exclusion just proved rules out the first alternative in Proposition 8.7; $7 < p$ rules out the second by the proof of Theorem 8.2. □

25.8 The remaining family with exponents six, six and four

Choose a braided counterexample of least dimension among the divisors of $p^6q^6r^4d$. All three primes must remain repeated by Dong's theorem. Lowering a sextic exponent gives the unordered box $(7, 5, 4)$ of Theorem 8.1; lowering the quartic exponent gives a cubic, covered by Theorem 4.16. No square-free prime survives: the table dimension Δ is square-full by Lemma 2.5; if a singleton two were removed in passing to a fermion quotient, its table would be odd, contrary to Proposition 4.8. The three exponents in the source are therefore exactly $6, 6, 4$. The quartic prime is the largest by Theorem 7.3. If the category is slightly degenerate then two occurs among the three primes, and is sextic. Theorem 26.3 excludes this possibility. The surviving category must be ordinary, integral, modular, perfect and fusion-simple.

For an even dimension the dyadic sextic theorem (Theorem 26.2) leaves only the odd pair $(3, 5)$, so the dimension is exactly 29160000. At this dimension 3 and 5 occur together in a twist order, and 2 occurs with at least one of them (Section 26.3 and Lemma 26.1). For odd primes the inequalities of Theorems 25.12 and 25.9 exclude whole families with two arbitrary companion exponents under the stated prime conditions. Some odd assignments still require separate twist configurations. For example, at $5^6 7^6 23^4$ both $5 \cdot 23$ and $7 \cdot 23$ must divide simple twist orders (Theorem 27.1); the witnesses can be different. These restrictions leave the unrestricted family open.

26 The dyadic sextic obstruction

On support $\{2, 3, 5\}$ we say a *triple* occurs if a twist has order divisible by 30. When order exactly 30 is needed, we state that condition explicitly.

26.1 Isolation at dyadic valuation six

Lemma 26.1. *Let an ordinary minimal table have precisely three dimension primes $2 < q < r$ and $v_2(D) = 6$. Some twist order is divisible by $2q$ or by $2r$.*

Proof. Suppose two is isolated, meaning that every label with nontrivial dyadic twist has trivial odd twist. There is at least one such label by categorical Cauchy. By Lemma 3.4, each has pure dyadic codegree, hence $D = 64n^2$ with n odd, and the active degrees are among $n, 2n, 4n$. (Codegree one would exhaust the norm at the unit.) Write P_2 for the principal two-block. By Proposition 6.3, applied to $\{2\}, \{q, r\}$, every degree of dyadic height less than three is in P_2. The integral column $u = 8W_2 v_0$ then satisfies

$$\|u\|^2 = 64, \qquad v_2(u_X) = v_2(d_X) \quad (8 \nmid d_X).$$

A field ramified at r is unramified at two by the same proposition.

Every row in P_2 hits every odd degree. Their number, including the unit, is a positive multiple of four by the dimension equation modulo four. Every nonunit row also hits distinct even degrees omitting q and r, respectively, by Lemma 2.5. Indeed a nonunit with two primes missing would have prime-power degree. Each nonzero character entry costs at least one on average over its Galois conjugates. These observations already exclude an active row of codegree four.

We record an orbit estimate. If an r-free label X has degree $2q^j$, cross it with a q-free witness of degree $2^h r^l$ in P_r (Lemma 4.14). The lcm-normalized entry has weight two, strict house less than two and residue two at every prime above r. By Theorem 4.13 it has a unique unsigned expression as a sum of two r-power roots, since $r \geq 5$. The roots are distinct by strictness. The tame stabilizer acts freely on the nonidentity roots, so its order is at most two. Thus the r-inertia orbit of X has length $t \geq (r-1)/2$.

Suppose first that there is an active Z of codegree sixteen, so $d_Z = 2n$. Every supported X lies in P_2 by Theorem 2.11. If $i = v_2(d_X) \geq 1$, then $b = \chi_Z(X)/2^{i-1}$ is integral by lcm divisibility, and has zero residue at two. Indeed it is a dyadic unit times $\chi_X(Z)$, which has the residue of $2n$. Each such nonzero orbit of the entries $\chi_Z(X)$ costs at least 4^i by Theorem 5.1; the assertion about all primes above two follows by Galois covariance within P_2. The vacuum therefore excludes $i \geq 2$. At an r-free even hit X the preceding orbit estimate now applies. Its field ramifies at r and is unramified at two, whereas the field of Z is contained in a dyadic cyclotomic field (Corollary 2.8). Thus $\chi_Z(X)$ is a nonzero even rational integer, equal on its r-inertia orbit. The t columns cost at least $4t$. The other even prime-omitting orbit costs at least four. Including the odd degrees gives

$$16 \geq 4 + 4t + 4 \geq 4 + 2(r-1) + 4.$$

Consequently $r = 5$, $q = 3$, and equality holds throughout. There are precisely four odd labels, one the unit, and every even hit of this row omits three or five. This applies to every active codegree-sixteen row.

If any odd label is dyadically active, let 2^s be the greatest order of its dyadic twist among odd labels, and let L count odd labels of that order. All such labels have degree n. Choose m which is 2^{s-1} on the dyadic part of the twist level and zero on the odd part. In the dyadic cyclotomic field

$$n^{-2}\sum_X d_X^2\theta_X^m = 64 - 2L + 4\sum_{Y\text{ active},\ d_Y=2n}(\theta_Y^m - 1).$$

The regular-object indicator $D^{-1}|\sum_X d_X^2\theta_X^m|^2$ is integral over $\mathbb{Z}$ by (3.3). Complex conjugation preserves the unique prime above two; with $v(2) = 1$ the displayed number thus has valuation at least three. Every summand in parentheses has positive valuation or is zero. It follows that $v(2L) > 2$, hence $4 \mid L$. There are only three odd nonunit labels, a contradiction.

All active labels now have degree $2n$, so the saturation of the row norm just proved gives $S_{ZZ'} = 0$ for any two of them. Let A be their coordinate span. The odd local factors act trivially, up to scalars, on A by isolation (as in the proof of Proposition 6.3). Thus $W_2A = sA \perp A$. The actual dyadic twist is the identity on $A^\perp$, and its W_2-conjugate is the identity on $(W_2A)^\perp$. In particular both preserve A. The corresponding upper and lower unipotents differ from these operators by scalars and generate the dyadic group. This would make $W_2A = A$, contradicting orthogonality.

It remains that every active degree is n. Let N count them. The same highest-order indicator calculation, now without its last sum, gives $N \geq 4$. Take an r-free even $X \in P_2$ by Proposition 4.2. Its degree 2^iq^j has $i = 1$ or 2 by Corollary 4.1. It hits all N active labels. On each of them the r-inertia conjugates of the X-row agree, since all active fields have dyadic conductor. Averaging those t orthogonal S-rows gives squared norm D/t. On the active columns the lcm scale is 2^in; averaging also over all Galois automorphisms gives squared mass at least $N4^in^2$, in addition to the vacuum. Thus $4^itN < 64$. First $i = 1$, then $t \geq (r-1)/2 \geq 2$ by the orbit estimate, giving $N \leq 7$. By Lemma 6.2 the modulus of the vacuum coordinate of u is

$$|u_0| = \tfrac{1}{8}\Big|64 - \sum_{Z\text{ active}}(1 - \theta_{Z,2})\Big| \geq (64 - 2N)/8 > 6.$$

Hence $u_0^2 \geq 49$. The N odd active coordinates cost at least four in total. The $t \geq 2$ even r-free conjugates cost at least eight, and a q-free even member of P_2 costs at least four, by its height one or two. These disjoint contributions exceed 64. □

Theorem 26.2. *Let $q < r$ be odd primes other than the pair $(3, 5)$, and let e be square-free and coprime to $2qr$. Every weakly integral braided fusion category of dimension dividing $2^6q^ar^be$ is weakly group-theoretical, for all $a, b \geq 0$. At $(3, 5)$ an ordinary minimal table of dyadic valuation six requires a mixed dyadic twist and a complementary principal crossing of weight four.*

Proof. It suffices to consider three-prime minimal tables. Pair an r-free nonunit of P_q with a q-free nonunit of P_r. Their degrees are $2^iq^j, 2^hr^l$, and $g = 2^{\min(i,h)} \geq q$ by Lemma 4.14. Square divisibility gives $g \leq 8$ in ordinary data and $g \leq 4$ in the positive fermion quotient. If $g \leq 4$, then $g = 4$, $q = 3$. For $r \geq 7$ the two principal residues and Theorem 4.13 give $1 + 3\eta$, with η a nontrivial r-power root. The two positions have at most three differences, so Lemma 4.20 excludes the crossing: its single nonzero opposite pair is moved by a tame unit and $3 < r/2$. Thus for $(q, r) \neq (3, 5)$ only ordinary data remain and every such pair has $g = 8$. Each witness then has odd codegree. A q-active column hits an r-free nonunit of P_q by Proposition 4.2, so cannot be dyadically active by (3.4). Interchanging q, r isolates two, contrary to Lemma 26.1; the presence of height three forces $v_2(D) = 6$. The same argument shows that in the ordinary $(3, 5)$ case at valuation six at least one crossing has weight four. □

26.2 Eliminating the slightly degenerate dyadic sextic

Theorem 26.3. *A divisor-minimal slightly degenerate obstruction cannot have dimension $2^6q^br^c$ for distinct odd primes $q<r$ and arbitrary b,c. Consequently the slightly degenerate branch of the unordered $(6,6,4)$ divisor family is empty.*

Proof. The positive fermion quotient has dimension $\Delta=2^5q^br^c$. Take an r-free nonunit of the principal q-block P_q and a q-free nonunit of P_r. The gcd of their degrees is a power of two at least q by Lemma 4.14 and at most four by square divisibility in the positive quotient. Thus $q=3$ and the gcd is four. If $r>5$, Theorem 4.13 gives $1+3\xi$ at the normalized crossing, where ξ has nontrivial r-power order. Theorem 24.1 excludes the resulting two-position character. Hence $r=5$. All complementary witnesses have degree $4\cdot 3^i$ or $4\cdot 5^j$ and positive dyadic defect one.

Take the integral minimal modular extension of Lemma 4.4, with fermion g and matrices $S,s=S/\sqrt{4\Delta}$. Put $\chi_X(Y)=S_{XY}/d_X$. Even and odd refer to monodromy with g. Let A,B be the symmetric and antisymmetric even spaces under tensoring by g, and C,D the corresponding odd spaces. The full Weyl interchanges this permutation and the monodromy sign, while odd local factors commute with both (Section 3). Thus the dyadic Weyl W preserves A,D and interchanges B,C. The dyadic upper unipotent $U=\lambda T_2$ interchanges A,B and preserves C,D since even mates have opposite twists and odd mates equal twists. Write $K=W|_A$, W_o for the product of odd Weyls and $\widehat{s}=s|_A=KW_o|_A$. The genuine congruence representation of Section 3 has $\lambda^8=1$ and trivial odd normalizing scalars. The dyadic group preserves full two-blocks and K preserves positive two-blocks by lifting their projectors (Lemmas 3.3 and 7.17).

Positive twists. The first assertion of Lemma 7.17 applies to each complementary witness, since $v_2(4\Delta)=7$. Thus its row sees only the two squared dyadic phases $+1,-1$. A pair of any other squared phase cannot be active at three or five, as Proposition 4.2 modulo the opposite prime would force a complementary hit. On each such extra phase fiber the odd local groups act trivially: they preserve the fiber, their upper unipotents act identically there, and conjugate unipotents generate. Their Weyls in particular fix both full lines of its even pairs (the actual dyadic twists are opposite). By Lemma 3.4 such a pair V has pure dyadic positive codegree; hence both b,c are even. Set $n=3^{b/2}5^{c/2}$ in this case. The column $Kv_V=\widehat{s}v_V$ has nonzero vacuum coefficient, so $V\in P_2$. It vanishes modulo two on even degrees. The estimate $f_V\geq[\mathbb{Q}(\chi_V):\mathbb{Q}]+8$ of Corollary 5.3 leaves only $f_V=32$ among $32,8,2$, giving $d_V=n$.

Dimension orthogonality on such a row modulo five gives a five-free nonunit hit X, necessarily of degree $2\cdot 3^h$ since $X\in P_2$ by Theorem 2.11, Corollary 4.1, Lemma 2.5. Crossing with a complementary $4\cdot 5^j$ witness gives a weight-two entry of strict house below two and residue two at five. Its unique unsigned five-primary word is a nonscalar sum of two roots (Theorem 4.13), so the tame stabilizer has order at most two. The five-inertia orbit of X has size $t\geq 2$. Every extra pair, being of degree n odd, is hit by every member of this orbit by principality. All extra pairs are five-unramified. The averaged norm of Lemma 2.4, retaining the vacuum and all L extra pairs and using $\operatorname{lcm}(n,d_X)=2n$, gives $4tL<32$.

Thus $L\leq 3$. If these phases occurred, take the nonempty set H of pairs whose squared phase has highest order 2^m, $m\geq 2$. The operator $J=(T_2^2)^{2^{m-1}}$ on A is $1-2P_H$ with P_H their coordinate projection. The trace of $JKJ^{-1}K^{-1}$ is an algebraic integer since the scalar of the powered unipotent cancels and the resulting group element has finite order. Its expansion is

$$\dim A-4|H|+\tfrac{1}{8}\sum_{V,Y\in H}|\chi_V(Y)|^2,$$

using $Kv_V=\widehat{s}v_V$. The sum is rational by Galois covariance and stability of H, hence belongs to $8\mathbb{Z}$. Its terms are congruent to one above two by principality; thus $|H|=2$. If both labels are individually fixed, the four entries are odd-integer squares. Otherwise their fields are the same quadratic dyadic field. The squared moduli form a matrix with diagonal y and off-diagonal σy, where σ interchanges the labels: use symmetry and Galois covariance in either argument. For

rational y the sum is four modulo eight by oddness. Otherwise y belongs to the real quadratic dyadic field and is $a + b'\sqrt{2}$ with a odd; the sum is $4a$, again impossible. Hence all even dyadic twists have order dividing four.

Lemma 7.17 now gives full dyadic level dividing eight and two equal-degree positive labels of opposite squared twists in each complementary block. It also gives Gaussian integer values on even objects for every purely dyadic odd character.

Odd spins. Put $Z = T_2^2|_A$, $F = KZK^{-1}$. These involutions anticommute, and U^4 is scalar on C, by Lemma 7.18. Over each complementary block the two odd orbits are free, and their actual dyadic twists have ratio $\pm i$. Their degrees satisfy $u^2 + v^2 = 2d^2$ and $v_2(u) = v_2(v) = 2$. For $d = 4 \cdot 3^i$, descent modulo three gives $u = v = d$; for $d = 4 \cdot 5^j$, reduction modulo three gives $3 \nmid uv$.

At its odd prime p each complementary witness has a full principal lift. Indeed, if Y is an odd label of p-unit degree, its square is even, so its character value is congruent to $\pm d_Y$. Tensoring the witness by g chooses the positive sign. For every odd V, evaluating the even product YV then gives the dimension residue on V. Such a Y is supplied by the opposite odd mates just found. This lift hits both odd mates of every opposite block by principality. As W preserves its full two-block and W_o preserves odd dyadic spin fibers, its row sees only the two odd spins of its own block. Thus the same pair $\delta, i\delta$ occurs in every complementary block. At each odd prime the full principal block has only even labels (evaluate at g) and excludes g (evaluate at an odd mate of unit degree). Therefore every odd label active at three or five hits a complementary witness by Proposition 4.2 modulo the other prime. All such labels have one of these two spins.

Some other odd spin occurs, since the two-spin conclusion of Lemma 7.18 would contradict perfectness of R_+.

By scalar U^4 all other odd orbits have spin $-\delta$ or $-i\delta$. Call their set E. The odd local groups fix their full lines, by the same fiber argument as above, so their full codegrees are dyadic by Lemma 3.4. This forces b, c even. Write $n = 3^{b/2}5^{c/2}$. For $V \in E$, $Wv_V = sv_V$ has nonzero vacuum coefficient, hence V is full two-principal. Its degree is $2^j n$, $j = 0, 1, 2$, by Corollary 4.1 at valuation seven.

The signed norm. At least eight odd-degree positive pairs lie in P_2. Its rank is even since Z, F restrict to anticommuting involutions. Its sum of squared degrees is divisible by 32 by Theorem 2.11 on the unit row. Write k_0, k_1 for the numbers of odd and even degrees. The even degrees have height one by Corollary 4.1. Modulo four, $4 \mid k_0$, so k_1 is even; modulo eight $k_0 + 4k_1 \equiv 0$. The vacuum gives $k_0 \geq 8$. Each of these k_0 values at $V \in E$ is nonzero by principality and has squared modulus at least one since it is Gaussian.

Compare the signed character of V with that of an odd mate of a $4 \cdot 5^j$ witness. These rows differ, since they have different spins. In their restricted orthogonality (Lemma 4.5), the vacuum survives modulo three and all three-bearing degrees contribute zero by lcm-integrality on the three-free mate. This gives a nonunit three-free hit X_3. A $4 \cdot 3^i$ mate gives a five-free hit X_5. Both belong to the full and positive principal two-block by Theorem 2.11, Lemma 4.5, with degrees $2 \cdot 5^l$ and $2 \cdot 3^h$, respectively, $l, h > 0$. Each $\chi_V(X)$ is a nonzero Gaussian integer vanishing above two. When its squared modulus is at least four, its Galois orbit costs at least four per pair. When it is two, $\chi_V(X)^2 = \pm 2i$ and $\chi_X(V^2) = (d_V/d_X)^2\chi_V(X)^2$. Since V^2 is even, i belongs to the positive character field of $[X]$. Five-inertia fixes i, so another coset of its orbit is disjoint from the first. Every conjugate there also has signed absolute square two, by Galois covariance, using full lifts and ignoring the fermion sign. The five-inertia orbit of $[X_5]$ has size at least two by the weight-two crossing above; for $[X_3]$ use the trivial lower bound one. The disjoint supports cost at least eight and four, respectively. Thus the signed codegree is at least $8 + 8 + 4 = 20$. By (4.1) it equals $32/4^j$ for a free orbit and $64/4^j$ for a fixed one. Hence $j = 0$. A fixed label would have even d_V^2, since VV^* would be g-invariant and all even simples occur in free pairs. Every $V \in E$ is therefore free of odd degree n.

Strict multiplicity. Take $X = X_5$ as above. Every $V \in E$ hits X, since X is two-principal and

d_V is odd. Lcm-integrality makes $\chi_V(X)$ a nonzero multiple of two in $\mathbb{Z}[i]$, of squared modulus at least four. The five-inertia orbit of $[X]$ has size $t \geq 2$. Its full orbit also has size t: the entries at the V are fixed by five-inertia by symmetry, equal degrees along the orbit, and pure dyadic codegree at V, whereas passage to gX negates them. All chosen representatives have the same dyadic twist.

Put $a_X = (v_X + v_{gX})/\sqrt{2}$, $b_X = (v_X - v_{gX})/\sqrt{2}$, and similarly c_V, d_V^- for symmetric and antisymmetric odd pair vectors. For $v = \sum_X b_X$ over the t representatives, $\|v\|^2 = t$. The coefficient of sb_X at c_V is $2S_{XV}/(\sqrt{128}\, n)$, independent of the representative and of squared modulus $|\chi_V(X)|^2/32 \geq 1/8$. Hence $t|E| \leq 8$. If $|E| = 4$, then $t = 2$ and equality gives $sv \in \langle c_V : V \in E\rangle$. The odd Weyls fix this space pointwise and commute with s, so $W_o v = v$. Applying U and using equal dyadic twist of the representatives gives $W_o a = a$ for $a = \sum_X a_X$. Thus $sa = Ka$ is supported on the positive P_2. A $4 \cdot 5^j$ witness Y lies outside P_2 by the dyadic height bound. Yet the Y-coordinate of sa is a nonzero common factor times $\sum_X \chi_Y(X) \equiv 2d_X \neq 0$ modulo a prime over five by five-principality. We have proved $1 \leq |E| \leq 3$.

Trace on the odd antisymmetric space. On D put $x = \lambda\delta$, $T = U/x$. Its eigenvalues lie in $\{1, i, -1, -i\}$. For the projection P_E onto $\langle d_V^- : V \in E\rangle$ one has

$$J = 1 - 2P_E = \tfrac{1}{2}((1-i)T + (1+i)T^{-1}).$$

We claim $\operatorname{Tr}_D(JWJW^{-1}) \in \mathbb{Z}$. Unitarity and cyclicity give

$$\operatorname{Tr}_D(JWJW^{-1}) = \Im \operatorname{Tr}_D(TWTW^{-1}) + \Re \operatorname{Tr}_D(TWT^{-1}W^{-1}).$$

The corresponding words in U, W in $\mathrm{SL}_2(\mathbb{Z}/8)$ have matrices

$$h = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \qquad k = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Both have order dividing six: $h^3 = -I$ and $k^3 = 5I$ modulo eight. Both are conjugate to their inverses in this group: the Weyl inverts k, and the coordinate swap inverts h while its product with $I + 2h$ has determinant one modulo eight. Their traces on D are consequently real algebraic integers in $\mathbb{Q}(\zeta_6)$, hence integers. Replacing U by T multiplies the first trace by $x^{-2} \in \mu_4$ and leaves the second unchanged, proving the claim. Expanding the projector gives

$$\operatorname{Tr}_D(JWJW^{-1}) = \dim D - 4|E| + 4 \sum_{V,Y \in E} |W_D(V,Y)|^2.$$

The odd Weyls fix each of these columns, so $W_D(V,Y)$ is the full s-entry in antisymmetric pair bases, $2S_{VY}/(\sqrt{128}\, n)$. It follows that

$$\sum_{V,Y \in E} |\chi_V(Y)|^2 \in 8\mathbb{Z}. \tag{26.1}$$

We prove an orbit congruence valid for any size of this packet. Its entries $\chi_V(Y)$ are dyadic units by principality and belong to $L = \mathbb{Q}(\zeta_8)$ by the pure codegrees and the level. The group $G = \mathrm{Gal}(L/\mathbb{Q})$ acts on the pairs: odd inertia fixes each full character, and the Galois square fixes its dyadic spin at level eight. Equal degrees and symmetry imply that conjugation in either index conjugates the entry, up to the sign from changing a fermion representative. Thus $y_{VY} = |\chi_V(Y)|^2$ is well defined on pairs and satisfies $y_{\sigma V, \tau Y} = \sigma\tau(y_{VY})$.

Partition E into G-orbits of sizes $n_i \in \{1, 2, 4\}$. On a rectangle $O_i \times O_j$ the sum of the y_{VY} is $n_i n_j m_{ij}$, where m_{ij} is their normalized absolute trace. Indeed averaging independently over two group elements averages their product uniformly over G. For an entry written $\alpha = \sum_{k=0}^{3} a_k \zeta_8^k$ with $a_k \in \mathbb{Z}$, the normalized trace of $|\alpha|^2$ is $\sum_k a_k^2$. It is odd since α is a dyadic unit and ζ_8 reduces to one. When either orbit is a singleton, conjugating that index acts on α only by sign.

The rational character lines in L are $\mathbb{Q}, i\mathbb{Q}, \sqrt{2}\mathbb{Q}, \sqrt{-2}\mathbb{Q}$; an integral dyadic unit on these lines is an odd integer times one or i. Then $m_{ij} \equiv 1 \bmod 8$. In every other diagonal rectangle $4 \mid n_i^2$, so oddness of m_{ii} gives $n_i^2 m_{ii} \equiv n_i^2 \bmod 8$. The remaining off-diagonal rectangles occur in transpose pairs of equal sum with $8 \mid 2n_i n_j$. Altogether

$$\sum_{V,Y \in E} |\chi_V(Y)|^2 \equiv \Big(\sum_i n_i\Big)^2 = |E|^2 \pmod 8.$$

Consequently (26.1) forces $4 \mid |E|$, contradicting $1 \le |E| \le 3$.

For the unordered box, an odd-dimensional obstruction is ordinary. When the dyadic valuation is at most five, Theorem 8.3 covers the reduced source, with the lower faces in Section 4. Every remaining even source has dyadic valuation six, just excluded. □

26.3 Restrictions on an ordinary dyadic obstruction

Write $d_i = \dim i$, $f_i = D/d_i^2$, $\chi_i(j) = S_{ij}/d_i$, P_u for the full principal u-block of ordinary integral modular data, and K_i for the cyclotomic field generated by χ_i. A label is u-active when its twist has nontrivial u-part. We work with a perfect fusion-simple category and the positive spherical structure.

Lemma 26.4 (Trace transfer across principal blocks)**.** *Let s, t be distinct primes. Choose $E = P_t$ or its intersection with the fiber of trivial prime-to-t twist. Suppose $X \in P_s$ is t-inactive, H is a finite group of cyclotomic automorphisms preserving its t-inactivity, and g is a positive integer. For each s-free nonunit $Y \in E$, suppose that $\gcd(d_X, d_Y) = g$ and that*

$$\sum_{\sigma \in H} \sigma(S_{XY}/\operatorname{lcm}(d_X, d_Y)) = \tau$$

for a common rational integer τ. If $v_s(g|H| - \tau) = 1$, every t-active J has $s \mid d_J$. More precisely, at every chosen prime over s there is an s-bearing $Y \in E$ with $v_s(\chi_J(Y)) < 1$, normalizing $v_s(s) = 1$. Consequently $1 \le v_s(d_Y) \le v_s(d_J)$. If J also lies in P_s, then K_X and K_J both ramify at s.

Proof. Indeed set $F(Y) = g \sum_{\sigma \in H} \chi_{X^\sigma}(Y) - \tau d_Y$. The characters transform by ordinary Galois action, with $d_{X^\sigma} = d_X$. Then $F(1) = g|H| - \tau$ and $F(Y) = 0$ at every s-free nonunit of E. Principality gives $F(Y) \equiv (g|H| - \tau) d_Y \equiv 0$ at every prime over s. Indeed $\sigma\chi_X(Y) = \chi_X(Y^\sigma)$ by symmetry, Galois covariance and invariance of integral dimensions. Thus the defining principal congruences are stable at a fixed prime under every cyclotomic automorphism (see also the proof of Theorem 2.11). This also gives the congruence at every conjugate prime. Restricted orthogonality gives

$$0 = \sum_{Y \in E} \chi_J(Y) \overline{F(Y)}.$$

For the full block this is Theorem 3.2. For its indicated fiber, both projectors commute with the t-local group and their product still does; conjugation by the global Weyl yields an operator commuting with the t-twist, exactly as in Proposition 4.2. The vacuum term has valuation one. Therefore, at some s-bearing Y, the product has valuation at most one. Since $\overline{F(Y)}$ has positive valuation, this forces the first assertion. The lcm divisibility of S_{JY} implies the height inequality. If $J \in P_s$, its character also has positive valuation at such Y. Unramifiedness of either field would make that factor divisible by s, since $\overline{F(Y)}$ lies in the abelian field K_X. This is impossible. □

The ordinary sixth dyadic valuation. Assume $D = 2^6 3^b 5^c$. If two mixes with three, every five-active label has degree divisible by three. If two mixes with five, every three-active label has degree divisible by five. More precisely, Lemma 26.4 applies with $(s, t) = (3, 5)$ and $(5, 3)$, respectively. For a column J mixing two and s, apply Proposition 4.2 at s, inside its trivial

complementary fiber and avoiding t. It produces a nonunit t-free $X \in P_s$ with $S_{XJ} \neq 0$ and $\theta_{X,t} = 1$. By Lemmas 3.4 and 4.14, $v_2(d_X) = 2$: it is at least two, and at height three the row misses J. Every s-free nonunit $Y \in P_t$ has dyadic height at least two by the same height bound, hence the gcd is four. For the normalized crossing, principality gives residue four above three and five and strict house less than four. Theorem 4.13 leaves $1 + 3\xi$, for a nonidentity five-power root, or $-\sum_{j=1}^{4} \eta_j \zeta_5^j$, in which the η_j are one copy of one plus one coset of the cube roots of unity. In the first case ξ has order five by the two-position relative-order result (Theorem 24.1), applied to the full-mass expression $(d_Y/4)(1 + 3\xi)$ for $\chi_X(Y)$. Take H to be the order-four tame subgroup of five-inertia in a common cyclotomic level, fixing the other primary factors. In the first case its trace is $4 - 3 = 1$; in the second its trace is $\sum_j \eta_j = 1$. Thus $g|H| - \tau = 15$ in both orientations, using *the same* five-inertia group in each.

Lemma 26.5 (Pure dyadic phase of a fusion product)**.** *In the ordinary modular table, if all constituents of XY have the same dyadic twist δ and no dyadically active row sees both X and Y, then $\delta = 1$. In particular the conclusion holds when either factor has odd codegree.*

Proof. Indeed the vector $v = [XY]$, written in the fusion basis, is an eigenvector of the dyadic upper unipotent with eigenvalue $a\delta$, where a is its common normalization scalar. On the other hand, $(Sv)_J = d_J \chi_J(X)\chi_J(Y)$ for the unnormalized S. Its support has trivial dyadic twist, so the conjugate generator $S^{-1}T_2S$ has eigenvalue a on v. These two conjugate unipotents generate the dyadic modular group, so that their actions on the common eigenline coincide. The odd-codegree specialization is Lemma 3.4. □

Applying Lemma 26.5 to the two ordinary weight-four phase profiles gives the following consequence. When both factors belong to their trivial complementary fibers and one has full dyadic height, the negative profile is excluded. Without the full-height hypothesis that profile forces a dyadically active row meeting both factors.

Complementary height under separation. Assume further $c = 4$ and that three and five never mix. Since an isolated quartic prime is excluded (Lemma 7.2), two mixes with five. Choose the resulting $Y \in P_5$ of dyadic height two and three-free degree as above. For every five-free nonunit $X \in P_3$ the gcd is four. All five-active labels have full three-height by Lemma 3.4; thus b is even, Y and X are five-inactive (Lemma 4.14 bounds X below full three-height). The negative profile is impossible: at most one of its four distinct five-parts permits a five-inactive constituent, and at most two of its four three-parts permit a three-inactive constituent. The positive profile puts total dimension $3\,\mathrm{lcm}(d_X, d_Y)$ on its nontrivial five-phase. It follows that $3^{b/2} \mid 3\,\mathrm{lcm}(d_X, d_Y)$ and hence $v_3(d_X) = b/2 - 1$, again by Lemma 4.14. In particular the height is exactly two at $(6, 6, 4)$.

Lemma 26.6 (Separated twists and inertia orbits)**.** *Let an ordinary perfect fusion-simple integral modular table have square dimension n^2, where $n = As^e r^c$, $e, c \geq 1$, s, r are distinct primes, s is odd, and $\gcd(A, sr) = 1$. Suppose s and r never divide the same twist order. Each s-active Z has degree $zs^h r^c$, with $z \mid A$ and $h < e$, by Lemma 3.4 and defect-zero vanishing. Suppose each such Z has a nonzero entry at a principal row $X \in P_s$ satisfying*

$$d_X = xs^{e-1}, \quad x \mid A, \quad x \geq M, \qquad K_X \text{ unramified at } s, \qquad |I_r X| \geq m,$$

where I_r is cyclotomic inertia at r. If every s-active Z with $h > 0$ has at least b Galois conjugates and $mbM^2 \geq A^2$, then $h = 0$ for all s-active labels and

$$\sum_{Z \text{ active at } s} z^2 < A^2 s^2/m, \qquad m\,\mathrm{rad}(A)s^e < 2As^2.$$

Proof. Since $r \nmid f_Z$, inertia at r fixes K_Z by Corollary 2.8. Average m distinct I_r-conjugate unnormalized S-rows of X. Their average has squared norm n^2/m by orthogonality. At every

conjugate Z' of Z its entry is $S_{XZ'}$, since the m entries coincide there. The vacuum contributes positively. At a nonzero entry divisible by an integer L in the algebraic integers, the mean squared modulus over conjugations is at least L^2, by the norm. These estimates still hold after conjugation since the Galois action permutes the rows and preserves degrees.

If $h > 0$, principality makes $\chi_X(Z)$ vanish at every prime above s; unramifiedness gives $s \mid \chi_X(Z)$. This and lcm divisibility make S_{XZ} a nonzero algebraic-integer multiple of $\operatorname{lcm}(x,z)s^e r^c$. Counting the Z-orbit and the vacuum therefore gives $mbM^2 < A^2$, a contradiction. Now fix one of the rows X. Since every s-active degree is prime to s, principality makes X meet all these columns. At each of them S_{XZ} is divisible by $zs^{e-1}r^c$. Counting all of them in the same averaged row gives the first bound.

In Lemma 6.2 select the s-part, killing the dyadic anomaly. Its vacuum entry gives the rational integer

$$L = \sum_{Z \text{ active at } s} z^2(1-\theta_{Z,s}) = A^2 s^{2e}(1-(W_s)_{00}).$$

It is positive since s is active by categorical Cauchy and each nontrivial phase has real part below one. Thus $L < 2A^2 s^2/m$. At the primes of A the same lemma gives $n(1-(W_s)_{00}) \in \operatorname{rad}(A)\mathbb{Z}$. Consequently $\operatorname{rad}(A)As^e \mid r^c L$, hence divides L, proving the second bound. □

Ruling out separation of three and five. Suppose three and five are separated at $D = 2^6 3^{2e} 5^4$, $e \geq 2$. For every three-active Z, Proposition 4.2 gives a five-free nonunit $X \in P_3$ hitting it. By the preceding complementary-height calculation $d_X = 2^i 3^{e-1}$ with $i = 2, 3$. Its weight-four crossing with the chosen degree-20 member of P_5 puts $\mathbb{Q}(\zeta_5)$ in K_X; thus $|I_5 X| \geq 4$ and K_X is unramified at three by Proposition 6.3. Lemma 26.6 with $(A, s, r, M, m, b) = (8, 3, 5, 4, 4, 1)$ gives $3^e < 18$, impossible for $e \geq 3$, and $\sum_Z 2^{2k} < 144$ for the three-active degrees $25 \cdot 2^k$.

If three is isolated, Lemma 3.4 gives $k = 3$ on every three-active label. The preceding sum allows at most two such labels. Proposition 6.3, applied to the partition $\{3\}, \{2, 5\}$, gives $3^e(W_3)_{00} \in \mathbb{Z}$. With $k = 3$ the vacuum identity becomes

$$3^{2e}(1-(W_3)_{00}) = \sum_{Z \text{ three-active}} (1-\theta_{Z,3}).$$

Its right-hand side is positive and at most four, whereas its left-hand side is a multiple of 3^e. This excludes isolation also for $e = 2$.

When two and five are separated. Assume $e \geq 3$ and that two and five never mix. Two must mix with three by the exclusion of an isolated dyadic prime (Lemma 26.1). Choose a five-free dyadic-height-two witness $X \in P_3$ for a column mixing two and three. Its twist has trivial five-part by the choice of complementary fiber. For any three-free nonunit $Y \in P_5$, five is inactive by trace transfer, and $v_5(d_Y) = 1$ by the principal bound. Neither factor is dyadically active, as a dyadically active label has full five-height by Lemma 3.4. The negative crossing profile would place all constituents in the nontrivial dyadic twist -1 while some have nontrivial five-part, contrary to separation. The unsigned profile has a nontrivial five-phase of total dimension $3\operatorname{lcm}(d_X, d_Y)$. Each constituent at this phase has full dyadic height. Since $v_2(d_X) = 2$, it follows that every such Y has $v_2(d_Y) = 3$ and hence $d_Y = 40$.

26.4 Principal witnesses and the first even core

A lower bound in the principal block at the larger prime. Let a perfect fusion-simple ordinary integral table have just three dimension primes p, q, r, with $q < r$ odd. Every p-free nonunit $Y \in P_r$ has $q^2 \mid d_Y$. Indeed, Lemma 4.14 gives $d_Y = q^j r^h$ with $j, h \geq 1$ and supplies an r-free nonunit $X \in P_q$ of degree $p^a q^k$, $a, k \geq 1$. If $j = 1$, put $z = S_{XY}/\operatorname{lcm}(d_X, d_Y)$. Then $\chi_X(Y) = r^h z$ and $\chi_Y(X) = p^a q^{k-1} z$. The principal congruences give $z \equiv q$ at every prime over q

and r, respectively. Balancing gives weight at most q, while fusion-simplicity and perfectness give strict house below q (see the proof of Lemma 4.14). This contradicts Theorem 4.13 at $g = q < r$.

Suppose now $v_q(D) = 6$, $v_r(D) = 4$ and a label J is active at both q and r. At each prime over p, Proposition 4.2, applied to J in the principal r-block and its fiber of trivial prime-to-r twist, gives a nonunit Y with $d_Y\chi_J(Y)$ a unit there. Thus $p \nmid d_Y$ and $S_{YJ} \neq 0$. Its degree is $q^j r$ by Lemma 4.14. If $j = 3$, square divisibility gives $q \nmid f_Y$, so (3.4) in Theorem 3.2 forces $S_{YJ} = 0$ because J is q-active. Hence $j \leq 2$, and the preceding lower bound gives $d_Y = q^2 r$. In particular, at the ordinary sextic core each jointly three- and five-active column is met in the indicated principal five-block fiber by an odd label of degree 45, with a unit character value at the chosen prime over two.

Lemma 26.7 (Prime avoidance with a field condition)**.** *In the ordinary integral setting of Proposition 4.2, let p be the avoided prime and fix a prime above p. The nonunit witness Y in the prescribed principal block or trivial complementary fiber may be chosen with $d_Y\chi_J(Y)$ a unit there, K_Y at most tamely ramified at p, and*

$$p \nmid [K_Y : F] \operatorname{Tr}_{F/\mathbb{Q}}(\chi_J(Y)), \qquad F = \mathbb{Q}(\chi_J(Y)) \subseteq K_J.$$

In particular the maximal subfield of K_Y of p-power degree lies in F.

Proof. Group restricted orthogonality (or the slice identity) by full Galois orbits. The orbit of Y contributes $c_Y d_Y [K_Y : F] \operatorname{Tr}_{F/\mathbb{Q}}(\chi_J(Y))$, where c_Y is the integer slice coefficient, equal to one in orthogonality. The vacuum term is a p-unit, so some nonunit orbit contributes a p-unit. This gives the asserted index and trace and a conjugate whose individual contribution is a unit at the specified prime. Wild p-inertia preserves the block and trivial complementary fiber by Galois covariance. Its orbits inside the chosen orbit have equal size (the Galois group is abelian), and their terms have equal residues within each wild orbit. A nontrivial wild orbit size would make the whole contribution zero modulo p. Thus K_Y is tame at p, and the prime-to-p index puts its maximal p-power subfield in F. At $p = 2$ tame cyclotomic inertia is trivial. □

Apply Lemma 26.7 at the avoided prime two to a three-active J in the ordinary core $D = 2^6 3^6 5^4$, now using P_3. The selected odd Y has degree $3^j 5^h$, $1 \leq j, h \leq 2$. If $h = 1$, cross it with any three-free nonunit in P_5, of degree $2^i 5$ by Lemma 4.14. The weight-five result above gives tame five-inertia of order divisible by four on K_Y. Because $[K_Y : F]$ is odd and $F \subset K_J$, its order-four tame inertia image survives in K_J. If J is also five-active, $h = 2$ is impossible by defect-zero vanishing, so every jointly three- and five-active J has the full quartic tame five-inertia image. In particular its orbit under five-inertia has at least $4 \cdot 5^{a-1}$ labels when the five-part of its twist has order 5^a: twist covariance already makes wild inertia have at least 5^{a-1} images on that orbit. Conversely, if J is three-active and lacks this quartic field inertia, the selected odd P_3 witness necessarily has degree 75 or 225. This applies, for example, to every jointly two- and three-active J when two and five are separated, since then $5 \nmid f_J$.

Lemma 26.8 (A common hit in a principal block)**.** *Let B be a principal t-block in integral ordinary modular data with a unique invertible label, and let $p \neq t$ be prime. Suppose J, K have distinct t-twists and that for each nonunit p-free $Y \in B$ at least one of S_{JY}, S_{KY} vanishes. Then p divides both d_J and d_K, and neither J nor K is p-principal. In fact there is a common hit $Y \in B$ with $p \mid d_Y$, $v_p(d_Y) \leq \min(v_p(d_J), v_p(d_K))$, at which both characters are units at the chosen prime above p (with conjugation on the second).*

Proof. Indeed the restricted orthogonality sum $\sum_{Y \in B} \chi_J(Y)\overline{\chi_K(Y)} = 0$ has vacuum term one. After reduction some p-bearing label must make a nonzero contribution. The height bound follows from lcm integrality. If either row were p-principal, its value on every p-bearing degree would vanish at the reduction prime. Principality holds above every prime over p by the Galois action on the argument; this also treats conjugation. □

For $D = 2^6 3^{2e} 5^4$, $e \geq 3$, suppose that two and five are separated. Take a two-active J and a five-active K. Then J is five-inactive. Every three-free nonunit of P_5 has degree 40 by the degree-forty argument in Section 26.3. It has odd codegree, and its row vanishes at J. Lemma 26.8 gives $3 \mid d_J, d_K$ and $J, K \notin P_3$. Thus every three-free degree and also every member of the principal three-block has pure ternary twist. In particular a dyadically active label has degree $25 \cdot 2^i 3^j$ with $j \geq 1$, and a five-active one has degree $8 \cdot 3^k 5^l$ with $k \geq 1$. This statement applies to both single and mixed active twists.

Proposition 26.9 (Simultaneous partial Gauss congruence)**.** *Let $D = n^2$ be an ordinary integral modular dimension, and T a nonempty set of its prime divisors. Set $c_t(Y) = 0$ when the t-twist of Y is trivial, $c_t(Y) = t/(t-1)$ when its order is t, and $c_t(Y) = 1$ at higher order. Then*

$$n \prod_{\substack{u \mid n,\ u \notin T \\ u \text{ prime}}} u \quad \Big| \quad L_T = \sum_Y d_Y^2 \prod_{t \in T} c_t(Y).$$

In particular L_T is a nonnegative rational integer, positive precisely when a single twist is active at all primes in T.

Proof. Indeed expand $\prod_{t \in T}(1 - \theta_{Y,t})$ by inclusion and exclusion. For $J \subseteq T$ choose m_J equal to one at the normalized local levels in J and zero on their complement; include the order of the dyadic anomaly scalar in the dyadic congruence. By Lemma 6.2, the Jth weighted sum divided by n equals $n(W_J)_{00}\xi^{m_J}$, where $n(W_J)_{00}$ is a rational integer divisible by every listed outside prime (for $J = \varnothing$, use n). The normalized trace of a dyadic root of unity is 0 or ± 1. On the other side the normalized trace factorizes over the coprime primary orders; the mean of $1 - \theta_{Y,t}$ is exactly $c_t(Y)$. □

For example, if both dyadic edges occur at $2^6 3^{2e} 5^4$, every simultaneously three- and five-active degree is $2^i 3^j 5$, with $1 \leq j < e$. Indeed the two degree transfers give the two odd prime divisors, and activity cuts off their full heights. Write $a_3 = 2c_3$, $a_5 = 4c_5$ on this nonempty joint sector. On cancelling its common factors, the congruence becomes

$$128 \cdot 3^{e-2} \quad \Big| \sum_{Y \text{ joint}} 4^i 9^{j-1} a_3(Y) a_5(Y).$$

At $D = 5400^2$ the positive integer on the right is therefore a multiple of 384.

Lemma 26.10 (Simultaneous quadratic-character balance)**.** *For an odd prime t choose a primitive tth root ζ_t. For a label of exact t-primary twist order t write $\epsilon_t(Y) = (a/t)$, the Legendre symbol for $\theta_{Y,t} = \zeta_t^a$. For any nonempty set T of odd primes,*

$$\sum_{\substack{Y:\, \mathrm{ord}(\theta_{Y,t}) = t \\ \text{for all } t \in T}} d_Y^2 \prod_{t \in T} \epsilon_t(Y) = 0.$$

Proof. Indeed the partial Gauss sum selecting T is rational by the same vacuum identity, as its exponent kills the dyadic normalization. Average its Galois conjugates with the product of the quadratic characters on the factors in T. The rational sum has average zero. For a primary root of order above t the average vanishes on the subgroup congruent to one modulo t, and a trivial root is killed by the quadratic character. At order exactly t the average is the sign $\epsilon_t(Y)$ times a fixed nonzero quadratic Gauss sum divided by $t - 1$. Independence of the primary factors gives the formula. □

Thus in the ordinary triangle the two signs $\epsilon_3 \epsilon_5$ carry equal squared-dimension mass among the labels with primary orders exactly three and five, regardless of their dyadic twist.

Corollary 26.11 (Sharpened outside divisor)**.** *In Proposition 26.9, suppose every label active at a prime of T has $v_u(f_Y) \le 2$, for a given $u \notin T$. Then one may replace the factor u in the divisor by $u^{v_u(n)}$.*

Proof. By Theorem 6.9 each W_t, $t \in T$, has entries integral at every prime above u. The same holds for their product W_J, $J \subseteq T$. Hence the rational integer $n(W_J)_{00}$ is divisible by $u^{v_u(n)}$ in the inclusion–exclusion sum. □

For example, if two and five mix at $2^6 3^{2e} 5^4$, every three-active label has degree $2^i 3^j 5^l$ with $l \ge 1$ by the sextic trace transfer. The sharpened singleton congruence gives

$$800 \cdot 3^e \;\Big|\; \sum_{Y \text{ three-active}} 4^i 9^j 25^{l-1}\, 2c_3(Y).$$

Proposition 26.12 (Minors on deficient rows and columns)**.** *Let an ordinary integral modular table have square dimension n^2 on three primes p, q, t, where p, q never divide the same twist order. Call x deficient if $k_x = v_p(n/d_x) > 0$. For any deficient rows A and columns B, the residue rank at p of $(\chi_x(y))_{A,B}$ bounds below the number of intermediate labels in either factor order $W_t V_p = V_p W_t$ between A and B, where $V_j = \gamma_j^{-3} W_j$. Only the full blocks and complementary twist fibers preserved by each factor need be counted.*

In particular, every deficient Z has a $Y \in P_p$ in its full t-block with the same prime-to-t twist and $(W_t)_{ZY} \neq 0$. If $t \nmid f_Z$, one may take $Y = Z$. Hence a t-full label deficient at both p and q belongs to both principal blocks.

Proof. The p-local translates of the nontrivial p-phase spaces span an invariant subspace on which the q-upper unipotent acts by its normalization scalar: separation gives this on each phase space, and the q-action commutes with its translates. The generating conjugates also act by scalars there, so $V_q = I$ there and $V_p = I$ on the orthogonal complement, as in Proposition 6.3. Thus $(V_p - I)(V_q - I) = 0$ and, up to a scalar root of unity, $S/n = W_t(V_p + V_q - I)$; a dyadic anomaly outside n acts by scalars. The q-active columns have p-unit codegrees, so V_q is p-integral by the lower-unipotent expansion in the preceding Gauss congruence. The single-column estimate in Theorem 6.9 bounds the valuation of column y of W_t below by $1 - k_y$ and, by unitarity applied to W_t^{-1}, that of row x by $1 - k_x$. Commutativity gives

$$v_p((W_t(V_q - I))_{xy}) \ge 1 - \min(k_x, k_y).$$

After scaling row x by n/d_x, the character minor therefore equals the scaled $W_t V_p$ minor in the residue field, up to a common unit. A nonzero residue minor is nonzero in characteristic zero; the factorization in either order gives the claimed count. For $A = \{Z\}$, $B = \{0\}$ the entry $\chi_Z(0) = 1$ is a unit. The intermediate label for $W_t V_p$ lies in P_p by the vacuum-column lemma and in the required block and fiber by W_t. The t-block is a singleton if $t \nmid f_Z$ by Theorem 3.2. □

A trace bound for several principal residues. Let Z be a nonunit in a perfect fusion-simple symmetric integral table with exactly three dimension primes a, b, t. Write H for any subset of these primes at which Z is principal. Then

$$f_Z \;\ge\; [K_Z : \mathbb{Q}] + \sum_{q \in \{a,b,t\}} \Bigg(\prod_{p \in H \setminus \{q\}} p \Bigg)^2 .$$

Indeed for each q, the dimension identity (2.5) in Lemma 2.4 has vacuum term one; reduction modulo q forces a nonzero absolute trace T_X at some nonunit q-free degree. That degree has both other prime factors by Lemma 2.5, so the three column orbits are distinct. Every conjugate of $\chi_Z(X)$ vanishes above each $p \in H$ dividing d_X, by principality and Galois covariance. Their

absolute trace is a rational integer therefore divisible by the product of these primes. The orbit norm identity (Lemma 2.4), (2.4) counts at least T_X^2 for each of these three orbits in addition to the vacuum contribution.

Consequently when a, b are separated, every t-full degree misses at least two powers of a or at least two powers of b relative to n. Otherwise let H be the set of deficient primes. The preceding Weyl argument puts Z in P_p for every $p \in H$, while $f_Z = (\prod_{p \in H} p)^2$. The term $q = t$ alone equals this value, a contradiction after counting the vacuum. Codegree one is impossible by the ordinary row norm as soon as a nonunit is present. This excludes in particular $4 \cdot 3^e \cdot 5$ when two and five are separated and $4 \cdot 3^{e-1} \cdot 25$ when two and three are separated.

26.5 Congruences and quartic fields at the dyadic sextic

Proposition 24.12 determines the full height at a missing prime. At $D = 2^6 3^{2e} 5^4$, in the absence of a twist whose order is divisible by 30, the joint $3, 5$ degrees are divisible by eight, the joint $2, 5$ degrees by 3^e, and the joint $2, 3$ degrees by 25. If all three pairs occur, Lemma 26.4 and its application at dyadic height six make every joint $3, 5$ degree equal to $8 \cdot 3^j 5$ with $1 \le j < e$. At $e = 3$ the possibilities are 120 and 360.

At $2^6 3^6 5^4$ there is an additional incompatibility. A dyadically active five-free P_3 witness and a three-active three-free P_5 witness cannot coexist without a twist divisible by 30. The same is true with their two activity conditions interchanged. Indeed their common weight is four: all relevant degrees have dyadic height at least two (Lemma 4.14), while a dyadically active degree cannot have height three. Neither witness can mix two and three in the absence of the triple, since such a label has full five-height, whereas the indicated five-heights are zero and one. Thus their dyadic and ternary twist products are both nontrivial. By Corollary 24.14 every constituent has these same two components, and the two different five-positions force some constituent also active at five.

More strongly, when both dyadic edges occur there are witnesses of dyadic height two on both sides, by Section 26.3. Every such label has trivial three-twist. Otherwise use $p = 2, T = \{5\}$ on its nontrivial three-fiber and pair it with any dyadic-height-two witness on the other side. Its codegree is even and the opposite degree divided by four is odd, whereas $3(\xi - \sigma_5 \xi)$ is a two-unit for a nontrivial five-unit acting on ξ. In particular each jointly two- and three-active column in this triangle requires a five-free nonunit of degree 12 or 36 in P_3 with completely trivial twist: use Proposition 4.2 in the trivial complementary fiber.

In the weight-five case with $d_X = 20$ the prefactor η equals one even when X has trivial dyadic twist. Use $p = 2, T = \{3, 5\}$ at X; the opposite degree divided by five is odd. If $\eta \neq 1$, the mixed difference of z equals $3(\eta - \sigma_3 \eta)(\xi - \sigma_5 \xi)$, a two-unit for appropriate local inertia. This contradicts the congruence.

By the block congruence in Section 24.7, at the even core any three-free degree-20 member X of P_5 lies in one of four distinct full two-blocks, each entirely three-inactive if the triple twist is absent. Choose an odd $Y \in P_3$ of degree 15 or 45 from a joint three- and five-active column. The preceding result gives $\chi_X(Y) \equiv \xi \pmod 2$, with ξ a primitive fifth root, because $d_Y/5$ is odd. The four tame five-conjugates have distinct residues in characteristic two.

Applications at the even core. Suppose $D = 2^6 3^{2e} 5^4$ with $e \ge 3$. If two and five are separated, every three-active J of dyadic height below three has an odd member Y of P_3 with full five-height in its trivial prime-to-three twist fiber, such that $d_Y \chi_J(Y)$ is a unit at the prescribed prime above two. Indeed in the restricted orthogonality of Proposition 4.2 even degrees contribute zero there. The other nonunit degrees under consideration are $3^j 5$ and $3^j 25$, $1 \le j \le e - 1$. Each of the former has a full tame five-inertia orbit of four by its weight-five crossing with a three-free member of P_5 (Section 24.4). The value $\chi_J(Y)$ is constant modulo two on this orbit by Proposition 24.13 applied to $\chi_J(Y)$, because $2 \mid f_J$ and the corresponding differences in the argument equal Galois differences of $\chi_J(Y)$ by symmetry and rationality of the degrees. Its orbit

contributes zero. The vacuum then forces the required full-height hit. At $e = 3$ its degree is 75 or 225.

26.6 Quartic fields on complementary twist-one witnesses

Lemma 26.13 (Complementary witnesses of weight four)**.** *Let the ordinary minimal table have dimension $2^6 3^{2e} 5^4$, $e \geq 3$, and a twist active at two and at one odd prime. There are complementary principal labels $X \in P_3$, $U \in P_5$, of pure ternary and pure quinary twist, respectively (allowing twist one), with $\gcd(d_X, d_U) = 4$. Put $z = S_{XU}/\operatorname{lcm}(d_X, d_U)$. If $z = 1 + 3\xi$ with ξ a primitive fifth root, both principal blocks have twist-one witnesses containing $\mathbb{Q}(\zeta_5)$ in their row fields. If $z = -\sum_{a=1}^{4} \eta_a \zeta_5^a$ with $\{\eta_a\} = \{1, 1, \omega, \omega^2\}$ and $\operatorname{ord}(\omega) = 3$, the same holds with $\mathbb{Q}(\zeta_{15})^+$. In either case the two witnesses lie in a common full two-block.*

Proof. At the odd endpoint of a dyadic edge, use Proposition 4.2 in its pure twist fiber, avoiding the other odd prime. The dyadic height is at least two by Lemma 4.14 and cannot be three because the row meets a dyadically active column. At the opposite odd prime choose a pure witness by the same proposition at an active column (categorical Cauchy supplies one). The degrees are $2^i 3^j, 2^k 5$ with $\min(i, k) = 2$. In the unsigned case a faithful quartic character ψ at five has nonzero Fourier coefficient on z, a nonzero multiple of $3\sum_{a=1}^{4} \psi(a)\xi^a$. In the signed case multiply ψ by the quadratic character at three. Ternary antisymmetrization leaves a nonzero multiple of $\zeta_5^i - \zeta_5^j$ at the distinct positions carrying ω, ω^2. The resulting coefficient is a nonzero Gauss sum times $\overline{\psi}(i) - \overline{\psi}(j)$. The two character kernels are unchanged under inversion at either odd prime. The former cuts out $\mathbb{Q}(\zeta_5)$; the latter, on $(\mathbb{Z}/15)^\times$, has kernel generated by simultaneous complex conjugation and cuts out $\mathbb{Q}(\zeta_{15})^+$. Apply Proposition 24.16 with $(u, v, w) = (3, 5, 2)$. □

Theorem 26.14. *In a perfect fusion-simple ordinary minimal table suppose $D = 2^6 3^{2e} 5^4$, with $e \geq 3$, and some column is active at two and at one of the odd primes. Suppose no twist order is divisible by 30. In each of P_3 and P_5 there is a simple Z with*

$$\theta_Z = 1, \qquad \mathbb{Q}(\zeta_5) \subseteq K_Z.$$

The two witnesses can be chosen in the same full two-block. Each of the principal blocks P_3 and P_5 thus contains the unit and at least four nonunit simples of twist one, including a non-self-dual pair.

Proof. For the pure inputs of Lemma 26.13, Corollary 24.14 gives $z = 1 + 3\xi$. Indeed, among four negative phases at most one cancels the fifth twist and at most two cancel the ternary twist, so a signed word forces a multiple of 30 at every ternary height. The lemma transfers the quartic field to both witnesses. Their orbits have size at least four, and complex conjugation gives distinct duals. □

Theorem 26.15. *At $D = 29160000$ the ordinary minimal table has twist-one labels $V_3 \in P_3$ and $V_5 \in P_5$ in the same full two-block whose fields both contain one of*

$$\mathbb{Q}(\zeta_5), \qquad \mathbb{Q}(\zeta_{15})^+.$$

If no twist has order exactly 30, the witnesses can be chosen with the first field. In particular each principal block contains at least four nonunit simples of twist one. The two chosen labels may coincide.

Proof. A dyadic edge exists by Lemma 26.1. At its complementary witnesses Proposition 24.3 excludes a wild cubic coset at the exact dimension 5400^2. The two remaining words are precisely the two cases of Lemma 26.13, proving the unconditional assertion. The rational vacuum is outside either quartic orbit.

In the signed case the same cancellation count gives a triple and hence both dyadic edges. The degree transfers of Section 26.3 then make every three-active degree five-bearing and every five-active degree three-bearing. Thus the five-free X and three-free U both have twist one. Their four signed monodromy phases are actual constituent twists (Proposition 24.2): the two with $\eta_a \neq 1$ have order exactly 30, the others order 10. Conjugating one input preserves twist one and conjugates the unique physical word, realizing all eight primitive thirtieth and all four primitive tenth roots. □

Which quartic field occurs. In the weight-four and weight-five classifications (Section 24.2 and Proposition 8.6), the entry field contains $K = \mathbb{Q}(\zeta_5)$ or $\mathbb{Q}(\zeta_{15})^+$, the first in the unsigned case. At weight five the tame five-trace is 5η in both alternatives (Section 24.4); division by this known ternary prefactor reduces to the same four-position calculation as at weight four, up to sign and an additive integer. For a signed word with a nontrivial coset of μ_3, the ternary average isolates the single fifth root with coefficient one. For the coset μ_3 itself the stabilizer calculation following Proposition 7.16 applies: nonopposite repeated positions force the stabilizer to fix ζ_5, and opposite ones leave exactly simultaneous conjugation.

Fix a joint three–five label J at the sextic core. Apply Lemma 26.7 at two to choose an odd $Y \in P_3$ in its pure ternary fiber. It has $d_Y = 3^j5$ by defect-zero vanishing. Its crossing with a three-free principal quinary witness puts the indicated K in K_Y. For $x = \chi_J(Y)$ and $F = \mathbb{Q}(x)$, both $[K_Y : F]$ and $\mathrm{Tr}_{F/\mathbb{Q}}(x)$ are odd. Hence $K \subseteq F \subseteq K_J$. Without a triple the word is unsigned, so K_J contains $\mathbb{Q}(\zeta_5)$ and J is not self-dual.

At $2^63^65^4$ Proposition 24.17 shows that every joint three- and five-active label requires a second full tame five-orbit in its own five-block with exactly its prime-to-five twist. Indeed the first orbit is full by Section 26.4. This applies also if the joint label is dyadically active. For a second application, in the unsigned weight-five crossing of that section write $z = \eta(2 + 3\xi)$, ξ of order five. If the ternary root η is nontrivial, $\mathbb{Q}(z)$ contains $\mathbb{Q}(\zeta_3, \zeta_5)$: the uniqueness of the two phases, of different multiplicities, forces any stabilizer to fix η and ξ. In a dyadically trivial fiber containing either witness, the $(3, 5)$-torus thus has at least four distinct orbits affording the product of the two quadratic characters, by Proposition 24.17.

The outside divisor for joint Gauss sums. In ordinary square dimension let T be a nonempty set of odd dimension primes, and let $u \notin T$ be another dimension prime. If no twist is active at all primes of $T \cup \{u\}$, then the simultaneous Gauss integer of Proposition 26.9 satisfies

$$u^{2v_u(n)} \mid L_T.$$

Here $L_T = \sum_X d_X^2 \prod_{t\in T} c_t(X)$, where $c_t = 0, t/(t-1), 1$ at primary orders $1, t, > t$, respectively. Indeed the raw value $\sum_X d_X^2 \prod_{t\in T}(1 - \theta_{X,t})$ is rational by the vacuum Weyl formula and inclusion–exclusion: all the CRT exponents kill the dyadic normalization. It is therefore equal to its normalized trace L_T. Every surviving term is divisible by $u^{2v_u(n)}$ in the cyclotomic integer ring because its label has full u-height (Proposition 24.12). At the even core the combined divisors give $43200 \mid L_{\{3,5\}}$. Equivalently, writing $d_J = 8 \cdot 3^j5$, $a_3 = 2c_3$, $a_5 = 4c_5$ on the joint sector,

$$24 \mid \sum_J 9^{j-1} a_3(J) a_5(J).$$

When two is included, the factors c_2, c_3 have denominators prime to five and c_2, c_5 have denominators prime to three. Therefore the omitted prime's full degree square divides the rational simultaneous sum directly in these two cases. Together with the outside divisor in Section 26.4, the three divisibilities in this triangle are

$$135000 \mid L_{\{2,3\}}, \qquad 145800 \mid L_{\{2,5\}}, \qquad 43200 \mid L_{\{3,5\}}.$$

If an order divisible by 30 occurs, both dyadic edges occur. One can then take X and U at the two dyadic edges so that $d_X = 12$ or 36 and $d_U = 20$. Both have twist one by the same degree transfers. Their fields already contain $\mathbb{Q}(z)$, which in the signed case is $\mathbb{Q}(\zeta_{15})^+$ or $\mathbb{Q}(\zeta_{15})$ according as the positions carrying the repeated ternary phase are opposite or nonopposite (Section 24.2). The two distinct degrees give two distinct nonunit orbits. In particular the triangle, with or without a triple twist, has at least nine twist-one simples, rising to seventeen for a nonopposite signed crossing.

Thirteen twist-one simples without a triple. Suppose all three edges occur and no twist order is divisible by 30. There are nonunit twist-one labels of each of the following degree types:

$$12 \text{ or } 36, \qquad 20, \qquad 15 \text{ or } 45.$$

Each can be chosen with character field containing $\mathbb{Q}(\zeta_5)$. For the first two use X, U from Lemma 26.13. Their unsigned crossing $1 + 3\xi$ gives the field on both sides. For the third, use Section 26.4 at a jointly three- and five-active column to obtain an odd $Y \in P_3$ in the pure three-twist fiber, of degree $3^j 5$ with $1 \le j \le 2$. By Section 26.5 its crossing with U has normalized word $2 + 3\xi'$ for a primitive fifth root. Its constituent packet at ξ' has dimension $3\,\mathrm{lcm}(20, d_Y)$. If Y had nontrivial ternary twist, every simple in this packet would be three- and five-active, since $\theta_U = 1$ and Y is pure ternary. All those degrees are divisible by eight in the absence of the triple (Proposition 24.12). But the packet dimension has dyadic valuation two. Thus $\theta_Y = 1$, and $\mathbb{Q}(\zeta_5) \subseteq K_Y$. The three degree types distinguish three Galois orbits, giving at least twelve nonunit twist-one simples in addition to the unit.

The quartic field throughout a dyadic residue block. There is also a block restriction valid when a twist is divisible by 30. More generally, take labels J, Y in ordinary integral data, put $x = \chi_J(Y)$ and $F = \mathbb{Q}(x)$, and suppose that $\ell \nmid \mathrm{Tr}_{F/\mathbb{Q}}(x)$. Let $F_{(\ell)}$ be the maximal subfield of F of ℓ-power degree. Every label in the full ℓ-block of J has character field containing $F_{(\ell)}$, and the orbit of that block under the cyclotomic Galois group has size divisible by $[F_{(\ell)} : \mathbb{Q}]$.

Indeed this Galois group acts on the full blocks at a fixed prime over ℓ: equivalence of two characters can be tested after permuting their arguments. If h stabilizes the block of J, then $gx \equiv ghx$ at that prime for every Galois element g, by using Y^g as the test argument. If the image of the block stabilizer on F had order divisible by ℓ, it would contain an element of order ℓ. Grouping the absolute trace into the orbits of this element would make it zero in the residue field. Thus that image has order prime to ℓ. It fixes $F_{(\ell)}$, since $F/\mathbb{Q}$ is abelian. The stabilizer of a label in the block is contained in the block stabilizer, proving both assertions.

For a joint three–five label J, use the Y, x, F, K chosen above. Since $K \subseteq F_{(2)}$, every label of its full two-block has field containing K. Its four tame five-conjugate blocks are distinct: five-inertia has image of order four on K. None is principal, and every label in them has five-height at most one by Corollary 2.8.

Retaining the quartic field on removing the five-twist. Let J be any jointly three- and five-active column at the ordinary even sextic core. Its full five-block contains a five-inactive label Z with exactly its dyadic and ternary twists and with

$$\mathbb{Q}(\zeta_5) \subseteq K_Z \qquad \text{or} \qquad \mathbb{Q}(\zeta_{15})^+ \subseteq K_Z.$$

Without an order divisible by 30, the first inclusion holds.

For the same Y, x, F, K, the integral trace $\mathrm{Tr}_{F/K}(x)$ lies outside the quadratic subfield of K since its absolute trace is odd. Consequently x has nonzero Galois Fourier coefficient for a faithful character of $\mathrm{Gal}(K/\mathbb{Q})$ inflated to the splitting field. Equivalently, for the corresponding character orbit vector v on the conjugates of Y we have $(sv)_J \neq 0$ (replacing the character by its inverse if required). All its coordinates are five-inactive.

Let E project onto the full five-block and complementary twists of J. Since $s = W_5 W_{5'}$ and E commutes with W_5, the vector $W_{5'}v$ has a nonzero coordinate in E. It is still five-inactive and still affords the same character: the local Weyls at two and three invert only a trivial or quadratic component. The stabilizer of a nonzero coordinate therefore fixes K, proving the claim.

In the absence of a 30-multiple, the same Z can moreover be chosen in P_3. The full dyadic height of J makes its two-block a singleton, so its row of W_2 is scalar on the J-line. From $(W_2 W_5 W_3 v)_J \neq 0$ we now get $EW_3 v \neq 0$. The initial orbit lies in P_3, which W_3 preserves. As before, $W_3 v$ is five-inactive and affords the faithful conductor-five character. In the no-30 triangle the resulting Z is even by the exclusion of ternary-active odd degrees $15, 45$ in P_3 proved by the packet argument above (which also applies if the odd label has nontrivial five-twist: dilating the primitive fifth root on the opposite witness makes the three-coefficient packet five-active): three-activity forces five-height at least one, ramification at five bounds it by one, and an odd principal degree would be 15 or 45. Thus each such joint label requires in its own full five-block four even non-self-dual, five-inactive principal ternary conjugates at its exact ternary phase.

26.7 Residue ranks on a separated path

We apply Proposition 26.12 with $(p, q, t) = (2, 3, 5)$; write $z_{xy} = S_{xy}/\operatorname{lcm}(d_x, d_y)$.

Now take $D = 2^6 3^{2e} 5^4$, $e \geq 3$, and suppose two and three are separated. Choose a pure five-primary $U \in P_5$ of degree 20 from a joint two-five column. For a joint three-five column choose an odd $V \in P_3$ of degree $3^j 5$ in its pure ternary fiber. In fact V has twist one, since a three-active degree is divisible by eight. Let H be tame five-inertia, taken of order four. The weight-five calculation in Section 26.5 gives $z_{UV} = 2 + 3\xi$ for a primitive fifth root ξ. Index the four H-conjugates of either label by the nonzero elements of $\mathbb{F}_5$, and append the vacuum with index zero. The residue character matrix between these two sets of five labels at a fixed prime above two is

$$(\chi_{U_a}(V_b) \bmod \mathfrak{p})_{a,b \in \mathbb{F}_5} = (\overline{\xi}^{ab})_{a,b \in \mathbb{F}_5}.$$

For nonzero a, b this follows from $\chi_{U_a}(V_b) = 3^j(2 + 3\xi^{ab})$; the vacuum entries are odd. The matrix is invertible by the Vandermonde determinant.

All five rows belong to P_5 and have pure five-primary twists. For every full two-block C containing b_C of the five columns, the forward factor order thus gives at least b_C twist-one labels in $P_5 \cap C$. In particular if $V \in P_2$ there are at least five twist-one labels in $P_2 \cap P_5$, since P_2 is Galois-stable. For the reverse factor order let B be the five-block of V. Five-inertia fixes B. The four blocks C_a of the U_a, $a \neq 0$, are distinct and nonprincipal by their residues at V. The reverse factor order at column V places a label in each $C_a \cap B$ with pure five-primary twist $\theta_{U_a,5}$: V_p preserves the complementary phases of the row U_a, and W_t preserves those of the column V. The vacuum row similarly gives a twist-one label in $P_2 \cap B$. Thus each such five-block meets all five of these specified two-blocks.

Every label Z of C_a has $\mathbb{Q}(\xi) \subseteq K_Z$. For any Galois automorphism h fixing Z, the two values $\chi_{U_a}(V^h)$ and $\chi_{U_a}(V)$ agree modulo the same prime above two, since one may substitute Z for U_a. The displayed crossing and prime-to-two injectivity of reduction give $h(\xi) = \xi$. In particular all four intersections $C_a \cap B$ contain non-self-dual labels with this precise subfield. This applies to each distinct five-block of an odd pure-ternary P_3 witness of five-height one.

There is also a second hit for the same joint J. In $E = P_3 \cap \{\theta_{3'} = 1\}$ set $F(Y) = \sum_{h \in H} \chi_{U^h}(Y) - d_Y$. All four rows and the unit are three-inactive, whereas J is three-active. Restricted orthogonality gives $\sum_{Y \in E} F(Y)\chi_J(Y) = 0$, using the dual of J in the conjugated row. Here $F(1) = 3$. At every odd nonunit of five-height one in E the crossing above gives $F(Y) = 0$. Other odd nonunits have full five-height and miss J. At $8 \mid d_Y$, lcm divisibility gives $F(Y) \in 2\mathcal{O}$. It follows, at every prescribed prime above two, that some even $Y \in E$ has both $F(Y)$ and $\chi_J(Y)$ nonzero in the residue field. Its two-height is one or two, its five-height is at most one, and Y itself has twist one by separation. Its tame five-inertia orbit has size four: otherwise the trace

defining $F(Y)$ counts each value an even number of times. This supplies distinct even and odd twist-one P_3 orbits of full tame five-inertia meeting every joint J.

The rank conclusions apply to this second orbit as well. Put $r_h = \overline{\chi_{U^h}(Y)}$ in the residue field at two, where here the bar denotes reduction. Its sum is nonzero by the choice of Y. The four-by-four matrix $(r_{ab})_{a,b\in H}$ is invertible: after inverting the column indices it is the regular multiplication matrix of $\sum r_h[h]$ in the group algebra of the cyclic group H of order four. Its augmentation ideal in characteristic two is nilpotent. Thus the forward count in each two-block holds separately for this even orbit. Its five-block also meets each C_a at the pure five-phase of U_a, using any nonzero entry in the corresponding row, and meets P_2 at twist one: the vacuum crossing with Y is nonzero by the two-sided valuation estimate after scaling by n/d_Y. If $v_2(d_Y) = 2$, the four conjugates of Y occupy four different full two-blocks, since $\chi_{Y^b}(U_a) = (20/d_Y)\chi_{U_a}(Y^b)$ differs by a two-unit from the residue matrix just shown invertible.

A height bound from the exceptional fifth-root adjoint. Suppose at nonunits X, Y the two monodromy packets have primitive fifth-root ratio, and their larger dimension is $\lambda d_X d_Y$, $1/2 \le \lambda < 1$. Suppose also $\theta_{Y,2} = 1$. Among the adjoints of X and of a tame five-conjugate X', there is a nonunit constituent Z and there is a constituent T of YZ for which the dyadic monodromy is exactly -1. Furthermore

$$d_Z \le d_X^2 - 1, \qquad d_T \le \lambda d_X^2 d_Y.$$

Indeed the adjoint character word at row Y has the full mass on $A = \{1, \xi, \xi^{-1}\}$, where ξ is the relative fifth root. On the identity coset of the five-primary positions put 1 on A and $-3/2$ on its two vacant positions, and put zero on every other coset. Tensor with normalized trace on the prime-to-five factor. This rational functional has value one only on A, and is less than one everywhere else except at the negatives of the two vacant roots, where its value is $3/2$. Indeed a complementary root other than -1 has normalized trace at least $-1/2$. Apply also the dilated functional to the conjugate with multiplier two at five, for which $A \cap A^2 = \{1\}$. If the two balancing expansions of the adjoints avoided their respective exceptional negative phases, equality at the full mass would force all phases of their common nonunit constituent into $\{1\}$. Strictness and the common-nonunit lemma in the proof of Theorem 24.1 prohibit this. The unit has no negative phase. For either adjoint every such T occurs in $(YX)X^*$ or its counterpart with X'. The first product YX splits into the two indicated packets, of which each simple has degree at most $\lambda d_X d_Y$. The same holds after moving just the X-label by tame inertia: the resulting two positive fifth positions are exposed by the two-mark functional in Section 24.2, so their weights persist.

Apply this to $X = U$ of degree 20 and any odd $Y \in P_3$ of degree $3^j 5$ in the pure ternary fiber. Here $\lambda = 3/5$ and Y has twist one. At least one of Z, T is dyadically active and therefore has degree at least $2 \cdot 3^e$, since it has full three-height and no nonunit has prime-power degree. As $399 < 1200 \cdot 3^j$, the two bounds imply $3^{e-j} \le 600$, hence $j \ge e - 5$. If $e \ge 5$, every Z in this argument is dyadically inactive since $2 \cdot 3^e > 399$. The actual dyadic component of T is then -1. Both the lower height bound and this prescribed phase hold uniformly without an upper bound on e.

The five-free P_3 degrees satisfy a similar bound. Write one of them $d_Y = 2^i 3^j$ with $i = 2$ or 3 and $j < e$ by the complementary-height and principality bounds. It is dyadically inactive by separation. The crossing with U is $1 + 3\xi$ by the unsigned weight-four result; here $\lambda = 3/4$. It follows that $2 \cdot 3^e \le 300 \cdot 2^i 3^j$, hence $e - j \le 5$ if $i = 2$ and $e - j \le 6$ if $i = 3$.

In particular the number of three-active labels on this path is bounded independently of e. Every such label meets a five-free member of P_3 by restricted orthogonality modulo five. These witnesses have degree at least $8 \cdot 3^{e-6}$ by the two height bounds, so their number is at most $625 \cdot 3^{12}$. Every three-active degree is divisible by 40 by separation and the five-bearing transfer. For a fixed witness and any such label it meets, the lcm of their degrees is at least $40 \cdot 3^{e-6}$. A

nonzero cyclotomic integer has mean squared conjugate modulus at least one. Averaging the ordinary row norm over embeddings thus bounds the number of such hits per witness by $25 \cdot 3^{12}$. Altogether there are at most $15625 \cdot 3^{24}$ three-active labels. A ternary phase of order 3^k yields 3^{k-1} different phases by the Galois square action, forcing $k \le 33$.

26.8 The upper ternary height when two and five are separated

Suppose $D = n^2 = 2^6 3^{2e} 5^4$, $e \ge 3$, in an ordinary perfect fusion-simple minimal table, with two and five separated and both joined to three. Fix the principal blocks P_3, P_5. Put

$$A = \{X \in P_5 : 3 \nmid d_X,\ X \ne 1\}, \qquad B_j = \{Y \in P_3 : v_5(d_Y) = 2,\ v_3(d_Y) = j\}.$$

Every $X \in A$ has degree 40 and $\mathbb{Q}(\zeta_5) \subseteq K_X$. Indeed the degree is the complementary degree theorem of Section 26.3, and the no-triple weight-five crossing of Corollary 24.14 with an odd degree-$3^j 5$ member of P_3 has the form $\eta(2 + 3\xi)$ with ξ primitive of order five. Its five-inertia trace recovers 5η. The odd member exists at a joint three–five column by prime avoidance. Also A contains a member X_0 tame at three, by Lemma 26.7 at a five-active column. Both sets above are Galois-stable.

Proposition 26.16. *Let C_j consist of the odd degree-$3^j 5$ members of P_3. Then*

$$4 \sum_j 9^j |B_j| + \tfrac{1}{4} \sum_j 9^j |C_j| < 9^e.$$

Proof. Take four conjugates of one $X \in A$ under the tame order-four subgroup of five-inertia. The average of their S-rows has squared norm $n^2/4$, including a positive vacuum contribution. At $Y \in B_j$ the entries coincide, since $5 \nmid f_Y$, and are nonzero by three-principality of Y and $3 \nmid d_X$. They are divisible by $200 \cdot 3^j$ by lcm integrality. At $V \in C_j$ the lcm is $40 \cdot 3^j$ and the normalized crossing $\eta(2 + 3\xi)$ above has tame average $5\eta/4$. Counting both sets in the Galois mean of the norm proves the inequality. □

An arithmetic form of the fusion moment test. For a rational row Y, all the eigenvalues $\lambda_X = d_Y \chi_Y(X)/d_X$ of multiplication by $[Y]$ are rational integers. For a prime ℓ with $a = v_\ell(D) > 0$, a residue $c \in \{0, \dots, \ell - 1\}$, and $0 \le t < a$, one has

$$\ell^{a-t} \;\Big|\; \sum_{\lambda_X \equiv c \pmod{\ell}} d_X^2 \big((\lambda_X - c)/\ell\big)^t.$$

In fact these conditions over all ℓ, c, t are equivalent to integrality of all vacuum moments $D^{-1} \sum_X d_X^2 \lambda_X^k$, $k \ge 0$. For necessity, put $m = \varphi(\ell^a)$ (or any multiple at least a). The polynomial $1 - (T - c)^m$ equals the indicator of the residue class c modulo ℓ^a on integers. Multiply it by $(T - c)^t$ and use integrality of the vacuum coefficient of every integer polynomial in $[Y]$. Conversely expand any monomial at each residue c using the binomial theorem. Terms of degree at least a in $T - c$ already vanish modulo ℓ^a, and the displayed conditions handle the others.

26.9 The principal three-block

Take an ordinary minimal table of dimension $2^6 3^{2e} 5^4$, $e \ge 3$. Separation of two and five forces both other edges by Section 26.3 and Lemma 26.1.

Theorem 26.17. *Suppose two and five are separated in an ordinary minimal table of dimension $2^6 3^{2e} 5^4$, $e \ge 3$.*

1. *No $Y \in P_3$ with $v_5(d_Y) = 2$ has $v_3(d_Y) = e - 1$.*
2. *Every member of P_3 of degree $4 \cdot 3^j$ or $5 \cdot 3^j$ has $j \le e - 2$. Both types occur.*

3. *At $e = 3$ the odd nonunit degrees in P_3 are exactly 15 and 75. Each jointly $2,3$-active column meets a degree-75 principal row, and each jointly $3,5$-active column meets a degree-15 principal row.*

We prove part (1) in this section and part (2) in Section 26.10. For part (3), a nonunit cannot have prime-power degree, and the principal block bound gives $v_3(d_Y) \le e-1$. Each odd nonunit in P_3 therefore has degree 3^j5 or 3^j25. Here a column meets a row if their S-entry is nonzero. A jointly $3,5$-active column meets an odd principal row by prime avoidance and misses full five-height rows by defect zero (Section 3, Proposition 4.2). For a jointly $2,3$-active column the full five-height odd hit follows from Section 26.5; the parity identity below also proves this directly.

Proof of part (1). Write $B = B_{e-1}$ and use A, X_0 from Section 26.8. Suppose $B \neq \varnothing$. For $Y \in B$ put $i = v_2(d_Y)$, $L = \operatorname{lcm}(40, d_Y) = n/3$ and $z = S_{X_0Y}/L$. By Corollary 2.8, $F = \mathbb{Q}(z)$ is unramified at two and five by the codegrees of X_0 and Y, respectively, and at most tame at three by the choice of X_0. Thus $F \subseteq \mathbb{Q}(\omega)$, $\omega = \zeta_3$. If $z = a + b\omega$ is nonrational, the nontrivial orbit Fourier coefficient in Proposition 24.15 gives

$$|b|\sqrt{3}\sqrt{[K_{X_0} : F][K_Y : F]} \le n/L = 3.$$

But K_{X_0} contains $\mathbb{Q}(\zeta_5)$, disjoint from F, so the first index is at least four. Hence $z \in \mathbb{Z}$. Principality gives $(8/2^i)z = \chi_Y(X_0) \equiv 40 \pmod 3$, while the trivial Fourier coefficient gives $|z|\sqrt{[K_{X_0} : \mathbb{Q}][K_Y : \mathbb{Q}]} \le 3$. Thus $z = (-1)^{i+1}$ and $[K_Y : \mathbb{Q}] \le 2$. For any $X \in A$ the crossing field is again unramified at two and five and has degree at most two inside K_Y, so the same argument applies. In particular $s_{XY} = (-1)^{i+1}/3$ for the normalized matrix $s = S/n$. The purely ternary twist of Y lies in μ_3: order at least nine would give at least three square-Galois conjugates. The codegree at $i = 3$ is nine, impossible by Corollary 5.3.

The constant unit vector on A contributes $|A|/9$ on each $Y \in B$ to the squared norm of its s-image, besides the vacuum. Thus $|A|\,|B| < 9$, and $4 \mid |A|$ by the quartic field. If $|A| = 8$, $B = \{Y\}$ and the Y-row is rational. Reduced dimension orthogonality at two and five requires distinct nonunit hits of odd and five-free degree, respectively. Both are three-bearing, since nonunit degrees are not prime powers. Each costs at least nine by three-principality. For $i = 1, 2$ this overspends the norm: $1 + 8(8/2^i)^2 + 18 > (24/2^i)^2$. For $i = 0$, lcm integrality makes the character even at even degrees. Its squared norm 576 modulo four requires at least three odd nonunit values besides the vacuum; each is divisible by three. The five-free hit has even degree and value divisible by six. These hits, the vacuum and A already cost $3 \cdot 9 + 36 + 1 + 8 \cdot 64 = 576$. At a two-active column three-avoidance gives a further three-free nonunit in P_2 outside A (its dyadic height is at most two); its entry is nonzero by three-principality. This contradiction gives $|A| = 4$. It is a single Galois orbit. Proposition 4.2 provides a member with pure five-twist by avoiding three, and all three-free labels have pure three-twist by Section 26.4; thus all four have twist one.

Put $a = \frac{1}{2}\sum_{X \in A} v_X$ and $w = sa$. The orbit is stable under duality, so w is real. On B every coordinate has absolute value $2/3$. The modular relation $sTs = \xi T^{-1} s T^{-1}$ and $Ta = a$ give

$$\left|\sum_Z w_Z^2 \theta_Z\right| = |\langle a, w\rangle|.$$

If there were two labels in B, their phases would have sum of modulus at least one since both lie in μ_3. Their combined contribution on the left has modulus at least $4/9$. The remaining mass is $1/9$, giving $|\langle a, w\rangle| \ge 1/3$. But $w_0 = 80/n > 0$ lies outside $A \cup B$, and hence $|\langle a, w\rangle| \le \|w|_A\| < 1/3$. Thus $B = \{Y\}$ and Galois invariance gives $K_Y = \mathbb{Q}$. In particular Y is self-dual.

Let $b_U = \chi_Y(U) \in \mathbb{Z}$, $i = v_2(d_Y)$, and $q = n/d_Y$. Thus $i = 0, 1, 2$, $q = 24/2^i$, and $|b_U| = 8/2^i$ on A. Let U be a five-free nonunit in P_3. Its degree is 2^k3^j with $k = 2$ or 3, $j \ge 1$ (Lemma 4.14).

If $b_U \neq 0$, then $3 \cdot 2^{k-i} \mid b_U$ by three-principality and lcm divisibility. For $k = 2$ its crossing with any member of A is $1 + 3\zeta$ after lcm normalization, with ζ a primitive fifth root: the signed weight-four crossing would produce a joint $2, 5$ twist (Section 26.3). Thus the Galois orbit of U has at least four members, all with the same b_U since Y is rational. For $k = 3$ a single label suffices. Either orbit costs at least $(24/2^i)^2 = q^2$ in $\sum_U b_U^2 = q^2$. This leaves no room for the unit. Hence $b_U = 0$ for every five-free nonunit U in P_3.

All members of P_3 have purely ternary twist (Section 26.4). If Y were three-active, prime avoidance at Y with avoided prime five (Proposition 4.2) would contradict the vanishing just proved. Thus $\theta_Y = 1$. Moreover every three-active J must have five-bearing degree. Indeed block orthogonality for distinct ternary twists gives

$$\sum_{U \in P_3} \overline{\chi_Y(U)} \chi_J(U) = 0.$$

The only five-free contribution is the unit. If $5 \nmid d_J$, every five-bearing U contributes zero modulo a prime over five by lcm divisibility, a contradiction.

The diagonal modular relation and $\theta_Y = 1$ give $|\sum_Z b_Z^2 \theta_Z| = q|b_Y|$. Since the row norm is q^2, its squared mass on twist one is at most $(q^2 + q|b_Y|)/2$.

At a dyadically active column use Proposition 4.2 on the principal two-block in the pure dyadic twist fiber, avoiding three. The resulting nonunit W is three-free, hence also has purely ternary twist and therefore twist one. It is outside A since its dyadic height is below three. Its degree is even, and $b_W \neq 0$ by three-principality of Y. By symmetry $\chi_W(Y) = d_Y b_W / d_W$ is an integer, congruent to d_Y modulo two by two-principality. Thus $|b_W| \geq 2$ for $i = 0, 1$, and $|b_W| \geq 1$ for $i = 2$.

By dimension orthogonality reduced modulo five there is at least one five-free nonunit Z with $b_Z \neq 0$. Every such Z has pure five-twist because it can be neither two-active (that requires full five-height) nor three-active (as just proved), and is five-inactive by defect-zero vanishing in the row of Y, since five does not divide q^2. Hence $\theta_Z = 1$. Write $d_Z = 2^k 3^j$ with $k, j \geq 1$, using the exclusion of nonunit prime-power degrees. Lcm divisibility and three-principality give

$$3 \cdot 2^{\max(k-i,0)} \mid b_Z.$$

The unit, A, Y and W give distinct twist-one contributions outside these positions. Subtracting them from the twist-one budget leaves at most

$$\frac{q^2 + q|b_Y|}{2} - 1 - 4(8/2^i)^2 - b_Y^2 - b_W^2 = \frac{68}{4^i} - 1 - b_W^2 - (|b_Y| - 6/2^i)^2.$$

For $i = 0, 1, 2$ this is at most $63, 12, 9/4$, respectively. Every five-free hit costs at least $36, 9, 9$, respectively. This excludes $i = 2$ outright. For $i = 0, 1$ exactly one hit remains and it has $k = 1$, since $k \geq 2$ costs at least $144, 36$, respectively. Uniqueness and the rationality of Y make this Z Galois-fixed, hence rational.

Now cross Z with any degree-40 label $X \in A \subset P_5$. The lcm-normalized entry $z = S_{XZ}/\operatorname{lcm}(d_X, d_Z)$ is rational since Z is rational, and integral by lcm divisibility. Since $\gcd(d_X, d_Z) = 2$ and $5 \nmid d_Z$, the congruence $\chi_X(Z) = (d_Z/2)z \equiv d_Z \pmod 5$ gives $z \equiv 2 \pmod 5$. Strictness gives $|z| < 2$, an impossibility.

At the first core the surviving degrees satisfy a common residue equation. More generally, take any jointly two- and three-active J at $D = 2^6 3^{2e} 5^4$ on the path separating two and five. For every prime above two,

$$\sum_{\substack{U \in P_3,\ d_U = 25 \cdot 3^j \\ 1 \leq j \leq e-2}} \chi_J(U) \equiv 1 \pmod{\mathfrak{p}}, \qquad \mathfrak{p} \mid 2.$$

Indeed block orthogonality gives $\sum_{U \in P_3} d_U \chi_J(U) = 0$. Only odd degrees survive reduction. Every odd nonunit degree is $3^j 5^t$, with $1 \leq j \leq e - 1$ and $t = 1, 2$, by Corollary 4.1 and the

exclusion of prime-power degrees. At $t = 1$ the weight-five crossing gives an orbit of length four under the order-four subgroup of tame five-inertia (Section 26.8). That subgroup fixes J, since two-activity requires full five-height. The character is constant on the four arguments, so the contribution vanishes modulo two. At $t = 2$ the upper ternary height has just been excluded. This proves the congruence. At $e = 3$ only degree 75 occurs in the sum; in particular every such J meets at least one degree-75 row. If d_J is odd, the same argument works modulo four after retaining the factors d_U: lcm divisibility makes $d_U \chi_J(U)$ a multiple of four at every even degree. Thus at $e = 3$ one has $\sum_{d_U=75,\ U\in P_3} \chi_J(U) \equiv 1 \pmod 4$ for such J, since $75 \equiv -1 \pmod 4$.

The degree-75 witness at $e = 3$ can be chosen with character field contained in $\mathbb{Q}(\zeta_{27})$. Indeed group the last residue sum by dyadic inertia. It is a two-group, and conjugate summands in an inertia orbit have the same residue. A nonzero contribution therefore comes from a fixed label. Its field is unramified at two, and at five by its codegree. It is consequently a ternary cyclotomic subfield. The weighted inequality for the sets B_j in Proposition 26.16 gives $4 \cdot 9|B_1| < 729$, so its entire orbit has size at most twenty. A ternary field of conductor at least 81 has inertia order divisible by 27. This proves the claim.

26.10 The local Weyl bound for a nontrivial tame character

Lemma 26.18 (Weyl compression in a tame torus character)**.** *Let $s = S/n$ in an ordinary integral modular category of square dimension $D = n^2$. For an odd prime p, take a character χ of the group of cyclotomic tori whose local factor ψ at p is inflated from $\mathbb{F}_p^\times$ and is nontrivial. In particular the diagonal torus is unsigned. Write P_0 for the projector onto p-inactive labels. On the χ-isotypic space,*

$$\|P_0 s P_0\| \le p^{-1/2}.$$

More precisely, let v be a p-inactive vector of torus type χ and E fix the complementary twist components, optionally inside one full p-block. When $\psi^2 \neq 1$, in Esv the sums of squared absolute coordinates at the p individual phases $1, \zeta_p, \dots, \zeta_p^{p-1}$ are all equal. Higher p-power phases may also occur.

Proof. Indeed put $w = EW_{p'}v$, using $s = W_pW_{p'}$. The projector E commutes with the local group, so w is upper-unipotent-fixed and has local torus type ψ. At local level p^a, let Q_1 average over the kernel of reduction mod p. The lower–diagonal–upper factorization of this kernel as in Section 24.10 gives $Q_1 w = \operatorname{avg}_{p|t} L(t)w$: the torus character is trivial on units congruent to one modulo p. Hence W_pQ_1w is precisely the restriction of $W_pw = Esv$ to phases in μ_p. Write $u = Q_1w$ in the level-one quotient and B for its upper Borel. If $t \neq 0$ in $\mathbb{F}_p$, $W_p^{-1}U(t)W_p \in BW_pB$. In the explicit Bruhat decomposition the two outer upper unipotents fix u, and the intervening torus factor has argument t or its inverse up to fixed sign. Thus the coefficient at u is a constant times $\psi(t)$ or $\psi(t)^{-1}$. Its sum over $t \neq 0$ is zero. Finite additive Fourier inversion at the zero frequency gives squared mass $\|u\|^2/p$ at the trivial phase. If $\psi^2 \neq 1$ the coefficient vanishes individually since W_p sends the torus type ψ to the distinct type ψ^{-1}. Each of the p masses is then $\|u\|^2/p$. Taking $E = 1$ proves the norm bound. The same proof in commuting local factors gives the norm bound $\prod_{p\in T} p^{-1/2}$ if input and output are inactive at all these primes and each local factor of χ is tame and nontrivial: after moving the other Weyls past the projectors, the local compressed operators preserve the other local character types. □

In particular, for two lists of p-inactive orbits the Fourier matrix of Proposition 24.15 in this mode has norm at most $n/\sqrt{p}$. Its (X, Y) entry remains the usual product of $\operatorname{lcm}(d_X, d_Y)$, the square root of $[K_X : F][K_Y : F]$, and $\sum_{g\in \mathrm{Gal}(F/\mathbb{Q})} \chi(g)g(z)$, where $z = S_{XY}/\operatorname{lcm}(d_X, d_Y)$ and $F = \mathbb{Q}(z)$, provided the character descends to F.

Two missing principal degrees on the separated path. Take $D = 2^6 3^{2e} 5^4$, $e \ge 3$, with two and five separated and both joined to three. Every member of P_3 and every three-free simple

is five-inactive by Section 26.4. Choose a three-free nonunit $U \in P_5$ tame at three by Section 26.4 at a five-active column, avoiding three. Its degree is 40 by Section 26.3. Let R range through Galois orbit representatives of members of P_3 of degrees $4 \cdot 3^{j_R}$ or $5 \cdot 3^{j_R}$. Each type occurs: apply prime avoidance at a column mixing two and three to omit five, and at a column mixing three and five to omit two, respectively (Section 26.3 and Section 26.4). Here the odd hit has five-height one because a full-five label misses a five-active column. In the even case the hit has dyadic height two because it sees a two-active column.

At the even type the normalized entry is $1 + 3\xi$, with ξ a primitive fifth root. At the odd type it is $\eta(2 + 3\xi)$ for a ternary root η (Corollary 24.14). Its tame five-inertia trace is 5η, hence $\eta \in K_U$. Tameness at three gives $\eta \in \mu_3$. The entry field F_R is $\mathbb{Q}(\zeta_5)$ for $\eta = 1$ and $\mathbb{Q}(\zeta_{15})$ otherwise, interpreting $\eta = 1$ for the even type. For a faithful quartic character at five alone the absolute Fourier coefficient on F_R is $3\sqrt{5}$ in every case: when η is primitive cubic its two conjugates sum to -1. The lcm is always $40 \cdot 3^{j_R}$. Dividing the improved column bound by $5 \cdot 9 \cdot 40^2$ gives

$$\sum_R 9^{j_R}[K_U : F_R][K_R : F_R] \le 9^{e-1}.$$

A single exponent $e-1$ would exhaust this budget, although the opposite degree type is compulsory. Thus both types satisfy $j_R \le e - 2$, with no hypothesis on the high full-five degrees. At $e = 3$ each of the two types exists, all such degrees are respectively $12, 15$, and the displayed sum of relative index products without the weights is at most nine. In particular the chosen degree-40 witness has field $\mathbb{Q}(\zeta_5)$ or $\mathbb{Q}(\zeta_{15})$ at the first dimension. Wild five-inertia would contribute a factor five to each of the two compulsory index products, exceeding nine. This field is already unramified at two and at most tame at three.

Pure fifth phases on the triangle. At the core with all three edges, take the twist-one $X \in P_3$, $U \in P_5$ of degrees 12 or 36, and 20 from Section 26.6. Their crossing has a nonzero Fourier coefficient for a character cutting out $K = \mathbb{Q}(\zeta_5)$ in the unsigned case and $K = \mathbb{Q}(\zeta_{15})^+$ in the signed case. Both characters restrict at five to a faithful quartic. Apply the equal-norm identity to the χ-orbit vector on X in the full principal five-block with trivial complementary phases. Its coordinate at U after applying s is nonzero. By equality every primitive fifth phase has a nonzero coordinate there. This fiber is globally Galois-stable, and s inverts the torus character, so the field of each such label contains K. Tame five-inertia inversion acts nontrivially on K and preserves each fifth phase by the square action. Hence at least two such labels occur at each of the four phases in P_5.

For the nonopposite signed word, both quartic characters have a nonzero coefficient. For the conductor-five character, the trace over three assigns coefficient 2 to the repeated positions and -1 to the others, up to global sign. The resulting nonzero coefficient has magnitude $3\sqrt{10}$ because the two repeated positions are nonopposite; the character twisted by the quadratic character at three was treated in Section 26.6. Each phase therefore has witnesses for both quartic fields. If the witnesses are in distinct Galois orbits each contributes at least two. If they are in the same orbit its field contains their compositum $\mathbb{Q}(\zeta_{15})$, and the orbit has at least eight labels at precisely two fifth phases with equal multiplicities. Thus each primitive fifth phase occurs at least four times in this case.

If the triangle has no triple, let $Z \in P_3$ be the third twist-one witness of Section 26.6. Write $d_X = 4 \cdot 3^j$, $d_Z = 5 \cdot 3^k$ with $j, k \in \{1, 2\}$, and $m_T = [K_T : \mathbb{Q}(\zeta_5)]$ for these three labels. Their two unsigned crossings against U have faithful-quartic coefficient of magnitude $3\sqrt{5}$. The compressed column norm gives

$$m_U(9^{j-1}m_X + 9^{k-1}m_Z) \le 36.$$

Consequently a degree-36 or degree-45 witness has relative-index product with U at most three. If both occur, $m_U(m_X + m_Z) \le 4$.

26.11 Which principal rows meet a joint three–five column?

Consider $D = 2^6 3^{2e} 5^4$, $e \geq 3$, on the ordinary path where two and three are separated. Let P_3 be the full principal three-block. A joint three–five column is a label whose twist order is divisible by 15. Every such column meets an odd member of P_3 of five-height one and an even member of five-height at most one (Section 26.7). We determine which even hits have five-height one and bound the number of distinct character values there.

Choose a joint two–five column. Apply the wild-inertia refinement of Section 26.4 to the principal five-block at this column, avoiding three. The resulting three-free label U has degree 20 by Section 26.3 and its field K_U is tame at three. Write $h = [K_U : \mathbb{Q}]$ and τ for normalized field trace to $\mathbb{Q}$. Its normalized orbit sum and eigenvalues are

$$B = \frac{8}{h} \sum_{U' \in \mathrm{Gal} \cdot U} [U'], \qquad b_Y = 8\tau(\chi_Y(U)), \qquad b_1 = 160.$$

The normalized-trace argument gives B in the three-local fusion ring and $b_Y \in \mathbb{Z}$. Indeed $3 \nmid h$ by tameness and the prime support $\{2, 3, 5\}$, and normalized traces of cyclotomic integers in this support lie in $\frac{1}{8}\mathbb{Z}$. For a nonunit Y one has $-160 < b_Y < 160$ by strictness. For $Y \in P_3$ one has $b_Y \equiv 160 \pmod 3$ by applying principality to each U'.

Proposition 26.19. *For every joint three–five J put $j = v_3(d_J)$. Among even $Y \in P_3$ with $v_5(d_Y) = 1$ and $S_{JY} \neq 0$, retain the distinct values $b = b_Y$ different from 10 and 40, forming L_J. Then*

$$\sum_{b \in L_J} v_3(160 - b) \;\geq\; 2e - j - 2.$$

In particular there are at least $\lceil (2e - j - 2)/5 \rceil$ distinct such values. At $e = 3$ either there are two distinct values or one is congruent to 160 modulo 9.

Proof. Let E be the idempotent for P_3, integral in the fusion ring over the local coefficient ring at three (Section 2). Let T be the set of distinct b_Y on nonunits of P_3 with $S_{JY} \neq 0$. Then

$$E[J] \prod_{b \in T} (B - b[1]) = d_J \prod_{b \in T} (160 - b)\, e_0,$$

as is seen on each character. The vacuum coefficient of the dimension idempotent e_0 is $1/D$. Its coefficient on the left is integral at three, so $2e \leq j + \sum_{b \in T} v_3(160 - b)$.

Every five-free nonunit $Y \in P_3$ has dyadic height at least two, and the lcm-normalized crossing with U equals $1 + 3\xi$ for a primitive fifth root ξ (Corollary 24.14). Hence $\chi_Y(U) = 5(1 + 3\xi)$ and $b_Y = 10$. At every odd member of five-height one, the degree is $3^k 5$ and the crossing is $2 + 3\xi$ (Section 26.5). Thus $\chi_Y(U) = 4(2 + 3\xi)$ and $b_Y = 40$. Labels of full five-height vanish on J by the defect-zero criterion (Lemma 3.4). The two discarded values contribute at most $v_3(150) + v_3(120) = 2$ in total. Finally $1 \leq 160 - b \leq 319$ and $j < e$ by three-activity, proving the counts. □

One can prescribe a second nonzero crossing. Fix any joint two–five column K. For the subset of L_J realized by the same Y with $S_{KY} \neq 0$, the valuation sum is at least $e - j - 2$. Indeed, use $E[J][K]$ in the proof and $v_3(d_K) = e$ by separation. Every resulting witness has dyadic height one or two by the defect-zero criterion at K. Its three-height is deficient by principality. Its dyadic and ternary twists are trivial by separation and the deficient heights.

26.12 The exponent at the even frontier

Consider an ordinary minimal table of dimension $D = 2^6 3^{2e} 5^4$, $e \geq 3$. A prime is active on a label if it divides the order of its twist. By Section 26.3 and Section 26.9, some twist order is divisible by 15 and at least one is divisible by 6 or 10.

Proposition 26.20. *A minimal obstruction has $e \le 35$. If some twist order is divisible by 10, it has $e \le 18$.*

Proof. Choose a five-active column, also two-active when possible. By the wild-inertia version of prime avoidance in Section 26.4, it meets a three-free $U \in P_5$ for which K_U is tame at three and $d_U = 2^i 5 \le 40$. If the column is two-active then $i \le 2$ by defect zero, so $d_U \le 20$. Our functional at three has $t = 2$. Theorem 16.6 now gives $2e \le v_3(79!) = 36$ whenever $d_U \le 20$.

It remains to consider $d_U = 40$ with no twist divisible by ten. Choose a jointly two- and three-active J. Every Y with $S_{JY} \neq 0$ has $v_2(d_Y) \le 2$ by defect zero. Thus $\chi_Y(U)$ is twice an algebraic integer by lcm divisibility, and $b_Y = F(\chi_Y(U))$ is even. Its vacuum value is 80. At nonunits strictness gives gaps $80 - b_Y = 2k$, $1 \le k \le 79$. In the three-local fusion ring,

$$[J] \prod_{k=1}^{79} (B - (80 - 2k)[0]) = d_J \, 2^{79} 79! \, e_0,$$

where e_0 is the dimension-character idempotent. Inactive eigenvalues of $[J]$ vanish and every remaining nonvacuum eigenvalue is killed by the product. The vacuum coefficient gives $2e \le v_3(d_J) + 36$. Three-activity implies $v_3(d_J) \le e - 1$, proving $e \le 35$. □

26.13 Odd ramification on the path separating two and three

Suppose an ordinary perfect, fusion-simple integral modular category has dimension $D = 2^6 3^{2e} 5^4$, with $e \ge 3$, and no twist order is divisible by six. Write P_ℓ for the full principal ℓ-block and $\chi_Y(U) = S_{YU}/d_Y$. By Section 26.11, a twist order divisible by ten occurs. Choose at such a column its degree-20 witness $U \in P_5$ whose character field K_U is tame at three. Use this same U for every J below.

Proposition 26.21. *Let $15 \mid \mathrm{ord}(\theta_J)$ and put $j = v_3(d_J)$. There is an even-degree $Y \in P_3$ with $v_5(d_Y) = 1$ and $S_{JY} \neq 0$ such that $F_Y = \mathbb{Q}(\chi_Y(U))$ ramifies at five, or $4 \mid d_Y$ and F_Y ramifies at three. If $2e - j - 2 > 5$, one can require ramification at five. In particular the first assertion applies at $e = 3$, and the second applies to every such J at $e \ge 7$.*

Proof. For every even-degree $Y \in P_3$ with $v_5(d_Y) = 1$ and $S_{JY} \neq 0$, put $x = \chi_Y(U)$ and $b = 8\tau(x)$, where τ denotes normalized field trace to $\mathbb{Q}$. By Proposition 26.19, the distinct b other than $10, 40$ satisfy

$$\sum_b v_3(160 - b) \;\ge\; 2e - j - 2.$$

All conjugates of x have modulus strictly below 20 and are congruent to 20 above three by $Y \in P_3$. They are also congruent to 20 above five, since $U \in P_5$ gives $\chi_U(Y) = (d_Y/20)x \equiv d_Y$ and $d_Y/20$ is a five-adic unit. Galois covariance gives these congruences at every conjugate prime.

Suppose F_Y is unramified at five. Then $z = x/5$ is an algebraic integer of conductor supported on two and three, by the character-field conductor bound. Hence $k = 2\tau(z) \in \mathbb{Z}$. Indeed the normalized trace of a dyadic root of unity is integral, that of a root of three-power order is half-integral, and the traces multiply at coprime orders. Roots of unity span the cyclotomic integer ring. Moreover $[F_Y : \mathbb{Q}]$ is prime to three since $F_Y \subseteq K_U$, the latter is tame at three, and the cyclotomic degrees at two and five are prime to three. Thus the congruences for the conjugates of z give $k \equiv 8 \pmod 3$. The bound $|k| < 8$ leaves $k = -7, -4, -1, 2, 5$, with $b = 20k$. The value $k = 2$ is discarded. The others contribute at most $1 + 1 + 2 + 1 = 5$ to the sum, proving the five-ramification assertion.

If F_Y is unramified also at three, its conductor is dyadic. Then $\tau(z)$ is integral, so k is even. Alternatively, if $v_2(d_Y) = 1$, lcm divisibility gives $x \in 2\mathcal{O}_{F_Y}$, hence $z \in 2\mathcal{O}_{F_Y}$ and again k is even. In both cases only $k = -4, 2$ remain. Their values $b = -80, 40$ contribute at most $v_3(240) = 1$. The three-activity of J gives $j \le e - 1$, so the required sum is at least $e - 1 \ge 2$. This proves the first assertion. □

27 The surviving odd twist path

27.1 Compulsory edges and orientation

Throughout this section a pair of primes occurs when their product divides some twist order. The larger prime is $r = 23$. The first result locates the two compulsory edges; the second fixes a common orientation for all complementary principal witnesses on the remaining separated path.

Theorem 27.1. *Suppose that an ordinary perfect fusion-simple minimal table has dimension $D = 5^6 7^B 23^4$. For every B some twist order is divisible by $7 \cdot 23$. At $B = 6$ some twist order is also divisible by $5 \cdot 23$.*

Proposition 27.2. *Consider an ordinary minimal table of dimension $5^6 7^6 23^4$ with separated twist primes five and seven. Let U range over all seven-free nonunits of P_5 and V over all five-free nonunits of P_7. A single $t \in \{5, 7\}$ ramifies in every one of these row fields; the other prime s is unramified in all of them. If $t = 5$ every such V has seven-height two, and if $t = 7$ every such U has five-height two. Every s-active degree has 23-height at most one. There is a twist-one orbit of size at least 44 among the V; when $t = 5$ the same holds among the U.*

The following notation applies throughout the separated path at $B = 6$.

Orientation	(s, t, r)	Complementary rows
Seven unramified	$(7, 5, 23)$	s-active rows have 23-height ≤ 1; excluded by Theorem 27.12
Five unramified	$(5, 7, 23)$	s-active rows have 23-height ≤ 1; the remaining orientation

Here t ramifies in the complementary principal row fields and s is the other endpoint. The twist-one orbit from Proposition 27.2 enters Corollary 27.13.

27.2 The remaining ordered triple at least prime five

Write P_t for the principal t-block. We use the following consequence of Proposition 24.4. At the ordinary $5, 7, 23$ table, a complementary P_7, P_{23} crossing of common weight 25 has four distinct 23-positions and trivial 7-parts if its entry is unramified at seven. Its actual masses are $4\ell, 7\ell, 7\ell, 7\ell$, where ℓ is the lcm of the degrees. Indeed Proposition 24.4 gives a unique expression in which the seven-parts are two copies of 1 and 23 copies of η. If the entry is unramified at seven, its unique seven-projection is invariant under seven-inertia, so $\eta = 1$. The word obstruction $W(23, 3)$ then forces four positions, and the congruence of the multiplicities modulo seven gives the stated masses. The tame 23-inertia orbit has length 22 by Corollary 24.5.

If a column is active at both five and 23, principal support (Proposition 4.2) at 23 modulo seven supplies a seven-free nonunit $C \in P_{23}$ hitting it. At $5^6 7^{2e} 23^4$ its degree is $25 \cdot 23$: its 23-height is one by quartic principality, its five-height is at least two by the complementary bound in Section 26.4, and it cannot be three by defect-zero vanishing on the chosen column.

Proposition 27.3. *In the frontier dimension $5^6 7^6 23^4$ the following constraints hold on the two possible separations involving five.*

1. *If five and 23 are separated, every seven-free nonunit $C \in P_{23}$ has degree $125 \cdot 23$ and a tame 23-inertia orbit of length 22.*

2. *If five and seven are separated, there is a seven-free $C \in P_{23}$ of degree $25 \cdot 23$. Its crossings with each five-free $B \in P_7$ of 23-height one and with each 23-free $X \in P_7$ of five-height two have only 23-primary monodromy phases. Both resulting 23-words have four positions.*

Proof. Five cannot be isolated, by Proposition 25.8. If five and 23 are separated, a five- and seven-active column therefore exists. A 23-free $X \in P_7$ hitting it has five-height exactly two: the complementary bound gives at least two and defect-zero vanishing rules out three. Each seven-free nonunit $C \in P_{23}$ has 23-height one and five-height at least two, as above. If its five-height were two, neither X nor C could be 23-active, since 23-activity forces full five-height. At their weight-25 crossing, Proposition 24.4 and Corollary 24.5 give a nonidentity 23-position with multiplicity $m \in \{7, 14, 21\}$. Its summands are 23-active, so $125 \mid m \operatorname{lcm}(d_X, d_C)$, impossible at common five-height two. Thus $d_C = 125 \cdot 23$, and the same crossing still has common weight 25 and a tame orbit of length 22.

Suppose instead that five and seven are separated. Five must mix with 23. Use the degree-$25 \cdot 23$ witness C above for that mixed column. Seven-active degrees have full five-height, so C and both indicated partners are seven-inactive. The weight-23 crossing with B has one common seven-part η by Proposition 8.7. If $\eta \neq 1$, all constituents would be seven-active, contrary to $125 \nmid d_B d_C$. At the weight-25 crossing with X, a nontrivial η in Proposition 24.4 would put total mass $23 \operatorname{lcm}(d_X, d_C)$ on seven-active constituents. This again is not divisible by 125. The words at both crossings are therefore pure 23-words on at most four positions, and $W(23, 3)$ forces all four. □

The middle-prime height of an outer witness. For any odd $p < q < r$, if $W(q, 1 + \lfloor q/p \rfloor)$ holds, then every r-free nonunit $U \in P_p$ has $q^2 \mid d_U$, independently of all three exponent caps. Choose a p-free nonunit $B \in P_q$. By Lemma 4.14 their degrees have the forms $p^i q^a, q^k r^b$, with all four powers positive. Suppose $a = 1$ and set $z = S_{BU}/\operatorname{lcm}(d_B, d_U)$. Since $\chi_B(U) = p^i z$, its residue at q is zero. Similarly $\chi_U(B) = q^{k-1} r^b z$ gives $z \equiv q$ above p. Its weight is q, so Lemma 8.4 and rescaling give the forbidden full q-word. In particular at $p = 5, q = 7$ every such U has seven-height at least two, since $W(7, 2)$ holds by Lemma 4.22. At $v_7(D) = 6$ this height is either two or three; in a nonzero hit on a seven-active label only two is possible by defect-zero vanishing.

27.3 The last odd assignment at least prime five

Proof of Theorem 27.1. Suppose seven and 23 are separated. Then $B = 2e$, $e \geq 3$, by Lemma 3.4 and Theorem 7.3. The quartic prime 23 is not isolated (Lemma 7.2), so it mixes with five and the degree-$25 \cdot 23$ witness $C \in P_{23}$ of Section 27.2 exists. Every 23-free nonunit $X \in P_7$ has degree $5^i 7^j$, where $i = 2, 3$ and $j < e$, by the complementary bounds. Its weight-25 crossing with C has 23-inertia orbit of length 22 by Corollary 24.5, so K_X is unramified at seven by Proposition 6.3. Neither factor is 23-active, as neither has full seven-height. Each nonidentity 23-phase has mass $7 \operatorname{lcm}(d_X, d_C)$ by Section 27.2, all on constituents of full seven-height; hence $j = e - 1$. Proposition 4.2 supplies such a row X hitting each seven-active Z. Odd-dimensional duality gives Z and Z^* as distinct Galois conjugates. Lemma 26.6 with $(A, s, r, M, m, b) = (125, 7, 23, 25, 22, 2)$ would give $22 \cdot 7^{e-2} < 50$, impossible for $e \geq 3$.

The separated endpoints five and 23. Now let $B = 6$ and suppose five and 23 are separated. Every seven-free nonunit of P_{23} has degree $125 \cdot 23$ by Proposition 27.3. Its row therefore vanishes on every five-active column. A five-active J and a 23-active K have distinct 23-twists. Lemma 26.8 on P_{23} at seven gives $7 \mid d_J, d_K$ and $J, K \notin P_7$. Thus the seven-free labels and all of P_7 have seven-primary twists.

There is an additional uniform restriction on crossings in this path. Choose a 23-free $X \in P_7$ meeting a joint five- and seven-active column; such a column exists since five is not isolated. Its degree is $25 \cdot 7^j$, $j = 1, 2$. Its crossing with a seven-free member of P_{23} has weight 25, so Corollary 24.5 makes K_X ramified at 23 and hence unramified at five. For every seven-free nonunit $Y \in P_5$, write $d_Y = 5^u 23^v$, $u, v \in \{1, 2\}$. Put $g = 5^u$, $L = \operatorname{lcm}(d_X, d_Y)$, and $z = S_{XY}/L$. Principality at seven gives $z \equiv g$ there. When $u = 2$, $\chi_Y(X) = 7^j z$ has zero residue above five,

while $z \in K_X$ is unramified at five. Thus $w = 5z/g$ is integral in both cases. It has a positive expression of weight five, strict house below five, and residue five at seven. By Lemma 8.5 that expression uses only seven-power roots and w is an unsigned integral sum of five such roots. Uniqueness at length less than seven and strictness give a nonzero number, at most five, of nonidentity terms. The tame stabilizer acts freely on them, so tame inertia at seven has orbit size at least two on w. In particular every such Y has a seven-ramified field, and K_X ramifies at both seven and 23.

A signed-length test for ramification. In an odd ordinary minimal table on three distinct primes s, p, q, $q \geq 5$, suppose a nonunit crossing z of weight $g = ms + d$ has positive residues g at s, p, where $m \geq 1$ and $0 \leq d < s$. Put $L = \lfloor (g + d)/s \rfloor \leq m + 1$. If $p > m + L$ and z is unramified at s, then

$$z = d + s \sum_{j=1}^{m} \eta_j$$

for p-power roots η_j. Indeed $(z-d)/s$ is integral in the complementary two-prime field; normalized s-trace and two-prime unimodularity give a signed integral word of length at most L. Erase its p-parts. The resulting q-primary word equals m modulo every prime above p, and p is unramified there. Its difference from m has house at most $m + L < p$, hence is zero. Every signed q-letter other than $+1$ has normalized trace at most $1/(q - 1)$. Two such letters would give trace less than m, since $L \leq m + 1$. A single exception must be rational and would require length at least $m + 2$. This proves the formula.

Strictness ensures a nonidentity position, so uniqueness of the word of length $m < p$ and the free tame action there give orbit size at least $(p - 1)/\max\{a \mid p - 1 : 1 \leq a \leq m\}$. In particular the test holds for all d if $p > 2m + 1$. When $g = 2s + d$, $0 < d < s/2$, one has $L = 2$ and $p \geq 5$ suffices. The two roots in this case are distinct and nonidentity: otherwise the displayed full-mass word, rescaled to the character value, contradicts strictness or the odd two-phase theorem.

Which outer prime ramifies. In fact every 23-free nonunit $U \in P_5$ and every label $E \in P_{23}$ of degree $49 \cdot 23$ ramify at 23 and are unramified at five. The assertion about E is nonvacuous: a joint seven- and 23-active column has such a hit on P_{23} by Section 26.4, applied modulo five. Write $d_U = 5^i 7^a$, where $i = 1, 2$ and $a = 2, 3$ by Section 27.2. Their lcm-normalized crossing z has weight 49, strict house below 49, and positive residue 49 at five and 23. If z is unramified at 23, apply the preceding test at $49 = 2 \cdot 23 + 3$ to get $z = 3 + 23(\eta_1 + \eta_2)$ for distinct nonidentity five-power roots. The tame stabilizer at five has order at most two by uniqueness. Thus the two witness fields ramify at a common outer prime, and separation prevents them ramifying at both. Since every such U can be paired with every such E, the prime is the same for all of them.

Suppose it were five. Fix a joint seven- and 23-active J. Then $d_J = 125\, 7^k 23^h$, $k = 1, 2$ and $h = 0, 1$, by separation and the preceding transfer. Its field is unramified at five and its orbit has at least 66 labels: the independent square actions on the seven- and 23-parts give at least $3 \cdot 11$ twist pairs of values, and duality pairs distinct labels in each fiber. Choose an E as above hitting J. If $h = 1$, E-principality and unramifiedness at 23 give $23 \mid \chi_E(J)$, and lcm divisibility makes S_{EJ} a nonzero multiple of $125 \cdot 49 \cdot 23^2$. Its Galois-mean cost on the J-orbit exceeds D because $66 > 49$. All joints therefore have $h = 0$ and every such E hits each one by principality. Fix E and set $L = 125 \cdot 49 \cdot 23$. Write $b = S_{EJ}/L$ and let $c = 7$ or 3 according as $k = 1$ or 2. Since $\chi_E(J) = 125b$, principality gives $b \equiv c \pmod{23}$ in the unramified field. At least two five-inertia conjugates of E agree on the J-orbit. Taking their average gives $M(b) < 161^2/(2 \cdot 66) < 197$, where M is the Galois mean square. If $b \neq c$, the integral element $(b - c)/23$ has mean square at least one; the triangle inequality then gives $M(b) \geq (23 - c)^2 \geq 256$. Therefore $b = c$. At $k = 1$ this makes $\chi_E(J) = d_J$, contrary to strictness. All joints have degree $125 \cdot 49$ and $b = 3$. If their number is N, their total cost gives $18N < 161^2$.

More precisely, if the primary twist orders of a joint are $7^u, 23^v$, its orbit length is a multiple of $66 \cdot 7^{u-1}23^{v-1}$ by the same square action and duality. Our bound thus gives $v = 1$ and $u = 1$ or 2. Write the counts in these two sectors $66A$ and $462B'$; then $1 \le A + 7B' \le 21$. The simultaneous Gauss congruence of Proposition 26.9 for $T = \{7, 23\}$ makes $5\sqrt{D}$ divide the positive integer

$$(125 \cdot 49)^2 \left(66A \frac{7}{6} \frac{23}{22} + 462B' \frac{23}{22} \right) = (125 \cdot 49)^2 \frac{7 \cdot 23}{2} (A + 6B').$$

In particular $23 \mid A + 6B'$, a contradiction. This eliminates the five-ramified alternative. In the surviving alternative z is unramified at five. Apply the converse test at $49 = 9 \cdot 5 + 4$ to get the unique word $z = 4 + 5 \sum_{j=1}^{9} \xi_j$ in 23-power roots. The 23-tame orbit length of every such U, E is at least eleven.

In particular any five- and seven-active Z whose degree is divisible by five has a 23-free hit of degree exactly $5 \cdot 49$ in P_5. Indeed take a hit U by prime avoidance. Its seven-height is two: it is at least two as above, and full height would miss Z. Write $d_Z = 5^h 7^k 23^2$ and $d_U = 5^i 49$. As K_U is unramified at five, $5 \mid \chi_U(Z)$, and S_{UZ} is a nonzero integral multiple of $5^{\max(h,i+1)} 49 \cdot 23^2$. The Z-orbit has at least twelve members, by the square twist action and odd duality. At least eleven inertia-conjugate U rows agree there. If $i = 2$, their norm bound would give $11 \cdot 12 < 49$. Hence $i = 1$ as claimed.

We exclude the second alternative. The preceding unsigned word on at most ten 23-positions is exposed among all phases: on a μ_{23}-coset with $b \le 10$ marks use value one on the marks and $-b/(23 - b)$ at unmarked positions, tensoring with normalized trace at complementary primes (Lemma 4.20). It is also linearly independent on its support by the cyclotomic coset relations. Thus every actual nonidentity balancing packet of U, E has dimension $5m \operatorname{lcm}(d_U, d_E)$ for a positive m, with the sum of these m at most nine. Both factors are 23-inactive since neither has full five-height, and every 23-active constituent has dimension divisible by 125. If $i = 1$, then $5 \mid m$ for every such packet. There would be at most one nonidentity position, contradicting strictness or Theorem 24.1. Hence all 23-free nonunits of P_5 have $i = 2$. The degree-$5 \cdot 49$ conclusion just proved rules out every five-bearing joint five- and seven-active label.

The remaining joint labels have degree $7^k 23^2$, $k = 1, 2$, by the seven-divisibility transfer and activity. Their set is nonempty since five is not isolated. Fix a 23-free $U \in P_5$ hitting one of them. Its degree must now be $25 \cdot 49$, and it hits every joint by principality. At such a joint J, put

$$L = 25 \cdot 49 \cdot 23^2, \qquad b = S_{UJ}/L.$$

Since K_U is unramified at five and K_J at 23, b lies in a seven-power field. The congruence $\chi_U(J) = 23^2 b \equiv 7^k 23^2$ above five therefore gives $b \equiv 7^k \pmod 5$. Counting at least eleven U-rows agreeing on the entire J-orbit as before, we get

$$M(b) < \frac{35^2}{11 \cdot 12} < 9.29.$$

For $k = 2$ this forces $b = -1$, since $(b + 1)/5$ is integral and a nonzero value would give $M(b) \ge (5 - 1)^2$. For $k = 1$ let $w = (b - 2)/5$. A nonzero integer in a seven-power field has mean square at least $5/3$ unless it is a signed seven-power root (Proposition 24.4). The reverse triangle inequality would then give $M(b) \ge (5\sqrt{5/3} - 2)^2 > 9.29$. For a signed seven-power root $M(b) = 29 + 20 \operatorname{Tr}_{\text{mean}}(w)$, and this is at most 9.29 only at $w = -1$. Thus at $k = 1$ we have $b = 2$ or -3.

All these entries are rational. The whole Galois orbit of U now agrees on every joint column. The 23-tame inertia orbit has size 11 or 22, so eleven divides the full orbit length. Duality has no fixed points on that orbit in odd dimension. Its full length is therefore divisible by 22. If N_k counts the joint labels of each degree, the averaged row gives $4N_1 + N_2 < 35^2/22 < 56$. Each joint orbit length is divisible by $12 \cdot 5^{u-1}7^{v-1}$, where its actual primary twist orders are $5^u, 7^v$.

Thus $u = v = 1$ throughout and, writing $N_1 = 12A$, $N_2 = 12B'$, we have $4A + B' \leq 4$ and $A + B' > 0$. The joint Gauss sum is now

$$L_{5,7} = \frac{5}{4}\frac{7}{6}\, 23^4(7^2 N_1 + 7^4 N_2).$$

It is divisible by $23\sqrt{D}$, hence $25 \mid A + 49B'$. Consequently $A \equiv B' \pmod{25}$, impossible in the indicated range. We have proved that a twist divisible by $5 \cdot 23$ is compulsory. Together with the first result of this section both edges to the quartic prime are compulsory; only separation of five and seven, or a complete three-edge graph, remains.

A bound on two crossing fields. In the ordinary $5^6 7^{2e} 23^4$ family, apply this to either of the two crossings of Proposition 24.6, at weights 23 and 25. Suppose $5^3 7^{e-1} 23 \mid L_{XY}$. By Lemma 24.7, the seven-part is trivial. In particular the entry field is $F = \mathbb{Q}(\zeta_{23})$. Thus for $n_X = [K_X : F]$ and $n_Y = [K_Y : F]$ we obtain

$$n_X n_Y \leq 3.$$

Indeed the three nonzero positions all have weight seven. At each of the four origins in the affine support classified by the enumeration in Proposition 24.6, Section 24.3 provides a nontrivial character at 23 sending those positions, after rotation, into an arc spanning at most four steps modulo 22, of length $4\pi/11$. A multiplicative Gauss sum of modulus $\sqrt{23}$ consequently gives a Fourier coefficient of modulus at least $21\sqrt{23}\cos(2\pi/11)$. On the other hand $\sqrt{D}/L_{XY} \leq 161$. Since $\cos(2\pi/11) > 5/6$ and $35\sqrt{23} > 161$, the single-entry bound in Proposition 24.15 excludes $n_X n_Y \geq 4$.

In particular one of the two character fields equals $\mathbb{Q}(\zeta_{23})$ and the other has relative degree at most three. Neither label admits wild inertia at $5, 7$ or 23. At most one of the labels can have a twist divisible by seven, since a label with seven-active twist has at least three distinct seven-inertia conjugates by the square action, all fixing F. If such a label occurs, its seven-part has order exactly seven, its field has relative degree three, and the other label has field F.

More explicitly, the relative field of nontrivial degree can only be obtained by adjoining to F one of

$$\sqrt{5}, \quad \sqrt{-7}, \quad \sqrt{-35}, \quad \zeta_7 + \zeta_7^{-1}.$$

Indeed after excluding wild inertia the ambient field is $\mathbb{Q}(\zeta_5, \zeta_7, \zeta_{23})$. A subfield containing the last factor splits as its compositum with a subfield of $\mathbb{Q}(\zeta_5, \zeta_7)$, by the product description of the Galois group. The trivial, quadratic and cubic subfields of this latter field give the list. The square Galois action now shows that both original twists divide $5 \cdot 7 \cdot 23$, that at least one has order dividing 23, and that neither of them has twist order divisible by 35. In fact any five- or seven-active twist forces an extra inertia image of order at least two or three, respectively, and activity at both forces their relatively prime orbit sizes to divide the same relative degree.

If five and seven are separated. Each of P_5 and P_7 contains a twist-one orbit of length at least 22 outside the vacuum. In P_5 its row field contains $\mathbb{Q}(\zeta_{23})^+$; in P_7 it contains $\mathbb{Q}(\zeta_{23})$. In fact take a seven-free $U \in P_5$ in the pure five-twist fiber hitting a joint five- and 23-active column, and a five-free $V \in P_7$ in the pure seven-twist fiber hitting a joint seven- and 23-active column, by Proposition 4.2. Their degrees are $5^i 23, 7^j 23$, respectively, $i, j \in \{1, 2\}$, by Lemma 4.14 and defect-zero vanishing at 23. Since U lacks full seven-height it cannot be five-active; likewise V cannot be seven-active. Thus both twists equal one.

Take $E \in P_{23}$ of degree $49 \cdot 23$ and $C \in P_{23}$ of degree $25 \cdot 23$ meeting the corresponding joint columns as above. The normalized crossing of U, E has weight 23, positive residue 23 at five by U-principality ($\chi_U(E) = 49z$), and residue zero at 23 by E-principality ($\chi_E(U) = 5^i z$). The odd prime-weight result in Lemma 8.4 gives a common five-part η on the actual monodromy, and

an r-word $h + 5\sum_{j=1}^{m} \xi_j$ after its removal, where $r = 23$, $h + 5m = 23$, and $1 \le m \le 4$ counts nonidentity positions with multiplicity in the sum. Both inputs are five-inactive by separation (their seven-heights are zero and two). If $\eta \neq 1$ then every constituent is five-active and has degree divisible by 7^3, whereas $7^3 \nmid d_U d_E$. Thus $\eta = 1$. Uniqueness in the prime-weight normal form shows that the order of the tame stabilizer divides m, so the tame 23-inertia orbit of the entry has size 11 or 22. Since the value is in a pure 23-power field, it generates a field containing the unique degree-eleven subfield $\mathbb{Q}(\zeta_{23})^+$. The orbit of U has even size by duality.

At the crossing of V, C, the same calculation uses $\chi_V(C) = 25z$, $\chi_C(V) = 7^j z$ with primes seven and 23. Both inputs are seven-inactive by separation, and $125 \nmid d_V d_C$ excludes any nontrivial common seven-part in Lemma 8.4. The resulting pure 23-word is $2 + 7(\xi_1 + \xi_2 + \xi_3)$: at most three nonidentity terms occur and $W(23, 3)$ requires four distinct positions including the identity. Its tame stabilizer order divides three and 22, so is one. This pure cyclotomic value therefore generates a field containing $\mathbb{Q}(\zeta_{23})$.

In fact all three principal blocks have large twist-one orbits here. At V, C, Proposition 24.6 puts the three distinct nonidentity positions in μ_{23}. The row C has pure 23-twist by separation and its deficient five- and seven-heights. For a faithful character at 23, the Fourier coefficient of this crossing is a nonzero Gauss sum times the sum of three 22nd roots of unity. The latter cannot vanish: three unit vectors with sum zero differ by cube roots, and $3 \nmid 22$. Proposition 24.16 gives twist-one members of P_7, P_{23} in a common full five-block, both with character field containing $\mathbb{Q}(\zeta_{23})$. Any additional dyadic factor acts by scalars since the actual twists have odd order. Similarly the pure 23-word at U, E is moved by the order-eleven part of tame inertia. At least one of its nonzero Fourier components has order divisible by eleven, and its character has 23-power conductor. Here E has pure 23-twist by its deficient five- and seven-heights. The same proposition gives members of P_5, P_{23} in a common full seven-block with twist one and fields containing $\mathbb{Q}(\zeta_{23})^+$. The two choices of member in P_{23} may differ.

27.4 Residues of mixed words at five, seven and twenty-three

Apply Proposition 24.13 at the odd assignment $(5, 7, 23)$ without assuming that a particular edge is missing. Suppose no twist is active at all three primes. At a weight-25 crossing of a degree $25 \cdot 23$ label $C \in P_{23}$ and a degree $25 \cdot 7^j$ label $X \in P_7$, both five-heights are two. At a weight-23 crossing of the same C with a five-free $B \in P_7$ of degree $7^j 23$, the argument is identical. In both cases take $p = 5, T = \{7, 23\}$ at C and note that the opposite degree divided by the common weight is a five-unit. Write the 23-positions as the identity and three nonidentity positions as in Proposition 24.6. If the seven-phase η is nontrivial, apply seven-inversion and any 23-dilation carrying one occupied nonzero position to a vacant one. At weight 23 the coefficient of η at the occupied position is seven. At weight 25, at most two of the letters have seven-phase one, so at least one of the three nonidentity positions has all seven letters of phase η. Use that position. The mixed difference has zero constant coefficient, and at the image position its coefficient is, up to sign, $7(\eta - \eta^{-1})$, a unit modulo five. It cannot be zero modulo $5\mathcal{O}$: the only relation on the 23 positions over the seven-primary integers, also after reduction modulo five, is the constant coefficient vector. Thus $\eta = 1$ in both cases. At the weight-25 crossing X is seven-inactive as well: if it were seven-active the singleton test with $p = 5, T = \{23\}$ at X would make all 23-dilations of that pure word congruent modulo five, which the same occupied and vacant positions disprove.

At $(5, 7, 23)$ the row C of degree $25 \cdot 23$ in either crossing just considered forces 22 distinct full five-blocks, each entirely seven-inactive in the absence of a joint twist. Its reduced character at the opposite label is a nonzero scalar modulo five times a pure four-position 23-word. The three marked nonidentity exponents have common coefficient $7 \not\equiv 0 \pmod 5$. The polynomial of a primitive 23rd root is irreducible modulo five since 5 has multiplicative order 22 modulo 23. A dilation stabilizing the reduction must preserve the three nonzero marked positions, by the constant-vector relation and the unchanged identity coefficient. Its order divides both three and 22, hence it is the identity. All 22 reduced values are distinct.

27.5 An orientation for two separated odd primes

Write P_s for the principal s-block. A positive expression of weight g is a sum of roots of unity with nonnegative rational coefficients adding to g. The balancing expression for $z = S_{UV}/\operatorname{lcm}(d_U, d_V)$ has weight $g = \gcd(d_U, d_V)$, and all conjugates satisfy $|\sigma z| < g$ at a pair of nonunits in a minimal table. Congruences above a prime are understood at all primes above it.

Lemma 27.4. *Suppose a cyclotomic integer z has a positive expression of integer weight g in roots supported at primes $\geq \ell$, and all its conjugates have modulus strictly below g. If $z \equiv g$ above the two distinct primes s, t and $st > g\ell/(\ell - 1)$, then $\mathbb{Q}(z)$ ramifies at s or t.*

Proof. If both primes are unramified then $(z - g)/(st)$ is integral. Since $st > g$, the number $1 + (z - g)/(st)$ is a convex combination of 1 and z/g with both coefficients positive. Its house is strictly below one, so its norm forces it to vanish. This gives $z = g - st < -g/(\ell - 1)$. The normalized trace of a root of order n is $\mu(n)/\varphi(n) \geq -1/(\ell - 1)$ under the stated prime restriction. Averaging the positive expression gives a contradiction. □

Lemma 27.5. *Let s, t, r be distinct odd primes and suppose z is a cyclotomic integer with a positive expression of weight r in roots supported on these primes. Assume $z \equiv r$ above s, t and $\mathbb{Q}(z)$ is unramified at s. Write $r = sm + c$, $0 < c < s$, and $L = \lfloor (r + 2c)/s \rfloor$. Suppose*

$$L < t, \qquad L + 1 < r, \qquad L < 2(t - m), \qquad L < m + 3.$$

Then $z = c + s\sum_{j=1}^{m} \eta_j$ for t-power roots η_j. This expression gives the multiplicities in every positive expression of weight r on the three primes provided

$$m + 1 < t, \qquad \frac{m + 1}{t - m - 1} < \min(s - 1, r - 1).$$

Proof. In a cyclotomic field of sufficiently divisible conductor on s, t, r, the fixed field of s-inertia is the cyclotomic factor on t, r. Thus $\alpha = (z - c)/s$ is integral in that factor. Average the positive expression over s-inertia. An s-power root contributes 1, $-1/(s - 1)$ or 0, according as its order is $1, s$ or larger. Give positive letters cost one and negative letters cost two. Subtracting c and dividing by s expresses α with rational signed cost at most $(r + 2c)/s$. The signed two-prime cyclotomic configuration has integral cost minimizers for integral values (Theorem 4.13 and Proposition 24.4). Choose an integral word of cost at most L.

Erase the t-parts of its letters, obtaining an r-primary word v. Reduction above t gives $v \equiv \alpha \equiv m$. At ambient r-power order r^e ($e \geq 1$), the relations modulo t are generated by the full μ_r-coset sums. Indeed reduction of $\Phi_{r^e}(x) = \sum_{j=0}^{r-1} x^{jr^{e-1}}$ gives precisely these relations; here the congruence holds at all primes over t and t is unramified. The coefficient vector of $v - m$ has support at most $L + 1 < r$. Thus each coefficient vanishes modulo t. Away from 1 the absolute values are at most $L < t$, so those coefficients vanish in $\mathbb{Z}$. The integer v lies between $-L/2$ and L and is congruent to m modulo t. The hypotheses give $v = m$. Each negative letter would add three to the cost above the net augmentation m, whereas $L < m + 3$. Hence all m letters are positive and have trivial r-part.

For the last claim mark 1 and the η_j in the t-primary factor. In each μ_t-coset with b marks assign the value 1 at the marks and $-b/(t - b)$ elsewhere. The coset sums vanish, so this defines a rational-linear functional. Tensor it with normalized trace on the s, r factors. This trace on a nonidentity root is strictly below one, and its negative values have magnitude at most $1/\min(s - 1, r - 1)$. By the last inequality the product functional attains one only at marked pure t-roots. Its value at z is r, forcing every positive expression of weight r onto those marks. They are linearly independent, since no full μ_t-coset is marked. This identifies their multiplicities. □

Proof of Proposition 27.2. The degrees are $5^i 23^a$ and $7^j 23^b$, respectively, with $i, j, a, b \in \{1, 2\}$ by square divisibility, positive defect in the principal block and the exclusion of prime-power degrees.

By Section 27.3 both sets contain twist-one witnesses at 23-height one, from the respective joint columns at five and 23, or seven and 23. Whenever $\min(a,b)=1$, the common crossing weight is 23. Principality gives $z \equiv 23$ above five and seven, since the multipliers $d_V/23$ and $d_U/23$ in $\chi_U(V)$ and $\chi_V(U)$ are units at five and seven, respectively. Lemma 27.4 applies since $35 > 23 \cdot 5/4$. Each row field contains the entry field, and separation prevents a row field from ramifying at both endpoints (Proposition 6.3). Pair the two height-one witnesses, then pair arbitrary members of either set with the opposite height-one witness. This forces the common t, which ramifies also in every entry field with $\min(a,b)=1$.

For $(s,t)=(7,5)$ the parameters (m,c,L) of Lemma 27.5 are $(3,2,3)$, and for $(s,t)=(5,7)$ they are $(4,3,5)$. Both satisfy its hypotheses, including exposure since $4<6$ and $5/2<4$, respectively. Strictness forces some $\eta_j \neq 1$. At a nonidentity phase the total dimension of the corresponding summands of $U \otimes V$ is $sm_x \operatorname{lcm}(d_U, d_V)$ for $1 \le m_x \le m < s$. Every t-active label has full s-height three by separation (Lemma 3.4). Both inputs have smaller s-height, so their t-twists are trivial. The packet therefore consists of t-active summands. Divisibility by s^3 proves $v_s(\operatorname{lcm}(d_U, d_V)) \ge 2$. Pairing again with the opposite height-one witness gives the degree assertion for each row.

Suppose an s-active H has 23-height two. Its degree is $t^3 s^k 23^2$, $0 \le k \le 2$, and its field lies in an s-power cyclotomic field by the codegree conductor bound. If $k>0$, choose a t-free principal s-row Y hitting H by Proposition 4.2. Its field is unramified at s, so S_{YH}, lying in both fields, is rational. Principality gives $s \mid \chi_Y(H)$, and Y has s-height two. By lcm divisibility, S_{YH} is then a nonzero integer multiple of $5^3 7^3 23^2 = \sqrt{D}$, impossible in its row norm. If $k=0$, choose the canonical 23-height-one Y in the same block. Principality makes it hit every conjugate of H, with rational entries divisible by $\sqrt{D}/s$. The orbit of Y has size at least 22 (Section 27.3). The orbit of H has size at least $s-1$ by the square action on its s-twist and free duality on each twist fiber. The averaged Y-row has squared norm $D/|\operatorname{Gal}\cdot Y|$ and agrees with Y on this entire orbit of H. This would give $22(s-1) < s^2$, impossible for $s=5,7$.

For the two chosen twist-one witnesses Section 27.3 gives $\mathbb{Q}(\zeta_{23}) \subseteq K_V$ and $\mathbb{Q}(\zeta_{23})^+ \subseteq K_U$. Their common nontrivial entry field F is t-primary by Lemma 27.5, hence disjoint from $\mathbb{Q}(\zeta_{23})$. This gives $[K_V:\mathbb{Q}] \ge 44$. If $t=5$ and F is real, then K_U contains the real compositum $F\mathbb{Q}(\zeta_{23})^+$, and free duality makes K_U nonreal, so its degree is at least 44. If $t=5$ and F is nonreal, its degree is divisible by four: the involution in the cyclic group of order $4 \cdot 5^h$ acts nontrivially on F. The compositum then already has degree at least 44. □

For the chosen height-one U, V, let $F=\mathbb{Q}(z)$ as in the proof. The t-inertia conjugates of each label occupy at least $[F:\mathbb{Q}]$ distinct nonprincipal 23-blocks. Indeed the opposite character value is a 23-unit rational factor times z. If the tested row were principal at 23, its opposite value would reduce to zero. Since F is unramified there, $z/23$ would be integral and of strict house less than one, whereas $z \equiv 23$ above s. If two conjugates have the same 23-block, evaluate on the entire Galois orbit of the opposite label. All conjugates of the difference of their z-values vanish above 23, and that difference belongs to $23\mathcal{O}_F$. In the t-primary words of Lemma 27.5, the coefficient differences after cancelling s range between $-m$ and m. Their residues are constant on each μ_t-coset by the cyclotomic relation modulo 23. Since $2m<23$, the integer differences themselves are constant there, giving equality of the two z-values. Finally t-inertia realizes every conjugate of F.

The complementary seven-blocks. On the separated five–seven path at $5^6 7^6 23^4$, Section 27.4 puts degree-$25 \cdot 23$ members of P_{23} in 22 distinct five-blocks, all seven-inactive. In the other direction, at least eleven distinct nonprincipal seven-blocks contain a degree-$49 \cdot 23$ member of P_{23}, and all their members are five-inactive.

Indeed Section 27.3 gives $E \in P_{23}$ of degree $49 \cdot 23$ and $U \in P_5$ of degree $5^i 23$, with $\chi_E(U) = 5^i(h + 5\sum_{j=1}^{m} \xi_j)$, where $1 \le m \le 4$ and the ξ_j are nonidentity 23-power roots, counted with multiplicity. If two tame 23-conjugates of E lie in the same seven-block, evaluate on all

conjugates of U. All conjugates of the difference of the two m-term sums vanish above seven, so the difference belongs to $7\mathbb{Z}[\zeta_{23^a}]$ at a common conductor 23^a. By the formula for Φ_{23^a} its coefficient vector is constant modulo seven on each coset of μ_{23}. Each coset has a vacancy in the combined supports, and the coefficients have absolute value at most four. Thus the multisets coincide. Their tame stabilizer has order dividing both m and 22, giving at least eleven blocks. The principal seven-block is Galois-stable, so none is principal. Each contains a positive-seven-defect label whose residue character is moved by 23-inertia. Since separation forbids a joint $5, 7, 23$ twist, Section 24.7 makes the whole block five-inactive.

For the labels U, V of Proposition 27.2, their entire 23-blocks are s-inactive. At height two the block is a singleton, and the proposition applies directly. At height one pair with an opposite height-one witness. Some t-inertia conjugate changes the t-word; after removing the common factor s, the difference has house at most $2m < 23$. It therefore remains nonzero modulo 23 in its unramified field. The character multiplier at each endpoint is a 23-unit, so its residue character moves (evaluate on all conjugates of the opposite witness). The same full-block criterion applies.

27.6 The two active twists on the separated odd path

Keep the separated path and the notation (s, t, r) from Proposition 27.2. A complementary principal s-row is a t-free nonunit of P_s.

Proposition 27.6. *Every label jointly active at s, r has degree divisible by s.*

Proof. A joint label H of s-free degree must have $d_H = t^3 r$: separation forces full t-height, activity precludes full r-height, and a nonunit cannot have prime-power degree. Fix the height-one witness $Y \in P_s$ of degree $s^2 r$ from the orientation. Its field is unramified at s and ramified at t, and its orbit has size at least 22 (Section 27.3 and Section 27.5). It hits every such H by principality. Write $S_{YH} = t^3 s^2 r z_H$. The cyclotomic integer z_H is unramified at s. Since $\chi_Y(H) = t^3 z_H \equiv t^3 r$ at every prime above s, $z_H - c \in s\mathcal{O}$ for the least absolute residue $c = \pm 2$ of 23 modulo s. If z_H is rational then $|z_H| \geq 2$. Otherwise $(z_H - c)/s$ is a nonzero algebraic integer, so the triangle inequality for the Galois mean-square norm gives $M(z_H) \geq (s-2)^2$.

All conjugate Y-rows agree on the rational entries, while at least two t-inertia conjugates agree on the remaining H since $t \nmid f_H$. Average the former set of rows on the rational coordinates and the latter on the other coordinates, and take Galois means. The two sets are Galois-stable. The number N_0 of these labels is consequently bounded by

$$N_0 < s^2 r^2 \left(\frac{1}{22 \cdot 4} + \frac{1}{2(s-2)^2} \right). \tag{27.1}$$

Here $D/(t^3 s^2 r)^2 = s^2 r^2$ and the vacuum gives strictness.

For a joint Galois orbit with primary orders s^u, r^v its size is $A(s-1)(r-1)s^{u-1}r^{v-1}/2$ with A a positive integer: the independent square actions are transitive on the possible twist pairs, and duality acts freely on each fiber. Set $k_p(1) = 1$ and $k_p(j) = (p-1)p^{j-2}$ for $j > 1$. The sum of $c_s(H)c_r(H)$ on that orbit, with c_p from the simultaneous Gauss congruence in Proposition 26.9, is $Asrk_s(u)k_r(v)/2$. Write B_0 for the sum of $Ak_s(u)k_r(v)$ on the s-free joint orbits. The congruence $s^3 \mid L_{s,r} = \sum_H d_H^2 c_s(H)c_r(H)$ applies since $s^3 \mid \sqrt{D}$. After multiplication by 2, each positive s-height orbit contributes a multiple of s^3. The remaining sum is $(t^3 r)^2 srB_0$, hence $s^2 \mid B_0$. Comparing the orbit size with $Ak_s(u)k_r(v)$ gives $N_0 \geq (s-1)(r-1)B_0/2$. A nonzero N_0 would therefore be at least 1100 at $s = 5$ or 3234 at $s = 7$; (27.1) gives respectively $N_0 < 886$ and $N_0 < 814$. □

Proposition 27.7. *Every label jointly active at s, r has twist of order exactly sr.*

Proof. By Proposition 27.6 write $d_J = t^3 s^h r^k$ for such a joint label, with $h = 1, 2$ and $k = 0, 1$. The degrees follow from separation and activity. By Proposition 4.2 choose a complementary

principal s-row Y hitting J with pure s-primary twist. Its twist is one, because s-activity would require full t-height. It has $d_Y = s^2 r$: the s-height is two by Section 27.5, and full r-height would miss J. Put $n = \sqrt{D}$ and $S_{YJ} = (n/r)b$. Then b is a nonzero algebraic integer. Indeed, $\chi_Y(J)$ is divisible by s because $Y \in P_s$, $s \mid d_J$ and K_Y is unramified at s; lcm divisibility supplies the other factors. The field of J is unramified at t and that of Y at s, so b belongs to an r-power cyclotomic field.

Suppose the primary twists of J have orders s^u, r^v. Its orbit has at least $(s-1)(r-1)s^{u-1}r^{v-1}/2$ labels: use independent square actions on the two primary twists and the distinct duals in each fiber. At least two t-inertia conjugates of Y agree on this whole orbit. Their average has squared row norm at most $D/2$, with a positive vacuum contribution. Writing $M(b)$ for the mean of $|b|^2$ over embeddings, this gives

$$(s-1)(r-1)s^{u-1}r^{v-1}M(b) < r^2.$$

Since $M(b) \geq 1$, the only exception to $u = v = 1$ could be $(s,t,u,v) = (5,7,2,1)$. In that case $|\operatorname{Orb}(J)| \geq 220$, and every nonzero b satisfies $M(b) < 529/440$.

Since $529/440 < 2(r-2)/(r-1)$, the prime-power mean-square bound in the proof of Proposition 24.4 makes b a signed r-power root.

For every hit Y here K_Y contains $\mathbb{Q}(\zeta_{23})^+$: pair Y with the degree-$49 \cdot 23$ row of P_{23} in Section 27.3. The proof there applies to every such Y, since both inputs are five-inactive and the nontrivial five-part would force $7^3 \mid d_Y d_E$. The remaining pure 23-word has at most four nonidentity letters after removing its common factor five, so its tame orbit has size at least eleven. If b were rational, the entire Y-orbit would agree on the J-orbit, giving $11 \cdot 220 < 529$. Nor can b have order divisible by 23^2, since then $[K_J : \mathbb{Q}] \geq 22 \cdot 23$, whereas $2|\operatorname{Orb}(J)| < 529$. Hence b is a signed primitive 23rd root. In particular K_Y contains $\mathbb{Q}(\zeta_{23})$ and also a nontrivial seven-primary field by Section 27.5. Its orbit size is therefore a multiple of 22 and at least 44.

Choose a faithful tame character ψ of $\mathbb{F}_{23}^\times$ and form its normalized orbit vector v on $\operatorname{Orb}(Y)$, giving the same coefficient on the fibers of restriction to $\mathbb{Q}(\zeta_{23})$. This is a unit vector of 23-torus type ψ supported at twist one. Let m_Y, m_J be the two orbit sizes. For each conjugate of J the Fourier coefficient of the normalized Weyl image Sv/n has modulus

$$\frac{\sqrt{m_Y}}{22 \cdot 23} \left| \sum_{a \in \mathbb{F}_{23}^\times} \psi(a)\zeta_{23}^a \right| = \frac{\sqrt{m_Y}}{22\sqrt{23}},$$

since b is a signed primitive 23rd root. Thus the J-orbit alone carries squared mass $m_Y m_J/(22^2 \cdot 23) \geq 20/23$. On the other hand its 23-twists occupy exactly eleven primitive phases by the Galois square action. The local Weyl formula in Section 26.10, also used in Proposition 28.3, bounds the mass of Sv/n on each one by $1/23$ because v is 23-inactive and $\psi^2 \neq 1$. This gives at most $11/23$, contradicting the lower bound. □

Proposition 27.8. *Suppose $s = 7$ and that J in Proposition 27.7 exists. Its orbit has size 66 and $(v_7(d_J), v_{23}(d_J)) = (1,1)$ or $(2,0)$. Precisely one orbit of five-free pure-twist labels in P_7 meets J. Each member Y is also in P_5, has orbit size 44, field $\mathbb{Q}(\zeta_{23}, \sqrt{5})$, and $S_{YJ}/(n/23)$ is a signed primitive 23rd root. For this Y and every seven-free nonunit $U \in P_5$, the lcm-normalized crossing is $9 + 7(\eta + \eta^{-1})$ with η of order five.*

Proof. Retain the notation b, m_Y, m_J from above. Now m_J is a multiple of 66, by the independent square actions and duality. Also m_Y is a multiple of 22 and at least 44: the pure prime-weight crossing with degree $25 \cdot 23$ in P_{23} puts $\mathbb{Q}(\zeta_{23})$ in K_Y, by the argument in Section 27.3, and the orientation gives a nontrivial five-primary subfield. Here $m_J < 529/2$, so $23 \nmid m_J$. Thus b belongs to $\mathbb{Q}(\zeta_{23})$. Write $b = c_0 + \sum_{a \neq 0} c_a \zeta_{23}^a$ with integer coefficients. For every character ψ of

$\mathbb{F}_{23}^{\times}$ with $\psi^2 \neq 1$, the same normalized orbit vector and local phase estimate just used give

$$m_Y m_J \left| \sum_{a \neq 0} c_a \psi(a) \right|^2 \leq 11 \cdot 22^2. \tag{27.2}$$

Indeed the orbit Fourier coefficient factors through the primitive Gauss sum, and all J-coordinates have the same modulus. This holds also for every Galois conjugate of the inner sum.

If $m_Y m_J > 11 \cdot 22^2$, all the indicated sums vanish by the algebraic norm. Then c_a is constant on squares and on nonsquares, so $b \in \mathbb{Q}(\sqrt{-23})$. A nonrational integer of that field has mean square at least six, whereas $M(b) < 529/132$. A rational nonzero b would give $m_Y m_J < 529$ by averaging the entire Y-orbit, also impossible. Thus $m_Y m_J \leq 5324$, forcing $m_J = 66$ and $m_Y = 44$: the only further multiple of 22 in range is 66, and 3 cannot divide m_Y since K_Y is supported on $5, 23$. Its two required subfields now give the asserted compositum.

At these orbit sizes the right hand side of (27.2) divided by $m_Y m_J$ is $11/6 < 2$. Any nonzero inner sum w is therefore a root of unity: $1 - w\overline{w}$ has all conjugates of modulus less than one. To read this condition, write $\mathbb{F}_{23}^{\times} = \{\pm a^j : 0 \leq j < 11\}$ with a of order eleven, and put $A_j = c_{a^j} + c_{-a^j}$, $B_j = c_{a^j} - c_{-a^j}$. The order-eleven Fourier coefficient of each vector is zero or a root of unity in $\mathbb{Q}(\zeta_{11})$. The only coefficient relation on the eleven positions is the constant relation. Thus each vector is constant apart from at most one change by ± 1. Since $A_j \equiv B_j \pmod 2$, any changes occur at the same index. Both constant would again place b in the quadratic field. Removing one signed term $\zeta_{23}^{\pm a^j}$ therefore leaves $\beta \in \mathcal{O}_{\mathbb{Q}(\sqrt{-23})}$.

Write $\beta = (x + y\sqrt{-23})/2$, $x \equiv y \pmod 2$. For a primitive 23rd root ζ, the quadratic Gauss sum gives

$$M(\beta \pm \zeta) \geq (x^2 + 23y^2)/4 + 1 - (|x| + 23|y|)/22.$$

For $y \neq 0$ this exceeds $529/132$ (use $|x| \geq 1$ when $|y| = 1$). Hence β is rational. The constant Fourier coefficient, by the two orbit norms, has absolute value at most $23/\sqrt{44 \cdot 66} < 1/2$. Since the mean of ζ is $-1/22$, this forces $\beta = 0$.

Choose the same faithful ψ for every such hit orbit. The resulting unit vectors are orthogonal, while their Weyl projections onto the J-orbit are collinear: that transitive orbit has at most one copy of their common full Galois character. Each projection has squared norm $44 \cdot 66/(22^2 \cdot 23) = 6/23$. For k' hit orbits a normalized linear combination therefore gives squared norm $6k'/23$ there, at most $11/23$ by the same phase estimate. Thus $k' = 1$.

Lemma 4.25 puts a member of the unique hit orbit in P_5 and hence puts all of it there. In its congruence the signed primitive roots over this orbit sum to ± 2. Since $d_Y = 7^2 23$ and $\chi_J(Y) = \pm 7^{3-h} 23^{1-k} \zeta$, it gives $2 \cdot 7^{5-h} 23^{2-k} \equiv \pm 1 \pmod 5$. Among $h = 1, 2$ and $k = 0, 1$ this leaves exactly the two asserted profiles.

Finally the pure five-word at Y, U in Section 27.5 is $2 + 7\sum_{j=1}^{3} \eta_j$. Its nontrivial field is contained in $\mathbb{Q}(\sqrt{5})$. The multiset of three roots is unique, since it has fewer than five positions in each defining coset. Every nonidentity orbit under the subgroup fixing $\sqrt{5}$ has size $2 \cdot 5^{e-1}$ at order 5^e. The three roots therefore consist of one identity and one inverse pair of order five. □

Corollary 27.9. *If $s = 7$ all jointly seven- and 23-active labels have the single degree $125 \cdot 7 \cdot 23$. Every seven-free nonunit of P_5 has degree $25 \cdot 23$, and every label of full 23-height has twist one.*

Proof. A seven-free nonunit $U \in P_5$ has degree $5^i 23^a$, $i, a \in \{1, 2\}$. Pair it with a 23-free nonunit $X \in P_7$ from Proposition 4.2. If $i = 1$, their lcm-normalized crossing z has weight five. Since $d_U/5$ is a seven-unit, X-principality gives $z \equiv 5$ above seven; K_U is unramified there by Section 27.5. The integral $1 + (z - 5)/7$ has strict house below one, so $z = -2$. The normalized trace of every root on $5, 7, 23$ is at least $-1/4$, giving a contradiction. Thus $i = 2$.

At least one joint exists by Section 27.3, hence so does its pure hit Y from Proposition 27.8. For every U above, the normalized entry there is $9 + 7(\eta + \eta^{-1})$ of Galois mean $11/2$. If $a = 2$,

then $n/\operatorname{lcm}(d_U, d_Y) = 35$. Averaging the 44 conjugate Y-rows and retaining just U would give $44(11/2)^2 \le 35^2$, impossible. Thus every such U has degree $25 \cdot 23$. Fix the particular twist-one U of Section 27.3. Its orbit has at least 44 members by Section 27.5. On all the pure Y hit orbits described above, the averaged U-row has the same normalized entry $11/2$. Now $n/\operatorname{lcm}(d_U, d_Y) = 805$, so the number b of these Y-orbits satisfies $44^2 b(11/2)^2 < 805^2$, hence $b \le 11$.

Each such Y meets at most one joint orbit of each 23-twist square class: its faithful orbit vector puts $6/23$ squared mass on each joint orbit it meets, whereas the eleven possible phases of one square class carry at most $11/23$. Thus there are at most 22 joint orbits. All have size 66 and exact twist order $7 \cdot 23$. Let A_{hk} count those of $7, 23$-degree heights (h, k). The only possibilities are $(1, 1)$ and $(2, 0)$. In $2L_{7,23}$ the two contributions together equal

$$7 \cdot 23 \cdot 5^6 (7^2 23^2 A_{11} + 7^4 A_{20}).$$

Since $23^2 \mid L_{7,23}$ by Proposition 26.9, we have $23 \mid A_{20}$. The orbit bound forces $A_{20} = 0$.

Finally suppose a five-active H has full 23-height. Its degree is $7^3 5^h 23^2$, $h \le 2$, and its field is unramified at 23. Choose a seven-free hit $U \in P_5$ of the now determined degree. Its 23-inertia orbit contains at least eleven labels by the pure word against degree $49 \cdot 23$ (Section 27.3). They agree on at least four conjugates of H, by its five-twist square action and duality. Since $\operatorname{lcm}(d_H, d_U) = n/5$, their averaged row would give $44 < 25$. Thus no such H exists; Section 27.5 excludes seven-activity at full 23-height, and full height itself excludes 23-activity. □

27.7 The unramified endpoint at seven in dimension $5^6 7^6 23^4$

Put $r = 23$, $n = \sqrt{D}$, and $m_Z = [K_Z : \mathbb{Q}]$. We first record the two estimates on pure seven-twist columns used at both possible r-heights. Throughout, a pure hit means a five-free $Y \in P_7$ with pure seven-primary twist and $S_{YH} \ne 0$. Such a Y has twist one by separation.

Fourier estimate on inactive orbits. Let Y, H be r-inactive, $22 \mid m_Y$, $m_Y \ge 44$, $6 \mid m_H$. If $S_{YH}/n = a + b/r$, with $a \in \mathbb{Q}$ and b an algebraic integer in an r-power cyclotomic field, then $b \in \mathbb{Z}$. Indeed, if b lies outside the quadratic field and has conductor r^c, Fourier inversion detects a character ψ of this conductor with ψ^2 of the same conductor (at $c = 1$ omit the trivial and quadratic characters). It factors through $\mathbb{Q}(b)$ and defines orbit vectors on both inputs. Writing $b = \sum_j d_j \zeta_{r^c}^j$ with integral d_j, the primitive Gauss formula of Proposition 24.15 gives their coefficient in absolute value as

$$\frac{\sqrt{m_Y m_H}\, r^{c/2}}{r\varphi(r^c)} \left| \sum_{(j,r)=1} d_j \overline{\psi(j)} \right|.$$

Proposition 24.20 bounds it by $r^{-c/2}$. Thus $m_Y m_H |\sum d_j \overline{\psi(j)}|^2 \le 484$ for every conjugate. The sum is a nonzero algebraic integer, whereas m_H has the additional divisor eleven if $c = 1$ or 23 if $c > 1$. This is impossible. In the quadratic case $b = (x + y\sqrt{-23})/2$, the orbit coefficient has modulus $\sqrt{m_Y m_H}|y|/(2\sqrt{23})$. Lemma 26.18 bounds it by $1/\sqrt{23}$, forcing $y = 0$.

Lemma 27.10 (Pure principal hits at seven)**.** *Suppose $(s, t, r) = (7, 5, 23)$. If H has pure seven-active twist and $7 \mid d_H$, each pure hit Y satisfies*

$$d_Y = 49r, \qquad S_{YH}/n = \pm 1/r, \qquad (m_Y, m_H) = (44, 6).$$

Thus $K_Y = \mathbb{Q}(\zeta_r, \sqrt{5})$ and the twist of H has order seven. The pure hits and the witnesses for joint columns in Proposition 27.8 together occupy at most eleven Y-orbits.

Proof. Separation, activity and Proposition 27.2 give $d_H = 125 \cdot 7^h r^k$, $h = 1, 2$, $k = 0, 1$, and $d_Y = 49r^a$, $a = 1, 2$. The lcm-normalized crossing z is divisible by seven: $\chi_Y(H) = 125z \equiv 0$ above seven and K_Y is unramified there. At $a = 2$ this would make S_{YH}/n a nonzero algebraic integer, exhausting the Galois-averaged norm. Thus $a = 1$, and $b = rS_{YH}/n$ is a nonzero cyclotomic integer, supported only at r since K_H is unramified at five. The crossing with degree $25r$ in P_r gives $\mathbb{Q}(\zeta_r) \subset K_Y$ (Section 27.3); ramification at five gives $m_Y \geq 44$. Seven-activity and duality give $6 \mid m_H$.

The preceding Fourier estimate makes b rational. The constant unit vector on the Y-orbit has squared mass $m_Y m_H b^2/529 < 1$ on the H-orbit after applying S/n, since the vacuum also contributes. As $3 \nmid m_Y$ (its conductor is supported at five and 23), this gives $b = \pm 1$ and $(m_Y, m_H) = (44, 6), (44, 12)$ or $(88, 6)$. The last two pairs cost $528/529$. Retain also the twist-one $U \in P_5$ of degree $25r$ and orbit size at least 44 from Corollary 27.9. At U, Y the word is $2 + 7\sum_{j=1}^{3} \eta_j$ in five-power roots (Lemma 27.5). By uniqueness wild five-inertia permutes the three letters, since $5 \nmid m_Y$, and hence fixes each. With t trivial letters the mean $(35t - 13)/4$ has absolute value at least $13/4$. The additional squared mass is at least $(44 \cdot 13/(4 \cdot 805))^2 > 1/529$, since $n/\operatorname{lcm}(d_Y, d_U) = 805$. Only $(44, 6)$ remains.

The square action on the seven-primary twist now gives exact order seven. The degree-44 field containing $\mathbb{Q}(\zeta_r)$ and ramified at five is $\mathbb{Q}(\zeta_r, \sqrt{5})$. The last argument of Proposition 27.8 thus identifies the three letters as one identity and an inverse pair. This gives mean $11/2$ against the fixed U both for every pure hit and for every joint-column witness of that proposition. By the averaged U-row their total number b' of orbits satisfies $44^2 b'(11/2)^2 < 805^2$, so $b' \leq 11$. □

Lemma 27.11 (Height of a seven-active label)**.** *In the orientation $(s, t, r) = (7, 5, 23)$ every seven-active H has $v_{23}(d_H) = 1$. At each such H there is a twist-one 23-free hit in $P_7 \cap P_{23}$, and W_7 has integral entries at every prime over 23.*

Proof. Full r-height is excluded by Proposition 27.2. If $r \nmid d_H$, Corollary 27.9 excludes a joint twist, so H is pure and $d_H = 125 \cdot 7^h$, $h = 1, 2$, by prime-power exclusion. A pure hit exists by Proposition 4.2. Its normalized crossing z is divisible by seven, whereas at $h = 1$ its house is strictly below seven. Thus $h = 2$.

By Lemma 27.10, each pure hit orbit costs $44 \cdot 6/529$ in the averaged H-row. At most two occur. Each one in P_5 contributes $\pm 44(49r)(7r) \equiv \pm 3 \pmod 5$ to the double-principal residue (Lemma 4.25). One cannot give -1 modulo five, so exactly two occur, both in P_5. Conversely each Y-orbit meets at most two such H-orbits by the same norm. The eleven-orbit bound gives at most eleven H-orbits. In L_7 each contributes $7(125 \cdot 49)^2$, and every r-bearing label contributes zero modulo r. Proposition 26.9 gives $r \mid L_7$, so the orbit count is divisible by 23 and must be zero. Now use the double-principal lemma with $(p, q) = (7, 23)$. Since $v_{23}(f_H) = 4 - 2 = 2$, Theorem 6.9 also gives the asserted integrality at every prime above 23. □

Theorem 27.12. *The separated five–seven path at $5^6 7^6 23^4$ cannot have the unramified endpoint $s = 7$. In any remaining separated table the complementary principal rows ramify at seven and are unramified at five.*

Proof. A seven-free seven-active H would have degree $125r$ by Lemma 27.11 and pure twist by Corollary 27.9. The orbit of 44 pure witnesses Y from a joint column meets every such H by seven-principality. Write $S_{YH}/n = z/(7r)$. As $K_Y = \mathbb{Q}(\zeta_r, \sqrt{5})$ and K_H is unramified at five, $z \in \mathcal{O}_{\mathbb{Q}(\zeta_r)}$. Principality gives $z \equiv r \equiv 2 \bmod 7$. Apply the inactive Fourier estimate with $a = 2/(7r)$, $b = (z - 2)/7$ to get $z \in \mathbb{Z}$, hence $|z| \geq 2$. The averaged Y-row bounds the total such labels by $N_0 < (7r)^2/(44 \cdot 4) < 148$.

For an orbit of exact seven-primary order 7^u write its size as $6 \cdot 7^{u-1} A$, $A \geq 1$, using the independent square action and duality. The orbit sum of c_7 is $7Ak(u)$ where $k(1) = 1$ and $k(u) = 6 \cdot 7^{u-2}$ for $u > 1$. In the singleton congruence $7^3 \mid L_7$ all seven-bearing orbits disappear

modulo 7^3. The remaining degrees equal $125r$, so 49 divides $\sum Ak(u)$ over the seven-free orbits. If nonempty this gives $N_0 \geq 6 \cdot 49 = 294$, a contradiction.

Every remaining seven-active degree is $125 \cdot 7^h r$, $h = 1, 2$. For pure twists Lemma 27.10 gives the double-principal contribution of each Y-orbit as $\pm 44 \cdot 49r \cdot 7^{3-h}$, congruent to ± 2 at $h = 1$ and ± 1 at $h = 2$ modulo five. At $h = 1$ at least two contributing orbits are needed and at $h = 2$ at least one.

Let A_h count the pure seven-active orbits of height h, and B the joint orbits. Each joint has height one and a pure witness by Proposition 27.8 and Corollary 27.9. For each of the at most eleven witness orbits, at most two pure orbits meet it by the row norm, and at most two joints by the primitive r-phase bound. Two pure orbits cost $528/529$ on its constant unit vector; a joint would cost another $44 \cdot 66/(22^2 \cdot 529) = 6/529$, since its signed primitive r-root has mean $\mp 1/22$. Thus the number of incident pure orbits plus half the incident joint count is at most two, giving

$$2A_1 + A_2 + B/2 \leq 22.$$

All seven-active twists have seven-component of exact order seven. Each of the two quadratic classes of primitive seventh phases occurs, by the quadratic Gauss balance of Lemma 26.10. At a fixed primitive phase α let P_α be its projector. Its squared-degree mass $M_\alpha = \sum_{\theta_{Y,7}=\alpha} d_Y^2$ is a rational integer. Since $sv_0 = (d_Y/n)_Y$ and the complementary Weyl is unitary and commutes with P_α, we have

$$\frac{M_\alpha}{D} = \|P_\alpha W_7 v_0\|^2 = \sum_{\theta_{Y,7}=\alpha} (W_7)_{Y0}\,\overline{(W_7)_{Y0}}.$$

Every summand is integral at all primes above $r = 23$ by the preceding all-primes bound and conjugation. The rational number M_α/D therefore belongs to $\mathbb{Z}_{(23)}$. Since $v_{23}(D) = 4$, we obtain $23^4 \mid M_\alpha$. Write $A_{h,\pm}, B_\pm$ for the orbit counts in the corresponding square class at seven. A pure orbit gives two labels over each of its three seventh phases, and a joint gives twenty-two over each of its three phases. Dividing the squared degree sum by $2(125 \cdot 7r)^2$ gives

$$529 \mid A_{1,\pm} + 49A_{2,\pm} + 11B_\pm.$$

Both quantities are positive, whence $A_1 + 49A_2 + 11B \geq 1058$. The joint quadratic balance at $7, 23$ from Lemma 26.10 gives equal mass at both values of the sign product. Every joint orbit has the same size and degree and one sign product, so the nonempty joint count B is even. But the incidence inequality gives

$$A_1 + 49A_2 + 11B \leq 1078 - 97A_1 - \tfrac{27}{2}B \leq 1051.$$

This contradiction proves the assertion. □

The opposite ramification orientation at this odd dimension, and the ordinary modular case at 5400^2, remain open.

Corollary 27.13. *Every non-WGT braided fusion category of dimension $D = 5^6 7^6 23^4$ has rank at least* 553. *The bound is* 1121 *if a twist order is divisible by $5 \cdot 7 \cdot 23$. In particular these bounds hold for every non-pointed simple integral modular category of dimension D.*

Proof. Every proper divisor of D divides one of $5^5 7^6 23^4$, $5^6 7^5 23^4$ or $5^6 7^6 23^3$. These three products are covered, respectively, by Theorems 8.3, 8.1 and 4.16. Lemma 2.1 therefore gives an ordinary simple perfect table, since D is odd. Conversely, a non-pointed simple integral modular category is non-WGT by the proof of Corollary 17.3.

Write S for the set of odd prime divisors of a twist order. Every singleton $S = \{p\}$ occurs. Indeed, the space on which all complementary primary twists are trivial contains the vacuum and is invariant under the local p-factor. If p were inactive there, its upper unipotent would act

trivially (the normalization is dyadic), as would the conjugate lower unipotent. They generate the local factor, so W_p would fix the vacuum, contrary to Lemma 3.4 since $p \mid f_{\mathbf{1}} = D$.

All three dimension valuations are even. In any nonempty stratum of fixed primary orders, each active p-coordinate admits at least $p-1$ phases with the other coordinates fixed (Proposition 19.1). Successively projecting away the coordinates gives at least $\prod_{p\in S}(p-1)$ distinct tuples. For $S \neq \varnothing$ each tuple occurs on a pair of distinct duals (Lemma 2.5). The three singleton supports therefore account for at least $2(4+6+22) = 64$ nonunits. Theorem 27.1 requires orders divisible by $5\cdot 23$ and $7\cdot 23$. Without a triple these give disjoint supports and at least $2(4\cdot 22+6\cdot 22) = 440$ further labels. A triple itself gives at least $2\cdot 4\cdot 6\cdot 22 = 1056$. Adding the vacuum gives the asserted triple bound.

In the remaining case the singleton and mixed supports account for 504 nonunits. If five and seven occur in a common twist, the support $\{5,7\}$ gives $2\cdot 4\cdot 6 = 48$ more labels. If they are separated, Proposition 27.2 gives an orbit of at least 44 nonunit labels of twist one. The rank is thus at least 553 or 549, respectively. Since every degree is odd, $R \equiv D \equiv 1 \pmod 8$, so both cases give $R \geq 553$. □

28 Degree and support restrictions at the first open dimension

28.1 Four disjoint orbits of witnesses

Say that a pair p, q occurs if pq divides some twist order. The pair $3, 5$ occurs and at least one of $2, 3$ and $2, 5$ occurs. We recall the following four disjoint sets, each of four labels of one degree.

- Complementary witnesses in P_3, P_5, of pure three- and five-primary twist, respectively, have degrees $a \in \{12, 24, 36, 72\}$ and $b \in \{20, 40\}$. Their weight-four crossing gives four distinct five-inertia conjugates of each (Theorem 26.15).
- At a jointly $3, 5$-active column, an odd witness in P_3 of pure ternary twist has degree $c = 15$ or 45. Its weight-five crossing gives four distinct five-inertia conjugates. The joint column itself has four such conjugates by Sections 24.4 and 26.4. Its degree d divides $360 = 2^3 3^2 5$, since both active primes are deficient in its degree (Lemma 3.4). The first three sets have pure twists and distinct degree types, so all four sets are disjoint.

28.2 Formal codegrees and the largest simple degree

Write $n = 5400$, $d_U = \mathrm{FPdim}(U)$, $b_U = S_{ZU}/d_Z$ for the character row at Z, and $f = f_Z = D/d_Z^2$. Recall that $d_U \mid n$, that every nonunit degree has at least two distinct prime factors, and that $S_{ZU}/\operatorname{lcm}(d_Z, d_U)$ is an algebraic integer (Section 2). Each nonunit row satisfies

$$b_0 = 1, \qquad \sum_U d_U b_U = 0, \qquad \sum_U |b_U|^2 = f, \qquad n = \left|\sum_U d_U b_U \theta_U\right|.$$

The last equality is the vacuum–Z entry of the modular relation $STS = p^+ T^{-1} S T^{-1}$, using $|p^+| = n$. Reducing the first orthogonality sum at each of $2, 3, 5$ gives a nonzero nonunit entry of degree prime to that prime. These three kinds of degrees are disjoint.

We give two details of the estimates on a row. Cyclotomic Galois conjugation of its entries permutes the argument U and preserves d_U. In an orbit O with $b_U \neq 0$, the mean of $|b_U|^2$ is at least one by the arithmetic–geometric mean inequality applied to its absolute norm. Thus $\sum_O |b_U| \leq \sum_O |b_U|^2$ by Cauchy–Schwarz, and the support has at most f labels. If a nonzero algebraic-integer value c in an orbit reduces to zero at a prime over p, its rational norm is divisible by p. Writing $t = [\mathbb{Q}(c) : \mathbb{Q}]$, its distinct conjugates have squared mass at least $tp^{2/t}$. For $p = 2$ this is at least 4, and for $p = 3$ at least 6, as follows by checking $t = 1, 2, 3$ and monotonicity

thereafter. For f a power of p, the row is supported in the principal p-block P_p (Theorem 2.11). On this block $v_p(d_U) \le v_p(n) - 1$ (Corollary 4.1) and $S_{UZ}/d_U \equiv d_Z$ at the chosen prime above p.

Codegree four is excluded by Lemma 2.6.

Theorem 20.8 excludes codegree nine in this ordinary fusion-simple perfect case.

Codegree sixteen. Here $d_Z = 1350$, and the support lies in P_2. For an even supported degree of two-height k, the number $b_U/2^{k-1}$ is integral by lcm divisibility. It has zero residue at two by two-principality of U: up to a dyadic unit it equals $\chi_U(Z)$. Its orbit costs at least $4 \cdot 4^{k-1}$. Consequently $k = 1$ at every even hit. In particular the three-free and five-free nonunit hits have degrees at most 50 and 54, respectively, and cost at least four each.

No two rows of degree 1350 have a nonzero mutual entry, including a row paired with itself. Otherwise symmetry and prime-power support put both in P_2. A row in P_2 meets every odd degree by principality. There are at least eight odd-degree labels: the unit and four odd conjugates described in Section 28.1 already give five, and their total number is divisible by four by reduction of $\sum d_U^2 = D$ modulo four. All odd positions therefore cost at least eight in total. The compulsory three-free and five-free nonunit orbits are even. Their additional cost exhausts sixteen before the proposed degree-1350 hit.

All remaining support degrees are at most 675. Indeed $k \le 1$, and the divisors of $2 \cdot 3^3 \cdot 5^2$ bigger than 675 consist only of 1350. In $\sum d_U |b_U|$, replace the sum of absolute values on each orbit by its squared mass, then reserve the unit and the two compulsory even orbits. This gives

$$n \le 675 \cdot 16 - (675 - 1) - 4(675 - 50) - 4(675 - 54) = 5142.$$

This contradiction excludes codegree sixteen. Codegree 25 is excluded by Proposition 5.4 at the quartic prime five. The degree 5400 would alone exhaust D. The only divisors of 5400 above 900 are $1080, 1350, 1800, 2700, 5400$, completing the degree bound.

28.3 Excluding a simple object of degree six

Theorem 28.1. *Neither an ordinary divisor-minimal table nor its positive reduced counterpart has a nonunit label of degree six. In particular every nonunit degree in either table is at least ten.*

Proof. Suppose $d_X = 6$ and write Δ for the table dimension, equal to the category dimension in the ordinary case and half of it in the reduced case. By Corollary 4.24 its prime support is $\{2, 3, 5\}$. By Lemma 4.14 a three-free nonunit in P_5 has five-height $j \ge 1$ with $2j + 2 \le v_5(\Delta)$. Thus in both tables $v_5(\Delta) \ge 4$.

Work in a common cyclotomic splitting field with Galois group G. Use prime avoidance at any five-active column in the full principal five-block, avoiding three. The wild-inertia argument of Section 26.4 gives a three-free nonunit U with K_U tame at three. Indeed at a five-active J the orthogonality relation $\sum_{Y \in P_5} d_Y \chi_J(Y) = 0$ holds in either table (Proposition 4.2). Wild ternary inertia preserves P_5 and permutes its arguments without changing their residues at three. Nonsingleton orbits contribute zero there. The vacuum term forces a singleton nonunit orbit with nonzero contribution. Its degree is $2^i 5^j$ with $i, j \ge 1$. The entry $z = S_{XU}/\operatorname{lcm}(6, d_U)$ has weight two and residue two above five, since $\chi_U(X) = 3z$. By Theorem 4.13, it is a sum of two five-power roots. They are distinct by strictness, and their ratio has order five by Theorem 24.1, applied to the full-weight word $3z$. Consequently $|\chi_U(X)|^2 = 9(2 + t)$, where $t = \zeta + \zeta^{-1}$ for a primitive fifth root ζ.

The unit occurs exactly once in XX^*, also in the positive quotient: for a lift X the fermion f does not occur in XX^* since $fX \ne X$. The other constituents have total degree 35. Each of these degrees is supported on $2, 3, 5$ and has at least two prime divisors. The only possible odd degree at most 35 is 15, occurring once. The remaining mass 20 can only be one degree 20 or two degrees 10. Write $XX^* = 1 + V + B$, with $d_V = 15$ and $d_B = 20$. Neither $v = \chi_U(V)$ nor $b = \chi_U(B)$ is rational. Indeed, if v is rational, $v/3$ is a rational integer of absolute value below

five by lcm integrality. Five-principality gives $v \equiv 0$ above five, whence $v = 0$. Now $b = 17 + 9t$ has a conjugate greater than 20. If b is rational, then $b \in \mathbb{Z}$, $|b| < 20$ and $b \equiv 0 \pmod 5$. Since $v = 17 + 9t - b \in 3\mathcal{O}$, where $\mathcal{O}$ is the ring of algebraic integers, $b \equiv 2 \pmod 3$. The possibilities $b = -10, 5$ both give a conjugate of v greater than 15.

For every $\sigma \in G$, the adjoints of X and X^σ share a nonunit (Theorem 24.1). If B is one simple or twice one simple, every σ fixes V or B. Their stabilizers are proper because v, b are irrational, and a group is never the union of two proper subgroups. Hence $B = W + W'$ with W, W' distinct of degree ten. No degree-ten simple can be fixed by G. Otherwise its character eigenvalues would be integers, the vacuum value 10 and all others strictly between -10 and 10. Applying $\prod_{k=-9}^{9}(T - k)$ to this simple in the fusion ring gives 19! times the vacuum idempotent, whose unit coefficient is $19!/\Delta$. This is impossible since $v_5(19!) = 3 < v_5(\Delta)$.

Suppose first that W, W' lie in different G-orbits. Then every σ fixes at least one of V, W, W'. If a finite abelian group is the union of three proper subgroups, they are the three subgroups of index two over their common intersection. To see this, each subgroup has an element in neither of the others. Adding such an element of the third subgroup to an element in the first two but outside the third would leave all three, so the three pairwise intersections coincide. Quotient by that intersection. Translation by a nonzero element of one subgroup embeds the nonzero part of another into that of the third. Consequently all have the same order n, and $n \mid 3n - 2$, so $n = 2$. Thus $K_V, K_W, K_{W'}$ are the distinct quadratic subfields of a biquadratic field. Their nonrational components span three independent rational lines. Since

$$v + \chi_U(W) + \chi_U(W') = 17 + 9t,$$

$\mathbb{Q}(\sqrt{5})$ is one of those subfields. If it is K_V, then b is rational; otherwise v is rational. Both alternatives were excluded.

It remains that W, W' lie in the same orbit. Identify the orbit with the additive group $A = \mathrm{Gal}(K_W/\mathbb{Q})$, taking W at 0 and W' at $s \neq 0$. Let $\pi : G \to A$ be restriction and H the stabilizer of V. For each $\sigma \notin H$, the translate by $\pi(\sigma)$ of $\{0, s\}$ meets $\{0, s\}$. Thus $\pi(G \setminus H) \subseteq \{0, s, -s\}$. If $\ker \pi \not\subset H$, every fiber has a point outside H, giving $|A| \leq 3$. Otherwise $H = \pi^{-1}(J)$ for a proper subgroup J and $A \setminus J \subseteq \{s, -s\}$, giving $|A| \leq 4$.

At $|A| = 2$ the pair sum is Galois-fixed. At $|A| = 3$ its character values at U are rational: K_W has conductor supported on $2, 3, 5$ by Corollary 2.8, and any such cubic field ramifies wildly at three. Indeed the dyadic and quinary cyclotomic factors have degrees prime to three, as does the tame ternary factor. Since K_U is tame at three, $K_W \cap K_U = \mathbb{Q}$. Both cases make b rational. Finally at $|A| = 4$, the complement of J must be the two different elements $s, -s$. Thus A is cyclic with generator s, and $F = K_V$ is its quadratic subfield. The displayed identity puts $t \in K_W$, hence $F = \mathbb{Q}(\sqrt{5})$ and $b \in F$. Write $\alpha = \chi_U(W)$ and γ for the generator corresponding to s. Galois covariance gives $b = \alpha + \gamma\alpha$. Since $\gamma^2 b = b$, the element $y = \alpha - \gamma^2\alpha$ satisfies $\gamma y = -y$ and $\gamma^2 y = -y$. These imply $y = 0$. Thus $\alpha \in F$ and $b = \mathrm{Tr}_{F/\mathbb{Q}}(\alpha)$ is rational, the final contradiction. Every integer between two and nine other than six is a prime power, already excluded in Section 2. □

Rank at the even core. At $D = 5400^2$ the rank is at most 46774. Write A, B, C for the sets of nonunits whose degrees omit two, three, five, respectively. They are disjoint, since a nonunit degree cannot be a prime power. For $X \in P_p$ the nonzero entries on all p-free nonunits cost at least $(d_Y / \gcd(d_X, d_Y))^2$ per label in the row norm D/d_X^2, after averaging on Galois orbits: division by this scale gives a nonzero algebraic integer of mean squared conjugate modulus at least one. The vacuum costs one separately.

Avoid five at a two-active column to obtain a five-free $X \in P_2$. Its degree is $2^i 3^j \geq 12$ by Theorem 28.1. The scale on A is at least five. If $d_X = 12$, the known degree-45 member of P_5 has scale 15 and costs an extra 200 above the baseline 25. Hence $25|A| + 200 \leq D/12^2 - 1$, giving $|A| \leq 8091$. If $d_X > 12$, then $d_X \geq 18$ and $|A| \leq 3599$. Since all degrees in A are at least 15, the

degree-45 witness gives

$$\sum_{Y\in A}(900-d_Y^2)\le 675\cdot 8091-1800.$$

In P_3 a five-free witness has degree at least 12, and gives scale at least five on B, so $|B|\le 8099$. All its degrees are at least 10, and the complementary P_5 quartet has degree 20 or 40. Thus the analogous sum over B is at most $800\cdot 8099-4\cdot 300$.

Use both the degree-45 row and a row of the complementary quartet of common degree $b=20$ or 40 in P_5 (Section 28.1). Put $s_m(d)=d/\gcd(d,m)$. For every five-free nonunit degree d,

$$35(900-d^2)\le 1557s_{45}(d)^2+172s_b(d)^2.$$

Indeed, the only such degrees below 30 are $12,18,24$. The pairs of squared scales are $(16,9),(4,81),(64,s_b(24)^2)$, respectively. The coefficients give equality in the first two cases, and the first term alone suffices in the third. Summing the two row-norm bounds gives

$$\sum_{Y\in C}(900-d_Y^2)\le\frac{1557(14400-1)+172(72900-1)}{35}=\frac{34957871}{35}.$$

All remaining nonunits have degree at least 30. Hence

$$\text{rank}\le 1+\left\lfloor\frac{D-1+5459625+6478000+34957871/35}{900}\right\rfloor=46774.$$

The degree-45 witness is supplied by Section 26.4.

28.4 The four primitive fifth phases on the path separating two and three

Lemma 28.2 (Fourier test for the pure fifth phases)**.** *In the ordinary minimal table of dimension 5400^2, suppose no twist order is divisible by six. If every degree-20 label of P_5 has primitive fifth twist, every jointly two–five-active column K meets a degree-20 label at each of the four primitive fifth phases.*

Proof. Write $\alpha=\theta_{K,5}$. Let E be the intersection of P_5 with the fiber with trivial complementary twists. In the local modular representation write $s=W_5W'$, $v=W'v_0$, and $w=W'v_K$. At five, v is fixed by the upper unipotents and the torus, and w has upper-unipotent phase α. The coordinate projector P_E commutes with this local group. For every unit u modulo the five-part of the level we have

$$H(u):=\sum_{Y\in E}d_Y\chi_K(Y)\theta_{Y,5}^u=C\alpha^{-u^{-1}}.$$

In fact $H(u)$ is D/d_K times $\langle v,P_EW_5^{-1}U(u)W_5w\rangle$. In the lower-unipotent Bruhat decomposition the left upper-triangular factor fixes v, and the right upper unipotent has parameter $-u^{-1}$. This gives the formula with a constant C independent of u; in particular $C=\alpha H(1)$ is integral. Also $H(0)=0$ by restricted block orthogonality.

Reduce at a prime above three. Every three-free nonunit in P_5 has degree 20 or 40 by Section 26.3; degree 40 misses K by full dyadic height. By hypothesis the degree-20 labels of P_5 have primitive fifth twist. Hence $\overline{H}$ is the discrete Fourier transform of masses on μ_5, with mass one at the trivial phase (only the vacuum contributes there). It is periodic modulo five and nonzero by Fourier inversion. Since $H(0)=0$, the constant C has nonzero residue. Comparing $u=1$ and $u=6$ gives $\alpha^5=1$, since reduction is injective on five-power roots. Thus α is primitive of order five. Fourier inversion at phase one gives $\overline{C}=-5$. At every other fifth phase δ, the mass is consequently

$$-\sum_{u\in\mathbb{F}_5^\times}\overline{\alpha^{-u^{-1}}\delta^{-u}}.$$

This sum is nonzero: Φ_5 is irreducible over $\mathbb{F}_3$, so a vanishing sum of four fifth roots would have all five multiplicities congruent modulo three. Their sum is four, so each would be at least two. Thus K meets degree-20 labels at each primitive phase. □

28.5 Counting active twists by a local torus character

Work in ordinary integral modular data of square dimension with unsigned cyclotomic tori. Let E be a coordinate subspace invariant under the local modular group G_p, and let H_p be its diagonal torus. Examples include the whole table and a full p-block restricted to specified complementary twist phases: the upper unipotent preserves the coordinates and the local Weyl preserves both conditions (Lemma 3.3 and Section 26.6).

Proposition 28.3. *Let $p \geq 5$ and let $L \subset E$ be an m-dimensional space supported on p-inactive labels, of torus type obtained by inflating a character ψ of $\mathbb{F}_p^\times$ (hence trivial on the wild subgroup $1 + p\mathbb{Z}/p^a\mathbb{Z}$ at level p^a), with $\psi^2 \neq 1$, $\psi(-1) = -1$. If the projector to local level at most p has rank r on L, then E contains at least $2(p-1)r$ labels of twist order exactly p at p, and at least $p(p-1)(m-r)$ of higher p-power order. They may be selected from contributing torus orbits, each of size divisible by the order of ψ. In particular, at five, m disjoint inactive orbits of size divisible by four force $8m$ active labels in equal-degree quartets within E.*

Proof. Write Q_1 for averaging over the kernel of reduction modulo p. For $v \in L$ the squared norm of the α-phase projection of $W_p v$ is $\|Q_1 v\|^2/p$ for every $\alpha \in \mu_p$ (Lemma 26.18). Each such projection has rank r. The unsigned involution $H_p(-1)$ fixes each phase and acts by minus one on the image, requiring two labels per independent vector. Sum over the $p-1$ primitive phases.

The Weyl image of the $(m-r)$-dimensional kernel is supported at higher orders. Torus type ψ^{-1} restricts to a space of dimension at most one on each orbit, with stabilizer in $\ker\psi$ whenever the restriction is nonzero. Its support thus meets at least $m-r$ orbits. At order p^a, the square action gives $p^{a-1}(p-1)/2$ phases, each containing at least two positions by oddness. This proves the second bound. At five each specified inactive orbit affords the same faithful tame quartic character. The contributing active orbits have constant degree and size divisible by four, giving the stated selection. □

Compulsory active orbits. Consider the four disjoint quartets in Section 28.1. The complementary rows $X \in P_3, U \in P_5$ have degrees $a \in \{12, 24, 36, 72\}$ and $b \in \{20, 40\}$, respectively. An odd $V \in P_3$ has degree $c \in \{15, 45\}$, and a joint three–five label J has degree dividing 360. Each has a local five-inertia orbit of size divisible by four, by Section 26.6 and Sections 24.4 and 26.4. The selected X, V have purely ternary twist and U purely quinary twist. For each five-inactive orbit of this kind the permutation representation of the cyclic group H_5 contains the same tame quartic character, since its orbit size is divisible by four. Distinct orbits thus count independently in Proposition 28.3.

Every five-active degree divides 1080 and is at most 540 by Section 28.2. We also use the five-inactive label Z of Section 26.6 in the full five-block of J, with the same dyadic and ternary twists. Its field contains $\mathbb{Q}(\zeta_5)$ or $\mathbb{Q}(\zeta_{15})^+$, giving a fourth-order five-inertia action and five-height at most one. If no triple occurs, one can take $Z \in P_3$; then its ternary height is at most two. If a triple occurs, take J triple-active and its ternary activity again bounds that height by two. In all cases $d_Z \mid 360$.

In the triangle X, U have twist one (Section 26.6). In the absence of a triple, that argument also supplies the odd V of twist one; Z is three-active and so distinct from all three quartets. With a triple, Z is both two- and three-active, while V has pure ternary twist. Thus X, U, V, Z give four distinct inactive quartets and at least 32 five-active labels. Their J quartet has degree at most 360.

If ten is missing, $(a, b, c) = (12, 40, 15)$ and the chosen U has twist one (Sections 26.9 and 26.10). Again there are at least 24 five-active labels. All have full two-height and three-bearing degree (Section 26.4); their degrees divide 1080, and the largest such degree below 1080 with full two-height is 360. There are also five-inactive odd labels of degrees $45, 75$ by Section 26.4 and Theorem 26.17. Counting the unit, these two and the four degree-15 labels gives seven odd labels; their total number is divisible by four, so there is another, of degree at most 675. At a

two-active column obtain a three-free nonunit $W \in P_2$ by prime avoidance; its degree divides 100 and is even, and it is five-inactive since it has deficient two-height. It is distinct from the degree-40 quartet by that same deficiency.

It remains that six is missing. Here $b = 20$. Every three-active label has full two-height by separation and is five-bearing because two and five occur together (Section 26.3). Hence the pure ternary X, V have twist one. The three-active Z is distinct from them and from U, and has full two-height and five-height one. In fact every degree-20 row of P_5 has pure quinary twist (its two- and three-heights are deficient) and full quartic five-inertia by its crossing with V. If even one is five-inactive, there are four inactive quartets and hence at least 32 active labels. Otherwise there are at least eight active degree-20 labels. Indeed a five-phase of order at least 25 gives an orbit of size divisible by 20, by the square action and the quartic inertia. If all their phases have order exactly five, apply Lemma 28.2: at a jointly two–five-active column the degree-40 rows miss by full two-height, so the only three-free terms are the vacuum and degree-20 rows at primitive fifth phases. That argument gives a hit at each primitive phase; its inertia involution supplies a second label of the same degree. Thus the three inactive quartets of X, V, Z force at least twenty-four five-active labels. A three-free witness $W \in P_2$ in the pure dyadic fiber has twist one: every two-active degree has full three-height by separation. Its degree distinguishes it from X, V, and its twist distinguishes it from Z. Take a joint two–five label K. Its full five-block contains a label T with the same dyadic twist and trivial five-twist, by removal of the primary twist (Proposition 24.12). Its ternary twist is trivial by separation. It has full three-height, deficient dyadic height, and five-height at most one: a full five-height label would be a singleton five-block and could not share that block with K. Thus $d_T \le 540$. Its nontrivial pure dyadic twist distinguishes it from the four witness orbits.

Now consider the full five-block of J at its exact ternary phase, denoted by E. Each label there has degree $40 \cdot 3^j$, $j = 0, 1, 2$: three-activity gives full two-height and a factor of five, and full five-height would make its block a singleton. If n_j count these three degrees then

$$n_0 + 9n_1 + 6n_2 \equiv 0 \pmod{25}.$$

Indeed the coordinate projector P_E commutes with W_5, so $\sum_{Y \in E} d_Y^2 = D\|P_E W_2 W_3 v_0\|^2$. On this nontrivial ternary phase W_2 is scalar: the dyadic upper and its conjugates are scalar there by separation. The operator W_3 is integral at every prime above five, by the enhanced Weyl estimate (Section 26.4), since every three-active degree has five-height at least one. Hence the sum is divisible by 5^4 and division by 40^2 gives the congruence.

Section 29.2 uses this congruence to give the size of each such fiber.

Intersections of principal blocks on the path missing ten. In the ordinary minimal table at $D = 5400^2$, suppose two and five are separated. Each five-active column has a degree-40 twist-one hit in $P_5 \cap P_3$, and each two-active column has a three-free twist-one hit in $P_2 \cap P_3$ of degree $10, 20, 50$ or 100. Indeed, in this graph three divides every two-active and every five-active degree (Section 26.4). Apply Lemma 4.25 with $q = 3$ and $p = 5, 2$, respectively. The three-free nonunits in P_5 have degree 40 by Section 26.3; the positive two-height in P_2 is at most two (Corollary 4.1), and a nonunit degree has at least two prime divisors. Both hits can be chosen tame at three: in the lemma's principal-block congruence group the terms by wild three-inertia. Every nonsingleton orbit contributes zero to the residue (Section 26.4). In particular the chosen degree-40 label has field $\mathbb{Q}(\zeta_5)$ or $\mathbb{Q}(\zeta_{15})$ (Section 26.10).

For each jointly three- and five-active column J, the same lemma at $(p, q) = (5, 2)$ gives a degree-45 twist-one hit in $P_5 \cap P_2$ whose field is unramified at two. In fact any odd nonunit $X \in P_5$ meeting this J must have degree $3^j 5$: it has positive ternary and five-height by the exclusion of prime-power degrees, and the five-height in P_5 is at most one (Corollary 4.1). The odd-prime-weight bound in Section 26.4 gives $j \ge 2$, while three-activity of J excludes $j = 3$ by defect-zero vanishing. Hence $j = 2$. Wild two-inertia fixes some contributing row in the resulting intersection by the same orbit argument (and the tame quotient at two is trivial).

For a jointly two- and three-active column the five-free hit from $(p, q) = (2, 5)$ lies in $P_2 \cap P_5$, has twist one, and has degree $12, 18$ or 36: the two-height and three-height are each at most two, and degree six is excluded by Section 28.3.

Four further principal labels when six is missing. At $D = 5400^2$ suppose two and three are separated. At any jointly three–five-active J, Lemma 4.25 gives an odd twist-one hit $V \in P_2 \cap P_3$. Its degree is $5 \cdot 3^j$, $j = 1, 2$: it is deficient at three by principality and at five since it meets J. At any jointly two–five-active column the reversed lemma gives a three-free twist-one $W \in P_2 \cap P_3$ of degree $2^i 5$, $i = 1, 2$, by the corresponding dyadic and quinary height bounds.

In fact $V \notin P_5$. Otherwise $z = S_{WV}/\operatorname{lcm}(d_W, d_V)$ has weight five and residue five above each of $2, 3, 5$. At two use $\chi_W(V) = 3^j z$ and $W \in P_2$; at three and five use $\chi_V(W) = 2^i z$ and $V \in P_3 \cap P_5$. The congruences hold at every conjugate prime. By Proposition 8.6, the residues at three and five and strictness leave just

$$z = \eta(2 + 3\xi) \quad \text{or} \quad z = \eta\left(1 - \sum_{t=1}^{4} \eta_t \zeta^t\right),$$

where η is ternary, ξ is a nonidentity five-power root, ζ is a primitive fifth root and $\{\eta_t\} = \{1, 1, \omega, \omega^2\}$ for a primitive cubic root ω. The first expression gives $\eta\xi = 1$ on reduction at two, impossible by injectivity on odd-order roots. For the second expression work at a common odd conductor. Two is unramified, so the congruences at every prime over two are equality modulo 2. The four powers ζ^t, $1 \le t \le 4$, are an integral basis over the ternary cyclotomic integer ring (coprime conductors), also after reduction. Substituting $1 = -\sum_{t=1}^{4} \zeta^t$ in the congruence shows that all η_t are equal modulo two, again impossible.

Consequently $P_2 \cap P_5$ contains at least four nonunit twist-one labels distinct from the V quartet. Indeed, Section 26.7 uses the four tame five-conjugates of V together with the vacuum as columns, and the analogous four degree-20 labels $U \in P_5$ together with the vacuum as rows. Their residue matrix at two is the invertible five-by-five Fourier matrix. All five columns now belong to P_2. The forward factorization in that section therefore requires five distinct twist-one labels in $P_2 \cap P_5$, including the vacuum. The X quartet is outside P_2 since $\chi_X(U) = 5(1 + 3\xi') \not\equiv d_U = 20$ above two, with ξ' primitive of order five. No three-free nonunit in P_5 belongs to P_2: its degree is 20 or 40, the latter has full dyadic height, and in degree 20 its value at V reduces to a primitive fifth root instead of the odd dimension residue. This separates the four new labels from W and U as well.

29 Twist fibers and rank at the first open core

At $D = 5400^2 = 29160000$, every nonunit degree divides 5400, lies between 10 and 900 and has at least two distinct prime factors (Section 28.3). A joint label is active at the indicated primes. Write m_z for the number of simples of degree z, R for rank, τ for normalized field trace and $\mathsf{M}(b) = \tau(b\bar{b})$.

Theorem 29.1. *A perfect fusion-simple integral modular category of dimension* 5400^2 *has rank* $101 \le R \le 46774$. *The last column gives the pages where the bounds for each twist configuration are proved.*

Twist orders	*Min. rank*	*Min. five-active*	*Rank without* 900	*Proof, p.*
None divisible by 6	141	60	≥ 141	*235*
None divisible by 10	101	40	≥ 127	*237*, *238*
All pairs, none divisible by 30	106	56	≥ 130	*236*
Some divisible by 30	112	56	≥ 142	*236*

At $R = 101$ at least 21 labels have degree 900, and at least 18 of their rows meet a label jointly active at two and three. At every rank $m_{675} \le 7$.

29.1 Degree caps used in the rank proof

For the remainder of this section let $\mathcal{C}$ be a perfect fusion-simple integral modular category with $D = 5400^2$. By Section 28.2 every simple degree divides 5400 and every nonunit degree is at most 900.

Corollary 29.2. *If no twist order is divisible by six at 5400^2, every simple degree is at most 675.*

Proof. A degree-900 row in Proposition 6.11 has $p = 2$, $q = 3$, $a = 2$, $b = 3$ and $f = 36$. A two-active degree is divisible by 27 and at most 900. Among divisors of 5400 with these constraints, the largest is 675. But $2 \cdot 675 \cdot 35 = 47250 < 48600 = 5400 \cdot 3^2$. The next possible degree after 900 is 675. □

Corollary 29.3. *At 5400^2, with at most three exceptions among all degree-900 simples, every such row has a nonzero entry on a label whose twist is jointly two- and three-active and five-inactive. Any exception has twist one and belongs to $P_2 \cap P_3$. It requires $\xi = \pm 1$, and the possible pairs $(k_{2,Z}, k_{3,Z})$ are $(3, -2\xi)$ and $(-3, 4\xi)$. In particular every orbit of at least four degree-900 simples meets the indicated joint sector at every member.*

Proof. A degree-900 row misses all five-active labels. If it also misses the indicated joint sector, the separated-sum equation $H_2 + H_3 = 5400\xi\theta_Z^{-1}$ holds for this row even without global separation. By Theorem 6.1, $3 \mid k_{2,Z}$ and $2 \mid k_{3,Z}$ since $f_Z = 36$. Neither integer can vanish, as the other would then have modulus one. Local Weyl vacuum columns belong to their respective principal blocks and have trivial twist at all complementary primes, by local commutation. This puts Z in both principal blocks at twist one. Each such label accounts for at least $9/36$ of the norm squared of $W_2 v_0$. Four labels would exhaust this norm on nonunit twist-one labels, impossible by the strict norm estimate following Proposition 6.11. Likewise $|k_{2,Z}| = 6$ is impossible as it would exhaust that norm on Z alone. Thus $k_{2,Z} = \pm 3$. Since $\gamma_3 = 1$, $\gamma_2^3 = \xi$ and $\theta_Z = 1$, the separated-sum equality becomes $k_{3,Z} = \xi(1 - k_{2,Z})$. This proves the assertions; the exceptional property is preserved by Galois. □

Proposition 29.4. *At 5400^2 write k for the total number of degree-675 labels. Then $k \le 7$. If any member of the full principal two-block has degree divisible by four, then $k \le 3$. Consequently, when $k \ge 4$ every even member of that block has dyadic height one.*

Proof. For $W \in P_2$ put $i = v_2(d_W)$. Since every row of degree 675 hits W by two-principality and $d_W / \gcd(d_W, 675) = 2^i$, Lemma 6.13 gives $ek4^i < 64$ for its five-inertia orbit size e. There exists a five-free nonunit $W \in P_2$ by Proposition 4.2. Its degree involves two and three. If $i \ge 2$ the bound gives $k \le 3$. Otherwise $d_W = 18$ or 54, since degree six is excluded. A three-free $U \in P_5$ has degree 20 or 40 (Section 28.1). Their crossing, of weight two, is a sum of two distinct five-power roots up to a scalar (Theorem 4.13). Its squared absolute value generates a field containing $\mathbb{Q}(\sqrt{5})$, so $e \ge 2$ and $k \le 7$. □

29.2 Local Weyl transitions in joint twist fibers

Write E for a full five-block restricted to fixed dyadic and ternary twists. Proposition 28.3 applies inside this fiber.

Two distinct ternary phases. Let J be jointly three- and five-active at 5400^2. There are two different nontrivial ternary phases with the *same* dyadic and quinary components as J at which a joint label occurs. Indeed at local order three the two primitive cubic phases have equal positive squared mass when the complementary phases are fixed (Section 24.10). At order $3^a \geq 9$, the ternary square action alone supplies 3^{a-1} phases. In each such phase choose a joint label, and take its full five-block and exact complementary phases. The resulting fibers E are disjoint. Each contains a five-inactive quartic orbit by Section 26.6. Consequently each contains at least eight five-active labels. Every active label there is jointly three- and five-active, so the local five-inertia involution moves it (Sections 26.4 and 26.6); in particular all active labels occur in quartets.

Equal mass at tame fifth phases. In a tame fifth fiber E as above suppose the active degrees are $d_0 b$ with b odd. Section 24.10 gives equal squared mass at each of the four primitive phases. After division by d_0^2, reduction modulo eight shows that the four label counts agree modulo eight. If all active orbits are quartets, each contributes twice its squared degree at each phase of its square class. Cancelling this factor two before reduction shows that the numbers of quartets in the two classes agree modulo eight. In particular, five quartets are impossible and four split two to each class. If in addition the two-term sums of the available b^2 are distinct up to order, the two classes have the same multiset of quartet degrees.

At least twenty-four labels in each joint fiber. Every such E contains at least 24 labels, including at least 16 active at five. All degrees in it are at most 360, since the two odd heights are deficient: the ternary phase is nontrivial, and full five-height makes a five-block a singleton. When the dyadic phase is nontrivial too, the upper degree bound drops to 180.

Write A for the number of five-active labels in E, all of them jointly three- and five-active. These labels occur in local five-inertia orbits of size divisible by four. If a phase has five-primary order $5^a \geq 25$, its orbit contains at least $2 \cdot 5^{a-1}$ distinct phases by the square action, with at least two labels over every phase by the inertia involution. Then $A \geq 20$. When $A < 20$ all active twists are tame and each active orbit is one quartet: the local group action factors modulo five because $U(5)$ acts trivially, and its conjugates generate the kernel of reduction modulo five. At any successive depth the conjugates of an upper elementary matrix span the trace-zero quotient over $\mathbb{F}_5$, which proves the normal generation by induction. The parity theorem (Section 24.9) says that the number of local five-orbits of tame order four in E is even. If $A = 8$, its two active quartets therefore require an even number of inactive quartets. There is at least one, so at least two, forcing $A \geq 16$ by the local Weyl count.

If $A = 12$, the three active quartets cover both square classes since the masses at all four individual fifth phases are equal by Section 24.10. Each quartet contributes twice its squared degree to each of the two individual phases in its class. The two quartets on one side and the single quartet on the other would therefore have degrees satisfying $d_1^2 + d_2^2 = d_3^2$. When two and three are separated or when two and five are separated, these degrees all have full dyadic height by separation. This also holds in the triangle without a triple (Proposition 24.12). After removing their common dyadic part the equation is impossible modulo four. For a triangle containing a triple, we choose the starting J triple-active, so all active labels in these fibers have five-height exactly one: positivity of the five-height is the ternary degree transfer and the upper bound is activity (Section 26.3). After dividing by five, every degree is supported on two and three. In a primitive Pythagorean triple the odd hypotenuse must have a prime divisor equal to one modulo four, since each of its prime divisors admits a square root of -1 modulo that prime. Thus the required equality is again impossible. The same height-one argument works for a dyadically trivial fiber whenever the ternary degree transfer is available.

It follows that $A \geq 16$. If $A = 16$, its four tame active orbits force an even number of inactive quartic orbits by the same parity theorem. Hence there are at least eight inactive labels in this case; if $A \geq 20$ we use the four already proved. This gives $|E| \geq 24$ as asserted.

On the path missing six every degree in E is $40, 120$ or 360 (Section 28.5). In fact $|E| \geq 25$. If equality held in the bound 24 with twenty active labels, the four inactive labels would form a single orbit. The orbit filtration (Section 24.10) then gives only orders dividing five on E. With sixteen active labels the same tameness is automatic. The normalized squares $1, 9, 81$ have distinct two-term sums by their base-nine digits. The tame count above excludes twenty active labels; at sixteen the degree multisets in the two square classes agree. The eight inactive labels are two quartets. The fiber mass divided by 40^2 is divisible by 25 (Section 28.5). Removing its common factor four, its residue would be $s + 2t$ with $s, t \in \{2, 7, 10, 12, 15, 18\}$, the sums of two residues from $1, 9, 6$. None is the negative of twice another. This excludes size twenty-four.

Two nontrivial ternary phases therefore supply at least 50 labels of these three degrees, including at least 32 jointly active at three and five. The twist-one quartets of X, V supply another 16 purely five-active labels by the local estimate. In addition, a jointly two- and five-active phase gives at least four jointly two- and five-active labels: at quintic order five use equal mass at the four primitive phases with the dyadic twist fixed, and a higher order already has at least ten square phases. With a ternary phase of order at least nine at the joint J, the corresponding three–five counts rise to 75 and 48.

When all three pairs occur. In the absence of a triple the other three inactive quartets X, U, V have twist one, giving at least 24 pure five-active labels outside the two fibers. In the presence of a triple, choose J triple-active, so the fibers share a nontrivial dyadic phase. The three inactive quartets again give 24 active labels outside those fibers: all three have trivial dyadic phase, even when the pure ternary V is active at three. There are consequently at least 56 five-active labels in the triangle. If a joint label at the chosen dyadic phase has ternary order at least nine, the corresponding argument gives at least 72.

Five inactive orbits on the path separating two and five. In an ordinary minimal table of dimension $2^6 3^{2e} 5^4$, $e \geq 3$, suppose the ternary prime occurs jointly with each of two and five, which are separated. There are at least five distinct five-inactive local five-inertia orbits of size divisible by four. Consequently there are at least forty five-active labels in equal-degree quartets. At $e = 3$ one can choose all five inactive orbits with ternary defect.

Let H_5 be local five-inertia. Inflate a character of order four of $\mathbb{F}_5^\times$ to the local torus and denote the result by ψ. For each five-inactive orbit $O_Y = H_5 Y$ whose size is divisible by four, put $\ell_Y = |O_Y| = 4m_Y$. Its stabilizer lies in $\ker \psi$ since the group of units at a power of five is cyclic. The input and output unit vectors are

$$v_Y = \frac{1}{\sqrt{\ell_Y}} \sum_{g \in H_5/\mathrm{Stab}(Y)} \overline{\psi(g)}\, v_{gY}, \qquad v_Y^- = \frac{1}{\sqrt{\ell_Y}} \sum_{g \in H_5/\mathrm{Stab}(Y)} \psi(g)\, v_{gY}.$$

We use the Hermitian inner product linear in its second variable. They form orthonormal bases of the five-inactive spaces L_ψ and $L_{\psi^{-1}}$ of the indicated torus types. Each space has one line per orbit. Every such orbit is dyadically inactive, since two-activity forces full five-height. The dyadic group acts scalarly: its diagonal upper generator is scalar on this invariant space and its conjugates generate the group. The ternary group acts on L_ψ. Section 26.10 gives three compulsory orbits U, X, V with $d_U = 40$, $d_X = 4 \cdot 3^j$, $d_V = 5 \cdot 3^k$, $1 \leq j, k \leq e - 2$. Their fields contain $\mathbb{Q}(\zeta_5)$; U is neutral and tame at three. For $Y = X, V$ the full-weight normalized entries are $1 + 3\zeta$ and $\eta(2 + 3\zeta')$, respectively, with primitive fifth roots ζ, ζ' and a ternary scalar η. The local five-orbit fixes η. In the double sum for $\langle v_Y^-, sv_U \rangle$, the entry at (gY, tU) depends on the tame product gt mod 5. Each of its four values occurs $4m_Y m_U$ times with the same character weight. Division by $\sqrt{\ell_Y \ell_U}$ leaves the factor $\sqrt{m_Y m_U}$. The constant term has zero quartic Fourier coefficient and the remaining Gauss sum has squared modulus $3^2 \cdot 5$. Since $n = 200 \cdot 3^e$ and $\mathrm{lcm}(d_U, d_Y) = 40 \cdot 3^{v_3(d_Y)}$, this gives

$$5|\langle v_Y^-, sv_U \rangle|^2 = 5^a / 9^h, \qquad 5^a = m_U m_Y, \quad h = e - v_3(d_Y) - 1 \geq 1.$$

The local five-block at each of two distinct nontrivial ternary phases contains an inactive quartic orbit. For $e = 3$ this was proved above. For varying e, the avoided-two trace selection of Section 26.4 in the full principal three-block of a joint label gives an odd hit of degree $5 \cdot 3^l$ with five-height one by defect-zero vanishing. Its crossing with U puts $\mathbb{Q}(\zeta_5)$ in its field. The selected value has odd relative field index and odd absolute trace, so a faithful quartic Fourier coefficient is nonzero (Section 26.6). Applying the complementary Weyls gives the required inactive orbit at the specified phase.

At $e = 3$ first set aside the coordinate orbits of full ternary height in each mode space. Each label then has ternary-unit codegree, hence a singleton three-block and trivial ternary twist. Its coordinate line affords the trivial ternary representation since the upper unipotent acts trivially and its conjugates generate the local group. These lines contribute no transitions between different ternary phases. For $e > 3$ retain all orbits.

Suppose the retained space has dimension at most four. It contains the three compulsory orbits at distinct degrees d_U, d_X, d_V. A ternary phase of order at least nine has at least three square conjugates at the same degree, on different five-orbits, forcing at least five orbits in total. Thus the local upper unipotent has cube one. Its representation factors through $\mathrm{SL}_2(\mathbb{F}_3)$. Indeed, in each congruence quotient of depths $j, j+1$ with $j \geq 1$, the image of the representation kernel contains the conjugates of E_{12}, from the upper unipotent at depth j. These span $\mathfrak{sl}_2(\mathbb{F}_3)$: interchanging the coordinate axes gives $-E_{21}$, and conjugation by a lower unipotent gives a nonzero diagonal component. Descending induction kills the entire first congruence subgroup.

Write $z = -I$ in this quotient. As an unsigned torus permutation it preserves degree and permutes the five-orbit lines. If a compulsory orbit is fixed setwise, the action on its labels commutes with H_5 and therefore is translation by an element of $H_5/\mathrm{Stab}(Y)$. Because K_Y contains $\mathbb{Q}(\zeta_5)$ and z fixes the fifth roots, this translation is wild, hence of odd order. The involution fixes that orbit pointwise. At most one additional orbit is available. If z exchanges it with a compulsory orbit the exchanged pair has a single negative eigenline, and otherwise only the extra line can be negative. On the negative eigenspace z has determinant one. Indeed the Weyl matrix and $\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ anticommute over $\mathbb{F}_3$, so their commutator is z; the eigenspace is stable under both. Its negative multiplicity is therefore even, hence zero. This works for either quartic mode. Both retained representations descend to A_4, with upper generator u a 3-cycle and Weyl element w a nonidentity element of the Klein four-group.

Let E_0 project onto the five-inactive coordinates, let $B = E_0 W_5|_{L_\psi} : L_\psi \to L_{\psi^{-1}}$, and let Q project onto local level at most five. Both operators here are defined on the full mode spaces, including any trivial lines set aside at $e = 3$. The five-phase estimate of Section 26.10, polarized, gives $5B^*B = Q$ on L_ψ. Both B and Q intertwine the ternary action. The compression of s between the two mode spaces equals BW_3 up to the dyadic unit scalar. At least one of X, V is nonneutral since both primitive cubic phases occur and at most four orbit lines are available. There is thus a nonzero transition from U to a different u-phase. Character summands cannot give this transition: w acts trivially on them and B intertwines u. Lemma 6.10 therefore gives a unique tetrahedral summand T on each side. The projection Q acts identically on the input T, since it commutes with the group and B is nonzero on T. Thus $\sqrt{5}B$ identifies the two tetrahedral summands unitarily. Any coordinate lines set aside at $e = 3$ were trivial constituents and do not affect this identification.

A transition from a simple u-phase to a different simple phase has squared modulus $4/9$ for $\sqrt{5}BW_3$, by the lemma. On either compulsory transition the actual value $5^a/9^h$ is one modulo four, where all denominators are odd. If the neutral phase were simple, X, V would therefore have to be the two orbit lines at one doubled nonneutral phase. The sum of their two squared transitions would be $4/9$, the squared norm of the projection onto that phase from the neutral tetrahedral line. The sum is two modulo four. This forces the neutral phase to be doubled, with basis $v_U, v_{U'}$ in the input; the two nonneutral phases are simple and the fourth constituent is

trivial. For a nonneutral $Y \in \{X, V\}$ we now have

$$\tfrac{4}{9}x = 5|\langle v_Y^-, sv_U\rangle|^2 = 5^a/9^h, \qquad x = \|P_T v_U\|^2.$$

Applying the last assertion of Lemma 6.10 in the input space gives

$$|\langle v_{U'}, W_3 v_U\rangle|^2 = \tfrac{16}{9}x(1-x) = \frac{5^a(4\cdot 9^{h-1} - 5^a)}{9^{2h-1}}.$$

The ternary torus fixes every label of O_U. Its wild part fixes U by tameness and commutes with H_5. Its involution preserves O_U because it acts identically on the retained mode space, and the translation argument above applies. The Galois formula of Lemma 6.2 sends W_3 to W_3 times a ternary torus element on the right. The columns of W_3 indexed by O_U are consequently rational. In particular, writing $G_Z = H_5/\mathrm{Stab}(Z)$, the two normalized input modes give

$$\sqrt{\ell_{U'}\ell_U}\,\langle v_{U'}, W_3 v_U\rangle = \sum_{g\in G_{U'},\ t\in G_U} \psi(g)\overline{\psi(t)}(W_3)_{gU',tU} \ \in \mathbb{Q}(i).$$

Its squared modulus is a Gaussian norm. So is $\ell_{U'}\ell_U = 16 m_{U'} m_U$, since $5 = |2+i|^2$. Hence the squared transition has even valuation at every prime congruent to three modulo four. But $4\cdot 9^{h-1} - 5^a$ is positive (it is nonnegative since $x \le 1$, and odd) and is three modulo four, so it has such a prime to odd valuation. Neither 5^a nor the square denominator changes this parity. This contradiction proves the five-orbit assertion. Proposition 28.3 gives the forty five-active labels.

29.3 Rank from compulsory simple degrees

For rank R, let m_d count simples of degree d. Suppose distinct positions have upper degree bounds $t_1, \dots, t_j$. For any cutoff b,

$$b^2 R + \sum_{d>b}(d^2 - b^2)m_d \ \ge\ D + \sum_{i=1}^{j}(b^2 - \min\{t_i, b\}^2). \tag{29.1}$$

Indeed the left side minus D counts $b^2 - \min\{d, b\}^2$ at every position. We use this with $b = 600$ or 675.

The following disjoint positions enter this budget. Here E_1, E_2 have distinct nontrivial ternary phases. A quartet comprises four equal-degree labels in one five-inertia orbit.

Twist graph	Positions counted before the remaining degree deficits
Missing 6	$\lvert E_1\rvert, \lvert E_2\rvert \ge 25$, degrees $40, 120, 360$; neutral X, V, Q quartets and the three-free W; at least 24 pure five-active labels; four joint two–five labels and one pure two-active label (first case).
Triangle	Unit, three inactive quartets ($d_X \le 72, d_U \le 40, d_V \le 45$), 24 five-active labels outside E_1, E_2 ($d \le 540$), and at least 48 positions in those fibers ($d \le 360$, or 180 on a triple); see the triangle fibers.
Missing 10	With at least two neutral quartets, count eight neutral positions, sixteen pure five-active labels and 48 in E_1, E_2. With only one neutral quartet reserve all five inactive quartets, ten active quartets and the unit, then the five outside labels in the separated case.

Different twist fibers are disjoint. Within a fiber the proofs below use distinct inertia orbits, blocks or degrees to separate the labels. An inactive quartet inside E_i contributes to its size only once.

The path separating two and three. Here $m_{900} = 0$ by Corollary 29.2. Put $g = 600^2$, $C(d) = g - d^2$ and $k = m_{675}$. By the compulsory active orbits and the double-principal-block argument, the following labels are distinct: the vacuum, neutral X, V quartets with degrees at most $72, 45$, a neutral three-free $W \in P_2 \cap P_3$ of degree 10 or 20, and four further neutral members of $P_2 \cap P_5$. There is also a pure two-active member of the five-block of a joint two–five label, of degree at most 540, and at least four jointly two–five-active labels with the same upper bound. The two joint ternary fibers have different ternary twists. The fiber count in this separated case gives at least 25 labels of degrees $40, 120, 360$ in each.

The four neutral members of $P_2 \cap P_5$ can be taken in a single tame five-inertia orbit Q. Indeed, let H be the order-four tame subgroup and adjoin the unit both to HU, for the pure five-primary $U \in P_5$ of degree 20, and to HV for the neutral $V \in P_2$. Section 26.7 gives a minor of $W_5 V_2$ between their five-dimensional coordinate spaces. After multiplying row Y by n/d_Y, its reduction at two is the Fourier matrix $(\bar{\zeta}^{ij})_{i,j \in \mathbb{F}_5}$, up to a scalar unit, for a primitive fifth root ζ. The complex minor is invertible. Its factorization through V_2 and W_5 only uses intermediate labels of twist one in $P_2 \cap P_5$, by block and complementary phase preservation. Both coordinate projections commute with H; V_2 commutes with H and W_5 intertwines its action with inversion. A faithful quartic eigenline in the domain survives in the intermediate space, giving the asserted orbit. The block distinctions separate Q from X, V, U and W.

The three neutral quartets X, V, Q supply at least 24 pure five-active positions, selectable in equal-degree quartets by the local Weyl count. The same conclusion holds if wild inertia enlarges one of these orbits: take its trivial-wild, faithful-tame character. The central involution acts negatively on the relevant Weyl images at both tame and wild phases. If a degree-20 row in P_5 is neutral, it gives a fourth neutral quartet, and we get 32 pure five-active positions. In the other case at least eight pure five-active labels have degree 20, in full quartets (Section 28.5); choose them among the 24. Since the joint ternary fibers supply 32 five-active labels and the two–five edge supplies four others, there are at least 60 five-active labels in either case.

For $Y \in Q$ let $i = v_2(d_Y)$. The inertia-orbit estimate of Lemma 6.13, with $z = 675$ and $p = 5$, gives $k4^i < 16$: every degree-675 row hits Y by two-principality. Hence the common degree of Q is bounded by $v = 540, 270, 135$ for $k = 0$, $1 \le k \le 3$, $4 \le k \le 7$, respectively. In the last case it is odd, of degree $15, 45$ or 135.

To count the two other five-free labels required by the dimension identity, put $x = d_X$. For $x = 12, 24, 36, 72$ the residues $1 + 4x^2$ modulo 25 lie in $\{2, 5, 10, 12\}$, none zero or the negative of a unit square. A five-free position outside Q and the X quartet contributes at least $540^2 - 216^2 = 244944$ beyond its previous allowance, whether filling a 540 slot or newly reserved. The joint ternary fibers, V, W and all degree-20 labels are five-bearing. If Q itself is five-free, its four degrees are at most 108 at $k = 0$ and at most 54 at $k > 0$. Accordingly the additional five-free contribution is at least $2 \cdot 244944$, $4(270^2 - 54^2)$, $2 \cdot 244944$ in the three ranges of k. In the middle range this is the cheaper of the two possibilities for Q.

For the branch with a neutral degree-20 quartet, the resulting lower bound on $gR + 95625k$ is the sum of D, these five-free contributions, and

$$C(1) + 4C(72) + 4C(45) + C(20) + 4C(v) + 5C(540) + 50C(360) + 4C(20) + 32C(540).$$

In the other branch replace the last two terms by $8C(20) + 16C(540)$, increasing the bound by 344000. For the first branch the baseline and its remaining allowance at $R = 140$ are

k	baseline	$140g + 95625k -$ baseline
0	48983451	1416549
1, 2, 3	49648299	847326, 942951, 1038576
4, 5, 6, 7	50076951	705549, 801174, 896799, 992424.

In any such fiber, if no inactive label has the minimum ternary height, at least 36 labels attain it. Indeed $\sum_E d_Y^2 = \sum_{Y \text{ inactive}} d_Y^2 k_{\{2,3\},Y}^2$. Divide by the square of the smallest degree in E

and reduce modulo nine: the number attaining it is a multiple of nine, and it is also a positive multiple of four since all such labels occur in equal-degree quartets. The two fibers E may have more than fifty labels or degrees below 360. Let $t = \sum_E (|E| - 25)$, and let a, b count their labels of degree $40, 120$ respectively. Since each fiber mass is divisible by $25 \cdot 40^2$ (Section 28.5), their extra contribution obeys

$$230400t + 128000a + 115200b, \qquad 6t - 5a + 3b \equiv 0 \pmod{25}.$$

Outside the fibers the previously reserved three-free positions are the unit, W of degree $w = 10, 20$, and four, respectively eight, positions of degree 20. Denote the number of further three-free positions outside by c. Each contributes at least 290000: a pure five-active three-free degree is at most 40 instead of its cap 540; an unreserved three-free degree is at most 200. No dyadically active label here is three-free, and none of the neutral nonunits in $P_2 \cap P_5$ is three-free (see the four further principal labels on the path missing six). None of these three-free savings coincide with a five-free one, by prime-power exclusion. In the first branch $a + c$ is a positive multiple of three: divisibility follows modulo three; if zero, modulo nine gives $1 + 4 \cdot 20^2 + w^2 = 0 \bmod 9$, hence $w = 10$. The unique degree-ten label would be rational, contrary to Section 24.11. In the second branch $a + c \equiv 2 \bmod 3$.

The extra cost is at least 870000, including the 344000 branch correction where applicable. If $a = 0$ use $c \geq 3, 2$, respectively. If $a > 0$, the only fiber cost below 870000 has $(t, a, b) = (0, 5, 0)$ and costs 640000. Indeed $t \leq 3$, $b \equiv -2t \bmod 5$ and $5a \equiv 6t + 3b \bmod 25$. At $t = 0$ test $b = 0, 5$; $b = 5$ forces $a \geq 3$. At $t = 1, 2$ the least b is $3, 1$, respectively, forcing $a \geq 3$ at these minima; at $t = 3$, already $b \geq 4$. In the surviving case the first branch needs $c \geq 1$; the second incurs the branch correction. This excludes $k = 1, 4, 5$.

The odd-degree count is divisible by four. At $k = 0$ three more odd labels are needed beyond the unit and V quartet. Each saves at least $540^2 - 135^2 = 273375$ in a capped slot or $C(225)$ outside. An odd degree distinct from 675 properly divides 675 and is at most 225; it is three- and five-bearing. At $k = 2$ at least one more odd is required. If Q is odd, its improved cap and the external five-free allowance already add $4(270^2 - 135^2) + (489888 - 279936) > 273375$; otherwise use the same odd saving outside Q. At $k = 6$ the unit, V, Q and the six degree-675 labels give fifteen odd labels, so one more is needed. These savings exclude $k = 0, 2, 6$. At $k = 3$, five-bearing Q increases the five-free allowance by $489888 - 279936$. Five-free Q has degree $d = 18$ or 54 and $1 + 4x^2 + 4d^2 \neq 0 \bmod 25$ for the four values of x, costing another 244944 outside the two quartets. Both exceed the allowance.

Finally $k = 7$ forces $w = 10$ by Proposition 29.4. Theorem 24.23 forces five-ramification on this separated path, supplying another neutral degree-ten conjugate outside the reserved positions. Its cost 359900 leaves less than 640000, so $a = 0$. We have $c \leq 3$; as the selected pure five-active positions are grouped in full quartets, none of these c positions lies there. Each besides the new degree-ten label therefore costs at least $C(200) = 320000$. This gives at least $359900 + 2 \cdot 320000$ in the first branch or $344000 + 359900 + 320000$ in the second. Both exceed 992424. This establishes $R \geq 141$.

All three pairs occur. Select 85 distinct positions: the vacuum, the three inactive quartets X, U, V of degrees at most $72, 40, 45$, twenty-four five-active positions outside the joint fibers capped at 540, and forty-eight labels in the two joint fibers. The complementary twist conditions separates these fibers from the other five-active positions. These last degrees are at most $u = 360$ if there is no triple, or $u = 180$ if a triple is used to define the fibers. At least two additional five-free labels besides the X quartet are required by the same residue argument. All degrees in the two chosen fibers are five-bearing, by the ternary degree transfer, and so are U, V. Each additional five-free label gains $540^2 - 216^2$ even if already counted among the twenty-four five-active labels. There is also a further three-free label: the vacuum and the U quartet give five, whereas their total number is a multiple of three. It differs from every five-free nonunit. If in a

joint fiber, its degree divides 40 by the five-defect; the generic five-active positions outside have three-bearing degrees by the degree transfer, and an unreserved three-free degree divides 200. This adds at least $u^2 - 40^2$. With a baseline G equal to 900^2 or 600^2, put

$$B(G,u) = D+(G-1)+4\sum_{t\in\{72,40,45\}}(G-t^2)+24(G-540^2)+48(G-u^2)+2(540^2-216^2)+(u^2-40^2).$$

Thus $900^2R \geq B(900^2,u)$, giving $R \geq 106,112$, respectively. In the absence of degree 900, (29.1) gives $gR + 95625k \geq B(g,u)$, whose right hand sides are $47123451, 51691851$. With $k \leq 7$ this proves $R \geq 130,142$, respectively.

The path separating two and five. At $n = 5400$, a five-inertia quartet of degree $12, 15, 20, 36$ or 45 meets at most eight degree-900 labels. A quartet of degree $24, 40$ or 72 meets at most two. Thus the three mandatory quartets of degrees $12, 15, 40$ on the path missing ten collectively meet at most eighteen such labels. Every other degree-900 row vanishes on all twelve columns.

There is a stronger bound on that path. Let $4a$ count all degree-12 labels in P_3. We have $1 \leq a \leq 8$: in the weighted sum of Section 26.10 a degree-12 orbit costs at least one quarter of its size, and the opposite degree-15 type costs at least one of the available nine units. At any jointly two- and three-active J, orthogonality on P_3 gives

$$1 + 12\sum_{d_X=12,\ X\in P_3}\chi_J(X) \equiv 0 \pmod{\mathfrak{p}}, \qquad \mathfrak{p} \mid 5.$$

Indeed a five-free nonunit of P_3 has degree 2^i3^j with $i = 2, 3$ (Lemma 4.14); full dyadic height misses J, and the other type has degree 12 by Section 26.10. The degree of J has full five-height by separation, so its row agrees on the four tame five-conjugates in each quartet. Write b_J for one quarter of the sum in this congruence. It is a rational integer: the full set of X is Galois-stable, and the sum is four times an algebraic integer. Thus $b_J \equiv -2 \pmod 5$. Applying s to the normalized indicator of the $4a$ labels gives

$$\sum_{J \text{ jointly two-, three-active}} d_J^2 b_J^2 < aD/4.$$

Hence the joint sector has squared mass strictly below $aD/16$. In particular fewer than $9a/4$ degree-900 simples can be joint, so there are at most 17 in any rank. At $a = 1$ there are at most two, and their ternary twist has order three by square Galois covariance.

Let $G = 675^2$, $g = 900^2$, and $m = \#\{Z : d_Z = 900\}$. The two joint ternary fibers from Section 29.2 contain at least 48 labels, all of degrees at most 360. The five-orbit argument gives five five-inactive quartets, including X, V, U of degrees $12, 15, 40$. A neutral quartet here means one with trivial ternary phase; all five have trivial dyadic phase. In particular U is neutral.

If at least two of the five quartets are neutral, their two orbits supply at least sixteen pure five-active labels in addition to the eight neutral labels and the 48 members of the joint fibers. The compulsory quartets have ternary defect by the five-orbit argument, so all these 72 degrees are at most 360. Reserving the vacuum, and the much smaller degrees of X, V, U whether inside this count or outside, gives squared mass at most

$$M = 1 + 4(12^2 + 15^2 + 40^2) + 60 \cdot 360^2$$

on 73 compulsory positions (if necessary replace unspecified positions by X, V, U). A further label of degree 75 lies outside: it has full five-height, so it is neither five-active, nor in a ramified quartet, nor in a five-block containing an active simple. Thus (29.1) with $b = 675$ gives $GR + (g - G)m \geq D + 74G - M - 75^2$. In particular $R \geq 101$.

It remains to treat the case where just U among the five is neutral. Select 40 five-active labels in ten equal-degree quartets: sixteen in each of the two joint fibers and eight in the pure

five-fiber of U. All have degree $24, 72, 120, 216$ or 360: five-activity gives full dyadic height and positive ternary height (Section 26.4). Together with the unit and the five inactive quartets this gives 61 labels. Write b, c for the degrees of the other two quartets besides X, V, U; they divide 360. The following five further labels are distinct and outside those 61:

$$45, \qquad 75, \qquad w \in \{10, 20, 50, 100\}, \qquad y \in \{12, 18, 36\}, \qquad o \in \{15, 45, 75, 135, 225, 675\}.$$

Indeed use the twist-one 45 and w from the double-block argument, and at a joint dyadic–ternary column the five-free twist-one witness $y \in P_2 \cap P_5$ from the same argument. All are distinct from the neutral 40 and the four nonneutral quartets. The witness of degree $75 \in P_3$ is five-inactive with full five-height, hence cannot lie in a five-ramified quartet. Finally the total odd count is divisible by four, and in the 61 selected labels the odd count is one modulo four. In addition to the $45, 75$ just retained a further odd o is therefore required. The multiplicity of degree 18 is even, as is that of 54: a degree-40 row in P_5 gives at each such label a sum of two distinct five-power phases after lcm normalization (Theorem 4.13). Its squared modulus puts $\mathbb{Q}(\sqrt{5})$ in the character field.

The resulting 66 labels have total squared mass at most

$$M_0 = 1 + 4(12^2 + 15^2 + 40^2 + 2 \cdot 360^2) + 40 \cdot 360^2 + 45^2 + 75^2 + 100^2 + 36^2 + 675^2 = 6703248.$$

In particular $GR + (g - G)m \geq D + 66G - M_0 = 52528002$.

At $R \leq 94$ the excess above D is at most 223248. Thus every remaining degree must be 900, $b = c = 360$, and $o = 675$: decreasing one of these upper bounds costs at least 354375, $4(360^2 - 180^2)$ or $675^2 - 225^2$, respectively. The value $y = 18$ is impossible by its even multiplicity. Modulo nine, the only three-free labels give $1 + 4 \cdot 40^2 + w^2 \equiv 0$, so $w = 50$. Modulo sixteen, the seven fixed odd squares give 7, the extra odd square gives 9, and w^2 gives 4, with all other contributions zero. This rules out $R \leq 94$.

With $b = 600$, the same 74 or 66 positions in (29.1) give respectively $D + 74b^2 - M - 75^2$ and $D + 66b^2 - M_0 + (675^2 - b^2)$: in the latter the cap on the reserved odd label becomes 600. Thus

$$360000R + 450000m + 95625k \;\geq\; 48010498 \quad \text{or} \quad 360000R + 450000m + 95625k \;\geq\; 46312377.$$

Here $k = m_{675} \leq 7$. For $m = 0$ they give $R \geq 132$ and $R \geq 127$ respectively. If $R \leq 105$ there are at least 18 degree-900 labels; if $R = 101$ there are at least 21. By Corollary 29.3 at least eighteen of these rows meet the joint sector.

29.4 The exact rank calculation

When two and five are separated, only the one-neutral case at ranks 95 through 100 remains. The integer recursions below enumerate the candidates. For each of the five survivors at rank 100, the certificates give both disjoint fibers and their norm witnesses. A separate calculation checks each pair. The argument on tame phases excludes all five.

Let $A = \{24, 72, 120, 216, 360\}$, $J = A \setminus \{216\}$, $g = 900^2$, and let $\mathcal{D}$ be the divisors of 5400 between 10 and 675 with at least two prime factors. The $R - 66$ additional degrees are 900 apart from a multiset L in $\mathcal{D}$. The ten selected five-active quartets contribute $2304S$, where S is a sum of ten squares from $\{1, 9, 25, 81, 225\}$. For the initial enumeration take $b \leq c$ from $\mathcal{D}$ dividing 360, $o \in \{15, 45, 75, 135, 225, 675\}$, $w \in \{10, 20, 50, 100\}$, and $y \in \{12, 18, 36\}$. Retain precisely the tuples satisfying

$$\begin{aligned} M_0 + (R - 66)g - D = {} & 4(2 \cdot 360^2 - b^2 - c^2) + (675^2 - o^2) \\ & + (100^2 - w^2) + (36^2 - y^2) + 2304(2250 - S) + \sum_{d \in L} (g - d^2). \end{aligned}$$

Here $|L| \leq R - 66$ and the multiplicities of $18, 54$ in $L \cup \{y\}$ must be even. An explicit enumeration subtracts $g - d^2$ from the remaining nonnegative budget, adding $d \in \mathcal{D}$ in decreasing order with

repetitions. Include the empty multiset. The least decrement is 354375. For the admissible S, start with $\{0\}$ and take ten successive sumsets with $\{1, 9, 25, 81, 225\}$. A tuple records S only once regardless of its number of representations.

R	95	96	97	98	99	100
initial tuples	0	1	43	384	3659	30100
$\max \#\{d \in L : d \in A\}$	–	0	1	2	3	5
after the two exact fiber norms	0	0	0	0	0	5

The maximum counts occurrences with multiplicity. Thus there are at most 45 five-active labels overall, with eight already reserved at U. Three distinct joint fibers would require at least 48 joint labels. All joint labels consequently lie in the two selected fibers, and their ternary phases have order three, since order nine or higher would generate a third fiber. Each fiber has 16 or 20 five-active labels, since its joint labels occur in quartets. The tame count in Section 29.2 excludes a fiber with twenty active labels, since every possible active degree divided by 24 is odd. A wild phase requires twenty label positions and, at that very fifth phase, the other nontrivial cubic phase also has positive mass. This would use at least $20 + 20 + 8 > 45$ five-active labels. Thus both fibers contain sixteen tame active labels.

The normalized squares $1, 9, 25, 225$ available in a joint fiber (activity excludes 216) have distinct two-term sums. Thus its quartet degrees are (u, u, v, v) by the same tame count. For a fixed fifth phase the two primitive cubic phases also have equal squared mass, so the very same u, v occur in both fibers. The ten selected quartet degrees altogether consist of four u, four v, and a pair $h, i \in A$ at U. We initially allow this last pair to be unequal.

Here is the exact second computation underlying the last line of the table. Take every $u \le v$ in J and $h \le i$ in A with

$$576S = 4u^2 + 4v^2 + h^2 + i^2.$$

Form a list of possible five-inactive positions in the two fibers. Include the four quartets $12, 15, b, c$ as groups of multiplicity four. From $L \cup \{o\}$ keep the degrees dividing 360. If $y = 18$ first remove its twist-one conjugate of degree 18 from L. Group the remaining 18's and 54's in pairs, and use multiplicity one for the others. This pairing is allowed because each five-inertia orbit at these two degrees has even size and the complementary phases and full five-block are invariant. The already reserved quartets and conjugate pair are themselves stable under tame five-inertia.

Take disjoint sublists for the two fibers. For each sublist I of pairs (m, d) require $\sum_I m \ge 8$, and put

$$F = 8(u^2 + v^2) + \sum_I m d^2, \qquad F = 25 \sum_I m d^2 x_{m,d}^2, \quad x_{m,d} \in \mathbb{Z}, \tag{29.2}$$

with a separate variable for each group, including repeated pairs. This identity follows by commuting the fiber projector through W_5 in the squared norm of the vacuum column: only five-inactive positions contribute on the $W_{\{2,3\}} v_0$ side. Theorem 6.1 gives the factor 25 because every degree in the fiber has positive five-defect. Indeed the unsigned five-torus fixes v_0 and commutes with $W_{\{2,3\}}$. Thus $W_{\{2,3\}} v_0$ and the degree are constant on each five-inertia orbit. This justifies one integer coordinate per selected quartet or pair, even when a larger orbit is split into such pieces: giving the pieces independent integers enlarges the feasible set. Equal degrees on different orbits impose no equality of coordinates. To test the equality, start with the single remainder $F/25$, discarding nonintegers. At each (m, d) replace each remainder t by the set of $t - md^2x^2$ for integers $0 \le x \le \sqrt{t/(md^2)}$. Keep the sublist if zero occurs after its last group. Two successful sublists must be disjoint, and unused positions are permitted. These recursions

give the following complete output, all at $R = 100$, $w = 50$, and $u = v = 360$.

(b, c)	o	y	L	S	(h, i)
$(15, 18)$	135	12	$360, 72, 12, 12$	2034	$72, 360$
$(15, 36)$	135	12	$360, 36, 12, 12$	2034	$72, 360$
$(18, 180)$	135	12	$72, 30, 12, 12$	2034	$72, 360$
$(24, 360)$	15	36	$600, 600, 270, 24, 12$	1882	$24, 216$
$(36, 180)$	135	12	$36, 30, 12, 12$	2034	$72, 360$

Eliminating the five remaining profiles. In each row at most two entries of L could be five-active. Hence the pure five-fiber at U contains at most ten active labels. The two selected quartets cannot come from a wild orbit, which would need twenty labels if it contained the faithful quartic mode. An additional wild orbit would itself require at least ten more. By the tame-phase count the four phase counts agree modulo eight. All are positive with sum at most ten, hence equal. There are just eight active labels, two quartets in opposite square classes, of equal degree. This contradicts every displayed pair (h, i) and proves the bound.

Proof of Theorem 29.1. The three row-norm estimates on p. 225 give the upper bound. Proposition 29.4 gives the multiplicity bound at 675.

For the lower bounds, the two disjoint ternary fibers each contain at least 24 labels. If 10 is missing, the five inactive orbits give 40 five-active labels. The degree budget and congruences in the second row's proof give $R \geq 95$ and $R \geq 127$ if $m_{900} = 0$. In Section 29.4, the rank enumeration tests the degree multisets and the two disjoint fiber norms (29.2). It leaves five profiles, all of rank 100. Their incompatible tame phases exclude all five. At $R = 101$ the same budget gives $m_{900} \geq 21$. At most three of those rows miss the joint sector by Corollary 29.3. □

30 The unbraided boundary in dimension 5400

Let $\mathcal{A}$ be a non-weakly-group-theoretical fusion category of dimension $n = 5400$. If $\mathcal{Z}(\mathcal{A})$ were WGT, then $\mathcal{A}$ would be WGT by [32, Proposition 4.1]. The center is modular of dimension 5400^2, and every proper divisor of 5400^2 is below the braided threshold in Theorem 17.1. Thus Lemma 2.1 applies to the center itself. Its modular alternative gives integrality, fusion-simplicity and perfection; Section 28.2 and Theorem 28.1 put its nonunit degrees in

$$\mathcal{D} = \{a : 10 \leq a \leq 900,\ a \mid n,\ a \text{ has at least two prime divisors}\}.$$

Dominance of the forgetful functor $F : \mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ implies that $\mathcal{A}$ is integral as well: a positive sum of square roots of integers is integral only when every summand is integral, by independence of distinct squarefree radicals.

Theorem 30.1. *Every proper fusion subcategory of $\mathcal{A}$ has dimension at most* 216. *Every formal codegree of $K_0(\mathcal{A})$ other than that of the dimension character lies between* 6 *and* 540. *The rank satisfies the following bounds.*

	$K_0(\mathcal{A})$ *commutative*	$K_0(\mathcal{A})$ *noncommutative*
arbitrary $\mathcal{A}$	$10 \leq \operatorname{rank} \mathcal{A} \leq 1512$	$12 \leq \operatorname{rank} \mathcal{A} \leq 1512$
$\mathcal{A}$ *simple and perfect*	$11 \leq \operatorname{rank} \mathcal{A} \leq 462$	$13 \leq \operatorname{rank} \mathcal{A} \leq 490$

30.1 The canonical induction algebra and character moments

Write I for the right adjoint of F. For each irreducible representation E of $K_0(\mathcal{A})_{\mathbb{C}}$, let $m_E = \dim E$ and f_E be its formal codegree. Ostrik's induction theorem identifies E with a

distinct simple summand of $I(\mathbf{1})$, of degree $a_E = n/f_E$ and multiplicity m_E [49, Theorem 2.13]. In particular

$$\sum_E m_E a_E = n, \qquad \sum_E m_E^2 = \operatorname{rank} \mathcal{A}, \qquad n\tau = \sum_E a_E \chi_E, \tag{30.1}$$

where τ takes the coefficient of the unit. The dimension character has $(m_E, a_E) = (1, 1)$, and all other a_E belong to $\mathcal{D}$. This proves the codegree assertion.

Indices and the length of induction. A subcategory of index e corresponds to a connected étale subalgebra of $I(\mathbf{1})$ of degree e [19, Theorem 4.10]. Hence $e \mid n$ and $e - 1$ is a sum from $\mathcal{D}$. Below 24 the only odd such sum is 15, whereas $16 \nmid n$. For the odd divisors $1 < e < 25$, namely $3, 5, 9, 15$, none of $e - 1$ is such a sum. Thus $e \geq 25$ for a proper subcategory. In particular the pointed subcategory has dimension $g \leq 216$, since a pointed category is weakly group-theoretical. All other degrees are at least two, so $\operatorname{rank} \mathcal{A} \leq g + (5400 - g)/4 \leq 1512$.

We next show that $L = \sum_E m_E \geq 10$. In the sum of the nonunit degrees of $I(\mathbf{1})$, equal to 5399, three different terms must omit two, three and five, respectively. Their upper bounds are $675, 200, 216$. Seven terms can contribute at most 4691. For eight terms the upper sum is 5591, leaving a deficit of 192. The other five terms must therefore equal 900 and the odd term 675. The two remaining terms sum to 224. The three-free term divides 200 and is 2 modulo three, hence is $20, 50$ or 200; only 200 leaves a divisor of 216. The three degrees $24, 200, 675$ occur once, giving three integer-valued linear characters h, u, v: each is unique at its codegree and thus Galois invariant. The sum of the remaining characters other than FPdim is also integer-valued, by the same Galois invariance and integrality of character values. For a simple X write $d = d_X$ and use h, u, v also for their values there. Applying (30.1) to X and XX^* gives, modulo 900,

$$d + 24h + 200u + 675v \equiv 0, \qquad d^2 + 24h^2 + 200u^2 + 675v^2 \equiv 0.$$

Modulo 25 the first gives $h \equiv d$. Modulo nine, eliminating u gives $3(d - h)^2 \equiv 0$, so $75 \mid d - h$. As $|h| \leq d$, every simple of degree at most 37 satisfies $h(X) = d_X$. The labels with this equality span a proper fusion subcategory: duality preserves it and equality in the triangle inequality forces it on every constituent of a product. Its inverse image under F is proper by dominance and contains the central simple of degree 24, contradicting fusion-simplicity of the center. This proves $L \geq 10$. By (30.1), rank is at least L, or $L + 2$ when some $m_E \geq 2$. If the center has no degree-900 simple, even nine nonunit terms sum to at most $7 \cdot 675 + 200 + 216 = 5141$. In this case $L \geq 11$ already without a simplicity assumption. This applies in particular when the center has no twist order divisible by six.

How many degree-three simples can occur. The second row of Theorem 30.1 assumes that the source $\mathcal{A}$ itself is simple and perfect. Under this additional hypothesis let n_j count its degree-j simples. A degree-two simple would produce a fusion subcategory of dimension 60 by the categorical Nichols–Richmond theorem [23, Theorem 3.4]; the other alternatives there require invertibles. Hence $n_2 = 0$. The following inequality holds in every simple perfect integral fusion category with $n_2 = 0$ and $n_3 \geq 2$:

$$n_3 \leq 2K, \qquad K = \min(4n_4, 5n_5) + 6n_6 + 9n_9. \tag{30.2}$$

Indeed put $P_X = XX^* - \mathbf{1}$ for $d_X = 3$. Its degree partition is $(8), (3, 5)$ or $(4, 4)$. For $X \neq Y$ of degree three the product X^*Y is either a sum of degree-three simples or has partition $(4, 5), (3, 6)$ or (9). In the first case its squared norm in the simple basis is at least three; in every other case it is at most two. Frobenius reciprocity gives $\|X^*Y\|^2 = 1 + \langle P_X, P_Y \rangle$. Call two partitions compatible if this last inner product is at least two, using actual simple labels. At most K partners $Y \neq X$ are incompatible with a fixed X, since XB contains at most d_B degree-three simples.

Suppose $n_3 > 2K$. A singleton (8) cannot occur since every other label is incompatible with it. Every remaining partition is compatible with itself, and any two occurring partitions must have a common compatible partner. A $(3,5)$ partition is compatible only with itself; if one occurs, all P_X equal $U+V$ with $d_U = 3, d_V = 5$. Here $U = U^*$ and $U^2 = \mathbf{1} + U + V$. Left multiplication by U preserves the nonempty set of degree-three labels other than U: for $Y \neq U$, the product UY has only degree-three constituents and $[UY : U] = [U^2 : Y] = 0$. This is impossible, since U generates the category and Y^*Y contains the unit.

For $(4,4)$ the only compatible pairs of different types are aa, ab. Two distinct repeated pairs, or two distinct pairs with unequal entries, are incompatible. A repeated pair aa and a pair bc with $a \notin \{b,c\}$ have no common compatible partner. Thus there is a pair compatible with all, and unique among occurring pairs of its kind (repeated or distinct): two repeated pairs aa, bb force the distinct type to be ab, and two different distinct pairs force a unique repeated type. If only one kind occurs, all pairs coincide. Choose X with this pair. Since $\|XX^*\|^2 = \|X^*X\|^2$, the type at X^* is the same and $P_{X^*} = P_X$. For $Y \neq X$ of degree three, X^*Y has only degree-three constituents and omits X^* since $[XX^* : Y] = 0$. Multiplying these constituents by X again gives only degree-three labels; it gives no X since $[XX^*Y : X] = \langle X^*Y, X^*X\rangle = 0$. Thus the nonunit constituents of XX^* and their duals preserve the degree-three labels other than X, contradicting generation as before. This proves (30.2).

Upper ranks. Set $H = \sum_{j\geq 4} j^2 n_j$ and $E = \sum_{j\geq 4}(j^2-9)n_j$. If $n_3 \leq 1$ the degree-square sum gives rank at most 338. Otherwise

$$5399 = 9n_3 + H \leq 18K + H \leq 60E/11.$$

The last inequality holds coefficientwise except at $4,5$ where it is $198\min(4n_4, 5n_5) \leq 244n_4 + 685n_5$. Since $E = 5399 - 9(\operatorname{rank}\mathcal{A} - 1) \equiv 8 \pmod 9$, this yields $E \geq 998$ and rank at most 490.

If the fusion ring is commutative, reduce the last identity of (30.1) at a prime above two. Distinct algebra characters over the residue field are linearly independent, so some $\chi \neq \mathrm{FPdim}$ with a_χ odd reduces to FPdim. In particular $a_\chi \geq 15$, $f_\chi \leq 360$. At each odd-degree label the character is nonzero. Its squared norm, and that of each Galois conjugate, is f_χ; a nonzero algebraic integer has mean squared conjugate modulus at least one. Thus $O = \sum_{j\geq 3,\ j \text{ odd}} n_j \leq 359$. From (30.2) we have $n_3 \leq 10n_5 + 12n_6 + 18n_9$. If $n_3 \geq 2$, comparison of the coefficients at $3,4,5,6,9$ gives

$$\operatorname{rank}\mathcal{A} - 1 \leq \frac{5399}{16} + \frac{61O}{176} + \frac{n_3 - 10n_5 - 12n_6 - 18n_9}{11} \leq \frac{5399}{16} + \frac{61\cdot 359}{176} < 462.$$

The two upper rank bounds are attained by degree multisets satisfying these numerical inequalities. In the general and commutative cases, respectively, they are

$$5400 = 1 + 401\cdot 3^2 + 48\cdot 4^2 + 38\cdot 5^2 + 2\cdot 6^2,$$
$$5400 = 1 + 328\cdot 3^2 + 100\cdot 4^2 + 31\cdot 5^2 + 2\cdot 6^2.$$

Their ranks are 490 and 462, and the second has $328 + 31 = 359$ odd-degree nonunits. These are arithmetic profiles. Any improvement of the upper bounds must use information beyond the inequalities they satisfy.

The two endpoint ranks. For rank ten, or for noncommutative rank twelve, the lower bound $L \geq 10$ and (30.1) force $L = 10$, respectively with all $m_E = 1$ or with exactly one $m_E = 2$. In fact for commutative rank ten the assertion is immediate; in the other case $\sum m_E(m_E - 1) \geq 2$ and equality is necessary. There are nine nonunit summands counted with multiplicity. Group the nondimension representations by their common $a = a_E$, and put $k_a = \sum m_E$, $b_a = \sum m_E^2$, $f_a = n/a$ within each group. For a nonunit simple X put $d = d_X$, $s_a = \sum \chi_E(X)$ and $t_a = \sum \chi_E(XX^*)$.

These are integers by Galois invariance; t_a is a sum of squared Hilbert–Schmidt norms in unitary realizations of the based-ring representations. The necessary row conditions are

$$\sum a s_a = -d, \quad \sum a t_a = n - d^2, \quad t_a \geq 0, \quad s_a^2 \leq k_a t_a, \quad |s_a| < k_a d. \tag{30.3}$$

For strictness, $\|\rho_E(X)\| \leq d$ by the spectral radius of XX^*. Equality in the trace bound would force $\rho_E(X)$ to act as a scalar of modulus d for each E in the block. Then every constituent of XX^* acts by its positive dimension, by equality in the trace bound for XX^*. A nonunit constituent and its dual generate the category, forcing E to be the dimension character. When $k_a = 1$, also $t_a = s_a^2 < f_a$ by character orthogonality and the vacuum contribution.

Including the unit row $(1, (k_a, k_a)_a)$, character orthogonality gives

$$\sum_X d^2 = n, \quad \sum_X d s_a = 0, \quad \sum_X t_a = b_a f_a. \tag{30.4}$$

Indeed the dimension vector is orthogonal to every non-dimension character, and $\sum_X XX^*$ acts on E by $m_E f_E$. Dual labels have the same row. A self-dual label has $s_a^2 - t_a \in 2\mathbb{Z}$, by taking the second elementary symmetric function of the eigenvalues in the block. Thus a row violating any such parity occurs an even number of times.

For reproducibility, enumerate nonincreasing lists of nine terms from $\mathcal{D}$ summing to 5399; there are 26. In the noncommutative case choose the block containing the two-dimensional representation, which gives 36 choices of a list and a block. For each, generate the rows (30.3) for $3 \leq d \leq 73$, and subtract the unit row's contribution from the target in (30.4). For a candidate row let v be its vector of moments in that equation and put $g = 2$ if the parity fails, $g = 1$ otherwise. Starting with the unit monomial, multiply the truncated geometric series $(1 - z^g \boldsymbol{x}^{gv})^{-1}$ over the row types. The coefficient at $z^{r-1}\boldsymbol{x}^{\text{target}}$ is zero in all 26 and 36 instances. Equivalently, recursively subtract g slots and gv in nondecreasing row order until the slots are zero. This integer calculation gives the two lower bounds in Theorem 30.1.

31 Decomposition over prime divisors

A modular category is *prime* if every braided Deligne decomposition has a trivial factor. For *tensor-prime*, allow all tensor decompositions. Write $\pi(D)$ for the prime divisors of D.

31.1 Splitting the group and its action

For an integral WGT modular category, a maximal Tannakian subcategory $\mathrm{Rep}(G)$ gives a faithful braided G-crossed extension of a pointed metric core $\mathcal{C}(A, \eta)$ whose equivariantization is the category [26, Proposition 4.56(i)–(ii), Definition 5.8 and Corollary 5.19], [43, Theorem 1.1]. Conversely a pointed core implies WGT by closure under extensions and equivariantizations. Thus

$$D = |G|^2 |A|, \qquad A = \perp_{\ell \| |A|} A_\ell, \qquad \rho_\ell : G \longrightarrow O(A_\ell, \eta_\ell).$$

There is no nonzero G-invariant isotropic subgroup in A: its equivariantization would enlarge $\mathrm{Rep}(G)$ inside its centralizer [26, Proposition 5.18]. The action may have kernel; a twisted double can have trivial core.

For a braided G-crossed extension of a pointed category with metric group (A, η), Etingof and Galindo [28, Lemma 4.2] show that each simple object in the g-component has Frobenius–Perron dimension $\sqrt{|(1 - \rho(g))A|}$. Their proof uses the stabilizer description of Davydov and Nikshych [20, Corollary 5.6 and Remark 5.7]. Equivariantization then gives the degrees and codegrees below.

Proposition 31.1 (Degrees and codegrees of a gauging)**.** *Let Y be simple in the g-component of a braided G-crossed extension of $\mathcal{C}(A,\eta)$. Write $H = G_Y$ for its inertia subgroup. The simple objects above the G-orbit of Y are indexed by the irreducible projective H-representations π of its equivariance cocycle. Their degrees and codegrees are*

$$d_{Y,\pi} = [G:H]\dim(\pi)\sqrt{|(1-\rho(g))A|}, \qquad f_{Y,\pi} = |A^g|\left(\frac{|H|}{\dim\pi}\right)^2.$$

In particular the gauging is integral exactly when $|(1-\rho(g))A|$ is a square for every g.

Proof. The A-orbits on the simples of $\mathcal{F}_g$ are module summands; indecomposability gives one orbit, say A/B. Its $|A/B|$ simples have equal degree, whose square is $|B|$ because the component has dimension $|A|$. To identify B, let σ be the polar form on A and β the stabilizer bicharacter. For all $a \in A$, $b \in B$,

$$(\rho(g)-1)a \in B, \qquad \beta(b,(\rho(g)-1)a) = \sigma(b,a)^{-1}$$

by [20, Corollary 5.6 and Remark 5.7]. The right side exhausts $\widehat{B}$ as a varies, by nondegeneracy and restriction of characters. Hence $B = (1-\rho(g))A$ and the component has $|A^g|$ simples. The degree formula now follows from equivariantization [13, Corollary 2.13], and $f = D/d^2$ gives the second formula. □

Lemma 31.2 (Coprime descent)**.** *Let $G = G_1 \times G_2$ have coprime factor orders. Let $\mathcal{B}_i$ be nondegenerate braided categories and $A_i = \mathrm{Inv}(\mathcal{B}_i)$. Suppose $\gcd(|G_1|,|A_2|) = \gcd(|G_2|,|A_1|) = 1$. If the action on $\mathcal{B}_1 \boxtimes \mathcal{B}_2$ in a faithful braided G-crossed extension is the product of homomorphisms $G_i \to \mathrm{Aut}_{\mathrm{br}}(\mathcal{B}_i)$ on isomorphism classes, then the extension and its equivariantization are external products of the corresponding constructions on the two factors.*

Proof. Put $A = A_1 \oplus A_2$, with the action induced by the given map on isomorphism classes. The opposite group G_{3-i} acts trivially on A_i, and its order is invertible on A_i. Averaging therefore kills its positive cohomology with these coefficients. The Lyndon–Hochschild–Serre sequence for $G_{3-i} \to G \to G_i$ identifies $H^n(G_i,A_i)$ with $H^n(G,A_i)$ by inflation. Its inverse is restriction to G_i. For scalar coefficients and $n>0$, restriction gives

$$H^n(G,\mathbb{C}^\times) \simeq H^n(G_1,\mathbb{C}^\times) \oplus H^n(G_2,\mathbb{C}^\times).$$

Indeed the Künneth tensor and Tor terms involving positive homology of both factors vanish by coprimality, and $\mathbb{C}^\times$ is divisible. Products of inflated scalar classes give the inverse.

The Picard-space classification identifies the two higher homotopy groups with A and $\mathbb{C}^\times$ [31, Theorem 5.2, Proposition 7.3 and Theorem 7.12]. External product includes each A_i as its summand and multiplies scalars. Apply the two cohomology splittings in the following order.

Stage	Group for G	Comparison with the two factors
First obstruction	$H^3(G,A)$	Restrict the zero obstruction of the given extension and project to A_i. Each factor admits a lower lift.
Choice of lower lift	$H^2(G,A)$	Choose lower lifts on the factors. Decompose the difference between their product and the prescribed lift, and add its two components to the respective choices.
Scalar obstruction	$H^4(G,\mathbb{C}^\times)$	At the corrected product lift the obstruction vanishes. Its restriction to G_i is the obstruction for the ith lift, since the opposite lift restricts to the constant map.
Choice of associator	$H^3(G,\mathbb{C}^\times)$	The difference from the prescribed associator is the product of two inflated classes. Twisting each factor by its component realizes the given extension.

The two choice groups act on torsors, so differences are taken only after fixing a lift at the preceding stage. Equivariantizing the external product by $G_1 \times G_2$ gives the product of the two equivariantizations. □

We use the Hall-factor criterion from the order graph of Erdős and Pálfy [27]. Partition $\pi(|G|)$ into two parts. If across the cut $s \nmid \ell^j - 1$ for $1 \le j \le v_\ell(|G|)$ in either direction, G is the product of its two Hall factors. Indeed, for even order every odd prime joins two. For odd order use solvability [33, p. 775] and induct on $|G|$. Choose a minimal normal subgroup $M \cong \mathbb{F}_\ell^e$. In G/M the opposite Hall factor is normal; its preimage is $M \rtimes H$ by Schur–Zassenhaus. The cut makes $|H|$ coprime to $|\mathrm{GL}_e(\ell)|$, so this preimage is $M \times H$ and its unique opposite Hall subgroup H is normal in G. A complementary Hall subgroup P commutes with H modulo M and $[P, H] \subseteq M \cap H = 1$.

Theorem 31.3 (Arithmetic and action components)**.** *For $D = \prod_\ell \ell^{a_\ell}$ form a graph $\Gamma(D)$ on the primes of D. Join distinct s, ℓ if in either direction*

$$s \mid \ell^{2j} - 1 \qquad (1 \le j \le \lfloor a_\ell/2 \rfloor).$$

An integral WGT modular category of dimension D is a braided product over the connected components Π of $\Gamma(D)$; the factor for Π has dimension $\prod_{\ell \in \Pi} \ell^{a_\ell}$. This conclusion holds for every integral modular category when all repeated primes lie in one component, including the square-free case. In particular this component formula applies whenever $2 \mid D$ or $3 \mid D$. It gives a nontrivial splitting precisely when the graph has at least two components.

For any modular category with pointed core, the same decomposition holds after adjoining the edges 2–ℓ for the odd primes with $a_\ell = 1$ and $v_\ell(d_X^2)$ odd for some simple X. When the action is known, one may instead use the following finer action graph on $\pi(|G||A|)$. Join s, ℓ if in either direction $s \mid |\rho_\ell(G)|$, or if $s, \ell \mid |G|$ and $s \mid \ell^j - 1$ for some $1 \le j \le v_\ell(|G|)$.

Proof. Across a cut in the finer graph, G splits by the Hall-factor criterion. Each factor acts trivially on the opposite primary summands of A. Lemma 31.2 thus splits the gauging into factors of dimensions $|G_i|^2|A_i|$. It remains to bound the edges of this graph.

The group edges are in $\Gamma(D)$ since $2v_\ell(|G|) + v_\ell(|A|) = a_\ell$. For an action edge choose an isometry of A_ℓ of prime order $s \neq \ell$. If s is odd, a primitive sth root occurs among its eigenvalues on $A_\ell/\ell A_\ell$: the kernel of reduction is an ℓ-group. Put $f = \mathrm{ord}_s(\ell)$ and $m = v_\ell(|A|)$. The composition factors of A_ℓ as a $\mathbb{Z}[C_s]$-module have total $\mathbb{F}_\ell$-dimension m. Their eigenvalues over $\overline{\mathbb{F}}_\ell$ are stable under Frobenius, and under inversion because the polar pairing identifies A_ℓ with its character dual as a C_s-module. If f is even the Frobenius orbit already has size f; if f is odd its inverse orbit is disjoint and adds another f. Thus $s \mid \ell^{2j} - 1$ with $j = f/\gcd(2, f) \le \lfloor m/2 \rfloor$. This works also at $\ell = 2$ and retains the rank bound $f \le \dim_{\mathbb{F}_\ell}(A_\ell/\ell A_\ell)$.

If $s = 2$ and $a_\ell \ge 2$, the edge belongs to $\Gamma(D)$. If $a_\ell = 1$, then $A_\ell = C_\ell$ has isometry group $\{1, -1\}$. The action is nontrivial exactly when some $(1 - \rho(g))A$ has ℓ-valuation one; this is exactly the odd-valuation test on d_X^2 by Proposition 31.1. In an integral gauging A_ℓ is therefore fixed whenever $|A_\ell| = \ell$ is odd, also when $\ell \mid |G|$, by the same moved-subgroup formula. More generally a pointwise fixed A_ℓ with $\ell \nmid |G|$ splits off by Lemma 31.2.

Finally, even without WGT all exponent-one primes in an integral modular category split off as pointed modular primary factors [14, Corollary 1.22]. If the repeated primes are in one component, every other component is an isolated exponent-one prime, giving the required decomposition. If 2 or 3 divides D, it joins each other repeated prime through $\ell^2 - 1$. □

Lemma 31.4 (Factors of a cyclic gauging)**.** *Let a cyclic group G act faithfully on a finite metric group A of coprime order. Join $s \mid |G|$ to $t \mid |A|$ when the Sylow subgroup G_s acts nontrivially on A_t. Suppose that each target has a neighbor and $A_t^{G_s} = 0$ on each edge. The action admits a modular gauging, and the connected components give the prime braided factors of every gauging.*

Proof. Coprimality gives $H^3(G, A) = 0$ and cyclicity gives $H^4(G, \mathbb{C}^\times) = 0$, so the two obstructions to a braided crossed extension vanish [31, Proposition 7.3 and Theorem 7.12]. Its gauging is modular [26, Proposition 4.56(ii)], and Lemma 31.2 splits it over the components. Conversely, its entire pointed subcategory is $\mathrm{Rep}(G)$: Proposition 31.1 excludes invertibles over $g \neq 1$ by faithfulness, and in the neutral component only $0 \in A$ is fixed by G. Any braided factorization therefore splits $\mathrm{Rep}(G)$ as the product of the pointed parts of its factors. This splits G as $G_1 \times G_2$ of coprime orders, since G is cyclic. De-equivariantizing factor by factor splits $A = A_1 \perp A_2$, with G_i acting trivially on A_{3-i}. If $G_s \leq G_i$ acts on A_t, the opposite summand of A_t lies in $A_t^{G_s} = 0$. Thus both ends of every edge belong to one factor. Since the factor dimensions are $|G_i|^2|A_i|$, a connected component admits no further splitting. □

31.2 Two exact criteria on a square-free set of primes

For a bipartition $S \sqcup T$ of a set of primes and a sign $\epsilon_t \in \{1, -1\}$ at each $t \in T$, draw the allowed edges

$$s \longrightarrow t \quad (s \in S,\ t \in T,\ s \mid t - \epsilon_t).$$

Here the signed graph concerns the *existence* of a prime category. By contrast, $\Gamma(D)$ bounds the interactions that the dimension permits, and the action graph of Theorem 31.3 uses a specified gauging. Connectivity in the following test refers to the underlying undirected graph. The signs belong to targets, not to individual edges.

Theorem 31.5. *Let L be composite and square-free.*

1. *A tensor-prime integral WGT fusion category of dimension L exists if and only if some bipartition has connected allowed graph with all signs $+1$. Moreover for any integral WGT category of this dimension, tensor-primeness is equivalent to primeness of its center.*

2. *A prime integral WGT modular category of dimension L^2 exists if and only if some bipartition and choice of target signs has connected allowed graph.*

Proof. Groups of square-free order. Every group H of square-free order has the form $F \rtimes E$, where $F = [H, H]$ and E are cyclic of coprime orders [50, Theorem 10.1.10]. Draw the actual nontrivial actions from the prime factors of E to those of F. Every edge $s \to t$ has $s \mid t - 1$. Subgroups belonging to distinct components commute and have coprime orders, so H is their direct product. Conversely a direct product must separate the prime factors without cutting an edge. Every connected all-positive allowed graph is realizable: let each C_s act nontrivially on every incident C_t through its cyclic automorphism group.

By [32, Proposition 9.1], an integral WGT category of dimension L has a group-theoretical presentation $\mathcal{C}(H, \omega, K, \alpha)$ with $|H| = L$. A coprime product $H = H_1 \times H_2$ splits this presentation. Indeed $K = K_1 \times K_2$ by taking powers to isolate each coordinate, and both ω and its trivialization on K can be chosen as products by the scalar cohomology calculation in Lemma 31.2. The twisted algebra and its category of bimodules then split. If H is directly indecomposable, its center is trivial: a nontrivial center is a central Hall subgroup and splits off by Schur–Zassenhaus, while a cyclic group of composite square-free order is decomposable. Its twisted double, the center of every presentation above, is prime by [40, Proposition 6.7(ii), p. 4215]: a decomposition would give commuting normal subgroups with product H and intersection in $Z(H) = 1$. The center of a tensor product is a braided product. This proves (i), using Vec_H for existence.

Modular categories of square-free root. For even L the all-positive star with source two is connected, and (i) supplies a prime center. Now let L be odd, and write $L^2 = |G|^2|A|$ as above. The order of G is square-free and coprime to $|A|$. Each primary summand A_t is weakly anisotropic, since an isotropic subgroup invariant under all its isometries would be G-invariant. As t is odd and $|A_t| = t^2$, A_t is a metric plane over $\mathbb{F}_t$ by [26, Theorem A.24(i),(iv)]. The odd image of G acts on it through the cyclic rotation group. The plane is anisotropic, since

rotations of a split plane preserve its isotropic lines, contrary to maximality of $\mathrm{Rep}(G)$. Thus for $G = F \rtimes E$ as above, F acts trivially on the planes. The prime factors of E act on the cyclic factors F_t through groups of order $t-1$ and on A_t through rotations of order dividing $t+1$. These are the positive and negative targets of a single signed graph. Across any cut G splits by its actual group actions, and its factors fix the opposite planes. Lemma 31.2 splits the category, so primeness implies connectivity.

Conversely, given a connected signed graph, take $G = \prod_{s \in S} C_s$ and one metric plane A_t for each target, split for sign $+1$ and anisotropic for sign -1. Its rotation group is cyclic of order $t - \epsilon_t$; choose the image of G there to have order the product of the incident primes. The action is faithful and every nonidentity rotation has no nonzero fixed vector. Lemma 31.4 gives a prime modular gauging; it is WGT, integral since each moved plane contributes t^2 to $|(1-\rho(g))A|$, and has dimension L^2. □

Both tests apply without a WGT hypothesis when L has at most five prime divisors by Theorem 15.3, using the center in (i). For three odd primes $p < q < r$, a connected graph is a star. All positive signs give $p \mid r-1$ and either $p \mid q-1$ or $q \mid r-1$. With arbitrary signs the criterion is

$$(p \mid q^2-1 \text{ and } p \mid r^2-1) \quad \text{or} \quad pq \mid r-1 \quad \text{or} \quad pq \mid r+1.$$

The first alternative has source p; the last two have target r. The hypothesis that L is composite matters in (i): Vec_{C_p} is tensor-prime for an odd prime p, but its center is the split metric plane.

Sharpness of the exponents and of integrality. Let C_s have prime order and act on a nonzero metric group A of coprime order with $A^{C_s} = 0$. The graph is a star, so the gauging is prime by Lemma 31.4. Its degrees and multiplicities are

$$(1, s), \qquad (s, (|A|-1)/s), \qquad (\sqrt{|A|}, s(s-1)).$$

The neutral contribution comes from the unit and the free orbits in A. Each nonidentity component has a single simple of dimension $\sqrt{|A|}$ and s equivariant structures, since $H^2(C_s, \mathbb{C}^\times) = 0$.

For distinct odd primes s, ℓ and $n = \ell^h$, use the norm on $\mathbb{F}_n \times \mathbb{F}_n$ or $\mathbb{F}_{n^2}$, followed by the trace to $\mathbb{F}_\ell$. Its polar form is nondegenerate by the trace pairing. The cyclic norm-one torus has order $n-1$ or $n+1$ and acts freely off zero. Either divisibility $s \mid n \pm 1$ therefore realizes the displayed type at $|A| = n^2$. For $(s, \ell, h) = (5, 7, 2)$ the first edge occurs at $\ell^4 - 1$; compare the anisotropic plane construction and its simple degrees in [34, Definition 3.1 and Proposition 4.2].

For $s = 2$, inversion on a cyclic odd metric group of order $M > 1$ gives the metaplectic categories [3, Theorem 3.2]. At $M = 35$ the two extra edges of Theorem 31.3 connect the isolated vertices of $\Gamma(140)$ through two. In dimension $4qr$ with q, r distinct odd primes, integral modular categories are pointed [6, Theorem 3.1]. Every fusion category of this dimension is WGT by applying Theorem 4.16 to its center.

31.3 Prime divisors of degrees and codegrees

Primes seen by simple dimensions. For integral modular $\mathcal{C}$, let $S_0 = \{p : p \mid d_X \text{ for some } X\}$. If $p \mid D$ and $p \notin S_0$, Burciu's categorical Itô–Michler theorem gives $p^{v_p(D)} \mid |U(\mathcal{C})|$ and splits off a pointed modular factor of dimension $p^{v_p(D)}$ [10, Proposition 1.3]. The divisibility also implies $p \notin S_0$ [12, Corollary 1.7]. Since $D = |U(\mathcal{C})|\,\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})$, it follows that $S_0 = \pi(\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}))$. Burciu extends this splitting to weakly integral braided categories using conjugacy-class dimensions [11, Theorem 1.1]. For WGT categories the graph of the remaining centralizer is $\Gamma(D)[S_0]$; removing only the exponent-one factors gives $\Gamma(Q(D))$. An exponent-one vertex may have joined two repeated primes. In particular, if all d_X are odd and $D = 2^\alpha M$ with M odd, the remaining odd-dimensional centralizer has dimension M. WGT for

it gives solvability of $\mathcal{C}$, for instance when $M = p^a q^b c$ with c square-free and coprime to pq [42, Theorem 7.4].

In the Frobenius–Perron prime graph [44, Definitions 2.1 and 4.1], which we call the simple-dimension graph, one joins distinct primes p, q if $pq \mid d_X$ for some X. In a product over arithmetic components Π, it is the join of the induced graphs on the nonempty $S_0 \cap \Pi$: witnesses from different factors can be tensored. Thus each connected component of its complement lies in one arithmetic factor, and with two nonempty factor supports the simple-dimension graph has diameter at most two. In the integral modular case, the class dimensions are the d_X^2, so the simple-dimension graph is also Burciu's class graph; if it is disconnected then the category is solvable [11, Theorem 1.2 and Lemma 6.2(1)]. A nonpointed prime integral modular category has $S_0 = \pi(D)$. Natale and Pacheco Rodríguez treat Frobenius groups with abelian kernel and complement in [44, Example 8.6], illustrating the calculation with $C_{341} \rtimes C_{10}$. Here take $G = C_{341} \rtimes C_5$, with faithful actions on C_{11} and C_{31}. The nonidentity fluxes have abelian centralizers of index 5 or 341, and the pure charges have degrees 1 or 5. Thus the simple dimensions of $\mathcal{Z}(\mathrm{Vec}_G)$ are $1, 5, 341$. The simple-dimension graph has edge 11–31; the arithmetic graph has edges 5–11 and 5–31.

Multiplicities and prime-power codegrees. For a weakly integral modular category let n_λ count the simples of degree λ, $N = \mathrm{ord}(T)$ and $K = \mathbb{Q}(S)$ for the unnormalized matrix. Then

$$\prod_{\substack{\ell \mid D \\ \ell \text{ odd}}} \frac{\ell - 1}{2} \;\Big|\; \frac{\varphi(N)}{|(\mathbb{Z}/N\mathbb{Z})^\times[2]|} \;\Big|\; [K : \mathbb{Q}] \;\Big|\; \prod_\lambda n_\lambda!.$$

By [8, Theorems 2.3 and 2.5(a), Lemma 2.6], $K \subseteq \mathbb{Q}(\zeta_N) \subseteq \mathbb{Q}(\zeta_n)$, where n is the order of a normalized twist matrix, and $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/K)$ has exponent at most two. The same holds for its quotient $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/K)$. Galois permutes the characters S_{ij}/d_j faithfully, since their values generate K (the unit column contains the d_i), and preserves the positive simple dimensions [8, Lemma 3.9]. The categorical Cauchy theorem [9, Theorem 3.9] identifies the primes of N and D. For instance $7 \mid D$ forces a repeated dimension of multiplicity at least three. If K is multiquadratic, then $N \mid 240$ by [8, Corollary 3.14]. Pointed core alone does not imply this: a nondegenerate cyclic metric group of order seven has $K = \mathbb{Q}(\zeta_7)$.

Proposition 31.6 (The smallest codegrees at prime-power dimension)**.** *An integral modular category of dimension p^n, $n \geq 4$, has no simple of formal codegree p^2.*

Proof. Such a category is nilpotent by successive cyclic gradings [30, Theorem 8.28], hence WGT, so use the pointed-core description above. For a simple above $Y \in \mathcal{F}_g$ with projective stabilizer representation π, the dimension formula gives

$$f = |A^g|\,|G_Y|^2/(\dim \pi)^2 \;\geq\; |A^g|\,|G_Y|.$$

Here $(\dim \pi)^2 \leq |G_Y|$ by the dimension of the twisted group algebra, and $g \in G_Y$: the crossed braiding identifies $Y \otimes Y^*$ with ${}^g(Y^*) \otimes Y$, which therefore contains the unit.

Suppose $f = p^2$. Then $p^n = f d_X^2$ is square, hence so is $|A| = p^n/|G|^2$. If $A \neq 0$, a p-group element fixes at least p elements of A by the orbit equation. Since $|(1 - g)A|$ is square by integrality, $|A^g|$ is square as well and is at least p^2. The inequality forces $G_Y = 1$, hence $g = 1$ and $|A| = p^2$. Every $h \in G$ then fixes all of A by the same argument. The neutral label $Y \in A$ has stabilizer G, giving $G = 1$, contrary to $n \geq 4$.

If $A = 0$ the crossed category has one invertible in each component, so the gauging is the twisted double of a p-group G of order at least p^2. Now $G_Y = C_G(g)$ has order at least p^2: for noncentral g it contains $Z(G)\langle g \rangle \supsetneq Z(G)$, and for central g it is G. Equality $f = p^2$ would force the twisted group algebra of $C_G(g)$ to be a single matrix algebra. For $g = 1$ its cocycle is trivial. For $g \neq 1$ its basis element at g is central: the centralizer cocycle is $\alpha_g(x, y) =$

$\omega(g,x,y)\omega(x,y,g)/\omega(x,g,y)$ for $x,y \in C_G(g)$ [44, (8.1)], so $\alpha_g(g,x) = \alpha_g(x,g) = \omega(g,x,g)$. Either way the center has dimension greater than one. □

Corollary 31.7 (Nilpotent modular categories and prime-power doubles)**.** *In an integral nilpotent modular category of dimension D, $v_p(D) \geq 4$ implies $v_p(f_X) \neq 2$. In particular every simple of $\mathcal{Z}(\mathrm{Vec}_P^\omega)$ has codegree divisible by p^4 when P is a p-group of order at least p^2.*

Proof. Use the prime-power decomposition [25, Theorem 1.1] and multiplicativity of codegrees. In a double of dimension $|P|^2$ all codegrees are even powers of p greater than one. □

The nilpotence hypothesis is necessary. The center of a Tambara–Yamagami category on an abelian group of order p^2, p odd, is integral modular of dimension $4p^4$. Its degree-p simples $Z_{\rho,\Delta}$ have codegree $4p^2$ and lie outside the principal p-block: their character values on $Y_{a,b}$, $a \neq b$, are zero rather than two modulo p [35, Proposition 4.1 and the S-matrix on p. 977]. Gelaki, Naidu and Nikshych credit Izumi with an earlier computation of these modular data [35, Remark 4.3].

Code, data and certificates

The verification release fixes every program, input and receipt cited here. Its manifest identifies the hypotheses, command, expected output and source labels for each calculation. The verification guide gives the commands and the statement index gives their printed numbers and pages. The file digests permit the same checks on an offline copy.

For the dimension bounds, the exponent comparison recovers the four thresholds in Theorem 17.1. For the boundary calculations, the terminal verifier checks seven trace obstructions and the two disjoint fiber assignments for each of the five numerical rank-100 survivors. The theoretical exclusion of those survivors is in Section 29.4.

For the general modular rank bound, the optimized verification guide documents the shared search. From its directory, run `python3 verify.py -families -jobs 4 -work family-results` to regenerate the complete family-pruned record. The same command with `-check certificate-families.json.gz` checks its source hashes, task partition and recorded totals. The option-free run reproduces the independent short rank proof. The rank guide gives the positive-quotient, nonsquare, odd and conditional square-dimension checks. The nonbraided argument uses `code/unbraided_small.py` and the odd rank-eleven verifier.

Acknowledgments

Sebastian Burciu was supported by a grant of the Ministry of Research, Innovation and Digitization, CNCS–UEFISCDI, project number PN-IV-P1-PCE-2023-0264, within PNCDI IV. Sébastien Palcoux is supported by the National Natural Science Foundation of China (NSFC), Grant no. 12471031.

GPT-6 Astra served as a research assistant in developing and drafting the arguments, identifying references, and checking finite calculations. The authors are continuing their review and have yet to reach a complete understanding of all the proofs.

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