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Prime factors of fusion-category dimensions: weak group-theoreticity and decomposition

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Abstract

Let D be the Frobenius–Perron dimension of a weakly integral fusion category and $Q(D)$ its square-full part. For primes $p<q<r<s$ and nonnegative integers a,b, we prove that a braided category of dimension D is weakly group-theoretical whenever $Q(D)\mid p^5q^ar^b$ or $Q(D)\mid p^3q^3r^as^b$. Without a braiding assumption, $D\mid pqr^as^b$ implies solvability. We also give criteria for adjoining a prime power of arbitrary exponent. For fixed m, the braided problem for dimensions dividing $\ell^e m$, with $\ell\nmid m$ prime and $e\geq 0$, reduces to finitely many dimensions.

A braided category of dimension D is weakly group-theoretical if $Q(D)<29160000$, and solvable if $Q(D)$ is odd and $Q(D)<6087156075$. Without a braiding assumption, the corresponding conclusions hold for $D<5400$ and for odd $D<121275$, respectively. We also obtain rank bounds and criteria for decomposing integral modular categories into braided factors.

1 Introduction

Let D be the Frobenius–Perron dimension of a fusion category over $\mathbb{C}$. The category is *weakly integral* if D is an integer, and *weakly group-theoretical* (WGT) if it is Morita equivalent to a nilpotent fusion category. Etingof, Nikshych and Ostrik asked whether every weakly integral fusion category is WGT [32, Section 11, Question 2]. The question is equivalent to its braided version: the Drinfeld center has dimension D^2, and weak group-theoreticity passes from the center to the original category [32, Proposition 4.1].

A prime is *repeated* if its exponent in D is at least two. Natale proved that at most two repeated primes suffice for weak group-theoreticity in the nondegenerate braided case, and obtained weak group-theoreticity for $D<1800$ and solvability for odd $D<33075$ [41, Theorems 7.4, 8.1 and 8.2]. Dong extended these results to arbitrary braided categories and obtained weak group-theoreticity for $D<120$ without braiding [22, Theorems 1.1–1.2, p. 556, and Corollaries 3.2–3.3]. A useful way to go beyond this theorem is to bound selected exponents while allowing the others to vary independently. In what follows, displayed primes are distinct and d denotes an arbitrary coprime square-free integer. We call N *covered* when every weakly integral braided fusion category of dimension dividing N is WGT.

Theorem 1.1 (Two unrestricted prime powers)**.** *For primes $p<q<r<s$, and arbitrary nonnegative integers a,b, the following hold.*

(i) *Every braided category of dimension dividing $p^3q^3r^as^bd$ is WGT.*

(ii) *Every fusion category of dimension dividing pqr^as^b is solvable.*

For three repeated primes, Theorem 8.3 gives the unrestricted family $p^5q^ar^bd$ at $p<q<r$. A quartic exponent at the middle prime also suffices (Theorem 7.3). Without any ordering condition on the primes, Theorem 8.1 covers $p^7q^5r^4d$. The five-prime theorem covers one quartic exponent, three cubic exponents and one square in any order, as well as the case of five cubic exponents (Theorems 15.1 and 15.5). These statements include every divisor of the indicated dimension. For an arbitrary number of primes, Theorem 4.17 bounds a prime outside a given

simple degree in terms of that degree with its largest prime-power factor removed. Lemma 16.1 uses two principal blocks to omit two designated primes. Proposition 16.2 consequently gives sufficient inequalities independent of both exponents. There is also a finite reduction in the opposite direction. For fixed complementary factor m the principal-block argument bounds both p and its exponent in a minimal table of dimension $p^a m$ (Theorem 16.5 and Corollary 16.6).

Write $Q(D) = \prod_{v_p(D)\geq 2} p^{v_p(D)}$. The odd condition on $Q(D)$ means precisely $4 \nmid D$; the dimension itself can be even. Comparing the family theorems gives the following strict thresholds (Theorem 17.1).

Table 1: Weakly integral fusion categories in the four dimension ranges.

Braiding assumption	Dimension hypothesis	Conclusion
Braided	$Q(D) < 29160000$	Weakly group-theoretical
Braided	$Q(D)$ odd and $Q(D) < 6087156075$	Solvable
None	$D < 5400$	Weakly group-theoretical
None	D odd and $D < 121275$	Solvable

The companion paper proves the braided bound $D < 5400^2$ directly, using the same principal-block and local Weyl arguments. Its treatment of the remaining dimensions uses short residue and norm calculations that can be checked by hand.

The first open braided core is $29160000 = 2^6 3^6 5^4 = 5400^2$. Using the cyclotomic estimates of Section 19, Sections 20–22 treat dimensions $p^6 q^6 r^4$ for arbitrary distinct primes, and Sections 23–25 treat the two boundary dimensions. The slightly degenerate branch is excluded throughout that family. At its least member a modular obstruction has rank between 101 and 46774. At dimension 5400 a non-WGT fusion category which is itself simple and perfect has rank between 11 and 462 when its fusion ring is commutative, and between 13 and 490 otherwise. In arbitrary dimension we exclude rank 15 for a nontrivial simple perfect integral modular category, giving rank at least 16. In odd dimension the bound 27 is due to Dong and Palcoux [24, Corollary 1.3]. A non-WGT perfect integral fusion category has rank at least 7, and every nontrivial odd-dimensional perfect integral fusion category has rank at least 15 (Corollary 18.2).

A different application concerns braided decomposition. For composite square-free L, Section 26 characterizes the existence of a prime integral WGT modular category of dimension L^2 by a signed divisibility graph. For example, at $L = 5 \cdot 7 \cdot 29$ the two possible edges require opposite signs at 29, since $5 \mid 29 + 1$ and $7 \mid 29 - 1$. Every integral modular category of this dimension therefore splits.

Proof strategy. Sections 2–6 reduce a divisor-minimal braided obstruction to one of two character tables. In the modular case the category is integral, perfect and fusion-simple. In the slightly degenerate case we use the positive table obtained by identifying the transparent fermion with the unit. On a full principal block, columns with different primary twists are orthogonal in the modular case. At 2 in the positive table, the corresponding assertion uses the squares of the twists. For three repeated primes, orthogonality gives two principal simples whose shared character value has prescribed residues (Sections 7–8). For four and five repeated primes, norms and inner products of local Weyl columns constrain the degrees (Sections 9–15). Section 16 treats additional prime powers. Section 17 compares the resulting exponent bounds to obtain Table 1. The even four-prime case uses the finite enumeration in Theorem 9.15. The programs and witnesses for the finite calculations are listed under Code, data and certificates.

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2 Reduction to a simple character table

Use the positive spherical structure of a weakly integral category over $\mathbb{C}$, so $D = \mathrm{FPdim}(\mathcal{C})$ and $d_X = \mathrm{FPdim}(X)$ are its categorical dimensions [30, Propositions 8.23 and 8.24]. Write $\mathcal{C}_{\mathrm{pt}}, \mathcal{C}_{\mathrm{ad}}, \mathcal{C}', U(\mathcal{C})$ for its pointed subcategory, adjoint subcategory, Müger center and universal grading group. We call a category *perfect* if its only invertible simple is the unit, and *fusion-simple* if its only fusion subcategories are trivial or the whole category. The corresponding ring conditions concern invertible basis elements and based subrings.

Lemma 2.1 (Minimal counterexample)**.** *Suppose every braided category of dimension a proper divisor of D is WGT. A counterexample $\mathcal{C}$ at D is integral, equals $\mathcal{C}_{\mathrm{ad}}$, and has one of two forms:*

(i) *$\mathcal{C}$ is modular, perfect and fusion-simple;*

(ii) *$\mathcal{C}' = \mathcal{C}_{\mathrm{pt}} = \mathrm{sVec}$, and the only fusion subcategories are $\mathrm{Vec}, \mathrm{sVec}, \mathcal{C}$. The ring modulo the transparent fermion is perfect and simple.*

Proof. A nontrivial universal grading reduces the question to its neutral component. A proper nontrivial nondegenerate subcategory splits off with its centralizer. A nontrivial Tannakian $\mathrm{Rep}(G)$ reduces to the neutral de-equivariantization, of dimension $D/(|G||H|)$ for grading support $H \leq G$. In each case the smaller dimensions divide D and WGT passes back by extension, equivariantization or tensor product [30, Propositions 8.15 and 8.20], [32, Proposition 4.1], [26, Theorems 3.13(i) and 4.44, Proposition 4.26], [41, Proposition 4.1]. Thus $\mathcal{C} = \mathcal{C}_{\mathrm{ad}}$ is integral by the dimension grading [36, Theorem 3.10], and its symmetric center is Vec or sVec. Every proper subcategory is integral and WGT, hence pointed in the absence of Tannakian subcategories [42, Theorem 3.1]. It thus lies in $\mathcal{C}_{\mathrm{pt}}$, which is trivial in the modular case by $(\mathcal{C}_{\mathrm{ad}})' = \mathcal{C}_{\mathrm{pt}}$, and is sVec in the slightly degenerate case by [41, Lemma 7.1]. In the latter case pull back based subrings from the fermion quotient. □

We use *ordinary modular table* for the character table of an integral modular category, and *positive reduced table* for the table of an integral slightly degenerate category after identifying its transparent fermion with the unit. A *minimal table* is one of the two categorical tables in Lemma 2.1, with the simplicity and perfectness conditions stated there. In particular, its twists, traces and indicators come from the category. Statements formulated for based rings or symmetric integral character data alone use the algebraic hypotheses specified in those statements.

Put $\Delta = D$ in case (i) and $\Delta = D/2$ in case (ii). In (ii) use the fusion ring $R = K_0(\mathcal{C})/(f - \mathbf{1})$ for the transparent fermion f. The twist gives $fX \not\cong X, X^*$. Duality-compatible orbit representatives containing $\mathbf{1}$ give structure constants $N_{XY}^Z + N_{XY}^{fZ}$ and unit coefficient δ_{Y,X^*}. Both rings have the following *symmetric integral character data*:

$$\begin{aligned} S_{\mathbf{1}X} = d_X, \quad S^\dagger S = \Delta I, \quad S_{X,Y^*} = \overline{S_{XY}}, \\ \chi_Y(X) = S_{XY}/d_Y, \quad f_Y = \|\chi_Y\|^2 = \Delta/d_Y^2 \in \mathbb{Z}. \end{aligned} \tag{2.1}$$

For the modular table these identities are standard [29, Sections 8.13–8.14]. For the reduced table, the normalized columns give all characters by [52, Proposition 3.3] (see [7, Proposition 2.7] for the unitary case). The ratio Δ/d_Y^2 is an algebraic integer by [52, Corollary 3.4], hence an integer here. Write $I_s = \{X : d_X = s\}$, $n_s = |I_s|$ and $\sigma\chi_X = \chi_{\pi_\sigma X}$. Complex conjugation acts by duality and commutes with cyclotomic Galois action. Thus $f_{\pi_\sigma X} = \sigma f_X = f_X$, the permutation preserves degrees, and symmetry gives

$$\sigma S_{XW} = S_{\pi_\sigma X, W} = S_{X, \pi_\sigma W}. \tag{2.2}$$

The characters take algebraic-integer values. The unit-coefficient trace is

$$\tau(B) = \sum_X \chi_X(B)/f_X. \tag{2.3}$$

Table 2: Hypotheses used in passing from character data to dimension and rank bounds.

Input	Consequence and use
Fusion ring with cyclotomic character values	A rational codegree f bounds the different and gives conductor dividing $2f$ (Theorem 2.7, Corollary 2.8).
Such a ring with symmetric integral character data	Unsigned Galois covariance in both indices and character orthogonality sharpen the conductor bound to f (Theorem 6.13).
Integral braided category	The LCM energy estimate of Proposition 2.2 includes the contribution of the symmetric center; the modular case uses the estimate of Dong and Palcoux [24].
Modular category or category with transparent fermion	Rotations and congruence give principal-block orthogonality (Theorem 3.1); at two the positive reduced table uses squared twists.
Integral modular category	The congruence action connects character fields to joint twist components (Theorem 19.12). At the first open dimension, its orbit and phase restrictions enter the rank bound (Theorem 24.1).

Proposition 2.2 (LCM energy and prime companions)**.** *Let $\mathcal{C}$ be integral braided, with its positive ribbon structure, and let S be its unnormalized matrix. For X simple set $h_X = \sum_{T\in\mathrm{Irr}(\mathcal{C}')} N_{XX^*}^T d_T$. Then*

$$\sum_{Y:S_{XY}\neq 0} \mathrm{lcm}(d_X, d_Y)^2 \leq D h_X.$$

If X centralizes $\mathcal{C}_{\mathrm{pt}}$, is not transparent, and $p \mid d_X$ while $p \nmid \mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}})$, there is a noninvertible Y with $p \nmid d_Y$, $S_{XY} \neq 0$, and $\mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}}) d_X^2 + \mathrm{lcm}(d_X, d_Y)^2 \leq D h_X$.

Proof. The Hopf-link characters $a_Y(X) = S_{XY}/d_Y$ are ring characters [29, Proposition 8.13.11]. Bézout applied to $a_Y(X)$ and $a_X(Y)$ shows $S_{XY}/\mathrm{lcm}(d_X, d_Y)$ is integral, as in the modular criterion of Dong–Palcoux [24, Lemma 2.7 and the proof of Theorem 1.1]. The regular element $R = \sum_Y d_Y[Y]$ satisfies $[T]R = d_T R$. Accordingly $a_T(R)$ is D when $a_T = d$ and zero otherwise. In the positive structure $a_T = d$ means precisely $T \in \mathcal{C}'$ by the double-braiding criterion [29, Proposition 8.20.5(i)]. Symmetry and $a_Y(X^*) = \overline{a_Y(X)}$ give

$$\sum_Y |S_{XY}|^2 = \sum_Y d_Y^2 a_Y(XX^*) = \sum_T N_{XX^*}^T d_T a_T(R) = D h_X.$$

Average this rational equality over a cyclotomic splitting field. Every nonzero integral normalized entry has mean squared modulus at least one by its norm. This proves the first assertion. For the second, $a_X(R) = 0$ and the invertibles contribute $\mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}})$. Reduction at a prime above p forces a noninvertible p-free hit; retain it and all invertibles in the first inequality. □

In a modular category $h_X = 1$. In a slightly degenerate category the same holds for the full table: $gX \not\cong X$ excludes the transparent fermion from XX^*. Passage to the reduced table halves the full energy sum and the number of invertible columns.

Lemma 2.3 (Divisibility in a tensor power)**.** *In an integral ribbon fusion category with positive spherical structure, the greatest common divisor of the dimensions of the simple constituents of $U^{\otimes k}$ divides d_U, for every simple U and every $k \geq 1$.*

Proof. Close the braid $\delta = \sigma_1 \cdots \sigma_{k-1}$ on k strands labelled by U. Successive partial traces give $d_U \theta_U^{k-1}$. On $\mathrm{Hom}(Y, U^{\otimes k})$, δ^k is the full twist and acts by θ_Y/θ_U^k. Ribbon twists have finite order (Vafa's theorem; see [29, Corollary 8.18.2(i)]), so the eigenvalues of δ are roots of unity. In the other evaluation of the trace each multiplicity-space trace is therefore an algebraic integer, weighted by d_Y. Dividing by the proposed gcd and by the root of unity θ_U^{k-1} makes a rational algebraic integer. □

For $k = 2$ one gets more than divisibility. If a_j are the eigenvalues of $c_{U,U}/\theta_U$ on the multiplicity spaces, and s_j the dimensions of their constituents (with repetitions), then

$$\sum_j s_j a_j = d_U, \qquad \chi_U(U) = \frac{\theta_U^2}{d_U} \sum_j s_j a_j^2. \tag{2.4}$$

Lemma 2.4 (Averaging over a character orbit)**.** *In ordinary or positive reduced symmetric integral data, let H be a subgroup of the cyclotomic Galois group, $e = |H \cdot Z|$ and $A_H(x) = |H|^{-1} \sum_{\sigma \in H} \sigma(x)$. Then*

$$f_Z/e = \sum_X |A_H(\chi_Z(X))|^2, \qquad 0 = \sum_X d_X A_H(\chi_Z(X)) \quad (Z \neq \mathbf{1}).$$

For the full Galois group write $K_X = \mathbb{Q}(\chi_X)$, $F_X = \mathbb{Q}(\chi_Z(X))$ and $T_X = \mathrm{Tr}_{F_X/\mathbb{Q}}(\chi_Z(X))$. Grouping columns into orbits gives

$$f_Z = \sum_{[X]} [K_Z : F_X][K_X : F_X] T_X^2, \tag{2.5}$$

$$0 = \sum_{[X]} d_X [K_X : F_X] T_X \qquad (Z \neq \mathbf{1}). \tag{2.6}$$

For any H, let A be a set of nonunits whose entries $z_X = S_{XZ}/\operatorname{lcm}(d_X, d_Z)$ are fixed by H. With $\mathsf{M}(z)$ the mean of $|\sigma z|^2$ over all conjugates, we have

$$e\left(d_Z^2 + \sum_{X \in A} \operatorname{lcm}(d_X, d_Z)^2 \mathsf{M}(z_X)\right) \leq \Delta.$$

Proof. The e conjugates of the row χ_Z are orthogonal with squared norm f_Z, and their average at X is $A_H(\chi_Z(X))$ by (2.2). Averaging preserves orthogonality to the dimension character. For the full group, the value at each of the $[K_X : \mathbb{Q}]$ conjugates of X is $T_X/[F_X : \mathbb{Q}]$. For the inequality multiply the norm identity by d_Z^2, retain the vacuum and A, then take the full Galois mean. The Galois group is abelian, so the conjugates of these entries are also fixed by H. □

Two choices of H will recur. For one nonunit X, take H fixing $L = \mathbb{Q}(z_X)$ and A equal to the full Galois orbit of X. Then $e = [K_Z : L]$ and the sum is $[K_X : \mathbb{Q}] \operatorname{lcm}(d_X, d_Z)^2 \mathsf{M}(z_X)$. For inertia at s, all columns with K_X unramified at s have fixed entries. If Z is principal at $t \neq s$ and $t \nmid d_X$, then $\chi_Z(X) \equiv d_X$ above t gives $z_X \neq 0$, hence $\mathsf{M}(z_X) \geq 1$ by the algebraic norm bound. This applies simultaneously to any set of such columns.

Lemma 2.5 (Zeros and square-fullness)**.** *In perfect symmetric integral data, every nonunit row contains a zero and Δ is square-full. For $s = d_X, t = d_Y$, $S_{XY}/\operatorname{lcm}(s,t)$ is an algebraic integer. In a simple perfect table $S_{XY} = 0$ whenever X, Y are nonunit and $(s,t) = 1$; in particular no nonunit dimension is a prime power. If Δ is odd then duality has no nonunit fixed point.*

Proof. Apply the Bézout and Galois-mean bounds of Proposition 2.2 to either character table. If a nonunit row of dimension s had no zero, averaging its norm would give

$$\Delta/s^2 \;\geq\; \sum_Y (d_Y/\gcd(s, d_Y))^2 \;>\; \sum_Y d_Y^2/s^2 = \Delta/s^2;$$

the strict inequality holds already at $\mathbf{1}$. For ordinary modular categories the row-vanishing theorem is due to Gelaki, Naidu and Nikshych [35, Theorem 6.1]. Square-fullness in the perfect integral modular case is [14, Corollary 10.10], and the coprime-degree and prime-power obstructions in the integral modular case follow from Etingof, Nikshych and Ostrik [32, Lemma 7.1 and Corollary 7.2]. For the fusion-ring vanishing theorem see [14, Theorem 1.2]. Now $P = \prod_X XX^*$

vanishes on every non-Perron character and hence, by (2.3), $\tau(P) = \prod_X d_X^2/\Delta$ is an integer. Since $d_X^2 \mid \Delta$, no prime can occur only once in Δ.

For a non-Perron character, the labels at which $|\chi(X)| = d_X$ form a based subring: equality in the triangle inequality for a product propagates to each constituent, and the set is closed under duality. Simplicity therefore makes the inequality strict off the unit, since equality everywhere would force codegree Δ. For coprime $s, t > 1$, all conjugates of $S_{XY}/(st)$ have modulus strictly below one; its norm forces it to be zero. On the other hand a nonunit row satisfies

$$1 + \sum_{Y \neq \mathbf{1}} d_Y \chi_X(Y) = 0. \tag{2.7}$$

Modulo any prime dividing Δ, the row must meet a nonunit label of degree prime to that prime. A prime-power row cannot do so.

In odd dimension all $1/f_X$ reduce to one at two; hence (2.3) gives $\tau(B) \equiv \mathrm{Tr}(N_B) \pmod 2$ for $B \in R$. Apply this congruence to $B = X^2$ and then to $B = X$, using $\mathrm{Tr}(N_X^2) \equiv \mathrm{Tr}(N_X) \pmod 2$. It gives $\delta_{X,X^*} \equiv \delta_{X,\mathbf{1}} \pmod 2$. □

Lemma 2.6 (Small ordinary codegrees)**.** *In a nontrivial simple perfect integral modular category every formal codegree is at least five.*

Proof. Use the unnormalized modular matrix S with $S_{0Y} = d_Y$. For a nonunit X put $n = d_X$, $f = D/n^2$ and $u_Y = S_{XY}/n$. Write $H = \{Y \neq \mathbf{1} : u_Y \neq 0\}$. Orthogonality gives $\sum_{Y \in H} d_Y u_Y = -1$ and $\sum_{Y \in H} |u_Y|^2 = f - 1$. Thus the degrees in H have gcd one. The $(X, 0)$ entry of the modular relation $STS = p^+ T^{-1} S T^{-1}$, where $|p^+| = \sqrt{D}$, gives

$$\sqrt{f}\, n \leq 1 + \sum_{Y \in H} d_Y |u_Y|.$$

Symmetry and algebraic integrality of the character values give $u_Y \in (d_Y / \gcd(d_Y, n))\mathcal{O}$, where $\mathcal{O}$ denotes the algebraic integers. Averaging the squared-norm identity over the cyclotomic Galois group, each nonzero term contributes at least $(d_Y / \gcd(d_Y, n))^2$, by the arithmetic–geometric mean inequality applied to its conjugates. If $f \leq 4$, we conclude that $|H| \leq f - 1$ and every d_Y in H divides n.

Perfection and the gcd condition require at least two terms. If there are two, both degrees are proper divisors of n. Cauchy's inequality then gives $\sqrt{f}\, n \leq 1 + (n/2)\sqrt{2(f-1)}$, impossible for $f = 3, 4$ and $n \geq 2$. For three terms we have $f = 4$, and all three entries are roots of unity: each has mean square one, so all its conjugates have modulus one. At most one of the degrees can be n, by the gcd condition. If one is n, the others cannot both equal $n/2$ since $n/2 > 1$. Ordering the proper divisors therefore gives $2n \leq 1 + 11n/6$. This forces $n \leq 6$. A nonunit degree in our category has at least two distinct prime factors by Lemma 2.5; hence $n = 6$, and its only divisor admissible as a nonunit degree is 6, contradicting the gcd condition. The vacuum codegree is D and is larger than four, since such a category contains a nonunit of degree at least six. □

2.1 Fields and residue classes of characters

For an irreducible representation E of a fusion ring R, put $K = \mathbb{Q}(\chi_E(X) : X)$ and let e_E be its central primitive idempotent. The formal codegree f_E is the Schur element of τ [16, Proposition 1.15]. Thus $\dim(E) f_E = \sum_X |\chi_E(X)|^2$ and [47, Lemma 2.3]

$$\sum_X \chi_E(X^*)[X] = f_E e_E. \tag{2.8}$$

For a categorified ring K is cyclotomic in the sense of lying in a cyclotomic field [30, Theorem 8.51]; the reduced characters also lift.

Theorem 2.7 (The different divides a rational formal codegree)**.** *For an irreducible representation of a categorified fusion ring, or of a fusion ring with cyclotomic character values, a rational formal codegree f is a positive integer. The different of its character field satisfies*

$$\mathfrak{D}_K \mid f\mathcal{O}_K. \tag{2.9}$$

Only primes dividing f ramify.

Proof. For $a \in \mathfrak{D}_K^{-1}$ the element $\sum_X \mathrm{Tr}_{K/\mathbb{Q}}(a\chi_E(X^*))[X]$ belongs to R. By (2.8) it equals $f\sum_{\sigma\in\mathrm{Gal}(K/\mathbb{Q})}\sigma(a)e_{\sigma E}$, with distinct conjugate characters. Its eigenvalue af is integral. Thus $f\mathfrak{D}_K^{-1} \subseteq \mathcal{O}_K$, and $a = 1$ gives the integrality of f.

□

Corollary 2.8 (Conductor of a character field)**.** *If a cyclotomic character field K has rational formal codegree f, its conductor divides $2f$.*

Proof. Work after a finite unramified extension of $\mathbb{Q}_p$, so the ramified part E of K lies in the totally ramified cyclotomic extension L of minimal conductor p^m. Let $e = [E : F]$, where F is the unramified base, and let d_E be the different exponent at E. Theorem 2.7 gives $d_E/e \le v_p(f)$. The derivative of Φ_{p^m} gives $d_L/[L : F] = m - 1/(p-1)$. For odd p and $e = e_0p^s > 1$, $e_0 \mid p-1$, cyclic wild inertia gives $m = s+1$ and L/E is tame. Thus $(d_E+1)/e = (d_L+1)/[L:F]$, or

$$\delta_p(e) := d_E/e = s + 1 - \frac{1-p^{-s}}{p-1} - \frac{1}{e}. \tag{2.10}$$

This also gives zero at $e = 1$. For $s = 0$, $e > 1$ it is positive, and for $s \ge 1$ it lies strictly between s and $s+1$. Hence $p^m \mid f$. At two, conductor four gives normalized different one. For $m \ge 3$, minimality excludes $1 + 2^{m-1}$ from $\mathrm{Gal}(L/E)$. Every nontrivial subgroup of $\langle 5\rangle$ contains that element, so $[L : E] \le 2$. The possible quadratic generators are -1 and $2^{m-1} - 1$. For either generator σ, a primitive 2^mth root ζ satisfies $v_L(\zeta - \sigma\zeta) = 2$; this is the relative different exponent. Transitivity therefore gives $d_E/e \ge m - 1 - 2/2^{m-1} > m - 2$, whence $2^{m-1} \mid f$. □

By symmetry the field of $S_{XY}/\operatorname{lcm}(d_X, d_Y)$ lies in both character fields; its different divides both codegrees by transitivity, and its conductor divides $2\gcd(f_X, f_Y)$. In particular, the field of a character of p-power codegree lies in a p-power cyclotomic field by Kronecker–Weber and the conductor bound (for odd p, adjoining a factor of two to an odd level does not change the field). Total ramification there gives a unique prime over p in the character field [50, Lemma 3.1.13 and Theorem 7.5.5].

Local projectors. Fix a prime $\mathfrak{p}$ of a splitting field over p. The full residue classes (or p-blocks) group the characters of the commutative order R with equal reductions on every basis element. Over a complete unramified coefficient ring $\mathcal{O}$ containing the residues in its residue field, the special-fiber idempotents lift. Hence for a union L of full blocks, $E_L = \sum_{X\in L} e_X$ lies in $R \otimes \mathcal{O}$, and for locally integral B,

$$\tau(E_L B) = \sum_{X\in L} \chi_X(B)/f_X \in \mathcal{O}. \tag{2.11}$$

Theorem 2.9 (Support in the principal residue class)**.** *In symmetric integral character data, for any label set $\mathcal{A}$ put $L = \{X : \chi_X(W) \equiv d_W \pmod{\mathfrak{p}}$ for every $W \in \mathcal{A}\}$. If $f_Z = p^a$, the support of χ_Z lies in L; it equals L when $\mathcal{A} = I_{d_Z}$ and $p \nmid d_Z$. For any Z and any Galois-stable $\mathcal{A}$, $\sum_{X\in L} |\chi_Z(X)|^2$ is an integer divisible by $p^{v_p(f_Z)}$.*

Proof. Write α for the sum. Symmetry gives $\alpha/f_Z = \tau(E_L Z Z^*) \in \mathcal{O}$. If $\mathcal{A}$ is Galois-stable, so is L at the fixed prime $\mathfrak{p}$: by (2.2), permuting X permutes the tests on W and preserves d_W. Thus $\alpha \in \mathbb{Z}$ with the asserted divisibility. Apply this to the set of *all* labels. Its principal block contains the unit, so $1 \le \alpha \le f_Z = p^a$ forces $\alpha = f_Z$. The support lies in this block and hence in every L. Conversely for $\mathcal{A} = I_{d_Z}$, $X \in L$ gives $\chi_X(Z) \equiv d_Z$, nonzero if $p \nmid d_Z$. □

Proposition 2.10 (Multiplicity at codegree a prime power times three)**.** *For the full principal class P at $\ell \neq 3$, suppose $t = \ell^e$, and a nonunit degree z prime to ℓ has codegree $3t$. If $\sum_{Y\in P} d_Y \chi_Z(Y) \neq 0$ for each $Z \in I_z$, then $n_z \leq 2t$, in either table. The trace hypothesis holds, for example, if all d_Y with $Y \in P \setminus \{\mathbf{1}\}$ share a prime divisor.*

Proof. Put $I = I_z$, $k = |P|$ and let $B = sE_P s^*$ be the conjugate of the coordinate projection onto P, $s = S/\sqrt{\Delta}$. The matrix $H = 3B_{I,I}$ has integral entries:

$$H_{XZ} = t^{-1} \sum_{Y\in P} \chi_X(Y)\overline{\chi_Z(Y)} = 3\tau(E_P X Z^*).$$

Indeed the sum is a rational integer by Galois stability, and the last expression is integral at ℓ by (2.11). Because B is a projection and $B_{X0} \neq 0$ by hypothesis, $[H(3\,\mathrm{id} - H)]_{XX} = 9\sum_{j\notin I} |B_{Xj}|^2 > 0$. Thus every diagonal is one or two and has at most one nonzero off-diagonal entry, necessarily ± 1. The resulting blocks have size at most two and positive rank. If all diagonals are two, H is invertible. All entries on $I \times P$ are nonzero by principality and $\ell \nmid z$; Galois averaging gives $k \leq tH_{XX} \leq 2t$. For $k \leq t$ we obtain $n_z/2 \leq \operatorname{rank} H \leq k$. Otherwise all diagonals are two and $n_z = \operatorname{rank} H \leq k$.

□

Blocks of defect one. After a sufficiently large finite unramified base change with fraction field F, integer ring $\mathcal{O}$ and residue field k, the field factors of a block correspond to its full inertia orbits and are totally ramified. An orbit is *tame* when wild inertia fixes it pointwise. The following observation explains why a single character of small defect can determine the entire block.

Lemma 2.11 (A block with a conductor of length one)**.** *Let A be a commutative local symmetric $\mathcal{O}$-order with normalization $M = \prod_i \mathcal{O}_i$, where all residue fields are k. Suppose its perfect trace on factor i is $\operatorname{Tr}_{E_i/F}/f_i$, and put $c_i = e_i v_p(f_i) - d_i$, where e_i, d_i are its ramification index and different exponent. A factor with $c_i = 0$ is the whole block and $A = \mathcal{O}_i$. If $c_i = 1$, there are precisely two factors, both with $c = 1$, and $A = \{(x, y) \in M : \bar{x} = \bar{y}\}$.*

Proof. Since A is self-dual, the conductor $(A : M)$ is the trace dual $M^{\#} = \prod_i f_i \mathfrak{D}_i^{-1}$. A unit conductor on one factor puts its idempotent in the local ring A. If $c_i = 1$, the conductor contains its maximal ideal $\mathfrak{m}_i$ on that factor alone. Writing B for the projection to the other factors, we have $A = \mathcal{O}_i \times_k B$: common residues are necessary, and the copy of $\mathfrak{m}_i$ lifts every compatible pair. There must be another factor, since otherwise $A = \mathcal{O}_i$ has conductor exponent zero. The perfect trace identifies A with $\operatorname{Hom}_{\mathcal{O}}(A, \mathcal{O})$, so A is Gorenstein. The fiber-product criterion [40, Fact 2.6 and Corollary 2.7] makes B a discrete valuation ring. The conductor of this two-factor order is the product of the two maximal ideals. □

Proposition 2.12 (The block of a defect-one character)**.** *In a local symmetric order as above, with cyclotomic characters and rational integral codegrees, suppose a character has defect $v_p(f) = 1$. Except for a single ramified quadratic branch at two, the block has two inertia orbits and is the fiber product of their valuation rings over k. In particular no unramified higher-defect character belongs to this block.*

For odd p both orbits are tame and of defect one. At $p = 3$ their lengths in symmetric integral character data are one and two. At two, if the given branch is unramified and the other has degree e, the block has rank $1 + e$ and the other defect is $1 + v_2(e)$. Consequently uniform defect one at two forces rank two; in an even-rank dyadic block of rank greater than two all defects are at least two.

Proof. At odd p the given branch is tame by Corollary 2.8, so $e_i - d_i = 1$; an unramified defect-one branch at two has the same conductor length. By Lemma 2.11 the other branch satisfies

$v_p(f) = (d_E + 1)/e$, which gives defect one if it is unramified. At odd p the right side for wild degree $p^s > 1$ lies strictly between s and $s + 1$ by (2.10); hence both branches are tame of defect one. At three their lengths are one or two. The integral identity trace $\sum e_i/f_i = \sum e_i d_{X_i}^2/\Delta$ forces the sum of the lengths to vanish modulo three, since after canceling the common powers of three each unit square is one there.

At two a ramified defect-one branch satisfies $2(e_i - 1) \le d_i \le e_i$ by the first two ramification groups and the different bound. Hence $e_i = d_i = 2$ and the conductor has length zero. In the two-branch case the identity trace $1/f_1 + e/f_2$ is integral; its first summand has valuation -1, so the other does too. Since e is a power of two, even rank $1 + e$ forces $e = 1$. A defect-zero branch is unramified with conductor length zero, hence has rank one by itself. □

3 Prime blocks and primary twists

Write $s = S/\sqrt{D}$, $T = \mathrm{diag}(\theta_i)$ and $N = \mathrm{ord}(T)$ for ordinary modular data. Let v_i denote a coordinate vector, e_i a character idempotent, and 0 the unit label. A twist decomposes uniquely into primary parts $\theta_{i,p}$; we call i *p-active* if $\theta_{i,p} \neq 1$.

For a weakly integral modular category use the lift $W = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \mapsto s$, $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \mapsto t = T/\gamma$, where $\gamma^3 = p^+/\sqrt{D}$. We may take γ of two-power order, equal to one for a center. The representation factors through $\mathrm{SL}_2(\mathbb{Z}/M)$, $M = \mathrm{ord}(t)$ [21, (1-10), Theorem II(i), Proposition 6.7(ii) and Theorem 6.10]. A subscript on W, U denotes its local factor under the Chinese remainder isomorphism.

For integral slightly degenerate $\mathcal{B}$ put $\Delta = \mathrm{FPdim}(\mathcal{B})/2$ and let g be its transparent fermion. With the positive spherical structure, $\mathcal{B}$ is pseudo-unitary, so the extension theorem of Johnson-Freyd and Reutter [37, Corollary 1.3] gives a minimal modular extension of dimension 4Δ, with matrices s, T. The Weyl matrix interchanges $v_i \mapsto v_{gi}$ and $v_i \mapsto \epsilon_i v_i$, with ϵ_i the fermion-monodromy sign. On their common $+1$ space, the vectors $(v_i + v_{gi})/\sqrt{2}$ give the normalized reduced matrix $\widehat{s}$. Even twist powers and odd local factors preserve this space. Their Bruhat and Galois identities therefore restrict, with the same local Galois sign since the two dimension normalizations differ by a rational factor.

Theorem 3.1 (Block orthogonality)**.** *For a full p-block B of ordinary modular data,*

$$\sum_{z \in B} \overline{s_{zi}} s_{zj} = 0 \qquad (\theta_{i,p} \neq \theta_{j,p}).$$

In reduced integral data the same holds for $\widehat{s}$ at odd p, and at two whenever $\theta_{i,2}^2 \neq \theta_{j,2}^2$. In particular, in the ordinary table

$$p \nmid f_z \in \mathbb{Z}, \quad p \mid \mathrm{ord}(\theta_j) \quad \Longrightarrow \quad S_{zj} = 0. \tag{3.1}$$

Proof. First consider ordinary data. Write $L_k^i = \nu_{k,1}^{(i,i^*)}$ for the indicator with insertion in $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$, extended linearly by [46, Corollary 5.5]. Its modular formula on a simple c is $\sum_{a,b} s_{ia} s_{ib} (\theta_a/\theta_b)^k N_{ab}^c$ [46, Proposition 6.1]. Pairing with $f_z e_z$ replaces the fusion coefficients by $\overline{\chi_z(a)\chi_z(b)}$; duality in the second factor gives

$$L_k^i(e_z) = |(sT^k s^{-1})_{iz}|^2. \tag{3.2}$$

A rotation congruence. Put $m = N_{p'}$. Given a genuine object A, choose $b \ge \max(1, v_p(N))$ with $p^b \equiv 1 \bmod m$, and $n \ge 2b$. Then

$$L_m^i(A^{p^n}) \equiv L_{mp^b}^i(A^{p^{n-b}}) \pmod{p^{n-2b+1}\overline{\mathbb{Z}}_p}. \tag{3.3}$$

Strand grouping identifies the two sides with traces of powers p^n, p^{n-b} of the same rotation on mp^n strands [46, Lemmas 2.5, 2.7 and Proposition 2.8(ii)]. Its eigenvalues are mp^nth roots, since

the insertion has twist one, and all power traces lie in $\mathbb{Q}(\zeta_N)$ by [46, Proposition 5.7], since $N = \mathrm{ord}(T) = \mathrm{FSexp}(\mathcal{C})$ by [44, Theorem 7.7]. Thus Galois orbits over that field have constant multiplicity. The two powers agree on the prime-to-p part and on roots of p-power order at most p^{n-b}. For order $p^t > p^{n-b}$, either power is constant on the subgroup orbit given by units $1 \bmod p^b$ on the p-factor. Its size p^{t-b} is divisible by p^{n-2b+1}, proving the congruence.

Separating blocks. In the mod-p fusion ring, residues with a given kernel and image $\mathbb{F}_{p^h}$ are Frobenius conjugate. Denote their blocks in Frobenius order by $B_0, \dots, B_{h-1}$ (all conjugates occur by local Galois symmetry). For $a \in \mathbb{F}_{p^h}^\times$, lift the Chinese-remainder element with residue a at this kernel and zero at all others to a genuine object, taking nonnegative basis coefficients. Choose $h \mid b, n$. As n tends to infinity, both powers of the object in (3.3) converge in character coordinates to $\sum_j [a]^{p^j} E_{B_j}$, where brackets denote Teichmüller lifts. Hence L_m^i and $L_{mp^b}^i$ agree on that sum. The characters $a \mapsto [a]^{p^j}$ of $\mathbb{F}_{p^h}^\times$ are distinct, so the functionals agree on each E_{B_j}. As $N \mid mp^b$, (3.2) gives

$$\sum_{z \in B} |(sT^m s^{-1})_{iz}|^2 = \mathbf{1}_{i \in B}. \tag{3.4}$$

Thus $sT^m s^{-1}$ commutes with the coordinate projector of B. Conjugating back gives the assertion because T^m distinguishes the p-primary twists. If $p \nmid f_z$, the idempotent e_z is locally integral, so $\{z\}$ is a block. Taking $i = 0$ gives (3.1).

Passage to a fermion quotient. In its minimal modular extension select all labels whose characters on the even subring have the prescribed residue of B. Their projector P is a sum of ordinary full-block projectors: equality of residues on the full ring implies equality on the even part. Since z and gz agree on the even subring, P preserves the positive fermion subspace, on which it restricts to the reduced block projector. Put $T_p = \mathrm{diag}(\theta_{i,p})$. The ordinary result gives $[P, sT_p s^{-1}] = 0$. Restrict this identity to the positive subspace at odd p, or its squared-operator version at two: s preserves that subspace and the members of a fermion pair have equal odd primary twists and equal squared dyadic twists. This gives precisely the two reduced assertions. □

Lemma 3.2 (Local modular action on residue blocks)**.** *Every full p-block subspace is invariant under the p-local modular group, in ordinary weakly integral data and also in positive reduced data at odd p. In particular W_p preserves each block.*

Proof. Its coordinate projector commutes with U_p, which is diagonal, and with $sU_p s^{-1}$ by Theorem 3.1, since U_p differs from T_p by a scalar. These elementary unipotents generate the local factor. □

3.1 When two primes never occur in the same twist

Lemma 3.3 (Twist primes seen by a row)**.** *In ordinary weakly integral modular data, if the local Weyl element W_p preserves the line of z then $p \nmid f_z$. In particular:*

1. *$p \mid f_z$ exactly when the z-row hits a p-active column;*
2. *if no twist contains both p and q, every p-active label has q-unit codegree.*

Every prime in the twist of z divides f_z.

Proof. We may assume $p \mid D$ and hence p divides the level by categorical Cauchy. Choose k divisible by its p-part and congruent to one on the complementary level. The identity $U(WUW^{-1})U = W^{-1}$ on the complementary factors identifies $sT^k s^{-1}$, up to diagonal phases on the left and right, with their inverse Weyl. If W_p preserves the line of z, the absolute value of the resulting $(0, z)$ entry is $|s_{0z}| = 1/\sqrt{f_z}$. Thus for $y = f_z e_z$, (3.2) gives $L_k^0(y) = 1$. Grouping p strands gives

$$L_k^0(y)^p \equiv L_{k/p}^0(y^p) = f_z^{p-1} L_{k/p}^0(y) \pmod{p\overline{\mathbb{Z}}}.$$

Indeed the pth power of a trace agrees modulo p with the trace of the pth power. This applies first to each simple, then by additivity and the Frobenius congruence in the commutative fusion ring to the integral coefficients of y. Thus $p \nmid f_z$.

Now if the row misses all active columns, then $sT_p s^{-1} v_z = v_z$. The local upper and lower unipotents both preserve the line; they generate the local group, so W_p also preserves it. On this line the two unipotents have equal scalars because they are conjugate. Comparing with $sT_p s^{-1} v_z = v_z$ gives $\theta_{z,p} = 1$. Together with (3.1) these facts give (1) and the twist assertion. For (2) the q-local group preserves every nontrivial p-twist fiber. There its upper unipotent and all its local conjugates act by scalars since the q-twist is trivial. Thus W_q preserves each coordinate line. □

Proposition 3.4 (Cauchy and vanishing at odd primes)**.** *For an integral slightly degenerate category and any odd $p \mid \Delta$, some simple has p-active twist. All such labels have zero S-entry on rows of p-unit codegree.*

Proof. If all even labels of a minimal extension were p-inactive, the p-local upper unipotent would fix the positive fermion subspace pointwise (the anomaly is dyadic). The lower unipotent would too, since s preserves this subspace. The fixed space of the local factor is invariant under the entire modular representation and contains $v_0 + v_g$, hence also $T(v_0 + v_g) = v_0 - v_g$ and sv_0. Every coordinate of sv_0 is nonzero. Thus the p-primary twist would be trivial everywhere, contrary to Cauchy's theorem in the extension [9, Theorem 3.9]. Vanishing follows from block orthogonality on the singleton block of a p-unit codegree. □

3.2 Twist orders and individual codegrees

Theorem 3.5 (Twist orders and formal codegrees)**.** *In a weakly integral modular category put $m_z = \operatorname{ord}(\theta_z)$ and $f_z = D/d_z^2$. Then*

$$(m_z)_{\mathrm{odd}} \mid f_z, \qquad m_z \mid \begin{cases} 4f_z & \text{in the integral case,} \\ 8f_z & \text{in general,} \end{cases} \qquad \operatorname{ord}(\theta_x/\theta_y) \mid 4\operatorname{lcm}(f_x, f_y), \qquad N \mid 4D.$$

The odd-part assertion holds also for an integral slightly degenerate category with reduced codegree $(D/2)/d_z^2$.

Proof. The lift $t = T/\gamma$ has $\operatorname{ord}(\gamma) \mid 8$ in the integral case and $\operatorname{ord}(\gamma) \mid 16$ in general [21, Proposition 6.7 and Theorem 6.10]. If p^k is the exact p-part of $\operatorname{ord}(t_{zz})$, each automorphism fixing K_z fixes z, so its square fixes t_{zz} [21, Theorem II]. At odd p its exponent is $\pm 1 \bmod p^k$; at two and $k \geq 4$ it is $\pm 1 \bmod 2^{k-1}$. Thus K_z contains the real subfield F at conductor p^a, with $a = k$ for odd p and $a = k - 1$ at two.

The conductor of F is p^a unless $F = \mathbb{Q}$. By Corollary 2.8, at odd p we obtain $p^k \mid f_z$; the exception $p^k = 3$ follows from Lemma 3.3. At two and $k \geq 4$ we get $2^{k-2} \mid f_z$. For an actual twist whose two-part exceeds $\operatorname{ord}(\gamma)$ this gives the same estimate. Otherwise use $2 \mid f_z$ for every nontrivial actual two-part, again by Lemma 3.3. This gives the two objectwise bounds.

For a ratio with two-part 2^k, $k \geq 4$, one of t_{xx}, t_{yy} has two-part at least 2^k, so $2^{k-2} \mid \operatorname{lcm}(f_x, f_y)$. For $1 \leq k \leq 3$ at least one actual twist has even order. Combined with the odd-part result this proves the ratio bound; taking $y = 0$ gives $N \mid 4D$ since the codegrees divide D. In a minimal modular extension each original label has codegree four times its reduced codegree, giving the last assertion. □

For modular categories of prime-power exponent, Czenky, Plavnik and Schopieray bound the exponent by the norm of the global dimension, up to a factor four at two [18, main theorem, p. 2]. Here integrality of D allows arbitrary exponent and gives an individual bound. Ising ($D = 4$, $d_z^2 = 2$, $m_z = 16$) attains the weakly integral factor eight and the global factor four. By passage to the center and [44, Theorem 5.5], the Frobenius–Schur exponent of any weakly integral category of dimension n divides $4n^2$, and its odd part divides n^2.

3.3 Mass of a primary twist sector

In an integral modular category write $\mu(A) = D^{-1}\sum_{X\in A} d_X^2$ for a set of labels A.

Lemma 3.6 (Mass and overlap of two label sets)**.** *For sets of labels A, B put $a = \mu(A)$, $b = \mu(B)$. Then*

$$D^{-1}\sum_{X\in A, Y\in B} d_X d_Y |s_{XY}| \;\geq\; ab - \sqrt{a(1-a)b(1-b)}.$$

In particular $S_{A,B} = 0$ implies $a + b \leq 1$, and hence the union of the A-row supports has mass at least a. For m labels H of common degree z and codegree $f = D/z^2$, taking $A = B = H$ gives $\sum_{X,Y\in H} |S_{XY}|/z \geq \sqrt{f}\, m(2m/f - 1)$.

Proof. The ranges of the projections $P = s^*E_A s$, $Q = TsE_B s^*T^*$ have orthonormal bases s^*v_X, Tsv_Y. Their cross inner products have moduli $|s_{XY}|$ by $sTs = \xi T^{-1}sT^{-1}$, $|\xi| = 1$. Expanding $\langle Pv_0, Qv_0\rangle$ in these bases bounds its modulus by the displayed sum. The vacuum coordinates of Pv_0, Qv_0 are a, b, equal to their squared norms. On their components orthogonal to v_0, Cauchy–Schwarz gives the claimed lower bound. □

Proposition 3.7 (Mass from a generating simple)**.** *Put $L_p = \sum_{p|\mathrm{ord}(\theta_i)} d_i^2$. If X generates $\mathcal{C}$ and $L_p > 0$, then $L_p \geq (p-1)D/(2pd_X)$. In a fusion-simple integral modular category one can take the smallest nonunit degree whenever $p \mid D$.*

Proof. Set $n = N_{p'}$, $t_i = \theta_i^n$. The indicator formula [44, Theorem 7.5] gives

$$\nu_n(X) = D^{-1}\sum_{i,j} N_{ij}^X d_i d_j\, t_i/t_j \in \mathbb{Z}:$$

it is an algebraic integer in both the p-power cyclotomic field of these phases and the prime-to-p field of the n-fold rotation. The weights sum to Dd_X. Equality $\nu_n(X) = d_X$ would force equal phases on every edge $i^* \to j$ of multiplication by X (duality preserves twists). Since X generates and X^* reverses the edges, every phase would be one. Thus $d_X - \nu_n(X) \geq 1$. The mean of a nontrivial p-power root is $-1/(p-1)$ at order p and zero otherwise. After Galois averaging, each unequal-phase pair contributes at most $p/(p-1)$ times its normalized weight to this deficit. Its first or second label is active; either choice has total weight $L_p d_X$ by reciprocity. This proves the bound. Categorical Cauchy gives $L_p > 0$ for $p \mid D$ [9, Theorem 3.9]. □

Proposition 3.8 (Odd-valuation slices)**.** *Suppose p and $v_p(D)$ are odd. Let B be a full p-block or the whole label set. Let x, y have trivial p-twists, with y fixed by p-inertia. For a complementary twist η put*

$$M_j = \sum_{\substack{i\in B,\ \theta_{i,p'}=\eta \\ \mathrm{ord}(\theta_{i,p})=p^j}} s_{xi}\overline{s_{yi}}.$$

Then $M_1 = (p-1)M_0$, also after summing over η. The same formula holds in integral positive reduced data with D replaced by Δ, provided the complementary fiber is taken up to the fermion sign. Thus on each such fiber the squared-dimension mass of order-p twists is $p-1$ times the inactive mass, and the active proportion is at least $1 - 1/p$. For the entire label set, activity at all primes in any set J of odd primes of odd valuation has proportion at least $\prod_{p\in J}(1-1/p)$, even conditional on fixed twist components outside J.

Proof. Let E project onto $B \cap \{i : \theta_{i,p'} = \eta\}$. It commutes with the local p-factor by Lemma 3.2 and commutation of distinct factors; so does $Q = sEs^{-1}$ since s normalizes that factor. Write $u = T_p, w, h_t$ for its upper unipotent, Weyl and torus. For a unit n modulo the local level p^a, the nth weighted moment of the p-twist on the fiber is

$$(Qwu^n w^{-1})_{xy} = (Qu^{-1/n}wh_{-n}u^{-1/n})_{xy} = \Big(\frac{-n}{p}\Big)(Qw)_{xy}.$$

The middle identity is Bruhat. The outer unipotents act trivially at x, y. Since inertia fixes y, the torus acts there by the Galois sign of $d_y/\sqrt{D}$ [21, Theorem II and Remark 4.5]; at odd $v_p(D)$ this is the displayed quadratic character, from $\sqrt{(-1)^{(p-1)/2}p}$. Average over n: the right side vanishes, while the left gives $M_0 - M_1/(p-1)$ by the means of p-power roots. For the reduced formula, use the restricted projector of Theorem 3.1 and the complementary twists modulo sign. The odd local group acts on the positive subspace, and its torus sign is the same since the two dimension normalizations differ by two. At $x = y = 0$ the weights are d_i^2/D or d_i^2/Δ. The active bound holds on each complementary fiber and thus on every union of them; iteration over J proves the product bound. □

Corollary 3.9 (The two masses in a defect-one block)**.** *Let B be a full p-residue block of an integral modular category, where p is odd, and assume that $v_p(f_i) = 1$ for every $i \in B$. Then B consists of one inertia orbit with trivial p-primary twist and one inertia orbit with p-primary twist of order p. If their respective lengths and dimensions are a, r and b, s, then*

$$bs^2 = (p-1)ar^2.$$

Proof. The codegree identity implies that $v_p(D)$ is odd. Proposition 2.12 gives exactly two inertia orbits; their lengths divide $p-1$ and the dimension is constant on each. Theorem 3.5 bounds all their p-primary twist orders by p. Inertia preserves these orders. Proposition 3.8, with the complementary condition omitted, says that both the trivial and the order-p masses are nonzero and have ratio $p-1$. Thus one orbit is of each kind and the displayed equation follows. □

Lemma 3.10 (The visible class at defect one)**.** *In an ordinary integral modular table, let t, s be distinct odd primes with $s \nmid t-1$. Suppose $\mathcal{U}$ is a nonempty set of s-free labels of t-defect one and $\mathcal{V}$ a nonempty set of t-defect-one labels in P_s. All these rows see only 1 and a single common square class of primitive tth roots as t-phases.*

Proof. A defect-one block has an inactive and an active orbit; the latter has one square class of primitive phases. Their lengths $a_0, a_1 \mid t-1$ and degrees d_0, d_1 satisfy $a_1 d_1^2 = (t-1)a_0 d_0^2$ (Proposition 2.12, Corollary 3.9). Thus the two degrees have equal s-valuation. In each block of a label in $\mathcal{U}$, all active representatives have s-unit degree, so every $V \in \mathcal{V}$ hits them by principality. Its row can see only phases in its own block: the local Weyl preserves the block and the complementary Weyl preserves t-phases. The active classes of every such pair of blocks therefore agree. □

4 Principal defects and fermion quotients

The conductor of a residue block bounds its principal degrees. Paired with orthogonality, this excludes a third repeated prime of exponent at most three. Two further tools will also be needed at higher exponents: comparison of the two fermion quotients at two, and positive residues of a common character value at distinct primes.

4.1 The first possible principal defect

Corollary 4.1 (Principal valuation bound)**.** *If $v_p(\Delta) \geq 2$, every ordinary or reduced principal-block row satisfies*

$$v_p(f_X) \geq 2, \qquad v_p(d_X) \leq \lfloor (v_p(\Delta) - 2)/2 \rfloor.$$

In particular the defect is at least three at odd valuation, and at valuation two or three every principal degree is p-free.

Proof. The dimension character is unramified of defect $v_p(\Delta)$. Proposition 2.12 excludes defect one; defect zero has an integral idempotent separating it from the dimension character. Now use $d_X^2 f_X = \Delta$. □

Proposition 4.2 (Prime avoidance on a principal block)**.** *Let P be a full principal p-block of ordinary integral modular data, or of positive reduced data at an odd prime. Let $E = P$; in ordinary data one may also intersect P with the fiber of trivial prime-to-p twist. Assume that the unit is the only invertible in E. Fix a label X and a prime $s \neq p$. If X is p-active, or if p and $v_p(\Delta)$ are odd and $s \nmid p-1$, then at each prime over s there is a nonunit $Y \in E$ for which $d_Y \chi_X(Y)$ has nonzero residue. In particular $s \nmid d_Y$.*

Proof. The block projector commutes with the local group (Lemma 3.2); so does the indicated complementary projector. Conjugation by the global Weyl normalizes the local group. For an active X the $X, 0$ entry of the conjugate projector is zero, whence $\sum_{Y \in E} d_Y \chi_X(Y) = 0$. For an inactive X in the second case the slice identity (Proposition 3.8) is

$$\sum_{\substack{Y \in E \\ \operatorname{ord}(\theta_{Y,p})=p}} d_Y \chi_X(Y) = (p-1) \sum_{\substack{Y \in E \\ \theta_{Y,p}=1}} d_Y \chi_X(Y).$$

In either identity the vacuum contribution is a unit at s, so a nonunit term must survive modulo each prime over s. □

In either setting of the proposition, at valuation two or three the nonunit degrees in P_p of a perfect table are p-free and have gcd one: apply prime avoidance to an active column at every other prime.

For larger defects an index inequality bounds the total ramification of a block.

Proposition 4.3 (A mixed-defect block bound)**.** *Let A be a commutative local symmetric order over an unramified p-adic integer ring, with separable cyclotomic generic algebra, rational positive integral codegrees, and a common character residue. After an unramified extension write its generic factors as E_j/K, $1 \leq j \leq t$, with totally ramified degrees $e_j = u_j p^{s_j}$, $u_j \mid p-1$. Put $a_j = v_p(f_j)$. Then*

$$\sum_{j=1}^{t} (p^{s_j} - (a_j - 1)e_j) \leq 2.$$

At two this becomes $\sum_j (2 - a_j)e_j \leq 2$.

Proof. Put $M = \prod_j \mathcal{O}_{E_j}$. The trace on E_j is $\operatorname{Tr}_{E_j/K}/f_j$, so the discriminant-index formula and self-duality of A give $2\operatorname{length}(M/A) = \sum_j (a_j e_j - d_j)$. Common residues impose $t-1$ independent conditions on M; the first two ramification groups give $d_j \geq e_j + p^{s_j} - 2$. Combining yields the bound, with $e_j = 2^{s_j}$ at two. □

4.2 The cubic exponent at two

Lemma 4.4 (An integral minimal modular extension)**.** *Let $\mathcal{B}$ be integral and slightly degenerate, with $\dim \mathcal{B} = D > 2$, pointed subcategory $\mathrm{sVec} = \langle g \rangle$, and no other proper nontrivial fusion subcategory. Every minimal modular extension has pointed subcategory sVec, adjoint subcategory $\mathcal{B}$, and precisely the proper fusion subcategories $\mathrm{Vec}, \mathrm{sVec}, \mathcal{B}$. Moreover one can choose an integral minimal modular extension. In such an extension, noninvertible simples of coprime dimensions have zero S-entry.*

Proof. By the extension theorem of Johnson-Freyd and Reutter [37, Corollary 1.3] there is a minimal modular extension $\mathcal{M}_0$, of dimension $2D$, with neutral component $\mathcal{B}$ for fermion monodromy. An odd invertible h would give a pointed subcategory of order four whose bicharacter has trivial radical: $b(g,g) = 1$, $b(g,h) = -1$, and in the cyclic case $b(h,h)^2 = -1$. Splitting it off gives $\mathcal{B} = \mathrm{sVec} \boxtimes \mathcal{N}$ with $\dim \mathcal{N} = D/2 > 1$, impossible. Hence $(\mathcal{M}_0)_{\mathrm{pt}} = \mathrm{sVec}$ and $(\mathcal{M}_0)_{\mathrm{ad}} = \mathcal{B}$. A proper subcategory crossing the grading has twice the dimension of its intersection with $\mathcal{B}$, hence dimension two or four. It is pointed or Ising. The pointed case has an odd invertible.

In the Ising case the subcategory is nondegenerate by [26, Corollary B.12] and splits off with fermion g, giving the same contradiction.

Normalize $t_X = \theta_X/\zeta$ by choosing a two-power root ζ with $\zeta^3 = p^+/\sqrt{2D}$ [21, Theorem 6.10]. At an odd prime any inertia element σ fixes t_0. The Galois image of the unit is invertible and its normalized twist is $\sigma^2(t_0) = t_0$; thus it is the unit, since $\theta_g = -1$ [21, Theorem II]. The vacuum-column ratios d_X are consequently unramified at every odd prime. Since d_X^2 is an integer, d_X is an integer or $\sqrt{2}$ times one.

In the nonintegral case the integral subcategory is exactly $\mathcal{B}$. Let $\mathcal{I}$ be Ising with fermion ψ and noninvertible σ. Condense the diagonal boson $g \boxtimes \psi$ in $\mathcal{M}_0 \boxtimes \mathcal{I}$: the local modules have dimension $(2D)4/2^2 = 2D$ and are nondegenerate [26, Propositions 4.26, 4.55(i) and 4.56(i)–(ii)]. Free modules on $X \boxtimes \mathbf{1}$, $X \in \mathcal{B}$, embed $\mathcal{B}$ braided fully faithfully since the nontrivial algebra summand contributes no morphisms by adjunction. These give all even pairs after identifying boson translates. An odd pair (Y, σ) has dimension $2n$; a free orbit gives one simple of dimension $2n$, and a fixed orbit gives two of dimension n since a cyclic stabilizer has only one-dimensional projective irreducibles. This extension is integral.

For noninvertible X in an integral extension, $\langle XX^* \rangle = \mathcal{B}$: containment in sVec gives $d_X^2 \leq 2$ since $1, g$ each occur at most once. A nonzero entry on coprime-dimensional noninvertibles would imply projective centrality [29, Proposition 8.22.5 and Lemma 9.15.1], hence membership of one in $\mathcal{B}' = \mathrm{sVec}$, impossible. □

Lemma 4.5 (The two fermion quotients)**.** *Let $\mathcal{M}$ be integral modular with fermion g and $\mathcal{B} = \langle g \rangle'$. Put*

$$R_\pm = K_0(\mathcal{B})/(g \mp 1).$$

These are symmetric integral orders with identical special fibers at two. Their blocks selected by the dimension residue have the same rank. Restrictions of modular characters have precisely the fermion orbits as fibers and belong to R_+ or R_- according as their labels are even or odd. For either quotient their codegrees are

$$f_X^\pm = \begin{cases} f_X/2 & gX = X, \\ f_X/4 & gX \neq X, \end{cases} \qquad f_X = \dim(\mathcal{M})/d_X^2. \tag{4.1}$$

A restricted character has the dimension residue if and only if the full modular character does.

Let P be the full principal two-block of $\mathcal{M}$. Then

$$\sum_{X \in P \cap \mathcal{B}} d_X^2 = \sum_{X \in P \setminus \mathcal{B}} d_X^2. \tag{4.2}$$

If $\dim \mathcal{B} = 2^{s+1}m$ with $s \in \{2, 3\}$ and m odd, the positive principal degrees are odd and its rank k satisfies $2^s \mid k$. Every full principal degree is odd or twice odd, and an even-dimensional principal label lies outside $\mathcal{B}$.

Proof. Choose duality-compatible orbit representatives in $\mathcal{B}$ ($gX \not\simeq X, X^*$ by twists). In either quotient the unit-coefficient trace satisfies $\tau(XY^*) = \delta_{XY}$, and the orders coincide modulo two, giving equal ranks of corresponding local blocks. If ϵ_Z is the fermion monodromy, then $\chi_{gX}(Z) = \epsilon_Z \chi_X(Z)$. Orthogonality with the even projector gives

$$\sum_{Z \in \mathrm{Irr}(\mathcal{B})} \chi_X(Z)\overline{\chi_Y(Z)} = \tfrac{1}{2} f_X(\delta_{XY} + \delta_{X,gY}). \tag{4.3}$$

This identifies the restriction fibers. Every character of the even subalgebra extends by semisimplicity, its value on g selects the quotient, and summing over representatives halves the displayed norm, giving (4.1). These rational codegrees are integral Casimir eigenvalues.

A restricted dimension residue extends to the full ring: each odd Y has Y^2 even and squaring is injective in the residue field. The principal labels $1, g$ have opposite dyadic twists, giving (4.2)

by block orthogonality. Corollary 4.1 at dimensions $2^s m$, $2^{s+2}m$ gives the claimed degree parities. Finally the positive block idempotent has integral trace $\sum_{Z\in P_+} d_Z^2/(2^s m)$ and each odd square equals one modulo 2^s for $s = 2, 3$. Thus $2^s \mid k$. □

Lemma 4.6 (Unramified signed characters from sign-only twists)**.** *Let $\mathcal{M}$ be an integral modular category with fermion g, and let $\mathcal{B} = \langle g\rangle'$. If the two-primary part of every actual twist of $\mathcal{B}$ is 1 or -1, then every character field of $K_0(\mathcal{B})$ is unramified at two. In particular this holds for the signed quotient $K_0(\mathcal{B})/(g+1)$.*

Proof. Let σ belong to inertia at two. The Galois identities [21, Theorem II] give $\sigma\chi_Y = \chi_{\widehat{\sigma}(Y)}$ and $t_{\widehat{\sigma}(Y)} = \sigma^2(t_Y)$, $t_Y = \theta_Y/\zeta$. Since the dimension character is rational, $\widehat{\sigma}(0) = 0$ and $\sigma^2(\zeta) = \zeta$. For even Y, Z the balancing formula expresses S_{YZ} in integral dimensions and actual even-sector twists, all in an odd cyclotomic field. Thus Y and $\widehat{\sigma}(Y)$ have the same restricted character and lie in one fermion orbit by Lemma 4.5. Their normalized twists agree, while Y, gY have opposite twists. Hence $\widehat{\sigma}(Y) = Y$. Symmetry of S and rationality of dimensions now give $\sigma(\chi_X(Y)) = \chi_X(Y)$ on all even Y for every X. Every character of the even subalgebra extends. □

Proposition 4.7 (Odd supports in the principal two-block)**.** *Choose an integral minimal extension $\mathcal{M}$ of a slightly degenerate minimal counterexample $\mathcal{B}$. Its positive reduced dimension is even. Two odd primes p, q of valuation two in $\dim\mathcal{B}$ cannot occur as a simple degree pq.*

If all principal two-block degrees are odd, their noninvertible supports intersect pairwise with empty total intersection. With exactly three odd primes each of their three pairs must occur as a support. When $v_2(\dim\mathcal{B}) = 3$ all principal degrees are indeed odd. Thus this two-valuation is excluded if the odd part has at most two prime divisors, or three with two valuations equal to two.

Proof. If $\Delta = \dim\mathcal{B}/2$ were odd, the positive dimension idempotent would be integral at two. Thus the full principal block would have only $1, g$ in its even sector by Lemma 4.5. Its odd mass would be two by (4.2), forcing an odd invertible, contrary to Lemma 4.4.

A label of degree pq has odd residue, hence every noninvertible principal two-block label meets $\{p, q\}$ in its support by coprime vanishing. Choose even p- and q-active columns Y, Z by reduced Cauchy. A p-bearing label has p-unit codegree and vanishes at Y; likewise at q, Z. Thus $1, g$ are their only common principal hits. Replacing Z by gZ makes the dyadic phases distinct while both contributions stay positive, contrary to Theorem 3.1.

If all principal degrees are odd, any two noninvertible labels meet by principality and coprime vanishing. They have no common prime: modulo that prime the mass identity (4.2) would read $2 = 0$, from the even invertibles $1, g$ alone. This also excludes an empty family. On two primes intersecting supports have a common prime; on three, empty total intersection forces all three pairs. The squared pair would then occur as a degree, contrary to the first assertion.

If $v_2(\dim\mathcal{B}) = 3$, Lemma 4.5 gives signed principal rank divisible by four. An even principal label would be odd-sector of degree twice odd and signed defect zero or one by (4.1), contradicting Proposition 2.12. □

4.3 Short cyclotomic sums and complementary principal labels

A cyclotomic integer has *weight at most g* if it is a nonnegative rational combination of roots of unity of total coefficient at most g.

Proposition 4.8 (Inertia forced by several positive residues)**.** *Let z be a cyclotomic integer, all of whose conjugates have modulus strictly less than a positive integer g. Suppose $z \equiv g$ at every prime above each prime in a finite set P. For every nonempty proper subset $R \subset P$ let H be the subgroup of $\mathrm{Gal}(\mathbb{Q}(z)/\mathbb{Q})$ generated by the inertia groups at $s \in R$. Then*

$$\prod_{s\in R} s < |H|g.$$

In particular any $s \geq g$ in a pair of prescribed positive residues ramifies. At a zero residue, a nonzero z must also ramify if $s \geq g$.

Proof. Put $e = |H|$, $M = \prod_{s \in R} s$ and $w = \sum_{\sigma \in H} \sigma z$. Its field is unramified at the primes in R, and $w \equiv eg$ at every prime above P. Therefore $(w - eg)/M$ is integral. If $M \geq eg$, every conjugate of $1 + (w - eg)/M$ has modulus strictly less than one, forcing $w = eg - M$. A prime of $P \setminus R$ would divide M. In the zero-residue case unramifiedness would make z/s integral of house less than one. □

Theorem 4.9 (Short cyclotomic sums with one or two positive residues)**.** *Let g be a positive integer and let the cyclotomic integer z have weight at most g. Suppose $z \equiv g$ at every prime over a prime $r > g$. If $r > 2g$, then z is an unsigned integral sum of exactly g roots of r-power order, unique at any common r-power level.*

Suppose also that $q < r$ is prime and $z \equiv g$ at every prime over q. At every $r > g$ one then has

$$z = h + q \sum_{i=1}^{m} \xi_i, \qquad h + qm = g,$$

with $h, m \geq 0$ and the ξ_i nonidentity r-power roots, except possibly when $r = g + 1$ and $q \mid g - 1$. In that exceptional case another possibility is

$$z = -\sum_{t \in \mathbb{F}_r^\times} \eta_t \zeta_r^t,$$

where the q-power roots η_t consist of one copy of 1 and $(g - 1)/q$ complete cosets of μ_q, with repetitions allowed. In particular $z = g$ if $g < q$, and $|z| = g$ if $g = q$. If $|z| < g$ and $q \leq g < 2q$, the field of z contains $\mathbb{Q}(\zeta_r)$ in the first case and at least $\mathbb{Q}(\zeta_r)^+$ in the exceptional case; in either case the subgroup of tame units of the r-power factor has orbit size $r - 1$ on z.

Proof. At level s^e the integral relations among powers of ζ_{s^e} are constant on each coset of μ_s, by Φ_{s^e}. This is the prime-power case of the Rédei–de Bruijn–Schoenberg theorem [38, Theorem 5.1]. We use this also modulo another prime. In particular an unsigned s-word of length $< s$ is unique, since in each coset its least coefficient is zero.

First suppose z is a signed word of length at most g in roots of order supported on the specified primes. At $r > 2g$ alone the signed augmentation is g modulo r and lies in $[-g, g]$, forcing g positive signs. For two primes erase the r-parts, obtaining w. Its field is unramified at r, so $(w - g)/r$ is integral. All conjugates of $1 + (w - g)/r$ lie in the unit disk, meeting its boundary only at 1; the norm therefore gives $w = g$ or $g - r$.

Erase instead the q-parts, retaining signed multiplicities c_i in the r-power roots. The erased sum is g modulo q in this unramified ring. Thus $c_i - g\delta_{i0}$ is constant modulo q on each μ_r-coset. If every coset has a vacant position after marking index zero, these constants vanish. The augmentation, also the residue of w above q, is then g modulo q; this excludes $w = g - r$. Equality $w = g$ forces every erased summand to equal 1, hence the original word is unsigned and r-primary. Each nonidentity multiplicity is divisible by q (for $q = 2$ signed and unsigned multiplicities have the same parity).

The only other possibility uses all $g = r - 1$ positions, once each, on the nonidentity rth roots. Their signs are $-g$ modulo q. This excludes $q = 2$ and a positive sign, and gives $q \mid g - 1$. As q is odd, the negative summands cannot give $w = g$. Hence $w = -1$ and $\sum_t \eta_t = 1$. The μ_q-coset relations show that, after removing one copy of 1, the multiplicities are nonnegative and constant on each coset.

It remains to justify this integral-word reduction. Write the level as MN with M supported on the specified primes, enlarged to include each, and $(M, N) = 1$. The signed Mth roots are a unimodular configuration [4, Theorems 3, 13]; minimizing total weight therefore gives an integral optimum for an integral point. Use $\mathbb{Z}[\zeta_{MN}] = \mathbb{Z}[\zeta_M] \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_N]$, by coprimality and the cyclotomic

degrees. Given an Nth root $a \neq \pm 1$, choose a basis in each primary factor of N by omitting one root from each prime-order coset. Keep 1 in every basis and, in a factor where the component of a is not a sign, keep that component as well. This is possible also at two, where the two chosen elements then lie in different cosets. The coefficient of 1 in the product basis defines λ with $\lambda(1) = 1$, $\lambda(a) = 0$ and $\lambda(\mu_N) \subset \{0, \pm 1\}$, since each omitted root is the negative sum of its remaining coset. The integral $\mathbb{Z}[\zeta_M]$-linear contraction $1 \otimes \lambda$ preserves the congruence at each retained prime s: the radical of (s) is generated by $1 - \zeta_{s^{v_s(M)}}$ at both levels. The contracted word has rational weight at most g; unimodularity and the preceding alternatives give minimum weight exactly g. Any positive coefficient of an original summand with complementary component a would give weight strictly below g after contraction. Thus every complementary component is a sign, and unimodularity applies to z itself.

In the unsigned expression, tame r-units stabilizing z permute the m nonidentity roots freely, counted after division by q. For $m > 0$ their stabilizer order divides m. In the exception, independence of the nonidentity rth roots over the q-power field makes η_t constant on the free orbits of a tame stabilizer. Its order divides $\sum_t \eta_t = 1$ and is therefore one. At $q \leq g < 2q$, strict inequality $|z| < g$ gives $m = 1$ in the unsigned case; in the exception $g = q + 1$, $r = q + 2$. Any automorphism fixing the exceptional sum permutes its one or two positions carrying coefficient 1, hence acts on ζ_r by 1 or -1. This proves the field assertions. □

Lemma 4.10 (Complementary principal degrees at three primes)**.** *Let a perfect fusion-simple ordinary or positive reduced table have three dimension primes t, u, v. In the reduced case assume u odd. The principal block P_u contains v-free nonunits. Every such degree is $t^i u^j$ with $i, j \geq 1$ and $2j + 2 \leq v_u(\Delta)$. Thus at a quartic u both complementary degrees ut^i, uv^k occur in P_u, each of defect two.*

When $u < v$ are odd, the t-parts of all v-free nonunits in P_u and all u-free nonunits in P_v are at least u.

Proof. Prime avoidance (Proposition 4.2) for a u-active column produces a v-free nonunit. It cannot have prime-power degree, and Corollary 4.1 bounds its u-height. At $v_u(\Delta) = 4$ interchange t, v. For $u < v$ odd pair opposite labels X, Y of degrees $t^i u^j, t^k v^l$. The lcm-normalized entry has weight $g = t^{\min(i,k)}$ by balancing and strict house below g. Principality gives residue g over both u, v, on all conjugates by Galois permutation of labels and symmetry. Theorem 4.9 therefore gives $g \geq u$. □

Corollary 4.11 (The two-adic height in a reduced table)**.** *For a slightly degenerate minimal counterexample of dimension $2^a q^b r^c$ with $q < r$ odd, one has $2^{\lfloor (a-1)/2 \rfloor} \geq q$. In particular $a \geq 5$.*

Proof. The positive table has two-valuation $a - 1$. Square-divisibility bounds all two-parts by $2^{\lfloor (a-1)/2 \rfloor}$, and Lemma 4.10 bounds some two-part below by q. Sources losing a repeated prime are covered by Dong. □

4.4 The uniform theorem and separation criteria

Theorem 4.12 (Three primes with a small exponent)**.** *Let p, q, r be distinct primes, $a, b, c \geq 0$, and let d be square-free and coprime to pqr. Every braided fusion category of dimension $p^a q^b r^c d$ is WGT if $c \leq 3$. Consequently every fusion category of dimension $p^a q^b r$ is WGT without a braiding hypothesis.*

Proof. A divisor-minimal failure has a perfect simple ordinary or reduced table of square-full dimension $\Delta = D$ or $D/2$ (Lemma 2.1); all three primes must remain by Dong's two-prime theorem. In ordinary data every prime valuation is at least four by Lemma 4.10. In reduced data that lemma applies at every odd prime, and Corollary 4.11 treats two. For an unbraided category pass to its center of dimension $p^{2a} q^{2b} r^2$ and use Morita invariance and closure under subcategories [32, Proposition 4.1]. □

Proposition 4.13 (The gcd of the prime-free degrees)**.** *Let R be a simple integral commutative based fusion ring having both r-divisible degrees and r-free nonunit degrees, for a prime r. If the latter have minimum m, maximum M and gcd g, then $mM \geq rg$. In particular $M \geq r$ if they form a divisibility chain.*

Proof. Suppose $mM < rg$ and take $d_U = m$. For each r-free $Y \neq U^*$, the r-divisible part of UY has dimension divisible by rg: all the other terms and the total degree are multiples of g. It is therefore absent (also when $Y = \mathbf{1}$). If absent from UU^* as well, multiplication by U, U^* preserves the span of the r-free labels, contrary to simplicity. Thus UU^* contains some r-divisible B with multiplicity h. Reciprocity makes hU^* the entire r-free part of U^*B, so $r \mid hm$ and $r \mid h$. Consequently $m^2 \geq 1 + hd_B \geq 1 + r^2$, whereas $mM < rg$ gives $m < r$ since $g \leq M$. For a divisibility chain $g = m$. □

Proposition 4.14 (Separation criterion)**.** *Fix distinct primes p, q, r, exponents $a, b \geq 2$, and $c \geq 0$. Put*

$$M = p^{\lfloor a/2 \rfloor} q^{\lfloor b/2 \rfloor}, \qquad h = \max(p, q).$$

Let u and v be the least positive residues satisfying $rpu \equiv -1 \bmod q$ and $rqv \equiv -1 \bmod p$. Then

$$(M/h)^2 - 1 < r(pu + qv) \tag{4.4}$$

implies WGT at $p^a q^b r^c d$ for every square-free d coprime to pqr. The residue-independent inequality $r > M^2/(hpq)$ also suffices. In both cases c is unrestricted. In (4.4), if p, q, r are all odd, one may instead use half the left side and the residues defined with $2rp, 2rq$.

Proof. In a minimal table all three primes remain. The principal block at the odd prime h has an r-free nonunit X by prime avoidance applied to an active column. Its degree s divides M/h by Corollary 4.1 and square-divisibility. Every r-free nonunit degree is a multiple of pq dividing M, and an r-divisible degree occurs since $\Delta \mid \prod_X d_X^2$. Proposition 4.13 gives $rpq \leq sM \leq M^2/h$.

The nonunit terms of XX^* have degrees divisible by pq, pr or qr. Thus $s^2 - 1 = pqA + rpB + rqC$ with nonnegative coefficients. As $pq \mid s$, reduction modulo q, p gives $B \geq u$, $C \geq v$. This contradicts (4.4). In an odd table pair dual constituents before the same reduction to obtain the variant. □

Positive defect and active columns. The perfect double $\mathcal{Z}(\mathrm{Vec}_{A_5 \times A_5})$ illustrates the role of defect zero in (3.1). Its invertibles are trivial since A_5 is centerless and perfect. Take the pure charge Z of degree $4 \cdot 3$ (standard and rotation representations), and flux Y of an involution in the second factor with centralizer character -1 on that involution. Then $d_Y = 15$, $\theta_Y = -1$ and $S_{ZY} = 4 \cdot 15 \cdot (-1) = -60$ by the charge–flux formula [39, Section 3.1, p. 4189], since a half-turn has rotation trace -1.

4.5 Common divisors and the primes of a minimal table

Balancing expresses $\chi_Y(X)$ as a positive rational sum of roots of unity of total weight d_X. In the reduced table use the actual simple lifts in the even ribbon category. Simple degrees are integral, so Galois symmetry gives $\chi_Y(X^\sigma) = \sigma\chi_Y(X)$ at the same row Y. Thus Galois permutes the based ring multiplicatively and preserves degrees and duals. In an odd ordinary table the twists, and hence all balancing phases, have odd order by the categorical Cauchy theorem [9, Theorem 3.9].

Lemma 4.15 (Common constituents and difference sets)**.** *Fix a nonunit row Y of a minimal table. Say a set of admissible phases is* exposed *if a rational linear functional on the common cyclotomic field takes value one on it and strictly less than one on the other phases. If the characters of UU^* and VV^* at Y have full-weight positive expressions supported on exposed sets*

E, F, for nonunits U, V, their adjoints share a nonunit whose balancing phases at Y belong to $E \cap F$.

In particular, suppose X is a nonunit and $\chi_Y(X)$ is a scalar root of unity times a positive rational sum of s-power roots of total weight d_X, for an odd prime s. Let A be their exponent set at a common s-power level and $T = A - A$. If each μ_s-coset contains fewer than $s/2$ positions of T, then $T \cap uT$ contains a nonzero exponent for every s-adic unit u. In an odd ordinary table, fewer than $2s/3$ positions per coset suffice.

Proof. Commutativity and rigidity give $\langle UU^*, VV^* \rangle = \langle UV, UV \rangle$. If only the unit were common, it would occur once in each adjoint, so UV would be simple. This also implies simplicity for a product of chosen lifts in the positive quotient. Its double braiding would be scalar, contrary to strictness at U, V. Take a common nonunit and use its same balancing expression in both decompositions. Each exposing functional attains the full weight, so every phase in that expression lies in both sets.

To expose a primary support assign one to its k marks in each μ_s-coset and $-k/(s-k)$ elsewhere. The coset sums vanish, so this defines a functional on the primary field. Tensor with normalized trace on the complementary field. Nonidentity complementary roots have mean strictly below one, and in odd order their mean $\mu(n)/\varphi(n)$ is at least $-1/2$. Thus the product is strictly below one off the marks when $k < s/2$, or $k < 2s/3$ in odd order. The full expressions for XX^* and $X^u(X^u)^*$ have supports in T, uT, since the scalar cancels. If zero were their only intersection, the common nonunit would have character equal to its degree at Y, again contrary to strictness. □

Lemma 4.16 (Descent on the number of differences)**.** *Fix a nonunit row Y in a minimal table. Suppose the character of a nonunit X at Y is, up to a scalar root of unity, a positive s-primary word of total weight d_X on at most $m+1$ positions. Set $e = m(m+1)/2$ and*

$$g(h) = \min\{k \geq 1 : k(k+1) \geq h\}, \qquad C(e) = 1 + \max_{1 \leq h \leq e} \frac{h(h-1)}{g(h)}.$$

If $1 + 2e < s/2$ (or $1 + 2e < 2s/3$ in odd ordinary data), then $(s-1)/2 \leq C(e)$.

Proof. Keep the row Y fixed and among all such words of nonunit labels at Y, with no size restriction, choose one minimizing the number h of opposite pairs of nonzero differences. This is a positive integer by strictness, and $h \leq e$. Let U be the chosen label and D its difference set. Work at a common s-power level. For each class u of tame units modulo sign put $k_u = |(D \cap uD) \setminus \{0\}|/2$. Both adjoint spectra are exposed since $|D| = 2h + 1 \leq 2e + 1$. Lemma 4.15 supplies a common nonunit with a full word on $D \cap uD = \{0, \pm a_1, \ldots, \pm a_{k_u}\}$. Its nonzero differences, modulo sign, lie among the $a_i, 2a_i$ and the $a_i \pm a_j$ for $i < j$. There are at most $k_u(k_u + 1)$ such pairs. Minimality gives $k_u \geq g(h)$.

Tame units modulo sign act freely on opposite pairs of nonzero exponents. Each ordered pair of such pairs in D therefore contributes to at most one k_u. Thus $\sum_u k_u \leq h^2$. Since $k_1 = h$, subtraction gives $((s-1)/2 - 1)g(h) \leq h(h-1)$. □

Theorem 4.17 (A prime bound from a common divisor)**.** *Let X, Y be nonunit labels in an ordinary or positive reduced minimal table. If Y belongs to the full principal block at an odd prime p and $p \nmid d_X$, then*

$$g = \gcd(d_X, d_Y) \geq 2, \qquad p \leq b(g), \qquad b(g) = \max\{5, (g-1)g^2\}.$$

Consequently, if x is any nonunit degree and L is its largest full prime-power factor, every prime dividing Δ either divides x or is at most $b(x/L)$.

Proof. The algebraic integer $z = S_{XY}/\operatorname{lcm}(d_X, d_Y)$ has weight at most g by balancing. Since $z = (g/d_X)\chi_Y(X)$ and Y is principal, $z \equiv g$ above p. This holds at every prime above p: Galois covariance permutes the test labels X and preserves their degrees. Since $p \nmid g$, the entry is nonzero, so $g \geq 2$ by Lemma 2.5. Assume $p > b(g)$, so $p > 2g$. Theorem 4.9 writes z as a sum of g positive p-power roots. Rescaling by d_X/g gives a positive word for $\chi_Y(X)$ of full weight d_X. A single position contradicts strictness. If $g = 2$, the two positions have difference $t \neq 0$ of p-power order, and $\{0, \pm t\}$ meets its double only at zero. Since $3 < p/2$, this contradicts Lemma 4.15.

For $g \geq 3$ put $j = g - 1$ and $E = j(j+1)/2$. In Lemma 4.16, write $k(v) = \min\{k \geq 1 : k(k+1) \geq v\}$. For $1 \leq v \leq E$ we have $(v-1)/k(v) \leq j$: use $v \leq k(v)(k(v)+1)$ when $k(v) < j$ and $v \leq E$ otherwise. Hence

$$2C(E) + 1 \leq 3 + 2Ej \leq j(j+1)^2, \qquad 2(1+2E) \leq j(j+1)^2.$$

Both bounds are strictly below p, contradicting Lemma 4.16.

For the last assertion only odd $p \mid \Delta$ with $p \nmid x$ need be considered. Write $L = \ell^{v_\ell(x)}$. At a p-active column, which exists by Cauchy's theorem (Proposition 3.4 in reduced data), Proposition 4.2 supplies a nonunit $Y \in P_p$ with $\ell \nmid d_Y$. Then $g \mid x/L$, and b is nondecreasing. □

Corollary 4.18 (Degrees six and ten)**.** *If a simple in an ordinary or positive reduced minimal table has degree* 6 *or* 10*, the prime divisors of* Δ *are exactly* $2, 3, 5$*.*

Proof. For either degree x one has $x/L = 2$. The theorem restricts the primes to $2, 3, 5$. At least three must occur: otherwise every nonunit degree would contain the same prime by Lemma 2.5 and $d_Y^2 \mid \Delta$, contradicting row orthogonality (2.7) modulo that prime. □

4.6 Two principal blocks at the same witness

Lemma 4.19. *In an ordinary integral modular table let* p, q *be distinct dimension primes such that* q *divides every* p*-active degree (in particular, this holds for separated primes). Every* p*-active label* J *has a nonunit* q*-free hit* $X \in P_p \cap P_q$ *of twist one. More precisely, in the full principal* p*-block with pure* p*-twist, the sum of* $d_X\chi_J(X)$ *over the* q*-free nonunits of the principal* q*-block is* -1 *modulo a prime above* q*.*

Proof. Write E for P_p with pure p-twist. Every q-free Z is p-inactive by hypothesis. Orthogonality of J, Z on E (Proposition 4.2) multiplied by d_Z gives, modulo q,

$$d_Z + \sum_{\substack{X \in E,\ X \neq 0 \\ q \nmid d_X}} d_X \chi_J(X) \overline{\chi_X(Z)} = 0.$$

For a q-bearing X, the omitted summand was zero since $q \mid \chi_Z(X)$ by lcm divisibility. The displayed identity now also holds for q-bearing Z, because each retained X is q-free. Thus this is a relation on the whole fusion-ring basis. The distinct ring characters in the residue field are linearly independent (equivalently apply the Chinese remainder theorem after extending scalars to the residue field). The coefficient of the dimension character must vanish, proving the congruence. Complex conjugation causes no change to this principal block, since $Z \mapsto Z^*$ permutes the basis and preserves dimensions. A q-free member of E is p-inactive, so its twist is one. □

Lemma 4.20 (Lifting a positive singleton at two)**.** *Let* g *be a fermion in an integral modular category* $\mathcal{M}$ *with its positive spherical structure and even sector* $\mathcal{B} = \langle g \rangle'$*. Suppose the pair* $[X] = \{X, gX\}$ *is alone in its block of* $R_+ = K_0(\mathcal{B})/(g-1)$ *at two. The full two-block over this residue consists of* X, gX, Y, gY *with* Y *odd,* $gY \neq Y$ *and* $d_Y = d_X$*. If* $S_{XV} \neq 0$ *and* V *is odd, then* $\theta_{V,2} = \theta_{Y,2}$*.*

Proof. In characteristic two the residue on $K_0(\mathcal{B})$ determines the residue on each odd V, since V^2 is even and squaring in the residue field is injective. Thus there is a single full block over this residue. The identical special fibers of R_+ and R_- have corresponding lifted block summands of equal rank (Lemma 4.5). The signed block therefore has one character, or equivalently one odd fermion orbit. Apply Theorem 3.1 on the full block to the columns $1, g$, whose dyadic twists are opposite. Since $S_{gZ} = \pm d_Z$ according to the sector, the odd squared mass equals the even mass $2d_X^2$. A fixed label would have $d_Y^2 = 2d_X^2$, impossible for integer degrees. The odd orbit is consequently free of degree d_X. In $s^{-1} = W_{2'}^{-1} W_2^{-1}$ the dyadic Weyl preserves the block and the complementary Weyl preserves sector and actual dyadic twist. The two odd mates have equal actual twists by balancing, which proves the support assertion. □

5 Ramification and higher defects

The defect of X at p is $v_p(f_X)$. At a quartic prime the complementary labels of Lemma 4.10 have defect two; at a quintic prime they have defect three. We describe the corresponding integral orders and bound the norm spent by their characters on a fixed residue.

5.1 Split blocks of uniform defect two

The discriminant permits arbitrarily large blocks at uniform defect two, but determines their order if all characters are unramified.

Lemma 5.1. *Let $A \subset M = \mathcal{O}^n$ be a split local symmetric order over an unramified p-adic integer ring, whose characters have common residue and whose trace is $p^{-2} \sum_i u_i x_i$ with u_i units. Then*

$$A = \mathcal{O}\mathbf{1} + pH_u, \qquad H_u = \{y \in M : \textstyle\sum_i u_i y_i \in p\mathcal{O}\}.$$

Proof. The identity trace gives $\sum_i u_i \in p^2\mathcal{O}$. Equal residues give $x = a\mathbf{1} + py$, and integrality of $\tau(x)$ gives $y \in H_u$. Since $\mathbf{1} \in H_u$, this inclusion imposes $n-1$ common-residue conditions and one further condition on y. The containing lattice thus has index length n in M, which equals the index of A by self-duality and the discriminant of M. □

Conversely, taking all $u_i = 1/u$ for $p \nmid u$ and $p^2 \mid n$ gives a local order with integral trace and index length n, hence a perfect trace and uniform codegree $p^2 u$. One may take $u \geq n/p^2$ to obtain $\tau(1) \leq 1$ even as n grows.

Lemma 5.2 (Incompatible unramified lifts)**.** *Let A_+ and A_- be commutative local symmetric orders over the integers $\mathcal{O}$ of a finite unramified extension of $\mathbb{Q}_2$. Suppose their generic algebras split, all character coordinates of each order have a common residue, and their perfect traces are*

$$\tau_+(x) = \frac{1}{8m} \sum_{i=1}^{n} d_i^2 x_i, \qquad \tau_-(x) = \frac{1}{4m'} \sum_{i=1}^{n} h_i^2 x_i,$$

with m, m', d_i, h_i odd integers and $n > 0$. Their special fibers are not isomorphic as algebras.

Proof. Let $M = \mathcal{O}^n$, $k = \mathcal{O}/2\mathcal{O}$. By Lemma 5.1, $A_- = \mathcal{O}\mathbf{1} + 2H$, $H = \{y : \sum y_i \in 2\mathcal{O}\}$. Since H is closed under coordinatewise squaring and $4M \subset A_-$, its special-fiber radical has

$$J_-^3 = 0, \qquad x^2 = 0 \quad (x \in J_-). \tag{5.1}$$

We show that the positive lift cannot have both properties. Write $A_+ = \mathcal{O}\mathbf{1} + 2L$, $L = \{y : 2y \in A_+\}$. The conductor and index give $4M \subset L$ and $\ell(M/L) = n/2 + 1$. Since $d_i^2 \equiv 1 \pmod 8$, trace integrality on $2L$ and on $4LL$ gives $\sum y_i \in 4\mathcal{O}$ and $\sum y_i z_i \in 2\mathcal{O}$ for $y, z \in L$. Thus $C = (L + 2M)/2M$ is self-orthogonal in k^n; the layer $(L \cap 2M)/4M$ embeds by division by two

in the sum-zero hyperplane. Their dimensions sum to $2n - \ell(M/L) = 3n/2 - 1$, so $\dim C = n/2$ and $C = C^\perp$.

If $J_+^3 = 0$, the trace of each $8yzw$, $y, z, w \in L$, is even. All triple dot products vanish in C, so $C = C^\perp$ is closed under coordinatewise products. This unital subalgebra consists of the functions constant on a partition. Self-orthogonality makes its $n/2$ blocks even, hence pairs. Lift a pair indicator to $y \in L$. If all radical squares vanished, $4y^2 \in 2A_+$ would give $y^2 \in L$, contradicting $\sum y_i^2 \equiv 2 \pmod 4$. □

5.2 Ramification and residue energy

For a cyclotomic integer x let $\mathsf{M}(x)$ be the mean of $|\sigma x|^2$ over its Galois conjugates and put $E(x) = [\mathbb{Q}(x) : \mathbb{Q}]\mathsf{M}(x)$. At a fixed positive degree Galois permutes the columns of a row. We bound the squared norm spent by one orbit of entries, using their common residue and ramification at p.

Theorem 5.3 (Energy at a prescribed prime residue)**.** *If a nonzero cyclotomic integer x vanishes at all primes over p, then $E(x) \geq 2p$. For odd p write the ramification index of $\mathbb{Q}(x)$ as $e = e_0 p^s$, $e_0 \mid p - 1$. Then*

$$\mathsf{M}(x) \geq W_p(e) := \begin{cases} p^2 & e = 1, \\ 2p/e_0 & s = 0,\ e_0 > 1, \\ 1 + (p-1)/e_0 & s > 0. \end{cases}$$

In particular $E(x) \geq p^2$ if $e = 1$ and $E(x) \geq p^{s+1}$ if $s > 0$.

For odd p, in a p-power cyclotomic field a prescribed integer residue $0 \leq k < p$ gives $\mathsf{M}(x) \geq k(p-k)/(p-1)$.

Proof. For $p = 2$, vanishing at a single prime already gives $|N(x)| \geq 2$, whence $E(x) \geq h\, 2^{2/h} \geq 4$ for $h = [\mathbb{Q}(x) : \mathbb{Q}] \geq 1$. Now take p odd. When $e = 1$, x/p is integral and its nonzero norm gives $\mathsf{M}(x) \geq p^2$.

In the tame case take $x \in F(\zeta_p)$ with F cyclotomic of conductor prime to p. In an integral expression $x = \sum_{j \bmod p} a_j \zeta_p^j$, the sum of the a_j vanishes at every prime of F over p, hence lies in $p\mathcal{O}_F$. Subtract its average from every coefficient. This gives a unique expression of coefficient sum zero, with $\mathsf{M}(x) = \frac{p}{p-1} \sum_j \mathsf{M}(a_j)$. Inertia fixing x has order $g = (p-1)/e_0$ and its nonzero-index orbits have length g. Each nonzero orbit of coefficients costs at least g by the norm. If there is just one such orbit with common value a, then $a_0 = -ga$, so the cost is at least $g + g^2 \geq 2g$. This proves the tame bound.

For $s > 0$ the inertia conductor is p^{s+1}. With $\zeta = \zeta_{p^{s+1}}$ and F as above, the relative integral basis over $F(\zeta_p)$ gives a unique expansion $x = \sum_{j=0}^{p^s-1} \zeta^j \gamma_j$. The terms are orthogonal for the relative trace, hence $\mathsf{M}(x) = \sum_j \mathsf{M}(\gamma_j)$. Inertia fixing x has order $g = (p-1)/e_0$; it acts freely on nonzero indices and preserves coefficient means, since the coefficients are conjugate up to pth roots of unity. Some nonzero-index orbit must occur by wild ramification. Two such orbits cost at least $2g \geq g + 1$, and one together with $\gamma_0 \neq 0$ costs $g + 1$. For a single orbit with $\gamma_0 = 0$, all its coefficients have the same residue at each prime over p. Their residues sum to zero, so each vanishes there since $p \nmid g$. The preceding tame estimate (or the unramified estimate) gives mean at least $2p/(p-1)$ for each. Their sum is at least $g + 1$. Finally $[\mathbb{Q}(x) : \mathbb{Q}] \geq e$ gives the energy bounds.

For the prescribed residue in $\mathbb{Q}(\zeta_p)$ choose integer coefficients c_j with $\sum c_j = k$. The same quadratic form gives $\mathsf{M}(x) = (p \sum c_j^2 - k^2)/(p-1) \geq k(p-k)/(p-1)$. At higher level the orthogonal expansion above adds the means of its coefficients. Their least residues sum to k modulo p, and $u(p-u) + v(p-v) \geq w(p-w)$ for $w \equiv u + v \bmod p$ in $[0, p)$. Iteration gives the claim. □

The bounds $2p$ and p^{s+1} are sharp. For any subgroup $H \le \mathbb{F}_p^\times$ of index $e > 1$ and distinct cosets aH, bH, $\sum_{j\in aH} \zeta_p^j - \sum_{j\in bH} \zeta_p^j$ vanishes above p. Its coefficient sum is zero, its stabilizer is H, and its energy is $2p$. For every $s \ge 1$ take the Teichmüller subgroup $H \subset (\mathbb{Z}/p^{s+1}\mathbb{Z})^\times$ of order $p-1$. The integer $1 + \sum_{j\in H} \zeta_{p^{s+1}}^j$ vanishes above p and has mean p, by the distinct orthogonal indices modulo p^s. Its nonconstant word has length $p-1$ and misses a slot on every μ_p-coset, so its stabilizer is precisely H. Its energy is p^{s+1}.

Lemma 5.4 (First tame shell and trace)**.** *Let p be odd and $0 \ne x \in \mathbb{Z}[\zeta_p]$ vanish above p. If $E(x) < 3p$, then $E(x) = 2p$. With H its stabilizer in $\mathbb{F}_p^\times$, x is either the difference of the sums on two nonzero H-cosets, or $\pm(1-\zeta_p^j)$ for $j \ne 0$. The former has trace zero; the latter has trace $\pm p$ and sum of conjugate squares p. At $p = 5$ the quadratic possibility in the first case is $\pm\sqrt{5}$.*

Proof. Take the unique coefficient vector of sum zero as in the theorem. For $h = |H|$, let c_C be its integer coefficient on each nonzero index coset of H and put $s = \sum_C c_C$. The coefficient at zero is $-hs$. The same trace formula gives

$$E(x) = p(hs^2 + \sum_C c_C^2), \qquad \mathrm{Tr}(x) = -ps.$$

Below $3p$ the bracket must be two: for $s = 0$ the nonzero entries are $1, -1$; for $s \ne 0$ one has $h = 1$ and a single entry ± 1. The sum of conjugate squares follows by summing $(1-\zeta_p^a)^2$. At five the difference of the two quadratic periods squares to five by the quadratic Gauss sum. □

Corollary 5.5 (Conductor and row budget)**.** *In a symmetric integral table fix an odd prime p and put $c = v_p(\Delta)$ and $t_X = v_p(f_X) = c - 2v_p(d_X)$. A row field has zero p-conductor exponent when $t_X = 0$ and conductor exponent at most t_X otherwise. The same holds for an entry field with t_X replaced by $t = \min(t_z, t_X)$. When $t \ge 1$ its wild inertia has order at most p^{t-1}.*

Suppose the entries $\chi_z(X)$ on p-bearing degrees vanish modulo p (at every prime above p). This holds if the block of z contains a p-unit degree. It also holds if $p \mid d_z$ and each nonzero entry on such a degree has $X \in P_p$ with $v_p(d_X) \ge v_p(d_z)$. For each p-bearing degree d let $u_d = (d/\gcd(d, d_z))_{p'}$. Partition the nonzero entries $\chi_z(X)/u_d$ at degree d into Galois orbits O counted with multiplicity. Write $h_O = |O|$, $e_O = e_{0,O}p^{s_O}$ for the degree and ramification index of the entry field. Then

$$\sum_{d:\, p\mid d} \sum_{O \text{ at } d} u_d^2 h_O W_p(e_O) \;\le\; f_z - 1. \tag{5.2}$$

Proof. Corollary 2.8 also shows that a codegree-four row takes values in $\mathbb{Q}(\zeta_8)$. The stated conductor bounds follow from the same result, applied to either row since their normalized entry lies in both fields. The bound from the different is attained at conductor p^t: the full cyclotomic field has normalized different $t - 1/(p-1)$.

On a p-bearing degree d, every character of p-unit degree vanishes above p by lcm integrality. The common residue of its block gives the same vanishing for χ_z. In the second case use symmetry: $\chi_z(X) = (d_X/d_z)\chi_X(z)$, with $\chi_X(z) \equiv d_z$ and d_X/d_z integral at p. In both cases Galois covariance gives vanishing on all conjugates. Division by the integral lcm scale u_d, prime to p, preserves both integrality and vanishing. Sum the energy bounds over the orbits and reserve the unit's squared contribution one. □

If all the scales under consideration are at least u, a nonzero orbit costs at least p^2u^2 when unramified, $2pu^2$ when nontrivially tame, and $p^{s+1}u^2$ at wild depth s.

Corollary 5.6 (Small budgets and forced hits)**.** *For a row satisfying this vanishing condition, if $f_z = p^2 m$, $p \nmid m$, at most $m-1$ nonzero orbits on p-bearing degrees are unramified or wild in total; the wild depth is at most one. For $m = 1$ all such orbits are nontrivially tame and number at most $(p-1)/2$.*

For any prime p, suppose there are just three dimension primes p, q, r, every nonunit degree involves at least two, and the nonunit row z has zero residue on its supported p-bearing degrees at every prime above p. Then

$$f_z \geq [K_z : \mathbb{Q}] + 2p^2.$$

Proof. For the first assertion divide the budget $p^2m - 1$ by p^2 or by $2p$ and use the conductor bound at $t_z = 2$. For the second apply (2.6) modulo q and r. Its vacuum term is one, so each reduction supplies an orbit of nonzero trace on a nonunit degree prime to the respective prime. The two orbits are distinct and both degrees are p-bearing. Each trace is a nonzero integer divisible by p. Their contributions to (2.5) together with the vacuum give the asserted bound. □

5.3 Degrees at a prime-square codegree

Proposition 5.7 (Degrees supporting a nonzero trace)**.** *Let $Z \neq \mathbf{1}$ have formal codegree p^2 in a perfect ordinary or positive reduced integral table. The nonunit degrees at which χ_Z has nonzero absolute trace all divide*

$$M = d_Z / \gcd(p, d_Z)$$

and have greatest common divisor one. If the table is fusion-simple, then M has at least three distinct prime divisors; when it has exactly three, each pair is the prime support of some such degree. In particular at a quartic prime of a three-prime minimal table there is no row of codegree p^2, for any p.

Proof. By (2.5) a nonzero trace T_X at $X \neq \mathbf{1}$ is prime to p: a nonzero multiple of p would exhaust the norm before counting the unit. Now $d_X^2 \mid \Delta = p^2 d_Z^2$, so $d_X / \gcd(d_X, d_Z)$ is one or p. In the latter case lcm integrality makes $\chi_Z(X)$, and hence its trace, a multiple of p. Thus $d_X \mid d_Z$. Moreover $X \in P_p$ by Theorem 2.9. If $v_p(d_X) = v_p(d_Z) > 0$, then $\chi_Z(X) = (d_X/d_Z)\chi_X(Z)$ vanishes above p by principality. This holds on all conjugates and again gives $p \mid T_X$. Hence $d_X \mid M$. The dimension identity (2.6) expresses -1 as an integral combination of the indicated degrees.

In the fusion-simple case no nonunit has prime-power degree. For each prime dividing M some indicated degree omits it; this proves both assertions about prime supports. Finally $v_p(\Delta) = 4$ gives $v_p(d_Z) = 1$, so with only two other dimension primes M cannot have three. □

5.4 The structure at defect two

Proposition 5.8 (Ramification of a principal defect-two row at two)**.** *Let a commutative symmetric integral order have cyclotomic characters, a perfect positive $*$-trace, rational positive codegrees, and the usual orthogonality of character rows. Let η be inertia-fixed at two with $v_2(f_\eta) \geq 3$. If a character χ of $v_2(f_\chi) = 2$ has the same residue as η, then for a sufficiently large finite unramified extension $K/\mathbb{Q}_2$ the local character field $K(\chi)$ is contained in $K(i)$.*

Proof. After unramified base change the conductor bound (Corollary 2.8) puts the local field inside $K(\zeta_8)$. The full extension has $(e, d) = (4, 8)$ and the quadratic fields $K(\sqrt{\pm 2})$ have $(2, 3)$. Their branch conductor lengths $2e - d$ are respectively zero and one. By Lemma 2.11 neither branch can coexist with the unramified η, whose conductor length is at least three. Only K and $K(i)$ remain. □

At odd p a defect-two field has at most one nontrivial wild layer by Corollary 2.8.

Lemma 5.9 (Wild coefficient vectors)**.** *Let χ be a character of rational codegree f, whose orbit under wild inertia at an odd prime p has length p. Suppose its p-primary conductor is at most*

p^2. *In a splitting field write* $\zeta = \zeta_{p^2}$ *and choose the wild generator* τ *with* $\tau\zeta = \zeta_p\zeta$. *There are integral tame vectors* A_j, $0 \le j < p$, *such that*

$$\chi = \sum_{j=0}^{p-1} \zeta^j A_j, \qquad \langle A_j, A_k \rangle = \delta_{jk}\, f/p.$$

If χ *belongs to the principal residue class, then* $\sum_j A_j$ *has the same residue as the dimension character. Every* A_j *is orthogonal to that character.*

Proof. After adjoining an unramified field and the tame cyclotomic factor, the integers are a free module over the tame integers with basis $1, \zeta, \dots, \zeta^{p-1}$. Expansion in this integral basis gives local integrality at p. Formula (5.3) divides only by p, so it also gives integrality at all other primes. Fourier inversion gives

$$\zeta^j A_j = \frac{1}{p} \sum_{t=0}^{p-1} \zeta_p^{-jt} \tau^t \chi. \tag{5.3}$$

The p characters on the right are distinct and orthogonal, each of norm f. Fourier orthogonality gives the stated pairings. Each is orthogonal to the dimension character, and hence so is A_j. Finally ζ has residue one, which gives the residue assertion. □

For example the primitive character of the pointed cyclic p^2 ring has codegree p^2. Its coefficient vectors are supported on the p residue classes of exponents and have norm p each. Their residues are the distinct indicator vectors of these classes. Only their sum has the dimension residue, so none gives a defect-one norm contradiction.

Proposition 5.10 (Trace lattices and the defect-two cotangent space)**.** *Fix a place over* p *and let* P *be the principal* p*-block of symmetric integral character data there. Write* $\mathcal{O}$ *for the integer ring of a finite unramified local coefficient field containing the inertia-fixed values below. For each full tame inertia orbit* O *of nonunit characters in* P *(wild inertia fixes every member), put* $e_O = |O|$, $A_O = e_O^{-1} \sum_{z \in O} \chi_z$ *and* $v_O = (A_O - d)/p$. *The coordinates of* A_O *and* v_O *belong to* $\mathcal{O}$, *and*

$$\langle v_O, v_Q \rangle = (\delta_{OQ} f_O/e_O + \Delta)/p^2. \tag{5.4}$$

If $v_p(\Delta) \ge 3$, *let* B *be the special fiber of the block over an unramified splitting base for these averages, with radical* J. *Each defect-two orbit gives* $u_O \in J \cap \mathrm{Ann}(J^{e_O+1})$. *There is a nonzero socle element* s *with*

$$u_O u_Q = \delta_{OQ} c_O s, \quad u_O s = s^2 = 0, \qquad c_O = f_O/(e_O p^2) \bmod p \neq 0. \tag{5.5}$$

The trace pairings are diagonal with entries c_O. *Together with* $1, s$ *the* u_O *are independent; those with* $e_O \le e$ *are independent modulo* J^{e+1}. *In particular the inertia-fixed rows give independent vectors in the cotangent space* J/J^2.

Proof. The character values are integral at the chosen place and e_O is prime to p. Inertia fixes their average, whose residue equals that of d because $O \subset P$. Thus $A_O - d$ has entries in the maximal ideal $p\mathcal{O}$. Averaging row orthogonality gives (5.4). In the block order put

$$a_O = \sum_X A_O(X^*)[X], \quad a_0 = \sum_X d_X[X], \quad U_O = (a_O - a_0)/p.$$

The dual coordinates belong to $\mathcal{O}$ as well, so U_O lies in the block order. The elements a_O are scalar multiples of sums of primitive idempotents; hence $a_O a_Q = \delta_{OQ}(f_O/e_O)a_O$, $a_O a_0 = 0$, and $a_0^2 = \Delta a_0$. Reduce these identities after dividing the products of differences by p^2 to obtain (5.5) with $u_O = \bar{U}_O$ and $s = \bar{a}_0$. Indeed $d(U_O) = -\Delta/p$, $xa_0 = d(x)a_0$, and $\tau(a_0) = 1$.

On the totally tamely ramified degree-e_O branch, a product of $e_O + 1$ radical lifts lies in $\mathfrak{p}_E^{e_O+1} = p\mathfrak{p}_E$; its trace lies in $p^2\mathcal{O}$, since traces of elements of $\mathfrak{p}_E$ are in $p\mathcal{O}$. Its dimension value

is also in $p^2\mathcal{O}$. Thus $\tau(U_O x) = (A_O(x) - d(x))/p$ vanishes modulo p on that radical power. Perfection gives $u_O J^{e_O+1} = 0$. The products and $\tau(s) = 1$ now give both independence assertions: pair a relation modulo J^{e+1} with each u_O of degree $e_O \le e$; for the relation with $1, s$, first take the residue, then multiply by each u_O. □

For a single fixed row the square and socle relations already occur in $k[t]/(t^3)$. Higher tame degrees give annihilation by a higher power of the radical.

Example 5.11 (Principal defect two in a categorified ring)**.** *For an odd prime p, let $G = \mathbb{F}_p^3 \rtimes C_p$, with the generator b acting by a single unipotent Jordan block of size three. This action has order p, including at $p = 3$, so $|G| = p^4$ and $|C_G(b)| = p^2$. Evaluation at b is a character of $K_0(\operatorname{Rep} G)$ with formal codegree p^2 and field exactly $\mathbb{Q}(\zeta_p)$: a linear character sends b to ζ_p. It is principal, since all eigenvalues of a p-element reduce to one. Its tame average satisfies*

$$\|A - d\|^2 = p^4 + p^2/(p-1).$$

Thus categorification does not itself eliminate principal defect two. This representation ring is not modular-table data: its characters are labelled by conjugacy classes, not by the simple objects.

For integral nilpotent modular categories the prime-power decomposition does exclude codegree valuation two whenever the dimension valuation is at least four (Corollary 26.5).

Principal defect two.

Question 5.12. *Let $\mathcal{C}$ be an integral modular category with $v_p(\operatorname{FPdim}\mathcal{C}) \ge 4$. Can its full principal p-residue block contain a character of formal codegree valuation two? Does the answer change under perfectness and fusion-simplicity? The corresponding reduced-table question at odd p is included.*

A negative answer for minimal tables would settle the largest-quartic case by Lemma 4.10. Section 7 gives further restrictions on those two complementary characters.

6 Local Weyl columns at separated primes

Lemma 6.1 (Local Weyl vacuum column)**.** *Let $\mathcal{C}$ be integral modular of dimension $D = n^2$ with positive spherical structure and normalized Weyl matrix $s = S/n$. Use the linear lift $t = T/\gamma$ of Lemma 3.3, with $\gamma^3 = \xi = p^+/n$ and γ of two-power order ($\gamma = 1$ for a center). For a set J of prime divisors of the level $\operatorname{ord}(t)$ put $W_J = \prod_{p\in J} W_p$. Choose a positive m congruent to one at the level factors in J and zero at the others. Then $w = W_J e_0 = W_J^{-1} e_0$ is rational, nw is integral, and*

$$A_J := n^{-2} \sum_i d_i^2 \theta_i^m = \xi^m (W_J)_{00}.$$

For $q \mid n$ outside J, $q \mid (nw_i)^2$ for every i. In particular w_i is q-integral if $v_q(n) = 1$. For $J = \{p\}$ its support lies in the principal p-block.

Proof. The Galois identity $\sigma_k(\rho(g)) = \rho(A_k g A_k^{-1})$, $A_k = \operatorname{diag}(k, 1)$, follows on the two generators from [21, Theorem II and Remark 4.5]. The torus permutations have positive signs and fix the vacuum because $s_{i0} = d_i/n$ are positive rational numbers. Conjugating W_J introduces only a torus factor on its right, proving rationality. The local torus at -1 fixes e_0 as well, giving the inverse Weyl assertion. In the factors in J apply $U(WUW^{-1})U = W^{-1}$. Its vacuum entry gives the displayed identity since the three scalar factors contribute γ^{-3m}. It also identifies the absolute entries of $sT^m s^{-1} e_0$ with those of w. The regular object $R = \sum_i d_i[i] = n^2 e_{\mathbf{1}}$ thus has indicator $L_m^i(R) = n^2 |w_i|^2$ by (3.2). This is both a rational number and an algebraic integer, so $nw_i \in \mathbb{Z}$. For $q \mid n$ outside J choose m also divisible by q. Grouping q adjacent strands, as

in Lemma 3.3 gives $L^i_m(R)^q \equiv L^i_{m/q}(R^q) \equiv 0$ above q: the insertion (i, i^*) has twist one, and $R^q = (n^2)^{q-1}R$. Therefore $q \mid (nw_i)^2$ in $\mathbb{Z}$. Finally, the projection onto the principal p-block commutes with the p-local factor by Lemma 3.2, so $W_p e_0$ belongs to that block. □

Proposition 6.2 (Weyl columns on separated prime sets)**.** *Partition the primes in the modular level of an integral modular category of dimension n^2 into sets J such that each simple has twist supported in at most one set. Write γ_J for the J-part of the anomaly normalization γ, n_J for the J-part of n, and $V_J = \gamma_J^{-3} W_J$. Then*

$$(V_I - 1)(V_J - 1) = 0 \quad (I \neq J), \qquad s = \xi\Big(1 + \sum_J (V_J - 1)\Big), \qquad n_J W_J e_0 \in \mathbb{Z}^{\mathrm{Irr}(\mathcal{C})}. \tag{6.1}$$

If $p \in J$ and $v_p(d_X) < v_p(n)$, then $v_p((n_J W_J e_0)_X) = v_p(d_X)$. If every other prime of J occurs to full height in d_X, then $X \in P_p$.

A character field ramifies at primes of at most one set J. More generally, in any integral modular category a character field cannot ramify at two distinct odd dimension primes that never occur together in a twist.

Proof. In factor J let U_J denote the represented upper unipotent. Its normalized twist $\gamma_J U_J$ has the actual J-phases. Let E_J be the span of the translates under the J-factor of the nontrivial phase spaces. On each of these spaces the complementary upper unipotents are scalar; their generating conjugates have the same scalars since the entire complementary group preserves the space. The same holds on its translates, by commutation. Hence on E_J each W_I, $I \neq J$, is the scalar γ_I^3 by $U(WUW^{-1})U = W^{-1}$. On $E_J^{\perp}$ the J-upper unipotent is scalar γ_J^{-1}, so the same reasoning gives $W_J = \gamma_J^3$. This proves the zero products and the formula for $s = \prod_J W_J$. In particular on E_J one has $s = (\xi/\gamma_J^3)W_J$.

Choose m selecting J as in Lemma 6.1. Up to diagonal phases and duality, the vacuum coordinates of W_J are the entries of $sT^m s^{-1} e_0$. Subtracting the untwisted coordinate leaves a sum of $(\theta_Y^m - 1)\chi_Y(X)/f_Y$. Only J-active labels occur; by Lemma 3.3 their codegrees are supported in J. Thus $W_J e_0$ is integral away from J, while $nW_J e_0$ is integral by Lemma 6.1. In the split vacuum column all terms except the J-term are p-integral, proving the valuation assertion by comparison with d_X/n. Full height at $q \in J \setminus \{p\}$ gives $q \nmid f_X$; the proof of Lemma 3.3 shows that W_q preserves the X-line. The nonzero X-entry of $W_J e_0$ therefore gives a nonzero entry of $W_p e_0$ there, proving principality.

For the ramification assertion take two distinct primes in different sets, or any two separated odd primes. The same moving-space argument gives $(P - I)(Q - I) = 0$ for their inertia torus permutations: the complementary torus acts as a scalar on an active phase space, hence as the identity there because it permutes its coordinate vectors. Here the upper unipotent at the odd prime acts trivially on the complement of its moving space, and so does its torus. The torus signs are positive in square dimension. In arbitrary integral dimension both separated odd primes have even valuation by Lemma 3.3 applied to the opposite active sectors, so their local Galois actions also fix $\sqrt{D}$ and have positive sign. If both permutations moved the label X, the diagonal entry at X of $(P - I)(Q - I)$ would be positive. □

Proposition 6.3 (Simultaneous singleton identities)**.** *Call t* isolated *if every t-active label has pure t-power twist, and a* singleton *if $v_t(n) = 1$. Let T be a nonempty set of odd isolated primes, each occurring once in n. Write $m_t = n/t$, $\epsilon = \xi$, $j_t = t(1 - (W_t)_{00})$ and $a_t(X) = \epsilon t (W_t)_{X0}$ for $X \neq 0$. Then $\epsilon \in \{1, -1\}$, $j_t > 0$ (and $j_t < t$ if $m_t > 1$), $m_t j_t \equiv -\epsilon \pmod t$, and $\sum_{t \in T} j_t/t \leq 1 + 1/n$. Every t-active label has degree m_t, and their number is at least $(t - 1)j_t$. One has*

$$t j_t = \sum_{Y \ t\text{-active}} (1 - \theta_{Y,t}). \tag{6.2}$$

For odd n, each j_t is even. Each a_t is integral, supported exactly on the nonunit t-free degrees, which belong to P_t. The column identities are

$$a_t(X) \equiv d_X/m_t \pmod t, \qquad \sum_{X\neq 0} d_X a_t(X) = (n+\epsilon)j_t,$$
$$\sum_{X\neq 0} a_t(X)^2 = 2tj_t - j_t^2, \qquad \sum_{X\neq 0} a_u(X)a_v(X) = -j_u j_v \quad (u\neq v). \tag{6.3}$$

In particular, after the active labels are removed the residual squared norm is at most $B_t = j_t(t+1-j_t)$.

Proof. Use the singleton clusters $\{t\}$, $t\in T$, and their complement H. A t-active label Y has $f_Y = t^2$, by complementarity and $v_t(f_Y) = 2-2v_t(d_Y) > 0$. The equality $s = \xi W_t$ on E_t gives $(W_t)_{Y0} = 1/(t\xi)$; rationality gives the common sign. The vacuum Gauss formula gives (6.2), so $j_t > 0$. The mean of a nontrivial t-power root is at least $-1/(t-1)$, proving the count. For odd n the active labels come in distinct equal-twist dual pairs, and t is odd.

Write $z_t = (W_t - I)e_0$. By (6.1), $s = W_H + \epsilon\sum_{t\in T}(W_t - I)$, with $W_H = I$ when H is empty and $\epsilon = 1$ in that case. Unitarity at the vacuum gives the deficit-sum bound. Multiply the split vacuum column by n and reduce modulo t to get both congruences. In particular $j_t \neq t$, so $j_t < t$ when $m_t > 1$. The scaled column is integral and its vacuum entry is nonzero, so every nonunit coordinate has absolute value $< t$. This proves the support and principality assertions. Finally $z_u \perp z_v$ for $u\neq v$, $\|z_t\|^2 = 2j_t/t$, and $\langle z_t, se_0\rangle = \epsilon\langle z_t, W_t e_0\rangle = \epsilon j_t/t$. Subtract the vacuum in these three identities to obtain (6.3); each active coordinate has square one. □

Lemma 6.4 (An isolated prime with unit vacuum deficit)**.** *Let $\mathcal{C}$ be integral modular of dimension n^2 with its positive spherical structure. Suppose $v\geq 5$ is an odd prime with $v_v(n) = 1$, and every v-active simple has v-primary twist (its prime-to-v twist is one). Put $m = n/v$ and suppose $j_v = v(1-(W_v)_{00}) = 1$. There is exactly one active label Y_u of twist ζ_v^u for each $u\in\mathbb{F}_v^\times$, and each has degree m. With $\xi = p^+/n = \pm 1$ one has*

$$\chi_{Y_u}(Y_t) = \xi\sum_{k\in\mathbb{F}_v^\times} \zeta_v^{-u/k-kt}.$$

All active labels are self-dual with field $\mathbb{Q}(\zeta_v)^+$, and all inactive label fields are unramified at v. The active rows agree on the inactive labels X, with integral values $b_X \equiv d_X \bmod v$. Omitting the vacuum from the inactive labels,

$$\sum_X b_X^2 = v, \qquad \sum_X d_X b_X = -1-\xi m. \tag{6.4}$$

Proof. Propositions 6.2 and 6.3 give $\sum_{Y\ \text{active}}(1-\theta_Y) = v$, with $s = \xi W_v$ on the active span. All active labels have degree m and are v-principal.

A square-Galois orbit of primitive v^eth phases, $e\geq 2$, has zero phase sum and size $v^{e-1}(v-1)/2 > v$, which the real part of this identity precludes. At level v the cyclotomic relation forces constant multiplicity on the nontrivial phases, hence multiplicity one. Since γ has two-power order, the v-part of $\mathrm{ord}(T/\gamma)$ is exactly v.

The $Y_u, 0$ entry of the local Bruhat identity $WU^kW^{-1} = U^{-1/k}Wh_{-k}U^{-1/k}$ for $k\neq 0$ gives $\sum_{t\neq 0}(\zeta_v^{kt}-1)\chi_{Y_u}(Y_t) = \xi v\zeta_v^{-u/k}$: the torus fixes the vacuum. Summing over $k\neq 0$ gives $\sum_t \chi_{Y_u}(Y_t) = \xi$; Fourier inversion gives the displayed entries. On the active coordinates their Gram matrix has diagonal $v(v-1)-1$ and off-diagonal $-v-1$ by Parseval. The full rows have Gram matrix $v^2 I$, so their inactive restrictions coincide, with squared norm $v+1$. Galois permutes the active rows, making their common inactive entries rational integers. Principality gives the congruences. Removing the unit from the norm and from orthogonality against the dimension character gives (6.4). Duality preserves each phase and so fixes its unique label; square

twist covariance gives precisely the indicated real field in $\mathbb{Q}(\zeta_v)$. On an inactive X, the vector $e_X - b_X e_0$ and its inverse Weyl image vanish on the active sector. Both local unipotents, and hence the local torus, fix it. The torus fixes e_0 as well, so X is inertia-fixed. □

Lemma 6.5 (Column norm on isolated row sets)**.** *In the setting of Proposition 6.3, let X be any nonunit label of degree d and $[d]_t$ its centered residue at t. For $k = |[d]_t|$ put $E_t = j_t k(t-k)/t^2$. Then $\sum_{t\in T} E_t < 1$. At an unramified t one may replace E_t by $(t-1)j_t[d]_t^2/t^2$. If also $t \geq 5$, every t-active Z has $\chi_Z(X) = [d]_t$. For such an inactive X,*

$$(W_t - I)e_X = [d]_t(W_t - I)e_0, \qquad a_t(X) = -\epsilon j_t[d]_t.$$

Any set L of nonunit inactive t-unramified labels therefore satisfies $1 + \sum_{X\in L}[d_X]_t^2 \leq \lfloor (t+1)/j_t \rfloor$. In particular, for every isolated odd singleton, $j_t \mid t-1$ or $j_t \mid t+1$.

Proof. Each active Z is t-principal, of codegree t^2. Its value on X lies in a t-power cyclotomic field and reduces to d at t. Theorem 5.3 and the active count give at least E_t in the Galois mean of the normalized X-column norm. Distinct active sectors are disjoint and the vacuum also contributes. Apply the estimate at the active degree $m_t = n/t$, with $k = |[m_t]_t|$. Then $0 < j_t k < 2t^2/(t+1) < 2t-1$ and $j_t k \equiv \pm 1 \pmod t$, so $j_t k$ is 1, $t-1$ or $t+1$.

If K_X is unramified at t, symmetry gives rational integers congruent to d, hence the sharper contribution. Distinct integers congruent modulo t have squares summing to at least $(t^2+1)/2$. Each occurs on an active orbit of at least $(t-1)/2$ labels by twist covariance, exceeding the norm budget for $t \geq 5$. The one value c obeys $(t-1)j_t c^2 < t^2$ and hence is $[d]_t$. If X is inactive, $e_X - ce_0$ and its inverse Weyl image both vanish on the active sector, so the upper and lower unipotents fix it. The vacuum entry gives the formula for a_t since $W_t^{-1}e_0 = W_t e_0$. On the active rows, columns $0, L$ now form a rank-one matrix with constant row $t^{-1}(1, ([d_X]_t)_{X\in L})$. Its operator norm is at most one. The active count and integrality of $1 + \sum [d_X]_t^2$ give the stated bound. □

Lemma 6.6 (Prime supports at an isolated prime)**.** *Let t be isolated in an ordinary minimal table of square dimension. Suppose a nonunit of codegree t^2 has t-valuation at most one in its degree. The proper nonunit t-free degrees intersect pairwise in prime support, have gcd one, and include at least three distinct degrees. Here proper means smaller than the t-free part of n.*

In particular, every isolated singleton t satisfies $t \geq 5$. Writing $m = n/t$, its nonunit t-free labels satisfy

$$\sum_X \left(a_t(X)^2 - \frac{d_X}{m} a_t(X) \right) = j_t(t - j_t) - \frac{\epsilon j_t}{m}.$$

Proof. Proposition 5.7 gives t-free nonunit degrees of gcd one. They divide the t-free part of n, so deleting that full degree preserves the gcd. All t-free labels are in P_t by Proposition 6.2. Two such nonunits hit one another by principality, so share a prime. Pairwise intersection with empty total intersection requires at least three distinct proper degrees and at least three distinct primes in them.

At a singleton a t-active row exists and has degree $m = n/t$ and codegree t^2 by complementarity. The active row hits the vacuum, the full degree and all three proper degrees by principality. Each costs at least one in its Galois-mean squared norm, excluding $t = 2$. For odd t subtract the dimension trace divided by m from the squared norm in (6.3). Each displayed summand is at least $1 - d_X/m \geq 0$. If $t = 3$, then $m \geq 2 \cdot 5 \cdot 7 = 70$ and the three distinct indices $m/d_X > 1$ are prime to three. Their reciprocals sum to at most $1/2 + 1/4 + 1/5$, so the three summands contribute at least $41/20 > 2 + 2/m$, contradicting the identity. □

Lemma 6.7 (Small-degree cutoff)**.** *At an ordinary minimal modular table of dimension n^2, partition the primes of n into clusters such that no twist meets two clusters. For a cluster J write $n_J = \prod_{v\in J} v^{v_v(n)}$, and put $B = \min_J n/n_J$. Every nonunit degree d satisfies $d^2 \geq B$.*

Proof. A label active in J has degree divisible by n/n_J, by Lemma 3.3 at every complementary prime. Thus all labels of degree below B have twist one. If $d_X^2 < B$, both X and all constituents of $X \otimes X$ have twist one. Balancing gives $S_{XX} = d_X^2$, contradicting strictness. □

Lemma 6.8 (An isolated quartic prime)**.** *Let q be an isolated prime in an ordinary minimal table of dimension n^2, where $n = q^2N$ and $q \nmid N$. If at least two simple degrees (counted with multiplicity) have q-valuation one, then at least $q(q-1)$ active labels have degree N, and some active J has degree qN. The row of each such J vanishes on all q-bearing degrees, and necessarily $q \geq 5$.*

Proof. Write A_0, A_1 for the active labels of degrees N, qN. These are all the possibilities by Lemma 3.3. Put $m = |A_0|$, $\delta_i = \sum_{Z \in A_i}(1 - \theta_{Z,q})$ and $b = q^2 W_q e_0$. Proposition 6.2 makes b integral, $v_q(b_X) = v_q(d_X)$ for $v_q(d_X) < 2$, and all height-one labels q-principal. The vacuum Gauss formula gives, for a root of unity λ,

$$\lambda b_0 = q^2 - \delta_0/q^2 - \delta_1.$$

For such a label X, every $Z \in A_0$ has $S_{XZ} \neq 0$ by principality, and $S_{XZ}/(qN)$ is integral by lcm divisibility. Their Galois-mean contribution to the normalized row norm is at least m/q^2 besides the vacuum; thus $m < q^2$. Since b_0 is a q-unit and δ_1 vanishes above q, the Gauss formula gives $m > 0$. All conjugates of the algebraic integer $1 - \delta_0/q^2$ are strict convex averages of 1 and nontrivial roots of unity, hence have modulus less than one. Therefore $\delta_0 = q^2$. The mean of $1 - \theta$ for a nontrivial q-power root is at most $q/(q-1)$, so $m \geq q(q-1)$. If A_1 were empty, $|b_0| = q^2 - 1$; with the two height-one coordinates this costs at least $(q^2-1)^2 + 2q^2 > \|b\|^2 = q^4$.

Finally let $J \in A_1$. Its row spends at least $m \geq q(q-1)$ on A_0 by principality. For height-one X, $\chi_J(X) = (d_X/d_J)\chi_X(J)$ vanishes above q, since $X \in P_q$ and d_X/d_J is a q-unit. This holds on all conjugates. A nonzero orbit would cost at least $2q$ by Theorem 5.3, exceeding the remaining norm $f_J - m \leq q$. Height two is excluded by defect-zero vanishing at the active column J. By Lemma 6.6 this row also hits at least three distinct proper q-free degrees in P_q. Together with A_0 and the vacuum they cost at least $q(q-1)+4$, forcing $q \geq 5$. □

6.1 A denominator bound for local Weyl columns

Let $\mathcal{C}$ be an integral modular category of dimension n^2 with positive spherical structure. Write $s = S/n$, $f_z = n^2/d_z^2$, and E_z for the primitive character idempotent and e_z for the coordinate vector of label z. Fix distinct primes q, p. For the congruence lift of Lemma 6.1, let U_q, W_q be the upper unipotent and Weyl element of the q-primary factor. The diagonal matrix T_q of actual q-primary twists differs from U_q by a scalar root of unity. Call z active at q if $(T_q)_{zz} \neq 1$.

Theorem 6.9. *Put $a = v_p(f_z)$. At every prime above p, normalized by $v(p) = 1$, each nonzero entry w_{iz} of $W_q^{\pm 1}$ satisfies*

$$v(w_{iz}) \;\geq\; \begin{cases} 0, & a = 0, \\ -\lfloor (a-1)/2 \rfloor, & a \geq 1. \end{cases}$$

If $v_p(f_z) \leq 2$ for every q-active label z, all entries of $W_q^{\pm 1}$ are integral at every prime above p.

Proof. Choose $m > 0$ congruent to one at the q-primary level and to zero at every other primary level, with $p \mid m$. Put $A = sT^m s^{-1}$ and $w = W_q^{-1}$. The identity $U_q(W_q U_q W_q^{-1})U_q = W_q^{-1}$ makes A equal to w up to diagonal root-of-unity factors. For $H = W_q^2$, the local torus at -1, Galois covariance of the linear lift gives $\overline{w} = w^{-1} = Hw$. The matrix H permutes the labels, since the torus signs in integral square dimension are positive. Moreover every cyclotomic automorphism with exponent one modulo the q-primary level fixes w: its action conjugates the group element by $\mathrm{diag}(k, 1)$, which fixes the q-component for these exponents; all other components of the group

element are trivial. Thus w_{iz} lies in a q-power cyclotomic field, unramified at p (Lemma 6.1 and its proof).

For labels i, j let $L_m^{ij} = \nu_{m,1}^{(i,j^*)}$ under $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$. The modular formula [46, Proposition 6.1] reads $L_m^{ij}(c) = \sum_{a,b} s_{ia} s_{jb} (\theta_a/\theta_b)^m N_{ab}^c$. Contracting with the coefficients of $y = f_z E_z = \sum_c \overline{\chi_z(c)}[c]$ replaces N_{ab}^c by $\overline{\chi_z(a)\chi_z(b)}$. Changing b to b^* in the second sum and using $s_{0z}^{-2} = f_z$ gives the mixed identity

$$L_m^{ij}(y) = f_z A_{iz} \overline{A_{jz}}.$$

For any central simple insertion X, abbreviate $L_m = \nu_{m,1}^X$. The rotation is of finite order since its mth power is the scalar θ_X^{-1}. For each fusion basis element c, the pth power of its trace is congruent at every prime above p to the trace of its pth power. Composition of rotations and grouping p strands identify the latter with $L_{m/p}(c^p)$ [46, Lemmas 2.5, 2.7 and Proposition 2.8(ii),(iii)]. The functionals are additive by [46, Corollary 5.5]. Thus for all integral cyclotomic coefficients a_c, with $y' = \sum_c a_c[c]$, the commutative fusion-ring congruence gives

$$L_m(y')^p \equiv \sum_c a_c^p L_{m/p}(c^p) \equiv L_{m/p}((y')^p) \pmod{\mathfrak{p}}.$$

In particular $L_m^{ij}(y)$ is an algebraic integer, and if $p \mid f_z$ its valuation is positive since $y^p = f_z^{p-1} y$.

Take $j = Hi$. Then $\overline{w_{jz}} = w_{iz}$. The mixed identity gives $a + 2v(w_{iz}) \geq 0$, strictly if $a > 0$. The valuation $v(w_{iz})$ is integral by unramifiedness, proving the bound also for $W_q = Hw$.

Work in a cyclotomic splitting field K and put $R = \bigcap_{\mathfrak{p}|p} \mathcal{O}_{K,\mathfrak{p}}$. The column bound puts $w_z = W_q e_z$ in $R^{\mathrm{Irr}(\mathcal{C})}$ for every q-active z when $v_p(f_z) \leq 2$. Complex conjugation permutes the primes above p, so the entries of w_z^* also lie in R. By unitarity,

$$W_q T_q W_q^{-1} = I + \sum_{z \text{ active at } q} (\theta_{z,q} - 1) w_z w_z^*.$$

Thus $W_q U_q W_q^{-1}$ has entries in R, since U_q differs from T_q by a root of unity. The unipotent identity above gives the assertion for W_q^{-1}, and taking the adjoint gives it for W_q. □

Lemma 6.10 (The tetrahedral Weyl matrix)**.** *Let u be a 3-cycle and w a nonidentity element of the normal Klein four-group K in A_4. A unitary representation of A_4 of dimension at most four is a sum of characters of A_4/K, or consists of one tetrahedral summand T and at most one such character. In the latter case T has one line at each u-eigenvalue. In a suitable orthonormal basis of these lines the matrix of w on T is $2J/3 - I$, where J has all entries one. On any character summand w acts by one. If T occurs and the u-fixed space is two-dimensional with orthonormal basis v, v', and $x = \|P_T v\|^2$, then*

$$|\langle v', wv \rangle|^2 = \tfrac{16}{9} x(1 - x).$$

Proof. The three nontrivial characters of K are cycled by u. Choose a unit vector b of the unique nontrivial K-character taking the value $+1$ on w, if this character occurs. Then $b, ub, u^2 b$ lie in mutually orthogonal character spaces and span an invariant copy of T. Its orthogonal complement has dimension at most one. If no such character occurs, K acts trivially throughout. On the displayed basis w has signs $(1, -1, -1)$. For $\omega = \exp(2\pi i/3)$, the vectors $3^{-1/2} \sum_{j=0}^{2} \omega^{-kj} u^j b$ diagonalize u. The projector onto $\mathbb{C}b$ has matrix $J/3$ in that basis, giving the formula for w. On the two-dimensional fixed space the compression of w is $I - 4P_T/3$, where P_T restricts there to a rank-one projection. Its off-diagonal entry gives the asserted square. □

6.2 Conductor, local Weyl coefficients and twist masses

For an integral modular category put $d_Z = \mathrm{FPdim}\, Z$, $f_Z = D/d_Z^2$, $s = S/\sqrt{D}$ and $\chi_Z(Y) = S_{ZY}/d_Z$. We use the positive spherical structure, and the lift of Lemma 6.1 with normalization

scalar γ of dyadic order and $\gamma^3 = \xi$, the multiplicative central charge. All uses of Frobenius–Schur indicators below concern actual categories. Let $\mathcal{C}$ be an integral modular category of dimension $D = n^2$. For any integer $m \geq 1$ put

$$M_m(Z) = \sum_Y d_Y \chi_Z(Y) \theta_Y^m.$$

For a set J of primes in the level write W_J for the product of its local Weyl operators in the normalized congruence representation, and $\theta_{Y,J}$ for the J-part of θ_Y.

Theorem 6.11. *For all m, Z, the number $|M_m(Z)|^2/D$ is an algebraic integer. For every J, Z,*

$$k_{J,Z} := \frac{n}{d_Z}(W_J e_0)_Z \in \mathbb{Z}.$$

More precisely, if m is one on the local levels in J and zero on the complementary levels, then $M_m(Z) = n\gamma_J^3 \theta_{Z,J}^{-1} k_{J,Z}$, with γ_J the J-part of the normalization scalar. Every prime q outside J dividing f_Z also divides $k_{J,Z}$.

Proof. Extend ν_m linearly to virtual objects with cyclotomic integer coefficients; its values are integral because each simple indicator is a finite-order trace. For every B, inserting Verlinde into the Ng–Schauenburg formula [44, Theorem 7.5 and Remark 7.6] gives $\nu_m(B) = \sum_Y \chi_Y(B^*)|(sT^m s e_0)_Y|^2$, where $*$ extends duality linearly. The two factors are conjugate since X, X^* have the same twist. For $B = \sum_X \chi_Z(X)[X]$ the coefficient at Y is

$$\sum_X \chi_Z(X)\chi_Y(X^*) = f_Z \mathbf{1}_{Y=Z}.$$

Thus $\nu_m(B) = f_Z|(sT^m s e_0)_Z|^2 = |M_m(Z)|^2/D$.

For the indicated m, $sT^m s e_0 = sT^m s^{-1} e_0$ since charge conjugation fixes the vacuum. Only the J-local upper unipotent occurs in the middle of the latter word, apart from the scalar γ_J. The matrix identity $WUW^{-1} = U^{-1}W^{-1}U^{-1}$ in each selected factor shows that its Z-coordinate is $\gamma_J^3 \theta_{Z,J}^{-1}(W_J^{-1} e_0)_Z$. The vectors $W_J^{-1} e_0 = W_J e_0$ are rational in square dimension (Lemma 6.1). Since the same coordinate equals $d_Z M_m(Z)/D$, the claimed expression for $M_m(Z)$ follows with rational $k_{J,Z}$. Its square is an algebraic integer by the first assertion, proving integrality.

For a prime q the rotation identity [45, (5.4) and proof of Theorem 5.1, pp. 75–76] gives $\nu_{qv}(X)^q \equiv \nu_v(X^{\otimes q})$ at a prime above q: trace to the qth power agrees in the residue field with trace of the qth power, and rotation by q places is rotation by one block of length q. This extends to every cyclotomic-integral virtual B by additivity and the Frobenius homomorphism in the commutative fusion ring modulo q. If $q \notin J$, choose the CRT parameter in the form $m = qv$. The above B equals f_Z times a primitive idempotent (at Z^*), hence $B^q = f_Z^{q-1} B$. For $q \mid f_Z$ the congruence now gives $k_{J,Z}^{2q} \equiv 0$ at that prime, proving $q \mid k_{J,Z}$. □

Proposition 6.12. *Let p, q be separated twist primes in a perfect fusion-simple integral modular category of square dimension. At a nonunit Z suppose $a = p^{v_p(n/d_Z)} > 1$, $b = q^{v_q(n/d_Z)} > 1$, and all other primes of n occur to full height in d_Z. Let L_p, L_q bound the degrees of p-active and q-active labels, respectively. Put $f = f_Z$. Then*

$$2L_p(f-1) \geq nb^2, \qquad 2L_q(f-1) \geq na^2.$$

Proof. Separation forces full q-height on p-active degrees and full p-height on q-active degrees. The deficiencies at both primes, together with full height elsewhere, therefore give $\theta_Z = 1$. Each other active prime misses the row Z by defect-zero vanishing. Write $b_Y = \chi_Z(Y)$ and put

$$H_t = \sum_Y d_Y b_Y(\theta_{Y,t} - 1) \quad (t = p, q).$$

By $\sum d_Y b_Y = 0$, Theorem 6.11 gives $H_t = n\gamma_t^3 k_{t,Z}$. Every nonzero hit with nontrivial twist has only one of the two components, hence $H_p + H_q = n\xi$ by the vacuum modular relation, where $|\xi| = 1$ and ξ is a root of unity. At p-active labels, $q^{v_q(n)} \mid d_Y$ and $b \mid b_Y$ by lcm integrality. Thus $b \mid k_{p,Z}$; similarly $a \mid k_{q,Z}$. Neither integer vanishes: the other would then have absolute value one, violating its divisibility. On the p-active labels, a union of Galois orbits, the integral entries b_Y/b satisfy $\sum |b_Y/b| \le \sum |b_Y/b|^2$. Indeed every nonzero orbit has mean square at least one by the absolute norm, followed by Cauchy–Schwarz. Consequently

$$nb \le |H_p| \le 2L_p \sum_{Y \ p\text{-active}} |b_Y| \le 2L_p(f-1)/b,$$

with one unit of squared norm reserved for the vacuum. Interchanging p, q proves the other inequality. □

A two-variable integer test. The magnitude bounds can be combined with the phase equation. Under the hypotheses of Proposition 6.12, there are nonzero integers $u \in b\mathbb{Z}$ and $v \in a\mathbb{Z}$ such that

$$u + v = 1, \qquad \frac{n}{2}\left(\frac{b|u|}{L_p} + \frac{a|v|}{L_q}\right) \le f - 1.$$

In particular $\xi = \pm 1$ in this situation. Indeed if both primes are odd, $\gamma_p = \gamma_q = 1$ so the phase equation is $k_{p,Z} + k_{q,Z} = \xi$, an integer of modulus one. If $p = 2$, it is $\xi k_{p,Z} + k_{q,Z} = \xi$; as $k_{q,Z} \neq 0$ both sides force ξ real. The other case follows on interchanging the primes. Absorb these signs into u, v to obtain $u + v = 1$. The estimate for H_p in the proof gives a squared row norm of at least $nb|u|/(2L_p)$ on the p-active sector, and likewise for q. The two sectors are disjoint and omit the vacuum. This proves the test. For fixed a, b the minimum on the left is elementary: solve $u \equiv 0 \pmod b$, $u \equiv 1 \pmod a$, and take the solution on each side of $[0, 1]$ nearest to it. Thus the test applies to prime decomposition forms without choosing individual simple labels. The same argument bounds the total number of deficient rows. Fix the primes and n in Proposition 6.12, but let Z range over all nonunit labels with the two indicated deficiencies and full height elsewhere. Since $k_{p,Z}$ is a nonzero multiple of $q^{v_q(n/d_Z)}$, the unit norm of $W_p e_0$ gives

$$\sum_Z p^{2v_p(d_Z)} < p^{2v_p(n)}, \qquad \sum_Z q^{2v_q(d_Z)} < q^{2v_q(n)}.$$

For strictness, equality would put $W_p e_0$ entirely on those twist-one labels. This vector and e_0 then have the same eigenvalue for the local upper unipotent U_p. Since $W_p^2 e_0 = e_0$ and $W_p^{-1} e_0 = W_p e_0$, the local identity $W_p U_p W_p^{-1} = U_p^{-1} W_p^{-1} U_p^{-1}$ applied to e_0 forces $W_p e_0$ proportional to e_0, contradicting its nonunit support. The strict bound for q follows in the same way.

The last dyadic conductor level. Write f_Z for the formal codegree and K_Z for the character field at a label Z.

Theorem 6.13 (Conductor at an individual label)**.** *Let a commutative fusion ring have symmetric integral character data satisfying (2.1) and (2.2), with cyclotomic algebraic-integer character values. Then the conductor of K_Z divides f_Z. In particular the conductor of the lcm-normalized XY crossing divides $\gcd(f_X, f_Y)$. In ordinary data, if the dyadic part of a twist has order 2^v with $v \ge 4$, then $2^{v-1} \mid f_X$ for its label X.*

Proof. The different bound (Corollary 2.8) already puts K_Z in a cyclotomic field of conductor dividing $2f_Z$. It remains to remove its last possible dyadic level. Put $m = 2^{v_2(f_Z)}$. There is nothing to check for $m = 1$. Take $L = F(\zeta)$ with F an odd cyclotomic field large enough and ζ primitive of order $2m$. Suppose the dyadic conductor is $2m$. In $G = (\mathbb{Z}/2m)^\times$ the stabilizer H of the row does not contain $e = 1 + m$, since the fixed field of e is $F(\zeta^2)$.

Write $b_Y = \chi_Z(Y) = \sum_{k=0}^{m-1} z_{kY}\zeta^k$. The coefficients belong to $\mathcal{O}_F$ by coprimality of conductors. The orthogonality of the m odd powers of ζ gives

$$z_{1Y} = m^{-1} \sum_{g \in G} \zeta^{-g} \chi_{gZ}(Y).$$

Different rows in this sum are orthogonal; the squared norm of each is f_Z. Thus

$$\sum_Y |z_{1Y}|^2 = \frac{f_Z}{m^2} \sum_{h \in H} \sum_{g \in G} \zeta^{g(h-1)} = \frac{f_Z}{m}.$$

Indeed the inner geometric sum vanishes unless $h = 1$ or $h = e$, and only 1 belongs to H. Lift e to the common splitting field, trivially on F. Its permutation of the argument labels sends z_{1Y} to $-z_{1Y}$. Every cycle on the nonzero coefficients therefore has even length and contributes to $2\mathcal{O}$ in the squared norm. This contradicts oddness of f_Z/m. Symmetry gives the crossing assertion. For the twist assertion, Theorem 3.5 puts $\mathbb{Q}(\zeta_{2^{v-1}})^+$ in K_X by the square action whenever $v \geq 4$. Its conductor is 2^{v-1}; the normalization scalar has order at most eight, hence does not affect this conclusion. □

Squared mass at a prescribed twist. Write $D = n^2$, $R = \operatorname{rank} \mathcal{C}$, $\tau_j = \sum_X d_X^2 \theta_X^j$, and $M_t = \sum_{\theta_X = t} d_X^2$. The following argument uses the genuine second indicator of the category. It applies to an integral modular category with its positive spherical structure.

Proposition 6.14. *Put $B = \sum_X d_X \nu_2(X)$. Then B is a nonnegative integer, $B \leq \sum_{X=X^*} d_X \leq n\sqrt{R}$, and for every unit complex number t,*

$$9M_t \leq 3D + 4\operatorname{Re}(\bar{t}\tau_1) + 2\operatorname{Re}(\bar{t}^2\tau_2) \leq 3D + 4n + 2n\sqrt{B}.$$

More generally, if the twist exponent is prime to all integers $1, \ldots, k-1$, then $M_t \leq (D + (k-1)n)/k$.

Proof. Multiply the Ng–Schauenburg formula for $\nu_2(X)$ by d_X, sum on X, and use $\sum_X N_{UV}^X d_X = d_U d_V$. This gives $DB = |\tau_2|^2$. The indicator bounds follow from its values 0 off the self-duals and ± 1 on them. Sum $d_X^2 |1 + w + w^2|^2$ with $w = \bar{t}\theta_X$. Its value at $w = 1$ is nine and every summand is nonnegative. Expanding proves the assertion since $|\tau_1| = n$. In the last assertion use $|1 + w + \cdots + w^{k-1}|^2$ in the same way. For each indicated j an automorphism takes τ_1 to τ_j, and τ_1 is n times a root of unity. □

Degree sums for a prime-power codegree. Let a row Z have formal codegree a power of p and $p \mid d_Z$. All its nonzero entries lie in P_p (Theorem 2.9). For every degree d with $v_p(d) \geq v_p(d_Z)$,

$$\sum_{d_Y = d} \chi_Z(Y) \in p\mathbb{Z}.$$

Indeed in P_p the algebraic integer $\chi_Y(Z)$ is zero modulo the prime above p; by symmetry the same holds for $\chi_Z(Y) = (d/d_Z)\chi_Y(Z)$. The sum over a degree is rational by Galois covariance. Applying any rational linear functional of value one at one to all the entries leaves this degree sum unchanged.

Lemma 6.15. *In ordinary integral modular data of dimension n^2, fix a nonunit degree z, set $f = (n/z)^2$, and suppose $p \nmid f$. All rows of degree z are unramified at p. For any nonunit Y, let e be its p-inertia orbit size and $c = d_Y / \gcd(z, d_Y)$. Then*

$$\#\{Z : d_Z = z,\ S_{ZY} \neq 0\} < \frac{f}{ec^2}.$$

Proof. Indeed after applying $s = S/n$ to the normalized indicator of that orbit its Zth coordinate is $\sqrt{e/f}\,\chi_Z(Y)$ at every Z of degree z. The positive vacuum coordinate gives strictness. Take Galois means and use lcm integrality and the norm inequality on $\chi_Z(Y)/c$. The set of hitting Z is a union of full Galois orbits. Thus the bound also applies to the field degree of each hit. □

7 Three ordered prime powers

Write $p < q < r$ for the three repeated primes. The two larger principal blocks impose a lower bound on the p-part of certain degrees. At a quartic prime the same bound separates twist sectors; at the largest possible p-height it forces both companion exponents to be even.

7.1 Quartic primes below the largest prime

Lemma 7.1 (Separation at a quartic prime)**.** *Let an ordinary perfect fusion-simple integral modular table have dimension $t^a u^4 v^b$, $u < v$, with three repeated primes. No twist order is divisible by uv. The same holds in a reduced minimal table for odd u, v.*

Proof. Lemma 4.10 gives $X \in P_u$ of degree ut^i. Every nonunit t-free $Y \in P_v$ has degree $u^h v^j$, $h = 1, 2$. If $h = 1$, $S_{XY}/\operatorname{lcm}(d_X, d_Y)$ has weight u, strict house below u and residue u at both u, v by principality (at u both are zero). This contradicts Theorem 4.9, also at $u = 2$. Thus all such Y have u-unit codegree and vanish on u-active columns. For a joint u, v-active column, reduction of its orthogonality on P_v modulo t would leave only the vacuum. At odd u, v the same argument uses positive balancing and reduced block orthogonality. □

Lemma 7.2 (An isolated quartic in a three-prime table)**.** *A quartic prime h cannot be isolated in an ordinary three-prime minimal table.*

Proof. By Lemma 3.3 isolation makes the dimension a square. The two height-one labels of Lemma 4.10 then give an active label of codegree h^2 by Lemma 6.8, contrary to Proposition 5.7 at the three primes. □

Theorem 7.3 (A quartic below a larger prime)**.** *Let $p < q < r$. Every braided category of dimension dividing $p^a q^b r^c d$, with d coprime and square-free, is WGT when $\min(a, b) \leq 4$.*

Proof. By descent and Theorem 4.12, only an exact quartic h below a larger prime k remains. An ordinary table with h least is already impossible: Lemma 7.1 separates h from both larger primes, contrary to Lemma 7.2. Corollary 4.11 disposes of a reduced table at $h = 2$. Thus suppose h, k are odd, and denote the third prime by t. In a reduced table $t = 2$ by Proposition 4.7.

We work in the ordinary table or its integral minimal extension. In the latter, separation of h, k in the even sector extends to all labels. Indeed the joint active space is invariant under the dyadic Weyl. This operator interchanges the fermion permutation F and the monodromy sign G. A space supported on odd labels contains no G-fixed vector, but contains the F-fixed vector $e_X + e_{gX}$ for each of its fermion pairs. Thus h, k are separated by Lemma 7.1 and no character field ramifies at both (Proposition 6.2).

There is a joint t, h-active column J: in the ordinary case otherwise h is isolated; in the extension choose an even h-active pair, whose opposite twists allow a dyadically active member. Choose $A \in P_h$ of degree hk^j and $B, C \in P_k$ respectively of degrees $h^v k^l, t^w k^m$, with all powers positive, by Lemma 4.10 and Proposition 4.2. Use full principal representatives in the extension: a principal character on the even ring has a common sign times the dimension on the odd sector modulo the odd prime, since each product of two odd labels is even; multiplication of the label by g changes the sign. Orthogonality on P_t modulo k now gives a k-free noninvertible hit V on J: the invertibles contribute one, or two in the extension since J is even. Defect-zero vanishing bounds its h-height by one. It hits both A and C by principality, so coprime vanishing gives $d_V = t^i h$, $i > 0$.

The two normalized entries $z_Y = S_{VY}/\operatorname{lcm}(d_V, d_Y)$, $Y = A, B$, belong to K_V and have strict house less than h. At A one has $t^i z_A = \chi_A(V) \equiv 0$ above h, while $z_A \neq 0$ since $V \in P_t$ and $t \nmid d_A$. At B the principalities of V, B give residue h at both t and k. The zero-residue and two-residue assertions of Proposition 4.8 force ramification of K_V at h and at $k > h$, respectively, contradicting separation. □

7.2 Twist phases at maximal height

We first record the common indicator computation for the odd and dyadic phase arguments. Fix a prime p, write W_o for the complementary Weyl, $w = W_o e_0$, and suppose a Galois-stable label set J satisfies $|w_j|^2 = (h/D)d_j^2 \mathbf{1}_J(j)$ for an integer h. Choose m zero on the p-level and one on the complement. The complementary upper–lower–upper identity gives $|(sT^m s^{-1})_{0j}| = |w_j|$. Thus

$$b_X = \nu_m(X) = \frac{h}{D}\sum_{j\in J} d_j^2 \chi_j(X), \qquad b = \sum_X b_X[X^*].$$

These indicators are algebraic integers by (3.2) and rational by Galois symmetry. The b_X are therefore integers, constant on character orbits by symmetry of S. Orthogonality gives $\chi_j(b) = h\mathbf{1}_J(j)$, so $b^2 = hb$. Also $b = b^*$, $b_0 = \|w\|^2 = 1$ and $\|b\|^2 = h$. In each application $0 \in J$, giving $\dim b = h$.

Lemma 7.4 (An odd-order phase cannot be missing)**.** *Let an integral perfect modular category have odd dimension D, with $v_p(D)$ odd. If all p-primary twists have order at most p, every pth root of unity occurs.*

Proof. Write U, W for the p-local upper unipotent and Weyl, P for the projection onto $\ker(U-I)$. Then $Pw = w$, $Ww = se_0$, and the local torus acts on w by the Legendre character as in Proposition 3.8. For $1 \le t < p$, Bruhat gives $PWU^tW^{-1}w = (\frac{-t}{p})PWw$. If ζ_p^k is missing, its Fourier projection vanishes. Conjugating by W and applying P gives $w = -GPse_0$ with $G = \sum_{t\neq 0}(\frac{-t}{p})\zeta_p^{-kt}$, $|G| = \sqrt{p}$. Apply the indicator computation with J the p-inactive set and $h = p$.

The coefficient vector $psPs^{-1}e_0$ of b is fixed by the local lower unipotent and transforms under the torus by the Legendre character. It therefore has an active coordinate Y, since invariance under both unipotents would give trivial torus action. Twist covariance gives $\mathbb{Q}(\zeta_p)^+ \subseteq K_Y$ and $Y \neq Y^*$ in odd dimension makes K_Y nonreal. Its orbit has at least $p-1$ labels. Their equal nonzero coefficients exhaust the nonvacuum norm $p-1$, so $p = \dim b = 1 \pm (p-1)d_Y$, contrary to perfectness. □

Lemma 7.5 (Dyadic twists seen by a defect-one row)**.** *In an integral modular category with $v_2(D)$ odd, a label X with $v_2(f_X) = 1$ has nonzero entries only at dyadic twists $1, \eta_X$ for a single primitive fourth root η_X. Both twists occur on labels of degree d_X in its full two-block.*

Proof. By Proposition 2.12 the full two-block B has at most two inertia orbits, and if X ramifies its orbit is the entire block.

Let E project onto B, and put $Q = sEs^{-1}$. Both commute with the dyadic group by block invariance. Since $s^{-1}\mathbb{C}^B = W_o^{-1}\mathbb{C}^B$ has a nonzero vacuum coordinate and W_o preserves dyadic twists, B contains a trivial phase. To detect order four, work modulo the dyadic part 2^a of $\mathrm{ord}(T/\gamma)$. This level is divisible by eight: $s_{00} = 1/\sqrt{D}$ belongs to the level field, and the square-free part of D is even, so the quadratic discriminant has two-part eight. Write $U(1) = \lambda T_2$, with $\lambda = \gamma^{-1}$ and $\lambda^8 = 1$ by integrality [21, Proposition 6.7(ii)]. For odd k put $l = 1 + 4k$ and $M(k) = \sum_{j\in B} |s_{0j}|^2 \theta_{j,2}^{-2k}$. The unit $l \equiv 5 \pmod 8$, extended by one on the odd part of the level, fixes the vacuum label and negates $\sqrt{D}$. Thus $H_l v_0 = -v_0$. As $L(b) = sU(-b)s^{-1}$, $(QL(2k))_{00} = \lambda^{-2k}M(k)$. Taking the vacuum entry after Q in the matrix identity $U(2)L(2k) = L(2k/l)H_l U(2/l)$ now gives

$$\lambda^{2-2k}M(k) = -\lambda^{(2-2k)/l}M(k/l). \tag{7.1}$$

All divisions here are by units modulo 2^a. The exponents agree modulo eight since their difference is $8k(1-k)/l$. The map $k \mapsto k/(1+4k)$ permutes the odd residues, with inverse $h \mapsto h/(1-4h)$, so the mean of $M(k)$ is zero. The mean of θ^{-2k} over odd k is 1 at orders one and two, -1 at

order four, and zero at higher dyadic orders. Thus the squared masses at orders one and two together and at order four are equal.

Inertia preserves the dyadic order and fixes each fourth root under the square action. Thus there are exactly two orbits, with phases $1, \eta_X$, and X is unramified by Proposition 2.12. On B both $U(4), L(4)$ are scalar; $U(1)L(4) = L(4/5)H_5U(1/5)$ makes H_5 scalar as well. The units 1 mod 4 consequently fix B pointwise. One orbit is a singleton and the other has at most two members. Equal squared masses and integral degrees force both to be singletons of degree d_X. The support assertion follows from $s^{-1}\mathbb{C}^B = W_o^{-1}\mathbb{C}^B$. □

Lemma 7.6 (Two dyadic twists force an invertible)**.** *In an integral modular category with $v_2(D)$ odd, if all dyadic twists are 1 or a single primitive fourth root η, then there is a nontrivial invertible.*

Proof. Put $Z = T_2^2$, $F = sZs^{-1}$ and $P = (I + Z)/2$. As above H_5 is scalar, hence $-I$ by its vacuum entry. Using $U(8) = L(8) = I$, the identity $U(2)L(2) = L(2/5)H_5U(2/5)$ gives $ZF = -FZ$, hence $PFP = 0$. With $a = (1 + \eta)/2$, $c = (1 - \eta)/2$, one has $T_2 = aI + cZ$. Up to scalar $W_2^{-1} = T_2(aI + cF)T_2$, so PW_2P acts on P by a scalar of modulus $1/\sqrt{2}$. Since $Pw = w$ and $W_2w = se_0$, the indicator computation applies on the trivial dyadic fiber with $h = 2$. The integral vector b has norm squared two, unit coefficient one and dimension two; its remaining entry is a positive invertible. □

Lemma 7.7 (A uniform odd-sector spin)**.** *Let $\mathcal{M}$ be an integral minimal modular extension as in Lemma 4.4. Its even dyadic twists cannot all be signs if its odd dyadic twists have a common value δ.*

Proof. Let F, G be the fermion permutation and sector sign. Write $A = (F = G = 1)$, $B = (F = -1, G = 1)$, $C = (F = 1, G = -1)$. The dyadic Weyl W interchanges F, G; W_o commutes with both. The dyadic upper unipotent $U = \lambda T_2$ exchanges A, B, squares to λ^2 there, and equals $c = \lambda\delta$ on C. Writing $J = W|_A$, evaluate $WUW = U^{-1}WU^{-1}$ on Ux, $x \in A$. Since $WUx \in C$ and W^2 is central, $cUJ^2x = \lambda^{-2}Ux$; thus J is scalar. Project $se_0 = Ww$ to A. The even pairs are symmetrized and w is supported on their representatives E of dyadic twist one; hence $|w_j|^2 = 4d_j^2/D$ there.

Use the indicator computation with $h = 4, J = E$, Galois-stable by twist covariance. The coefficients of b on the even basis equal those of $1 + g$: the characters of each pair j, gj agree there, and the sum over the full even sector is the projection onto $\chi_j(g) = 1$. Therefore $v = b - 1 - g$ has odd support, $gv = v$, $v^2 = 2(1 + g)$ and $\dim v = 2$. Its squared norm is two. A free pair gives an odd invertible; otherwise the odd simples have degree at least two, so $v = Y - Z$ with two fermion-fixed simples. Now vY has norm squared four and coefficients one at $1, g$ by reciprocity. Its remaining part is $\pm(A' + gA')$ for an even noninvertible A'. By reciprocity vA' hits Y; it is orthogonal to v and has norm squared two. Thus $vA' = Y + Z$ by positive dimension, whence $d_Y, d_Z = d_{A'} \pm 1$. Coprime vanishing in Lemma 4.4 gives $\chi_{A'}(Y) = \chi_{A'}(Z) = 0$, contradicting $\chi_{A'}(v)^2 = 4$. □

Lemma 7.8 (Twist activity in an invariant fiber)**.** *Let s be odd, with $v_s(\Delta)$ odd in ordinary or positive reduced integral modular data. Every nonempty coordinate subspace invariant under the s-local group contains an s-active label. This applies to each fiber of a complementary twist, in either fermion sector of an integral minimal extension, and to each squared dyadic twist fiber in the positive quotient.*

Proof. On an inactive subspace the s-local upper unipotent is the identity, as are its conjugates there. They generate the local group, yet the torus of a nonsquare unit at s acts by minus a permutation since its Galois automorphism negates $\sqrt{\Delta}$. Complementary twists and the fermion grading commute with the odd local group. □

7.3 The two companion exponents

Theorem 7.9 (A prime-power codegree at maximal least-prime height)**.** *Let an ordinary minimal table have dimension $p^a q^b r^c$, $p < q < r$, $k = \lfloor a/2 \rfloor \geq 2$ and $p^{k-1} < q$. Some p-active label Z has $f_Z = p^{a-2j}$ and $d_Z = p^j \sqrt{q^b r^c}$. In particular b, c are even. If p is odd, then*

$$(p-1)(r-1) < H_p(r) p^{a-2}, \qquad H_x(r) = \max\{e : e \mid r-1,\ e \leq x\}.$$

Proof. For $s = q, r$ each s-active column hits a complementary label of P_s with trivial prime-to-s twist, by prime avoidance on the trivial complementary fiber (Proposition 4.2). Call these labels witnesses. By Lemma 4.10 every witness has p-height exactly k.

We find a nontrivial p-phase missed by all witness rows. If $a = 2k$ the witnesses have p-unit codegree, so any active phase works. If $a = 2k+1$, they have defect one. For odd p, apply Lemma 3.10 with $t = p$, $s = q$, taking the q-free witnesses of P_r and the witnesses in P_q. Since $q > p$, their visible p-phases are contained in 1 and one common primitive square class. A deeper phase, or the other square class by Lemma 7.4, is missed. At $p = 2$, Lemma 7.5 gives 1 and one primitive fourth root with a representative of the witness degree at both phases. Opposite witnesses hit these representatives by principality, so the nontrivial phase agrees. Lemma 7.6 and perfectness supply a phase outside the two allowed. In all cases the missed phase has a nonempty fiber on which no q- or r-active column can occur. The two companion groups act trivially there, since their generating upper unipotents and their conjugates do. Thus Lemma 3.3 gives the asserted codegree for any Z of this fiber.

Now suppose p is odd. The orbit $\mathcal{O}$ of Z has at least $p-1$ elements, since K_Z contains $\mathbb{Q}(\zeta_p)^+$ by twist covariance and is nonreal by oddness. If $r \leq 2p^{k-1}$ the bound follows from $H_p(r) \geq 2$. Otherwise reduce the orthogonality of the Z-column to the vacuum on P_p at r. It gives a hit X of degree $p^h q^i$, $1 \leq h \leq k-1$ by principality. Pair it with a $p^k r^l$ witness in P_r. The normalized entry has residue p^h at $r > 2p^h$ and strict house below p^h. Theorem 4.9 gives a unique unsigned r-word. Its tame stabilizer acts freely on its nonidentity roots and has order at most $H_{p^h}(r)$; hence the r-inertia orbit of X has size $e \geq (r-1)/H_{p^h}(r)$.

Now K_Z is unramified at r by its codegree. At the nonzero entry S_{ZX} all e inertia conjugates of X therefore agree on Z. The field-energy bound (Lemma 2.4) and $|\mathcal{O}| \geq p-1$ give

$$e(p-1)\operatorname{lcm}(d_Z, d_X)^2 < \Delta, \qquad (p-1)(r-1) < H_{p^h}(r) p^{a-2h}.$$

Here $d_Z = p^j \sqrt{q^b r^c}$ and $d_X = p^h q^i$, so the first inequality gives $e(p-1) < p^{a-2\max(h,j)}$. For $h = 1$ the second inequality is the assertion; for $h \geq 2$ its right-hand side is at most $p^{a-h} \leq p^{a-2}$. □

Proposition 7.10 (A fermion core with an odd companion valuation)**.** *Let $2 < q < r$, $k \geq 2$ and $2^{k-1} < q$. A slightly degenerate divisor-minimal counterexample of dimension $2^{2k+1} q^b r^c$ has both b, c even.*

Proof. Call a companion of odd valuation q, interchanging q, r if needed. In the reduced table all complementary principal labels have height k by Lemma 4.10 and square divisibility. Their two-codegree is a unit and their two-block is a singleton. Their rows vanish on nontrivial squared dyadic twists by reduced block orthogonality. Every q-active column hits one by orthogonality on P_q modulo r. Every squared dyadic twist fiber contains a q-active label by Lemma 7.8. Hence all even dyadic twists in the extension are signs.

Choose full principal representatives of the complementary labels as above. By Lemma 4.20, the odd orbit over each witness is free of the same degree, and the witness sees only its mate's actual dyadic phase on odd columns.

The odd mate of an opposite r-witness has q-unit degree; every complementary q-witness must hit it, so all their odd hits share one phase. Evaluating at g makes every full q-principal label even, and $g \notin P_q$ by evaluating on that odd mate. Thus each odd q-active column hits a complementary q-witness by orthogonality modulo r. Every nonempty odd-sector dyadic fiber contains such a column by the same lemma. A common odd phase contradicts Lemma 7.7. □

Theorem 7.11 (Lifting the companion exponents)**.** *Let $p < q < r$ and $M = p^A q^B r^C d$, with d square-free and coprime to pqr. Put $k = \lfloor A/2 \rfloor$. If $q > p^k$, every braided category of dimension dividing M is WGT. If $q > p^{k-1}$, the same conclusion holds as soon as it holds on every divisor of*

$$p^A q^{2\lfloor B/2 \rfloor} r^{2\lfloor C/2 \rfloor} d.$$

Under this second gap one may take $B = 5$ and arbitrary C. One may also take arbitrary B, C for odd p and $A \geq 4$ when $(p-1)(r-1) \geq H_p(r) p^{A-2}$. Without a gap hypothesis the caps $A = B = 5$ always suffice.

More precisely, if $p^a q^b r^c$ is the square-full part of a divisor-minimal counterexample, then $q \leq p^{\lfloor a/2 \rfloor}$, and $q > p^{\lfloor a/2 \rfloor - 1}$ forces both b, c even.

Proof. Take a least counterexample among the divisors of M. Its three exponents a, b, c satisfy $a, b \geq 5$, $c \geq 4$ by Theorems 4.12 and 7.3. With $l = \lfloor a/2 \rfloor$ one has $q \leq p^l$ by Lemma 4.10, proving the first assertion. We claim that $q > p^{l-1}$ forces b, c even. In the ordinary table this is Theorem 7.9. In a reduced table $p = 2$ by Proposition 4.7; an even a contradicts Corollary 4.11, and for odd a apply Proposition 7.10.

Now $l \leq k$: a gap at $k-1$ makes both actual exponents even, so the rounded caps suffice. For $B = 5$ the rounded q-exponent is four, covered by Theorem 7.3. When p is odd, the table is ordinary by Proposition 4.7, and Theorem 7.9 forces $(p-1)(r-1) < H_p(r) p^{a-2}$ with $a \leq A$. Finally at $A = B = 5$ we have $k = 2$ and $q > p$ automatically, so every uncovered choice has $A + B \geq 11$. □

7.4 The largest quartic prime

At $v_q(\Delta) = 5$, reducing unit orthogonality on P_q at the omitted prime forces degrees qp^i and qr^j. More precisely, for any X of degree $q^j r^b$ with $j, b > 0$, some $Y \in P_q$ of degree qp^a satisfies, with $z = S_{XY} / \operatorname{lcm}(d_X, d_Y)$,

$$r \nmid [K_Y : \mathbb{Q}(z)] \operatorname{Tr}_{\mathbb{Q}(z)/\mathbb{Q}}(z). \tag{7.2}$$

Indeed use block orthogonality if X is q-active and the odd-valuation slice of Proposition 4.2 otherwise. The respective vacuum coefficients $1, -(q-1)$ are units at r. Some r-free orbit in P_q thus contributes nontrivially modulo r; the principal height bound gives degree qp^a. Its contribution before the slice coefficient is $qp^{2a}[K_Y : \mathbb{Q}(z)] \operatorname{Tr}(z)$ by (2.2). For $j = 1$, $r^b z = \chi_Y(X) \equiv 0$ at every prime over q, so the trace is a nonzero multiple of q and $\mathbb{Q}(z)$ ramifies there (otherwise z/q is integral of house less than one). If $r \nmid f_X$, then $\mathbb{Q}(z) \subseteq K_X$ is unramified at r and (7.2) excludes wild r-ramification in K_Y. Finally, when $v_r(\Delta) = 4$, a p-free nonunit in P_r has degree $q^2 r$: degree qr paired with $qp^a \in P_q$ violates Theorem 4.9 at $g = q$.

Proposition 7.12 (Complementary fields at a quintic middle prime)**.** *Let $k \geq 3$ and consider $D \mid p^{2k} q^5 r^c d$, $p < q < r$, with d coprime and square-free. Suppose $p^{k-2} < q$ and put $g = p^{k-1}$. In a divisor-minimal counterexample the source exponents at p, q are exactly $2k, 5$, respectively, and $q \leq g$. In its table some $X \in P_q$ has $d_X = gq$; every q-free nonunit $Y \in P_r$ has $g \mid d_Y$. If $q \leq g < 2q$ and $r > g$, the two fields contain $\mathbb{Q}(\zeta_r)$, except that when $r = g + 1 = q + 2$ the guaranteed common subfield is $\mathbb{Q}(\zeta_r)^+$.*

Proof. Theorem 7.11 forces $b = 5$ and excludes every $a \leq 2k - 1$. In the reduced table $p = 2$ and p-heights are at most $k - 1$. In the ordinary table a p-active fiber contains a q-active column by Lemma 7.8. Its r-free hit on P_q has p-height at most $k - 1$ by defect-zero vanishing. In either case Lemma 4.10 and $q > p^{k-2}$ force height $k - 1$ there and at least $k - 1$ on every opposite label. The principal q-height bound gives $d_X = gq$. Apply Theorem 4.9 to each opposite pairing with weight g; for general $r > g$ the proof gives tame orbit size at least $(r-1)/\lfloor g/q \rfloor$. □

At $2k = 6$ the exceptional equality $p^2 = q + 1$, $r = q + 2$ forces $(p, q, r) = (2, 3, 5)$. In the exceptional word, if 1 occurs once, its position forces each stabilizer to fix ζ_5. Otherwise

the coefficients are $1, 1, \omega, \bar{\omega}$ for a primitive cube root ω. Unless the two positions of 1 are opposite modulo five, every stabilizer fixes ζ_5; if opposite, its only nonidentity element is complex conjugation. Thus the common entry field contains $\mathbb{Q}(\zeta_5)$ or equals $\mathbb{Q}(\zeta_{15})^+$, and always contains $\mathbb{Q}(\sqrt{5})$.

The forced degree $p^{k-1}q$ also satisfies the tensor-square inequality of Proposition 4.14. For $p = 2$ and $k = 3$, Theorem 21.2 covers all odd pairs except $(3, 5)$, with both companion exponents unrestricted.

A related descent observation holds at any odd prime r. Every WGT braided category of dimension divisible by r^3 contains a nontrivial Tannakian subcategory. Indeed, a category without one would equal its core. By [42, Theorem 1.1] it would then be pointed or a product of a pointed category and an Ising category. The pointed r-primary part would have order at least r^3. Its quadratic form has a nonzero isotropic element: a cyclic subgroup of order at least r^2 has an isotropic subgroup of order r, and in the elementary abelian case a quadratic form over $\mathbb{F}_r$ in at least three variables has a nonzero zero. This gives the required Tannakian subcategory. In particular this applies to the unordered $(6, 5, 4)$ family by Theorem 8.1 whenever an odd cube survives in the dimension. For arbitrary companion exponents in $p^A q^B r^4 d$, a Tannakian $\mathrm{Rep}(G)$ with $r \mid |G|$ suffices for WGT by descent: the neutral de-equivariantization has r-valuation at most three, so Theorem 4.12 applies.

7.5 Dyadic blocks of a fermion centralizer

Let $\mathcal{M}$ be an integral modular category of dimension $D = 4\Delta$ with its positive spherical structure, a fermion g and even sector $\mathcal{B} = \langle g \rangle'$. Write $R_+ = K_0(\mathcal{B})/(g-1)$. An even pair $[Y] = \{Y, gY\}$ has opposite actual dyadic twists and hence a well-defined squared twist $\epsilon_Y = \theta_{Y,2}^2$. Odd mates have the same actual dyadic twist. The positive space A has orthonormal basis $(v_Y + v_{gY})/\sqrt{2}$ on even pairs. Let B be the antisymmetric even space and C the symmetric odd space. In the integral congruence lift, $U = \lambda T_2$, $\lambda^8 = 1$, exchanges A, B and preserves C. The dyadic Weyl W interchanges B, C and preserves A, because it interchanges fermion permutation and monodromy sign. Put $K = W|_A$ and write $W_o = W_{2'}$ for the complementary Weyl.

Lemma 7.13 (Positive defect one and the dyadic level)**.** *Suppose $v_2(D)$ is odd, and let E be a positive two-block containing a character of dyadic defect one. Its squared twists are $+1, -1$, with equal sums of squared degrees at the two signs. Its rows vanish on pairs of every other squared twist. If the actual dyadic twists of* all *even labels have order dividing four, the full dyadic level divides eight, and E consists of two labels of equal degree, both unramified at two. Under this twist bound, each full character whose field is contained in a dyadic cyclotomic field takes values in $\mathbb{Z}[i]$ on even objects.*

Proof. The residue on the even ring determines the full residue by squaring. The projector onto the lifted block commutes with the dyadic group by Lemma 3.2. Its restriction to A projects onto E and commutes with K. Thus $\widehat{s}^{-1}\mathbb{C}^E = W_o^{-1}\mathbb{C}^E$ has nonzero vacuum projection, and E contains squared phase $+1$. Apply the Bruhat mass identity (7.1) from Lemma 7.5 on A. the operators $U(2), L(2j), H_{1+4j}$ (j odd), the projector and its $\widehat{s}$-conjugate all preserve A. Here $L(t) = WU(-t)W^{-1}$. The torus negates the symmetrized vacuum ($v_2(D)$ is odd and both vacuum labels have rational characters). Its weights on E are d_Y^2/Δ. Averaging $\theta_{Y,2}^{-2j}$ over odd j gives $+1, -1, 0$ at squared twist $+1, -1$, or higher order, respectively. Thus the two signed masses are equal and nonzero. By Proposition 2.12, E has at most two inertia orbits. The two signs already occupy distinct orbits, so no higher order occurs. Since W_o preserves squared dyadic twist, the equality of subspaces above gives the support assertion.

Now assume the global bound on the even twists. Balancing puts even-even S-entries in $\mathbb{Q}(i, \zeta_m)$ for some odd m. A dyadic Galois unit $l \equiv 1 \pmod 4$, trivial on the odd factor, fixes the restricted character of every even label. Each even fermion pair is consequently fixed as a set by Lemma 4.5. The square action on actual twists fixes each representative's twist, and

opposite twists distinguish the representatives. Symmetry of S and rationality of degrees now fix the values of every character on even objects. In particular an odd fermion orbit is fixed as a set. Its members have equal twists, so the unit $l = 5$ gives $\theta_{Y,2}^{24} = 1$ also on odd labels. The normalization scalar cancels in the square action since the vacuum is fixed. All actual dyadic twists have order dividing eight, as does U since $\lambda^8 = 1$. For a purely dyadic character the fixed field of the units 1 mod 4 lies in $\mathbb{Q}(i)$, giving the asserted integrality. All positive characters have dyadic ramification index at most two. The two-branch defect-one block has an unramified branch by Proposition 2.12. The other has length one or two, and length two would equate an integer square to twice another integer square by the mass equality. Thus each sign has one label, of the same degree. □

Lemma 7.14 (Signed blocks at dyadic level dividing eight)**.** *Suppose $v_2(D)$ is odd and all even actual dyadic twists have order dividing four. Then U^4 is scalar on C. If a positive two-block contains a defect-one character of degree d, the full block over it contains two odd fermion orbits, both free. Their actual dyadic twists have ratio $\pm i$ and their degrees u, v satisfy*

$$u^2 + v^2 = 2d^2, \qquad v_2(u) = v_2(v) = v_2(d).$$

If the entire odd sector has just two dyadic twists with ratio $\pm i$, then R_+ has a nontrivial invertible basis element.

Proof. The level bound just proved uses only the bound on even actual twists. Put $Z = T_2^2|_A$, $F = KZK^{-1}$. These are involutions. The torus H_5 fixes the even labels as a permutation and negates $\sqrt{D}$, hence $H_5 = -I$ on A. Modulo eight the identity $U(2)L(2) = L(2/5)H_5U(2/5)$ gives $ZF = -FZ$. Since $H_3 = H_5H_{-1} = -K^2$ on A, it commutes with both. The factorization

$$(WU)^{-1}U^2(WU) = U(-1)L(-2)U(1) = L(-2/3)H_3U(2/3) = L(2)H_3U(6)$$

gives the exact restriction

$$(WU)^{-1}U^2(WU)|_A = \lambda^4 FH_3Z. \tag{7.3}$$

Its square is scalar since $H_3^2 = I$, and WU maps A onto C.

Over a positive defect-one block, Lemma 7.13 gives two labels of degree d, both unramified at two. Their full block contains four even labels, and H_3 acts scalarly on the two-dimensional positive space. Since WU preserves the full block, its odd symmetric space also has dimension two. The invertible operator FZ anticommutes with Z, so its two eigenvalues are opposites. By (7.3) the squares of the two odd twists differ by a sign.

The two corresponding signed characters have equal dyadic defect if inertia interchanges them. If inertia fixes both, their local order over the completed maximal unramified base $\mathcal{O}$ is a suborder of $\mathcal{O}^2$ containing the diagonal. The ideal of coordinate differences is $2^h\mathcal{O}$ for some $h \geq 1$. Thus the order consists of pairs congruent modulo 2^h, with conductor 2^h on each branch. By self-duality the conductor on each unramified branch is its codegree ideal (Lemma 2.11). Equal codegree valuations in (4.1) give the same fixed/free type by parity, then $v_2(u) = v_2(v)$. Column orthogonality of $1, g$ on the full block equates its odd mass to $4d^2$. If both labels were fixed, $u^2 + v^2 = 4d^2$ would have odd two-valuation on the left and even on the right. Thus both orbits are free and $u^2 + v^2 = 2d^2$, which also gives $v_2(u) = v_2(d)$.

If the only odd spins are δ_1, δ_2 with ratio $\pm i$, on C one has $U = aI + bU^2$, where, writing $x = \lambda\delta_1$, $y = \lambda\delta_2$, one has $a = xy/(x + y)$, $b = 1/(x + y)$ and $|a| = |b| = 1/\sqrt{2}$. Restrict $WUW = U^{-1}WU^{-1}$ to UA, and multiply by U^{-1} on the left. Since $WUA = C$ and W^2 is central, on A this gives

$$U^{-2}K = aK^2 + b\,U^{-1}L(-2)UK^2.$$

Here $U^{-2} = \lambda^{-2}Z$, and (7.3) gives $U^{-1}L(-2)U = cFH_3Z$ with $|c| = 1$. Since $H_3 = H_5H_{-1} = -K^2$ and $K^4 = I$, it follows that $K = a'ZK^2 + b'F$ with $|a'| = |b'| = 1/\sqrt{2}$. For the projection $P = (1 + Z)/2$ this yields $PKP = a'PK^2P$.

Put $w = W_o e_{[1]}$ in A. We have $Pw = w$, while $K^2 w$ is a unit scalar multiple of w by the torus action on the vacuum and commutation of local factors. Since $\widehat{s}e_{[1]} = Kw$, $|w_{[j]}|^2 = 2d_j^2/\Delta$ on $\epsilon_j = 1$ and vanishes otherwise. Lift back to the full vacuum $w^{\mathrm{full}} = W_o e_1$. This is supported on the even labels with actual dyadic twist one: W_o preserves monodromy sector and commutes with T_2. On every even pair the reduced coefficient is $w_j^{\mathrm{full}} + w_{gj}^{\mathrm{full}}$ because W_o commutes with the fermion permutation. Exactly one representative of each positive-sign pair can contribute, so its squared weight is again $2d_j^2/\Delta$.

Take m zero on the dyadic level and one on the odd complementary level. The indicator formula at the beginning of Subsection 7.2 gives, for even Y,

$$\nu_m(Y) = \sum_{j \in \mathrm{Irr}(\mathcal{M})} |w_j^{\mathrm{full}}|^2 \chi_j(Y) = \frac{1}{\Delta} \sum_{[j]} (1 + \epsilon_j) d_j^2 \chi_{[j]}([Y]).$$

This is the coefficient at $[Y]^*$ of the reduced element $1 + J$, where $J = \sum_{[j]} \epsilon_j e_{[j]}$. It is an algebraic integer by the cyclotomic integrality of indicators [45, Theorem 5.1] and rational by Galois covariance of the sign. Thus J has integral coefficients. It is self-dual with $J^2 = 1$, so has squared based-ring norm one, and its dimension is $\epsilon_1 = 1$. This gives a positive invertible basis element in R_+. Both signs occur by anticommutation, so it differs from the unit. □

8 Three repeated primes with a quartic or quintic exponent

Theorem 8.1 (Three bounded exponents)**.** *Let p, q, r be any distinct primes, and let d be square-free and coprime to pqr. Every weakly integral braided fusion category of dimension dividing $p^7 q^5 r^4 d$ is WGT.*

The exponents in this theorem may be assigned to the primes in any order. Throughout this section, we write $p < q < r$ unless stated otherwise. A quartic below a larger prime is already covered by Theorem 7.3. Our focus is a quartic at the largest prime, and d always denotes a coprime square-free factor.

For $m \geq 1$ set $e = m(m+1)/2$. Define

$$g(h) = \min\{k \geq 1 : k(k+1) \geq h\}, \qquad C(e) = 1 + \max_{1 \leq h \leq e} \frac{h(h-1)}{g(h)}.$$

We say the odd prime s satisfies the *support bound* at m if

$$1 + 2e < 2s/3, \qquad (s-1)/2 > C(e). \tag{8.1}$$

Theorem 8.2. *Let $p < q < r$ be odd primes and set $m = \lfloor r/q \rfloor$. If $m + 1 < p$ and r satisfies (8.1), every braided fusion category of dimension dividing $p^a q^b r^4 d$ is WGT for all $a, b \geq 0$.*

In particular, this holds for all odd primes $p < q < r < 2q$, and for $5 \leq p < q < r < 3q$. Indeed $C(1) = 1$ and $C(3) = 4$; in the additional interval $2q < r < 3q$, $q \geq 7$ gives $r \geq 17$. For $p \geq 5$ and $q \geq 11$ the range extends to $r < 4q$, since $C(6) = 16$ and $r > 3q$ suffices for the two tests at $m = 3$. More generally, for each p, the whole interval $q < r < (p-1)q$ is covered once q is sufficiently large in terms of p. A quintic at the least prime requires no size condition on the other primes.

Theorem 8.3 (A quintic and two unrestricted exponents)**.** *For any primes $p < q < r$ and any $b, c \geq 0$, every weakly integral braided fusion category of dimension dividing $p^5 q^b r^c d$ is WGT.*

Section 8.5 proves this theorem. For a middle quintic and odd primes, Theorem 8.20 handles $p^A q^5 r^4 d$ whenever $A \geq 6$ and $q > p^{\lfloor A/2 \rfloor - 2}$. This holds automatically for $A = 6, 7$. The even case combines the ordinary and fermion reductions in Sections 8.6–8.8. The unrestricted quartic problem remains open.

In a divisor-minimal obstruction all three repeated primes survive and the square-free factor disappears. A third exponent at most three is already covered (Lemmas 2.1, 2.5, Proposition 4.7 and Theorem 4.12). Thus $v_r(\Delta) = 4$ in the quartic problem, including the positive table, whose dimension is half the categorical dimension. A q-free or p-free nonunit in P_r then has degree rp^i or rq^j with $i, j > 0$, by the principal height bound and prime avoidance (Lemma 4.10). We use *house* for the largest absolute value among all conjugates.

8.1 Character values and tensor products

Lemma 8.4 (An odd word of prime weight)**.** *Let $t < r$ be odd primes. Suppose the cyclotomic integer z is a nonnegative rational combination of odd-order roots of total weight r, has strict house below r, and has respective residues $r, 0$ over all primes at t, r. Then for a t-power root η,*

$$z = \eta \left(h + t \sum_{i=1}^{m} \xi_i \right), \qquad h + tm = r,$$

with $h \geq 1$, $m \geq 1$ and nonidentity r-power roots ξ_i. In fact every root in the original positive expression has this same t-part, and the multiplicities of its distinct r-parts are uniquely determined after averaging coincident roots.

Proof. On signed words in t, r-primary roots impose cost one on $+$ and $3/2$ on $-$. Every integral word with the prescribed residues has cost at least r. Indeed, a word of length at most $r - 1$ gives, on erasing r-parts, a signed t-primary sum w divisible by r in the unramified integer ring. Its strict house bound $< r$ gives $w = 0$. Erase instead t-parts; the signed coefficients $c_i - r\delta_{i0}$ are constant modulo t on each μ_r-coset. If every coset has a vacant position after marking the identity, all constants vanish, and the signed augmentation is both r and 0 modulo t. Therefore all $r - 1$ nonidentity positions in the identity coset occur. They occur just once, with a common sign e since t is odd. Then $t \mid r - 1$ by $w = 0$, and $e \equiv -r \equiv -1 \pmod{t}$. This word costs $3(r - 1)/2 > r$.

For each complementary prime-power level u^a contract the cyclotomic lattice by a random integral functional. On μ_u give 1 the value one, a uniformly selected nonidentity root the value -1, and all the others zero. On every other μ_u-coset give all positions zero. The coset relations make this well defined. Tensor all the functionals, leaving the t, r-level alone. The two prescribed residues are preserved since the radicals are generated by a primary cyclotomic uniformizer in the retained factor. When an original positive summand has a nontrivial complementary part its expected signed cost is at most $3/(2(u - 1)) < 1$ for any nontrivial component u. The trivial complementary part always costs one. If any complementary part occurred, some contraction would thus cost less than r. Its value is integral, and unimodularity of the signed two-prime configuration [4, Theorems 3 and 13] gives an integral point attaining the minimum of the same asymmetric linear cost. This contradicts the preceding paragraph.

It remains to minimize on the unsigned two-prime roots alone. Again the configuration is unimodular and an integral minimum exists. Each optimum has length r. Erasing r-parts now gives a sum w of r unsigned t-power roots, divisible by r. It is nonzero since a vanishing unsigned sum has length divisible by t. The integral norm makes w/r a root of unity. Equality in the triangle inequality makes all the t-parts the same root η. The remaining unsigned r-word has augmentation r and coefficients $c_i - r\delta_{i0}$ constant mod t on each μ_r-coset. A full nonidentity coset makes the word zero. If all $r - 1$ nonidentity positions of μ_r are occupied, the extra term cannot be outside the coset or duplicate one position, by these congruences. The complete coset again gives the prohibited zero at t. All constants therefore vanish. Strict house excludes the word supported only at one. This proves the formula. Every optimal vertex has the same η, since the ratio of two such roots lies in $\mathbb{Q}(\zeta_{t^a}) \cap \mathbb{Q}(\zeta_{r^b}) = \mathbb{Q}$. The remaining coefficients are unique: every nonzero coset has a vacant position, and a reduced unsigned r-word has a unique coefficient vector. The original expression is a convex combination of those optimal vertices. □

Lemma 8.5 (A single residue in odd order)**.** *Let z be a cyclotomic integer expressed as a positive rational combination of roots of total weight $g \in \mathbb{Z}_{>0}$. Suppose every prime of their orders is at least $\ell \geq 3$, and $z \equiv g$ at all primes above $s > g$ in the common field. If $(s-g)(\ell-1) > g$, all terms of this expression have s-power order, and z has an integral unsigned s-word of length g.*

Proof. Choose a cost c with $\max(1, g/(s-g)) < c < \ell - 1$ for negative roots; positive roots cost one. At each complementary prime contract by the random integral functional of Lemma 8.4. Every nontrivial complementary part has expected cost at most $c/(\ell-1) < 1$; a trivial part costs one. The contracted integral target has residue g at s. An integral signed s-word of cost at most g has augmentation g mod s. A negative augmentation hence costs at least $c(s-g) > g$, while augmentation g forces exactly g positive terms. Integral optima exist for the signed one-prime configuration by unimodularity. Thus any loss of cost contradicts its minimum, proving the assertion on the original terms and the integral word. □

8.2 A prime-weight crossing with one zero residue

Write μ_{s^∞} for the roots of s-power order. Let a word of weight r mean a positive rational sum of roots of unity whose coefficients sum to r. All residue conditions below are at every prime over the indicated rational prime.

Proposition 8.6. *Let $q < r$ be odd primes with $q \nmid r-1$. Suppose that a cyclotomic integer z has weight r, has strict house less than r, and satisfies $z \equiv r$ above q and $z \equiv 0$ above r. Then for some $\eta \in \mu_{q^\infty}$ one has one of the following expressions.*

1. *$z = \eta(h + q\sum_{j=1}^{m} \xi_j)$, where $h + qm = r$, $h, m \geq 1$, and $\xi_j \in \mu_{r^\infty} \setminus \{1\}$.*
2. *$r \equiv 2 \pmod q$ and $z = \eta(1 - \sum_{t=1}^{r-1} \eta_t \zeta_r^t)$. Here the multiset of the η_t consists of one copy of 1 and $(r-2)/q$ complete cosets of μ_q in μ_{q^∞}.*

Every weight-r expression has exactly the phases and the masses in the indicated word, after combining coincident phases.

Proof. First work with a signed integral word of length at most r in products of q- and r-power roots. Erase the r-parts to obtain w. It is divisible by r in the q-cyclotomic integers, since this field is unramified at r. If $w \neq 0$, Kronecker's theorem makes w/r a root of unity, since its house is at most one. Equality in the triangle inequality then gives exactly r terms with one common signed q-part. If $w = 0$, their signed augmentation is zero modulo q.

Let c_t be the signed coefficients obtained by erasing the q-parts, indexed at a common r-power level in additive notation. On each μ_r-coset, the numbers $c_t - r\delta_{t0}$ have a common residue modulo q, by the cyclotomic coset relations. A coset with a vacant position after marking zero has common residue zero. Any entirely occupied coset other than the identity coset would need all r letters; the empty nonzero positions of the identity coset would then contradict the residue at zero. If all common residues vanish, the augmentation is r modulo q, so $w \neq 0$, the uniform sign is positive, and each nonidentity multiplicity is divisible by q. This gives the first possibility; strictness excludes $m = 0$.

It remains to have occupied all $r-1$ nonidentity positions of the identity coset. In the case of a common signed q-part no such word can occur. With the remaining letter at zero the erased r-word is a multiple of the complete cycle and has residue zero at q. At any already occupied position its coefficient would differ by one in absolute value from the others, against the coset relation. Finally a letter on a new coset violates the zero coefficient congruence there. Thus $w = 0$. Put $s = c_0$. The signed augmentation modulo q is $r + r(s-r)$, since the sole possible nonzero coset residue is $s - r$. Consequently $s \equiv r - 1 \pmod q$. There is room for at most one letter at zero. Since $q \nmid r-1$ and $r \not\equiv 0 \pmod q$, we obtain $s = 1$ and $r \equiv 2 \pmod q$. The other signs are all negative. After division by the q-part η of the positive letter, $w = 0$ reads $\sum_t \eta_t = 1$.

Subtracting the copy at 1, the integral coefficients must be constant on each μ_q-coset. These constants are nonnegative, since $q > 1$. This is the second possibility. In particular no signed integral word of length below r has the required residues.

We justify the integral-word reduction. Apply the contraction used in Theorem 4.9 onto the q, r-primary field. A rational integral-basis functional on the complementary field fixes 1 and takes values $0, \pm 1$ on roots. It preserves both residue conditions. The signed primary configuration is unimodular, so an algebraic integer with a rational expression of mass at most a there has a signed integral expression of length at most a. The preceding lower bound applies also without strict house (only $m > 0$ used strictness). For each specified complementary root different from a sign the functional can be chosen to kill it, as in the same proof. Any such letter in the original positive expression would then leave mass strictly below r. Hence all complementary parts are signs, and unimodularity supplies the asserted integral word at z itself.

In the first possibility there are at most $1 + \lfloor (r-1)/q \rfloor < r/2$ distinct primary positions. They are exposed by Lemma 4.15 and independent by the coset relations. For the second case divide by η. On the coset containing μ_r, give the functional at $t = 0$ the normalized trace T on the complementary field and at $t \neq 0$ the functional $y \mapsto T(y/(-\eta_t))$. Their sum is zero because $\sum \eta_t = 1$. Put zero on other cosets. Its maximum one is attained precisely at the r indicated roots in any common conductor. They are linearly independent over $\mathbb{Q}$: after using $1 = -\sum_{t \neq 0} \zeta_r^t$, a rational dependence would require the same rational number to be a rational multiple of every η_t, including a nonreal one. Exposure and independence fix the masses in every positive expression. □

8.3 Proof for arbitrary exponents

There are two possibilities for a putative minimal table, even without a size condition on the primes. The second uses the *canonical positive word* in a single prime-power field: in each prime-order coset subtract the least coefficient. We use additive notation for its positions, with zero the identity.

Proposition 8.7 (The unrestricted odd case)**.** *Take $p < q < r$ odd and $v_r(\Delta) = 4$, with arbitrary valuations at p, q. A minimal table has at least one of the following normalized entries $S_{XY}/\operatorname{lcm}(d_X, d_Y)$, of common weight r.*

1. *A p-free nonunit of P_q has r-height one. An entry at this label has the form $\eta(h + q\sum_{j=1}^{n} \xi_j)$ with $h + qn = r$, $n > 0$, η of q-power order and each ξ_j of r-power order.*

2. *All p-free nonunits of P_q have r-height two. The q- and r-active labels are disjoint, and a normalized entry belongs to a p-power cyclotomic field. Its canonical positive coefficient vector (c_i) has length $L \leq r$ with $L \equiv r \pmod{p}$. On every μ_p-coset the entries $c_i - r\delta_{i0}$ are constant modulo q.*

Proof of Proposition 8.7 and Theorem 8.2. Suppose a minimal counterexample exists. Its table is ordinary of odd dimension. Choose $X \in P_r$ of degree rp^i. Every p-free nonunit $Y \in P_q$ has degree $q^j r^v$ with $v = 1$ or 2. Suppose $v = 1$. The normalized entry $z = S_{XY}/\operatorname{lcm}(d_X, d_Y)$ has positive odd-order weight r and strict house less than r. Its residues are r at both r and q: use $\chi_X(Y) = q^j z$ at r, and $\chi_Y(X) = p^i z$ at q. Lemma 8.4, with $t = q$, gives at most $m + 1$ positions of r-power order up to translation. This proves the first alternative when such a label exists. Under the hypotheses of Theorem 8.2, the support bound holds at r. Lemma 4.16, applied after rescaling by $\operatorname{lcm}(d_X, d_Y)/d_Y$, gives a contradiction.

We may therefore assume every such Y has $v = 2$. Its formal codegree $f_Y = \Delta/d_Y^2$ is an r-unit, so its row vanishes on r-active columns (Theorem 3.1). If a column were both q- and r-active, orthogonality to the vacuum on P_q, reduced modulo p, would give a p-free nonunit hit (Proposition 4.2). This proves that q- and r-activity are disjoint. By Proposition 6.2 no character field then ramifies at both q and r.

If r were also separated from p, it would be an isolated quartic prime, excluded in three-prime ordinary data by Lemma 7.2. Fix a joint p, r-active column J. Orthogonality on P_p on the trivial prime-to-p twist fiber, reduced modulo q, gives a q-free nonunit $V \in P_p$ hitting J. It has degree $p^e r$: both primes divide the degree by the exclusion of prime powers, and r^2 is impossible at a hit of J by defect-zero vanishing.

Take $A \in P_r$ of degree rq^i and a p-free nonunit $B \in P_q$, hence $d_B = r^2 q^j$. At V, A the normalized entry has weight r and is nonzero by $V \in P_p$ since $p \nmid d_A$. Its residue at each prime above r is zero by $A \in P_r$, since $\chi_A(V) = p^e S_{VA}/\operatorname{lcm}(d_V, d_A)$. If this value were unramified at r, division of the normalized entry by r would give a nonzero algebraic integer of house less than one. Thus $K_V = \mathbb{Q}(\chi_V)$ ramifies at r and is unramified at q. The field K_B is unramified at r by its codegree (Theorem 2.7).

Now form $z' = S_{VB}/\operatorname{lcm}(d_V, d_B)$. It lies in $K_V \cap K_B$, so is unramified at both q and r. It belongs to a field of p-power conductor: the entry fields are cyclotomic with no other conductor primes, and the inertia groups of the other two factors fix z'. It has weight r, strict house less than r and residue r at both p and q by the principalities of V, B.

Average its balancing expression over the q, r-cyclotomic factor. The negative mean of a complementary root has magnitude at most $1/(q-1)$. Replace a negative coefficient on a p-power root by positive coefficients on the other $p-1$ positions of its μ_p-coset. This gives a positive rational p-word of weight at most r. Subtract the minimum coefficient on each coset. The remaining coefficients c_i are nonnegative integers, since z' also has an integral expression and two such vectors differ by a constant on each coset. Write $L = \sum c_i \le r$. Reduction at p gives $L \equiv r \pmod p$. Reduction at q makes $c_i - r\delta_{i0}$ constant modulo q on each μ_p-coset.

This proves the second alternative, without any size hypothesis. To finish the WGT theorem assume (8.1). Suppose zero is the only vacant position in the identity coset and put $\rho = r - mq$. Each of its $p-1$ other coefficients is $q - \rho$ modulo q, and elsewhere all coefficients are multiples of q. Thus $L = (p-1)(q-\rho) + qn$ with $0 \le n \le m$. Reduction at p gives $n \equiv m+1 \pmod p$, impossible since $m+1 < p$ by hypothesis. There is therefore a vacant nonidentity position on the identity coset as well. All $c_i - r\delta_{i0}$ now vanish modulo q. Consequently $L \equiv r \pmod{pq}$; since $r < (m+1)q < pq$ we have $L = r$.

The equality $L = r$ also identifies the prime support of the *actual* double-braiding phases of V, B. Any physical term with nontrivial prime-to-p part would have lost weight in passing to the positive p-word: a positive normalized trace on a nontrivial complementary root is strictly less than one, and a negative trace costs at most $(p-1)/(q-1) < 1$ after its conversion to positive p-roots. This would give $L < r$. Thus all phases have p-power order. This deduction holds for every p-free nonunit $B \in P_q$.

By the choice of the trivial complementary fiber, V has p-primary twist. Both V and B are r-inactive, the latter by its r-unit codegree. Since all the phases of their double braiding are p-primary, every constituent of VB is r-inactive. Write $v = [VB]$ in the fusion basis, and $s = S/\sqrt{\Delta}$. The r-local upper unipotent fixes v. The coordinates of its Fourier transform are

$$(sv)_T = d_T \chi_T(V) \chi_T(B)/\sqrt{\Delta}.$$

They vanish at all r-active T because B has r-unit codegree. Thus the conjugate lower unipotent also fixes v. The scalar normalizing the twist has trivial r-part, as in Section 3. These two unipotents generate the local modular group, so its torus fixes v. Since $v_r(\Delta) = 4$, r-inertia fixes $\sqrt{\Delta}$; the corresponding torus operators are the unsigned Galois permutations. As B is unramified at r, this proves

$$BV^\sigma = BV \qquad (\sigma \in I_r). \tag{8.2}$$

Take any p-free nonunit $A \in P_r$, of degree rq^i. The earlier ramification argument applies to *each* such A: $\chi_A(V)$ is moved by some $\sigma \in I_r$. Evaluating (8.2) at A therefore forces $S_{AB} = 0$ for every p-free nonunit $B \in P_q$. Now A is q-inactive: q, r are separated, and its r-codegree is

divisible by r (Lemma 3.3). Choose any q-active column J. Block orthogonality on P_q yields

$$0 = \sum_{W \in P_q} \chi_A(W)\overline{\chi_J(W)}.$$

At p-free nonunits the first factor vanishes by the preceding argument. At p-bearing W it is divisible by p by lcm integrality, since $p \nmid d_A$. The unit contributes one. Reduction modulo a prime above p gives the contradiction. □

The first ratio outside the length argument. Suppose p, q, r are odd, $m = p - 1$, and r satisfies (8.1). Then every q-active simple of a minimal table has degree divisible by pr^2. To see the additional factor p, retain V, A above and put $c = r - pq$. At a p-free nonunit $B \in P_q$, either $L = r$, in which case $S_{AB} = 0$ by (8.2), or the canonical word has a vacancy only at zero in the identity coset. Indeed in the other case $L \equiv r \pmod{pq}$ and $r < pq$. In the exceptional case $n \equiv m + 1 = 0 \pmod{p}$ and $0 \le n \le p - 1$, so $n = 0$. The sum of the nonidentity pth roots is -1, and therefore $z' = -(q - \rho) = c$. At least one such rational entry must occur, since a full zero rectangle was just excluded.

For each p-free $A \in P_r$ the normalized entry at A, V has weight r and residues r at p, r. Lemma 8.4 shows that its actual phases have trivial q-part. Both labels are q-inactive, hence every constituent of AV is q-inactive. For a q-active J, apply block orthogonality to the fusion vector $r[AV] - cd_V[A]$ against J on P_q. By symmetry and multiplicativity, after dividing by the common degree factors this gives

$$0 = \sum_{W \in P_q} \frac{\chi_A(W)}{d_W}(r\chi_V(W) - cd_W)\overline{\chi_J(W)}.$$

The p-free nonunit terms vanish by the two alternatives at B. If $p \nmid d_J$, then at p-bearing W the first factor is p-integral since it equals $\chi_W(A)/d_A$, the bracket has strictly positive p-valuation by $V \in P_p$, and the last factor is in $p\mathcal{O}$. This is incompatible with the vacuum term pq of valuation one. The r^2 factor follows from separation.

8.3.1 Common constituents and intersections of difference sets

Proposition 8.8 (A finite test at one row)**.** *Fix a nonunit row in either type of minimal table. Suppose that some nonunit has, up to a root of unity, a full-weight positive word on at most k positions of odd prime-power order s^a. Put $E = \binom{k}{2}$, $H = (s-1)/2$, and use $\ell = 2$ in either table or $\ell = 3$ in odd ordinary dimension. Assume $\ell(1 + 2E) < (\ell - 1)s$. Let L be the least $1 \le h \le E$ for which $(H - 1)g(h) \le h(h - 1)$, where $g(h) = \min\{j : j(j + 1) \ge h\}$. If L does not exist, there is no such word. Otherwise, with $b = \binom{k-1}{2}$, the inequality*

$$\max\{E^2,\, sb^2 + (E - b)^2\} < sHg(L)$$

forces the initial support into a translate of $\mathbb{F}_s$. In this case it suffices to examine k-subsets of $\mathbb{F}_s$ containing $0, 1$. For each such set A, start with the difference set $A - A$. If D, F are any two symmetric difference sets already obtained, each with $\ell|D|, \ell|F| < (\ell - 1)s$, then for every unit u one may adjoin $T - T$, where $T = D \cap uF$. Discard the initial set if any resulting difference set has fewer than L nonzero pairs modulo sign.

Proof. The minimization in Lemma 4.16 runs over all full words on this row, with no bound on their lengths. Its minimum number of opposite nonzero differences is therefore at least L. If the initial positions meet several cosets of μ_s, then at most b opposite differences have order s. Write x for their number. At order s^j the units act transitively on $s^{j-1}H$ opposite pairs. Thus the mean number of opposite nonzero differences common to the initial spectrum and a unit multiple is at most $x^2/H + (E - x)^2/(sH)$. Convexity bounds this by the larger of its values at $0, b$. On

the other hand every such intersection contains at least $g(L)$ pairs, by the common-constituent argument of Lemma 4.16. This proves the conductor assertion. After translation and scaling one may pad even a smaller true support to a k-set containing $0, 1$. Strictness rules out a one-point support.

Inductively each retained D contains the adjoint phases of some nonunit at this row. Apply Lemma 4.15 to that nonunit and a Galois translate of the one for F. When both sets are exposed the common constituent has phases in T, so its adjoint has phases in $T - T$. This proves the induction even for padded supports, and the minimum L justifies rejection. □

Corollary 8.9. *For odd primes $7 \le p < q < r$ and arbitrary $a, b \ge 0$, all divisors of $p^a q^b r^4 d$ are WGT dimensions if $r < 5q$, or if $q \ge 17$ and $r < 6q$. More generally, for fixed $M \ge 2$ the entire strip $M < p < q < r < Mq$ is covered when q is sufficiently large in terms of M.*

Proof. For the smaller endpoints, the support enumeration gives the following table. The commands for all listed primes are recorded under Code, data and certificates. In each row k is the number of positions and the second column lists the primes above the indicated cutoff where the scalar inequality alone still permits a word. The other primes above the cutoff are excluded by Lemma 4.16.

k	s	cutoff	surviving supports at listed primes
2	none	3	0
3	none	7	0
4	29, 31	23	0
5	47, 53, 59, 61	43	0
6	83, 89, 97, 101, 103, 107	79	0

For a direct reproduction at a listed prime, enumerate the $\binom{s-2}{k-2}$ sets starting with $0, 1$. For A take its opposite differences. For each pair of retained sets and each multiplier modulo sign form T in the proposition. Reject at once if T has fewer than $g(L)$ opposite nonzero pairs or if $T - T$ has fewer than L. Otherwise add the dilation class of $T - T$ whenever exposed and new. Continue until rejection or exhaustion of the finite list. Use $\ell = 3$ for $k \le 5$ and $\ell = 2$ for $k = 6$. If the program reaches its work limit before rejecting a seed, it counts that seed as surviving.

In Proposition 8.7 the first alternative has $k = 1 + \lfloor r/q \rfloor$ positions, while $k < p$ excludes the second as in Theorem 8.2. For $k = 4, 5$ the inequalities $r > (k-1)q$ with $q \ge 11$ give the corresponding prime cutoffs; for $k = 6$ use $q \ge 17$. For general M, choose T_k large enough that Lemma 4.16 excludes k positions at primes $s > T_k$; such a choice exists since k is fixed. It suffices that $q \ge \max_{2 \le k \le M} \lceil T_k/(k-1) \rceil$. □

8.4 Small exponents and even dimensions

Theorem 8.10 (A bounded exponent and a large companion prime)**.** *Let t, u, r be distinct primes with r odd, and $c \ge 4$. Put $w = t^{\lfloor (c-2)/2 \rfloor}$ and $e = w(w-1)/2$. If*

$$1 + 2e < r/2, \qquad (r-1)/2 > C(e),$$

then every braided fusion category of dimension dividing $t^c u^A r^B d$ is WGT for arbitrary A, B and square-free d coprime to tur. For three odd primes one may replace $r/2$ by $2r/3$.

Proof. We work in a minimal table with all three primes repeated. Except when $t = 2$ and the table is reduced, take an r-free nonunit of P_t and a u-free nonunit of P_r. Their degrees have shapes $t^i u^j$ and $t^h r^k$ and their gcd g is at most w by the principal valuation bound at t. Their normalized entry has residue g at r and weight g. The stated inequalities imply $r > 2w$ (for odd data $w \ge 3$). Thus the single-residue theorem gives an unsigned r-word on at most w positions, contrary to Lemma 4.16.

In the reduced dyadic case exchange u, r if necessary to make r the larger, preserving the inequalities since w is fixed. Use complementary witnesses in P_u and P_r; their common 2-part satisfies $g \le 2^{\lfloor (c-1)/2 \rfloor} \le 2w$. It gives both positive residues g at u, r. Here the general phase inequality implies $r > 2w+1$ since $w \ge 2$. Hence Theorem 4.9 gives the unsigned r-word with at most $1 + \lfloor g/u \rfloor \le 1 + \lfloor 2w/3 \rfloor \le w$ positions. The same lemma finishes. □

For $t = 3$ and $c = 5$ this gives all $r \ge 17$ in either parity, and all $r \ge 11$ for three odd primes: $w = 3$, $e = 3$ and $C(3) = 4$.

The first separation in the proof above also gives a parity reduction when the second separation is unavailable. We record a version that includes a dyadic least prime.

Proposition 8.11. *Let $p < q < r$ be primes, $q \ge 5$, $m = \lfloor r/q \rfloor$ and $e = m(m+1)/2$. A minimal counterexample at quartic r has even q-valuation under either of the following hypotheses:*

$$\begin{aligned} p > 2: &\quad 1 + 2e < 2r/3, \quad (r-1)/2 > C(e), \\ p = 2: &\quad q \nmid (r-1)(r-2), \quad 1 + 2e < r/2, \quad (r-1)/2 > C(e). \end{aligned}$$

In particular both tests hold when $q + 2 < r < 2q$. In the applicable range the WGT assertion for all divisors of $p^a q^b r^4 d$ follows from the one with b rounded down to an even integer. Every divisor of $p^a q^5 r^4 d$ is therefore covered.

Proof. If a column were both q- and r-active, it would be hit by a p-free nonunit $Y \in P_q$ by principal orthogonality modulo p. Its degree is $q^j r$: a second factor of r would give defect zero and forbid the hit. Choose a q-free nonunit $X \in P_r$, of degree $p^i r$ by the principal height bound at a quartic. The normalized crossing has weight r and residues $r, 0$ at q, r respectively. In odd data Lemma 8.4 gives a translated positive r-word on at most $m+1$ positions. For $p = 2$ Proposition 8.6 gives the same conclusion because the signed alternative requires $q \mid r-2$. Scaling to the character gives a full word, excluded in both cases by Lemma 4.16. Thus q and r are separated. At odd q-valuation Lemma 7.8 would produce q-activity on a nonempty r-active fiber. Descent gives the rounding assertion, and Theorem 7.3 covers middle cap four. In the stated interval $m = 1$, $C(1) = 1$, and $r \ge 11$. □

There is also a finite-largest-prime test valid in either parity with no bound on the middle exponent. For a cap a put $G = p^{\lfloor a/2 \rfloor}$. If $q > G$, all divisors of $p^a q^b r^4 d$ are covered (Theorem 7.11). Otherwise, with $m_0 = \lfloor G/q \rfloor$, $e_0 = m_0(m_0+1)/2$, they are covered whenever

$$r > G + 1, \qquad 1 + 2e_0 < r/2, \qquad (r-1)/2 > C(e_0).$$

Indeed take complementary witnesses of degrees $p^i q^j$ in P_q and $p^t r$ in P_r. Their common weight $g = p^{\min(i,t)}$ is at most G. The two-residue theorem (Theorem 4.9) writes the normalized entry as a positive r-word on at most $1 + \lfloor g/q \rfloor$ positions when $r > g + 1$. Lemma 4.16 gives the contradiction. In odd dimension the same test may interchange p and q, using the ordinary principal block at p.

Corollary 8.12 (A middle-prime test independent of the largest prime)**.** *Let $p < q$ be odd primes, let $G = p^{\lfloor A/2 \rfloor}$, and put $M = \lfloor G/q \rfloor$. If $M + 1 < p$ and $q \ge (M+1)^2$, then every braided category of dimension dividing $p^A q^B r^4 d$ is WGT for* every *prime $r > q$, with B arbitrary.*

Proof. If $M = 0$, use Theorem 7.11. For $M > 0$, the preceding large-prime test and Theorem 8.2 both use r-words on $1 + m$ positions for $m = \lfloor \min(r, G)/q \rfloor$: if $r < G$, the latter theorem applies; if $r > G$, the former applies since the odd numbers r, G differ by at least two. Thus $1 \le m \le M$ and $r > mq$. For $m = 1$ the bound at r is immediate as $r \ge 7$. For $m \ge 2$ set $e = m(m+1)/2$. At $h \le e$ one has $(h-1)/g(h) \le m$: if $g(h) < m$ this follows from $h \le g(h)(g(h)+1)$, and otherwise use $h \le e$. Hence $2C(e) + 1 \le 3 + 2em \le m(m+1)^2$. The exposure condition $2(1 + 2e) \le m(m+1)^2$ also holds. Now $r > mq \ge m(m+1)^2$ verifies both inequalities. The equality $r = G$ is impossible for the distinct primes. □

Proposition 8.13 (Transport of deep quartic twists)**.** *Let an ordinary or positive reduced minimal three-prime table at $p < q < r$ have $v_r(\Delta) = 4$. If the r-part of some twist has order greater than r^2, then for every complementary witness $X \in P_r$ of degree $p^i r$ and every occupied nonidentity coset of μ_{r^2} in the r-primary twists, that coset contains a label Z with $2\gcd(p^i, d_Z) \geq r$. The same holds upon exchanging p, q. In particular the r-primary level divides r^2 if some witness has $2p^i < r$ or $2q^j < r$.*

Proof. Fix a nonidentity coset. Each of its labels Z is r-free by Theorem 3.5 (the odd twist part divides the ordinary or reduced codegree). Put $g = \gcd(d_X, d_Z)$ and suppose $2g < r$. Principality gives $S_{XZ}/\operatorname{lcm}(d_X, d_Z) \equiv g$ above r, since d_Z/g is a unit there. The one-residue short-word theorem (Theorem 4.9) provides an unsigned r-primary word of length g, and the functional in Lemma 4.15 identifies its roots with the actual monodromy phases. Wild inertia over the r^2-cyclotomic level fixes the sum: the row field of X has conductor exponent at most two because its r-defect is two (Corollary 2.8). Uniqueness of the unsigned word and $g < r$ force this inertia to fix each root, since any nontrivial orbit has length at least r. Thus all phases lie in μ_{r^2}, as does the r-twist of X by twist–codegree divisibility. Every constituent of XZ remains in the original coset by balancing. The odd primary twist is independent of the representative of a fermion pair, so this also holds on the positive quotient. If $2g < r$ throughout the coset, its coordinate span is invariant under multiplication by X and by X^* (another witness of the same degree). It is then invariant under every constituent of their tensor words, by positivity. These generate the entire based ring by simplicity, whereas Z^*Z contains the unit, which lies outside the coset. This is impossible. □

8.5 A quintic valuation

The proof in odd dimension uses two consequences of the local modular action. Filtering a row by the trivial twist at the quintic prime selects a trace-contributing orbit. The same local action limits the field in which that trace is taken. We write W_v, U_v and $H_{v,a}$ for the local Weyl, upper unipotent and diagonal torus operators of Section 3, and $S_0 = S/\sqrt{\Delta}$. In this section $D = \Delta$ denotes the ordinary table dimension whenever we work with an ordinary table. At an odd prime v the eigenvalues of U_v are the actual v-primary twists, since the anomaly normalization has two-power order.

Lemma 8.14 (Gauss sums at a quadratic class)**.** *Let $D = p^{2k+1}n^2$, $k \geq 1$, $p \nmid n$, be the dimension of an ordinary integral modular category, with p odd. Fix one of the two square classes of primitive pth roots. Suppose every label whose p-part of the twist is neither 1 nor in that class has degree $p^i n$ for some $i \geq 0$. Let L_i count the degree-$p^i n$ labels at orders p^a with $a \geq 2$, and let M_i count such labels at one fixed primitive root of the other class. Then*

$$p^{k+1} \;\Big|\; \sum_i p^{2i}(L_i + pM_i).$$

Each L_i is divisible by p. In odd dimension L_i and M_i are even.

Proof. Square Galois covariance preserves degrees. A square-unit orbit of primitive roots of order p^a, $a \geq 2$, is a union of full μ_p-cosets, so its sum is zero and its size is divisible by p. At order p, multiplicity is constant within each square class. In odd dimension every nonunit has a distinct dual of the same twist (Lemma 2.5).

Write $L = \sum_i p^{2i} L_i$, $M = \sum_i p^{2i} M_i$ and put $A = (p^{2k+1} - L)/p$. By the odd-valuation slice (Proposition 3.8), the squared-dimension mass at twist one is $n^2 A$, and the mass at order p is $(p-1)n^2 A$. In units of n^2 the mass per primitive phase is therefore M on the other square class and $2A - M$ on the chosen class. The sums of the roots in these classes add to -1 and differ by a quadratic Gauss sum of squared modulus p. Thus for $T = \sum_X d_X^2 \theta_{X,p}$ we have $|T|^2 = pn^4(A - M)^2$. The regular-object indicator at an exponent selecting the p-primary twist

is $|T|^2/D$, a rational algebraic integer by [44, Theorem 7.5] summed over the simples with their dimension multiplicities. It follows that $p^k \mid A - M$, as asserted. □

Corollary 8.15 (Exceptional labels and principal inertia)**.** *In Lemma 8.14, suppose either that all exceptional labels have degree n, or that $k \le 2$. Then $p^{k+1} \mid L_0 + pM_0$. If $L_0 + M_0 > 0$, the number N of degree-n exceptional labels satisfies $2N \ge c(p-1)p^k$, where $c = 2$ in odd dimension and $c = 1$ otherwise. For $U \in P_p$ of p-height j, an inertia orbit on U of length e at a prime $s \ne p$ then satisfies*

$$c(p-1)ep^{2j} < 2p^{k+1}.$$

Proof. Higher strata in the congruence vanish under either hypothesis: $p \mid L_i$ gives $p^3 \mid p^{2i}(L_i + pM_i)$ for $i \ge 1$. Thus a nonzero $L_0 + pM_0$ is at least cp^{k+1}, by parity in odd dimension. Its contribution to N is $L_0 + (p-1)M_0/2 \ge (p-1)(L_0 + pM_0)/(2p)$. Each of these labels has codegree p^{2k+1}, so its field is unramified at s. Lemma 2.4 gives $eN(p^j n)^2 < D$. □

Lemma 8.16 (A trace on the trivial twist fiber)**.** *Let an ordinary odd minimal table at three primes have dimension $D = t^5 n^2$, with t prime and $t \nmid n$. Suppose the t-primary twist level is t and one quadratic class of primitive tth roots occurs only on rows of codegree t^3 and field $\mathbb{Q}(\zeta_t)$. If that class occurs, then for each prime $s \mid n$ at least one s-free nonunit in P_t is fixed by s-inertia.*

Proof. Fix a label Z in that class and an outer prime $s \mid n$. Its degree is tn. The two complementary local groups act by scalars on each twist fiber of this class. Indeed the fibers are preserved by each complementary group, their labels have complementary unit codegrees, and the upper unipotents and their conjugates generate these groups. Thus the t-local Weyl W agrees with S_0 up to a root of unity on each such coordinate vector. The same holds for the torus translates of Z.

Put $U = U_t$. For $a \in \mathbb{F}_t$ set

$$M_a = \sum_V d_V \overline{\chi_Z(V)} \theta_{V,t}^a = \frac{D}{d_Z} (S_0 U^a S_0^{-1})_{0Z}.$$

Orthogonality gives $M_0 = 0$. For $a \ne 0$ use the local Bruhat identity

$$W U^a W^{-1} = U^{-1/a} W H_{t,-a} U^{-1/a}.$$

It gives M_a equal to a root of unity times $\sqrt{D}$. Indeed the diagonal factors contribute phases, and the vacuum entry of W at the torus translate has modulus $d_Z/\sqrt{D}$ and agrees with that of S_0 up to a root of unity. Since $\sqrt{D} = t^2 n\sqrt{t}$ and $s \mid n$, summing over a gives

$$\sum_{\theta_{V,t}=1} d_V \overline{\chi_Z(V)} \equiv 0 \pmod{s\overline{\mathbb{Z}}}.$$

This set of labels is Galois stable. Its vacuum contribution is one. Grouping by Galois orbit therefore supplies an s-free nonunit X with trivial t-twist and nonzero absolute trace of $x = \chi_Z(X)$. Its row meets Z, so $X \in P_t$ by the prime-power codegree support theorem (Theorem 2.9).

Suppose every such s-free nonunit is moved by s-inertia. Let A be the span of the s-local translates of the s-active coordinates. It is t-locally invariant. The chosen quadratic class is absent from A, since it is absent on s-active coordinates and the s-group preserves their t-phases. On $A^\perp$ the s-upper unipotent and all its conjugates are the identity. Let P project onto the U-fixed space. For a nonzero torus eigenvector $v \in PA$ with eigencharacter λ, the Fourier projector at a missing phase ζ_t^c vanishes on $W^{-1}v$. The same Bruhat identity gives

$$0 = \sum_{a \bmod t} \zeta_t^{-ca} PWU^a W^{-1} v = v + \left(\sum_{a \ne 0} \zeta_t^{-ca} \lambda(-a) \right) PWv.$$

Consequently $PWv \neq 0$, and the parenthesized sum has the same nonzero value for every c in that class. Replacing c by cu^2 multiplies it by $\lambda(u^2)^{-1}$. Thus every square torus acts trivially on PA. On $PA^\perp$ the s-group acts trivially, so on the entire t-inactive space

$$(H_{t,u^2} - 1)(H_{s,b} - 1) = 0.$$

Both tori act by unsigned Galois permutations since their automorphisms fix $\sqrt{D}$. Take b moving X. If H_{t,u^2} also moved X, the coefficient at X in the mixed difference of e_X would be positive. Hence all square t-units fix X. The entry field $F = \mathbb{Q}(x) \subseteq K_X \cap K_Z$ has degree at most two, since $K_Z = \mathbb{Q}(\zeta_t)$. Also $[K_X : F] \geq 2$, since s-inertia fixes K_Z and moves X.

Write u' for the other outer prime. The s-free nonunit X has degree $t(u')^i$: the principal valuation bound gives t-height at most one and a nonunit prime-power degree is impossible. The t-valuation of d_Z is also one. Since $X \in P_t$, symmetry gives $x = (d_X/d_Z)\chi_X(Z) \equiv 0$ above t. Its nonzero absolute field trace is an integer multiple of t. In the trace-norm identity (Lemma 2.4) this orbit contributes

$$[K_Z : F][K_X : F] \operatorname{Tr}_{F/\mathbb{Q}}(x)^2 \;\geq\; (t-1)t^2.$$

Reducing the ordinary dimension–trace identity at u' supplies a second, distinct orbit, of degree ts^j, with a nonzero trace also divisible by t. It contributes at least t^2. Together with the positive vacuum term this exceeds $t^3 = f_Z$. □

Proposition 8.17 (An odd quintic with two complementary witnesses)**.** *Let an ordinary odd minimal table have three dimension primes t, u, s, where $s > \max(t, u)$ and $v_t(D) = 5$. Suppose the s-free nonunits of P_u and the u-free nonunits of P_s all have t-height two. Some s-free nonunit of P_t is fixed by s-inertia.*

Proof. Suppose all s-free nonunits of P_t are moved. Call the two sets in the hypotheses the witnesses. They exist by prime avoidance. Each witness has t-defect one. Lemma 3.10 gives a single visible quadratic class for both sets. Each s-active or u-active label hits a corresponding witness by principal-block orthogonality modulo the omitted prime (Proposition 4.2). All their t-phases are therefore 1 or in the chosen class. There is a label outside these phases: either a deeper twist occurs or Lemma 7.4 gives the opposite primitive class. On every such fiber both complementary local groups act by scalars. By Lemma 3.3 every label Z in it has pure t-power codegree. It follows that $D = t^5 n^2$ and the possible codegrees t, t^3, t^5 have degrees $t^2 n, tn, n$, respectively. Each exceptional Galois orbit contains at least $t - 1$ labels: square covariance gives at least $(t-1)/2$ distinct phases, and both duals occur at each phase in odd dimension.

The row of Z is supported on P_t. The dimension–trace equation reduced at each outer prime gives separate nonunit orbits of degrees tu^i and ts^j with nonzero entry trace. Each contributes at least one to the trace norm, while the vacuum contributes $[K_Z : \mathbb{Q}] \geq t - 1$. Hence $f_Z \neq t$. If an exceptional label had degree n, take an s-free nonunit $U \in P_t$. It has degree tu^i and inertia orbit length $e \geq 2$ by assumption. Corollary 8.15, with $k = 2$, $j = 1$ and $c = 2$, would give $(t-1)e < t$. Every exceptional row therefore has codegree t^3.

To determine its field, take the space A spanned by the s-local translates of the s-active labels as in Lemma 8.16. On A the only t-phases have order at most t. Thus $U_t(t)$ and its conjugates are trivial there. The upper–lower factorization

$$U_t(1)L_t(b) = L_t(b/(1+b))H_{t,1+b}U_t(1/(1+b))$$

for the lower unipotent $L_t(b)$ at $b = tj$ gives $H_{t,1+tj} = 1$ there: $U_t(1/(1+tj))$ agrees with $U_t(1)$ and the lower parameters are multiples of t. On $A^\perp$ the s-group is trivial. The mixed-difference argument of Lemma 8.16 shows that every label moved by s-inertia is fixed by wild t-inertia; both tori again fix $\sqrt{D}$. Apply this to the s-free trace-contributing orbit above. Its entry field F is tame at t, and its nonzero trace is divisible by t since $d_Z = tn$. If K_Z were wild at t, then $t \mid [K_Z : F]$ and this orbit alone would cost t^3, before the vacuum term. The codegree makes K_Z

a field of t-power conductor (Theorem 2.7 and Corollary 2.8). Tameness now gives $K_Z \subseteq \mathbb{Q}(\zeta_t)$, with equality by its degree bound. The local Galois unit $1+t$ fixes Z, and its square fixes the twist of Z. Hence that twist has t-part of order at most t. This holds on every exceptional row, and the other rows already have t-phase of order at most t. The global t-primary level is thus t, and Lemma 8.16 gives the required contradiction. □

Corollary 8.18 (A least quintic in odd dimension)**.** *For odd primes $p<q<r$ the conclusion of Theorem 8.3 holds for all $b,c\geq 0$.*

Proof. In a divisor-minimal failure all three primes survive and the least valuation is five by Theorem 7.3. Opposite r-free nonunits of P_q and q-free nonunits of P_r have common p-part at least q by Lemma 4.10, and at most p^2 by square divisibility. Hence every such witness has p-height two. An r-free nonunit of P_p has degree pq^i. Pair it with a q-free witness of P_r. The normalized entry has weight p and residue p at r. All twist primes are at least p, and $(r-p)(p-1)>p$. Lemma 8.5 gives an unsigned r-word of length p. It is unique and has a nonidentity term by strictness. As the tame units act freely on nonidentity r-power roots, its stabilizer has order at most $p<r-1$. Thus r-inertia moves every r-free nonunit of P_p, contradicting Proposition 8.17. □

8.5.1 A quintic at the middle prime in odd dimension

Let $p<q<r$ be odd primes. There are two ways to control a middle quintic. The first uses short words at q, independently of the two other exponents. The second uses the exponent at p to shorten a word at r.

Proposition 8.19 (Short words at the middle prime)**.** *Put $j=\lfloor q/p\rfloor$. Suppose*

$$1+j(j+1)<2q/3.$$

Then every divisor of $p^Aq^5r^4d$ gives WGT for every A. If $(r-q)(p-1)>q$, the two exponents A and C may both be arbitrary in $p^Aq^5r^Cd$.

Proof. In a minimal table with three primes the middle valuation is exactly five by Theorem 7.3. First a p-free nonunit $A\in P_r$ has q-height two: height one would give weight q with a complementary $qp^i\in P_q$, with residue q at both q,r. By Theorem 4.9 the house would be q, contradicting strictness.

Suppose now that an r-free nonunit $U\in P_p$ has degree p^iq. Compare two character values on this one row. Choose $A\in P_r$ as above, of degree q^2r^l, and a p-free $B\in P_q$ of degree qr^k. Each crossing has common weight q. At U,A the residues at p,r are both q, so Theorem 4.9 gives an unsigned r-word on at most $1+j$ positions; its signed alternative would require $r=q+1$. At U,B the residues at p,q are $q,0$. Lemma 8.4 gives, up to a scalar, an unsigned q-word on at most $1+j$ positions. Rescaling each normalized crossing gives the corresponding character value of full weight d_A or d_B.

The adjoints have full words on r-power and q-power roots, respectively, each on at most $1+j(j+1)<2q/3$ positions. Both supports are exposed and meet only at 1, contradicting Lemma 4.15. Thus every r-free nonunit of P_p has q-height two.

For any r-free nonunit $X\in P_q$, $d_X=qp^i$. Pair it with such a p-free witness of P_r. Their common weight is q and the residue at r is q. If $(r-q)(p-1)>q$, Lemma 8.5 gives an unsigned r-word of length $q<r$. It has a nonidentity term and, by uniqueness, its tame stabilizer has order at most $q<r-1$. Thus r-inertia moves every such X, contradicting Proposition 8.17 with $t=q$, $s=r$. For $C=4$ the complementary range has $r<2q$ and is covered by Theorem 8.2. □

Theorem 8.20 (A least-prime exponent bound for a middle quintic)**.** *For $p<q<r$ odd and $A\geq 6$, suppose $q>p^{\lfloor A/2\rfloor-2}$. Every weakly integral braided fusion category of dimension dividing $p^Aq^5r^4d$ is WGT. In particular $A=6$ and $A=7$ work for all odd prime triples.*

Proof. A divisor-minimal failure has table of dimension $p^\alpha q^5 r^4$, by the smaller-quartic and cubic-exponent theorems (Theorems 7.3 and 4.12). Theorem 8.3 gives $h = \lfloor \alpha/2 \rfloor \geq 3$. Consider opposite nonunits of P_q, P_r, respectively r-free and q-free. Their common p-part is at least q by Lemma 4.10. Since $q > p^{h-2}$, their individual p-heights are $h-1$ or h.

They cannot all have height h. At $\alpha = 2h$ all these witnesses would have p-unit codegree. But a p-active fiber contains a q-active column by the odd q-valuation, and orthogonality on P_q modulo r gives a witness hitting it. At $\alpha = 2h+1$ every witness would have p-defect one. As in Theorem 7.9, its two orbit degrees agree: their squared ratio is a_0 or $2a_0$ with $a_0 \mid p-1$, and the primes of the degrees are all at least p. The opposite witnesses thus see the same primitive quadratic class, as in Proposition 8.17. The missing-phase argument gives a row of pure p-codegree, which would force the q-valuation to be even.

Consequently there exists an opposite pair of common weight $G = p^{h-1}$. Write $M = \lfloor G/q \rfloor$, so $1 \leq M < p$. For $r > G$ the two-residue theorem gives an r-word on at most $1+M$ positions: $r = G+1$ is impossible for two odd numbers. For $r < G$ the odd q-valuation gives a column active at both q and r; the first step of the proof of Proposition 8.7 then gives an r-word on at most $1 + \lfloor r/q \rfloor$ positions. Explicitly, that column hits a p-free member of P_q of r-height one, and one pairs it with a q-free member of P_r of r-height one. Lemma 8.4 applies. Thus in either range we have a word on at most $1+m$ positions, with $m = \lfloor \min(r, G)/q \rfloor \leq M$.

If $q \geq (M+1)^2$, then $r > mq \geq m(m+1)^2$. For $m \geq 2$ the estimates of Corollary 8.12 give both support inequalities at r, and for $m = 1$ they hold since $r \geq 7$. This contradicts Lemma 4.16. In particular $h \geq 4$ is excluded, since then $q > p^2 \geq (M+1)^2$.

It remains that $h = 3$ and $q < (M+1)^2$. Since $q \geq 5$ we must have $M \geq 2$, so $q < p^2/2$. For $j = \lfloor q/p \rfloor$ this gives $j \leq (p-1)/2$. We claim $1 + j(j+1) < 2q/3$. For $j = 1$ use $q \geq 5$. For $j \geq 2$ we have $p \geq 5$ and $2p - 3(j+1) \geq (p-3)/2 \geq 1$. Consequently $3 + 3j(j+1) < 2(pj+1) \leq 2q$. Proposition 8.19 now gives the contradiction. In particular its unrestricted-A conclusion holds throughout $2q < p^2$. □

8.5.2 The ordinary table at dyadic valuation five

Lemma 8.21 (The mass outside two dyadic phases)**.** *Let an integral modular category have dimension $D = 2^{2k+1} n^2$, with n odd and $k \geq 1$. Fix a primitive fourth root i. If every label with two-primary twist outside $\{1, i\}$ has dimension n and such a label exists, there are at least 2^k such labels.*

Proof. For each exponent m, the regular-object indicator $D^{-1} |\sum_X d_X^2 \theta_X^m|^2$ is an algebraic integer. Suppose the largest dyadic twist order is $2^s \geq 8$, and let L count labels at that order. Choose m equal to 2^{s-1} on the dyadic level and zero on its odd part. The Gauss sum is $n^2(2^{2k+1} - 2L)$. Its indicator is rational, so $2^k \mid L$.

It remains to consider twists of dyadic order dividing four. The vacuum calculation in the proof of Lemma 7.5, on the whole label set, equates the squared-dimension mass at orders one and two together with the mass at order four. Each therefore equals $D/2$. Let u, v count the exceptional labels at $-1, -i$, respectively. Choosing m one on the dyadic level and zero elsewhere gives the Gauss sum $n^2((2^{2k} - 2u) + i(2^{2k} - 2v))$. Its integral indicator gives $2^{2k-1} \mid u^2 + v^2$. Hence u, v are divisible by 2^{k-1}, and their quotients have the same parity: this follows from $v_2(x^2 + y^2) = 2\min(v_2(x), v_2(y))$ when the valuations differ, and $2v_2(x) + 1$ when they agree. Thus $2^k \mid u + v$ and $u + v \geq 2^k$ whenever an exceptional phase occurs. □

Proposition 8.22 (An ordinary quintic at two)**.** *There is no ordinary minimal table of dimension $2^5 q^b r^c$ for odd primes $q < r$ and any b, c.*

Proof. Apply the proof of Theorem 7.9 with least prime two and height two. It gives a fourth root i for which the set of dyadic phases outside $\{1, i\}$ is nonempty, and every label there has pure two-power codegree. Thus b, c are even. Put $n = q^{b/2} r^{c/2}$. Every such Z has degree $2^j n$

and codegree 2^{5-2j}, $0 \le j \le 2$. Its row is supported on the principal two-block by Theorem 2.9. Reduce the dimension-trace equation (Lemma 2.4) at q and at r. The vacuum forces two distinct nonunit trace orbits in this block, of degrees $2q^h$ and $2r^l$: a prime-power degree is impossible, and an even principal degree has two-height at most one. Each orbit costs at least one in the trace norm, already excluding codegree two. At codegree eight one has $j = 1$. For a representative U of either orbit, $\chi_U(Z)$ vanishes above two by principality. By symmetry the same holds for $x = \chi_Z(U)$ since d_U/d_Z is a two-adic unit. Principality holds at all primes above two by Galois stability. The trace of x from its entry field therefore belongs to $2\mathbb{Z}$. The two nonzero orbit traces now cost at least $4 + 4$ besides the vacuum by Lemma 2.4. Consequently all exceptional labels have degree n and codegree 32.

Take an r-free $U \in P_2$ of degree $2q^h$ and a q-free nonunit $Y \in P_r$ of degree $4r^l$ by the complementary principal-degree lemma. At their crossing the common weight is two and the residue above r is two. As $r \ge 5$, the single-residue word theorem gives a strict unsigned r-word with two summands. The stabilizer of the word among tame units has order at most two, since it acts freely on the nonidentity roots in this unique unsigned word. The r-inertia orbit of U therefore has length $e \ge (r-1)/2 \ge 2$. Every exceptional Z has odd degree n and r-unramified field. Lemma 2.4 bounds their number N by $4eN < 32$, hence $N \le 3$, contrary to Lemma 8.21. □

8.5.3 The fermion quotient at dyadic valuation five

Lemma 8.23 (Sign twists and the principal two-block)**.** *Let an integral slightly degenerate category be the even sector of an integral modular category of dimension $D = 4\Delta$. If the dyadic twist of every even label is a sign, then $v_2(\Delta)$ is even. If also $v_2(\Delta) \ge 3$, no member of the positive principal two-block has dyadic defect two.*

Proof. Write g for the fermion, F for tensoring by g, G for the monodromy sign, $U = \lambda T_2$ for the normalized dyadic upper unipotent and W for its Weyl operator. Put $A = (F = G = 1)$, $B = (F = -1, G = 1)$ and $C = (F = 1, G = -1)$. Dyadic inertia fixes the even labels individually by Lemma 4.6. The torus W^2 therefore acts on A, B by the common Galois sign ϵ of $\sqrt{D}$. It acts by the same sign on $C = WB$. Let $J = W|_A$ and $P = WU : A \to C$. This is a bijection, since U exchanges A, B. Applying $WUW = U^{-1}WU^{-1}$ to Ux, using $U^2 = \lambda^2$ on A, B, gives $UPx = \epsilon\lambda^{-2}PJx$. Since $J^2 = \epsilon$, the dyadic phases of odd labels differ only by a sign: each odd fermion orbit has a vector in C of its dyadic phase. Denote the two possible odd-sector dyadic phases by $\pm\delta$. The even contributions to the positive Gauss sum cancel in fermion pairs. Hence $p^+/\delta = (\xi/\delta)\sqrt{D}$ lies in an odd-conductor cyclotomic field K, with ξ the multiplicative central charge. Squaring shows that $(\xi/\delta)^2 \in K$, so the odd-order part of ξ/δ belongs to K and its dyadic part has order at most four. Thus $\sqrt{D} \in K(i)$. Were $v_2(D)$ odd, the quadratic discriminant of $\mathbb{Q}(\sqrt{D})$ would be divisible by eight, impossible in this cyclotomic field of conductor dividing four times an odd integer. We conclude that $v_2(D)$ and $v_2(\Delta) = v_2(D) - 2$ are even.

Choose k zero on the dyadic level and two on the odd level. In the modular indicator sum for even V, the two fusion inputs belong to the same fermion sector, so the dyadic twist ratio is a sign and $\nu_k(V) = \nu_2(V)$. For $b = \sum_Y \nu_k(Y)[Y^*]$, (3.2) gives $\chi_x(b) = f_x|(sT^k s^{-1})_{0x}|^2$. Odd-local Bruhat at the unit parameter -2 identifies this absolute entry with that of the vacuum column w of the odd Weyl, up to a torus permutation. For clarity, let d_o be the product of odd primes t with $v_t(D)$ odd and let $d_o^* = \prod_t (-1)^{(t-1)/2} t$ over the same primes. The Galois identity used in Lemma 6.1 conjugates the odd Weyl by $\mathrm{diag}(a, 1)$, introducing on its right the odd torus. On the vacuum this torus has the sign of the corresponding odd inertia action on $\sqrt{D}$, namely $\sigma_a(\sqrt{d_o^*})/\sqrt{d_o^*}$. Thus $w_i \in \sqrt{d_o^*}\,\mathbb{Q}$ and $|w_i|^2$ is d_o times a rational square. Since $v_2(D)$ is even, $d_o f_x = d_o D/d_x^2$ is a rational square. Also w has even support and trivial actual dyadic twist, by commutation with the grading and the dyadic unipotent. It follows that $gb = b$ and that for even x at most one of $\chi_x(b), \chi_{gx}(b)$ is nonzero. Consequently $W^+ = \sum_{[V]} \nu_2(V)[V^*]$ in the

positive quotient has eigenvalues $w'_x = (\chi_x(b) + \chi_{gx}(b))/4$ that are integer squares. Here the average first selects the even terms, then divides out the two representatives of each pair; $gb = b$ gives $\nu_2(gV) = \nu_2(V)$.

Modulo two, $(W^+)^2$ equals the Casimir $\sum_{[V]}[V][V^*]$. Nonselfdual pairs cancel, and any self-dual pair has a self-dual representative by the opposite twists of gV and V^*. Thus $4 \mid w'_x$ whenever f_x^+ is even. If x were principal of defect two, sign-only balancing makes the positive fields unramified at two. Hence $v(V) = (\chi_x(V) - d_V)/2$ is locally integral. Orthogonality gives $\sum_{[V]} v(V)v(V^*) = (f_x^+ + \Delta)/4 \equiv 1 \bmod 2$. After cancellation of nonselfdual pairs, Frobenius in the residue field gives $\sum_{[V]} \nu_2(V)v(V^*) \neq 0 \bmod 2$. But this sum is $(w'_x - w'_0)/2$, divisible by two. □

Proposition 8.24 (The reduced quintic at two)**.** *There is no slightly degenerate divisor-minimal counterexample of dimension $2^5q^br^c$ with $2 < q < r$ and b, c arbitrary.*

Proof. Write Δ for the positive quotient dimension and $D = 4\Delta$ for the dimension of the integral minimal modular extension from Lemma 4.4. Let g be the transparent fermion of the even sector. By Theorem 7.11, b, c are even. Write $\Delta = 16n^2$ with n odd. The complementary r-free nonunits of P_q and q-free nonunits of P_r all have two-height two: comparison across the two blocks gives two-part at least q, while square divisibility gives at most four. Thus $q = 3$ and the respective degrees are $4q^i, 4r^j$.

Call an even pair deep when its squared dyadic twist is not one. Every complementary witness has odd positive codegree and hence a singleton positive two-block. Orthogonality on this singleton, using the nonzero vacuum entry, gives vanishing on all deep pairs. Orthogonality on P_q modulo r forces each q-active pair to hit an r-free witness. The same holds with q, r interchanged. Consequently the fibers of each deep squared dyadic twist are inactive at both odd primes. Their normalized local actions are scalar on each such fiber, by invariance and generation by conjugate unipotents. If Z lies there, the odd Weyls preserving the symmetrized pair line preserve its two full lines separately by commuting with the dyadic unipotent (the actual twists are opposite). By Lemma 3.3, Z has pure dyadic positive codegree. It is 16: codegree one would exhaust the dimension and four is excluded at the quartic valuation by Proposition 5.7. Thus $d_Z = n$.

If a deep pair exists, orthogonality with the vacuum on the positive principal two-block modulo r gives an r-free nonunit X hitting it. Its degree is $2q^i$ by the principal valuation bound and exclusion of prime-power degrees. Pairing it with a complementary $4r^j$ witness gives a normalized entry of weight two, residue two at $r \geq 5$, and strict house below two. By Theorem 4.9, it is an unsigned sum of two r-power roots. Its row therefore has r-inertia orbit size $e \geq (r-1)/2 \geq 2$: uniqueness of the unsigned word bounds the tame stabilizer by two. Every deep pair has odd degree n and field unramified at r. Lemma 2.4 gives $4eN < 16$ for the number N of deep pairs. Thus the deep pair is unique. In the symmetrized positive space A, the dyadic Weyl $K = W_2|_A$ preserves each positive two-block by the full lift (two-residues on even objects determine them on odd objects by squaring). The odd local Weyls preserve the line of e_Z, so Ke_Z agrees up to phase with the reduced Weyl column of Z and has nonzero vacuum entry. Thus $Z \in P_2$ and $\chi_Z(Z)$ is a unit at every prime over two. Squared dyadic twists on A form $R = 1 + (u-1)P$, $u \neq 1$, for the rank-one coordinate projection P at Z. The product $RKR^{-1}K^{-1}$ has finite order since the scalar lifts of the local even unipotents cancel. Expanding its trace shows $(2 - u - u^{-1})|K_{ZZ}|^2$ is an algebraic integer. But $|K_{ZZ}|^2 = |\chi_Z(Z)|^2/16$ has valuation -4 at two, and $v_2(2 - u - u^{-1}) \leq 2$ for a nontrivial dyadic root. This excludes all deep pairs.

All even dyadic twists are therefore signs. The preceding lemma excludes defect two in the positive principal two-block. Since $v_2(\Delta) = 4$, the principal valuation bound now makes every degree of that block odd.

By Lemma 4.20, a complementary singleton has an odd mate of the same degree, and its row sees only the mate's dyadic phase on odd columns. Opposite principal witnesses hit these mates,

giving a common phase δ. Orthogonality modulo the omitted prime supplies a complementary witness on every q- or r-active odd column: evaluation at g makes all full principal labels even and an opposite odd mate excludes g itself. Thus every odd fiber of phase different from δ is inactive at both odd primes, and any simple Y in it has pure dyadic full codegree as above. Such a fiber exists by Lemma 7.7.

Take an odd mate T of a complementary r-free witness, so $r \nmid d_T$ and its phase is δ. The restrictions of T, Y to the even ring give distinct signed characters. Orthogonality over even fermion-pair representatives, reduced modulo r, supplies a nonunit r-free even pair V hit by Y: the vacuum contributes one and $r \mid \chi_T(V)$ at every r-bearing degree by lcm integrality. Since f_Y is a power of two, Theorem 2.9 puts V in the full principal two-block, hence its pair is in the positive principal block. Its degree is odd by the preceding paragraph, so it is a power of q, impossible in the perfect simple quotient. □

Conclusion of Theorem 8.3. For odd p use Corollary 8.18. For $p = 2$ Propositions 8.22 and 8.24 exclude the ordinary and reduced tables. Smaller least-prime valuations already give WGT by Theorem 7.3, and the two-prime theorem of Dong [22, Theorem 3.1].

8.6 A quintic at three in even dimension

Lemma 8.25 (Weight three at two and three)**.** *In an ordinary or positive reduced minimal table with dimension primes $2, 3, r$, $r > 3$, let $\gcd(d_V, d_B) = 3$. Set $z = S_{VB}/\operatorname{lcm}(d_V, d_B)$. If $z \equiv 1$ at every prime above two and $z \equiv 0$ above three, then $z = \lambda\sqrt{-3}$ for a dyadic root of unity λ.*

Proof. A signed integral word of length at most three represents z: at a common level $2^a 3^b r^c$ the root lattice on the odd factors has a totally unimodular root configuration by [4, Theorems 3 and 13]. A basis of the 2-power lattice consists of one member of each opposite pair, so tensoring gives disjoint signed copies of the same unimodular matrix. Integral minimum length follows.

Erase the 3-parts of the word. The remaining sum w is divisible by three in its unramified ring and has house at most three. If $w = 0$ the only possibility is a cancelling pair (a vanishing triangle would require the prime three in the complement). Thus $z = \lambda(\alpha - \beta)$ with λ prime-to-three and α, β of 3-power order. If $w \neq 0$, Kronecker and equality in the triangle inequality give $z = \lambda(\alpha + \beta + \gamma)$ with the same conventions. In either case λ has trivial r-part: invert its r-part while fixing the other factors and reduce both values modulo two. They are both 1 and the reduction is faithful on the odd-order roots.

Erase the dyadic factor modulo two. In the pair case the vector of $\alpha, \beta, 1$ modulo two is a union of complete μ_3-cosets by the cyclotomic relations. Its odd cardinality is at most three, so it is the identity coset and $\{\alpha, \beta\} = \{\omega, \omega^{-1}\}$ for a primitive cube root ω. In the triple case the vector of $\alpha, \beta, \gamma, 1$ has even cardinality at most four and is a union of complete triples. Thus it is empty and the word is $\lambda(1 + 2\xi)$ for a 3-power root ξ. Strict house excludes $\xi = 1$. At order at least nine, the adjoint support $1, \xi, \xi^{-1}$ has one mark in each occupied μ_3-coset. Its dilation by the 3-adic unit four meets it only at one. Lemma 4.15 after scaling to the character value at B gives a contradiction. Thus $\xi = \omega^{\pm 1}$ and in both branches $z = \pm\lambda(\omega - \omega^{-1})$. □

Proposition 8.26 (Transfer at two and three)**.** *In an ordinary or positive reduced minimal three-prime table at $2, 3, r$, a row $U \in P_2$ of degree $2^i 3$ forces every odd-degree label to have the same 3-primary twist as U. No such row exists when $v_3(\Delta)$ is odd. The same implications hold in a full integral modular table if there is a subset $E \subseteq P_3$ with the following properties. It is invariant under ternary inversion ι, contains the vacuum as its only invertible, and supports restricted orthogonality at unequal ternary phases and the odd-valuation mass identity. At each odd-degree nonunit of E, the crossing with U is negated by ι and is nonzero.*

Proof. Let ι invert the 3-power roots and fix the other primary factors, and put $H = \chi_U + \chi_{U^\iota}$. The two row labels have the same 3-twist by Galois square covariance. Each odd nonunit

$B \in E = P_3$ in the minimal tables has degree $3^h r^j$ with $h, j > 0$. The normalized crossing has weight three and residues $1, 0$ at $2, 3$, respectively, by principality. Lemma 8.25 therefore gives the required negation and nonvanishing. In all cases $H(B) = 0$ at each such B, and $B^\iota \neq B$. On every even B both terms in H vanish modulo any prime above two by principality. If an odd-degree J has different 3-twist, block orthogonality gives $\sum_{B \in E} \chi_J(B)\overline{H(B)} = 0$. The vacuum contributes two. On every even B, lcm divisibility makes $\chi_J(B)$ divisible by two, so all other nonzero contributions have dyadic valuation greater than one. This proves the transfer; in particular every odd-degree label is 3-inactive by the vacuum. Now the odd labels of E number one modulo two: all except the vacuum are paired by ι. At odd $v_3(\Delta)$ the principal-block mass identity of Prop. 3.8 reads

$$\sum_{B \in E,\ \mathrm{ord}(\theta_{B,3})=3} d_B^2 = 2 \sum_{B \in E,\ \theta_{B,3}=1} d_B^2.$$

The left is zero modulo four whereas the right is two. □

Lemma 8.27 (Three-primary phases at odd valuation)**.** *If an ordinary or positive reduced minimal table has odd $v_3(\Delta)$, then some 3-twist lies outside $\{1, \eta\}$ for each primitive cube root η.*

Proof. If all phases are in $\{1, \eta\}$, the first paragraph of the proof of Lemma 7.4 and its immediately preceding indicator construction (with $p = h = 3$) give an integral self-adjoint virtual element b with $b^2 = 3b$, $b_1 = 1$, $\dim b = 3$. In the ordinary table this uses the nontrivial local Legendre sign from $v_3(\Delta)$ odd, and applies also in even dimension; odd dimension in that lemma is used subsequently for the nonreal character orbit. For clarity, the missing Fourier coefficient gives $w = -GPse_0$ on the inactive phase projector, $|G|^2 = 3$. The indicator construction gives $\chi_j(b) = 3\mathbf{1}_{\theta_{j,3}=1}$, rational integral coefficients and the stated normalization. For a positive reduced table we obtain the same element as follows. In the integral minimal modular extension let g be the fermion and A the positive space spanned by $(e_Y + e_{gY})/\sqrt{2}$ for even fermion pairs. Write W_o for the product of the local Weyls away from three and $w = W_o e_0$. It interchanges the fermion permutation and its monodromy sign. Therefore the coordinates of $W_o(e_0 + e_g)/\sqrt{2}$ in the pair basis are $2w_Y$, with $w_Y = w_{gY}$ for even Y. For m zero on the ternary level and one on its complement, the complementary Weyl identity and the ordinary indicator formula give, for even V,

$$\nu_m(V) + \nu_m(gV) = \sum_{[Y]} |2w_Y|^2 \, \widehat{\chi}_{[Y]}([V]).$$

Indeed the odd-sector contributions cancel and each even pair contributes twice the same character. If all ternary phases in the positive table lie in $\{1, \eta\}$, the proof of Lemma 7.4 applies its ternary Bruhat identity on A and gives $|2w_Y|^2 = 3d_Y^2/\Delta$ at inactive pairs and zero elsewhere. The displayed sum is rational by Galois invariance and is an algebraic integer as a sum of indicators. It therefore defines the same integral virtual element in the positive ring, with norm squared three and dimension three.

Let $M = N_b$; $M^2 = 3M$ and M is symmetric integral. At a diagonal $a = 1$ or 2 the off-diagonal squared row norm is $a(3 - a) = 2$, so the row has two neighbors with signs. Rows at $0, 3$ are isolated. The unit component is a cycle of length three or four: a longer cycle has nonneighbors with a unique length-two path, contradicting the matrix equation. A triangle has trace three (the trace is a multiple of three and the unit diagonal is one), hence rank one and all three diagonals one. Its positive dimension eigenvector forces equal dimensions, contradicting perfectness. On a four-cycle let A oppose the vacuum. Adjacent diagonals sum to three, so the two neighbors Y, Z have diagonal two. Their dimension equations give $d_Y, d_Z \in \{d_A - 1, d_A + 1\}$. Coprime vanishing gives $\chi_A(Y) = \chi_A(Z) = 0$, impossible because $b = 1 \pm Y \pm Z$ and the only eigenvalues of b are zero and three. □

Theorem 8.28 (A pure ternary codegree)**.** *In an ordinary minimal table of dimension* $2^a3^5r^4$ *with* $r > 3$*, the exponent* a *is even. For some primitive cube root* η*, the set of labels with* 3*-primary twist outside* $\{1, \eta\}$ *is nonempty, and every such label has formal codegree* 27 *or* 243*.*

Proof. By prime avoidance, P_2 contains r-free nonunits. Their degrees have positive 2- and 3-heights, and hence 3-height exactly two by the transfer proposition. The 2-free nonunits of P_r likewise have degree $9r$ (Section 7.4).

All these witnesses have ternary defect one. By Lemma 3.10 their rows see only 1 and a common primitive cube root η. By principal orthogonality modulo the omitted prime, each 2-active or r-active column hits one of the witnesses. Their 3-phases therefore lie in $\{1, \eta\}$. The preceding lemma gives a nonempty 3-phase fiber outside the pair. It has neither 2- nor r-activity and is invariant under both corresponding local groups. Their upper unipotents act by scalars there, hence their conjugates and local Weyl operators do too. By the complement lemma every row Z in that fiber has 2- and r-unit codegree, so a is even, and $f_Z = 3^{5-2j}$ for $0 \le j \le 2$. The value three is impossible: orthogonality to the vacuum reduced at each of $2, 3, r$ requires three distinct nonunit hits, since a hit omitting two primes would have prime-power degree. The Galois-averaged squared norm of each nonzero algebraic integer is at least one, so the vacuum and these three hits already exceed three. (Equivalently average the entire row norm over the splitting field.) □

Lemma 8.29 (The trace restriction at codegree 27)**.** *Suppose there is an ordinary minimal table of dimension* $2^a3^5r^4$ *and a label* Z *of pure codegree* 27*, so* a *is even and* $d_Z = 2^{a/2}3r^2$*. There is a unique Galois orbit of* r*-free nonunits with a nonzero absolute trace on this row. For its representative* U *write* $d_U = 2^i3$*. Its field equals* K_Z*, is of* 3*-power conductor, and*

$$9 \cdot 2^i \equiv \pm 1 \pmod{r}.$$

There is likewise exactly one odd nonunit orbit contributing nonzero trace, of degree $3r^j$*,* $j = 1, 2$*, with field* K_Z*.*

Proof. Every hit lies in P_3 by Theorem 2.9. The integral trace–dimension relation of Lemma 2.4 at Z reduced modulo r and modulo two gives a separate trace-contributing r-free and odd nonunit orbit. Every such label U has exactly 3-height one by the principal valuation bound and absence of prime-power nonunit degrees. Then $\chi_U(Z)$ vanishes above three by principality, and $\chi_Z(U) = (d_U/d_Z)\chi_U(Z)$ does too since the ratio is a 3-unit. This holds at every prime above three, so its absolute field trace is an integer multiple of three (reduce the sum of all conjugates). For $F = \mathbb{Q}(\chi_Z(U))$ the exact contribution to the orbit trace norm is $[K_U : F][K_Z : F] \operatorname{Tr}_{F/\mathbb{Q}}(\chi_Z(U))^2$ by Lemma 2.4. A nonzero contribution is at least nine. Three such orbits are impossible by the positive vacuum term, and either relative field index exceeding one would likewise cost at least $18 + 9 + 1 > 27$. Thus for each of these two orbits the entry field equals both row fields, and the corresponding absolute trace is ± 3. The dimension trace identity modulo r now reads $1 \pm 3d_U = 0 \bmod r$ for the unique r-free orbit. □

Proposition 8.30 (The odd witness at the ternary quintic)**.** *Let an ordinary minimal table have dimension* $2^a3^5r^4$*, with* $r > 3$*. If* $r \ge 11$*, no such table exists, for any* a*. If* $r = 7$*, all exceptional ternary phases of Theorem 8.28 have codegree* 243*; every* r*-free nonunit of* P_3 *has tame* r*-inertia orbit of size exactly two, and every* r*-active label has even degree.*

Proof. Take any r-free nonunit $A \in P_3$ and any odd nonunit $Y \in P_r$. Their degrees have the forms $3 \cdot 2^i$ and $9r$. Their normalized crossing z has weight three and strict house less than three, and $z \equiv 3$ above r by r-principality. If K_A is r-unramified and $r \ge 7$, the algebraic integer $(z - 3)/r$ has house less than one. This would give $z = 3$, which is impossible. In particular the r-free trace witness of Lemma 8.29 cannot exist, so every exceptional label has codegree 243.

For $r \geq 7$ the single-residue theorem gives an unsigned word of length three in r-power roots. It is unique at a common level, and it has at least one nonidentity term. The stabilizer in tame r-inertia acts freely on its nonidentity roots, so its order is at most three. Its orbit therefore has length at least $(r-1)/3$. By Corollary 8.15, applied with $j = 1$ and $k = 2$, the length is strictly less than three. This excludes every $r \geq 11$.

At seven every such A has full and tame inertia orbit size two. The word at any odd nonunit $Y \in P_7$ therefore consists of a single order-three orbit of roots; wild inertia fixes the word, and its orbits on the three terms are singletons, so all terms have order seven. Thus z is one of the two quadratic periods, whose sum is -1. Let A' be the conjugate under inversion at seven. Every label in P_3 is 7-inactive: an exceptional row of codegree 243 has degree prime to three, hence hits all of P_3 by principality and only 7-inactive labels by its 7-unit codegree. Thus A, A' are 7-inactive. The function $H = 3(\chi_A + \chi_{A'}) + \dim$ equals 7 at the vacuum and vanishes on every odd nonunit $Y \in P_7$ since $\chi_A(Y) + \chi_{A'}(Y) = -d_Y/3$. Orthogonality on P_7 to any 7-active row J now precludes d_J odd: at even Y the factor $\chi_J(Y)$ would be even by lcm integrality, leaving the odd vacuum term. □

Theorem 8.31 (A sextic dyadic and a quintic companion)**.** *Let $2 < q < r$ be primes. Every weakly integral braided fusion category of dimension dividing $2^6 q^5 r^4 d$ is WGT, for any square-free d coprime to $2qr$.*

Proof. In a divisor-minimal counterexample, Proposition 7.12 forces the first two source exponents to be six and five and $q \leq 2^2$, so $q = 3$. The r-valuation is four by the reduction in Section 8. Proposition 8.35 excludes a slightly degenerate obstruction at ternary valuation five. It suffices to exclude an ordinary table of dimension $2^6 3^5 r^4$ with $r < 11$, by the preceding proposition.

There is a row $U \in P_3$ of degree 12, by the same Proposition 7.12. Every 3-free nonunit of P_r has degree $4r$ or $8r$. Paired with any degree-12 row of P_3, it gives a normalized entry of weight four, of strict house less than four and of residue four at both three and r. The two-residue theorem (Theorem 4.9) gives a tame r-inertia orbit of size $r-1$ on the entry, also in its signed alternative. In particular every degree-12 principal row is ramified at r, and the orbit of U has size at least $r-1$.

A codegree-27 row would, by Lemma 8.29, require an r-unramified, r-free row of P_3 of degree $3 \cdot 2^i$, with $9 \cdot 2^i \equiv \pm 1 \pmod r$. Its dyadic part is at least four by Lemma 4.10 applied against a 3-free nonunit of P_r, and at most eight by square divisibility. Degree 12 has just been excluded for this row. Degree 24 would require $r \mid 71 \cdot 73$, impossible as both numbers are primes larger than 11. Thus Theorem 8.28 supplies a nonempty set of exceptional phases, all of degree $n = 8r^2$ and codegree 243. Apply Corollary 8.15 with $k = 2$, $j = 1$ and the row U. It would give $9(r-1) \leq 27$, whereas $r \geq 5$. □

8.7 An even-dimensional reduction by the fermion

We prove a further reduction for arbitrary odd b. Its proof works for any largest-prime exponent c, which we retain throughout this section. In the quartic application take $c = 4$. Let $\mathcal{B}$ be a divisor-minimal slightly degenerate counterexample of dimension $2^{2k} q^b r^c$, $2 < q < r$, $k \geq 3$, and $2^{k-2} < q$. Assume b odd and write R for the positive fermion quotient and $\Delta = \dim(\mathcal{B})/2$. We use the integral minimal modular extension $\mathcal{M}$ supplied by Lemma 4.4. Its even sector is $\mathcal{B}$ with transparent fermion g, and $\dim \mathcal{M} = 4\Delta$. Write e_Y for the primitive character idempotents. The local upper unipotent, Weyl and torus have the conventions of Section 3. A complementary witness is an r-free nonunit of P_q or a q-free nonunit of P_r in R.

Proposition 8.32. *All even-sector dyadic twists have order dividing four, and all twists of $\mathcal{M}$ have dyadic order dividing eight. Put $\epsilon_Y = \theta_{Y,2}^2 \in \{1, -1\}$ on fermion pairs. Each complementary witness X has a unique mate X' in its reduced two-block. They have equal degrees and opposite squared dyadic twists.*

Proof. Complementary witnesses exist on both sides. For opposite witnesses their dyadic gcd is at least q by Lemma 4.10. Since $2^{k-2} < q$ and $d_X^2 \mid \Delta$, every witness has dyadic height $k-1$, hence positive dyadic defect one.

Lemma 7.13 confines all witness rows to the two squared phases $+1, -1$, since $v_2(4\Delta)$ is odd. Every nonempty squared dyadic twist fiber contains a q-active label by oddness of b. It has a nonzero entry on a complementary witness in P_q by principal-block orthogonality modulo r. Thus all actual even dyadic twists have order dividing four. The second assertion of the lemma now bounds the full level and gives the equal-degree mate. □

Theorem 8.33 (Odd middle valuation in a reduced table)**.** *Under the hypotheses of the preceding proposition there is no slightly degenerate minimal counterexample. In particular this excludes every reduced obstruction with source dimension $2^6 3^b r^c$, b odd.*

Proof. All complementary witnesses have positive defect one and the even twists satisfy the order bound of Lemma 7.14, by Proposition 8.32. Over each witness of degree d there are two free odd orbits whose actual dyadic twists have ratio $\pm i$ and whose degrees u, v satisfy $u^2 + v^2 = 2d^2$ and $v_2(u) = v_2(v) = k-1$.

Choose a complementary witness of P_r, so $d = 2^{k-1} r^j$. The odd parts of u, v involve only q, r by square divisibility. After canceling 2^{k-1} and the common r-power r^t, their squares sum to $2r^{2(j-t)}$. If $t < j$, neither remaining integer is divisible by r, as the other would also have to be divisible by r. They are therefore powers of q, at least one equal to one. This gives $1 + q^{2e} = 2r^{2(j-t)}$, where $q < r$ forces $e \geq 2$. But q^e is then a nontrivial perfect power among the first coordinates of solutions of $x^2 - 2y^2 = -1$. This is excluded by [17, Theorem 6.1]: the associated Pell sequence contains no perfect power greater than one. Thus $t = j$ and both odd mates have degree d. At $q = 3$ just reduce $u^2 + v^2 = 2d^2$ modulo three to see that both degrees are q-units, which suffices for what follows. Denote their dyadic twist phases by δ_1, δ_2. Any full q-principal representative of a complementary witness in P_q hits both odd orbits by principality. Its row can only see odd phases from its own block, since W preserves the block and W_o preserves both the fermion sector and the dyadic twist. As this block has only two odd phases, those phases must be δ_1, δ_2 for every such representative. Finally the full principal q-block contains only even labels by evaluation at g. Choose full principal representatives of positive pairs at q by the sign argument in the proof of Theorem 7.3. The full principal block excludes g, because g takes value minus the dimension on a q-unit odd mate. Every odd dyadic fiber contains a q-active column by oddness of b. Its orthogonality on this principal block modulo r supplies a nonunit r-free hit, necessarily a complementary witness lifting P_q. Thus δ_1, δ_2 are the only dyadic phases in the entire odd sector.

Lemma 7.14 now gives a nontrivial invertible in the perfect positive quotient, a contradiction. □

Proposition 8.34 (Forcing the top dyadic height)**.** *Suppose $2 < q < r$, b is odd, and a slightly degenerate minimal counterexample has source dimension $2^{2k+1} q^b r^c$, with $k \geq 2$. Put $G = 2^{k-1}$. If $G < 2q$ and $r > G + 1$, such a counterexample cannot occur.*

Proof. In the positive table, pair an r-free nonunit of P_q with a q-free nonunit of P_r. Such witnesses exist by prime avoidance, and the common weight g is a power of two between q and 2^k by Lemma 4.10 and square divisibility. If either witness has dyadic height less than k, then $q \leq g \leq G < 2q$ and $r > g + 1$. The positive residues at q, r and strict house force the normalized entry to be $g - q + q\xi$ with ξ a nonidentity r-primary root by Theorem 4.9. This two-point word has three differences, and the difference-set argument (Lemma 4.15) excludes it for $r \geq 7$: its support is exposed since $3 < r/2$, and a tame dilate moves the unique nonzero opposite pair. Here $r \geq 7$ holds whenever $g \geq q$ is possible, since $g \geq 4$ and $r > g + 1$.

All complementary witnesses therefore have height k and singleton positive two-blocks. The singleton rows vanish on all nontrivial squared phases. Every squared phase contains a q-active

pair, which hits a witness by principal-block orthogonality modulo r. Thus the even twists are signs. By Lemma 4.20, opposite principal witnesses see the same odd phase, and an odd q-active column hits such a witness by the same orthogonality on the full block (the opposite q-unit odd mate excludes g). Every odd dyadic fiber has a q-active label by Lemma 7.8, contrary to Lemma 7.7. □

8.8 A quintic valuation at three in the fermion extension

Proposition 8.35. *A slightly degenerate minimal obstruction cannot have dimension $2^a 3^5 r^c$ for any prime $r > 3$ and positive exponents a, c.*

Proof. Write $\mathcal{B}$ for the category, $\mathcal{M}$ for an integral minimal modular extension, and g for its fermion. An *even label* belongs to $\mathcal{B}$, the neutral fermion-monodromy component. An *odd degree* refers to the integer d_X. The only invertibles of $\mathcal{M}$ are $1, g$. For each noninvertible X of $\mathcal{M}$, the constituents of XX^* generate $\mathcal{B}$. Projective centralization would put the other simple in $\mathcal{B}' = \langle g \rangle$. Thus strict house and coprime vanishing apply to all pairs of full noninvertibles (Lemma 4.4 and its proof). The case of a missing repeated prime follows from the two-prime reduction.

At each odd prime s, a positive principal pair has a full principal representative by the sign argument in the proof of Theorem 7.3. It follows that no full noninvertible can have prime-power degree t^i. Choose $s \in \{3, r\}$ distinct from t. Prime avoidance on positive P_s supplies a t-free nonunit pair by Propositions 3.4 and 4.2. Its full s-principal lift must hit the degree t^i, contradicting coprime vanishing.

We first exclude $U \in P_2(\mathcal{M})$ of degree $2^i 3$, $i > 0$. Put

$$E = P_3(\mathcal{M}) \cap \{B : \theta_{B,3'} = 1\},$$

where $3'$ denotes the entire prime-to-three factor. The only invertible in E is the vacuum, since $\theta_g = -1$. Evaluation at g makes every label of $P_3(\mathcal{M})$ even. The projector onto E commutes with the local ternary modular group, as do its two factors. Block orthogonality at unequal ternary phases and the odd-valuation mass identity hold on E by Lemma 3.2, Proposition 3.8 and the proof of Proposition 4.2.

An odd-degree nonunit $B \in E$ has degree $3r^j$ with $j > 0$, by the exclusion of prime powers and the principal height bound at valuation five. The number $z = S_{UB}/\operatorname{lcm}(d_U, d_B)$ has weight three, residue one above two by $U \in P_2$, and residue zero above three by $B \in P_3$. The weight-three argument of Lemma 8.25 gives $z = \lambda\sqrt{-3}$ with λ a dyadic root of unity. To justify its use in the full extension, the cyclotomic calculation requires only the three level primes and strict house. Its remaining difference-set step uses the adjoints of B and its Galois translate, evaluated at U. Both fusion labels are even, and tensoring by g switches the sign of their twists, so neither BB^* nor $B^\sigma(B^\sigma)^*$ contains g. The vacuum is their only possible invertible constituent. All noninvertible constituents satisfy the strictness just proved. These are the two uses of simplicity in Lemma 4.15.

Ternary inversion preserves E by square covariance and negates these nonzero crossings. Proposition 8.26 on E now excludes U, since $v_3(\dim \mathcal{M}) = 5$.

Every r-free noninvertible $U \in P_2(\mathcal{M})$ consequently has $v_3(d_U) = 2$. Indeed its degree has both primes $2, 3$ and $d_U^2 \mid \dim \mathcal{M}$. We need a full r-principal even label V of odd degree with $v_3(d_V) = 2$. Take a 2-free positive nonunit from P_r and lift it as above. If its degree were $3r^j$, compare it in the positive table with an r-free nonunit of P_3, necessarily of degree $3 \cdot 2^k$. The normalized crossing has weight 3 and residue 3 above both 3 and r. Indeed if R is the P_3 witness the entry is $\chi_R(V)/r^j$, whose denominator is a unit at three. The equality case of Theorem 4.9 gives modulus three, contrary to strictness.

Both V and the r-free noninvertibles of $P_2(\mathcal{M})$ have ternary defect one. Lemma 3.10 supplies a common visible primitive phase η.

In every even pair J, gJ at least one representative has nontrivial dyadic twist. For it, orthogonality to the vacuum on the full P_2 yields an r-free noninvertible hit, since modulo r the invertibles $1, g$ together contribute two. Its row sees only $1, \eta$ at three. The two representatives have the same ternary twist, so every positive label has ternary phase 1 or η. This contradicts Lemma 8.27 in the positive table. □

Proof of Theorem 8.1. If the quartic is below a larger repeated prime, Theorem 7.3 applies. For a largest quartic and a least quintic use Theorem 8.3. The remaining ordering has a middle quintic. In odd dimension Theorem 8.20 applies at exponent seven. For a dyadic least prime and $q \geq 5$, use Theorem 7.11 with $2^{\lfloor 7/2 \rfloor - 1} = 4 < q$: rounding down the middle cap leaves a quartic below the largest prime, covered by Theorem 7.3. For $q = 3$, Theorem 8.31 covers dyadic exponents at most six. At exponent seven Theorem 8.28 excludes the ordinary table by dyadic parity, and Proposition 8.35 excludes the fermion case. Each reduction allows divisors and the coprime square-free factor. □

8.9 A principal-block criterion for individual categories

Fix a place over an odd prime r in a common splitting field. Write P_r for the principal character block there, using the positive quotient in the slightly degenerate case.

Proposition 8.36 (Principal blocks under gauging of order prime to r)**.** *Let $\mathcal{C} = \mathcal{E}^G$ be an integral modular or slightly degenerate gauging of $\mathcal{D} = \mathcal{E}_e$, with $r \nmid |G|$ and $\mathrm{Rep}(G) \cap \mathcal{C}' = \mathrm{Vec}$. Every member of $P_r(\mathcal{C})$ has neutral flux. The G-orbits supporting at least one member of this block are exactly the orbits in $P_r(\mathcal{D})$. In particular one block contains a degree divisible by r if and only if the other does.*

Proof. The equivariant braiding on a homogeneous object of grade g acts on a pure charge π by its equivariant structure at g [26, Section 4.4.4 and proof of Proposition 4.56(i)]. Thus evaluation of a charge at that flux gives its ordinary character at g, up to inversion. Since the character table of the r'-group G is invertible at r, a principal simple has neutral flux. For a neutral simple Z over U and neutral equivariant object W we have $\chi_Z(W) = \chi_U(FW)$, where F forgets equivariance. The terms in the underlying orbit give equal normalized traces because FW is invariant. Every simple in $\mathcal{E}$ occurs under an equivariant simple, whose degree is its degree times an r-unit [13, Corollary 2.13]. In fact its orbit size divides $|G|$, as does the stabilizer order, and a projective irrep of the stabilizer lifts to an irrep of an r'-central extension. In particular its degree has no factor r, since irreducible degrees divide the group order. The integral block idempotent of $P_r(\mathcal{D})$ is G-invariant and lifts to a neutral element of the local Grothendieck algebra by $|G|^{-1}\,\mathrm{Ind}$. Evaluation shows that a principal Z lies over $P_r(\mathcal{D})$.

Conversely, note first that every homogeneous simple has integer degree: an equivariant lift has integer degree and its ratio to the original degree is an integer, so the latter is a rational algebraic integer. Put $K = \{g : \mathcal{E}_g$ contains a simple of r-unit degree$\}$. Products, duals and conjugation make K a normal subgroup. For a pure charge π, its row at a simple of flux g is $d_W \,\mathrm{tr}_\pi(g)/\dim \pi$ up to inversion. Thus π is principal exactly when its character is congruent to $\dim \pi$ on K. Taking scalar products with nontrivial irreducibles of K shows their multiplicity in $\pi|_K$ would be divisible by r. By Clifford theory any occurring multiplicity divides $\dim \pi$, an r-unit. Hence the principal charges are exactly $\mathrm{Irr}(G/K)$.

Let F_0 forget equivariance and then project to the neutral grade; it is an integral linear map. Fix any neutral lift Z' above U. For the primitive character idempotents e_Z, orthogonality gives

$$\chi_U(F_0(e_Z)) = \frac{d_Z}{|G| d_{Z'}} \sum_{\pi \in \mathrm{Irr}(G)} \dim(\pi) N_{\pi Z'}^{Z}.$$

Indeed the left side is $f_Z^{-1} \sum_{W \text{ neutral}} \overline{\chi_Z(W)} \chi_{Z'}(W)$. The indicator of neutral flux at W is the regular charge character $|G|^{-1} \sum_\pi \dim(\pi) \chi_W(\pi)$. Use multiplicativity of χ_W and symmetry of S.

The displayed sum is zero unless Z lies above the orbit of U. For $c = F_0(\sum_{Z \in P_r(\mathcal{C})} e_Z)$, evaluating at the unit gives $\chi_{\mathbf{1}}(c) = |G|^{-1} \sum_{\pi \in \mathrm{Irr}(G/K)} (\dim \pi)^2 = |G/K|/|G|$, an r-unit. If $U \in P_r(\mathcal{D})$ its value on the r-integral element c is congruent to that same unit, proving some lift principal. In the slightly degenerate case take the quotient by the transparent fermion in both rings. The fermion belongs to the neutral sector and its forgetful image is the transparent fermion of $\mathcal{D}$, so F_0 descends to the based quotients. Averaged induction commutes with this quotient. All sums and orthogonality identities above are now on fermion pairs. □

For passage from an integral modular category to the centralizer of a fermion, each positive principal pair has a full principal lift at r by the same sign argument (proof of Theorem 7.3).

Theorem 8.37 (Recognition on the original principal block)**.** *Let $\mathcal{C}$ be integral modular or slightly degenerate, of dimension $p^a q^b r^4$, with distinct p, q, r and r odd. If all degrees in $P_r(\mathcal{C})$ are prime to r, then $\mathcal{C}$ is weakly group-theoretical. The same conclusion holds with dimension dividing this number.*

Proof. Induct on dimension, using the known result at r-valuation at most three. A Tannakian $\mathrm{Rep}(G)$ with $r \mid |G|$ suffices by de-equivariantization, since its neutral dimension has lower r-valuation and weak group-theoreticity passes back. If $r \nmid |G|$, Proposition 8.36 applies directly. The smaller neutral component is integral, since a lift of each simple has dimension an integral multiple of its degree; its center is trivial or the image of the same transparent fermion. We may therefore assume that there is no nontrivial Tannakian subcategory.

A nontrivial proper nondegenerate subcategory splits off. The principal blocks of a tensor product are exactly the products of the principal blocks, so induction applies to both factors. If $\mathcal{C}$ is modular and has a fermion, pass by the preceding even-sector argument to $\langle f \rangle'$ and use the faithful two-grading. An ordinary category still remaining has no proper nontrivial subcategory: its center would be either trivial, Tannakian, or contain a fermion, by the classification of symmetric fusion categories. The pointed case is immediate; otherwise it is perfect and fusion-simple.

If $\mathcal{C}$ is slightly degenerate, any proper subcategory strictly between sVec and $\mathcal{C}$ is itself nondegenerate over sVec, again by the absence of Tannakian subcategories. It splits off with its centralizer in the tensor product relative to sVec: the usual centralizer dimension formula gives the factorization. The positive based ring is the product of the two positive based rings (the simples of the relative product are the simultaneous fermion orbits of pairs). Induction applies to these smaller factors; weak group-theoreticity passes through their product and condensation of the diagonal boson. With no such intermediate subcategory, the positive table is simple, and either pointed or perfect. In each remaining perfect fusion-simple table at the three dimension primes, Lemma 4.10 forces an r-bearing degree in P_r, contrary to hypothesis.

□

8.10 Equality in the multiplier criterion

Let $W(r, m+1)$ have the meaning of Lemma 4.16: a nonunit at a nonunit row of an odd ordinary minimal table cannot have a full positive r-primary word on at most $m+1$ positions, up to a common root of unity. Put $E = m(m+1)/2$, $H = (r-1)/2$ for an odd prime r, and assume $1 + 2E < 2r/3$. Failure of W requires an integer $h \le E$, $h > 0$, satisfying

$$h(h-1) \ge (H-1)k, \qquad k = \min\{j \ge 1 : j(j+1) \ge h\}.$$

Equality imposes two more restrictions.

Proposition 8.38. *At equality $h(h-1) = (H-1)k$, the following hold.*

1. *If $k \ge 2$, then $h < k(k+1)$.*

2. *Write $n = h - k$. For every $b > 1$ dividing H and every prime $\ell \nmid b$ for which -1 is a power of ℓ modulo b, $v_\ell(n)$ is even.*

Proof. Use the minimizing word and its h opposite difference pairs A from Lemma 4.16. The tame multiplier group modulo sign is cyclic of order H, acts freely on the pairs, and satisfies $|A \cap uA| \geq k$ for $u \neq 1$. Equality in the count forces every nonidentity overlap to equal k and all pairs to lie in one free orbit: pairs from different orbits would contribute to no overlap. Identify the orbit with the group G. Then A is a cyclic (H, h, k) difference set.

Suppose $h = k(k+1)$ and fix $u \neq 1$. Let $B = A \cap uA$. The common word on $\{0\} \cup \{\pm b : b \in B\}$ has at least h differences. The upper bound is exactly $k(k+1) = h$, with the possible differences given by $b, 2b$ and $b \pm b'$ for different members of B. Thus all these pairs are distinct and are realized; write A' for the resulting h-pair mask. It too is minimizing and lies in a single free orbit. Since it contains B, that orbit is the original one. Comparing the two words yields $c_g = |A' \cap gA| \geq k$ for every $g \in G$. The difference-set equations and double counting give

$$\sum_g (c_g - k) = h - k, \qquad \sum_g (c_g - k)^2 = (h - k)^2.$$

Indeed $\sum_g c_g = h^2$ and $\sum_g c_g(c_g - 1) = kh(h-1)$. All excesses are nonnegative, so one equals $h - k$. In particular $A' = gA$ for some g.

Both B and $2B$ lie in A'. Multiplication by two on the original free orbit is consequently one tame multiplier v: at one pair it agrees up to sign modulo its order with a tame unit, and every pair on this orbit has the same order. Here $v \neq 1$ since $r > 3$. Therefore $B \subseteq A' \cap v^{-1}A'$ and equality follows from the common cardinality k. Every intersection of two distinct translates of A is consequently a translate of $A \cap v^{-1}A$.

The H translates of A are the blocks of a symmetric design with H points and H blocks, and any two points lie in exactly k blocks. Let t be the stabilizer size of a block intersection and s the number of blocks containing that intersection. All intersections form one orbit of length H/t. Two blocks containing the intersection meet exactly in it. Counting ordered pairs of distinct blocks gives $s(s-1) = t(H-1)$. But two points of the intersection give $s \leq k$, while $H - 1 = h(h-1)/k > k(k-1)$. This proves the first assertion.

For the second, choose a character χ of G of order b and put $\alpha = \sum_{a \in A} \chi(a)$. The difference-set equations give $\alpha\overline{\alpha} = n$. In $\mathbb{Q}(\zeta_b)$, ℓ is unramified and the decomposition group of each prime over ℓ contains the Frobenius ℓ. Thus it contains complex conjugation. At that prime the valuations of $\alpha, \overline{\alpha}$ agree and the valuation of n equals $v_\ell(n)$. □

9 Two squared primes and their local columns

Theorem 9.1 (Heights at the two higher primes)**.** *Let $p < q$ be the two primes with exponent caps A, B and let u, v be two other primes. Put $h = \lfloor A/2 \rfloor$, $k = \lfloor B/2 \rfloor$. If a divisor of $p^A q^B (uv)^2 d$ supports a braided counterexample to WGT, where d is coprime and square-free, then*

$$p < \min(u, v), \qquad h, k \geq 2, \qquad h + k \geq 6, \qquad h \geq 3 \text{ if } p \text{ is odd.}$$

In particular $p^7 q^5 (uv)^2 d$ is covered in either order of the two higher primes. When $p < q$ are odd, $p^5 q^B (uv)^2 d$ is covered for every B.

The numbers h, k bound the exponents of p, q in a simple degree. In an odd minimal root a twist involving both p and q forces its two heights to lie in the rectangle $\mathcal{H}(h, k) = \{(i, j) \in \mathbb{Z}^2 : 0 < i < h,\ 0 < j < k\}$. For $h, k \geq 2$ the inequality $h + k \geq 6$ says that this rectangle has at least three lattice points. For even roots, the local columns lead to the finite trace calculation in Theorem 9.15. Section 10 excludes an odd least prime of height two with any companion height. The only remaining root needed for the theorem is the odd root $p^3 q^2 uv$, treated in Section 12. Section 17 then obtains the four dimension cutoffs by comparing the remaining profiles.

Work in a minimal table as in Lemma 2.1. We determine which primes are isolated, then apply the simultaneous column identities of Proposition 6.3. We write P_t for the principal t-block and $[x]_t$ for the centered residue at an odd prime.

Lemma 9.2 (Three pair degrees)**.** *In an ordinary table let a, b, c be distinct primes of valuation two in D, and suppose the degrees ab, ac, bc all occur. On the union of the a-, b- and c-active labels the ℓ-primary twist is one for every other prime ℓ.*

Proof. A nonunit in P_ℓ must hit ab, ac, bc by principality, hence has at least two of their three primes by coprime vanishing. A b-active and a c-active column therefore share only the vacuum hit on P_ℓ, by (3.1). Their ℓ-primary phases agree by block orthogonality. All three sectors are nonempty by Cauchy, so cycling the pair gives a common phase. It is one: the complementary Weyl W' keeps e_0 in the vacuum U_ℓ-eigenspace. Its image has a nonzero b-active projection since W_ℓ preserves that sector and $W_\ell W' e_0 = s e_0$ has full support. □

Proposition 9.3 (Isolation at two squared primes)**.** *Let an ordinary minimal table have dimension $p^A q^B u^2 v^2$, $A, B \geq 2$, with four distinct primes. Both u, v are isolated and at least five, and the dimension is a square n^2. At $t \in \{u, v\}$ all active labels have degree n/t. Writing $w = uv/t$, both a degree $p^i w$ and a degree $q^j w$ occur, with $i, j > 0$. Moreover every p-free nonunit of P_q is divisible by uv, and such a label exists. Every q-active label has degree divisible by puv, and conversely every p-active label has degree divisible by quv.*

Proof. A t-active column hits only t-free degrees. Reducing its orthogonality to the vacuum modulo p or q, and excluding nonunit prime-power degrees, gives the two asserted degrees. A p-free nonunit of P_q must hit both $p^i u$ and $p^{i'} v$, hence is divisible by uv. Orthogonality on P_q of a q-active column modulo p supplies such a label. If the column were also t-active, the required hit would have t-unit codegree, a contradiction. This separates u, v from p, q. A column active at both u, v has all nonunit hits divisible by pq, contrary to vacuum orthogonality modulo p.

Complementary localization now gives codegree t^2 on the t-active labels; indeed its only prime is t and its positive valuation is $2 - 2v_t(d)$. This gives even A, B and degree n/t. Lemma 6.6 gives $t \geq 5$.

Compare a q-active column Z with a u-active column on P_q. Their q-phases differ. Any nonunit common hit is u-free by defect zero and hence divisible by p, by the preceding description of p-free nonunits in P_q. If $p \nmid d_Z$, lcm divisibility makes every nonunit contribution vanish modulo p, contradicting the vacuum term one. Complementary localization gives $uv \mid d_Z$. Interchange p, q for the other assertion. □

If three positions are squared, applying the proposition to each pair isolates all four primes and gives all three pair degrees.

Lemma 9.4 (Two squared primes in the reduced table)**.** *A slightly degenerate minimal counterexample cannot have dimension $2^a q^b u^2 v^2$, with q, u, v distinct odd primes, $a \geq 1$, $b \geq 2$.*

Proof. In the reduced table choose a u-active column by Proposition 3.4. Its nonunit hits are u-free; orthogonality modulo two gives an odd one, of degree $q^i v$, $i > 0$. Likewise $q^j u$ occurs. Lift both to the integral minimal extension with fermion g. A noninvertible q-free label of its principal two-block hits both lifts, hence has $uv \mid d_X$ by coprime vanishing. Take a u-active even label Y, choosing between Y, gY to make its two-primary phase nontrivial. On the full principal two-block its vacuum orthogonality modulo q has no noninvertible contribution: a q-free one has u-unit codegree. The two invertibles contribute one each, contradicting $q \nmid 2$. □

For four surviving repeated primes, this excludes the reduced case with two specified squared source positions: a squared two would leave two-valuation one in the reduced dimension, and a square-free two leaves an odd reduced dimension (Lemma 2.5 and Proposition 4.7).

9.1 The two singleton columns

For the rest of this section, a *common* label is a nonunit whose degree omits both specified singletons. For four primes write $n = Cuv$, $C = p^h q^k$. We call the degrees $p^i w, q^j w$ of Proposition 9.3 the *axes at* w.

Lemma 9.5 (Inertia from principal residues)**.** *Let $z = S_{XY}/\operatorname{lcm}(d_X, d_Y)$ be a normalized entry on two nonunits, $g = \gcd(d_X, d_Y)$, m the least dimension prime, and $b = m/(m-1)$. Suppose $z \equiv g$ at every prime above t, as occurs for $Y \in P_t$ and $t \nmid d_X$. If e is the ramification index of $\mathbb{Q}(z)$ at t, then $t \le beg$. More generally $st \le beg$ when $z \equiv g$ also above a distinct prime s unramified in $\mathbb{Q}(z)$. The t-inertia orbit of each label has length at least*

$$L(t,g) = \begin{cases} (t-1)/\max\{e : e \mid t-1,\ e \le g\}, & t > 2g, \\ 1, & t \le 2g. \end{cases}$$

Proof. Take $s = 1$ for the first assertion. The sum w of the inertia translates of z has $(w - eg)/(st)$ integral: $(z-g)/s$ is integral, and its inertia trace vanishes at every prime over t in the unramified fixed field. If $st > eg$, strict house $|z| < g$ at every embedding gives $|1 + (w-eg)/(st)| < 1$ at every embedding, so $w = eg - st$. The balancing expression for z has positive rational weights summing to g. By Cauchy its roots have only dimension primes in their orders, so their Ramanujan means are at least $-1/(m-1)$. Thus $eg - st \ge -eg/(m-1)$. For $st \le eg$ the bound is immediate.

For $t > 2g$ the single-residue case of Theorem 4.9 gives a unique unsigned word of g roots of t-power order. The tame stabilizer permutes its terms, freely on every nontrivial root; at least one is present by strictness. Its order divides $t-1$ and is at most g; both character fields contain z. □

For a four-prime root with $h, k \ge 1$, write I for the nonunit labels of degrees prime to uv. Each such degree is $d = p^i q^j$, $i, j > 0$, and its field is unramified at one of u, v by Proposition 6.2. At that endpoint t, the opposite axes are t-principal, and their gcds with d are at most p^i, q^j. Lemma 9.5 yields the first bound below. The unramified estimate in Lemma 6.5 gives the second:

$$t \le \frac{m}{m-1} \min(p^i, q^j), \qquad (t-1) j_t [d]_t^2 < t^2. \tag{9.1}$$

We use a_t, j_t, B_t as in Proposition 6.3.

Proposition 9.6 (A congruence obstruction on the two repeated prime powers)**.** *Let $n = Crs$, $C = xy$, with $x = p^A, y = q^B$ positive powers of two distinct primes and r, s two further odd primes. Suppose r, s are isolated in an ordinary perfect fusion-simple modular table of dimension n^2. If*

$$C \equiv \pm 1 \pmod{rs}, \qquad H := (1 - x^{-1})^2 + (1 - y^{-1})^2 > 1 + C^{-1},$$

no such table exists. For a center and $C \equiv 1 \pmod{rs}$ the weaker hypothesis $H \ge 1$ suffices. In particular the obstruction applies whenever $A = B = 2$ and $C \equiv \pm 1 \pmod{rs}$.

Proof. Put $m = sj_r + rj_s$ and let ρ be the least positive residue of ϵC^{-1} modulo rs. By the two-column identities, $m \equiv -\rho \pmod{rs}$ and $1 < m \le rs + 1/C$, so $m = rs - \rho$. The complementary Weyl column $W_C e_0$ has vacuum entry $1/n + \epsilon m/(rs)$. Its coordinate on an axis $p^i s$ has modulus $r^{-1}|p^i/C - a_r| \ge (1 - 1/y)/r$, since a_r is a nonzero integer. Summing over the four distinct axes gives

$$H(r^2 + s^2) \le (\rho - \epsilon/C)(2rs - \rho + \epsilon/C).$$

If $C \equiv \pm 1$, then $m > 1$ forces $\rho = 1$ and the right side is strictly below $2rs(1 + C^{-1})$ (below $2rs$ for a center, where $\epsilon = 1$). Use $r^2 + s^2 \ge 2rs$. For $A = B = 2$, $H \ge (1 - 1/4)^2 + (1 - 1/9)^2 > 1 + 1/36 \ge 1 + 1/C$. □

Proposition 9.7 (Bounds for the other three primes)**.** *Under the hypotheses and notation of the two-column proposition, let $u < v$, $L = \min(p^h, q^k)$ and $b = m/(m-1)$. Then*

$$u \le bL, \qquad v \le bLB_u, \qquad \max(p,q) \le b\min(vB_u, uB_v), \qquad \min(\max(p,q), v) \le bu. \tag{9.2}$$

In particular the least prime is one of p, q. For the family with $h, k \le 2$, specifying any one of its four primes leaves only finitely many choices of the other three in an ordinary minimal table.

Proof. The mixed identity makes I nonempty; (9.1) gives $u \le bL$. The comparison of a common label at v with the axis of smaller prime-power part has gcd at most L and inertia length at most $|I| \le B_u$. Lemma 9.5 gives $v \le bLB_u$.

Compare a $p^i v$ axis with a p-free label in P_q; by Proposition 9.3 the latter has degree $q^a uv$, $a \ge 0$. The gcd is v, and every q-inertia translate of the axis spends a unit of B_u (its degree differs from n/u). This gives $q \le bvB_u$ and, by both interchanges, the third bound. An axis $p^i u$ compares both with $q^j u \in P_v$ and with $q^a uv \in P_q$. Both gcds equal u, so if $q, v > bu$ its field would ramify at v and q, contrary to isolation. This gives the last bound by symmetry. If $u = m$, every other prime is at least $u + 2 > bu$, impossible.

A prescribed repeated prime with bounded exponent bounds L, then u, v and the other prime by (9.2). A prescribed v bounds u and both other primes. For fixed u, either $v \le 2u$ or both repeated primes are at most $2u$; bounded exponents then bound L and v. □

Lemma 9.8 (Multiplicity at an isolated odd singleton)**.** *Suppose an ordinary minimal table has odd root n and an isolated prime $t \ge 5$ of root-valuation one. If H distinct proper nonunit t-free degrees occur, then*

$$H \min\{2j_t^2,\, 2^{v_2(t-1)}\} \;\le\; B_t = j_t(t+1-j_t).$$

In particular, no isolated singleton of an odd minimal square root can be five.

Proof. An unramified label and its distinct dual cost at least $2j_t^2$ by Lemma 6.5. A ramified label has a pure t-power field by isolation and Proposition 6.3. Its field is nonreal since the dual is distinct. Its stabilizer in the cyclic cyclotomic Galois group omits the unique involution, so its orbit size is a multiple of $2^{v_2(t-1)}$. Every translate costs at least one in squared norm. Sum these bounds on the H proper degrees. At $t = 5$ Lemma 6.6 gives $H \ge 3$. Both even deficits $j_t = 2, 4$ would require twelve units, exceeding $B_t = 8$. □

Lemma 9.9 (A unique complementary degree)**.** *Let $u, v \ge 5$ be distinct isolated odd singletons in an ordinary minimal table of square root n. Suppose $D = n/(uv)$ is its only nonunit degree prime to uv, and the signed coordinates of these labels are centered residues at u, v. If $u < v$ then $D \equiv \pm 1 \pmod v$. If its multiplicity M exceeds one, then $M = j_u$, $v = uj_v \pm 1$, and all common labels are unramified at v. For odd n the centering and multiplicity conditions are automatic.*

Proof. For the moment name the endpoints t, w without ordering. The signed entries are the centered inverses $A = [w^{-1}]_t$, $B = [t^{-1}]_w$. The integer $wA + tB$ is one modulo tw and has absolute value at most $tw - (t+w)/2$, so $wA + tB = 1$. The mixed column identity gives $M|AB| = j_t j_w$. If $M = 1$ the unique label is Galois-fixed; an active row at each endpoint gives $1 + [D]_t[D]_w = 0$ by orthogonality and Lemma 6.5.

Assume $M > 1$. At most one endpoint ramifies at each label. At an unramified endpoint its deficit divides its signed coordinate; both $j_t \mid A$ and $j_w \mid B$ would force $M \le 1$. Thus one endpoint, say w, is unramified at every common label. A t-active row Z and a w-active row now give $1 + [D]_w \sum_{d_X = D} \chi_Z(X) = 0$. Integrality gives $[D]_w = \pm 1$, hence $|B| = j_w$ and $j_t = M|A|$. Reducing at t gives $MD \equiv \pm 1 \pmod t$.

Since $M \mid j_t$ and $j_t \mid t-1$ or $t+1$ by Lemma 6.5, $M \mid t \pm 1$ and the centered residue of D has magnitude at least $(t-1)/M$. If $|A| \ge 2$, the active rows at t charge at least

$$|A|(t-1)(t-(t-1)/M)/t^2 \;\ge\; 1 - 1/t^2$$

to the D-column norm. At w, $|[D]_w| = 1$ charges at least $j_w(w-1)/w^2$. This is greater than $1/t^2$ since $tj_w = |1 - wA| \geq 2w - 1$. The two sets of active rows are disjoint.

Thus $|A| = 1$, $M = j_t$, and $w = tj_w \pm 1$. Parity gives $j_w \geq 2$ even and $w > t$. For odd n, a common label and its distinct dual share their signed coordinate by rationality; its square is at most $B_t/2$, which forces centering at each endpoint. □

Corollary 9.10 (Three omitted primes)**.** *Let $u < v < w$ be isolated odd singletons at least five. If for both $t = u, v$ the only common nonunit degree at t, w is $n/(tw)$, and both signed coordinates are centered, the table cannot exist.*

Proof. Put $C = n/(uvw)$. The preceding lemma gives $Cu, Cv \equiv \pm 1 \bmod w$, hence $u \equiv \pm v \bmod w$, impossible for distinct odd primes $u < v < w$. □

Proposition 9.11 (Full valuations under isolation)**.** *Let $n = p^h q^k uv$, $h, k \geq 1$, and suppose all four primes are isolated. Every occurring degree $p^i t$ or $q^j t$ at $t = u, v$ has $i = h$ or $j = k$, respectively. If uv occurs, then all common degrees are $C = p^h q^k$. If n is odd and $k = 2$, the degree uv occurs.*

Proof. By Proposition 6.2, full isolation puts each label in the principal block of every prime at which its degree is deficient.

An axis $p^i t$ with $i < h$ would have to hit the coprime opposite axis $q^j w$. A deficient common degree would likewise have to hit uv if it occurs.

For n odd and $k = 2$, suppose uv were absent. A p-free nonunit of P_q is divisible by uv by Proposition 9.3. Defect zero excludes q^2, leaving quv. Together with its distinct dual it gives, by Lemma 6.8, a q-active row vanishing on every q-bearing degree. Its vacuum orthogonality modulo p would require a nonunit hit of degree uv. □

Proposition 9.12 (One unrestricted prime and three squares)**.** *For any four distinct primes ℓ, a, b, c, every weakly integral braided fusion category whose square-full part divides $\ell^A(abc)^2$ is weakly group-theoretical.*

Proof. Descend over source divisors. Losing a repeated prime leaves Theorem 4.12 or Dong's theorem. With all four retained, the reduced case was excluded above. In the ordinary table Proposition 9.3 leaves $n = \ell^h abc$ with four isolated primes, a, b, c odd, and degrees ab, bc, ca present. For the pair b, c, the preceding proposition applies with higher powers ℓ^h, a: the degree bc occurs, so the only common degree is $n/(bc)$, and it occurs by (6.3). The same holds for the other pairs. The other proper coordinates force centering since $B_t \leq (t+1)^2/4$. Apply Corollary 9.10. □

9.2 Trace and norm tests for two higher powers

The two Weyl columns have prescribed norms and scalar product. Keeping the two axis sets separate gives congruences in addition to the norm bounds.

Lemma 9.13 (Trace and Gram constraints on degrees)**.** *For a root $n = p^h q^k uv$ as above, let m_d be the multiplicity of each degree d and $A_t(d) = [d/(n/t)]_t$ for $t = u, v$. Put $w = uv/t$. For $b = p, q$ set $D_t(b) = \{b^i w : 1 \leq i \leq v_b(n)\}$, and put $G = \{p^i q^l : 1 \leq i \leq h, 1 \leq l \leq k\}$. The two axis sets at t each contain an occurring degree. One has*

$$\sum_{d \in D_t(b)} m_d d A_t(d) \equiv \epsilon j_t \pmod{pq/b},$$

$$\sum_{d \in D_t(p) \cup D_t(q) \cup G} m_d A_t(d)^2 \leq B_t, \qquad \sum_{x \in G} m_x A_u(x) A_v(x) = -j_u j_v. \tag{9.3}$$

An occurring degree has $j_t k(t-k) < t^2$, $k = |[d]_t|$. An occurring $x = p^i q^l \in G$ has an unramified endpoint t satisfying (9.1) and $A_t(x) = -\epsilon j_t [x]_t$; if $m_x = 1$ both endpoints do.

For finer multiplicity bounds, write $\mu = 2$ for odd n, $\mu = 1$ otherwise. An occurring axis $d = b^i w$ has $m_d \geq \max(\mu, L(t,w), L(pq/b, w))$. An occurring common degree $x = p^i q^l$ has $m_x \geq \max(\mu, L(u,g), L(v,g))$ at $g = \min(p^i, q^l)$. Any degree occurring once has unramified field.

Proof. The two axes and a common degree all occur by Proposition 9.3 and the mixed identity (6.3). Each shares the norm $B_t \leq (t+1)^2/4$ with the other two, so its signed entry must be the centered residue $A_t(d)$. The pq/b-free nonunit degrees in the t-column are precisely the occurring members of $D_t(b)$. Reduce the dimension trace in (6.3) at pq/b; the other two assertions in (9.3) are the single and mixed column norms. Lemma 6.5 gives the energy and unramified tests. Finally an axis $b^i w$ has gcd w with both the opposite t-free axis and a b-free label of $P_{pq/b}$ as supplied by Proposition 9.3. A common label compares at each endpoint with an axis of gcd at most g. Apply Lemma 9.5 and in odd dimension count distinct duals. □

Lemma 9.14 (Two trace divisors)**.** *For $h, k \geq 2$ in the preceding setting take $u < v$ and $U \geq 2$ such that $U \geq \lfloor (u+1)/j_u \rfloor - 1$. One can always take $U = u$, and for odd n one can take $U = (u-1)/2$ since j_u is even and $u \geq 5$. For a finite set of positive integers E prime to u, define the test $\mathcal{T}_{u,U}(P, a, E)$ by*

$$P \mid 1 + \sum_{d \in E} m_d \, d[d]_u, \qquad \sum_{d \in E} m_d [d]_u^2 \leq U, \qquad m_d \in \{0\} \cup \{L(P,a), L(P,a)+1, \ldots\}. \tag{9.4}$$

Here the m_d are nonnegative integers to be chosen. Then

$$\begin{aligned} q > 2v \quad &\Longrightarrow \quad \mathcal{T}_{u,U}(q, v, \{p^i v : 1 \leq i \leq h\}), \\ v > 2p^h \quad &\Longrightarrow \quad \mathcal{T}_{u,U}(v, p^h, \{p^i q^j : 1 \leq i \leq h,\ 1 \leq j \leq k,\ u \leq 2\min(p^i, q^j)\}). \end{aligned}$$

In particular $u \leq 2p^h$, $v \leq p^h U + 1$, $q \leq vU + 1$, and $q \leq 2u$ when $v > 2u$.

Proof. Both tests are the signed dimension trace at u, reduced at the omitted prime P. Only its P-free nonunit terms survive. For $P = q > 2v$ these are the axes $p^i v$: comparison with a q-principal p-free label of degree $q^j uv$ has gcd v. For $P = v > 2p^h$ these are the common degrees; comparison with a v-free p-axis has gcd at most p^h. In either case positive multiplicity forces P-ramification by Lemma 9.5, hence u-unramifiedness by isolation. Thus $a_u = -\epsilon j_u [d]_u$, and in the second case $u \leq 2\min(p^i, q^j)$ by (9.1). Cancel $j_u < u < P$ in the trace. The inertia-orbit bound and the unramified rank bound of Lemma 6.5 give (9.4). Some multiplicity is positive, whence $L(P,a) \leq U$ and $P \leq aU + 1$ (also if $P \leq 2a$). The other bounds follow from (9.1) and (9.2). □

Theorem 9.15 (Total height at two higher primes)**.** *In an ordinary minimal table of dimension $(2^h q^k uv)^2$, where q, u, v are distinct odd primes and $h, k \geq 2$, either $h + k > 6$ or $h = k = 3$.*

Proof. Order $u < v$, put $C = 2^h q^k$, $n = Cuv$, and use Lemma 9.13. At $t = u, v$ call d admissible when $j_t |[d]_t| (t - |[d]_t|) < t^2$. Write $F_t(d)$ if $A_t(d) = -\epsilon j_t [d]_t$ and $(t-1) j_t [d]_t^2 < t^2$. The primes and signs to compare come from Lemma 9.14:

$$5 \leq u \leq 2^{h+1}, \quad u < v \leq 2^h U + 1, \quad 3 \leq q \leq vU + 1, \qquad U = \max(2, \lfloor (u+1)/j_u \rfloor - 1),$$

with $q \leq 2u$ if $v > 2u$. For $t = u, v$ use $1 \leq j_t < t$, $j_t \mid t-1$ or $j_t \mid t+1$, $C(uv/t) j_t \equiv -\epsilon \pmod t$, and $0 < n - \epsilon - C(v j_u + u j_v) < n$.

For $a = 2, q$ and $w = uv/t$ choose an admissible axis $d = a^i w$. When present it costs at least $H_t(d) A_t(d)^2$, where

$$H_t(d) = \max\{L(t,w), L(2q/a, w), 1 + \mathbf{1}_{\neg F_t(d)}\}.$$

Write $c_t(a)$ for the smallest such cost (infinite for an empty axis). The negative mixed sum requires some common degree $x = 2^i q^l$ admissible at both ends with $A_u(x)A_v(x) < 0$. Put $g = \min(2^i, q^l)$ and count the endpoints with $t \le 2g$ and $F_t(x)$. This count f is positive. The multiplicity is at least $M = \max\{L(u,g), L(v,g), 3 - f\}$, hence at each endpoint

$$MA_t(x)^2 + c_t(2) + c_t(q) \le B_t.$$

The table gives all surviving choices. The vacuum counts are 12, 14, 92, 18 and 365 at heights $(2,2)$, $(2,3)$, $(3,2)$, $(2,4)$ and $(4,2)$, respectively. To obtain the rows, run $1 \le i \le h$, $1 \le l \le k$ in the last inequality for each vacuum class. At the indicated t the 2-axis has budget $K = B_t - MA_t(x)^2 - c_t(q)$. The final column contains all its nonempty trace residues modulo q within K.

h	k	ϵq	u	v	j_u	j_v	(i,l)	t	K	residues
3	2	-13	7	11	2	3	$(3,2)$	11	10	$9, 12$
4	2	3	5	11	1	2	$(4,2)$	11	8	1
4	2	11	5	7	2	1	$(2,1), (2,2)$	5	2	$4, 10$
4	2	11	7	13	2	2	$(4,2)$	13	12	$4, 5, 10$
4	2	13	11	17	1	3	$(4,2)$	11	6	$2, 4, 5, 7, 10, 12$
4	2	-41	13	17	3	2	$(4,2)$	13	16	15
4	2	23	13	17	3	2	$(4,2)$	13	15	$11, 20, 21$

Indeed each positive multiplicity m_d costs $m_d A_t(d)^2$ and contributes $m_d d A_t(d)$ to the trace. Adjoining one admissible $d = 2^i w$ at a time therefore adds these pairs for $m_d = 0$ or $H_t(d) \le m_d \le K/A_t(d)^2$, discarding total costs above K. For instance, in the first row only two copies of 14 or one of 56 fit. In table order the required residues $\epsilon j_t \bmod q$ are $10, 2, 2, 2, 1, 38, 3$; each is absent. The enumeration produces these seven terminal rows. Their trace residues can also be checked independently. □

For other heights, the general trace and Gram calculation uses the same additions to impose all of (9.3). At each axis retain the cheapest nonempty choice in the required trace class and subtract its cost. For each common degree add $m_x(A_u(x)^2, A_v(x)^2, A_u(x)A_v(x))$ to the two costs and the mixed sum, retaining costs within both remaining budgets. The final mixed sum must equal $-j_u j_v$. At an odd least prime p use even multiplicities and $t \le (p/(p-1))g$ in the unramified test.

10 Mixed twists and intermediate heights at four primes

Let the ordinary minimal table have odd root $n = Cuv$, where $C = p^h q^k$, $h, k \ge 2$ and $u < v$ are isolated singletons. A common degree omits both u, v; an axis at $w = u, v$ has degree $p^i w$ or $q^l w$. We use the signed columns a_t, deficits j_t, principal blocks P_t, and centered residues $[x]_t$ of Section 9. Put $b = m/(m-1)$ for the least dimension prime m. In what follows the crucial distinction is whether a label's character field ramifies at a singleton.

10.1 The local level and its two moving orbits

Here the local results hold for *any* isolated singleton t in an odd ordinary minimal root, independently of the other exponents. By Lemmas 6.6 and 9.8 one has $t \ge 7$. Put $j = j_t$ and write W for its Weyl operator. On the nontrivial t-local space $s = S/n = \epsilon W$ by Proposition 6.2. Each active label has codegree t^2. The field of every active or t-ramified label is pure at t and nonreal, and local $-I$ exchanges its label with its dual. Since the anomaly is two-primary, the t-part of the level is the largest order of a t-twist.

Lemma 10.1. *The local level is t. Every active row field is $\mathbb{Q}(\zeta_t)$ and there are exactly j free Galois orbits of active labels.*

Proof. For an active label of maximal twist order t^a, square Galois covariance puts $\mathbb{Q}(\zeta_{t^a})^+$ in its character field. This field is nonreal and contained in $\mathbb{Q}(\zeta_{t^2})$ by Corollary 2.8 and $f = t^2$; thus $a \le 2$. If $a = 2$, the field is the full $\mathbb{Q}(\zeta_{t^2})$ of degree $h = t(t-1)$. The vacuum leaves just $t^2 - h = t$ in the trace budget of Lemma 2.4. A wild value orbit has trace divisible by t by total ramification, so a nonzero trace costs at least t^2. A tame value orbit has degree dividing $t-1$, so a nonzero trace costs at least $h/(t-1) = t$. Consequently at most one nonunit orbit can have nonzero trace. The traced dimension equation would give 1 as a multiple of its nonunit degree, a contradiction.

At level t purity puts every active field in $\mathbb{Q}(\zeta_t)$. It contains the real subfield and is nonreal, hence equals $\mathbb{Q}(\zeta_t)$. The rational twist sum forces equal multiplicity on the nontrivial phases; (6.2) gives j labels per phase and thus j full orbits. □

Lemma 10.2 (The trace interval for a singleton column)**.** *In a centered singleton column, in either parity, put $m = n/t$, $j = j_t$. The multiplicity F of the full degree m satisfies*

$$tj - j(1 - j/t) + \frac{(t+1)\epsilon j}{mt} \le F \le tj + j(1 - j/t) + \frac{(t-1)\epsilon j}{mt}.$$

Thus $F > (t-1)j$ if $jm > t+1$, and $F = t$ if $j = 1$ and $m > t+1$.

Proof. For a proper t-free nonunit degree d put $I = m/d$ and $A = [I^{-1}]_t$. Since $I > 1$ and $IA \equiv 1 \pmod t$, positive IA is at least $t+1$, and negative IA is at most $1-t$. Hence

$$-A^2/(t-1) \le A/I \le A^2/(t+1).$$

Removing the F full labels leaves trace $(t + \epsilon/m)j - F$ and norm $2tj - j^2 - F$ by Proposition 6.3. Summing the two bounds over the remaining labels gives the interval. □

Theorem 10.3 (The two ramified orbits at an isolated singleton)**.** *Let t be an isolated singleton in an odd ordinary minimal root n, and write $m = n/t$, $j = j_t$. Exactly tj labels have degree m. The inactive t-ramified labels form two orbits of size j, one of degree m and one of a proper degree $d_* < m$. Their character field is the degree-j subfield of $\mathbb{Q}(\zeta_t)$, with j even and $(t-1)/j$ odd. The evaluations of a fixed active character on either orbit are the conjugates of an integer plus a signed Gaussian period, of trace $-\epsilon a_t(X)$ with $a_t(X) \equiv \pm 1 \pmod j$.*

Put $A_* = a_t(d_*)$ and sum over the unramified proper t-free nonunit labels in dual pairs. Then

$$t - j = A_*^2 + 2j\sum [d]_t^2, \qquad A_* d_* - 2\epsilon \sum d[d]_t = \epsilon, \qquad A_*^2 \equiv 1 \pmod{2j}.$$

If k_t distinct proper t-free nonunit degrees occur, then $(t-1)/j \ge 2k_t - 1 \ge 5$. In particular the odd part of $t-1$ is at least five. Any two distinct such degrees x, y satisfy $t \le b \gcd(x,y)$, where $b = \ell/(\ell-1)$ for the least dimension prime ℓ. For a label Y of degree d_* one has $S_{XY} \in \mathbb{Q}$ whenever $d_X \notin \{d_*, m\}$.

Proof. The local representation. Let R count the ramified inactive labels, P the average over the upper unipotents, and H the diagonal torus of $\mathrm{SL}_2(\mathbb{F}_t)$. If $Px = x$ and the H-average of x vanishes, then $PW^{-1}PWPx = x/t$. Indeed the zero parameter gives x/t and Bruhat decomposition expresses the sum over the nonzero parameters as a full torus sum between P and W, which commute with and normalize H, respectively. On the negative eigenspace of $-I$ the dimension is $D = ((t-1)j + R)/2$ and the upper-fixed rank is $b_0 = R/2$ (conjugate pairs contribute one each). For the unipotent averages P_L attached to the $t+1$ lines, the identity and transitivity on ordered distinct lines give $\mathrm{Tr}(P_L P_{L'}) = b_0/t$ for $L \ne L'$. Thus the sum of these projections has trace $(t+1)b_0$ and squared trace norm $2(t+1)b_0$. Cauchy–Schwarz yields $R \le 2j$.

The period lattice. For a ramified inactive orbit O let $e = |O|$, $K = K_X$ for $X \in O$ and $x_0 = e^{-1}\sum_{Y\in O} e_Y$. The nonreal field K has odd index in $\mathbb{Q}(\zeta_t)$ and e is even. First assume the index exceeds one, so $e \le (t-1)/3$. Choose representatives Z_i of the j active orbits and form $\alpha_i = \chi_{Z_i}(X) \in \mathcal{O}_K$ and $L_i = \mathrm{Tr}_{K/\mathbb{Q}}\,\alpha_i$. The projection identity applied to $x = e_X - x_0$ gives $\|(I-P)sx\|^2 = (1-1/t)(1-1/e)$. Its coordinates on active orbit i are the conjugates of $(\alpha_i - L_i/e)/t$, equally often. Thus, writing Var for the Galois-mean squared distance from the mean,

$$\sum_i \mathrm{Var}(\alpha_i) = t(1-1/e), \qquad \sum_i L_i^2 < et^2/(t-1).$$

For the strict bound apply Lemma 2.4 to the X-row. On the $t-1$ labels in each active orbit, $\chi_X(Z_i) = (\sqrt{f_X}/t)\alpha_i$. Their traced cost is $(t-1)f_X L_i^2/(et^2)$, besides the vacuum. Since $e \le (t-1)/3$, the sum of the L_i^2 is $< t^2/2$, while the L_i are all ed_X modulo t by principality. Hence they have the same value L.

Let B fix K in $\mathbb{F}_t^\times$. The periods $\eta_a = \sum_{b\in aB}\zeta_t^b$ form an integral basis of K of trace -1, by grouping the permuted basis $(\zeta_t^b)_{b\neq 0}$. Write $\alpha_i = \sum_a c_{ia}\eta_a$. Each row sums to $-L$ and Fourier orthogonality gives

$$\sum_{i,a}(c_{ia}+L/e)^2 = e-1, \qquad \text{since } \mathrm{Var}(\alpha_i) = (t/e)\sum_a (c_{ia}+L/e)^2.$$

If $e \mid L$, each centered row is integral of sum zero and even squared norm, contradicting the odd right side. Otherwise, for $r \equiv -L \bmod e$, $0 < r < e$, each row has squared norm at least $r(e-r)/e \ge (e-1)/e$, with the minimum attained when its coefficients differ by at most one. Thus $j \le e$; this also holds if $e = t-1$. There are at most two ramified inactive orbits. At least one of the three distinct proper degrees of Lemma 6.6 therefore supplies an unramified dual pair; the rank budget gives $3j \le t+1$. This excludes full degree $e = t-1$ when $t > 5$ by $R \le 2j$, and $t = 5$ is already excluded by Lemma 9.8.

Parity in the vacuum column. The vector $x_0 - (L/e)e_0$ and its inverse Weyl image vanish on active coordinates, so it is locally invariant. Its vacuum coordinate gives $ea_t(X) = -\epsilon jL$. Also $\sum c_{ia}^2 = e - 1 + jL^2/e \equiv -jL \equiv 0 \pmod 2$. Hence jL^2/e is an odd integer. Since $|jL^2/e| = |La_t(X)|$, both factors are odd and $\nu_2(j) = \nu_2(e) = \nu_2(t-1)$.

Each active signed coordinate is one by Proposition 6.3; the $(t-1)j$ active labels thus leave exactly $j(t+1-j)$ in the inactive squared norm.

Dividing by j, unramified terms $j[d_X]_t^2$ are even and each ramified orbit contributes the odd integer $ea_t(X)^2/j = jL^2/e$. Since $t+1-j$ is even, the number of ramified orbits is zero or two. In the latter case $R \le 2j$ and $e \ge j$ force $e = j$ on both. Equality in the row minima forces $r = 1$ or $e-1$ and one coefficient in each row to differ from the others by ± 1. The periods sum to -1, so the evaluations are integers plus signed periods, with trace $L = -\epsilon a_t(X)$ congruent to $\pm 1 \bmod j$.

The full and proper degrees. Lemma 6.6 supplies at least three distinct proper t-free nonunit degrees. None is active and at most two can ramify, by the orbit count above. Choose a label X of an unramified one. Its normalized diagonal entry is unramified at t and has positive residue d_X there because $X \in P_t$. Lemma 9.5 gives $t \le bd_X \le bm/3 \le m/2$, using the odd index $m/d_X > 1$. Thus $jm > t+1$. Lemma 10.2 supplies a full label beyond the $(t-1)j$ active ones. An inactive full label has codegree t^2 and nonreal character field unramified away from t, so it ramifies at t. The orbit count therefore gives two inactive ramified orbits of size j. If both were full, the lower trace bound would give $j(1-\epsilon/m) \le j(t-1-j)/(t-1)$, again impossible. Thus exactly one is full. Subtracting the tj full labels from the two identities and using $a_t(d) = -\epsilon j[d]_t$ on unramified dual pairs gives the displayed residual identities. Their norm and the oddness of $(t-1)/j$ give the congruence and the bound on k_t. Since $k_t \ge 3$, the odd index $(t-1)/j$ is at least five. For distinct proper degrees x, y, at least one of the labels is unramified at t. Their normalized entry has the positive principal residue $\gcd(x,y)$ at t; Lemma 9.5 gives the bound.

All labels of degree d_* belong to the ramified orbit: their common coordinate is odd while an unramified coordinate is divisible by the even j. Finally K_Y is pure at t; if $d_X \notin \{d_*, m\}$ its character field is unramified at t (by defect zero when $t \mid d_X$). The two fields intersect in $\mathbb{Q}$, giving the rational entry. □

Proposition 10.4 (Two residues at an isolated prime)**.** *In an odd ordinary minimal table let s be an isolated singleton. Put $b = m/(m-1)$ for the least dimension prime m. Suppose X, Y are s-free, s-inactive nonunits whose normalized entry z has positive residue $g = \gcd(d_X, d_Y)$ at every prime over a second prime $a \neq s$. Then*

$$\min(a, s) < g, \qquad a \leq bg\frac{s-1}{2s}.$$

In particular, for $n = p^h q^k uv$ with u, v isolated singletons, if a common degree D belongs to P_q, every axis $p^i w$ ($w = u, v$) satisfies $q < \frac{b}{2} p^{\min(i, v_p(D))}$, independently of the other higher exponent.

Proof. Principality gives the second positive residue, at s. Purity implies that z is unramified at a or s, so Proposition 4.8 gives $\min(a, s) < g$. By Lemma 9.5, $as \leq beg$ whenever one prime is unramified and e is the inertia degree at the other. If z ramifies at s, its field is pure there and $e \leq j_s \leq (s+1)/3 \leq (s-1)/2$ by Theorem 10.3. If it ramifies at a, at least e translates of X are s-unramified and cost at least e in the rank budget, so $e \leq (s+1)/j_s - 1 \leq (s-1)/2$. If neither prime ramifies take $e = 1$. The last assertion uses $s = uv/w$ and the q-principality of D.

□

Lemma 10.5 (A simultaneous two-part constraint)**.** *For two isolated singleton primes u, v of an odd ordinary minimal table, some common orbit ramifies at one endpoint. At either endpoint t one has $\nu_2(j_t) = \nu_2(t-1)$, and the odd part of $t-1$ is at least $2z+1$ when z distinct proper degrees there support unramified labels.*

Proof. The proper ramified orbit has odd signed coordinate by Theorem 10.3. An unramified label has even coordinate $-\epsilon j_t [d]_t$. Since the coordinate depends only on the degree, the ramified degree is distinct from all z others. Theorem 10.3 gives odd index $(t-1)/j_t \geq 2z+1$. If all common labels were unramified at both ends, each dual pair would contribute an even integer to the mixed Gram sum divided by $j_u j_v$, whose value is -1. □

Corollary 10.6 (Parity of the signed coordinates)**.** *For a proper t-free degree the signed coordinate is odd precisely when its labels ramify at t. Some common degree consists of one ramified orbit at one endpoint t, with odd centered residue at the other.*

Proof. The first assertion follows from the preceding proof. A common degree ramified at t consists of the sole proper ramified orbit there, of size j_t, and is unramified at $w = uv/t$. It contributes $\pm a_t(d)[d]_w$ to the divided mixed sum. At least one such contribution is odd. □

Corollary 10.7 (Odd parts of singleton orders)**.** *An isolated singleton in an odd ordinary minimal root can be neither $2^a + 1$ nor $3 \cdot 2^a + 1$. For four odd primes, the braided source family $p^A q^B (uv)^2 d$ is WGT on all divisors when u or v has either form, with any coprime square-free d.*

Proof. The local assertion is Theorem 10.3. For divisor descent an odd reduced dimension is impossible; losing a repeated prime leaves the three-prime small-exponent theorem, and height one is covered by Proposition 9.12. Otherwise Proposition 9.3 isolates both u, v. □

Proposition 10.8 (Degrees supporting mixed twists)**.** *In an odd ordinary minimal table of root $n = p^h q^k uv$ with $h, k \geq 2$, every p, q-mixed label Z has*

$$d_Z = p^i q^l uv, \qquad 1 \leq i < h, \quad 1 \leq l < k.$$

For any such mixed column, at each $t = u, v$ and $s = p, q$ there is a nonunit t-free hit in P_s. Both higher valuations of the hit are deficient, and the higher prime other than s divides its degree.

Proof. The lower bounds follow from Proposition 9.3; any active prime divides the formal codegree, giving the strict upper height bounds. For the final assertion reduce the orthogonality of the mixed column with the vacuum on P_s modulo t. The hit omits t, cannot have full height at either active prime by defect zero, and cannot omit the other higher prime by the same two-axis argument. □

Lemma 10.9 (A single common degree)**.** *Let D be the only common nonunit degree at the two isolated singletons of an odd four-prime root. Its labels form one ramified orbit at one endpoint t, and are unramified at $w = uv/t$. Moreover $a_t(D) = \pm 1$, $[D]_w = \pm 1$, $j_t D \equiv \pm 1 \pmod t$, and $j_w(1 + j_t) \le w + 1$.*

Proof. Write M for the multiplicity. Since $M|a_u a_v| = j_u j_v$ and $M \ge 2$, at some endpoint t we have $j_t \nmid a_t$. Every D-label there ramifies, so M is a multiple of j_t by Theorem 10.3. At the other endpoint $|a_w| = j_w|[D]_w|$. Cancelling in the mixed identity gives $(M/j_t)|a_t[D]_w| = 1$, proving $M = j_t$ and both unit assertions. The vacuum congruence gives $j_t D \equiv \pm 1 \bmod t$, and the unramified rank budget gives the final inequality.

□

10.2 Fusion transport and bounds at arbitrary heights

Return to $m = \min(p, q, u, v)$, $b = m/(m-1)$. For distinct unramified, inactive t-free nonunit degrees x, y each has a distinct dual pair; hence Lemma 6.5 gives

$$\left|j_t[x]_t[y]_t\right| < t/4, \qquad -\epsilon j_t[x]_t[y]_t \equiv xy/(Cw) \pmod t, \quad w = uv/t. \tag{10.1}$$

The integer on the left of the congruence is even. In particular $xy/(Cw)$ cannot be an odd integer k in $(0, 3t/4]$.

Lemma 10.10 (A forbidden square residue)**.** *At a singleton $t \ge 7$ with nonzero inactive ramification, no t-free inactive nonunit Z satisfies $j_t^2 d_Z \equiv \pm 1 \pmod t$.*

Proof. Let $j = j_t$ and $H = (t-1)/j$. The two-orbit theorem and $3j \le t + 1$ give H odd, $H \ge 3$, j even. First suppose Z is unramified at t. Choose the sign of $c = \pm[d_Z]_t$ giving $c \equiv H^2 \pmod t$. The dual-pair rank budget gives $2c^2 \le (t+1)/j - 1 \le H$, hence $|c| \le (H-1)/2$. Write $H^2 - c = kt$. Since $t \ge 2H + 1$ and $|c| \le (H-1)/2$, we have $0 \le k \le (H-1)/2$ (the quotient is strictly below $(H+1)/2$). Reduction modulo H gives $k + c \equiv 0$; the bounds force $k = -c$. Thus $H = kj$, contrary to parity. If Z ramifies, its signed vacuum coefficient must instead have absolute value H: it is centered and satisfies $a_t(Z) \equiv -\epsilon j_t d_Z \equiv \pm j_t^{-1} \pmod t$. The other ramified orbit and an unramified dual pair cost at least $j + 2j^2$ in the same budget. Such a pair exists among the three compulsory distinct proper degrees, of which at most two have ramified orbits. Thus $H^2 \le jH + 1 - 3j$, forcing $H < j - 1$, whereas the signed period trace gives $H \equiv \pm 1 \pmod j$ and $H \ge 3$. □

Lemma 10.11 (Transport from an inactive singleton orbit)**.** *Let $t \ge 5$ be one of the two isolated singletons of an odd ordinary minimal table with root $p^h q^k uv$, $h, k \ge 2$. Suppose X, Y are t-inactive, K_X is pure at t, and S_{XY} is moved by t-inertia. For any t-unramified label Z, put $c = [d_Z]_t$, allowing $c = 0$. Then*

$$\chi_Y(Z) \ne c \quad \Longrightarrow \quad d_X d_Z \ge (t-1)n/t.$$

Proof. On every active row T one has $\chi_T(Z) = c$ by the unramified-column formula, including $c = 0$ by defect zero. If XZ has no active constituent, then $F = XZ - cX$ and $s^{-1}F$ vanish on active coordinates, the latter since $\chi_T(F) = 0$. Thus both elementary unipotents fix F, hence so does the local torus. Its unsigned permutations act on the ring since $\chi_T(\sigma X) = \sigma\chi_T(X)$ and characters

separate elements. As $\sigma Z = Z$, evaluating $\sigma F = F$ at Y gives $(\chi_Y(\sigma X) - \chi_Y(X))(\chi_Y(Z) - c) = 0$. The first factor is nonzero for some σ.

If $\chi_Y(Z) \neq c$, there is therefore an active constituent. Since $S_{XZ} \in K_X \cap K_Z = \mathbb{Q}$, balancing XZ puts the sum of its active phases, counted with multiplicity, in the complementary cyclotomic field and thus in $\mathbb{Q}$. The nontrivial tth roots form a permuted basis, so all $t - 1$ occur. Each active constituent has degree n/t. □

Proposition 10.12 (The two squared primes at arbitrary heights)**.** *Let $n = Cuv$, $C = p^h q^k$, be an odd ordinary minimal root with $h, k \geq 2$, $u < v$, $b = m/(m-1)$ for the least prime m, and $J = \max(p^{h-1}, q^{k-1})$. Some common degree differs from C, and*

$$v \leq b \min(u, J).$$

In fact $v \leq b \min(p^i, q^l)$ for each occurring common degree $p^i q^l$. A strict reverse of the displayed inequality excludes counterexamples at divisors of $p^A q^B (uv)^2 d$ for four odd primes, $h = \lfloor A/2 \rfloor, k = \lfloor B/2 \rfloor \geq 2$, and coprime square-free d.

Proof. The compulsory axes $p^i u, q^l u$ at v have gcd u. Their degrees are distinct, proper and v-free, so Theorem 10.3 gives $v \leq bu$. Comparing a common degree with each of these axes gives $v \leq b \min(p^i, q^l)$ for its own exponents i, l. If C alone occurred, Lemma 9.9 would give $v = uj_v \pm 1 > bu$, since $j_v \geq 2$, $u \geq 5$ and $b \leq 3/2$. Some common degree therefore has a prime-power factor at most J.

Under divisor descent, an odd reduced table is excluded by Proposition 4.7. Loss of a repeated prime leaves the three-prime small-exponent case, and a higher root height below two is covered by Proposition 9.12. Otherwise the singletons persist and J only decreases. □

10.3 Isolation of a quartic prime

Theorem 10.13 (Full isolation at four odd primes)**.** *Let $n = p^h q^k uv$ be an odd ordinary minimal root, $h, k \geq 1$. If all four primes are isolated then uv is absent and neither h nor k equals two.*

Proof. Proposition 9.12 excludes height one. With $h, k \geq 2$, full isolation and the occurrence of uv would make $p^h q^k$ the sole common degree by Proposition 9.11, contrary to Proposition 10.12. Either height equal to two forces uv by the same isolation proposition. □

10.4 The two higher prime powers in increasing order

Close singletons rule out the full common degree, independently of the two higher exponents. This elementary estimate has two applications: it treats a quartic valuation at the least prime with arbitrary companion height, and later it applies to the odd sextic root.

Theorem 10.14 (The full common degree is absent)**.** *Let $n = Cuv$ be an odd ordinary minimal root with C the product of two prime powers of exponents at least two. Let p be the least of the four primes and $b = p/(p-1)$. If $u \leq bp^2$ and $u < v \leq bu$, the full degree C is absent.*

Proof. For $p = 3$, $u \leq 13.5$ leaves only $u = 11$ by Corollary 10.7 and Lemma 9.8, and no admissible $v \leq 3u/2$ remains. Let $p \geq 5$ and $\delta = v - u$. Since $\delta \leq u/(p-1) \leq p^3/(p-1)^2 < p + 3$, the even gap satisfies $\delta \leq p + 1$.

If C occurs, its signed coordinates are the centered inverses $a_u = [v^{-1}]_u$, $a_v = [u^{-1}]_v$; $va_u + ua_v = 1$. At the endpoint t with odd coordinate the C-labels ramify. One orbit costs $j_t a_t^2$ in B_t, so $|a_t| < \sqrt{t} \leq bp < p + 2$. Also $\delta a_t \equiv \pm 1 \bmod t$. Since δa_t is even and $t \geq (p-1)\delta$, $|a_t| \geq (t-1)/\delta > p - 2$. Oddness gives $|a_t| = p$. Hence $t > p^2$, $\delta = p + 1$, and $t = p(p+1) \pm 1$: here $t - 1 \leq p\delta < 2t - 1$. If $t = v$, then $u = p^2$ or $p^2 - 2$, contrary to primality and $u \geq p^2 - 1$. The choice $t = u = p^2 + p - 1$ gives composite $v = p(p+2)$. The remaining inverse pair is

$$(u, v) = (p^2 + p + 1, (p+1)^2 + 1), \qquad (a_u, a_v) = (-p, p+1).$$

The C-orbit is v-unramified, so $j_v \mid p+1$. Since $v-1=(p+1)^2$, the two-orbit theorem excludes all inactive v-ramification, contrary to the compulsory inactive full orbit at v in Theorem 10.3. □

Theorem 10.15 (A quartic exponent at the smaller higher prime)**.** *Let $q<p$ be odd primes, and let u,v be two further distinct primes. For every $A\geq 0$ and every square-free d coprime to $pquv$, all braided fusion categories of dimension dividing $q^5p^A(uv)^2d$ are WGT.*

Proof. After divisor descent it suffices to exclude an ordinary odd root $n=Cuv$, $C=q^2p^h$, $h\geq 2$, in a minimal table. Indeed, losing a repeated prime leaves at most two or three with an exponent at most two. In a reduced table the dimension either is odd or has two-valuation one at a dyadic singleton, contrary to Proposition 4.7 and Lemma 2.5. Proposition 9.3 makes the ordinary higher valuations even and the singletons odd; Proposition 9.12 excludes a root height below two. Order $u<v$. By (9.2) the least prime is q, and $b=q/(q-1)$.

At an unramified endpoint t, a common degree x satisfies $t\leq bq^{v_q(x)}$ by (9.1). Since $t>bq$, write $x=q^2p^i$. For $i<h$ the full q-height and deficient p-height put x in P_p by Proposition 6.2. Then degree uv is absent by coprime vanishing. A p-free nonunit of P_q contains uv by Proposition 9.3; its q-height is below two by defect zero, leaving degree quv. The comparison of x with quv has gcd q and positive residue at $p>bq$, hence ramifies there. Every proper common label is therefore unramified at both singletons. The mixed Gram identity has a ramified common label by Lemma 10.5, so C occurs.

Proposition 10.12 gives $u\leq v\leq bq^2$ and $v\leq bu$, so Theorem 10.14 excludes C. □

Proposition 10.16 (An unrestricted power at the least odd prime)**.** *Let $n=p^hq^2uv$ be an odd ordinary minimal root, for distinct primes with p least, $u<v$, $h\geq 3$. Set $P=p^{h-1}$ and $b=p/(p-1)$. Then*

$$u\leq\begin{cases}(1+b^2/2)P, & p\geq 5,\\ (9P+2)/4, & p=3.\end{cases}$$

Consequently the strict reverse bound suffices for WGT for the braided source family $p^Aq^B(uv)^2d$, $h=\lfloor A/2\rfloor$, $4\leq B\leq 5$, with any coprime square-free decoration d.

Proof. Suppose the asserted bound fails, so $u>b^2P$. By (9.1) every common degree is $L=p^hq$ or $C=p^hq^2$. Proposition 10.12 forces L to occur. It is q-principal since the p-valuation is full and the vacuum denominator has a q-part. Hence uv is absent. By (9.1), $q\geq u/b>bP$. A q-free nonunit in P_p has degree p^iuv, $1\leq i<h$ (compare both q-axes); such a label exists by reducing principal-block orthogonality at q. Pairing with L forces q-inertia of size at least $q/(bP)>1$ at every L-label. They are therefore unramified at both singletons. At each t their comparison with both opposite axes gives $t\leq bq$ and full p-axis height since $t>bP$. The mixed column from Theorem 10.13 has, by Proposition 10.8, a t-free hit in P_q with deficient heights. Both common degrees and all p-axes have full p-height, so the hit has degree $Y=p^lqw$, $1\leq l<h$, $w=uv/t$. Comparing it with p^ht at q gives q-inertia of size at least $q/(bP)$ here too. The disjoint unramified orbits of L,Y then give $j_t|[L]_t[Y]_t|<bPt/(2q)\leq b^2P/2$. But $LY/(Cw)=p^l$ is odd, so the even representative in (10.1) has modulus at least $t-p^l\geq u-P>b^2P/2$ (also for $p=3$). The bounds increase with h; smaller roots with $h'\leq 2$ or $k'\leq 1$ descend by Theorem 10.15 or Proposition 9.12. □

Corollary 10.17 (Reduction to an odd sextic root)**.** *Let $p<q$. A divisor-minimal counterexample to WGT among the divisors of $p^Aq^B(uv)^2d$ would have an ordinary table of square root $n=p^iq^juv$, with $p<\min(u,v)$ and $i,j\geq 2$. If p is odd, then $i\geq 3$. If $i+j<6$, the only possibility is an odd root p^3q^2uv with p least.*

Proof. All four repeated primes must survive: loss of one leaves at most two, or three with an exponent at most two, covered by Theorem 4.12 and Dong's theorem. In a reduced table a dyadic singleton would leave valuation one, contrary to Lemma 2.5; an odd reduced dimension is

also impossible by Proposition 4.7. These observations exclude two outside the higher positions, and Lemma 9.4 excludes two in a higher position. In the ordinary table Proposition 9.3 makes the higher valuations even. A root height below two is covered by Proposition 9.12. Lemma 6.6 excludes a dyadic singleton, so an even root has $p = 2$ and $i + j \geq 6$ by the dyadic enumeration in Theorem 9.15. In an odd root the least prime cannot be a singleton by (9.2); Theorem 10.15 excludes height two there, at every companion height. Now $i \geq 3$ and $j \geq 2$ leave only $(3, 2)$ below total height six. □

11 Lifting the smaller prime exponents to cubes

11.1 Odd cubic primes and complementary supports

Lemma 11.1 (A degree at an odd cubic prime)**.** *In an ordinary or positive reduced minimal table suppose that an odd prime p has $v_p(\Delta) = 3$. If a nonunit degree has prime support $\{p, q\}$, then $q \mid p - 1$. The same conclusion holds throughout an integral minimal modular extension of a slightly degenerate counterexample, including its odd sector.*

Proof. Write X for the indicated simple. Every label $Z \in P_p$ has p-free degree by Corollary 4.1. For a noninvertible Z either $q \mid d_Z$ or $S_{XZ} = 0$ by coprime vanishing (Lemmas 2.5 and 4.4). Thus $d_Z \chi_X(Z)$ vanishes above q in either case. Apply Proposition 4.2; in the extension intersect P_p with the trivial prime-to-p twist fiber to remove the extra invertible fermion. The restriction preserves the vacuum and the p-local action, and $v_p(4\Delta) = v_p(\Delta)$. □

Proposition 11.2 (Small exponents at four repeated primes)**.** *At four repeated primes a divisor-minimal counterexample cannot have both a square and a cube among its prime powers. If u is an odd cubed prime and another prime v is cubed, the product of the two remaining primes divides $u - 1$.*

Proof. Write p, q for the cube and square. We seek a column J active at both primes. In the ordinary table q-activity exists by categorical Cauchy. Separation would give a q-active label with p-unit codegree (Lemma 3.3), impossible at odd p-valuation of the dimension. In the reduced table with p, q odd, a q-active fiber contains a p-active label by Lemma 7.8. A squared two would leave valuation one after reduction. For a cubed two pass to the integral minimal extension: choose an even q-active label, then choose from its fermion pair a dyadically active J.

In every case each noninvertible degree on P_p omits p, by Corollary 4.1, or by Proposition 4.7 for the extension at two. A hit on J also omits q by defect-zero vanishing. Consequently the degrees of the proper hits are divisible by both remaining primes. Their contributions to $\sum_{X \in P_p} d_X \chi_J(X) = 0$ cannot cancel the unit modulo an odd one of those primes. In the extension the two invertibles contribute two since J is even, giving the same contradiction. This proof also applies to a square and cube in ordinary perfect fusion-simple integral modular data and, at odd p, q, in positive reduced data.

For two cubes u, v let p, q denote the other primes. If v is odd or the table is ordinary, prime avoidance for a v-active column on P_v, first at q and then at p, gives two v-free degrees with supports $\{u, p\}$ and $\{u, q\}$. If $v = 2$ in a reduced counterexample, the same two supports occur in the full extension by Proposition 4.7. Lemma 11.1 now gives $pq \mid u - 1$ in both cases. The prime-avoidance proof applies equally at a second prime of valuation two whenever that prime is ordinary or odd in the reduced table. □

Theorem 11.3 (Lifting squared primes to cubes)**.** *Let Π be a set of four primes, let $a_p \geq 0$ be integers, and fix $S \subseteq \Pi$ of size at least two. Put*

$$M = d \prod_{p \in \Pi \setminus S} p^{a_p} \prod_{s \in S} s^2, \qquad d \text{ square-free and coprime to } \prod_{p \in \Pi} p.$$

Suppose every braided category of dimension dividing M is WGT. Then this remains true after raising any proper subset of the designated squares to cubes. All of them may be raised if $|S| \geq 3$, or if some odd $s \in S$ satisfies $\prod_{p \in \Pi \setminus S} p \nmid s-1$. In particular every divisor of $\ell^H (abc)^3 d$ is covered for arbitrary $H \geq 0$.

Proof. Descend to a least counterexample among the divisors. Losing a repeated prime leaves at most two, or three with an exponent at most three. With four surviving primes, the preceding proposition forbids a mixture of exponents two and three on S. All twos are covered by hypothesis, so every member of S must have been raised and must have exponent three. If $|S| \geq 3$, choose two odd members $u < w$ and a third member v. The proposition at u, v would give $w \mid u-1$. Thus $|S| = 2$ and the same proposition makes the product outside S divide $s-1$ for each odd $s \in S$. For the final assertion use Proposition 9.12 as the starting family. □

The same lift applies to the two squares of Theorem 9.1, in particular for higher caps $(7, 5)$.

12 Four odd primes at root heights three and two

Here $n = p^3 q^2 uv$ is the odd root left by Corollary 10.17, with p least and $u < v$ the isolated singletons. Put $b = p/(p-1)$, $\delta = v - u$ and

$$A = p^2 q, \qquad B = p^2 q^2, \qquad L = p^3 q, \qquad C = p^3 q^2.$$

A nonunit degree is common if it omits both u and v. The general singleton bound $v \leq bu$ leaves just one pair when $p = 3$. For $p \geq 5$ the full degree C is excluded by combining the close-singleton obstruction of Section 10 with a bound on u. The last step compares the possible fields when A is absent and when it occurs.

As $v - u \geq 2$, one has $u \geq 2p - 1 > bp$. The bound in Proposition 10.12 therefore restricts common degrees to A, B, L, C. A common label with full p-valuation and deficient q-valuation is q-principal, and symmetrically with p, q exchanged, by Proposition 6.2 for the cluster $\{p, q\}$. Proposition 9.3 supplies a p-free nonunit in P_q whose degree contains uv; since its q-height is deficient, that degree is quv whenever uv is absent. Theorem 10.13 supplies a mixed p, q-active row Z.

We use one trace test throughout. Let T be active at $t \in \{u, v\}$ and let $w = uv/t$. Suppose all proper common hits of Z, T, on the whole table or on P_q, have degree d. Then

$$[d]_t = \pm 1 \quad \text{if unramified at } t, \qquad a_t(d) = \pm 1 \quad \text{if ramified at } t.$$

Indeed ramification status is constant on a proper degree by Corollary 10.6. In the first case $\chi_T(X) = [d]_t$ on every hit. In the second, K_X is pure at t and K_Z is unramified there, as purity would contradict the square Galois action on its nontrivial q-twist ($q > 3$). Thus $\chi_Z(X)$ is integral rational and constant on the orbit. The orbit sum of $\chi_T(X)$ is $-\epsilon a_t(d)$ by Theorem 10.3, and the principal block is Galois stable. In both cases orthogonality makes the indicated integer divide the vacuum term one.

By Proposition 10.12, $v \leq bu$ and at least one of A, B, L occurs.

Lemma 12.1 (Axis heights forced by a ramified common degree)**.** *If a label of degree B or L ramifies at a singleton t, every p-axis has full height. For L the q-axes are full as well, no degree pqs, $s = u, v$, occurs, and $qw \leq bp^3$, $w = uv/t$.*

Proof. The gap bound gives $u \geq (p-1)\delta + 1 \geq 2p - 1 > bp$; also $q > bp$. Write D for the ramified label. It is principal at $a = p$ for $D = B$ and at $a = q$ for $D = L$. For a label X of degree ds, $s = t, w$, consider its normalized pairing z with D. For the axis and witness degrees considered below, ds differs from both d_D and $Cw = n/t$. Thus z is rational by Theorem 10.3. Now let $s' = uv/s$. If $D = B$ and $d = p^i$, $i \leq 2$, then $p \mid z$ by p-principality and $z/p \equiv p^{i-1} \pmod{s'}$,

impossible since $s' > bp$ by the strict house and balancing mean. If $D = L$ and $d = p^i$ $(i = 1, 2)$, both q and s' divide $p^i - z$, giving $qs' \le bp^i$. For $d = p^i q$ $(i = 0, 1)$, $q \mid z$ and $z/q \equiv p^i \pmod{s'}$ give $s' \le bp^i$. Again $q, s' > bp$ exclude these possibilities. In particular the rational pairing of L with the compulsory $p^3 t$ gives $qw \le bp^3$. □

Corollary 12.2 (The least prime three)**.** *An odd ordinary minimal root $27q^2uv$ is impossible.*

Proof. Propositions 10.16 and 10.12 give $u \le 20$, $u < v \le 3u/2$, and an odd part at least five in both $u - 1$ and $v - 1$. The only possible pair is $(u, v) = (19, 23)$. At an unramified endpoint (9.1) forces the 3-part of every common degree to exceed nine, leaving only L, C. If L ramified at either endpoint, Lemma 12.1 would give $qw \le 81/2$, impossible for $q \ge 5$ and $w \ge 19$. The mixed parity constraint therefore forces C to occur. Its coordinate at 19 is $[23^{-1}]_{19} = 5$, hence odd and ramified. The residual norm of Theorem 10.3 would give $25 \le 19 - j_{19}$. □

12.1 Common degrees when the least prime is at least five

For the remainder let $p \ge 5$. We first exclude C and then use the ramification of a proper common degree to eliminate the remaining patterns. A mixed row on P_q supplies the same two possible witness heights in both steps.

Lemma 12.3. *If $n = p^3q^2uv$ with $p \ge 5$ least, then $u \le bp^2$.*

Proof. Assume $u > bp^2$. The only possible common degrees are L and C by (9.1), and L occurs by Proposition 10.12. Comparison of L at an unramified endpoint gives $q \ge u/b > p^2$. An L-label ramified at t would give $qw \le bp^3$ by Lemma 12.1, contrary to $q > p^2$ and $w > bp$. Thus L is unramified at both singletons; then $v \le bq$ by the q-axis comparison. The mixed Gram identity and even multiplicity exclude L alone, so C occurs. The centered inverse coordinates $A_u = [v^{-1}]_u$ and $A_v = [u^{-1}]_v$ on C satisfy $vA_u + uA_v = 1$. Hence C is ramified at the odd coefficient endpoint t, and unramified at the even endpoint w. If $q \le bp^2$, its orbit cost gives $A_t^2 \le t + 1 - j_t \le t - 1$, and $\delta A_t \equiv \pm 1 \pmod{t}$ with δ even implies $\delta \ge \sqrt{t-1} > p$. Yet $q \le p^2 + p + 1$ and $q \le bp^2 < u$, so $v - q \le q/(p-1) \le p + 2 + 3/(p-1) < p + 3$. Both $v - q$ and $u - q$ are positive and even, giving $\delta \le p - 1$.

Suppose $q > bp^2$. The comparison of L with the p-axes forces all of them to have full height since $t > bp^2$. On P_q a mixed column, reduced at t, therefore needs a hit $Y_l = p^l qs$ with $1 \le l \le 2$, $s = uv/t$. Comparing it at q with $p^3 t$ forces both to be q-ramified, hence unramified at t. The even quantity in (10.1) for L, Y_l is congruent to p^l. As $t > p^2$, this requires $p^l > 3t/4$, forcing $l = 2$ and $v < 4p^2/3$. Now $\delta \le u/(p-1) < 2p$ so $p \nmid \delta$. At the even endpoint w, the identity $pY_2 = sL$ shows that $p[Y_2]_w \pm \delta[L]_w$ is a multiple of w. It is nonzero because $0 < |[L]_w| < \sqrt{w}/2 < p$. Write $c = [L]_w$, $d = [Y_2]_w$, and $a = [s^{-1}]_w$, where $s = uv/w$. The two unramified dual pairs and at least two full-degree labels give $2j_w(c^2 + d^2) + 2a^2/j_w \le w + 1$ by the unramified column norm. Since $|a| \ge (w-1)/\delta$ by parity, Cauchy–Schwarz and the arithmetic–geometric mean inequality give

$$p - 1 \le w/\delta \le \sqrt{p^2 + \delta^2}\,\sqrt{c^2 + d^2}/\delta \le \frac{\sqrt{p^2 + \delta^2}(w+1)}{4(w-1)} < \frac{7\sqrt{5}}{24}p < p - 1.$$

Here $w > p^2 \ge 25$. This excludes the second alternative. □

These bounds give $\delta \le u/(p-1) \le p^3/(p-1)^2 < p+3$. Thus $\delta \le p+1$, $u \ge (p-1)\delta + 1 \ge 2p-1$, and Theorem 10.14 excludes C. We determine which of A, B, L must carry the ramification required by the mixed Gram identity. We shall use the following consequence of the gap bounds. For $s = u, v$ and $\delta z \equiv \pm p \pmod{s}$ one has

$$|z| \ge \frac{s-p}{\delta} > \max\left\{\frac{\sqrt{s}}{2}, \frac{s+1}{\delta+3}\right\}. \tag{12.1}$$

The first inequality follows from parity. For the square-root bound $(s-p)/\sqrt{s}$ increases with s. At $s_0 = (p-1)\delta + 1 \le p^2$ one has $2(s_0 - p)/\delta = 2(p-1)(1-1/\delta)$, greater than p for $\delta \ge 4$ and equal to $p - 1 > \sqrt{2p-1}$ for $\delta = 2$. The second bound says $3s > (p+1)\delta + 3p$. Using $s \ge s_0$ gives this for $\delta \ge 4$; at $\delta = 2$ equality could occur only for $p = 5, s = 9$, which is composite.

Proposition 12.4 (The degree p^2q cannot occur alone)**.** *A cannot occur as the sole common degree; in particular uv is absent.*

Proof. Put $F = C/A = pq$. Suppose A occurs alone. Use the two endpoints and signed coordinates from Lemma 10.9. The coordinate congruence $a_t \equiv A/(Cw) = 1/(Fw) \pmod t$ now gives $F \equiv \pm\delta^{-1} \pmod t$. Since $A = pF$ we obtain $pj_t \equiv \pm\delta \pmod t$. At w one has $C = A^2/p \equiv 1/p$ by $[A]_w = \pm 1$. The vacuum congruence $j_w Ct \equiv \pm 1 \pmod w$ therefore gives $\delta j_w \equiv \pm p \pmod w$. Since $pj_t \ge 2p > \delta$, the first congruence and parity give $pj_t \ge 2t - \delta > p\delta$, so $j_t \ge \delta + 2$. At w, (12.1) gives $j_w(\delta + 3) > w + 1$, contrary to $j_w(1 + j_t) \le w + 1$. Finally, if uv occurred, full-height principality would exclude both p^2q^2 and p^3q and force A. The exclusion of the full common degree C would then make A the sole common degree, as just ruled out. □

Proposition 12.5 (Consequences of a common degree p^2q)**.** *Let $n = p^3q^2uv$ be an odd ordinary minimal root, $p \ge 5$ least, $u < v$, $b = p/(p-1)$. If $A = p^2q$ occurs, so does $B = p^2q^2$, and*

$$q \ge uv/(bp) > bv.$$

All labels of degree p^iu or p^iv ramify at q and are unramified at both singletons. If $L = p^3q$ also occurs, every A- and L-label is unramified at both singletons, and some B-label is singleton-ramified.

Proof. We first record a restriction common to $D = B, L$. If an A- or D-label ramifies at a singleton t, with partner w, then its normalized pairing $z = S_{A,D}/D$ with any label of the other degree is rational. Indeed the degrees are distinct, neither is full at t, and the rational-entry assertion of Theorem 10.3 applies.

For $D = L$ a rational pairing is itself impossible. The integer z is divisible by q since $L \in P_q$ and $\chi_L(A) = z$. Both labels are principal and units at u, v, so $z/q \equiv p^2 \pmod{uv}$. The house and balancing mean give $-p^2/(p-1) \le z/q < p^2$, forcing $uv \le bp^2$, whereas $u, v \ge p + 2$. Hence, if L occurs, no A- or L-label can ramify at either singleton.

Some common label is singleton-ramified by the divided mixed Gram identity and even orbit sizes. The full degree C is absent. If B were absent, the preceding paragraph would exclude coexistence with L, and A as the sole common degree was already ruled out. Thus B occurs. In the presence of L, it carries the compulsory ramification; in its absence at least one of A, B does. Choose this ramified label and a label of the other degree. Their pairing $z = S_{A,B}/B$ is rational by the first paragraph. Since $B \in P_p$, $p \mid z$. The singleton congruences give $z/p \equiv pq \pmod{uv}$, and the same strict house and mean bounds yield $uv \le bpq$. Also $v - u \ge 2$ and $v \le bu$ give $u \ge 2p - 1 > b^2p$, so $q \ge uv/(bp) > bv$.

Finally $B \in P_p$ excludes the coprime degree uv; the complementary p-free witness in P_q has degree quv. Comparing it at q with p^is for $s = u, v$ gives q-inertia at least $q/(bs) > 1$ on the corner. Isolation of either singleton then excludes ramification there.

□

For the rest of the proof we will repeatedly use two consequences at an endpoint s, with $s' = uv/s$. Once the p-axes are full, all proper common hits of Z and an s-active row on P_q have degrees

$$A, \qquad Y_1 = pqs', \qquad Y_2 = p^2qs'.$$

Indeed the two heights are deficient, p-free nonunits carry uv, and a hit omits s. A Y_1-hit is s-unramified: its comparison in P_q with p^3s has gcd p, and $q > bp$ forces q-ramification. Also if a

common label of degree x is s-unramified, an s-unramified Y_l is excluded whenever $xY_l/(Cs')$ is an odd integer at most $3s/4$, by (10.1). For $x = A$ these quotients are $1, p$; for $x = L$ they are p, p^2. Throughout $p < 3s/4$.

Proposition 12.6 (The degree p^2q^2 cannot occur alone)**.** *The sole common degree cannot be* $B = p^2q^2$.

Proof. Under the sole-degree hypothesis, Lemma 10.9 provides endpoints t, w with $a_t(B) = \pm 1$, $[B]_w = \pm 1$, $j_t B \equiv \pm 1 \pmod t$ and $j_w(1 + j_t) \le w + 1$. As $C = pB$, the first congruence gives $pw \equiv \pm 1 \pmod t$. All p-axes have full height by Lemma 12.1. Since B is the sole common degree, the proper P_q-hits at t are Y_1 (unramified) or Y_2.

If a Y_2-hit were unramified at t, its dual-pair budget would give $0 < j_t[Y_2]_t^2 < t/2$. This even integer is congruent to $j_t(pq)^2 \equiv \pm 1 \pmod t$ since $pw \equiv \pm 1$, impossible.

Nor can Y_2 ramify there: B is already the unique proper ramified degree. Thus only Y_1 is hit, and the trace test gives $|[Y_1]_t| = |[q]_t| = 1$. As $p\delta \equiv \pm 1$ and $j_t(pq)^2 \equiv \pm 1 \pmod t$, we get $j_t \equiv \pm\delta^2 \pmod t$. Here $j_t < t/2$ and $\delta^2 < 3t/2$ since $\delta \le p + 1$, $t > (p - 1)\delta$, and $p \ge 5$. Their sum and absolute difference are even and less than $2t$, forcing $j_t = \delta^2$.

Finally $p\delta \equiv \pm 1 \pmod t$ and $0 < p\delta \le v < 2t - 1$, so $t = p\delta + \sigma$ and $w = t + \eta\delta$, with signs σ, η. Thus $pt \equiv h := \delta + \eta\sigma \pmod w$. From $C = pB$, $[B]_w = \pm 1$ and the vacuum congruence, $j_w h \equiv \pm 1 \pmod w$. Thus $j_w(\delta + 1) \ge j_w h \ge w - 1$ by parity. But $j_w(1 + \delta^2) \le w + 1$ by the sole-degree budget, impossible since $j_w(\delta^2 - \delta) \ge 4$. □

12.2 The character fields of the common labels

Proposition 12.7 (Mixed witnesses when A is absent)**.** *The degree* $A = p^2q$ *must occur. Consequently every* L*-label is unramified at both singletons and its field is ramified only at* q.

Proof. Suppose A is absent. Then L occurs by the exclusion of a sole B. If L is unramified at both singletons, mixed Gram parity forces B to ramify at one. Thus in either case Lemma 12.1 gives full p-axes. At a singleton s the proper P_q-hits all have degree $Y_2 = p^2qs'$. Indeed if L ramifies anywhere the lemma excludes Y_1 outright. Otherwise a Y_1-hit is unramified and its quotient with L is p, also excluded. Write $Y_s = p^2qs'$.

Whenever L is unramified at s, so is its whole degree. If Y_s were unramified there, the trace test would give $[Y_s]_s = \pm 1$. But $pY_s = s'L$ would give $\delta[L]_s \equiv \pm p \pmod s$, contradicting (12.1) and $|[L]_s| < \sqrt{s}/2$. Hence Y_s ramifies there, with $a_s(Y_s) = \pm 1$ by the same test; equivalently $pq \equiv \pm 1 \pmod s$.

If L is unramified at both endpoints, Y_s is the unique proper ramified degree at each. This contradicts the ramification of a common degree. Thus L ramifies at t and Y_w at $w = uv/t$. If L alone is common, Lemma 10.9 gives $[L]_w = \pm 1$. As $L = p^2(pq)$ and $pq \equiv \pm 1 \pmod w$, one has $p^2 \equiv q^2 \equiv -1 \pmod w$: the positive sign would give $p \equiv \pm 1$, impossible for $1 < p < w - 1$. The compulsory axis p^3t then has signed coordinate $[q^{-2}]_w = -1$ and must ramify at w, where the distinct degree Y_w already ramifies.

Therefore B occurs. At t all Y_t-hits are unramified since L is the proper ramified degree. The trace gives $Y_t \equiv \pm 1 \pmod t$. Since $Cw = pqY_t$, the vacuum yields $j_t^2 B \equiv 1 \pmod t$, contrary to Lemma 10.10. □

Lemma 12.8 (The odd sextic root)**.** *No ordinary minimal table has root* p^3q^2uv *with* $p \ge 5$ *least.*

Proof. We now have both A, B and $q \ge uv/(bp) > bv$ (Propositions 12.7 and 12.5). All p-axes are unramified at both singletons and have full height. Indeed if B ramifies at either singleton this follows from Lemma 12.1; otherwise apply (10.1) to $B, p^i r$ at $s = uv/r$ for $r = u, v$. At $i = 1, 2$ the quotient $p^{i-1} \le p < 3s/4$ is forbidden.

We claim that A is unramified at both endpoints. Suppose instead it ramifies at t. All A-labels are then pure at t, unramified at $w = uv/t$, and L is absent by Proposition 12.5. None can belong to P_q: its rational normalized pairing with p^3t would give two positive residues p^2 at q, w, hence $qw \le bp^2$, contrary to $q > bv$. The P_q-witnesses are now Y_1, Y_2 at w. Since A is unramified there, the quotient test excludes all unramified Y_l, including every Y_1. Only ramified Y_2 remain.

Their trace criterion gives $a_w(Y_2) = \pm 1$, or $pq \equiv \pm 1 \pmod{w}$. Hence $|[A]_w| \ge p - 1$ as $w \ge 2p - 1$. But its unramified dual-pair budget gives $|[A]_w| < \sqrt{w}/2 \le bp/2 < p - 1$, a contradiction.

Thus A and any L are unramified at both singletons. By mixed Gram parity B ramifies at some t. At t the proper P_q-hits are A or ramified Y_2 by the same quotient test and the exclusion of Y_1.

Since B is the unique proper ramified degree at t, no ramified Y_2 occurs. The remaining hits are unramified of degree A, so $[A]_t = \pm 1$ by the trace criterion.

Since $C = A^2/p$, the vacuum gives $\delta j_t \equiv \pm p \pmod{t}$ and therefore $j_t(\delta + 3) > t + 1$ by (12.1). The p^3w-axis has at least δ labels, all t-unramified. By the q-principal comparison with quv its multiplicity is at least $(q - 1)/w$ for $q > 2w$; for $q \le 2w$ this bound follows from duality. Thus its even multiplicity is at least $t/(bp) - 1/w > \delta - 2$: use $t > (p - 1)\delta$ if $\delta \le p - 1$, and $t \ge p^2 + 2$ by primality if $\delta = p + 1$. The vacuum, this axis and an A dual pair give $j_t(\delta + 3) \le t + 1$ in the unramified budget, a contradiction. □

Proof of Theorem 9.1. The root heights i, j in Corollary 10.17 satisfy $i \le h$, $j \le k$. Its even case uses the dyadic enumeration of Theorem 9.15. The corollary gives each asserted bound except possibly $h + k \ge 6$ at an odd root p^3q^2uv. Such a root is excluded by Corollary 12.2 at $p = 3$ and by Lemma 12.8 at $p \ge 5$. □

13 Four repeated primes with an unrestricted exponent

Theorem 13.1. *Let P, Q, u, v be distinct primes with Q greater than at least one of P, u, v. For every nonnegative integer A, the integer $P^5Q^Au^3v^2$ is covered.*

Write a four-prime ordinary minimal root as P^hQ^kuv, with $u < v$ the isolated singletons. Here P, Q, u, v are distinct primes and $h, k \ge 2$. Proposition 9.3 supplies degrees $P^au, Q^bu, P^{a'}v, Q^{b'}v$, with positive powers. The same proposition gives $Q \mid d_J$ for every P-active label J. Consequently for $k = 2$ Theorem 6.9 gives Q-integrality of W_Pe_0. In particular $P^hW_Pe_0$ is integral: the singleton denominators are already removed by Lemma 6.1. This holds for arbitrary h and arbitrary prime positions.

The finite trace and Gram test of Theorem 9.15 excludes the two dyadic minimal root profiles with height pairs $(4, 2)$ and $(2, 4)$.

13.1 An arbitrary exponent at the larger of two higher primes

For four distinct primes P, Q, u, v with $P < Q$, arbitrary $A \ge 0$, and square-free e coprime to $PQuv$, every weakly integral braided fusion category whose dimension divides

$$P^5Q^Au^3v^2e$$

is weakly group-theoretical. Either one of the two designated squared exponents may be raised to three. Both can be raised if $PQ \nmid s - 1$ for some odd $s \in \{u, v\}$. For odd P, Q, the base conclusion with two squares is Theorem 10.15. The cube assertions then follow from Theorem 11.3. It remains to treat $P = 2$ in the two-square base. Write $q = Q$; the remaining three primes are odd. It suffices to exclude ordinary minimal roots $n = 4q^kuv$ with $k \ge 1$, $u < v$. Indeed, losing any repeated prime leaves at most three repeated primes with one exponent at most two, or fewer,

covered by Theorem 4.12 and Dong's theorem. With four repeated primes the reduced case is excluded by the fermion-pair lemma of Section 9, and the ordinary table has square dimension. Its 2-valuation is even and at most five, hence the root height at two is at most two. If two has root height one, the case with three squared positions is covered by the one-unrestricted-prime proposition of Section 9. The square-free decoration is permitted by divisor descent. An odd exponent at q in the source merely bounds $2k$ from above and does not change the argument.

Apply the finite trace and Gram test in the proof of Theorem 9.15 with $h = 2$. Only u, q, v are bounded. The level sets of the signed coordinates depend on $q^k \bmod uv$. For all positive i the signatures of q^i needed for the two axis tests and the common Gram equation are

$$(q^i \bmod 2uv,\ \min(q^i, 4)).$$

Starting at $i = 1$, multiply the first component by q modulo $2uv$ and cap the second component after multiplication. Stopping at the first repeated pair is exhaustive by determinism. We permit all these signatures at any k, relaxing the condition $i \le k$. The cap is exact for $\min(2^a, q^i)$, $a = 1, 2$. In each axis group with identical squared cost and trace residue, and in each common group with identical signed coordinate pair, the total multiplicity is zero or at least the smallest allowed positive multiplicity in that group. This aggregation preserves every possible table.

At each endpoint t, put $m_t = 4q^k uv/t$. Since $v \le 4u+1$ and $4q^k u > v+1$, the full-multiplicity bound of Lemma 10.2 reserves a full label beyond the $(t-1)j_t$ full labels already reserved. Its signed coordinate is one. The residual squared budget is $j_t(t+1-j_t)-1$ and the degree trace is $(j_t-1)m_t + \epsilon j_t$. Further full labels remain allowed.

The recurrence for the nonempty axis sets minimizes the sum of the squared costs at each residue modulo the omitted repeated prime. For the common degrees, enumerate their shared multiplicities in the remaining two norm budgets and impose their mixed Gram sum $-j_u j_v$. The recurrence gives 279 vacuum residue rows, 53 after both axis budgets, and two after the common Gram sum. Both rows have $q = 3$, $u = 5$, $v = 7$, $j_u = 1$, $\epsilon = -1$. Their $(j_v, 3^k \bmod 35)$ values are $(2, 17)$ and $(1, 27)$.

One calculation rules out both rows. At five the remaining squared norm is four and the degree trace is -1. Both axis types and a common degree must still occur, so there are exactly four remaining labels, each of signed coordinate $[d_X]_5 = \pm 1$. A 3-free one can only have degree 14: the alternative 28 has $|[28]_5| = 2$. Its signed trace contribution is -14. At least one such axis is compulsory, and the trace modulo three requires a second one. There are exactly two since the other axis and a common degree are also compulsory. The other axis has degree $7 \cdot 3^b$ with b odd because $|[7 \cdot 3^b]_5| = 1$. Every inactive label is unramified at this unit-deficit singleton. A common degree $2^i 3^a$ must therefore satisfy $5 \le 2 \cdot 2^i$ by the opposite axis, hence $i = 2$. Its residue has magnitude one precisely for even $a \ge 2$. The trace would give

$$\pm 4 \cdot 3^a \ \pm 7 \cdot 3^b = 27.$$

As the exponents have opposite parity, the 3-valuation gives $\min(a, b) = 3$. Hence $b = 3$, $a \ge 4$, and $4 \cdot 3^{a-3}$ would have to be 6 or 8. This is impossible.

For any specified dyadic root height $H \ge 2$, replace four by 2^H in the cap and window and allow all powers 2^i with $1 \le i \le H$. Again $2^H q^k u > v + 1$ forces the extra full label. The algorithm thus tests all opposite heights at once for each input H.

If $Q < P$ and Q exceeds a singleton, then the least of the four repeated primes is a singleton, which is excluded by Corollary 10.17. The same divisor descent and cube lift apply. This proves Theorem 13.1 in its stated generality.

The two dyadic exponent boxes. By the four-prime descent used above, the dyadic boxes $2^9 q^5 (uv)^2 e$ and $2^5 q^9 (uv)^2 e$ reduce to ordinary roots of higher heights at most $(4, 2)$ or $(2, 4)$. The trace calculation in Theorem 9.15 excludes every such root. A dyadic singleton is excluded by Corollary 10.17. Thus for $(8, 4, 2, 2)$ the remaining roots are odd, of form $P^4 Q^2 uv$ with P least

and P, Q mixed (Theorems 10.13, 10.15 and Section 12). The complementary-prime integrality above applies to these roots.

14 Residue conditions at two isolated primes

We use the exact full multiplicity and residual trace and norm of Theorem 10.3. Write $m = n/t$, $j = j_t$, $[\cdot]_t$ for centered residue, and ϵ for the common anomaly. Let k_t count the distinct proper nonunit t-free degrees.

Consider an ordinary odd four-prime minimal root $n = P^h Q^k uv$ with $h, k \geq 2$ and isolated $u < v$.

The two mandatory axes and a common degree from the mixed Gram equation give $k_t \geq 3$. Consequently $(t-1)/j_t$ is odd and at least five, and in particular the odd part of $t-1$ is at least five at *each* of u, v. Thus $P^A Q^B (uv)^2 e$ is covered for all four primes odd, arbitrary A, B, and coprime square-free e whenever the odd part of $u-1$ or $v-1$ is at most three. Losing a repeated prime descends by 4.12, a higher position of root height one descends by 9.12, and 4.7 excludes an odd reduced minimal table. 11.3 permits either one of the two designated squares to be cubed and both under its stated divisibility test. For $(A, B) = (8, 4)$ the result also holds for arbitrary prime assignments with either specified odd singleton satisfying the condition, by the already established even-core theorem.

The residual norm also restricts the pair of singletons. Write $a(t) = 2^{\nu_2(t-1)}$ and $o(t) = (t-1)/a(t)$. At some endpoint t there is a common proper degree ramified there (10.5). These labels are unramified at the other endpoint w by separation of inertia. They contribute at least j_t unramified labels there. The two compulsory axis degrees at w are distinct from the common degree and each other, and at least one supplies an additional unramified dual pair. The norm formula therefore gives

$$H_w = (w-1)/j_w \geq j_t + 3.$$

Since H_t is an odd divisor of $t-1$ and $a(t) \mid j_t$, a necessary condition on the two primes is

$$o(v) \geq a(u) + 3 \quad \text{or} \quad o(u) \geq a(v) + 3.$$

Furthermore $t - 1 = j_t H_t \leq (o(w) - 3)o(t)$, hence

$$u \leq o(u)o(v) - 3\min\{o(u), o(v)\} + 1, \qquad v \leq 3u/2.$$

Thus prescribed odd parts give only finitely many possible pairs, independently of both higher primes and exponents. In particular $o(u) = o(v) = 5$ forces $u \leq 11$ and then $u = 11$. The interval $11 < v \leq 33/2$ contains no prime $1 + 5 \cdot 2^k$ $(k \geq 1)$. More generally if both odd parts are at most seven, the single-end obstruction leaves only $5, 7$. If $o(u) = o(v) = 7$, the bound gives $u \leq 29$ and $v \leq 43$. There is just one prime with odd part seven in this range, namely 29. If the odd parts are five and seven, $u \leq 21$ and $v \leq 31$. The only possible pair is $11, 29$, and it violates the closeness bound. Thus both odd parts at most seven are impossible.

14.1 A residue criterion for the two higher exponents

Fix distinct odd primes P, Q, u, v with u, v the designated squares. For a divisor-minimal obstruction to $P^A Q^B (uv)^2 e$, loss of a repeated prime is covered by Theorem 4.12 or Dong's theorem; the reduced table and a third root height one are excluded by Propositions 4.7, 9.3 and 9.12. Thus its ordinary root is $P^h Q^k uv$, $h, k \geq 2$. At $t = u, v$ set $w = uv/t$ and $H_t = (t-1)/j_t$. These are odd indices at least five, with $3j_t \leq t + 1$ and $j_u + 3 \leq H_v$ or $j_v + 3 \leq H_u$, as proved above. The vacuum congruence reads

$$P^h Q^k w \equiv \epsilon H_t \pmod{t}.$$

In $G = \mathbb{F}_u^\times \times \mathbb{F}_v^\times$ let E be the set of pairs $(\epsilon H_t/w)_t$ for all such indices and both common signs. Every surviving exponent pair satisfies $P^h Q^k \in E$; the same necessity holds after any group homomorphism. We use its quadratic characters to constrain both parities and cyclic quotients to eliminate an unrestricted exponent.

Quadratic characters. Write χ_t for the Legendre symbol modulo t. For $t \equiv 1 \pmod 4$ and any odd divisor H of $t-1$, quadratic reciprocity gives $\chi_t(H) = (t/H) = 1$, where the right side is the Jacobi symbol. At such an endpoint every residual four-prime root therefore satisfies

$$\chi_t(P)^h \chi_t(Q)^k = \chi_t(w). \tag{$*$}$$

The same character argument at a ramified endpoint also applies with any number of other primes.

Lemma 14.1. *In an odd ordinary minimal table of dimension n^2, at most one isolated singleton $t \equiv 1 \pmod 4$ satisfies $\chi_t(n/t) = -1$. The other prime factors of n may have any heights.*

Proof. Among any two isolated singletons, a common inactive orbit ramifies at one endpoint t (Lemma 10.5). There $H_t = (t-1)/j_t$ is odd by Theorem 10.3, and the vacuum congruence gives $n/t \equiv \epsilon H_t \pmod t$. If $t \equiv 1 \pmod 4$, both factors on the right have character one. □

For each endpoint congruent to one modulo four, equation $(*)$ is linear over $\mathbb{F}_2$ in the parities of h, k. If these equations are inconsistent, every box $(A, B, 3, 2)$ is covered, for arbitrary A, B, by descent and the four-prime cube lift (Theorem 11.3). The cubed companion can be chosen freely. In particular it suffices that $\chi_t(P) = \chi_t(Q) = 1$ and $\chi_t(w) = -1$. The fixed primes $P = 13$, $Q = 17$, $t = 101$, $w = 103$ illustrate the test since $35^2 \equiv 13$, $44^2 \equiv 17 \pmod{101}$, and $103 \equiv 2$ is a quadratic nonresidue modulo 101.

If $\chi_t(P) = 1$, $\chi_t(Q) = -1$ and $\chi_t(w) = (-1)^\delta$, $\delta \in \{0, 1\}$, the possible source exponents at Q are $2k \equiv 2\delta \pmod 4$. Hence for $K = 4b + 2\delta$, $b \geq 0$, one has the fixed-prime lift

$$B(A, K, 2, 2) \quad \Longrightarrow \quad B(A, K+3, 3, 2).$$

Indeed, the largest allowed Q-valuation at most $K+3$ is at most K; first apply this with both squares intact, then apply the cube lift. The output $3, 2$ can again be permuted. More generally the admissible parities of both higher positions can be used together. For each admissible pair (α, β), round A down to the largest nonnegative integer $\leq A$ congruent to $2\alpha \pmod 4$, and do the same to B using 2β. A pair with no such corner is discarded. Coverage of each resulting corner with the two squares implies coverage of $(A, B, 3, 2)$.

At root heights $(4, 2)$ both higher powers are squares, so $\chi_t(w) = -1$ alone excludes that root. At every lower corner of the source $(8, 4, 2, 2)$, the higher root heights either sum to at most five or one is at most one, unless they are exactly $(4, 2)$. Those smaller cases are covered by Sections 9 and 12. Thus this quadratic nonresidue criterion already covers the complete divisor box $(9, 5, 3, 2)$ for every such assignment.

The same argument also handles two endpoints $u, v \equiv 3 \pmod 4$. For $m_t = (t-1)/j_t$ one has $\chi_t(j_t) = (-1)^{(m_t+1)/2}$ by the reciprocity sign. Putting $C = P^h Q^k$, multiplication of the two vacuum characters cancels the common sign $-\epsilon$, and reciprocity at u, v gives

$$\chi_u(C)\chi_v(C) = -\chi_u(j_u)\chi_v(j_v).$$

If all prime factors of $(u-1)(v-1)/4$ are one modulo four, then m_u, m_v are both one modulo four. Thus the right side is -1, excluding any square C, in particular root heights $(4, 2)$. If in addition $\chi_u(P) = \chi_v(P)$ and $\chi_u(Q) = \chi_v(Q)$, every pair of heights is excluded. For example take $(P, Q, u, v) = (13, 17, 443, 467)$. Here $(u-1)/2 = 13 \cdot 17$ and $(v-1)/2 = 233$. Modulo 13 the two singletons give $1, -1$, and modulo 17 they give $1, 8$. All four residues are squares, so reciprocity gives the two character equalities and the all-height conclusion applies. For either odd residue class, the image of E under (χ_u, χ_v) tests the four possible parity pairs with the same sign at both endpoints. The corner rule then applies to every choice of exponent caps.

Cyclic quotients and jumps of a higher exponent. Project G, E to either endpoint or retain both.

In $G/\langle P\rangle$, let d be the order of the image of Q and set

$$R = \{i \in \mathbb{Z}/d\mathbb{Z} : Q^i\langle P\rangle \text{ meets } E\}.$$

Then every surviving k satisfies $k \bmod d \in R$, simultaneously for all h. Conversely, those are exactly the $k \geq 2$ allowed by this residue test because the powers with $h \geq 2$ exhaust $\langle P\rangle$.

Suppose all divisors of $P^A Q^B (uv)^2 e$ are covered, where e is any coprime square-free factor. Set $k_0 = \min\{k \geq 2 : 2k > B,\ k \bmod d \in R\}$, with $k_0 = \infty$ if R is empty. Then $P^A Q^{2k_0-1} u^3 v^2 e$ and all its divisors are covered, with ∞ denoting an unrestricted cap. On the two-square box the higher valuation of a divisor-minimal obstruction is exactly $2k$. Theorem 11.3 then lifts either singleton to a cube. In particular one can start with $B = 0$ and arbitrary A.

At one endpoint put $G_t = \mathbb{F}_t^\times/\langle -1, P\rangle$. If the odd part of $t-1$ is a prime $\ell \geq 5$, then $H_t = \ell$. Thus $\bar{Q}^k = \bar{\ell}/\bar{w}$ in G_t, which has no solutions or one class $k = c \pmod d$, where $d = \mathrm{ord}_{G_t}(\bar{Q})$. In the latter case every $k_0 \geq 2$ in that class gives the lift

$$B(A, 2k_0, 2, 2) \implies B(A, 2k_0 + 2d - 1, 3, 2)$$

for the same prime assignments.

For example, let $u = 113$, $v \equiv 14$, $P \equiv 1$, and $Q \equiv 3$ modulo 113. The only allowed H_u is 7. The element 3 has order 112, with $3^{44} = 56$ and $3^{100} = 57$ modulo 113. Thus $14 \cdot 3^k \equiv \pm 7$ is equivalent to $k \equiv 44 \pmod{56}$. Consequently the box $(A, 87, 3, 2)$ is covered for every A, and for every $b \geq 0$ one obtains the transfer

$$(A, 88 + 112b, 2, 2) \implies (A, 199 + 112b, 3, 2).$$

The primes $(P, Q, u, v) = (227, 3, 113, 127)$ realize this assignment, with Q least and $127/113 < 3/2$.

The two-endpoint test can be sharper. If $u = 449$, $v = 463$ and P, Q have the residues of $23, 13$ modulo uv, the calculation in the finite quotient gives $d = 168$ and $R = \{42, 46, 126, 130\}$, proving $(A, 83, 3, 2)$ for arbitrary A. If Q instead has the residues of 67, then R is empty. Already at 449 the only allowed H is 7 and $23^{112}, 67^{112}, 7^{112}, 463^{112}$ are $1, 1, 1, -1$ respectively. Also $(-1)^{112} = 1$. Thus the two higher exponents are both unrestricted in this family. The set R can be computed from discrete logarithms at the selected endpoints and elementary linear congruences. This eliminates h by the generalized Chinese remainder theorem and never enumerates source dimensions or any finite range of the two heights.

Arbitrarily prescribed heights. Fix an odd prime Q, an integer $L \geq 2$ and a distinct odd prime t. Fix the residue of P modulo t, of multiplicative order m. For all but at most $2m(L-1)\tau(t-1)$ nonzero residues of the other singleton w modulo t, the box $(A, 2L+1, 3, 2)$ is covered for every A, for distinct odd primes P, Q, t, w. Here τ counts positive divisors and the cube may be put at either singleton. Indeed, for $2 \leq k \leq L$ the vacuum congruence restricts w to the residues $\epsilon H P^{-h} Q^{-k}$, with $H \mid t-1$ odd, $H \geq 5$, and h taken modulo m. This proves the count. The divisor descent above gives the asserted box.

The residue criterion excludes prime assignments even within the ratio bound of Proposition 10.12. Fix Q, L and let $t > Q$ tend to infinity through primes. Take $P \equiv 1 \pmod t$, so that $m = 1$. The prime number theorem gives $\sim t/((Q-1)\log t)$ primes w in $(t, Qt/(Q-1)]$, all in distinct classes modulo t. At most $2(L-1)\tau(t-1) \leq 4(L-1)\sqrt{t}$ are exceptional. The proportion covered by this test therefore tends to one in this interval. One may choose any prime $P > w$ in the class 1 modulo t by Dirichlet's theorem. Then Q is the least prime and $w \leq Qt/(Q-1)$ as required. Other necessary conditions may also exclude some or all of these assignments.

14.2 Finite windows with one unbounded height

The odd least-prime window. For the trace and Gram test at odd least prime P and root P^hQ^2uv, all three primes Q, u, v are free. For $h \geq 3$ use the window of Propositions 9.7 and 10.16, Lemma 9.14. Its first limit is

$$u \leq \begin{cases} (9P^{h-1}+2)/4, & P = 3, \\ (1 + P^2/(2(P-1)^2))P^{h-1}, & P \geq 5. \end{cases}$$

Put $U = \max(2, \lfloor (u+1)/j_u \rfloor - 1)$; then $v \leq P^hU + 1$ and $Q \leq vU + 1$. Remove $5, 7, 13$ as isolated singleton positions by Lemma 9.8 and Corollary 10.7.

Proposition 10.12 gives $v \leq Pu/(P-1)$. The deficits are even, with $3j_t \leq t+1$ and $(t-1)/j_t$ odd (Theorem 10.3); each proper degree occurs in dual pairs. At height $h = 4$ the two budgets and the sign of one mixed contribution suffice. For given u, v, j_u, j_v, ϵ, solve

$$Q^2 \equiv -\epsilon(P^4 v j_u)^{-1} \pmod{u}, \qquad Q^2 \equiv -\epsilon(P^4 u j_v)^{-1} \pmod{v}.$$

Combine all pairs of roots by CRT and retain prime representatives within the proven window. At each endpoint t, with $w = uv/t$, both a P-axis P^iw and a Q-axis Q^lw must occur. Each has at least two labels by odd duality. From the norm budget $j_t(t+1-j_t)$ subtract twice the least possible $A_t(P^iw)^2$ and twice the least possible $A_t(Q^lw)^2$, using admissible degrees. The remainder must be positive, since the mixed Gram sum is $-j_uj_v < 0$. Some common degree $y = P^iQ^l$ must contribute negatively to that sum. Its two signed costs $2A_u(y)^2, 2A_v(y)^2$ must fit the two budgets. It must be admissible at both endpoints and pass the common-degree unramifiedness test of Lemma 9.13 at one endpoint. The enumeration implements these necessary tests and gives

P	3	5	7	11	13	17	19
vacuum classes	14	59	181	1338	2706	8065	12977
positive axis budgets	0	3	0	1	0	1	1
negative common term	0	0	0	0	0	0	0

Thus the other primes are unrestricted at each of these least octic primes. The test can be applied at any other odd P and any root height $h \geq 3$ in P^hQ^2uv, replacing fourth powers by hth powers and using the corresponding proven window.

A general prime-size bound and the dyadic case. Let P, Q, u, v be distinct primes, $H \geq 2$ and $A \geq 0$. Set $b = P/(P-1)$ and $c = 1$ for $P = 2$, $c = 2$ for odd P. The bounds on the two axes imply that

$$Q > 1 + \frac{b^3}{c}P^{2H} \quad \Longrightarrow \quad P^{2H+1}Q^Au^3v^2e \text{ is covered.}$$

For the two-square base, Corollary 10.17 gives a putative ordinary minimal root P^hQ^kuv, with P least and $2 \leq h \leq H$, $k \geq 2$. After ordering $u < v$, Proposition 9.7 gives $u \leq bP^h$ and $\min(Q, v) \leq bu$. Lemma 9.14 gives $Q \leq vU + 1$, where U can be taken at most u/c. If $Q > bu$, then $v \leq bu$ and hence $Q \leq bu^2/c + 1$. If $Q \leq bu$ this bound holds as well. Substitution gives the contradiction, and Theorem 11.3, raises either of the two squared positions to a cube.

For $H = 3, 4$, consider all ordinary minimal roots 2^Hq^kuv, with q, u, v distinct odd primes and $k \geq 2$. Run the periodic vacuum, two axis trace and signed Gram tests of Section 13, with cap 2^H and all positive powers of q. Record just the q values among the surviving classes, denoting their set by S_H. The finite recurrence gives

H	vacuum	axes	Gram	$\lvert S_H \rvert$	$\max S_H$
3	3967	582	98	9	37
4	87793	5704	1638	39	283

Here $S_3 = \{3, 5, 7, 11, 13, 17, 19, 31, 37\}$ is contained in S_4. Two independent calculations give the same sets. The first enumerates residues by multiplication until repetition. The second reconstructs the vacuum residue by CRT, checks membership in the cyclic group generated by q, and retains minimal costs for each partial Gram sum. Allowing exponent signatures for all positive exponents includes every signature with actual exponent at most k. Thus an omitted q rules out all k and all prime assignments to u, v.

It follows that, for arbitrary $A \geq 0$, every divisor of $2^{2H+1}q^A u^3 v^2 e$ is covered if $q \notin S_H$ and e is square-free and coprime to the four primes. Indeed, for the two squared companions a divisor-minimal failure retains all four repeated primes. Its reduced case is excluded by Lemma 9.4, and its ordinary table is square by Proposition 9.3. Lower dyadic heights are one, covered by Theorem 11.3, two, covered by Theorem 13.1, or, at $H = 4$, three, where $S_3 \subseteq S_4$. A single opposite root height is covered by the same one-high-prime theorem. On loss of a repeated prime, there are at most two, or three with a squared companion. Finally Theorem 11.3 raises either designated square to a cube.

Unbounded opposite height at three. There is an odd version of the same projection with no bound on the opposite height. Consider an odd ordinary minimal root $P^h Q^k uv$, P least, $u < v$ isolated, $h, k \geq 2$. Put $b = P/(P-1)$. Use $u \leq bP^h$, $v \leq bu$ and $Q \leq v \max(2, \lfloor (u+1)/j_u \rfloor - 1) + 1$ (Propositions 9.7, 10.12 and Lemma 9.14). Remove singleton values $5, 7, 13$ (Lemma 9.8 and Corollary 10.7). Theorem 10.3 leaves the squared budget $j_t(t - j_t)$ for proper degrees and exactly one proper ramified orbit of size j_t. All proper multiplicities are even by duality; an odd signed coordinate, when used, has multiplicity exactly j_t (Corollary 10.6). Here j_t is even, has odd index in $t - 1$, and satisfies $3j_t \leq t + 1$.

At least one common degree is ramified at one of the endpoints (Lemma 10.5). Branch on that endpoint. Its two axes must be unramified and the mixed Gram sum must use a ramified degree there.

For clarity, the finite calculation in `odd_periodic.py` uses all positive powers of Q, represented by $(Q^i \bmod m, \min(Q^i, P^h))$ until repetition. The moduli are uv for the vacuum and the common degree tests, and Pt for a Q-axis at t (signed coordinate modulo t, trace modulo P). At the P-axis the exact powers $P, \ldots, P^h$ suffice. With $w = uv/t$ the axis degree is $D = P^i w$ or $Q^i w$. For the opposite higher prime L, minimize $\sum m_D A_t(D)^2$ subject to $\sum m_D D A_t(D) \equiv \epsilon j_t \pmod{L}$. Use the coordinate tests of Theorem 9.15, inertia floors and the multiplicities above. The mixed test uses the remaining two budgets and target $-j_u j_v$. For each pair of signed coordinates it keeps minimal v-cost for fixed u-cost, mixed sum, and a flag for the required ramified degree. Combining signatures can only allow additional solutions. The periodic calculation first runs the relaxed test, then tries the two choices of ramified endpoint for each survivor. Its counts are

(P, h)	window	vacuum and P-axes	both axes	mixed	remaining Q
$(3, 3)$	4376	9	4	2	11
$(3, 4)$	284978	23	20	11	$5, 7, 11, 19, 29$

The two remaining sign classes at height three both have $(u, v, j_u, j_v) = (23, 31, 2, 2)$.

A second congruence removes every remaining class. At the endpoint t carrying a proper ramified common orbit, Theorem 10.3 gives the exact trace $\sum A_t(D)D = \epsilon j_t$ and squared cost $\sum A_t(D)^2 = j_t(t - j_t)$ on the proper labels, counted with multiplicity. Both identities may be halved by duality.

For $m = Q^s$ the trace calculation retains all degrees $D = P^i Q^a w^\delta$, $0 \leq i \leq h$, $a \geq 0$, $\delta = 0, 1$, having at least two distinct prime factors, as required by the prime-power exclusion of Lemma 2.5. The signatures use residues modulo mtw (or the least common multiple) and the same height cap. An odd coordinate is allowed only at a common degree. For a common degree the two signed coordinates must be admissible and their gcd bound must permit unramifiedness at one endpoint (Lemma 9.13). Choose t by Corollary 10.6, so that its proper ramified common orbit

also has odd centered degree at w. The row cost of a dual pair is $A_t(D)^2$ and its trace increment $DA_t(D)$ mod m. The recurrence uses each signature zero or more times within the exact cost $j_t(t-j_t)/2$, with three flags recording the two mandatory axes and the ramified common orbit. The latter must occur exactly once, contributing $j_t/2$ pairs of just one signature. An accepted state requires all three flags and residue $\epsilon j_t/2$ mod m. For each tested modulus both endpoints are allowed. Retesting surviving rows gives the following counts:

h	mixed	Q^2	Q^3	Q^4
3	2	0	0	0
4	11	5	1	0

The single class after Q^3 is $(Q,u,v,j_u,j_v,\epsilon)=(7,103,149,2,4,-1)$. Thus neither height admits an odd ordinary minimal root with $P=3$, for any other odd prime Q or any $k\geq 2$. The height-two case with $P=3<Q$ is covered by Theorem 10.15. By the same reduced-case exclusion, one-high-prime descent and cubic lift as in the dyadic projection, every divisor of $3^9Q^Au^3v^2e$ is covered for every odd prime $Q>3$, arbitrary $A\geq 0$, any further distinct primes u,v, and coprime square-free e. An even singleton cannot yield an ordinary minimal obstruction by Corollary 10.17.

Consequently an ordinary obstruction to the unordered $(8,4,2,2)$ family must be odd, with the octic prime least and at least 23. The four dimension thresholds do not require the finite windows of this section.

15 Five repeated primes and local Weyl columns

Theorem 15.1. *For arbitrary distinct primes p,a,b,c,d, the integer $p^4(abcd)^2$ is covered. Any three of the four squared exponents can be raised to three.*

The square of a product of five primes gives the starting case. Its proof also yields the unbraided solvability assertion of Theorem 1.1. We then pass to the higher exponents by studying the supports of principal witnesses.

15.1 The square of a product of five primes

Theorem 15.2. *Let n be the product of five distinct primes. Every weakly integral braided fusion category of dimension dividing n^2d, with d square-free and coprime to n, is WGT.*

Proof. At a least counterexample among its divisors, losing a repeated prime is covered by Theorems 4.12 and 11.3. The reduced case has either two-valuation one or odd reduced dimension, excluded by Lemma 2.5 and Proposition 4.7, respectively. Thus the table is ordinary with square root n.

Every isolated singleton is at least five by Lemma 6.6.

Suppose first some twist contains two primes a,b. Put $C=ab$ and let $u<v<w$ be the others. Hits on that column omit a,b by defect-zero vanishing. Reducing its vacuum orthogonality on P_a and P_b at each of u,v,w forces the three pair degrees uv,uw,vw in both blocks. Lemma 9.2 separates u,v,w from a,b. A twist meeting two of u,v,w would share only the vacuum hit with the mixed column. Thus the last three primes are isolated, and are odd. For any two of them s,t with partner x, a nonunit degree prime to st must contain both a,b: otherwise it must hit degree st in the corresponding principal block, contrary to coprime vanishing. It cannot be C, since C and st would be coprime x-free degrees in P_x. Thus its only possible value is Cx, and the mixed column norm ensures it occurs. At s the distinct proper degrees tx,Ct,Cx share the norm $j_s(s+1-j_s)\leq (s+1)^2/4$, so each signed coordinate is centered. Corollary 9.10 now rules out this triple of isolated primes.

If all primes are isolated they are odd. Each proper degree occurs with its distinct dual, so again its signed coordinates are centered. Form the graph whose edges st are the two-prime degrees that occur. Two disjoint edges are impossible: their degrees would be coprime and principal at the fifth prime. If $\{s,t\}$ meets every edge, the only common nonunit degree is $n/(st)$ by the mixed norm identity. It occurs at least twice by duality. No vertex w has two distinct smaller partners of this kind, by Corollary 9.10. This excludes a graph supported on at most three vertices: every pair in a containing triple meets every edge. Otherwise the pairwise-intersecting edges form a star with at least three leaves. For its center h every pair $\{h,t\}$ meets every edge, so at most one prime is smaller than h. Choose larger leaves c,d and a third leaf b. A label of degree hb is unramified at one of c,d, say c, by Proposition 6.2. Pair it with a label of degree hd. Both degrees omit c and the fifth prime, hence both labels are principal at each by Proposition 6.3. Their normalized entry has strict house below h and residue h at both primes, contradicting Proposition 4.8 since $c > h$. □

Corollary 15.3 (Solvability without a braiding assumption)**.** *Every fusion category of square-free dimension with at most five prime divisors is solvable; if integral it is group-theoretical. For four distinct primes with* $\min(u,v) < \min(p,q)$*, every fusion category of dimension dividing* uvp^aq^b *is solvable, with arbitrary* $a,b \geq 0$*. In particular this includes* $2pq\ell^h$ *for any distinct odd primes* p,q,ℓ*.*

Proof. Pass to the Drinfeld center, whose dimension is the square of the given dimension. The five-square theorem applies in the first case. In the second, Corollary 10.17 excludes every minimal obstruction, since the least prime would have to be p or q. WGT descends from the center by Morita invariance and closure under subcategories [32, Proposition 4.1]. In both cases $4 \nmid \mathrm{FPdim}(\mathcal{C})$. To prove solvability, and for later use at the odd cutoffs, recall the group-extension argument. A WGT category is Morita equivalent to a nilpotent category built by extensions; the group orders multiply to its dimension. If 4 does not divide the dimension, each group has odd order or order twice odd. In the latter case the sign of the regular permutation representation gives an odd-order kernel of index two. The odd-order theorem [33, p. 775] and [32, Proposition 4.5(i)] now give solvability. In integral square-free dimension use also [32, Proposition 9.1]. □

Proof of Theorem 1.1. For $p < q < r < s$, Corollary 10.17 covers all divisors of $p^2q^2r^as^bd$: the least prime of an obstruction would have to be r. Theorem 11.3 raises both smaller powers to cubes, since q is odd and $rs \nmid q-1$. The assertion without braiding follows from Corollary 15.3 with p,q as the two singletons. □

15.2 Prime support of principal witnesses

Work in a perfect fusion-simple ordinary integral modular table with dimension Δ and prime support P. A squared prime has valuation two in Δ. A witness (s,A) means that the principal s-block P_s contains a nonunit whose degree is supported in $A \subseteq P$. If $|A| < 2$, no such witness exists. If $(s,A),(t,B)$ are witnesses with $A \cap B = \varnothing$ and at least one of $s \notin B$, $t \notin A$ holds, their entry vanishes by coprimality and is nonzero by principality (Lemma 2.5). For increasingly ordered odd primes, a two-element $A = \{u,v\}$ with $u < v$ and $v_u(\Delta) = 3$ is also impossible, since $v \mid u-1$ by Lemma 11.1.

Lemma 15.4. *Call two primes mixed if one column is active at both. Every prime of odd valuation in* Δ *mixes with every other prime. The following rules produce witnesses.*

1. *Let* J *be* s*-active and let* $q \neq s$*. In a witness* (s,A) *one can omit* q *and every squared prime at which* J *is active. One can also omit* s *if* $v_s(\Delta) \leq 3$*.*

2. *Let* J *be* s*-active, and* $w \neq s$ *a squared prime. There is a witness on* P_s *omitting* w*, every squared prime active at* J*, and one prescribed prime* $q \nmid d_J$*. One may instead take* $q = s$ *whenever* $v_s(d_J) \leq 1$*, in particular if* $v_s(\Delta) \leq 4$*.*

Proof. Categorical Cauchy gives an active column at each prime. If no column were active at both s, u, every u-active column would have s-unit codegree (Lemma 3.3). This is impossible if $v_s(\Delta)$ is odd since $f_J = \Delta/d_J^2$. For the first rule apply prime avoidance on P_s at q (Proposition 4.2). A hit on J has degree prime to every squared active prime by defect-zero vanishing (Theorem 3.1). The extra omission follows from Corollary 4.1.

For the second rule, choose a w-active, s-inactive column K as follows. The product of the complementary Weyl elements sends the vacuum into the trivial s-twist fiber. Its projection to the w-active sector is nonzero, since the s-local Weyl preserves that sector and completes the product to the global Weyl, whose vacuum column has full support. The J, K columns are orthogonal on P_s, with vacuum contribution one. If $q \mid d_X$ and $q \nmid d_J$, then $\chi_J(X) = (d_X/d_J)\chi_X(J)$ vanishes modulo a prime above q. For $q = s$ this still holds if $v_s(d_J) = 1$: the quotient is locally integral and $\chi_X(J) \equiv d_J \equiv 0$ by principality. Thus there is a q-free common nonunit hit. Defect-zero vanishing omits w and all squared primes active at J. Finally $s \mid f_J$ by Lemma 3.3, so $2v_s(d_J) \leq v_s(\Delta) - 1$.

□

Allowing more than two active primes on one column. The first rule of Lemma 15.4 applies to every squared prime simultaneously active on the chosen column. In an ordinary minimal table with $k \geq 3$ dimension primes, choose $k-2$ primes of which one, s, has valuation two or three and all the others have valuation two. No column can be active at all $k-2$ primes. Otherwise apply prime avoidance on P_s at one of the two complementary primes. The required nonunit omits that prime, s by the principal valuation bound, and each other chosen prime by defect-zero vanishing. Its degree is supported at most at the last complementary prime, contrary to Lemma 2.5. Both complementary valuations are arbitrary.

15.3 A cube and two squares among five primes

Theorem 15.5 (Two squares beside a cube)**.** *A divisor-minimal braided obstruction with exactly five repeated primes cannot have one valuation three and two other valuations two. The two remaining valuations may be arbitrary. Moreover, for five distinct primes $p_1, \ldots, p_5$, every weakly integral braided fusion category of dimension dividing*

$$(p_1p_2p_3p_4p_5)^3 d$$

is weakly group-theoretical, where d is square-free and coprime to the five primes.

Proof. Denote the cubed prime by p, the squared primes by u, v, and the others by a, b. In the ordinary case p mixes with u, v by Lemma 15.4. Its first rule, applied to P_p with additional omitted prime b or a, gives respective witnesses $(p, \{v, a\})$ and $(p, \{u, b\})$. Their disjoint supports contradict principality.

For a positive reduced table, a squared prime two would leave two-valuation one in the reduced dimension, contradicting its square-fullness. If p is odd, then u, v are odd and the same prime avoidance and defect-zero arguments apply on the positive table. The joint columns exist because a nonempty u- or v-active fiber contains a p-active pair at odd p-valuation (Lemma 7.8). The argument also permits $a = 2$ or $b = 2$.

If $p = 2$, pass to the integral minimal modular extension with fermion g. Its full principal two-block has only odd degrees. Its noninvertible degree supports intersect pairwise and have empty total intersection (Proposition 4.7), so none is a singleton. At each $w = u, v$ take an even w-active label by reduced Cauchy and choose its fermion mate if necessary to make its actual dyadic twist nontrivial. In the full block orthogonality the two invertibles $1, g$ contribute two because the column is even. Reducing modulo the odd primes b, a in turn therefore gives the same disjoint supports as above. Coprime vanishing holds in the full extension by Lemma 4.4, contrary again to two-principality.

For the divisor assertion, a drop to four or fewer repeated primes is covered by Theorems 4.12 and 11.3 and the two-prime theorem. If all five survive and at least two are squared, either a cube occurs or the five-squares theorem (Theorem 15.2) applies. If precisely one square remains, there are four cubes and one square w. Choose three distinct odd cubed primes p, u, v with $u < v$. A joint p, w-active column exists as above (a squared two in a reduced source is already impossible). Orthogonality on P_p modulo the fifth prime forces a degree of prime support $\{u, v\}$. By Lemma 11.1, the odd cubic valuation at u requires $v \mid u - 1$, a contradiction.

It remains possible to cube the fifth prime as well. Suppose every valuation is three and write h for the least prime. No degree can have exact support $\{a, b\}$ for two odd primes $a < b$ among the five, by Lemma 11.1. Fix distinct primes $p, s \neq h$. Every s-free nonunit in P_p has degree divisible by h: it omits p, and omitting h as well would leave at most the two remaining odd primes. The same holds with p, s exchanged.

In an ordinary table all these witnesses have h-defect one. If h is odd, since it is least, their two h-block orbits have equal degrees; only 1 and one square class of primitive hth roots are visible on each row, as in Theorem 7.9. At $h = 2$ the same conclusion uses one primitive fourth root in place of the class (Lemma 7.5). A witness in P_s hits all the s-free active mates of the witnesses in P_p, so both witness sets share one allowed class. Prime avoidance now places every p- or s-active label in the allowed h-phases. A phase outside one and this class occurs by Lemma 7.4 or 7.6 (for odd h the table has odd dimension). On that fiber no p-active label occurs. The p-local group preserves the fiber and its generating conjugate unipotents are scalar there. Its Weyl therefore preserves every label line in the fiber, forcing p-unit codegree by Lemma 3.3. This is impossible since $v_p(\Delta) = 3$.

A reduced table in the all-cubed case has $h = 2$, with positive two-valuation two. For any two odd primes p, s, the same complementary witnesses are thus of dyadic height one, with singleton positive two-block. We detail the consequence, since the argument works with the other odd primes present. Every p-active positive pair hits an s-free witness in P_p; such a witness vanishes on every nontrivial squared dyadic twist. By odd p-valuation each occupied squared dyadic fiber contains a p-active pair, so all actual even dyadic twists are signs. For each such witness the full block has one free odd orbit of the same degree by Lemma 4.20. A full p-principal representative hits the odd mates of each opposite witness because they are p-free. A witness sees only the phase of its own odd mate on the odd sector, by local Weyl invariance, hence this phase is common to both sets. The full p-principal block contains only even labels and omits the fermion: evaluate at the fermion and at an opposite odd mate, respectively. Thus every odd p-active column hits a complementary witness by orthogonality modulo s. Since the odd valuation at p puts such a column on every nonempty odd dyadic fiber, all odd dyadic phases agree. This contradicts Lemma 7.7. □

15.3.1 Four squares beside an unrestricted power

Theorem 15.6 (Five-prime lifting)**.** *Fix five distinct primes p, a, b, c, d, a nonnegative integer H, and a square-free integer e coprime to them. Suppose every weakly integral braided category of dimension dividing $p^H(abcd)^2e$ is WGT. Then the same holds for every divisor of*

$$p^H(abc)^3d^2e.$$

Any one of the four companions may be assigned to the squared position. All four may be cubed if their least member h is less than p and at least one of the other three, say v, satisfies $p \nmid v - 1$.

Proof. After the loss of any repeated prime use Theorems 4.12 and 11.3: at most one of the four surviving primes has an unrestricted exponent. With all five present, four exact squares are covered by hypothesis. If there is a cube and at least two squares among the designated positions, Theorem 15.5 applies. If there are three cubes and one square d, choose two odd cubed primes $u < v$ and call the third t. On P_t, use a joint t, d-active column and reduce vacuum

orthogonality modulo p. As in the proof of Theorem 15.5, every hit omits t, d, so the reduction forces a nonunit with support exactly $\{u, v\}$. If $t = 2$ in the fermion case use the full two-block: the two invertibles contribute two and p is odd. A squared two is impossible in a reduced source. In every case Lemma 11.1 would give $v \mid u - 1$. This proves the first assertion.

For the last assertion it suffices to exclude four exact cubes. Denote by q, s the two cubed primes other than h, v. An s-free nonunit of P_q omits q. If it also omitted h, it would have support $\{v, p\}$; since v is an odd cubic prime, Lemma 11.1 would force $p \mid v - 1$. Thus every such nonunit is h-bearing, as is every q-free nonunit of P_s. These are precisely the witness properties used in the all-cubed part of Theorem 15.5: h is the least prime of the minimal table, h, q, s have odd valuation three, and the opposite witnesses have h-height one. In the ordinary case their defect-one blocks confine the h-twists on the q-active columns; a missing phase then forces q-unit codegree despite cubic valuation. In the reduced case $h = 2$, and their positive singleton blocks force a common dyadic phase on the odd sector by the same full-extension argument. Neither argument uses the valuation at p. □

15.4 The slightly degenerate case of one quartic and four squares

If the quartic prime is odd, a squared two would give $v_2(\Delta) = 1$, contrary to square-fullness. If two is absent or occurs in the coprime square-free factor, the source or its positive quotient would have odd dimension, excluded by Proposition 4.7. Thus it suffices to consider a quartic at two.

Proposition 15.7 (A quartic two and four odd squares)**.** *There is no reduced minimal obstruction of source dimension $2^4 n^2$ for n a product of four distinct odd primes.*

Proof. Write $\Delta = 8n^2$, let P be the positive principal two-block, and choose the integral minimal modular extension. By Lemmas 4.4–4.5 and Proposition 4.7 the degrees in P are odd, its rank is divisible by eight, and no two distinct primes of n can occur as their exact product in a simple degree. The positive quotient is perfect and simple, hence has no nonunit prime-power degrees. Each nonunit of P therefore has degree n or n/a for some prime $a \mid n$.

Call an even pair deep when its squared dyadic twist is nontrivial. If a deep pair were also a-active for $a \mid n$, its row would vanish on every a-bearing degree by defect-zero vanishing. On P its row is orthogonal to the vacuum by reduced dyadic block orthogonality. Reducing there modulo a different $b \mid n$ leaves only the unit: no nonunit in P omits both a, b. Thus no odd prime is active on a deep pair.

A deep pair exists: otherwise the even dyadic twists are signs, contrary to Lemma 8.23 at odd $v_2(\Delta)$. On each deep squared-twist fiber the odd local groups act by scalars, since they preserve the fiber, their upper unipotents are scalar, and their conjugates generate. The symmetrized pair line preserved by each odd Weyl lifts to two invariant coordinate lines, since the odd group commutes with the dyadic unipotent and the two actual even twists are opposite. The complement lemma (Lemma 3.3) now makes the positive codegree pure dyadic. It is 2 or 8. A codegree of two leaves room for at most one nonunit hit besides the vacuum by the mean-square bound. The dimension orthogonality equation then implies a nonunit degree divides one. Hence every deep pair Z has $d_Z = n$, $f_Z = 8$. Its row hits every label of P by principality and oddness of n. Since the row norm is eight and $|P|$ a positive multiple of eight, $|P| = 8$; every entry there has Galois mean squared modulus one and is a root of unity, and the row vanishes elsewhere. This last assertion uses the arithmetic–geometric mean inequality and Kronecker's theorem for each nonzero algebraic integer.

Let K be the dyadic Weyl restricted to the positive space. It preserves each positive two-block, as follows by lifting the block to the full extension using odd squares in the even ring (Lemma 4.5). Thus each deep Z itself belongs to P: the odd Weyls preserve its line, so Ke_Z is its global reduced Weyl column up to phase and has nonzero vacuum entry. On the row of Z, dimension orthogonality modulo each $a \mid n$ requires an n/a-degree label. There are therefore at most three labels of degree n in P besides the four required degrees and the vacuum.

Take the power of the squared dyadic twist operator which gives $R = 1 - 2E$ on the positive space, where E projects onto all pairs of maximal squared dyadic twist order. Let m be its rank, so $1 \le m \le 3$. The scalar lifts cancel in $RKR^{-1}K^{-1}$, making its trace an algebraic integer since the local modular group is finite. On the m indicated columns K agrees up to phases with the reduced Weyl. Each such column has squared modulus $1/8$ on each of the m coordinates, by the preceding row equality. Expanding the trace gives

$$\operatorname{Tr}(RKR^{-1}K^{-1}) = \dim A - 4m + 4\operatorname{Tr}(EKEK^{-1}) = \dim A - 4m + m^2/2,$$

where A is the positive space. Thus $m = 2$.

On each degree n/a occurring once in P, both maximal-twist characters take the same value, 1 or -1. Indeed P is Galois stable, so its unique label of that degree has rational character field. The sign is determined by reduction of the dimension equation modulo a. At least three of the four degrees n/a occur just once because P has size eight and contains the unit and the two maximal-twist labels. If all four occur once, the two rows agree on at least five of their eight unit-modulus coordinates, violating orthogonality. Otherwise they agree on four coordinates including the unit, and hence they are opposite on each of the other four. In particular their sums on the two labels at the repeated degree n/a are negatives of one another. Both sums are congruent to $-(n/a)^{-1}$ modulo a by dimension orthogonality. Since a is odd, this is impossible. □

15.5 One quartic and four squares in the ordinary case

Write $n = p^2abcd$ and $\Delta = n^2$, and let $\beta = m/(m-1)$ for the least prime $m \mid n$. Two primes *mix* if some twist is active at both; a prime with no partner is *isolated*. In this section a *singleton* is a prime of valuation one in n.

We recall three facts about an isolated singleton t from Section 6. Each t-free label lies in P_t. Its signed coordinate in the local Weyl column is a nonzero integer off the vacuum, and the sum of their squares over *proper* degrees (smaller than n/t) is at most $B_t = j_t(t+1-j_t) \le (t+1)^2/4$, for a positive integer j_t. On proper t-free labels unramified at t we have the stronger bound

$$1 + \sum_X [d_X]_t^2 \le (t+1)/j_t,$$

where $[d]_t$ denotes the centered residue. There are at least three distinct proper t-free degrees. Consequently all signed coordinates at these degrees are centered: an uncentered coordinate would exhaust B_t by itself. An isolated singleton is at least five. For two isolated singletons $u < v$, if the only nonunit degree omitting both is $n/(uv)$, then $v \mid n/(uv) \pm 1$. For three such primes $u < v < w$, this unique-degree property cannot hold at both pairs u, w and v, w (Lemma 9.9 and Corollary 9.10). Their mixed column norm always supplies at least one common degree.

Lemma 15.8. *Suppose X has degree Ax, where x is a singleton and A is a power of p. For each singleton $q \ne x$ there is a normalized crossing of X with common weight at most A and positive residue equal to that weight at q. Thus $q > \beta A$ forces K_X to ramify at q. If $y \ne q, x$ is an isolated singleton and K_X is unramified there, then $q \le \beta Ay$.*

Proof. Choose an x-free nonunit of P_q by prime avoidance. It hits X by q-principality. If e is the inertia index at q of the normalized entry field, Lemma 9.5 gives $q \le \beta e \gcd(d_X, d_Y) \le \beta eA$. All inertia translates of X are proper y-free labels. If unramified at y, there are at most y of them by the displayed bound. Their number is at least e because the entry lies in K_X. □

15.5.1 Which primes can mix

Proposition 15.9 (The unique possible mixed pair)**.** *In an ordinary minimal table with square root p^2abcd, at most one pair of distinct primes can mix. If p, a mix, the other three primes*

$b < c < d$ are isolated and all their pairwise products occur in P_p. For any two of them with third prime x, the only possible common nonunit degrees are px and p^2ax, and at least one of pb, pc occurs. The quartic prime cannot be least when all five primes are odd.

If two squared primes u, v mix, write c, d for the remaining squared primes. Then p, c, d are isolated. The sole nonunit degree omitting both c, d is p^2uv. At each $t \in \{c, d\}$, writing $w = cd/t$, the sole p-free nonunit degree in P_t is uvw. Both p^2c, p^2d occur in each of P_u, P_v.

Proof. If p, a mix, the second rule of Lemma 15.4 supplies a witness on P_p omitting p, a and any specified one of b, c, d. Prime-power degrees are impossible, so all three pair degrees bc, bd, cd occur on P_p.

A second partner would force a disjoint pair of these p-free degrees on P_p, contrary to principality and coprime vanishing. In the one-partner case the other three primes are isolated. Indeed, an occurring degree uv at two squared primes forces every nonunit of P_t, $t \notin \{u, v\}$, to carry u or v, by t-principality. A joint t, u-active column would yield a v-free nonunit hit on P_t by prime avoidance, whereas each hit omits u. The same holds for v. Applying this to bc, bd, cd proves isolation. For any two of the three isolated primes with third member x, a common nonunit degree can only be px or p^2ax. Indeed on omitting the two endpoints a p-free degree cannot hit their coprime product in P_p. A degree p^ia misses x and would have to hit the same product by x-principality. A degree p^2x belongs to P_a by full p-height in the cluster $\{p, a\}$, and a degree pax belongs to P_p by full a-height. Thus both encounter the same coprime product. This exhausts the possible degrees. If px occurred for neither $x = b$ nor $x = c$, the pairs c, d and b, d would have unique common degrees $n/(cd), n/(bd)$. At each endpoint another proper degree (the product of the other two isolated primes) ensures centering, since one uncentered coordinate exhausts the residual budget. This contradicts Corollary 9.10.

Choose one such px. At each of the other three companion primes $q > \beta p$ its field ramifies by Lemma 15.8. These primes belong to distinct twist clusters, so at most one can ramify. If p were least and odd, each companion would exceed $\beta p = p^2/(p-1)$, a contradiction.

Now suppose u, v are squared and mix. The preceding case forbids either from mixing also with p. On the mixed column J every hit omits u, v by defect-zero vanishing. Vacuum orthogonality on P_u reduced successively at p, c, d forces supports cd, pc, pd in P_u, since the lesser supports have prime-power degree. The same holds on P_v. For any prime $t \notin \{c, d\}$ every nonunit of P_t hits the t-free degree cd, so contains c or d. A t, c-mixed column would require a d-free P_t-hit, by vacuum orthogonality, whereas all its hits omit c. The same argument separates d from t. Finally a c, d-mixed column would share only the vacuum hit with J. Thus c, d are isolated and at least five (Lemma 6.6).

The mixed norm at the isolated columns c, d gives a common nonunit omitting both by Proposition 6.3. Any such X must contain u, v: otherwise it would have to hit a coprime cd from the corresponding principal block. If its p-height were less than two, the complementary cluster $\{p, u, v\}$ puts X in P_p by full height at u, v (Proposition 6.2), again impossible at the p-free degree cd. Thus $D = p^2uv$ is the unique common degree.

The prime p is isolated by the first case and the isolation of c, d. A witness of degree p^ic on P_u or P_v cannot have $i = 1$: it would then belong to P_p and have to hit a coprime p-free nonunit of P_c. The same argument at d supplies the two squared axes in the assertion. Finally, a p-free nonunit of P_t must meet both p^2t witnesses in the opposite principal blocks whenever it omits u or v. Thus it contains both u, v, and it must also meet p^2w by t-principality. This forces uvw. Prime avoidance gives its existence. □

15.5.2 A quartic at two

Proposition 15.10 (Separation among four odd squares)**.** *Let an ordinary minimal table have dimension $16M^2$, with M the product of four distinct odd primes. No two odd primes occur in the same twist. No degree has the form $2a$ for $a \mid M$ prime. A degree $4a$ can occur only if at*

most one of the other prime factors s of M satisfies $s \geq 11$ or mixes with two and satisfies $s \geq 5$.

Proof. Suppose a column is active at odd primes $a < b$, and write $c < d$ for the other two. Every hit on this column omits a, b by defect-zero vanishing. On each of P_a, P_b, reducing vacuum orthogonality successively at $2, c, d$ forces all three supports $cd, 2d, 2c$, since a nonunit cannot have prime-power degree. Cross a label of support $2c$ in P_a with one of support $2d$ in P_b. Their lcm-normalized entry z has strict house below its weight $g \in \{2, 4\}$ and has residue g at both a, b. The short-word theorem (Theorem 4.9) forces $a = 3$ and $g = 4$. If $b \geq 7$ it gives $z = 1 + 3\xi$ for a nonidentity b-power root. Its three difference positions $1, \xi, \xi^{-1}$ are exposed since $3 < b/2$, and a tame dilate intersects them only at one. The difference-set lemma (Lemma 4.15) excludes this. Hence $b = 5$ and K_{4c} ramifies at five by the two-residue inertia bound (Proposition 4.8). In particular all pairs of odd primes at least five have already been separated.

Here $d \geq 11$. Choose a c-free nonunit of P_d by prime avoidance. Its crossing with the label of degree $4c$ has common weight $h \leq 4$ and residue h above d. If it were unramified there, $(z' - h)/d$ would be an algebraic integer of strict house less than $2h/d < 1$, contradicting strictness. Thus K_{4c} also ramifies at d. This contradicts separated inertia (Proposition 6.2) at $5, d$ and proves separation.

Apply the same argument to a putative label of degree $2a$: each $s \neq a$ dividing M with $s \geq 5$ gives an a-free witness in P_s, now of common weight at most two. There are at least two such s, forcing simultaneous ramification. For $4a$ the ordinary prime-avoidance argument applies at each $s \geq 11$. When s mixes with two, use a joint active column for this avoidance; every hit on it has two-height at most one, so $s \geq 5$ already suffices. □

Corollary 15.11. *In an ordinary minimal table of dimension $16(abcd)^2$, every prime is isolated.*

Proof. The four odd primes are pairwise separated by Proposition 15.10. If two had a partner, the one-partner proof would force a degree $2x$ for one of the other three singletons. The same proposition rules this out. □

15.5.3 All five primes isolated

If two were a singleton it could not be isolated. Thus either $p = 2$ or n is odd. We first show that a degree cannot contain p to exponent exactly one in either case.

Lemma 15.12 (No intermediate height at an isolated odd quartic)**.** *In an ordinary minimal table of odd dimension n^2, suppose h is isolated and $v_h(n) = 2$. Then every simple degree has h-valuation zero or two.*

Proof. Any label of h-valuation one and its distinct dual give two such labels. The Gauss calculation in the proof of Lemma 6.8 then gives

$$\sum_{\substack{X \ h\text{-active} \\ d_X = n/h^2}} (1 - \theta_X) = h^2.$$

The summation set is a union of dual pairs with equal twists, so its sum is twice an algebraic integer. This is impossible for odd h. □

The intermediate height at two. Put $M = abcd$. We first exclude all degrees of two-valuation one. Two such labels are already impossible by Lemma 6.8. Suppose there is just one, X. The proof of that lemma up to the Gauss identity still gives $\delta_0 = 4$ and $2 \leq |A_0| < 4$, where A_0 consists of the dyadically active labels of odd degree M and $\delta_0 = \sum_{Z \in A_0}(1 - \theta_Z)$. Indeed the upper bound uses only X and the mean square bound on its crossings with A_0; the lower bound uses only the Gauss identity. If a label J of degree $2M$ were dyadically active, the final paragraph of that proof applies without a second height-one label: its row vanishes on all even

degrees and hits the vacuum, A_0, and three distinct proper odd degrees, impossible at codegree four. All active labels therefore have degree M. The Gauss identity gives $|(4W_2e_0)_0| = 3$. The integral vector $4W_2e_0$ has squared norm 16. The unique height-one label costs at least four, the vacuum nine, and each odd nonunit contributes at least one, by Proposition 6.2. Besides A_0 there is room for at most one proper odd nonunit label Y. On the row of an active Z all degree multiples of four vanish. Reducing dimension orthogonality at each prime of M thus forces that prime to be missing in X or Y, since all remaining nonunit degrees on the row equal M. If Y is absent then X has pure dyadic degree; if present then $\gcd(d_X, d_Y) = 1$, impossible because $X \in P_2$ and d_Y is odd.

In both cases the degrees are now subproducts of the five pairwise coprime integers $H = p^2, a, b, c, d$. Join two positions if their product occurs as a degree. Two disjoint edges would be coprime rows on the principal block of the fifth prime, where principality forces them to hit each other. If two singleton vertices s, t meet every edge, the sole possible nonunit degree omitting both is $n/(st)$, and it occurs by the mixed column norm. By the three-singleton result recalled above, no singleton vertex has two smaller singleton partners with this property.

At least one degree Hx occurs. Otherwise all edges are on the four singleton vertices. If supported on at most three of them, every pair in a containing triple would meet every edge. Otherwise the pairwise-intersecting edges form a star on four vertices. Its center h has two larger leaves c, d; let b be the remaining leaf. A label of degree hb is unramified at one of c, d, say c, by separated inertia. Its crossing with a label of degree hd has weight h, is strictly smaller in house, and has positive residues at c and p by principality. Since $c > h$, Proposition 4.8 forces ramification at c, a contradiction.

For $p = 2$ there cannot be two edges $4a, 4b$. Their weight-four crossing would ramify at both other singleton primes, each at least five, by the same two-residue result. As these primes are isolated this is impossible. Thus $4a$ is the unique edge on four and every edge meets a. At most one other odd prime is smaller than a by the three-singleton result. On the other hand Proposition 15.10 allows at most one of the other three to exceed seven, so the other two are $5, 7$, both smaller than a. This excludes the dyadic quartic.

15.6 Three isolated primes in odd dimension

For the rest of the odd five-prime argument put $H = p^2$ and $\beta = m/(m-1)$, where m is the least prime. The following calculation allows an arbitrary complementary factor G.

Lemma 15.13 (Centered Weyl column for a separated set of odd primes)**.** *In an odd ordinary minimal table of square root $n = HN$, let $H > 1$ be the full part of n supported on a nonempty set of odd dimension primes which is separated from its complement: every twist is supported on one side or the other. Put $J = H(1 - (W_H)_{00})$ and $a_H(X) = \epsilon H(W_H)_{X0}$ for nonunits, with $\epsilon = \xi$. Then J is a positive even integer and*

$$JN \equiv -\epsilon \pmod{H}, \qquad a_H(X) = [d_X/N]_H.$$

In particular its nonunit support is exactly the degrees deficient at some prime of H. No assumption on the internal prime powers or internal separation in H is needed.

Proof. The split Weyl column is integral after scaling each local column by its part of n (Proposition 6.2); reduction of the vacuum and nonvacuum entries modulo H gives the displayed congruences before centering. Every H-active label Z has degree Nk_Z with $k_Z \mid H$ by complementarity; on its active subspace $s = \epsilon W_H$, so $a_H(Z) = k_Z$. Here $k_Z < H$ since $f_Z = 1$ is impossible in a nontrivial table, and thus $k_Z \le H/3$. Choose an even exponent projecting the modular level to its odd H-part. The vacuum Gauss identity of Lemma 6.1 gives

$$HJ = \sum_{Z \ H\text{-active}} k_Z^2(1 - \theta_Z).$$

Since H is odd, equal-twist dual pairs show J is even. Its positivity follows by taking real parts (a nonidentity phase exists by Cauchy). Let $m \geq 3$ be the least prime of H and $\beta = m/(m-1)$. The Galois mean of each $1 - \theta_Z$ is at most β by the Ramanujan mean formula. Subtracting the vacuum and the active coordinates from the norm H^2 leaves at most

$$2HJ - J^2 - HJ/\beta \leq (2 - 1/\beta)^2 H^2/4.$$

Distinct duals have the same signed coordinate (the Weyl column is rational and invariant under duality). Each inactive nonunit coordinate therefore has magnitude at most $(2-1/\beta)H/(2\sqrt{2}) < H/2$, since $\beta \leq 3/2$. Together with the active bound, this proves centering. □

Proposition 15.14 (Three isolated prime positions with full complementary degrees)**.** *Let $n = QGcd$ be an odd ordinary minimal square root, where $Q = p^2$, p, c, d are distinct isolated primes, $c < d$ are singletons, $G > 1$, and $\gcd(G, pcd) = 1$. For each pair of the three positions Q, c, d, suppose that every nonunit degree prime to both equals the full complementary divisor: QG, Gd, Gc for the pairs $(c, d), (Q, c), (Q, d)$, respectively. Then degree Qc cannot occur.*

Proof. For $H = Q, c, d$ let $z_H = (W_H - I)e_0$. Its vacuum coordinate is nonzero and its other coordinates are supported precisely on the degrees deficient at H, by Lemma 15.13. For $H = Q$ this means precisely the p-free degrees, by Lemma 15.12. The vectors for distinct positions are orthogonal by (6.1), so every pair has a common nonunit coordinate to cancel the vacuum product. The pairs (c, d) and (Q, d) therefore supply the degrees QG and Gc, respectively.

At c, d the only common nonunit degree is $QG = n/(cd)$. Since n is odd, Lemma 9.9 gives $d = cj_d \pm 1$, with j_d even and positive by Theorem 10.3. Hence $d \geq 2c - 1 > 3c/2$. If Qc occurred, Qc and Gc would be distinct proper d-free nonunit degrees, with greatest common divisor c. The same theorem gives $d \leq \ell c/(\ell - 1) \leq 3c/2$ for the least dimension prime ℓ, a contradiction. □

Corollary 15.15 (Odd roots with five repeated primes)**.** *In an odd ordinary minimal root $n = p^2uvcd$ at five distinct primes, no two squared primes mix. Under full isolation there cannot be exactly two degrees of the form p^2x with x a singleton.*

Proof. If u, v mix, put $G = uv$ and order the isolated singletons $c < d$. The mixed-square structure supplies p^2c, p^2d, Gc, Gd and the sole common degree p^2G at c, d. For $\{t, w\} = \{c, d\}$ every p-free t-free nonunit is t-principal by isolation and has degree Gw by the same structure. This is precisely the excluded configuration.

In full isolation the possible degrees are subproducts of the five coprime factors p^2, u, v, c, d (no intermediate p-height). Write p^2c, p^2d for the two stated degrees, with $c < d$. Any two-factor degree on the ordinary primes must meet both of these edges, so only cd is possible. For each pair of positions in p^2, c, d, a nonunit degree omitting the pair is therefore the full product of the other three factors: one factor is forbidden by the exclusion of prime-power degrees, and a two-factor degree would be a further p^2-edge or an ordinary edge other than cd. The same proposition again contradicts the degree p^2c. □

15.7 Full isolation in odd dimension

Proposition 15.16 (Two axes at an isolated quartic)**.** *Let an ordinary minimal table of odd square root $n = p^2abcd$ have all five primes isolated. Put $H = p^2$. There are at least two distinct companion primes s, t for which Hs, Ht occur as simple degrees. Consequently at least two of the four companions are strictly smaller than $\beta p^2/2$, where $\beta = m/(m-1)$ and m is the least prime dividing n.*

Proof. By the no-intermediate-height lemma, all degrees are subproducts of H, a, b, c, d. The graph whose edges are occurring two-factor degrees has pairwise intersecting edges. The argument in Section 15.5.3 shows that an H-edge occurs. Suppose it is uniquely Hs. Every edge then

meets s. For each other companion w the only nonunit degree omitting s, w is $n/(sw)$, and it occurs. The unique-complementary-degree lemma applies at all these pairs.

Corollary 9.10 permits at most one companion smaller than s. For each of the at least two companions $w > s$, Lemma 9.9 gives signed s-coordinate ± 1 at $n/(sw)$. This odd coordinate forces ramification at s. These are distinct proper degrees, whereas Theorem 10.3 allows only one proper ramified orbit. This proves the first assertion.

Finally choose any two H-edges. Their normalized crossing has weight H and positive residues at the two omitted singleton primes. The two-residue estimate at an isolated prime, applied in both orders, puts each omitted prime below $\beta H/2$. □

Proposition 15.17 (Crossings of two quartic axes)**.** *In an odd, fully isolated minimal root p^2abcd, let Hx, Hy be two occurring degrees, where $H = p^2$. For the complementary companions z, w one has $zw \le \beta H$. In particular p cannot be least.*

Proof. The crossing has weight H and positive residue H at z, w. If nonrational, its field ramifies at a prime t common to the two character fields. By their codegrees $t \in \{z, w\}$. Both degrees would belong to the sole proper t-ramified orbit of Theorem 10.3, impossible since they differ. The crossing is therefore an integer $H - kzw$, $k \ge 1$, and its balancing mean is at least $-H/(m-1)$, where m is the least prime (Lemma 9.5). This yields the inequality. If p is least, $zw \ge (p+2)^2 > p^3/(p-1) = \beta H$, whereas Proposition 15.16 supplies two axes. □

Proposition 15.18 (No remaining quartic axes in odd dimension)**.** *An odd ordinary minimal table of root p^2abcd cannot have all five primes isolated and at least three occurring axes p^2t.*

Proof. Put $H = p^2$ and $N = abcd$. No ordinary pair can be an edge: it would miss one of the at least three H-edges. For each companion t, prime avoidance on P_t forces a p-free nonunit degree $E_t = N/t$, since no proper subproduct of the other three companions can occur. This is the only nonunit degree omitting both p, t.

Proposition 15.17 excludes p least. Order the companions $a < b < c < d$, so a is least, $a \ge 11$ and $\beta = a/(a-1)$. For every axis Hx and companion $w \ne x$, the degrees Hx, E_w are distinct and proper at w, with gcd x. Theorem 10.3 gives $w \le \beta x$. In particular Ha cannot occur since $b \ge a+2 > \beta a$. The three axes are Hb, Hc, Hd and $d \le \beta b$; from $c \ge b+2$ it follows that $b \ge 2a-1$.

Fix a companion t. Its unique proper ramified orbit is given by Theorem 10.3. An E_t-label has codegree $(Ht)^2$, so its nonreal character field ramifies at p or t, and at only one by isolation. Its signed coordinate $B_t = [H^{-1}]_t$ has odd parity exactly in the latter case (Corollary 10.6); the choice is common to all its labels.

If Ht occurs, its nonreal field ramifies at another companion u, by its codegree. Were E_t ramified at t, the unique proper ramified degrees at t, u would be E_t, Ht respectively. The former contains u and the latter t, leaving no common ramified orbit at either endpoint, contrary to Lemma 10.5. Consequently all E_b, E_c, E_d ramify at p.

Choose a p-active row Z and a t-active row T. Their only possible common nonunit hits have degree E_t, by defect-zero vanishing and full heights. The rows are orthogonal with vacuum contribution one. If E_t ramifies at p, the constant evaluation on T is $[E_t]_t$; it therefore divides one and $|B_t| = j_t$. If E_t ramifies at t, its labels form one orbit. The evaluations on Z are constant integers, since the two character fields are pure at distinct primes; the evaluations on T sum to $-\epsilon B_t$ by Theorem 10.3. Thus $|B_t| = 1$. Since $HB_t \equiv 1 \pmod t$, we have $Hj_t \equiv \pm 1 \pmod t$ in the first case and $Hj_t \equiv \pm j_t \pmod t$ in the second.

The fully separated Weyl sum on the vacuum, with the trivial auxiliary two-primary factor, gives

$$H = J + \sum_{t \in \{a,b,c,d\}} Hj_t/t + \epsilon/N, \qquad J = H(1 - (W_p)_{00}) \in \mathbb{Z}$$

(Lemma 15.13). If E_a also ramifies at p, an integer equals $\pm 1/a \pm 1/b \pm 1/c \pm 1/d + \epsilon/N$. This has magnitude less than one, so it vanishes. Multiplication by the odd N gives four odd terms and the extra odd ϵ, an impossibility. If E_a ramifies at a, the integer instead equals $\pm j_a/a \pm 1/b \pm 1/c \pm 1/d + \epsilon/N$. It again has magnitude less than one, since $3j_a \le a+1$, $a \ge 11$ and $b \ge 2a-1$. But it is nonzero: $j_a/a \ge 2/a > 3/(2a-1) + 1/N$. This exhausts the possibilities. □

15.8 The remaining odd configuration

Proposition 15.19 (The three isolated primes beside a mixed quartic)**.** *Let the odd ordinary minimal table have root p^2abcd, with a mixing with p and $b < c < d$ isolated. Then a is the least of all five primes. Put $\beta = a/(a-1)$. All three low degrees pb, pc, pd occur, and $2a - 1 \le b < c < d \le \beta b$.*

Proof. Put $\beta = m/(m-1)$ for the least prime m. By Proposition 15.9, the three pair products bc, bd, cd occur, as does at least one of pb, pc. Whenever px occurs, take any other singleton t and let y be the third. At t the degrees px, xy are distinct and proper with gcd x; hence $t \le \beta x$ by Theorem 10.3. Conversely, if px is absent, the only common nonunit degree of the other two singletons $r < t$ is $n/(rt)$. Lemma 9.9 gives $t = rj_t \pm 1 \ge 2r - 1$.

If pb were absent, pc would occur and give $d \le \beta c$, whereas $d \ge 2c - 1 > \beta c$. Thus pb occurs and $d \le \beta b$. Absence of either other low degree would force $d \ge 2b - 1 > \beta b$. The least prime cannot be b, since $d \ge b+2$, or p, by Proposition 15.9; hence it is a. Finally $b+2 \le d \le ab/(a-1)$ and parity give $b \ge 2a - 1$.

□

A divisor-minimal failure of the five-prime theorem must retain all five repeated primes. On losing one, Theorems 4.12 and 11.3 apply. If the quartic drops, use Theorem 15.5. Section 15.4 excludes the reduced table when all exponents survive. Thus for the square case the ordinary table has dimension $(p^2abcd)^2$. The quartic prime two was excluded in Section 15.5.3. In the odd case, Corollary 15.15 rules out mixing two squared primes and Propositions 15.16–15.18 rule out full isolation.

15.9 The odd five-prime root

Suppose all five primes are odd. Permute the companion primes so that $a < b < c < d$. By Corollary 15.15 and Propositions 15.16–15.18 and 15.19, p mixes with a, a is the least of the five primes, and the remaining three primes are isolated. Mixing means that the order of some simple twist is divisible by both primes. Put $H = p^2a$ and $\beta = a/(a-1)$. We use the constraints

$$11 \le b < c < d, \qquad d \le \beta b.$$

For an isolated pair r, t with third prime x, the only possible common nonunit degrees (those prime to rt) are px and $Hx = n/(rt)$. We call these the low and high degrees of the pair. All six degrees bc, bd, cd, pb, pc, pd occur (Propositions 15.9 and 15.19).

For $t \in \{b, c, d\}$ put $m_t = n/t$. The signed Weyl coordinate of a nonunit t-free degree h is $A_t(h) = [h/m_t]_t$, with $m_t j_t \equiv -\epsilon \pmod t$. At proper degrees an odd coordinate detects ramification; an unramified label has $A_t(h) = -\epsilon j_t [h]_t$. We use the trace and norm of Proposition 6.3, in particular the mixed sum $-j_r j_t$ away from the vacuum and the bound $j_t(1 + \sum_{X \in L} [d_X]_t^2) \le t+1$ for any set L of nonunit inactive t-unramified labels.

15.9.1 The three-prime cycle

Theorem 10.3 gives one inactive full orbit and one proper ramified orbit at each t, both of size j_t, and $\sum_{\text{proper}} A_t(d_X) d_X = \epsilon j_t$. Each pair needs a ramified orbit at an endpoint by Lemma 10.5. Its common degrees px, Hx, indexed by the third vertex x, are proper, and different pairs have

disjoint degree sets. Orient each pair from its ramified endpoint: the three arrows form a directed cycle. All other proper degrees at t are unramified there, in particular the product of its two neighbors.

For a high degree on $\{u, v\}$ the two signed coordinates are the centered inverses $[v^{-1}]_u$ and $[u^{-1}]_v$. Indeed, the Bezout identity $v[v^{-1}]_u + u[u^{-1}]_v = 1$ follows by reducing modulo uv and using centering. Consequently exactly one coordinate is odd, and its endpoint ramifies. Thus whenever the high degree occurs, it accounts for the unique proper ramified orbit on its edge. If the low degree also occurs there, all of its labels are unramified at both endpoints. At t the compulsory degree rs, where r, s are the other vertices, is also unramified: its degree differs from both eligible degrees on each incident edge and only one proper orbit ramifies.

When the high degree is absent from an edge $u \to v$, only the low degree px contributes to its mixed Gram equation. Its multiplicity is j_u, its ramified coordinate has magnitude one, and $|[px]_v| = 1$; indeed the product of these last two integers has magnitude one after dividing the mixed equation by $j_u j_v$. In particular

$$pav \equiv \pm 1 \pmod{u}, \qquad px \equiv \pm 1 \pmod{v}.$$

Two consecutive low-only arrows $u \to v \to z$ would therefore give $pz \equiv \pm 1$ and $paz \equiv \pm 1$ modulo v. This is impossible since $1 < a < v - 1$. Thus at least two edges carry a high degree.

All three high degrees are forced. On a high edge $u < v$, the odd coordinate z at its ramified endpoint t satisfies $z^2 \le t + 1 - j_t \le t - 1$ by the singleton norm, hence $z^2 \le t - 2$ by parity. The even integer $(v - u)z$ is ± 1 modulo t, so

$$(v - u)^2 \ge (t - 1)^2/(t - 2) > t \ge b.$$

The window $d - b \le b/(a - 1)$ now gives $b > (a - 1)^2$.

If $t \to r$ is low-only, the incoming edge $s \to t$ carries a high degree. Its low degree pr is unramified at t. The outgoing congruence $par \equiv \pm 1 \pmod{t}$ gives $a|[pr]_t| \ge t - 1$, since the only integer multiple of a congruent to ± 1 with magnitude less than $t - 1$ would be ± 1. Its dual-pair budget gives $2j_t[pr]_t^2 \le t + 1 - j_t \le t - 1$. Thus $t \le 1 + a^2/(2j_t) \le 1 + a^2/4$. But $t \ge b > (a - 1)^2 > 1 + a^2/4$ for $a \ge 3$. It follows that every edge carries a high degree.

The cyclic divisibility contradiction. At $s \to t \to r$ let

$$A = [s^{-1}]_t, \qquad B = [r^{-1}]_t, \qquad w = [rs]_t.$$

The incoming high degree Hr has even coordinate A at t and multiplicity j_s; the outgoing high degree Hs has odd coordinate B at t and multiplicity j_t.

The degree rs contributes at least $2j_t^2 w^2$ to the residual norm at t. Pairing this cost first with the incoming high cost $j_s A^2$ and then with the outgoing high cost $j_t B^2$ gives

$$|Aw| \le \frac{t + 1 - j_t}{2\sqrt{2j_s}} < \frac{t}{4}, \qquad |Bw| \le \frac{t + 1 - j_t}{2\sqrt{2j_t}} < \frac{t}{4}.$$

Here $j_s, j_t \ge 2$. The vertex window gives $2/3 \le r/t, s/t \le 3/2$, strictly inside $(1/4, 7/4)$. For $x = r, s$ the only representative of x modulo t that can have magnitude less than $t/4$ is $x - t$: all $x + kt$ with $k \ge 0$ exceed $t/4$, and all those with $k \le -2$ are less than $-t/4$. As $Aw \equiv r$ and $Bw \equiv s$ modulo t, this gives $Aw = r - t$ and $Bw = s - t$. As A is even, B odd, and $w \ne 0$,

$$\nu_2(r - t) > \nu_2(s - t).$$

Applied at the three vertices of the directed cycle this gives a strict cyclic inequality. The odd minimal table is impossible.

15.10 A squared prime two

The remaining root has a squared two in the dimension and an odd quartic prime p. Two primes *mix* if the order of some simple twist is divisible by both. There is one mixing pair and it contains two. Its partner is p or another squared prime u (Proposition 15.9 gives at most one pair, and Lemma 6.6 excludes an isolated squared two). Let P_ℓ be the principal residue block at ℓ. In a perfect fusion-simple table nonunit labels of coprime degrees have zero S-entry, while a label in P_ℓ hits every label whose degree is prime to ℓ (Sections 2–4).

The odd denominator. Suppose first that two mixes with p. Write $n = 2p^2bcd$ and order the three isolated primes as $b < c < d$. For an isolated pair the only possible common nonunit degrees are px and $2p^2x$, where x is the third isolated prime. Each product of two isolated primes occurs, and at least one of pb, pc occurs (Proposition 15.9 and its proof). Fix such a low degree pt, and let r, s be the other isolated primes. If a dyadically active J had $p \nmid d_J$, the second rule of Lemma 15.4, with principal prime two, $w = t$ and $q = p$, would give a nonunit on P_2 of degree supported on r, s. Its degree must be their product, coprime to the odd degree pt it must hit by principality. Thus $v_p(f_J) = 4 - 2v_p(d_J) \le 2$ on the dyadically active sector.

If two mixes with a squared prime u, the quartic prime p is isolated, so every dyadically active label has p-unit codegree by Lemma 3.3.

In both cases Theorem 6.9 makes W_2e_0 p-integral. Its other odd denominator primes have height one in n and are removed by Lemma 6.1. The same lemma gives rationality and nW_2e_0 integral. Thus

$$x = 2\epsilon W_2 e_0 \quad \text{is integral,} \qquad \|x\|^2 = 4,$$

where $\epsilon = \xi = \pm 1$ by the presence of an odd isolated singleton (Proposition 6.3). Choose the anomaly lift $\gamma = \epsilon$.

The four positions. The vector x lies in P_2 and hence has odd degrees on its support. The complementary Weyl factor commutes with T_2 and carries W_2e_0 to se_0. Thus the mass $\sum_{i:\theta_{i,2}=\eta} d_i^2/n^2$ of each dyadic phase equals $\sum_{i:\theta_{i,2}=\eta} x_i^2/4$. In particular x cannot be supported at a single position: the vacuum and a dyadically active label (by Cauchy) have different phases and both have positive mass. Hence its four nonzero coordinates are signed units. Let K be the number of active positions. By the vacuum Gauss identity (Lemma 6.1), with an odd exponent selecting the dyadic part,

$$4 - K + \sum_{i=1}^{K} \eta_i = 2x_0.$$

Here $\eta_i \ne 1$ are the phases on those positions.

At any odd isolated singleton t separated from two, $(\epsilon W_2 - I)e_0$ and $(W_t - I)e_0$ are orthogonal. Indeed ϵW_2 fixes the t-moving subspace and W_t fixes its orthogonal complement, by the proof of Proposition 6.2. Their vacuum coordinates are $x_0/2 - 1$ and $-j_t/t$, both nonzero. The singleton column therefore forces a nonunit t-free position of x. This position is dyadically inactive, since all dyadically active labels have full t-height by separation.

When two mixes with p, no position can serve two different isolated primes. Of the only possible common nonunit degrees $px, 2p^2x$, the second is even and the first is coprime to the occurring odd degree given by the product of the opposite pair. Neither can lie in P_2. Thus three distinct nonvacuum positions are inactive. An active position must also occur, so $K = 1$ and $x_0 = 0$. The Gauss identity would say $3 + \eta_1 = 0$, which is impossible.

When two mixes with u, the isolated singletons c, d again require two distinct positions, since their only common nonunit degree is $2up^2$. Both positions have p-divisible degrees: for $t = c, d$ and $w = cd/t$, every p-free t-free nonunit has degree $2uw$ (Proposition 15.9 and its proof). Every active position also has degree divisible by p, by separation from p. If $x_0 = 0$, the Gauss identity and $K \le 2$ give $K = 2$ with both phases -1. If $x_0 \ne 0$, exactly one active position remains, and

$3 + \eta_1 = 2x_0$ gives $x_0 = 1$, $\eta_1 = -1$. All nonvacuum positions have now been accounted for, and their degrees are multiples of p. The phase-mass identity implies $T_2^2 = I$ on the entire table.

Put $Q = s(I - T_2)s^{-1}/2$. As $U_2 = \epsilon T_2$, the local level divides two. The Weyl in $\mathrm{SL}_2(\mathbb{F}_2)$ is an involution, and the unipotent identity of Theorem 6.9 gives

$$\epsilon W_2 = T_2(sT_2s^{-1})T_2, \qquad 4Qe_0 = 2e_0 - T_2x.$$

The vacuum coefficient of $4Qe_0$ is K by the phase masses. Its dimension is zero, since the normalized dimension row is e_0^*s, s^2 fixes e_0, and $T_2e_0 = e_0$. Reducing the dimension modulo p gives $p \mid K$, impossible for $K = 1, 2$. This proves the assertion for four squares in Theorem 15.1. The three cubic companions follow by Theorem 15.6.

16 Extending the support by new prime powers

Throughout, Δ is the dimension of the ordinary or positive reduced table attached to a divisor-minimal obstruction (Lemma 2.1).

16.1 Two unrestricted exponents

Lemma 16.1. *Let r, p, t divide Δ, with r odd, $p \neq r$ and $t \neq r$. In the reduced case assume also that t is odd. The primes p, t may coincide. There exist nonunit labels $X \in P_t$ and $Y \in P_r$ with $r \nmid d_X$ and $p \nmid d_Y$. Every such pair satisfies*

$$r \leq B(g) := \max\{5, (g-1)g^2\}, \qquad g = \gcd(d_X, d_Y) \geq 2.$$

In particular its common degree factor is prime to both p and r. At t, its height is at most $\lfloor v_t(\Delta)/2 \rfloor - 1$.

Proof. Categorical Cauchy supplies active columns at t, r, using its odd-prime form on the positive table (Section 3). Apply prime avoidance (Proposition 4.2) first on P_t at r and then on P_r at p. The valuation assertion follows from Corollary 4.1. No principal block at two is used in the reduced case, even when $p = 2$.

For $g = \gcd(d_X, d_Y)$ the principal crossing at r gives $r \leq \max\{5, (g-1)g^2\}$ by Theorem 4.17, in the ordinary and reduced cases alike. The case $g = 1$ is impossible: coprimality forces the entry to vanish, whereas $Y \in P_r$ and $r \nmid d_X$ give $\chi_Y(X) \equiv d_X \not\equiv 0 \pmod{\mathfrak{r}}$.

□

Proposition 16.2. *For every $k \geq 0$ and integers $C_1, \ldots, C_k \geq 2$, the procedure below produces sufficient prime inequalities for every divisor of*

$$P^A R^B \prod_{i=1}^{k} t_i^{C_i} e$$

to be a braided weakly group-theoretical dimension. Here all the displayed primes are distinct, $t_1 < \cdots < t_k$, R is odd, $A, B \geq 0$, and e is coprime and square-free. Any of the C_i may be indeterminates. The inequalities are independent of A, B, and admit prime specializations for every choice of the C_i.

Proof. Enumerate the subsets S of displayed primes retained as repeated primes in a hypothetical divisor-minimal source. If $|S| \leq 2$, use Dong's theorem. For $|S| = 3, 4, 5$ terminate when respectively at most $2, 1, 0$ of the surviving caps can exceed three, by Theorems 4.12 and 11.3 and Theorem 15.5. Count P, R and every symbolic cap as possibly exceeding three. For every other support consider an ordinary table and all possible reduced tables. A reduced table requires a surviving repeated dyadic prime, either P or t_1, whose source cap permits a valuation at

least three. This exhausts the possibilities by square-fullness of Δ and evenness of the reduced dimension (Proposition 4.7).

On such a branch take $r = R$ if it belongs to S and otherwise take the largest surviving companion. This r is odd: a nonterminal branch has at least three surviving primes, and P is the only available noncompanion if R is absent. Choose $p \in S \setminus \{r\}$, taking $p = P$ when P survives. Choose $t \in S \setminus \{r\}$ odd in the reduced case. Such a t exists since at most one surviving prime is two. In Lemma 16.1 the gcd is divisible by neither r nor p. For every other surviving companion use the bound $h_i = \lfloor C_i/2 \rfloor$ on its height, subtracting one at t if it is such a companion. For a numerical cap at the reduced dyadic prime one can use $\lfloor (C_i - 1)/2 \rfloor$ instead. The product W of the resulting prime powers bounds the gcd. If $W = 1$ the branch is impossible. Otherwise record $r > B(W)$; one may choose the smallest convenient monomial among the allowed pairs (p, t). Every factor of W is a companion preceding r in the order $t_1, \dots, t_k, R$. Thus the hypotheses can be satisfied successively by choosing primes.

The certificate records the terminal theorem or the pair (p, t) for every branch. A hypothesis $r > B(V)$ covers a recorded W whenever $V \geq W$ throughout the prescribed order and heights. For monomials V, W, write V/W as a product of powers of t_1 and of the ratios t_i/t_{i-1}. The exponents in this expression are the successive suffix sums of the exponent differences. Nonnegative suffix sums suffice; for affine exponents with independent $h_i \geq 1$ this is checked by checking their slopes and their values at $h_i = 1$. Replaying these checks for every branch proves the divisor assertion by Lemma 16.1 and the minimal-counterexample reduction, without substituting individual primes. □

Corollary 16.3. *Let P, R, u, v, w be distinct primes, with $u < v < w$ and R odd. If $R > (uv - 1)(uv)^2$, every weakly integral braided fusion category whose dimension divides*

$$P^A R^B (uvw)^3 e$$

is weakly group-theoretical, for arbitrary $A, B \geq 0$ and any square-free e coprime to the five primes.

Proof. Apply Proposition 16.2 with three companion caps equal to three. On any nonterminal branch both P, R and at least two companions survive: otherwise the terminal theorem for at most four primes applies. In either table take $r = R$, $p = P$ and t the largest surviving companion, necessarily odd. Its valuation is at most three, so the certificate removes its whole height; each other companion has height at most one. Hence $W \leq uv$ and $B(W) \leq B(uv) = (uv-1)(uv)^2$. □

16.2 One new prime of arbitrary height

Proposition 16.4. *Let $p_1 < \cdots < p_k < r$ be primes, where $k \geq 2$, and let $a_i \geq 2$. Suppose $N = \prod_{i=1}^k p_i^{a_i}$ is covered, and put $T = \prod_{i=1}^k p_i^{\lfloor a_i/2 \rfloor}$. If*

$$r > \frac{T^2}{p_1 p_k},$$

then Nr^b is covered for every $b \geq 0$. For $k = 2$ the weaker inequality $r > T^2/(p_1 p_2^2)$ suffices.

Proof. Fix b, e and take a counterexample of least dimension D among their allowed divisors. Its ordinary or positive reduced table has dimension $\Delta = D$ or $D/2$ and a perfect, simple, integral based ring. Moreover Δ is square-full, $d_Y^2 \mid \Delta$ for each label Y, and $\Delta \mid \prod_Y d_Y^2$ by Lemmas 2.1 and 2.5. If $v_r(D) \leq 1$, include the possible factor r in the square-free decoration of the covered prefix. Thus $r^2 \mid \Delta$, since r is odd. Dong's two-prime theorem applies unless at least two prefix primes are repeated in D. All repeated prefix primes remain in Δ, where valuation one is impossible. Their largest, h, is odd. Put $T' = \prod_i p_i^{\lfloor v_{p_i}(\Delta)/2 \rfloor}$.

Choose an h-active column by categorical Cauchy, using (Proposition 3.4) in the reduced case. Prime avoidance produces an r-free nonunit X in the principal h-block, so $d_X \le T'/h$ by Corollary 4.1 and Proposition 4.2. Every r-free degree is at most T'. Some degree is divisible by r since $\Delta \mid \prod_Y d_Y^2$. Each such degree is at least rp_1, since nonunit degrees cannot be prime powers (Lemma 2.5). The bound $T'^2/h \le T^2/p_k$ holds also when $h < p_k$, since then $T' \le T/p_k$. Under the displayed inequality, therefore, $d_X T' < rp_1$. For every r-free label Y, every constituent of XY is r-free by its degree. The same holds for X^*Y. All words in X, X^* then have only r-free constituents, contradicting simplicity in either based ring.

For $k = 2$ both prefix primes survive. Each r-free nonunit degree is divisible by p_1p_2. The gcd bound of Proposition 4.13 gives $rp_1p_2 \le d_X T' \le T^2/p_2$. This recovers precisely the residue-independent clause of Proposition 4.14. □

Starting from Dong's theorem on two repeated primes, repeat Proposition 16.4 along a chain of increasing primes. The new exponent is unrestricted at every step. It may enter the thresholds at subsequent steps. For each finite list of exponent values the resulting inequalities admit infinitely many prime tuples by successive choice of the primes. Any previously covered prefix can shorten the chain.

16.3 Bounding a prime exponent

Theorem 16.5. *In a minimal table let p divide Δ and write $\Delta = p^a m$ with $a = v_p(\Delta) \ge 1$. Put*

$$u(m) = \prod_{\ell \mid m} \ell^{\lfloor v_\ell(m)/2 \rfloor}, \qquad c = 2^{\mathbf{1}_{3 \mid m}}, \qquad t = (p-1)c.$$

There is a nonunit X in the full principal p-block with $p \nmid d_X [K_X : \mathbb{Q}]$. Every such X satisfies

$$a \le v_p((2td_X - 1)!) < 2c\, d_X \le 2c\, u(m).$$

The bound $a \le v_p((2td_X - 1)!)$ also holds for a specified nonunit X whose field is tame at odd p, or unramified at $p = 2$. If K_X is unramified at odd p, one may use $t = c$ in the factorial.

Proof. The unit coefficient of the p-local principal-block idempotent is $\Delta^{-1} \sum_{X \in P_p} d_X^2$. Thus $p^a \mid \sum_{X \in P_p} d_X^2$, so this sum vanishes modulo p. The block is Galois-stable and the unit orbit contributes one. Some other orbit has degree and size prime to p. Its field is tame at odd p, and unramified at two if $p = 2$, since the wild inertia or the entire dyadic inertia, respectively, is a p-group. Also $d_X \mid u(m)$.

Fix a primitive ℓth root η_ℓ for every $\ell \ge 5$ dividing m. On the ℓ-power roots prescribe a linear functional L_ℓ taking 1 at 1, -1 at η_ℓ, and zero elsewhere. These values sum to zero on each coset of μ_ℓ, exactly the cyclotomic relations at a prime power. Form their tensor product with the normalized trace on the dyadic factor, $(p-1)$ times the normalized trace on the p-factor for odd p, and twice the normalized trace on a complementary three-factor. Include the dyadic factor just once if $p = 2$. Call the result F. Then $F(1) = t$, and F is integer-valued on cyclotomic integers: check on roots, using the normalized traces $1, -1/(\ell-1), 0$ at orders $1, \ell, \ell^k$ with $k > 1$. For unramified odd p, use the prime-to-p conductor of K_X and extend by normalized trace at p to evaluate balancing words. Integrality on K_X is preserved and $t = c$.

On K_X the functional is a sum of embeddings with coefficients integral at p. Indeed, take a cyclotomic field containing K_X and the marks, with only the tame p-level, omitting the dyadic level at $p = 2$. Such a field exists by the local inertia description and the conductor bound in Corollary 2.8. On this field L_ℓ is normalized trace after multiplication by $(\ell-1)(1-\eta_\ell^{-1})/\ell$. The factor $\ell - 1$ cancels the p-part of the degree of the ℓ-primary cyclotomic field. At a complementary three the factor 2 does this for $p = 2$. The remaining local degrees are p-units. One may evaluate F on any higher balancing level by normalized relative trace; the smaller field suffices

for the coefficients of the embeddings. Galois covariance therefore gives a p-locally integral linear combination B of the $[X^\sigma]$ with eigenvalues $b_Y = F(\chi_Y(X))$.

We prove $|b_Y| < td_X$ for $Y \neq 0$. On roots $|F| \leq t$. A root attaining t is a product of a subset of the $-\eta_\ell$; a root attaining $-t$ is the negative of such a product. Indeed all unmarked odd coordinates must be trivial, and the signs of the marked coordinates determine the dyadic sign. Equality at Y forces every phase of a positive balancing expression to attain the same endpoint. Its product with its conjugate gives a full-mass word for XX^* supported on

$$T = \left\{ \prod_\ell v_\ell : \ v_\ell \in \{1, -\eta_\ell, -\eta_\ell^{-1}\} \right\}.$$

There is a rational functional taking 1 exactly on T and strictly smaller values on every other root. At ℓ give 1 the value 1, give $\eta_\ell^{\pm 1}$ the value -1, and give the other ℓth roots the value $1/(\ell - 3)$. Give higher-order roots the value zero. The coset sums are zero. Multiply these functionals and the normalized traces at all remaining factors. Since $\ell \geq 5$, the only possible coordinates of absolute value one are the three marked positions and the dyadic signs. The sign making the product equal to one is precisely the sign in T.

Take a Galois multiplier σ squaring each η_ℓ and fixing the other primary factors. Then $T \cap \sigma T = \{1\}$, since $\{\pm 1\} \cap \{\pm 2\}$ is empty modulo every $\ell \geq 5$. By Lemma 4.15 the two adjoints have a common nonunit V, including for the chosen lifts in the positive quotient. Their exposing functionals put every phase of the same balancing expression for $\chi_Y(V)$ in $T \cap \sigma T$, contradicting strictness.

Let $d = d_X$. The polynomial with roots all integers in $[-td + 1, td - 1]$ takes B to $(2td - 1)!$ times the dimension-character idempotent. Its unit coefficient $(2td - 1)!/\Delta$ is integral at p. This proves the factorial inequality; the estimate on its valuation follows from Legendre's formula. □

16.4 A finite reduction for a fixed complementary factor

Set $L(m) = 2 \cdot 2^{\mathbf{1}_{3|m}} u(m)$. Let $w(m)$ be $u(m)$ divided by its largest full prime-power factor, with $w(m) = 1$ if $u(m) = 1$, and put

$$B(m) = \max\{5, (w(m) - 1)w(m)^2\}.$$

Corollary 16.6 (Fixed complementary factor)**.** *If a minimal table has dimension $p^a m$ with $a > 0$ and $p \nmid m$, then $p \leq B(m)$ and $a < L(m)$. Consequently, for a fixed m and any prime $p \nmid m$, to prove weak group-theoreticity for all braided fusion categories of dimensions $p^j t$ with $t \mid m$ and $j \geq 0$, it suffices to check the divisors of m and the dimensions*

$$p^j t, \qquad t \mid m, \quad p \leq B(m), \quad 1 \leq j < L(m) + \mathbf{1}_{p=2}.$$

This is also a finite reduction when p varies.

Proof. The exponent bound is Theorem 16.5. Choose the p-free nonunit X supplied by that theorem. For $p \geq 7$, Theorem 4.17 gives $p \leq b(d_X/L)$ with L the largest full prime-power factor of d_X. Since $d_X \mid u(m)$, deleting its largest primary factor leaves $d_X/L \leq w(m)$. Indeed, deleting instead the prime component corresponding to the largest factor of $u(m)$ can only increase that quotient. Thus $p \leq B(m)$, also immediate for $p \leq 5$.

Now choose an obstruction of least dimension in the divisor family and write $p^j t$ for its dimension. The case $j = 0$ is among the divisors of m. Otherwise the part m' of its table dimension prime to p divides m. Both B and L are nondecreasing under divisibility. If p occurs in the table, the two bounds give $p \leq B(m)$ and table exponent at most $L(m) - 1$. Passing to the category adds one only for a positive fermion quotient at $p = 2$. If p is absent from the table then necessarily $p = 2$ and $j = 1$, also in the displayed range. This proves the finite reduction. □

The normalized trace projector. The unmarked normalized trace gives another form of the projector in Theorem 16.5. If X is a specified nonunit, $h = [K_X : \mathbb{Q}]$, and H clears the normalized traces of its cyclotomic integers with H/h integral at p, take $B = (H/h)\sum_{X' \in \mathrm{Gal}\cdot X}[X']$. Its eigenvalues are $H\tau(\chi_Y(X))$, rational integers of absolute value strictly less than Hd_X at nonunits. The same projector therefore gives $a \le v_p((2Hd_X - 1)!)$. One admissible trace denominator is the product of $\ell - 1$ over primes of a conductor containing K_X, by the normalized trace of a primary root.

17 The four dimension bounds

Theorem 17.1 (Uniform dimension bounds)**.** *The four bounds in Table 1 hold. Every strictly weakly integral fusion category of dimension less than* 10800 *is weakly group-theoretical.*

Proof. We compare the prime exponents of a divisor-minimal braided obstruction with those allowed by the family theorems. With a braiding, the square-full part of its dimension divides $Q(D)$. Without a braiding, the Drinfeld center has dimension N^2, so a valuation e in a minimal obstruction requires valuation at least $\lceil e/2 \rceil$ in N. At most two repeated primes are covered by Dong's theorem [22, Theorem 3.1]. The following lower bounds use only the families needed for this comparison.

# repeated primes	$Q(D)$	$Q(D)$ odd	N	N odd
3	$2^6 3^6 5^4$	$3^6 5^6 7^4$	5400	165375
4	6300^2	121275^2	6300	121275
5	$2^3\, 2310^2$	$3^3\, 15015^2$	9240	135135
≥ 6	30030^2	255255^2	30030	255255

At three primes $p < q < r$, the exponents must be at least $(6, 5, 4)$ in that order, by Theorems 4.12, 7.3 and 8.3. The unordered box $(7, 5, 4)$ is covered by Theorem 8.1. Thus the braided comparison costs at least $p^6 q^5 r^4 \min(p^2, q, r)$; for the center use the ceilings of half the three original lower exponents. Both costs increase with the primes.

At four primes write the exponents in decreasing order as a, b, c, d. Theorem 11.3 gives $b \ge 4$. If $c \ge 4$ or $c = d = 3$, the respective lower profiles are $(4, 4, 4, 2)$ and $(4, 4, 3, 3)$. Their minimum is attained by assigning the larger exponents to the smaller primes. Otherwise $d = 2$ and $c \le 3$. Theorems 9.1 and 11.3 give $\lfloor a/2 \rfloor + \lfloor b/2 \rfloor \ge 6$. This leaves $(6, 6, 2, 2)$ or $(8, 4, 2, 2)$ as lower profiles. For odd primes the least of the four products is $(3^2 5^2 7^2 \cdot 11)^2 = 121275^2$. For even support the only profile with minimum below $(2^2 3^2 5^2 \cdot 7)^2 = 6300^2$ is $(8, 4, 2, 2)$. When two is in a higher position the dyadic $(9, 5)$ box at the two higher primes and the lift of one square to a cube cover all $a \le 9, b \le 5$. The two placements follow from the dyadic enumeration in Theorem 9.15 and the lift in the paragraph The two dyadic exponent boxes. The next possible exponents have $a \ge 10$ or $b \ge 6$. When two is in a lower position the product is already at least $(2 \cdot 3^4 5^2 \cdot 7)^2$. Taking square roots after rounding odd valuations upwards gives the same lower limits for N.

At five primes let P be their product and $p < q$ the two least primes. Theorems 15.5 and 15.1 cover the profiles $(3, 3, 3, 3, 3)$ and $(4, 3, 3, 3, 2)$ in every order. Hence one valuation is at least five, two are at least four, or a quartic is accompanied by four valuations at least three. All three alternatives cost at least $p^3 P^2$. For centers we may test the even exponent caps in N^2 directly, since the covered families include every divisor. A prime then occurs to height at least three in N, or two primes occur to height at least two, so $N \ge p^2 P$. Six primes give the last row immediately.

Taking column minima gives 29160000, 6087156075, 5400 and 121275. A Python check reproduces each entry of the table by enumerating exponent vectors in order of increasing product and, for the two center columns, testing the doubled exponents. WGT descends from the center, and $4 \nmid D$ when $Q(D)$ is odd, so the solvability argument of Corollary 15.3 applies to both odd

columns. For a strictly weakly integral category the canonical dimension grading has integral neutral component of dimension at most $N/2$ [36, Theorem 3.10]. The bound 5400 applies to this component, and WGT passes to its extension. □

The first exponent profiles left by this comparison are $2^6 3^6 5^4$ and $3^5(5 \cdot 7 \cdot 11 \cdot 13)^2$ in the braided columns. The boundary dimensions for centers are $2^3 3^3 5^2$ and $3^2 5^2 7^2 \cdot 11$.

17.1 Solvability

For a WGT category of dimension D, the group-extension argument of Corollary 15.3 proves solvability if no order of a nonabelian finite simple group divides D. Braiding gives a smaller integer to test. For $m \geq 1$ put $R(m) = \prod_p p^{\lfloor v_p(m)/2 \rfloor}$, the square root of its largest square divisor.

Proposition 17.2 (Solvability from the square divisors)**.** *Let $e = 1$ in a nondegenerate weakly integral braided category of dimension D and $e = 2$ in a slightly degenerate one. If the category is WGT, it is solvable whenever no order of a nonabelian finite simple group divides $R(D/e)$. The condition $R(D/e) < 60$ suffices even without the WGT hypothesis.*

Proof. For $e = 1$ the WGT assertion is [42, Corollary 5.2]: the integers u with $u^2 \mid D$ are exactly the divisors of $R(D)$, and a nonsolvable finite group has a nonabelian simple composition factor. For either value of e, each Tannakian $\mathrm{Rep}(G)$ satisfies $e|G|^2 \mid D$ by the centralizer formula: its centralizer contains the product of $\mathrm{Rep}(G)$ with the symmetric center. Thus $|G| \mid R(D/e)$. Under the first hypothesis every such G is solvable, and [42, Theorem 5.1] applies. Under the second, use $Q(m) \leq R(m)^3$ and $Q(2m) \leq 4Q(m)$ to obtain $Q(D) \leq 4 \cdot 59^3 < 29160000$. Theorem 17.1 first gives WGT; the smallest nonabelian simple group has order 60. □

In particular $Q(D) < 3600e$ suffices for solvability. Indeed the dimension bound first gives WGT. A nonsolvable Tannakian group would have order at least 60 divisible by four, so $e|G|^2$ would itself be square-full and divide $Q(D)$. Likewise $16e \nmid D$ suffices once WGT is known. The center of Vec_{A_5} and its product with sVec attain the respective thresholds and contain $\mathrm{Rep}(A_5)$. Together with the classification of Xu and Dong [51, p. 292, abstract], the dimension bound also gives the following small-range classification: every nontrivial perfect weakly integral modular category of dimension < 14400 is a twisted center $\mathcal{Z}(\mathrm{Vec}^{\omega}_{A_5})$, of dimension 3600.

For an odd-dimensional modular category put $\mathcal{E} = \mathcal{C}_{\mathrm{ad}} \cap \mathcal{C}_{\mathrm{pt}} = \mathrm{Rep}(H)$, $g = \mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}})$ and $e_0 = \mathrm{FPdim}(\mathcal{E})$. The symmetric center of $\mathcal{C}_{\mathrm{ad}}$ is $\mathcal{E}$, so the modular quotient $(\mathcal{C}_{\mathrm{ad}})_H$ has dimension $D/(ge_0)$ [26, Proposition 4.26 and Corollary 4.31(i)]. Solvability passes in both directions [32, Proposition 4.5(i), (iv)]. Iteration ends at a perfect category, which is trivial precisely in the solvable case since every nontrivial solvable category has a nontrivial invertible object.

18 Dimension and rank bounds for simple perfect categories

The dimension thresholds also give bounds for simple perfect categories. We first state consequences that hold in arbitrary rank and then give lower bounds on rank. Here and below a simple perfect category is understood to be nontrivial.

Corollary 18.1 (Dimension in arbitrary rank)**.** *Every nontrivial simple perfect integral modular category has dimension at least* 29160000*, and at least* 6087156075 *in odd dimension. Without a braiding assumption every non-WGT weakly integral fusion category has dimension at least* 5400*, and at least* 121275 *in odd dimension. The odd lower bound also holds for every nontrivial perfect weakly integral fusion category.*

Proof. A nontrivial simple modular category has no nontrivial Tannakian subcategory: simplicity would force that subcategory to be the whole category, whereas a nontrivial symmetric category is degenerate. An integral WGT braided category with no nontrivial Tannakian subcategory is

pointed by [42, Theorem 3.1]. Perfection rules out a nontrivial pointed category. The first two assertions now follow from Theorem 17.1 and $Q(D) \leq D$. The same theorem gives the unbraided bounds, since solvability implies weak group-theoreticity and a nontrivial solvable category has a nontrivial invertible object [32, Proposition 4.5(iv)]. In the unbraided statement the non-WGT assumption matters even for simple perfect categories: $\mathrm{Rep}(A_5)$ is such a category of dimension 60 and rank five. □

Corollary 18.2. *A nontrivial simple perfect integral modular category has rank at least* 16*, and at least* 27 *in odd dimension. Without a braiding assumption, a non-WGT perfect integral fusion category has rank at least* 7*. A nontrivial odd-dimensional perfect integral fusion category has rank at least* 15*.*

Proof. Alekseyev, Bruns, Palcoux and Petrov excluded nontrivial perfect integral modular categories through rank 13, and through rank 24 in odd dimension [2, Theorems 1.5 and 12.2]. The modular lower bounds 15 and 27 are established in [24, Corollaries 1.2 and 1.3]. For the additional step at rank 15, we use as input the exhaustive list of 2341 remaining perfect dimension types from [24, Section 4.3] (arXiv:2510.10311v1). For a type (d_i), its smallest codegree is $\sum_i d_i^2/(\max_i d_i)^2$. For the values $2, 3, 4, 5$ the respective counts are $459, 24, 1857, 1$.[1] Lemma 2.6 excludes the first three groups. The last type has dimension 720 and contains a simple of degree two, contrary to Lemma 2.5.

In the unbraided case, integral Grothendieck rings of rank at most six are commutative apart from $\mathbb{Z}S_3$, and odd-dimensional ones are commutative through rank thirteen [1, Theorems 1.5 and 1.8]. In the commutative case the formal codegrees f_i divide D, $\max f_i = D$, and $\sum 1/f_i = 1$. In a perfect category all $f_i \geq 3$: for a character other than the dimension, orthogonality to the latter requires at least two nonzero entries away from the unit. Indeed, a unique such entry would be $-1/d_Y$, which is not integral. Each nonzero entry has conjugate mean square at least one. The reciprocal-sum enumeration gives candidate dimensions at most 1806 for rank at most five, and at most 903 for odd dimension and rank at most seven [1, Sections 7.1 and 8.2.2].

Here D/f is an integer by Ostrik's induction formula: it is the dimension of a simple in the center, whose image under the forgetful functor has integral dimension [48, Theorem 2.13]. At rank six, the only $D \geq 5400$ from this reciprocal sum are

$$6240, \quad 7140, \quad 7320, \quad 8268, \quad 8364, \quad 12324, \quad 14520, \quad 24492.$$

After squaring their exponent shapes are $(a, 2, 2, 2)$, $(6, 4, 2, 2)$ and $(4, 2, 2, 2, 2)$ up to permutation. The four- and five-prime theorems therefore apply to the center.

In odd dimension duality fixes only the unit. Its trace on the based ring is one, so it fixes just one primitive idempotent, that of the dimension character. Thus $\chi \neq \overline{\chi}$ for every other character. Its support outside the unit is a union of dual pairs and the gcd of their degrees is one. Perfection requires at least two pairs. If $f \leq 5$, the mean-square bound forces $f = 5$ and precisely two pairs, whose four values are roots of unity. Write these as $u, \bar{u}, v, \bar{v}$ and their respective degrees as d, d, e, e. Set $s = u + \bar{u}$, $t = v + \bar{v}$. Orthogonality to $\overline{\chi}$ and to the dimension character gives

$$s^2 + t^2 = 3, \qquad ds + et = -1.$$

Rational s, t would be integers, which is impossible in the first equation. Otherwise s has degree two and its trace is $-2d/(d^2 + e^2)$, by eliminating t. This lies strictly between -1 and zero, contradicting integrality. All formal codegrees are odd divisors of D, hence are at least seven.

In ranks nine and eleven the codegrees consist of a distinguished D and equal pairs [1, Proposition 8.5]. With minimum seven, their largest possible D is respectively 105 and 22155,

[1] The file `RemainingPerfect.txt` contains one type per line. Our codegree count uses a copy of that list. It checks its SHA-256 digest against the value recorded in the data record, verifies that the 2341 types are distinct, and computes their smallest codegrees. Exhaustiveness of the input is the cited classification result.

below the odd threshold in Theorem 17.1. The reciprocal enumeration for ranks six, nine and eleven proceeds as follows. If the outstanding sum is a/b and the remaining total weight is h, the next nonfinal denominator x of weight w satisfies $wb/a < x \le hb/a$ and leaves $(ax - wb)/(bx)$. Take denominators in increasing order and retain earlier denominators dividing the final D, which is fixed by the last reciprocal. Six weights one and minimum three give the displayed eight values above the threshold. In odd dimension take odd denominators, paired weights two, and last weight one. At minimum seven this gives 3 tuples of length nine and 30 of length eleven (counting repeated denominators). Odd dimension forces odd rank, so only rank thirteen remains. To handle it we need a restriction on codegree seven. In a commutative perfect odd-dimensional integral fusion ring arising from a category, a character χ of codegree seven has nonreal field contained in $\mathbb{Q}(\zeta_7)$ by Corollary 2.8. Its support consists of the unit and two or three dual pairs, by the preceding norm and gcd estimates. With three pairs all six nonunit values are roots in μ_{14}. Orthogonality to $\bar{\chi}$ says that the seven squares of the entries sum to zero. The cyclotomic relation forces these squares to run through μ_7. Orthogonality to the dimension character is then a relation on all seventh roots: after writing each value as a signed seventh root, the powers are distinct and the signed coefficients are its simple degrees. The vacuum coefficient is one, so every coefficient equals one, contrary to perfection.

With two pairs, all Galois conjugates of χ have the same five-label support and are mutually orthogonal, so the field has degree at most five. The only nonreal proper subfield of $\mathbb{Q}(\zeta_7)$ is $\mathbb{Q}(\sqrt{-7})$. The two nonzero squared absolute values are positive integer norms summing to three. Thus on a pair A the character is $\epsilon = \pm 1$ and on a pair B its values have norm two and sum $\delta = \pm 1$ by the norm form of that field. Their degrees satisfy $1 + 2\epsilon d_A + \delta d_B = 0$, so $d_B > d_A$. Associate the ordered pair $A \to B$ to $\{\chi, \bar{\chi}\}$. Two such pairs have distinct heads, because $\chi - \bar{\chi}$ is supported just at the head, and two differences with that same support would be proportional and orthogonal. Their tails too are distinct: at a common tail the inner product of the two character sums $\chi + \bar{\chi}$ would be 4 ± 8. For distinct heads and tails this inner product is four from the unit and ± 4 per head–tail coincidence. Its vanishing therefore requires exactly one such coincidence. Three pairwise consecutive directed edges with distinct heads and distinct tails would form a cycle, which strict degree increase precludes. There are at most two complex-conjugate pairs of codegree seven.

At rank thirteen the recursion with six paired denominators at least seven and one final D gives 520 tuples. Of these, 17 have $D \ge 121275$ and at most two paired denominators equal to seven. Testing their squares by the exponent comparisons of Theorem 17.1 leaves the paired denominators and dimensions

$$(7, 7, 11, 11, 31, 4785; 1038345), \qquad (7, 7, 9, 11, 95, 575; 7571025).$$

In both, $v_7(D) = 1$ and no other paired denominator is divisible by seven. In $D\tau = \sum_\chi (D/f_\chi)\chi$, reduce at a prime above seven. The two codegree-seven pairs give two distinct reduced characters: the two quadratic-conjugate values agree at the unique prime of $\mathbb{Q}(\sqrt{-7})$ over seven, and the values on each pair A, B are nonzero modulo seven. The two supports differ, since their edges are consecutive and directed by strictly increasing degrees. The surviving relation contains only the dimension character and these two distinct characters, each of the latter with nonzero coefficient $2D/7$. This contradicts independence of distinct characters over the residue field, even if one is the dimension character. Hence rank thirteen is impossible.

In the nonpointed odd-dimensional case with noncommutative fusion ring the lower bound is 23 by [1, Theorem 1.8]. □

For comparison, the Anyon Atlas [5, Section 3] uses a numerical search to construct modular data through rank 22. The proof above uses the exact rank-15 list.

19 Cyclotomic support and local ramification

Throughout these sections a minimal table is one of the two tables in Lemma 2.1. The dimension of a positive fermionic quotient is half the dimension of its category. Write P_t for the principal block at a chosen prime above t, and call X *t-active* when the t-part of θ_X is nontrivial. The twist graph has the dimension primes as vertices and an edge t, u when a simple twist is divisible by tu. An edge need not have the same witness as another edge. We call t, u *separated* when this edge is absent. In the ordinary square-dimensional case put $n = \sqrt{D}$, $s = S/n$, $f_X = D/d_X^2$ and $K_X = \mathbb{Q}(\chi_X)$. A nonzero S-entry will be called a hit. A full word at nonunits X, Y is a positive rational combination of phases, of total mass d_X, equal to $\chi_Y(X)$ up to a common root of unity. Write $W(t, k)$ for the impossibility of such a word on at most k positions of t-power order in odd ordinary minimal data. All assertions about divisor families allow a coprime square-free factor d.

19.1 Minimum positive mass and exposed faces

Fix a prime s, an integer $a \geq 1$, $L = s^a$, and ζ primitive of order L. Identify a coset of μ_s with its indices modulo L/s. Choose m coprime to s. At each j mod L permit the phases $\zeta^j \eta$ where η runs over the primitive roots of orders in a specified subset H_j of the divisors of m. These sets may vary with j and may be empty. Let μ be the Möbius function and φ Euler's totient. For $\epsilon = \pm 1$ put

$$b_j^{\epsilon} = \min\{\varphi(n) : n \in H_j,\ \mu(n) = \epsilon\}, \qquad \min \varnothing = +\infty.$$

Representations have nonnegative rational coefficients. Their mass is the sum of these coefficients.

Theorem 19.1. *For rational c_j and $x = \sum_j c_j \zeta^j$, the least mass of a representation by permitted phases is*

$$\sum_{C \in \mu_L/\mu_s} \min_{t \in \{c_j : j \in C\}} \sum_{j \in C} \left[b_j^+ (c_j - t)_+ + b_j^- (t - c_j)_+ \right].$$

Here $0 \cdot \infty = 0$, and an infinite minimum means that no representation exists. Every finite minimum is attained and gives an explicit word.

Proof. Average a representation over the complementary Galois group; it fixes x, preserves mass, and acts transitively on the primitive roots of each allowed order. Their normalized trace is $\mu(n)/\varphi(n)$. If the resulting rational coefficient at j is $u > 0$, its minimum cost is $b_j^+ u$. Indeed no orbit contributes more per unit mass than $1/b_j^+$; equality comes from a minimizing orbit, since the sum over it is 1. For $u < 0$ use an orbit of sum -1 and obtain $b_j^-(-u)$. An orbit of sum zero cannot reduce the minimum. The only rational relations among the L indicated powers of ζ are the sums of the μ_s-cosets: these L/s relations have disjoint supports and $L - \varphi(L) = L/s$. Hence the possible coefficients on C are exactly $c_j - t$. The nonnegative piecewise linear cost attains any finite minimum at a breakpoint c_j, since feasibility endpoints also occur there. Different cosets are independent. □

Say that $E \subset \mu_L$ is exposed if some rational linear functional on the cyclotomic field takes value one at the indicated pure roots and strictly less than one on all other permitted phases. Require $1 \in H_j$ at every marked position.

Theorem 19.2. *E is exposed if and only if in every coset C,*

$$|E \cap C| < \sum_{j \in C \setminus E} b_j^- .$$

Proof. Average an exposing functional over the complementary Galois group. It then takes the value $t_j \mu(n)/\varphi(n)$ on the primitive order-n orbit above j. The rational values t_j satisfy precisely $\sum_{j \in C} t_j = 0$, by the coset relations and linear disjointness. At a marked root $t_j = 1$ works,

since every nonidentity orbit has mean less than one. At an unmarked position with finite b_j^-, strictness requires $t_j > -b_j^-$. Summing gives necessity, including the case of a completely marked coset. Conversely choose $t_j \le 0$ on the unmarked positions with total $-|E \cap C|$. With a finite, positive total capacity, distribute this negative sum proportionally to the capacities. If an infinite capacity occurs, place it all there. For an unmarked coset take zeros throughout. The strict inequality makes every unmarked value less than one, including orbits with zero or positive mean. Composing the resulting functional on $\mathbb{Q}(\zeta)$ with normalized relative trace defines the required functional on the whole field. □

For H_j the full set of divisors of $m > 1$, $b_j^- = \ell - 1$ for ℓ the smallest prime dividing m. The criterion specializes to $|E \cap C| < s(\ell - 1)/\ell$. When $m = 1$, it says that each coset contains an unmarked root. For an arbitrary finite alphabet lacking complementary Galois invariance, choose power-basis coordinates modulo Φ_N and maximize ε subject to value one on the marked roots and value at most $1 - \varepsilon$ on every other permitted root. Exposure is equivalent to a feasible positive gap.

19.1.1 Fusion witnesses on a nonexposed face

If the criterion in Theorem 19.2 fails on a coset with k marked positions, let v count the unmarked positions with $b_j^- < h$ for an integer $h \ge 2$. Since every capacity is at least one,

$$v \;\ge\; \max\left\{0, \left\lceil \frac{h(s-k)-k}{h-1} \right\rceil\right\}. \tag{19.1}$$

Each counted position admits an order n with $\mu(n) = -1$ and $\varphi(n) < h$. These possible orders are finite and explicit: $n \le 2\varphi(n)^2$ by multiplication of the prime-power factors. If the alphabets are obtained by closing the *actual* phases of a chosen expression under complementary Galois conjugation, every such order at that position occurs in the original expression. Adding the marked pure phases for the comparison creates no negative order at an unmarked position.

Proposition 19.3. *Suppose $\chi_Y(U)$ at a fixed nonunit row has a full-mass positive expression, up to a scalar root, on primary support $A \subset \mu_L$. Set $D = A - A$ in additive exponents. If $D \cap uD = \{0\}$ for a unit multiplier u, then at least one of the two expanded adjoints of U, U^u has a coset failing Theorem 19.2, relative to its own marked set D, uD. On that coset (19.1) holds.*

Proof. Take actual simple-constituent balancing expressions in both expanded adjoints at Y. Independently in each expression close the allowed complementary orders at each primary position as above; extend the primary level if needed. New cosets with no marked root impose no restriction. If both marked sets were exposed, positivity and equality of total mass would put every constituent's phases in its respective marked set. By Lemma 4.15 the adjoints share a nonunit, also for the chosen lifts in the positive quotient. Using the same expression for that constituent on the fixed row puts all its phases in $D \cap uD = \{0\}$, contrary to strictness. □

Corollary 19.4. *If a full-mass two-position word of a nonunit at a nonunit row has relative phase order s^a for an odd prime s, then $s^a = 3$ or 5. In the latter case a constituent balancing expression in one of the two adjoints in Proposition 19.3 has a phase of exact order ten. In odd ordinary data only order three remains possible.*

Proof. Absorb a scalar and write the positions as $0, d$, where d has order s^a. If $s^a > 3$, the sets $\{0, \pm d\}$ and $\{0, \pm 2d\}$ intersect only at zero. If $a > 1$, the three positions of each are in distinct order-s cosets, and every capacity is at least one. If $a = 1$ and $s \ge 7$, a coset with three marked positions has at least four unmarked ones. Both cases force exposure, contrary to the proposition. At $s = 5$ the only coset which can fail has three marked positions and two unmarked. Formula (19.1) at $h = 2$ forces an unmarked position to admit a complementary root of order two. The primary part at that position is a nonidentity fifth root. The order of the actual phase is therefore ten. Odd ordinary data admit only odd-order phases (Section 8.1). □

A test from the unresolved family. At $2^6 3^6 5^4$ one surviving complementary crossing can have the normalized word $z = 1 + 3\zeta_5$. This word has strict house below four, residue four over three and five, and tame trace one at five. On the alphabet μ_{30} the pure coefficient vector for $|z|^2$ is $(10, 3, 0, 0, 3)$. The minimum in Theorem 19.1 occurs at $t = 3$ and is thirteen:

$$|z|^2 = 7 + 3(-\zeta_5^2) + 3(-\zeta_5^{-2}).$$

The required adjoint mass is sixteen. Given *any* $\omega \in \mu_{30}$, add $\omega(1 + \zeta_3 + \zeta_3^2) = 0$ to the right. This gives a positive integral expression of mass sixteen containing ω. Thus even the sharpest one-entry support bound at this alphabet retains every phase. Nevertheless Corollary 19.4 forces the indicated order-ten event once this two-position alternative is selected: after scaling, the crossing gives a full character word. It is a disjunction about specified products. The joint restrictions on the surviving modular data are developed in Section 24.

19.2 Two phases of arbitrary relative order

In this subsection exposure is taken among all roots at any common multiple of the conductors of the expressions compared.

Theorem 19.5. *If a full-weight word has exactly two distinct phases at a pair of nonunits in a minimal ordinary or positive reduced table, their ratio has order* $2, 3, 4, 5$ *or* 6*. In odd ordinary data only order* 3 *is possible.*

Proof. Write the ratio as ρ and its order as n. We prove that for $n \geq 7$ the set $D = \{1, \rho, \rho^{-1}\}$ is exposed at every common conductor. We may choose any primitive nth root for ρ by Galois symmetry. We first give a construction that handles all but six orders at once.

Split off a prime-power factor s^b of the common conductor, writing each root as αy, with α of s-power order and y of order coprime to s. At a marked primary position write y_i for the complementary component of its marked root. The three primary positions will always be distinct in this construction. Prescribe there the functional $f_i(y) = \mathrm{Tr}_{\mathrm{norm}}(y/y_i)$, normalized absolute trace on the complementary field. It takes the value 1 on a root only when $y = y_i$, and lies in $[-1, 1]$ at all roots. On each μ_s-coset of primary positions with k marks, put $-\sum f_i/(s-k)$ at each vacant position. These assignments define a rational functional by linear disjointness and the primary cyclotomic relation that the values sum to zero on each coset.

If $s \mid n$ and $s \geq 7$, then $k \leq 3 < s/2$, giving strictness off the marks. If $s^2 \mid n$ for an odd prime s, the three positions lie in three distinct cosets, so $k \leq 1 < s/2$. If $8 \mid n$, use $s = 2$. The three marked cosets are again distinct, and every complementary root has odd order and normalized trace at least $-1/2$. Indeed the trace of a primitive dth root is $\mu(d)/\varphi(d)$ by summing its conjugates. Thus the vacant position opposite a mark has value at most $1/2$.

The remaining $n \geq 7$ divide 60. At $m = \mathrm{lcm}(n, 2)$ the table gives an integer functional L in the basis $1, z, \ldots, z^{\varphi(m)-1}$, where z is a primitive mth root. Its value on D and its maximum on the other mth roots are recorded in the last two columns.

n	$(L(z^i))_{0 \leq i < \varphi(m)}$	on D	elsewhere
10	$(2, 2, 1, -1)$	2	1
12	$(2, 2, 1, 0)$	2	1
15	$(2, -1, 2, 1, -1, 1, -1, 1)$	2	1
20	$(2, 2, 1, 0, -1, -1, -1, -1)$	2	1
30	$(4, 4, 2, 3, 3, 1, 1, -1)$	4	3
60	$(4, 4, -1, -1, 1, 2, -1, 3, -1, 1, 1, 1, -1, -1, -1, -1)$	4	3

The entries can be checked by the recurrence $L(z^{j+d}) = -\sum_{i<d} a_i L(z^{j+i})$ for $\Phi_m(x) = x^d + \sum_{i<d} a_i x^i$. Normalize by the value on D. To extend the functional, compose with normalized relative trace to $\mathbb{Q}(\mu_m)$. A root outside μ_m has relative trace zero if it uses a higher power of a

prime of m. Otherwise a new odd prime contributes zero or a factor of absolute value at most $1/2$. The normalized functional has absolute value at most one on μ_m since m is even. Thus all additional roots have value strictly less than one.

The adjoint of the two-phase word gives a full-weight positive expression supported on D. Choose a Galois multiplier u with $u \not\equiv \pm 1 \pmod{n}$; such a unit exists for every $n \geq 7$ and lifts to the common conductor. Then $D \cap uD = \{1\}$. As in Lemma 4.15, the two adjoints share a nonunit. Exposure forces its balancing phases at Y to lie in this intersection, contrary to strictness. For odd ordinary data Corollary 19.4 excludes order five, and the categorical Cauchy theorem excludes even orders. □

19.3 Fusion support of the exceptional cyclotomic word

For an ordinary or positive reduced crossing of nonunits X, Y, write $\ell = \operatorname{lcm}(d_X, d_Y)$, $g = \gcd(d_X, d_Y)$ and $z = S_{XY}/\ell$. Choose actual even lifts in the reduced case. A constituent V of XY gives its monodromy phase $\theta_V/(\theta_X\theta_Y)$ mass $N_{XY}^V d_V$. The total mass is $g\ell$.

Proposition 19.6. *Suppose the signed alternative in the two-residue test of Theorem 4.9 occurs at distinct odd primes $q < r$. Thus $g = r - 1$ and $k = (r-2)/q$ is a positive integer. Its $r-1$ phases $-\eta_t\zeta_r^t$ each have mass ℓ in XY. All constituents have dyadic twist $-\theta_{X,2}\theta_{Y,2}$, and total dimension at least $(r-3-k)\ell$ comes from constituents active at both q and r. In particular the signed alternative requires a column mixing the two odd primes.*

Proof. The stated alternative writes $z = -\sum_{t=1}^{r-1} \eta_t\zeta_r^t$, where the q-power roots η_t consist of one copy of 1 and k complete cosets of μ_q. Write $\alpha_t = -\eta_t$, so $\sum_t \alpha_t = -1$. Split an arbitrary common conductor into its r-part and a coprime part, enlarging the latter to be even. Let T be normalized rational trace on the coprime cyclotomic field. At the $r-1$ marked positions define $f_t(y) = T(y/\alpha_t)$, and at the remaining position use $f_0 = -\sum_t f_t = T$. Use zero on other μ_r-cosets of the r-power part. The values sum to zero in each coset and therefore define a rational functional. Each marked functional reaches one only at α_t. At the zeroth position f_0 initially reaches one at 1. Choose j for which α_j is nonreal, and put $L(y) = T(y) - T(\alpha_j)f_j(y)$. Replace f_j by $f_j + L/8$ and f_0 by $f_0 - L/8$. The coset relation is preserved, while $L(\alpha_j) = 0$ and $L(1) = 1 - T(\alpha_j)^2 > 0$. At every root other than the former maximizer the old functional is at most $1/2$: for a root of order $m > 1$ the normalized trace is $\mu(m)/\varphi(m)$. Since $|L| \leq 2$ on roots, the new maximum one occurs precisely at the prescribed phases. The nonidentity rth roots are linearly independent over the coprime cyclotomic field, so the prescribed phases are rationally independent. Applying the functional to the balancing expression forces every phase into the displayed set; independence fixes each mass. At most one phase cancels the r-part of $\theta_X\theta_Y$, and at most $k+1$ cancel its q-part, since each coset uses each position at most once. The remaining number is at least $r - 3 - k > 0$, as $q \geq 3$ and $r \geq 5$. □

The unsigned alternatives of Theorem 4.9 also determine all monodromy phases and masses. At one residue with $r > 2g$, at most $g < r/2$ positions are used. At two residues with $g < r$, write $z = h + q\sum_{j=1}^m \xi_j$ with $h + qm = g$. If $h = 0$, the number b of distinct positions satisfies $2b \leq 2m \leq g$. If $h > 0$ then $2b \leq 2 + 2m \leq g + 1 \leq r$. In both cases $b < r/2$, since r is odd. For each μ_r-coset of r-primary positions with a marks, place the normalized trace on the coprime field at each mark and $-a/(r-a)$ times that trace at each vacant position. This exposes precisely the marked roots, and their independence follows from the coset relations. Thus each monodromy phase in the displayed integral word has mass ℓ times its multiplicity and every constituent has the same prime-to-r twist as $\theta_X\theta_Y$.

At $g = 4$, assume the residues at $3, 5$ are four and the house is strictly less than four. The signed profile puts mass ℓ at each of its four phases. The only other expression in Theorem 4.9 is $1 + 3\xi$ with ξ of five-power order. Since $\chi_Y(X) = (d_X/4)z$ has full weight, the two-phase theorem gives $\operatorname{ord}(\xi) = 5$. Their masses are $\ell, 3\ell$, respectively.

Let H be tame five-inertia, of order four, acting trivially on the prime-to-five factor at a common conductor. In both cases $\sum_{\sigma\in H}\sigma z = 1$: the four conjugates of a primitive fifth root sum to -1, and $\sum \eta_t = 1$. The proof identifies the constituent masses before applying the trace; thus it applies at arbitrarily large common conductor.

Proposition 19.7 (A single coset of wild phases)**.** *Let s, r be distinct odd primes, let ω, ζ be primitive sth and rth roots, and let θ have s-power order. Write $e(i,j) = \omega^i\zeta^j$ on $\mathbb{F}_s \times \mathbb{F}_r$. Let T be a set of at least two points with distinct first coordinates and distinct, nonzero second coordinates. Suppose that*

$$(i,j),(i',j') \in T - T,\ j = j' \quad\Longrightarrow\quad i = i'.$$

If, up to a scalar root, the normalized character $\chi_Y(U)/d_U$ at nonunit labels U, Y in a minimal ordinary or positive reduced table is a convex combination with positive rational coefficients of $\{1\} \cup \theta e(T)$, then $\theta^s = 1$.

Proof. On $\mathbb{Q}(\omega,\zeta)$, a rational linear functional is specified by its values on the s-by-r array $e(i,j)$, subject precisely to zero row and column sums, by the geometric-sum relations and the degree $(s-1)(r-1)$. Consider any set of h marked entries using each column at most once, with k_i marks in row i and $2\max_i k_i < h$. It is exposed in the signed array. To see this, take two copies of the h marks and match them while forbidding pairs in the same row. Every allowed pair extends to a perfect matching: after deleting the pair, a source subset from two or more rows sees all remaining targets, and a single row satisfies Hall's inequality since $2k_i \le h-1$. Average these matchings uniformly. In each used column put 1 at its mark and at every other row put the negative of the probability that the match ends in that row. All row and column sums vanish, and at least three rows contain marks, so every unmarked absolute value is strictly below 1. In unused columns put zero. Two marks in different rows and columns are also exposed: extend them in two different ways to three marks in different rows and columns. The extensions exist because s, r are distinct odd primes. Average the functionals exposing these triples.

Suppose θ has order s^t, $t \ge 2$. The ratios of the word form the union $D = C \cup H \cup H^{-1}$, where $C = e(T-T)$ and $H = \theta e(T)$. In the direct sum decomposition of $\mathbb{Q}(\theta,\zeta)$ over $\mathbb{Q}(\omega,\zeta)$ by cosets $\theta^a\mu_s$ of the s-primary parts, these sets occupy three different summands. Indeed $[\mathbb{Q}(\theta,\zeta) : \mathbb{Q}(\omega,\zeta)] = s^{t-1}$, with basis θ^a, $0 \le a < s^{t-1}$. In C only the identity has row zero. The other row sizes satisfy $k_i = k_{-i}$ by symmetry, so $2\max k_i < |C|$. The preceding construction therefore exposes C and, using prefactors $\theta^{\pm 1}$ on their summands, H, H^{-1}. Use zero on unused summands. This exposes D at conductor $2rs^t$. To extend to any larger multiple of this conductor compose with normalized relative trace. At any phase outside μ_{2rs^t} this trace is zero or a root in μ_{2rs^t} times a rational of absolute value at most $1/2$: an increased old prime-power level gives zero average, and a new odd prime ℓ contributes zero or $-1/(\ell-1)$.

Let σ fix all s-power roots and invert ζ. It preserves each of the three summands. If (i,j) belongs to both $T-T$ and its image under column negation, then $(-i,j)$ also belongs to $T-T$ by central symmetry. Uniqueness in column j gives $i = 0$, hence $j = 0$. In T the uniqueness of the row forces a common point with its image to have $j = -j$, impossible. Thus $D \cap \sigma D = \{1\}$. The adjoints of U and σU have a common nonunit constituent. By exposure its balancing expression at Y uses only the intersection, contradicting strictness. □

The same proof with $D = C$ shows that a word supported on $e(T)$ alone, up to a scalar, is impossible. In this assertion columns of T may include zero, since the argument uses only differences. For example, the proposition applies to $T = \{(j, b+j) : 0 \le j < m\}$ whenever $2 \le m \le s$, $r \ge 2m-1$, and all its columns are nonzero. The column of a difference uniquely determines its integer step in $[-m+1, m-1]$. For the signed four-term crossing, take ζ to be the fifth root at the distinguished singleton. In order of the remaining exponents $2, 3, 4$, the ternary parts are $\theta, \theta\omega, \theta\omega^2$, on choosing one of the primitive cube roots as ω. Division by the singleton gives the progression $s = 3, r = 5, m = 3, b = 1$, hence $\theta^3 = 1$.

At the remaining level $\theta^3 = 1$ there are exactly two Galois orbits of signed words with the four equal weights. Indeed, the two positions with ternary part 1 are either opposites in $\mathbb{F}_5^\times$ or a pair with quotient 2 up to inversion. Multiplying the positions by a unit is transitive on each type, and ternary inversion exchanges the assignments on the other two positions. The four nonidentity fifth roots are independent over $\mathbb{Q}(\omega)$, so the stabilizer of the word is exactly the stabilizer of its array. For an opposite pair it is generated by complex conjugation, giving $\mathbb{Q}(z) = \mathbb{Q}(\zeta_{15})^+$; for a nonopposite pair it is trivial, giving $\mathbb{Q}(z) = \mathbb{Q}(\zeta_{15})$. Thus if either label is self-dual, the opposite signed profile is forced, since the unsigned value $1 + 3\xi$ is also nonreal. For both orbits the adjoint word contains $1, \xi, \xi^{-1}$ for a primitive fifth root ξ, from the repeated ternary part. The identity $1 + \xi + \xi^{-1} = -\xi^2 - \xi^{-2}$ replaces an equal positive amount at these three positions with strictly less total weight in the signed alphabet. The saved weight can be distributed uniformly over all roots of the common conductor since their sum is zero. Thus neither orbit has a proper exposed adjoint face in the full signed alphabet.

19.4 Two positive residues just above the larger prime

Write μ_{v^∞} for the group of all v-power roots of unity. In the normalized crossing of two integral simples, a positive expression of weight g is a sum of roots of unity with nonnegative rational coefficients totaling g. The weight from balancing is $g = \gcd(d_X, d_Y)$. The following result also determines the monodromy phases of the product.

Proposition 19.8. *Let $s > q \geq 5$ be odd primes, $g = s + t$, $1 \leq t < q$, and suppose every prime in the phase orders is at least $\ell \geq 3$. Let z be a cyclotomic integer with a positive expression of weight g, congruent to g at all primes above q and s. For a residue a modulo q with representative $0 \leq a_0 < q$, put $w(a) = \min(a_0, c(q - a_0))$. Suppose some real $1 < c < \min(\ell - 1, q - 1)$ satisfies*

$$\left(1 + \frac{2t}{s}\right)^2 < \frac{2(q-2)}{q-1}, \quad c(s-t) > g, \quad w(g) > t, \quad (s-1)w(a) + w(g+a) > g \quad (a \neq 0).$$

Then every positive weight-g expression for z has the same phase multiplicities. They are nonnegative integers. The q-parts of the terms consist of t copies of 1 and s copies of one q-primary root η. All remaining parts are s-primary. At every s-position x its multiplicity n_x satisfies $n_x \equiv g\, 1_{x=1} \pmod q$.

Proof. Give positive roots cost one and negative roots cost c. First consider an integral signed word in q- and s-primary roots with the two residues and cost at most g. Erase the q-parts and let m_x be the signed multiplicity at s-position x. Modulo q, the numbers $m_x - g1_{x=1}$ are constant on each coset of μ_s by the cyclotomic relations. The identity coset cannot have a nonzero constant a by the last cost inequality. A nonidentity coset with nonzero constant would cost at least s, besides the identity position's cost $w(g) > t$. Thus all constants vanish and the signed augmentation is g modulo q.

Erase instead the s-parts, obtaining a signed q-primary word of value v. Its residue over s gives $\alpha = (v - t)/s$ integral, as s is unramified in the q-power field. Each conjugate of α has absolute value at most $1 + 2t/s$. A nonzero cyclotomic integer in a q-power field of mean square less than $2(q-2)/(q-1)$ is a signed q-power root. Indeed, write it as an integer sum of q-power roots. Cassels's trace identities [15, (3.4) and (3.16), pp. 114–115] make different μ_q-cosets orthogonal for normalized trace and give the contribution $\sum_{i<j}(b_i - b_j)^2/(q-1)$ for a coset with coefficient vector (b_i). A nonconstant coset contributes at least one. In a single coset, two coefficient levels at distance one give numerator $k(q-k)$; below $2(q-2)$ one of their multiplicities must be one. If a pair differs by at least two, its contribution and those with the remaining $q-2$ points total at least $4 + 2(q-2)$. This proves the assertion. Also $v = t$ would give signed augmentation t modulo q, contradicting $g \equiv t + s$. Hence $v = t \pm s\eta$ for a q-power root η.

For the minus sign, prescribe value $-c$ at η and $c/(q-1)$ elsewhere in its μ_q-coset, zero on other cosets. This is a rational-linear functional with real values, bounded by the assigned

costs on signed roots. Its value at $t - s\eta$ is at least $c(s-t) > g$. For the plus sign, mark 1 and η, assigning value 1 at each marked position and $-k/(q-k)$ at the other positions of a coset containing k marks. The resulting functional is strictly below cost on every signed letter other than the marked positive letters. Its value at $t + s\eta$ is g. Thus cost at most g forces an unsigned word of length exactly g with precisely the stated q-parts, by linear independence of the one or two marked roots.

It remains to pass from integral signed words to arbitrary positive expressions and possibly additional primes. We use the contraction and two-prime unimodularity of Theorem 4.9 and Lemma 8.4. In each complementary primary factor, the random integral functional fixes 1, sends a uniformly chosen nonidentity root of prime order to -1, and sends the remaining positions to zero. These maps preserve both residues. A nontrivial complementary factor in a positive term has expected signed cost at most $c/(\ell - 1) < 1$. Its presence would give a contracted expression of cost below g. Linear optimization on the signed two-prime cyclotomic configuration has an integral minimum, contradicting the first part. Thus all original phases have only q- and s-parts. Their expression is a convex combination of integral optima, since the optimal face is bounded and its vertices are integral by the same unimodularity.

For uniqueness it suffices that the union of the s-positions of two optima have fewer than s points in each μ_s-coset. Each optimum has at most $1 + \lfloor g/q \rfloor$ positions, including 1 since $w(g) > 0$. Hence the union has size at most $1 + 2\lfloor g/q \rfloor < s$ (use $q \geq 5$, $t < q < s$). These s-roots are independent over the q-power field, so the two coefficient sums of q-roots agree separately at each position. The union of their q-roots is contained in $\{1, \eta, \eta'\}$; these are independent if distinct since $q \geq 5$. All multiplicities therefore coincide. □

For $g = 25$, $q = 7$, $s = 23$, $t = 2$, take $c = 2$ and $\ell = 5$. The inequalities are $(27/23)^2 < 5/3$, $42 > 25$, $w(25) = 4 > 2$. For $a = 1$ one has $22 + w(26) = 26$, and for other nonzero residues $22w(a) \geq 44$.

Corollary 19.9. *At any nonunit crossing of the two-residue type just described, tame* 23*-inertia has orbit size* 22 *on the normalized entry* z*. In particular both fields of the two labels ramify at* 23*. This applies to all opposite complementary rows of the principal seven- and* 23*-blocks with common* 5*-part* 25*, for arbitrary exponents of the other degree factors.*

Proof. The 23-projection contains 1 and at most three further positions. If it is just 1, then $z = 2 + 23\eta$. If $\eta = 1$ this violates strictness; otherwise the full-mass two-position character has relative order at least seven, excluded by Theorem 19.5. By uniqueness every stabilizer in the order-22 tame group preserves the nonzero positions (in additive 23-exponents). Its order divides their number, since that group acts freely there. Only order two could survive. In that case the projection is $A = \{0, a, -a\}$. The full positive adjoint word on the same row has 23-components in $A - A = \{0, \pm a, \pm 2a\}$ and 7-components in $\{0, \pm b\}$, writing b for the exponent of η. This entire product of sets is exposed. In each primary factor mark the prescribed positions, give them value one, and use $-k/(v-k)$ elsewhere in a μ_v-coset with k marks. Each density is less than one half, so the product functional (with normalized trace on any complementary factor) exposes the prescribed set. An independent 7-unit moves the nonzero pair off itself, and a tame 23-unit moves the two nonzero opposite pairs off themselves (at most four of eleven multipliers can give a match modulo sign). The original and conjugate adjoints would then have a common nonunit whose balancing phases are all 1, contradicting strictness (Lemma 4.15). This rules out the last stabilizer. □

Proposition 19.10. *In the* $g = 25$*,* $q = 7$*,* $s = 23$ *crossing of Proposition 19.8 in an ordinary minimal table on* $5, 7, 23$*, the four* 23*-positions belong to* μ_{23} *and, in additive exponents, are affinely equivalent to* $\{0, 1, 3, 14\} \subset \mathbb{F}_{23}$*. Their multiplicities are four at zero and seven at each of the other three. At most two letters have* 7*-part different from* η*. The joint inertia orbit of the*

entry at $7, 23$ *has size* $22\varphi(\operatorname{ord}\eta)$. *With* $v_{23}(23) = 1$,

$$v_{23}(z - 2 - 23\eta) = 1/22$$

at every prime above 23. *At the prime-weight crossing* $z = \eta(2 + 7\sum_{i=1}^{3}\xi_i)$ *of Proposition 8.7 at* $7, 23$, *the same affine classification gives* $v_{23}(z) = 1/22$.

Proof. Write A for the 23-projection of the weight-25 word, $D = A - A$ and $B = \{0, \pm b\}$ for the differences of the two possible 7-parts. The rectangle $B \times D$ exposes the adjoint word even when the two components are correlated. Use the product of the two coset functionals in Corollary 19.9, and the normalized trace on the remaining primary factor. Indeed the negative values in the 7- and 23-factors have magnitudes bounded by $3/4$ and $13/10$, respectively, since $|A| \le 4$. Their product is at most $39/40 < 1$. In the remaining factor the trace is at least $-1/4$. Thus the functional attains one precisely on this rectangle. An independent tame 7-unit moves the nonzero pair in B off itself. For every 23-unit u, comparison with the conjugate rectangle consequently gives a common nonunit whose entire word on the fixed row is pure 23-primary on $D \cap uD$. Likewise comparison of the rectangle with any subsequently forced pure mask uses precisely their 23-coordinate intersection.

Every pure word on this row has at least five opposite pairs of nonzero differences. Otherwise a minimizing one with $h \le 4$ has exposed differences and violates $10\min\{j : j(j+1) \ge h\} \le h(h-1)$ (Lemma 4.16). Thus each of the intersections just obtained contains at least two opposite pairs. If A meets two order-23 cosets, at most three of the at most six opposite pairs in D have order 23. With l such pairs the average overlap over all units is at most $(l^2 + (6-l)^2/23)/11 < 2$, a contradiction. Hence $A \subset \mu_{23}$ since it contains the identity. The remaining finite calculation takes place in $\mathbb{F}_{23}$. Normalize two marked positions to $0, 1$ and take the other two in increasing order. For each of the $\binom{21}{2} = 210$ supports, start with the set B of its nonzero differences modulo sign. Store one representative per dilation class. Given two stored sets B, C and $u \in \mathbb{F}_{23}^{\times}$, put $H = B \cap uC$. Reject if $|H| < 2$ or if the nonzero differences of $\{0\} \cup (\pm H)$ comprise fewer than five opposite pairs. Otherwise add these differences if the resulting support L is exposed, i.e. $3(1+2|L|) < 46$. Repeat until rejection or no new dilation class appears. The preceding rectangle argument justifies each use of the initial set, and later sets are pure. Of the 18 distinct initial dilation classes, the surviving supports among the 210 are exactly the 12 affine translates and dilates of $\{0, 1, 3, 14\}$ normalized in this way. The closure calculation with `check 23 4 -ell 3 -classes` reproduces these counts (the least conductor prime in this exposure test may be taken to be three). A starting mask with fewer than five opposite pairs is likewise impossible, since its mean overlap over nonidentity tame units is at most $4 \cdot 3/10 < 2$. Thus the same four-position proof gives precisely the indicated affine class. Fewer than four positions give at most three difference pairs. The congruences in Proposition 19.8 now fix the four multiplicities.

The full 23-unit group has trivial stabilizer of these positions: a stabilizer acts freely on the three nonzero ones and has order dividing 22. Under a joint action the 23-projection is still preserved by uniqueness, so the same argument applies. With the 23-action then trivial, the 7-component must fix η since it has multiplicity 23. This gives the orbit size.

Write $0, a_1, a_2, a_3$ for the projected positions. Their affine centroid is outside the support, since the centroid of the displayed representative is 16 modulo 23. In particular $U = 7\sum_{j=1}^{3} a_j \not\equiv 0 \pmod{23}$. Color two letters with 7-part one and the others with 7-part η. This convention works also when $\eta = 1$. Let V be the sum of the exponents of the two colored letters. For $\pi = \zeta_{23} - 1$, the linear coefficient in $z - (2 + 23\eta)$ is $V + (U - V)\eta$ modulo π. It is nonzero: if $\eta = 1$ it equals U, and otherwise the reduction of η has 7-power order greater than one and hence lies outside $\mathbb{F}_{23}$. This proves the first valuation. For the pure prime-weight word the linear coefficient after division by η is U, and its constant term is 23. The same argument applies. □

Lemma 19.11. *Let* X, Y *be integral labels in a normalized minimal table of dimension* D, *let* $L = \operatorname{lcm}(d_X, d_Y)$ *and* $z = S_{XY}/L$. *Let* G *be the Galois group of the abelian field generated by* z.

For every character ψ of G,

$$L\left|\sum_{u\in G}\psi(u)u(z)\right|\le\sqrt{D}.$$

Proof. Choose a Galois lift σ_u of every $u\in G$. The labels X^{σ_u} are distinct since their entries at Y are distinct. The same holds for Y^{σ_u}. The submatrix of S/L on these rows and columns has entries $(uv)(z)$ because degrees are integral. Testing on a unit vector with coordinates $\psi(v)/\sqrt{|G|}$ gives the claimed inequality by whole-table orthogonality. □

In either crossing of Proposition 19.10, if $\eta\neq 1$ and $Q=\operatorname{ord}(\eta)$, then

$$\frac{\sqrt{D}}{L}\ \ge 19\cos(2\pi/11)\sqrt{23Q}.$$

Indeed $G=(\mathbb{Z}/Q)^\times\times\mathbb{F}_{23}^\times$ by the proven orbit size and conductor, also for the pure prime-weight word by the same stabilizer argument. Use a primitive character at Q and a nontrivial character χ at 23. Summing first at Q kills each term with 7-part one and gives a Gauss sum of modulus $\sqrt{Q}$ for every remaining term. Summing at 23 kills the zero position and yields a common Gauss sum of modulus $\sqrt{23}$ times $\sum_{j=1}^{3}m_j\overline{\chi(a_j)}$, where m_j counts the letters with 7-part η at a_j. Thus $\sum_j m_j\ge 19$. For each origin $c=0,1,3,14$ in the affine representative found by the four-position enumeration, use the character sending the primitive root 5 to $\exp(2\pi ik/22)$ with $k=18,7,11,9$, respectively. At the three other points shifted by $-c$, its exponents modulo 22 are respectively $\{0,2,4\},\{10,11,14\},\{11\},\{2,4,5\}$. An overall dilation rotates all three values together. Projection onto the bisector of the resulting arc gives the indicated bound. In particular if $D=5^6 7^{2e}23^4$ and $5^3 7^{e-1}23\mid L$, then $\eta=1$ without a separation hypothesis: otherwise the upper bound is 161 while $19\cos(2\pi/11)\sqrt{23\cdot 7}>19\sqrt{161/2}>161$.

19.5 Crossings of weight five at the primes three and five

Consider an ordinary minimal table with dimension primes $2,3,5$. For a jointly 3- and 5-active column, suppose $v_5(D)=4$. Proposition 4.2 gives a two-free nonunit $Y\in P_3$ meeting the column with $d_Y=3^j5$; the height at five cannot be two by defect-zero vanishing. Take any three-free nonunit $X\in P_5$ of degree 2^i5. The normalized crossing $z=S_{XY}/\ell$, $\ell=\operatorname{lcm}(d_X,d_Y)$, is congruent to 5 above three and to 0 above five by principality. Apply Proposition 8.6 with $(q,r)=(3,5)$. Its actual phase masses are either $2\ell,3\ell$ at $\eta,\eta\xi$, for ξ primitive of order five by Theorem 19.5, or precisely ℓ at each of $\eta,-\eta\eta_t\zeta_5^t$, $t=1,2,3,4$. In particular, tame five-inertia has trace 5η in either case. In the second case there are four phases at which the dyadic monodromy equals -1 and a fifth at which it is 1.

Both witnesses may be chosen in the fibers of trivial prime-to-principal twist, by Proposition 4.2, so their actual dyadic twists are trivial. In the signed case at most one of the $r-1$ negative letters cancels the r-part of $\theta_X\theta_Y$, and at most $k+1$ cancel its q-part, where $k=(r-2)/q$. Thus at least $(r-3-k)\ell$ of the total dimension belongs to summands active at all three primes $2,q,r$. In particular, at the sextic frontier, absence of a twist of order divisible by 30 forces the first spectrum for every such crossing.

There is also an exact local valuation at the sextic frontier: at every prime above five, $v_5(z)=1/4$, with $v_5(5)=1$. This is immediate in the first case from $2+3\xi=5+3(\xi-1)$. In the second case expand in $\pi=\zeta_5-1$. The constant coefficient vanishes and, up to a unit, the linear coefficient is $\sum_{t=1}^{4}t\eta_t$. If $\theta\mu_3$ is the ternary coset of the three nondistinguished letters, this reduces to $a+\theta B$, $1\le a\le 4$, $B\in\mathbb{F}_{25}$. When $\theta^3\neq 1$ its residue is outside $\mathbb{F}_{25}$, since $3\|25-1$. Hence the coefficient is nonzero. If $\theta^3=1$, there are indices i,j with ternary part one and k,l with parts ω,ω^2 respectively, with ω primitive of order three. The linear coefficient is $(i+j-l)+(k-l)\omega$, nonzero at five as $k\neq l$ and $\omega\notin\mathbb{F}_5$. Tame five-inertia consequently gives at least four different conjugates of this crossing and of each witness. Indeed it fixes the ternary

coefficients, and the leading residue of π is multiplied by each fourth root of unity in the residue field.

19.6 The conductor of a cyclic character subfield

Lemma 19.11a (Local ideals on a joint twist fiber)**.** *Let V have a basis (e_i) and commuting actions of $G_p = \mathrm{SL}_2(\mathbb{Z}/p^{b_p}\mathbb{Z})$ for a set S of odd primes. Write $U_p(x), L_p(x)$ for the upper and lower unipotents, and $H_{p,a} = \mathrm{diag}(a, a^{-1})$ for the diagonal torus element or its action. Suppose $U_p(1)e_i = \theta_{i,p}e_i$. For $0 \le k_p < b_p$ and units $a_p \equiv 1 \pmod{p^{k_p}}$ put*

$$E = \mathrm{span}\{e_i : \theta_{i,p}^{p^{k_p}} \neq 1 \text{ for every } p \in S\}, \qquad A = \prod_{p\in S}(H_{p,a_p} - I).$$

Then $AV \subseteq \sum_{g\in\prod_{p\in S} G_p} gE$. For $S = \varnothing$ take $E = V$ and $A = I$.

Suppose also that $G_2 = \mathrm{SL}_2(\mathbb{Z}/2^{b_2}\mathbb{Z})$ commutes with the odd factors and satisfies $U_2(1)e_i = \delta\theta_{i,2}e_i$ for a common scalar δ. For $0 \le k < b_2$, let $a \equiv 1 \pmod{2^k}$ be a unit and let P_a be a scalar multiple of $H_{2,a}$ that permutes the basis vectors. If $\theta_{i,2}^{2^k} = 1$ on all basis vectors of E, then $(P_a - I)A = 0$.

Proof. Work first in one factor $G = \mathrm{SL}_2(\mathbb{Z}/p^b\mathbb{Z})$, including $p = 2$. Modulo the normal subgroup generated by $U(p^k)$ all $U(x)$ and $L(x)$ with $x \in p^k\mathbb{Z}/p^b\mathbb{Z}$ are trivial, by taking powers and conjugating by the Weyl element. For a unit $t = 1 + x$ the identity

$$U(x)L(1) = L(1/t)H_tU(x/t)$$

then gives $H_t = 1$ in the quotient because $1 - 1/t = x/t$. Consequently $H_t - I$ belongs to the two-sided ideal of $\mathbb{C}[G]$ generated by $U(p^k) - I$. Indeed the latter ideal contains $h - I$ for each conjugate h of $U(p^k)^{\pm1}$, and hence for every product of such conjugates.

Set $B_p = U_p(p^{k_p}) - I$. Expand each $H_{p,a_p} - I$ as a sum of terms $c\,g_pB_ph_p$, with $g_p, h_p \in G_p$ and $c \in \mathbb{C}$. Commuting distinct factors puts every term of A in the form $c'(\prod g_p)(\prod B_p)(\prod h_p)$. The diagonal operator $\prod B_p$ has image exactly E. This proves the inclusion and also shows that G_2 preserves E whenever present.

If $E \neq 0$ and all its dyadic phases have order dividing 2^k, then $U_2(2^k)|_E = \delta^{2^k}I$. Its image in $\mathrm{PGL}(E)$ is the identity. Applying the normal-subgroup calculation there shows that $H_{2,a}|_E$, and hence $P_a|_E$, is scalar. A scalar that permutes a nonempty set of basis vectors is one. Commutation now gives $P_a = I$ on every odd translate gE. If $E = 0$ the first inclusion gives $A = 0$. □

For example, take dyadic level eight and a character of dyadic conductor eight. It is nontrivial at $a = 5$, the generator of the kernel of reduction of units modulo four. Here $k = 2$ and $5^{-1} = 5 \pmod 8$, so the displayed factorization is $U(4)L(1) = L(5)H_5U(4)$. If every dyadic phase on a nonzero E has order at most four, then $U(4) = \delta^4 I$ there and both $U(4)$ and its Weyl conjugate $L(-4) = L(4)$ act trivially in $\mathrm{PGL}(E)$. Since $L(5) = L(1)L(4)$, H_5 is projectively trivial, and its unsigned permutation fixes E pointwise. For a concrete joint realization, label the simples of $\mathcal{Z}(\mathrm{Vec}_{C_{24}})$ by $(x, y) \in (\mathbb{Z}/24)^2$, where y indexes the character sending 1 to ζ_{24}^y. The label $(1, 0)$ has trivial twist and its row contains ζ_{24}. Take $F = \mathbb{Q}(\sqrt{6})$, of conductor 24. Here E is the space of labels with nontrivial ternary twist, and $(1, 1)$ has twist ζ_{24}, of order 8 at two and 3 at three.

Theorem 19.12. *Let X be simple in an integral modular category with its positive spherical structure. Let F be a cyclic subfield of its character field K_X, with conductor f. There is a simple V such that $\mathrm{lcm}(f, \mathrm{ord}\,\theta_X)$ divides the order of the actual twist θ_V. At every prime p with $v_p(f) \le v_p(\mathrm{ord}\,\theta_X)$ one may simultaneously prescribe $\theta_{V,p} = \theta_{X,p}$.*

Proof. Use the lift $U(1) \mapsto T/\gamma$ of Section 3 at its congruence level L [21, Theorem II]. The scalar γ has two-power order, so the eigenvalues of $U_p(1)$ are the actual p-primary twists for

odd p, and those of $U_2(1)$ are $\gamma^{-1}\theta_{i,2}$. The tori act by Galois permutations with common sign $\epsilon_D(a) = \sigma_a(\sqrt{D})/\sqrt{D}$, as follows by applying Galois symmetry to $s_{i0} = d_i/\sqrt{D}$. Take the torus convention in which the unsigned permutation corresponds to σ_a (replacing all characters by their inverses gives the other convention).

Inflate a faithful character of $\mathrm{Gal}(F/\mathbb{Q})$ to a character ψ of $(\mathbb{Z}/L\mathbb{Z})^\times$. Its conductor is f because its kernel cuts out F. Let J consist of the primes for which $c_p = v_p(f) > v_p(\operatorname{ord}\theta_X)$, and put $k_p = c_p - 1$. Take V spanned by the labels with the same actual twist components as X outside J. Each local group at $p \in J$ preserves V because it commutes with the complementary upper unipotents. On the orbit of X under the product of the local tori in J, the unsigned action is the permutation action on a quotient of this abelian product. Its stabilizer fixes every $\chi_X(Y)$ and hence F. Thus ψ restricted to the product occurs on the orbit, and the represented tori have a common eigenline of character $\epsilon_D\psi$ in V.

For each $p \in J$ choose a local unit $a_p \equiv 1 \pmod{p^{k_p}}$ with $\psi(a_p) \neq 1$, extending it by one in the other factors. Such a unit exists by the local conductor of ψ. Among the odd primes in J set aside those for which $c_p = 1$ and $v_p(D)$ is odd. Call this set R, and write S for the other odd primes in J. For $p \in S$ the local quadratic sign is trivial at a_p. Indeed it is trivial on the entire local factor when $v_p(D)$ is even and on the principal units at every odd prime. Hence the eigenvalue of $A = \prod_{p\in S}(H_{p,a_p} - I)$ on the chosen line is $\prod_{p\in S}(\psi(a_p) - 1) \neq 0$.

The space E of Lemma 19.11a is therefore nonzero. Its labels satisfy $p^{c_p} \mid \operatorname{ord}\theta_i$ for every $p \in S$. If $2 \in J$, the unsigned permutation $P_{a_2} = \epsilon_D(a_2)H_{2,a_2}$ has eigenvalue $\psi(a_2)$ on the same line. Thus $(P_{a_2} - I)A \neq 0$. The dyadic assertion of the lemma supplies a label of E whose dyadic twist order is at least 2^{c_2}.

Fix the actual twist components of the resulting label outside R. Proposition 3.8 gives a proportion at least $\prod_{p\in R}(1 - 1/p) > 0$ of squared-dimension mass active at every $p \in R$ in this fiber. Since $c_p = 1$ for these primes, any such label has all required orders and retains the prescribed twists outside J. □

Corollary 19.13. *If* $\operatorname{ord}\theta_X$ *is maximal under divisibility among the orders of all simple twists, then* $K_X \subseteq \mathbb{Q}(\theta_X)$.

Proof. For every cyclic subfield of K_X, the theorem and maximality force its conductor to divide $\operatorname{ord}\theta_X$, so it lies in $\mathbb{Q}(\theta_X)$. Their compositum is K_X, since the characters of the finite abelian group $\mathrm{Gal}(K_X/\mathbb{Q})$ separate its elements. □

Corollary 19.14. *If three primes ramify in the field of a single row, and at least one of their inertia images has order greater than two, some simple has twist order divisible by their product. More precisely, absence of such a twist forces the three inertia images to be the distinct subgroups of order two in a Klein four group.*

Proof. Let $G = \mathrm{Gal}(K_X/\mathbb{Q})$ and let I_p, I_q, I_r be the three nontrivial inertia images in G. By Theorem 19.12 every character of G is trivial on at least one of these subgroups if the stated twist is absent. Thus the three proper annihilators cover $G^\vee$. Recall the elementary three-subgroup argument. None of these subgroups is contained in the union of the other two: a group covered by two subgroups equals one of them. Each two of them therefore have the same intersection N as all three; for if x were common to the first two but outside the third, adding x to an element only in the third gives one in none. Modulo N, the pairwise intersections are zero. Choose a only in the first and b only in the second. For any nonzero a' in the first, both $a + b$ and $a' + b$ are in the third, whence $a - a' = 0$. By symmetry all three images have size two. Thus the quotient is a Klein four group; dualizing gives the claimed inertia images. □

A second consequence uses the actual twist of the given row. If q and r ramify in K_X and θ_X is p-active for a third prime p, some twist is active at all three. Indeed the two proper annihilators of the inertia images cannot cover the dual of $\mathrm{Gal}(K_X/\mathbb{Q})$. Choose a character outside both: its cyclic field has conductor divisible by qr, and the theorem retains the p-activity of X.

For example, in the absence of a twist of order divisible by 30, each complementary witness whose tame five-inertia orbit has length four is unramified at two or at three. This applies to the weight-five crossing of Proposition 8.6 as well as the complementary weight-four crossing of Section 19.3.

More directly, if the twist of J is active at two of the three dimension primes and its row field ramifies at the remaining prime, choose a cyclic subfield ramified there. The theorem produces a twist active at all three. Hence, at the ordinary sextic core in the absence of a twist of order divisible by 30, every jointly active row is unramified at the third prime. In particular every jointly two- and three-active column has a five-unramified field. The principal hit chosen above at two in the trivial prime-to-three twist fiber of P_3 therefore has odd degree 75 or 225, and its twist has three-power order (possibly one). This applies even when both two and three mix separately with five.

19.7 Changing the primary components of a twist

Work first in ordinary integral modular data with positive spherical structure. Write $s = S/\sqrt{D}$, e_X for the coordinate vector of a simple, and W_I for the product of the local Weyl operators at the primes of a subset I of the congruence level. We use the linear lift of Dong–Lin–Ng [21, Theorem II], with the normalization of Section 6. Thus $s = W_I W_J$ when I, J partition the level primes. A common scalar convention for s does not affect any conclusion below. Different local groups commute.

Theorem 19.15. *Let A, B be sets of labels such that their coordinate subspaces are invariant under W_I, W_J, respectively. Then the submatrix $s_{A,B}$ has exactly $|A \cap B|$ nonzero singular values, all equal to one. In particular*

$$\sum_{X \in A, Y \in B} |S_{XY}|^2 = D\,|A \cap B|.$$

If $S_{XY} \neq 0$, any choice of the I-components of θ_Y and the complementary components of θ_X occurs as a simple twist. If $I = \{p\}$, the realizing simple can be chosen in the full p-block of X.

Proof. Write E_A, E_B for the coordinate projectors. The two Weyl operators are unitary, so the invariance gives

$$E_A s E_B = W_I E_{A \cap B} W_J.$$

For the second statement take A to prescribe the J-components of X, and B the I-components of Y. The nonzero entry forces a nonempty intersection. At a singleton I we may intersect A with the full p-block of X by Lemma 3.2. □

Proposition 19.16. *Fix a simple X, a prime $p \mid D$, its full p-block and all of its prime-to-p twist components. In this set there is a simple Z with trivial p-twist and $(W_p)_{XZ} \neq 0$. There is a simple Y in the same set with nontrivial p-twist and $(W_p)_{YX} \neq 0$ if and only if $p \mid f_X$. If X is itself p-active, the inactive Z has $p \mid f_Z$.*

Proof. Use the proof of Theorem 19.15 with Y there the tensor unit. Since $s_{X0} = d_X/\sqrt{D} \neq 0$, a term $(W_p)_{XZ}(W_{p'})_{Z0}$ is nonzero, giving the first assertion. If $p \nmid f_X$, its locally integral primitive idempotent makes its full p-block a singleton, and it is inactive by Theorem 3.1 and Lemma 3.3. Conversely suppose $W_p e_X$ has only inactive coordinates. The local upper unipotent is a scalar multiple of the actual p-twist, so both it and its conjugate by W_p^{-1} preserve the X-line. They generate the local group; hence W_p preserves the line. By Lemma 3.3 this implies $p \nmid f_X$. Every coordinate reached by W_p is in the stated block and fiber. An inactive Z with $p \nmid f_Z$ would lie in a singleton block and could not be joined by W_p to an active X. □

A simple with twist order maximal under divisibility satisfies $\operatorname{rad}(f_X) = \operatorname{rad}(\operatorname{ord}\theta_X)$. Indeed Proposition 19.16 would enlarge the twist order at any missing prime of f_X; the converse inclusion follows from Lemma 3.3. More generally, if no twist is divisible by $p\prod_{t\in T} t$, then every label active at all primes of T has p-unit codegree and its row vanishes on p-active labels. For square $D = n^2$, this

is precisely $v_p(d_X) = v_p(n)$.

The conductor-by-conductor bounds for inactive labels in the same fiber are proved in Proposition 19.22.

19.8 A congruence for primes missing from joint twists

Proposition 19.17. *Let p be any prime and let T be a nonempty set of other odd primes in a square-dimensional integral modular category. Prescribe some twist components outside $\{p\}\cup T$. Suppose no label in that fiber is active at every prime of $\{p\}\cup T$. If X is in the fiber and $p \mid f_X$, then for every label U and arbitrary independent local t-inertia automorphisms σ_t,*

$$\prod_{t\in T}(1-\sigma_t)\chi_X(U) \in p\mathcal{O},$$

where $\mathcal{O}$ is the ring of integers of a common cyclotomic field. The ideal here is generated by p, including any ramification.

Proof. All indicated local groups preserve the fiber. By Lemma 19.11a at $k_t = 0$, the image of $P = \prod_{t\in T}(I - H_t)$ lies in the span of T-local translates of its simultaneous T-active coordinate space: the ideals generated by $I - U_t(1)$ contain the respective torus differences and the factors commute. The p-local modular factor preserves the joint space, on which its upper unipotent acts by the normalization scalar, hence the Weyl W_p acts by γ_p^3 there and on its translates (Proposition 6.2). Thus $W_pPe_X = \gamma_p^3 Pe_X$. Write $B = \prod_{u\neq p} W_u$ over the complementary local level factors, so $s = BW_p$. If $k = v_p(\sqrt{f_X}) \geq 1$, every entry of the X-column of B has p-valuation at least $1-k$. This is the proof of Theorem 6.9 for the product of the complementary factors: choose an exponent divisible by the full p-level and equal to one on its complement. The Bruhat identity identifies $sT^m s^{-1}$ with B^{-1} up to diagonal phases and a scalar unit. The mixed indicator and the p-strand congruence give $2v_p(B_{iX}) + 2k > 0$, using the local torus at -1 on the other insertion as in that theorem. The entries of B are unramified at p by Galois covariance, so the valuation bound follows. It applies to every Galois conjugate of X. Multiply $sPe_X = \gamma_p^3 BPe_X$ by $\sqrt{f_X}$ and use Galois covariance of characters. □

Corollary 19.18. *Suppose there is no twist of order divisible by 30 in an ordinary minimal table on $2,3,5$. At every complementary P_3, P_5 crossing of weight four with five-free and three-free degrees, respectively, the lcm-normalized word is $1+3\xi$, $\operatorname{ord}\xi = 5$. At every crossing of an odd $Y \in P_3$ of degree 3^j5 with a three-free $X \in P_5$ of degree 2^i5, it is $\eta(2+3\xi)$ with η ternary. If X is dyadically active in the latter case, $\eta = 1$.*

Proof. Section 19.3 and Proposition 8.6 classify both words without bounds on the other exponents. In their signed alternatives the four nonidentity five-positions have coefficients $-\eta\eta_a$, $a = 1,2,3,4$, with the multiset of the η_a one copy of one plus one coset of the cubic roots in the ternary roots. At weight four $\eta = 1$ and the constant coefficient $c_0 = 0$; at weight five $c_0 = \eta$. If both witnesses are dyadically inactive, all four negative packets are dyadically active. At most one cancels the five-twist of the product of the factors, and at most two cancel its three-twist. This forces a twist divisible by 30.

If the P_3 witness is dyadically active, apply the preceding congruence with $p = 3, t = 5$ in its dyadic twist fiber. Its 3-defect is positive by principality, and the opposite degree divided by the weight is a 3-unit. Thus all five-inertia conjugates of the normalized word agree modulo $3\mathcal{O}$. A tame generator permutes the four nonzero five-positions transitively and fixes the coefficient at

zero. Over the ternary cyclotomic integers the only relation on these five roots modulo three is the constant vector, by linear disjointness and the monic cyclotomic basis. The four nonzero coefficients would agree modulo $3\mathcal{O}$. Distinct three-power roots differ by an element of 3-valuation at most $1/2$. This rules out the signed alternative.

If the P_5 witness is dyadically active, use $p = 5, t = 3$. The opposite degree divided by the weight is a five-unit, hence $c_0 - c_a$ is invariant modulo five under ternary inertia for every a, by the same integral-basis argument now reduced modulo five. At weight four this would fix all the distinct η_a under ternary inversion. At weight five one η_a equals one, so this first fixes 2η, whence $\eta = 1$, and then every η_a, again impossible. Reduction modulo a prime over five is injective on three-power roots. In the unsigned weight-five case the difference $(\eta - \sigma\eta)(2 + 3\xi)$ is divisible by five; the second factor has five-valuation $1/4$, forcing $\eta = 1$ as asserted. □

The congruence extends to whole residue blocks. In the absence of a joint p, t, r twist, let B be a full p-block containing a label of positive p-defect. If local t-inertia moves its residue character, every label in B is r-inactive. Indeed each member has positive p-defect, for one of defect zero is a singleton block. An r-active member would have t-invariant residue character by the singleton congruence proved above. Galois covariance in the *argument* shows that this applies to the entire t-orbit of B even at a fixed prime above p.

19.9 Character fields at twist one

Let S be ordinary symmetric integral modular data of dimension D, with $SS^* = DI$. Let G be the Galois group over $\mathbb{Q}$ of a cyclotomic field containing all the character fields $K_X = \mathbb{Q}(\chi_X)$. For a character ψ of G, use the same symbol for its descent to a quotient whenever it is trivial on the kernel.

Proposition 19.19. *Choose representatives X and Y of any two lists of distinct Galois orbits (the two lists may intersect). Form a matrix indexed by the representatives for which ψ descends to their character fields. Put $L_{XY} = \operatorname{lcm}(d_X, d_Y)$, $z_{XY} = S_{XY}/L_{XY}$ and $F_{XY} = \mathbb{Q}(z_{XY})$. The following matrix has operator norm at most $\sqrt{D}$:*

$$A_{XY} = L_{XY}\sqrt{[K_X : F_{XY}][K_Y : F_{XY}]} \sum_{u \in \mathrm{Gal}(F_{XY}/\mathbb{Q})} \psi(u)u(z_{XY}) \qquad \text{if } \psi \text{ descends to } F_{XY}.$$

The entry is zero otherwise. In particular every row and every column has squared norm at most D.

Proof. On the orbit of Y, take the unit vector whose coordinates are $\psi(v)/\sqrt{[K_Y : \mathbb{Q}]}$, with v ranging over $\mathrm{Gal}(K_Y/\mathbb{Q})$. On the orbit of X take the analogous vector with $\overline{\psi}$ in place of ψ. These are orthonormal families on each side. The matrix coefficients of S between them are the double averages of $L_{XY}\psi(u)\psi(v)(uv)(z_{XY})$. Here lifts from the two fields give the same answer by symmetry and the covariance of characters, since the degrees are rational. Summing on the fibers of restriction to F_{XY} gives zero if ψ is nontrivial on a kernel; otherwise it gives the displayed coefficient. Thus A is a compression of S between two isometries. □

Proposition 19.20. *In ordinary integral modular data with positive spherical structure let u, v, w be three distinct primes. Suppose the other local groups act by scalars. Let $X \in P_u$ have u-power twist and $Y \in P_v$ have v-power twist, allowing twist one in either case. Write G for the Galois group of the cyclotomic splitting field. Let ψ be a character of G. For $j = u, v$ suppose that inverting just the j-primary component of ψ leaves its kernel unchanged. In particular this holds for a character of prime-power conductor. If*

$$\sum_{a \in G} \psi(a)\, a(S_{XY}) \neq 0,$$

then P_u and P_v have twist-one members in a common full w-block. Their character fields both contain the cyclic field cut out by ψ.

Proof. By Proposition 19.19 the orbit vectors affording $\overline{\psi}$ at X and ψ at Y have a nonzero matrix coefficient for s. Now s is a scalar multiple of $W_uW_wW_v$. Move W_u^* onto the first vector and W_v onto the second. For completeness, when D is nonsquare the diagonal torus is $H_a = \epsilon_D(a)P_a$, with P_a the permutation matrix and $\epsilon_D(a) = a(\sqrt{D})/\sqrt{D}$ (Theorem 19.12). This quadratic character is unchanged by inversion in any subset of the local factors. Thus the conjugation rule for a Weyl and a torus also holds with P_a. A local Weyl inverts the torus on its own factor, so both new characters have the original kernel. The new row vector is supported on pure u-power twists in P_u, and the new column vector on pure v-power twists in P_v. A nonzero W_w-entry between their supports preserves the complementary twists and the full w-block (Lemma 3.2). Hence the two twists are one. The stabilizer of any nonzero coordinate of an eigenvector is in the kernel of its character, giving the two field inclusions. □

19.10 Multiplicities of quadratic inertia characters

In square-dimensional ordinary integral modular data let H_p be the group of local diagonal tori at an odd prime p, acting on the simple labels by the Galois permutations. Use the linear lift of Section 6. For a set T of odd primes dividing the level, consider a coordinate subspace E stable under the local groups at every $p \in T$. For example, one may prescribe the complementary primary twist components, and for a single p one may also prescribe its full p-block. Write $H = \prod_{p\in T} H_p$ and let $\lambda = \prod_{p\in T}(\frac{\cdot}{p})$ be its product of quadratic characters. Retain the H-orbits in E on which λ is trivial on the stabilizer; write $m_1, \dots, m_b$ for their sizes.

Proposition 19.21. *One has $2^{|T|} \mid b$. Moreover, putting $K = \mathbb{Q}(\sqrt{p} : p \in T)$,*

$$\prod_{i=1}^{b} m_i \ \in \operatorname{disc}(K)^{\,b/[K:\mathbb{Q}]}\, N_{K/\mathbb{Q}}(K^\times)\,(\mathbb{Q}^\times)^2.$$

In particular, in every full five-block and specified prime-to-five twist fiber, the number of five-inertia orbits of tame order four is even, and so is the number of tame order two.

Proof. The rational λ-eigenspace V has the orthogonal basis of signed orbit sums on the retained orbits. Its positive form thus has Gram matrix $\operatorname{diag}(m_1, \dots, m_b)$. Write $p^* = (-1)^{(p-1)/2}p$. On V the Weyl operator W_p preserves the character, since it inverts the local torus and the latter acts by signs. For a cyclotomic automorphism σ_a the congruence representation gives $\sigma_a(W_p) = H_{a,p}W_p$, with the opposite torus convention producing the same signs. This follows by conjugation by $\operatorname{diag}(a, 1)$ in Lemma 6.1. Consequently $A_p = \sqrt{p^*}\,W_p$ has rational matrix on V. Here $W_p^2 = H_{-1,p} = (\frac{-1}{p})I$. Unitarity gives $A_p^* = A_p$ and $A_p^2 = pI$. These commuting operators make V a vector space over the stated multiquadratic field K, of degree $2^{|T|}$. The zero-dimensional case is immediate.

The rational symmetric form B is balanced over K since all A_p are self-adjoint. Nondegeneracy of the field trace defines a symmetric K-bilinear form B_K by $\operatorname{Tr}_{K/\mathbb{Q}}(uB_K(x, y)) = B(ux, y)$ for all $u \in K$. Diagonalize it over K. Taking the trace of a one-dimensional summand with coefficient c gives determinant $\operatorname{disc}(K)N_{K/\mathbb{Q}}(c)$ up to rational squares, proving the formula. For $p = 5$, the retained orbit sizes have the form $2 \cdot 5^a$ or $4 \cdot 5^a$. The prime-to-five unit of every nonzero norm from $\mathbb{Q}(\sqrt{5})$ is a square modulo five: in $x^2 - 5y^2$ the two summands have valuations of opposite parity, and -1 is a square modulo five. The discriminant is 5 and the unit part of a rational square is a square. Since two is a nonsquare, the number of tame-order-two orbits is even. Subtraction from the even dimension proves the assertion at tame order four. □

19.11 The conductor filtration inside a twist fiber

Let $D = n^2$ in ordinary integral modular data. Fix an odd prime p, one full p-block (or all blocks), and the exact prime-to-p twist components. Denote the resulting nonempty fiber and its coordinate projector by E. Let Γ_E be the subgroup of all diagonal tori preserving E, and let b count its orbits on the p-inactive labels of E. Write

$$m(\alpha) = \sum_{X \in E:\, \theta_{X,p} = \alpha} d_X^2, \qquad J = \{j \geq 2 :\ m(\alpha) > 0 \text{ for some } \alpha \text{ of order } p^j\}.$$

Proposition 19.22. *All nonidentity pth roots have the same mass m_*. One has*

$$m_* \leq (p+1)m(1), \qquad b \geq |J| + \mathbf{1}_{m_*>0} + \mathbf{1}_{m_*<(p+1)m(1)} \geq |J| + 1.$$

For the number of inactive labels, replace $|J|$ by $\sum_{j\in J} |G_j^2|$, where $G_j = (\mathbb{Z}/p^{\lfloor j/2 \rfloor})^\times$. At odd p one has $|G_j^2| = \varphi(p^{\lfloor j/2 \rfloor})/2$. If all inactive labels lie in one orbit and the fiber is active, its only twists at p are the pth roots, with $m_* = (p+1)m(1)$ and total mass $p^2 m(1)$. In particular $m(1) = d_Y^2$ for a singleton Y.

Proof. Write U, L, H, W for the local upper, lower, torus and Weyl operators. Put $v = nEW_{p'}e_0$. The upper unipotents and diagonal torus at p fix v, as does Γ_E, since conjugation across $W_{p'}$ only inverts the complementary tori and all tori fix the vacuum. The vector Wv has entries d_X on E and zero elsewhere.

At local level p^a let P_j average over the kernel of reduction modulo p^j, for $1 \leq j \leq a$. Its lower–diagonal–upper factorization is unique, the diagonal entries being 1 mod p^j. Averaging the factors in this order therefore gives $P_j v = \mathrm{avg}_{p^j|t}\, L(t)v$. Conjugation by W shows that WP_jv is exactly the projection of Wv onto p-twists of order dividing p^j. (Take $j = 1$ to mean the identity projection when $a = 1$.) The mutually orthogonal vectors $(P_j - P_{j-1})v$, $2 \leq j \leq a$, are thus nonzero precisely at $j \in J$. Each is upper-fixed and Γ_E-fixed, because its projectors commute with the local group and the other local factors. They lie in the space of invariant vectors on the inactive labels, of dimension b.

Fix such a nonzero layer $v_j = (P_j - P_{j-1})v$, on which the action factors through level p^j. Put $l = \lceil j/2 \rceil$ and $d = j - l$. The kernel of reduction modulo p^l is abelian, of elements $I + p^l A$ with $A \in \mathfrak{sl}_2(\mathbb{Z}/p^d)$. As v_j is upper- and torus-fixed, the characters in its decomposition on this kernel have the form $A \mapsto \exp(2\pi i c A_{21}/p^d)$. Some c is a unit: otherwise $L(p^{j-1})$ fixes v_j, although Wv_j has exact twist order p^j. Every square-unit multiple of this c occurs by torus invariance. Their character components are orthogonal, and are upper-fixed since upper conjugation fixes A_{21}. They give the indicated number of independent vectors on the inactive labels within this layer; different layers are orthogonal.

It remains to calculate in the quotient $G = \mathrm{SL}_2(\mathbb{F}_p)$ on $u = P_1 v$; its Weyl image gives all masses at orders one and p. Let B be its upper Borel, P_B its average, and $A = P_B W P_B$ on B-fixed vectors. Then

$$A = P_G - p^{-1}(I - P_G).$$

Indeed the two double cosets are B and BWB, of relative sizes $1 : p$. Averaging over G therefore gives $P_G = (I + pA)/(p+1)$ on this space. Write $u = u_0 + u_1$ with $u_0 = P_G u$ and $u_1 = u - u_0$. The mass at one equals $\|Au\|^2$, since Wu is torus-fixed. For $t \in \mathbb{F}_p^\times$, the element $W^{-1}U(t)W$ belongs to BWB, so the tth moment of the masses is $\langle u, Au \rangle$. Finite Fourier inversion gives

$$m(1) = \|u_0\|^2 + p^{-2}\|u_1\|^2, \qquad m(\alpha) = (p+1)p^{-2}\|u_1\|^2 \quad (\alpha \in \mu_p \setminus \{1\}).$$

Each nonzero u_i supplies a further orthogonal invariant vector on the inactive labels. There is at least one such vector: $v \neq 0$ is upper-fixed, hence there are inactive labels, so $m(1) > 0$. □

The gap between the two masses also gives a signed integer identity at every local level:

$$m(1) - m_* = n \sum_{X \in E:\, \theta_{X,p}=1} d_X \, (W_{p'} e_0)_X .$$

Indeed, for a unit t in the local ring, $W^{-1}U(t)W \in BWB$ for its upper triangular subgroup B. Its coefficient at v equals $\langle v, Wv \rangle$. Averaging such coefficients over the units kills roots of order at least p^2 and averages a primitive pth root to $-1/(p-1)$. The entries $n(W_{p'}e_0)_X$ are rational integers divisible by p by Lemma 6.1.

The same result holds at $p = 2$ whenever the dyadic anomaly scalar γ is 1 or -1. If $\gamma = -1$, tensor the local lift with the sign character of $\mathrm{SL}_2(\mathbb{F}_2) \cong S_3$: the normalized upper generator then becomes the actual dyadic twist, the diagonal tori are unchanged, and the Weyl changes only by a sign. In particular this applies if a nonempty exact odd-twist fiber has only dyadically inactive labels. The dyadic group preserves this fiber, so its upper acts there by γ^{-1}, and Bruhat gives $W_2 = \gamma^3 I$ there. Its square is an unsigned torus permutation, hence $\gamma^6 = 1$; the lift has dyadic γ, giving the assertion.

At $D = 2^6 3^6 5^4$ a joint three–five column whose five-twist has order 25 consequently requires two Galois-inequivalent five-inactive labels in its full five-block with the same dyadic and ternary twist components (inequivalence is under Γ_E). This strictly strengthens a count of two labels when those labels are conjugate. There is a joint three–five column at this dimension, so absence of a twist of order divisible by 30 supplies the odd-twist fiber just used and gives the dyadic assertion on every block and complementary fiber. In particular, if $b = 1$ there and the fiber is dyadically active, its actual dyadic twists are $1, -1$ and the mass at -1 is precisely three times the mass at 1.

Simultaneous orders and distinct twist-one orbits. Apply the filtration simultaneously at all primes, assuming the dyadic normalization above when two occurs. Write $M_{\boldsymbol{i}}$ for the squared-degree mass at twists of exact order $\prod_p p^{i_p}$. On an array indexed by these exponents apply the following operation in each prime coordinate, with the other indices fixed:

$$(M_0, M_1, M_2, \ldots) \longmapsto \left(M_0 - \frac{M_1}{p^2-1},\, \frac{p^2 M_1}{p^2-1},\, M_2, \ldots \right).$$

All entries of the resulting array V are nonnegative, and the number of its positive entries is at most the number of distinct Galois orbits of twist-one labels.

In fact put $Q_{p,0} = P_{G_p}$, $Q_{p,1} = P_1 - P_{G_p}$, and $Q_{p,j} = P_j - P_{j-1}$ for $j \geq 2$. Then $V_{\boldsymbol{j}} = n^2 \| \prod_p Q_{p,j_p} e_0 \|^2$. To see this, compress each local twist-order projector, conjugated by its Weyl, to the space of upper-Borel fixed vectors. At order one its compression is $Q_{p,0} + p^{-2} Q_{p,1}$, at order p it is $(1 - p^{-2}) Q_{p,1}$, and at order p^j, $j \geq 2$, it is $Q_{p,j}$. These are the squared-norm formulas in the proof above, applied to an arbitrary upper-Borel fixed vector and polarized. Distinct local factors commute, so their compressed projectors can be multiplied inside the vacuum coefficient. Inverting the resulting two by two tame relation gives the stated array. Its nonzero vectors are orthogonal, fixed by every torus and every upper unipotent. They are consequently supported on twist-one labels and constant on their Galois orbits.

In the forward relation a profile n' can contribute to order m only if $n' = md$ with d square-free and coprime to m. Every positive V-profile also occurs as an actual twist order. Thus every occurring m for which no such strict extension occurs gives its own positive entry. At the even core without a twist order divisible by 30, this applies to *every distinct mixed order*, even when one mixed order divides another. Hence their number is bounded by the number of Galois orbits at twist one. If all three pairs occur there are at least three such orbits, of which at least two are inequivalent nonunit orbits.

There is an unconditional form on the branch admitting an order divisible by 30. Fix $a \geq 0$ and take E to be the union of all dyadic phases of exact order 2^a. Work only with the local

groups at three and five, with initial vector nEW_2e_0. It is fixed by both upper Borels and by all tori, and its two local Weyls give the degree vector on E. The same array argument shows that

$$\#\{(b,c) \in \mathcal{Z}^2_{>0} : 2^a 3^b 5^c \in \mathcal{T}\} \;\le\; \#(\{X : \mathrm{ord}(\theta_X) = 2^a\}/\,\mathrm{Gal}).$$

Here $\mathcal{T}$ is the set of actual twist orders. Indeed each profile with both odd exponents positive already specifies every odd prime, so no other profile can contribute to it. This holds for each a, with arbitrary dyadic central charge, and with the same proof for any prescribed set of odd primes in a Galois-stable union of complementary twist fibers.

Primitive odd components at a fixed dyadic phase. The filtration can equally be applied at just three and five, fixing the dyadic twist. In the next formulas the total masses sum over both possible signs at three and all four possible primitive phases at five, as appropriate. Fix an individual nontrivial dyadic phase δ at $D = 2^6 3^6 5^4$. Write M_{ij} for the mass with $\theta_{X,2} = \delta$ and odd primary orders $3^i, 5^j$, $i, j \in \{0,1\}$. Then

$$M_{11} \le 8M_{01}, \qquad M_{11} \le 24M_{10}, \qquad M_{10} \le 8M_{00}, \qquad M_{01} \le 24M_{00},$$

and

$$192M_{00} - 24M_{10} - 8M_{01} + M_{11} \ge 0.$$

Every one of the eight individual phase pairs (ζ_3^u, ζ_5^v), $u = 1, 2$ and $1 \le v \le 4$, has mass $M_{11}/8$. This follows by applying the individual balance successively, fixing the other primary twist each time.

If this mass is positive, the degree transfers and activity height bounds give the triple degrees $2^i 3^j 5$, $0 \le i \le 2$, $1 \le j \le 2$. The full quartic tame five-inertia result for jointly three- and five-active labels shows that the unique involution of tame five-inertia moves every such label. It fixes each of these eight phases, by the square action on the fifth roots, and preserves the degree. Thus each degree occurs with even multiplicity at each phase. Since every such degree is divisible by 15,

$$3600 \mid M_{11}.$$

There are at least sixteen labels across the eight phases, and at least four distinct full Galois orbits: the square classes of the two odd components are unchanged under the square Galois action. All these conclusions concern exact odd primary orders 3 and 5.

Opposite dyadic phases for arbitrary central charge. Write $v_E = \sum_{X \in E} d_X e_X$ for the degree vector restricted to a fiber. Let $p = 2$, fix the odd twist components and optionally one full two-block, and write $m(\delta)$ for the mass of an individual dyadic phase in the resulting set E. For the multiplicative central charge ξ one has

$$m(\delta) = m(-\delta) \qquad (\delta \ne \pm\xi).$$

In fact the integral lift has $\gamma^3 = \xi$, $\gamma^8 = 1$, and the actual dyadic twist operator is $\gamma U(1)$. Conjugating the upper unipotents on the vacuum shows that $L(y)v_E = \gamma^y v_E$ for the standard lower unipotents. The dyadic diagonal tori $H(a)$ fix v_E. For odd u the identity

$$U(u) = L(u^{-1})H(-u)WL(u^{-1}), \qquad W = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

therefore gives

$$\sum_{X \in E} d_X^2 \theta_{X,2}^u = \gamma^{u+2u^{-1}} v_E^* W v_E = \xi^u v_E^* W v_E.$$

The exponent calculation uses $u^{-1} \equiv u \pmod 8$ (enlarge the local modulus to include 8 if necessary). The coefficient $v_E^* W v_E$ is an integer by the local Weyl vacuum column formula. Thus

$\sum_{X\in E} d_X^2\theta_{X,2}/\xi$ is rational. In the power basis of a dyadic cyclotomic field, opposite roots cancel and one representative of each opposite pair gives a basis. Rationality forces equality of the two coefficients except in the pair $\{1,-1\}$.

If ξ is nonreal, every nonempty odd-twist fiber in a full two-block has the same positive mass at its dyadic phases 1 and -1, since it contains a dyadically inactive label by Weyl routing. Consequently any joint ternary and quintic phases, including arbitrary higher orders, then occur with dyadic phase exactly -1 in that block. At any central charge, when a nonexceptional δ admits a triple with odd primary orders 3 and 5, the opposite phase $-\delta$ contains the same mass at each of the eight individual odd-phase pairs.

Wild inertia at an inactive prime. Work in ordinary integral modular data of square dimension. At a prime p let $U(t)$, $L(t)$, $H(a)$ be the upper, lower and diagonal subgroups of the local modular representation at level p^e, and W_p its Weyl element. Torus actions are the unsigned Galois permutations. At two a common normalization scalar shifts the phase labels; we formulate the local statement for the actual representation and use its categorical consequences only at odd p.

Proposition 19.23. *Suppose v is fixed by $U(t)$ and has torus character ψ. Let p^b be the conductor of ψ^2, $b\ge 1$. The squared masses of W_pv on the distinct upper-unipotent phases are constant on each coset of μ_{p^b} in μ_{p^e}. In particular the squared mass on any single phase is at most $p^{-b}\|v\|^2$. The assertion also applies to the global normalized S-image of a vector with these local properties, since the complementary Weyl is a commuting unitary.*

Proof. If $j=v_p(t)<b$, choose a unit a congruent to one modulo p^j with $\psi(a)^2\neq 1$ (at $j=0$ there is no restriction). Take y satisfying $ty=a^{-1}-a$. For $B=\left(\begin{smallmatrix} a & y\\ 0 & a^{-1}\end{smallmatrix}\right)$ the matrix $L(-t)BL(t)$ is upper triangular with first diagonal entry a^{-1}. Its character on v is the inverse of that of B. Unitarity gives $\langle v, L(t)v\rangle = 0$. The Fourier coefficients of the phase masses of W_pv are these lower-unipotent coefficients, up to the sign of t. They vanish off $p^b\mathbb{Z}/p^e$, proving the asserted invariance by finite Fourier inversion. □

For odd p, let an inactive label have local inertia representation of conductor p^b with $b\ge 2$ on its orbit (equivalently wild inertia has order p^{b-1} there). The permutation representation of its cyclic local orbit admits a character ψ of that conductor. Squaring leaves its wild conductor unchanged. A character vector supported on this orbit supplies v in Proposition 19.23. Every occupied phase coset contains at least p^b-1 nonidentity phases and at least $(p-1)p^{b-1}$ of order $\ge p^b$. If the local involution $H(-1)$ acts nontrivially on the orbit, take ψ faithful, so $\psi(-1)=-1$. The Weyl image is odd under the same involution, which preserves each phase. Each occupied phase now requires two different labels. The same doubling holds for every nonselfdual inactive label, irrespective of its local tame group: choose a character of its full Galois orbit with p-primary part faithful on wild inertia, and independently with 2-primary part nontrivial on complex conjugation. Global duality commutes with the Weyl and pairs the resulting coordinates at every fixed twist.

19.12 Ramification at half a self-dual degree

Theorem 19.24. *Let X be self-dual of degree $2p$ in an ordinary or positive reduced minimal table, where $p\ge 5$ is prime, and put $a=v_p(\Delta)$. If K_X is unramified at p, then $a\le 5$. When $a\ge 4$, some nonunit Y in the full principal p-block has $\chi_Y(X)/p$ a root of T^2-T-1 or T^2+T-1. In particular $\sqrt{5}\in K_X\cap K_Y$. Consequently every self-dual simple of degree ten in a minimal table with $v_5(\Delta)\ge 4$ ramifies at five.*

Proof. Set $a_Y=\chi_Y(X)/p$. Principality gives residues zero above p at every conjugate, and K_X is unramified there, so a_Y is an algebraic integer. It is totally real by self-duality and has strict

house below two if Y is a nonunit. The roots of $T^2 - a_Y T + 1$ are algebraic integers all of whose conjugates have absolute value one. By Kronecker $a_Y = \xi + \xi^{-1}$ for a root of unity ξ. The full-weight word $p\xi + p\xi^{-1}$ has two distinct terms. Thus $\operatorname{ord}(\xi^2) \in \{2,3,4,5,6\}$ by Theorem 19.5. The resulting values, respectively, are 0, ± 1, $\pm\sqrt{2}$, the roots of $T^2 \mp T - 1$, and $\pm\sqrt{3}$.

Let $m(z) = \sum_{Y \in P_p,\, a_Y = z} d_Y^2$. The principal block is Galois-stable, hence conjugate values have equal masses; $m(2) = 1$. For $0 \le j < a$, the unit coefficient of $E_{P_p}[X]^j$ is integral at p, giving

$$\sum_z m(z) z^j \equiv 0 \pmod{p}.$$

If neither golden quadratic occurs, $z^3 - z$ vanishes at $0, \pm 1$ and sums to zero on each of the conjugate pairs $\pm\sqrt{2}, \pm\sqrt{3}$. The vacuum contributes six. Thus $a \ge 4$ would imply $p \mid 6$. This proves the quadratic-field conclusion and rules out the case $p = 5$.

For $p \ge 7$ use $P(T) = 3T^5 - 10T^3 + 7T$. It is odd, vanishes at $0, \pm 1$, and sums to zero on the two roots of $T^2 - T - 1$ since the traces of T, T^3, T^5 there are $1, 4, 11$. The other quadratic has their negatives as roots. Therefore $\sum_z m(z)P(z) = P(2) = 30$. If $a \ge 6$ all three moments give $p \mid 30$, a contradiction. □

Theorem 19.25. *Let $d_X = 2p$ with $p \ge 5$ prime. Suppose K_X is unramified at p and at five, and $v_p(\Delta) \ge 4$. Then $v_p(\Delta) = 4$, and there is a p-free nonunit $Y \in P_p$ with*

$$\chi_Y(X) = \pm p(1 - \omega), \qquad \omega \text{ a primitive cube root.}$$

In particular $\mathbb{Q}(\sqrt{-3}) \subseteq K_X \cap K_Y$. In ordinary data all three pairs from $\{2,3,p\}$ occur in twist orders. For $p = 5$, unramifiedness at five alone gives all these conclusions.

Proof. For $d_X = 2p$, Theorem 4.17 gives $\operatorname{supp}(\Delta) \subseteq \{2,3,5,p\}$ since $b(2) = 5$. For $p = 5$ the support is exactly $\{2,3,5\}$ by Corollary 4.18, and Lemma 4.10 gives $v_5(\Delta) \ge 4$ in either table.

The hypotheses and Corollary 2.8 give $K_X \subset L = \mathbb{Q}(\zeta_{2^u 3^v})$, where $u, v \ge 1$ may be enlarged. For $Y \in P_p$, write $\alpha_Y = \chi_Y(X)/p$; its integrality follows from its residues as in Theorem 19.24. It has a signed word of length at most two in L. Indeed the balancing expression has rational weight two; normalized relative trace into L does not increase its signed weight (on omitted prime factors use the rational normalized trace). For completeness, choose one of each pair of opposite dyadic roots for the dyadic integral basis. In each dyadic coordinate the relations on ternary powers are precisely the constant coefficient shifts on μ_3-cosets. An integral representation exists since α_Y is integral, and subtracting a median on each coset minimizes total absolute coefficient over rational shifts while retaining integrality.

Let $H = \operatorname{Gal}(L/\mathbb{Q}(\zeta_3))$ and put $\beta_Y = |H|^{-1} \sum_{s \in H} s\alpha_Y$. On signed roots this trace fixes the sixth roots and kills all others. Its denominator $|H|$ is prime to p. For a nonunit Y, $|\beta_Y| < 2$ by strictness, so β_Y is zero or an Eisenstein integer of squared modulus one or three. The latter possibilities follow directly from adding at most two sixth roots. The vacuum value is $\beta_0 = 2$.

Use the p-local orbit average $B = |H|^{-1} \sum_{s \in H} [X^s]$, whose eigenvalues on P_p are $p\beta_Y$. Applying the local coefficient identity (2.11) to monomials in B, B^* gives, for every integral polynomial P in $z, \overline{z}$ of total degree less than $v_p(\Delta)$,

$$\sum_{Y \in P_p} d_Y^2 P(\beta_Y, \overline{\beta_Y}) \equiv 0 \pmod{p}.$$

Take $P = (z + \bar{z})(z\bar{z} - 1)$. The vacuum contributes 12 and all nonunits of squared modulus zero or one contribute zero. Thus some p-free nonunit has squared modulus three and nonzero real part. Its value is $\pm(1 - \omega)$. Both terms of its signed word survived the trace, so $\alpha_Y = \beta_Y$. To bound the p-valuation, put $x = z + \bar{z}$, $t = z\bar{z}$ and take the degree-four polynomial

$$x^4 - 2x^2 t - 2t^2 - 3x^2 + 6t.$$

The nonvacuum pairs (x,t) are $(0,0)$, $(\pm1,1)$, $(\pm2,1)$, $(0,3)$ and $(\pm3,3)$; the polynomial vanishes on each. The vacuum $(4,4)$ gives 72, so $v_p(\Delta)\le 4$.

In an ordinary table, XY has exactly two twist values of relative order six, each carrying dimension pd_Y. To see this, the two phases in the displayed word for $\chi_Y(X)$ are adjacent vertices of the regular hexagon. A rational functional on $\mathbb{Q}(\zeta_6)$ is one at these two vertices and strictly less at the others (assign the value one on the two adjacent vertices, which form a rational basis). Normalized relative trace extends it to every common conductor: a root outside μ_6 has zero trace at a higher dyadic or ternary level, or an additional odd factor of normalized trace of absolute value at most $1/4$. On the hexagon the functional has absolute value at most one. Equality for the full-weight balancing expression forces the stated support; the two adjacent roots are linearly independent, so their masses agree with those in the word.

At least one fiber is two-active and at least one is three-active. If either prime is separated from p, that fiber must have full p-height (Lemma 3.3), impossible since its mass has p-height one. If two and three are separated, each constituent of a two-active fiber has full ternary height. Since $p\neq 3$, its mass pd_Y then forces full ternary height in d_Y, so K_Y is unramified at three by Corollary 2.8. This contradicts $\omega\in K_Y$. □

Corollary 19.26. *In the degree-ten case of Theorem 19.25, the principal five-block contains at least seven distinct five-free nonunits.*

Proof. In the notation of the proof, write $\beta=a+b\omega$ with $a,b\in\mathbb{Z}$, and reduce (a,b) modulo five. The nonvacuum points satisfy $a^2-ab+b^2<4$. They are distinct after reduction. The six lines through the vacuum $(2,0)$ with slopes $\infty,0,1,2,3,4$ contain respectively $1,3,1,2,3,3$ of them, as one sees from the three norm values $0,1,3$. Suppose at most six points have nonzero squared mass modulo five. They can be covered by at most three lines avoiding the vacuum. Indeed place at most two in each of three groups, using distinct slopes in a group: list the points in consecutive blocks of equal slope and deal cyclically into the groups, since each block has length at most three. A pair determines a line missing the vacuum, and a singleton permits a choice. The product of the corresponding affine forms is nonzero at the vacuum and zero on all contributing nonunits. Each coordinate is a linear combination of $\beta,\bar\beta$ over the five-local integers of $\mathbb{Q}(\omega)$, since $\omega-\bar\omega$ is a five-unit. This contradicts the cubic moment identities in the preceding proof. A nonzero squared mass modulo five requires some five-free label. □

20 Two unrestricted prime powers and a sextic prime

20.1 Forcing a mixed column at the largest quartic prime

Throughout this section $p<q<r$ are odd primes. Proposition 4.7 rules out a positive reduced table when these are the repeated primes.

Theorem 20.1. *Let $h\ge 2$, $U=p^{h-1}$, $G=p^h$, and $m=\lfloor G/q\rfloor$. Suppose*

$$(p-1)q>U,\qquad W(s,j+1)\text{ holds for every }1\le j\le m\text{ and every prime }s>jq.$$

For every prime $r>q$ and every $B\ge 0$, each braided fusion category of dimension dividing $p^{2h}q^Br^4d$ is weakly group-theoretical.

Proof. In a remaining divisor-minimal table the r-valuation is four, while square divisibility bounds every p-height by h. Lemma 4.10 gives an r-free nonunit in the principal q-block and a q-free nonunit in the principal r-block. For every such pair the degrees are p^iq^j and p^tr, and the weight of the lcm-normalized entry is $g=p^{\min(i,t)}\ge q$. Thus $m=0$ is impossible.

If $r\le U$, then $j=\lfloor r/q\rfloor$ satisfies $1\le j\le m$, $r>jq$ and $j+1<p$. In Proposition 8.7, the first alternative gives a word excluded by $W(r,j+1)$. The proof of Theorem 8.2 rules out the second alternative using $j+1<p$; its scalar test is used only for the first alternative.

Suppose $r > U$. If any of the above pairs has $g \le U$, then $r \ge g+2$ since both are odd. Theorem 4.9, applied to the two principal residues, expresses the normalized entry as a positive r-word of weight g on at most $1 + \lfloor g/q \rfloor$ positions. Multiplying by $\mathrm{lcm}(d_X, d_Y)/d_Y$ gives a character of full weight d_X. Since $r > g \ge q\lfloor g/q \rfloor$, the word hypothesis excludes this pair. Consequently every opposite pair has $g = G$. All its witnesses have p-height exactly h, forcing the dimension's p-valuation to be exactly $2h$ and their codegrees to be p-units.

Every q-active column has a nonzero entry at an r-free nonunit of the principal q-block by Proposition 4.2, applied modulo r. This row has p-unit codegree, hence vanishes on every p-active column by Theorem 3.1. Thus p and q never occur together in a twist. The same argument on the principal r-block separates p from r. By Lemma 7.2, the quartic r cannot be isolated, so some column is both q- and r-active.

At this column apply Proposition 4.2 on the principal q-block modulo p. The resulting p-free nonunit has degree $q^j r^v$ for $v \in \{1, 2\}$ by Lemma 4.10 and square divisibility. At $v = 2$ its row would vanish on this column by its r-unit codegree. Thus $v = 1$, exactly the hypothesis that gives the first word in Proposition 8.7. Its support has at most $1 + \lfloor r/q \rfloor$ positions, with no restriction on p in this construction. If $r < G$, this contradicts $W(r, 1 + \lfloor r/q \rfloor)$. If $r > G$, apply Theorem 4.9 to an opposite pair of weight G: the resulting word contradicts $W(r, 1 + m)$. Equality $r = G$ is impossible. □

Corollary 20.2. *The word hypothesis in Theorem 20.1 holds whenever $q \ge (m+1)^2$. In particular the theorem applies whenever $h \ge 3$ and $q > p^{h-1}$. For $h \ge 6$ it applies throughout $(p-1)q > p^{h-1}$.*

Proof. With $E = j(j+1)/2$, put $g(v) = \min\{t \ge 1 : t(t+1) \ge v\}$ and $C(E) = 1 + \max_{1 \le v \le E} v(v-1)/g(v)$ as in Lemma 4.16. At $j = 1$ the two-position theorem applies. For $j \ge 2$, the scalar estimates in the proof of Theorem 4.17 bound both $3(1+2E)/2$ and $2C(E)+1$ by $j(j+1)^2$. This is strictly less than s since $s > jq$ and $q \ge (m+1)^2$. If $h \ge 3$ and $q > p^{h-1}$, then $m + 1 \le p$ and $q > p^{h-1} \ge p^2$. Finally, if $h \ge 6$ and $(p-1)q > p^{h-1}$, then $m + 1 \le p(p-1)$ and $q > p^{h-2} \ge p^4 > (m+1)^2$. □

The all-r criterion in Corollary 8.12 required $m + 1 < p$. For instance at $p = 3$, $q = 11$, $h = 3$ one has $m + 1 = 3$. Theorem 20.1 covers $3^6 11^B r^4 d$ for every B and prime $r > 11$. This includes $r = 23$, where the ratio test of Theorem 8.2 also has $1 + \lfloor r/q \rfloor = 3$. Direct use of the strict scalar inequalities gives examples below $q = p^{h-1}$: the same theorem covers $5^6 23^B r^4 d$ and $7^6 47^B r^4 d$ for all B and all r above the middle prime. Indeed their respective values of m are five and seven, and for $j = 1, \ldots, 7$ the exact values of $2C(j(j+1)/2) + 1$ are $3, 9, 33, 63, 108, 193, 1527/5$. Each required inequality follows from $s > jq$. These examples illustrate the criterion; no prime value is used in its proof.

Proposition 20.3. *Let $h, k \ge 2$, $U = p^{h-1}$, $V = q^{k-1}$, and $m = \max\{\lfloor U/q \rfloor, \lfloor V/p \rfloor\}$. For any prime $r > \max(U, V)$ for which $W(r, 1+m)$ holds, every braided fusion category of dimension dividing $p^{2h} q^{2k} r^4 d$ is weakly group-theoretical.*

Proof. Repeat the opposite-pair argument of Theorem 20.1 first with p as the common degree prime. Every common weight $g \le U$ would force a word on at most $1 + \lfloor U/q \rfloor$ positions, so all complementary degrees have top p-height. The codegree and prime-avoidance argument isolates p. Now use the principal blocks at p and r instead, so the common degree prime is q. Lemma 4.10 gives for every such pair a common weight $g = q^a \ge p$. At $g \le V$ we have $r \ge g + 2$ and Theorem 4.9 gives a word on at most $1 + \lfloor V/p \rfloor$ positions. Thus all those witnesses have upper height k, forcing valuation $2k$ and q-unit codegrees. Applied on the principal blocks at p, r, prime avoidance now isolates q. No twist can then contain r and either other prime, contrary to Lemma 7.2 at the quartic r. □

20.2 An isolated prime with unramified principal witnesses

This obstruction depends only on the exponent at the least prime. It does not require a quartic at either companion.

Lemma 20.4 (Trace and mass at an isolated prime)**.** *Let p be isolated in an odd ordinary minimal table with $v_p(D) = 2h$, $h \geq 3$. Write $D = p^{2h}n^2$. Each p-active Z has degree $p^j n$ with $0 \leq j < h$, and K_Z has p-power conductor.*

1. *If $j > 0$, no $X \in P_p$ of height $h-1$ can have nonzero absolute trace T_X in the Z-row of (2.5).*
2. *If all active labels have degree n, their number N satisfies $N \geq 2p^{h-1}(p-1)$. In that case P_p has no nonunit of height $h-1$.*

In particular some active column has no hit on a principal row of height $h-1$ unramified at p.

Proof. The descriptions of d_Z and K_Z follow from isolation, Lemma 3.3 and Corollary 2.8. In (1), $x = \chi_Z(X)$ satisfies $x/p^{h-1-j} \in \mathcal{O}_{K_Z}$ by lcm divisibility. By symmetry this quotient equals $\chi_X(Z)d_X/(p^{h-1-j}d_Z)$, whose second factor is a p-unit. Principality gives zero residue above p. Since K_Z is totally ramified at p, the rational trace of the quotient from $F_X = \mathbb{Q}(x) \subseteq K_X \cap K_Z$ is divisible by p. A nonzero T_X would thus have square at least $p^{2(h-j)} = f_Z$, leaving no room for the unit.

For (2), choose a CRT exponent m equal to one on the p-level and zero on the complementary level, including the dyadic anomaly. Lemma 6.1 and Proposition 6.2 give

$$p^{-h} \sum_{Z \ p\text{-active}} (1 - \theta_{Z,p}) = p^h(1 - (W_p)_{00}) \in \mathbb{Z}.$$

The integer is positive by taking real parts and even since distinct dual labels have the same twist and p is odd. Each summand on the left before division has normalized trace at most $p/(p-1)$, giving the lower bound. Any $X \in P_p$ hits every one of these labels by principality. If $v_p(d_X) = h-1$, then $S_{XZ}/(p^{h-1}n)$ is a nonzero algebraic integer by lcm divisibility. Its Galois mean squared modulus is at least one, by the norm and the arithmetic–geometric mean inequality. Averaging the row norm and counting the unit gives $N < p^2$. Finally, for an unramified X the intersection $K_X \cap K_Z$ is rational, so every nonzero hit has nonzero trace; (1) and (2) give the last assertion. □

Theorem 20.5 (Trace budget for two companion primes)**.** *Suppose p is isolated in an odd ordinary minimal table of dimension $p^{2h}n^2$, $h \geq 3$, with n supported on exactly two primes $q < r$, both different from p. Put $\ell = \min(p, q)$. For each $s = q, r$ and $1 \leq t \leq h-2$, call t eligible at s if*

$$s > p^t, \qquad (s - p^t)(\ell - 1) > p^t.$$

At an eligible height put

$$b_s(t) = \frac{s-1}{\max_{a \mid s-1,\, a \leq p^t} a/\gcd(a,2)}, \qquad c_s(t) = (p-1)b_s(t).$$

At every other height set $b_s(t) = c_s(t) = 1$. Let

$$A_s = \min_t p^{2(t-1)} c_s(t),$$

omitting eligible t for which $W(s, p^t)$ holds, and interpreting an empty minimum as infinity. Then $A_q + A_r < p^{2h-4}$.

Proof. Suppose first that Z is p-active of degree $p^j n$, $j > 0$. For each companion s, the unit term of (2.6) is one, so reduction modulo s supplies an s-free nonunit X with nonzero trace T_X on the Z-row. It belongs to P_p by Theorem 2.9. Write $d_X = p^t u^i$, where u is the other companion and $t, i > 0$ by Lemma 4.10. Defect-zero vanishing excludes $t = h$, and Lemma 20.4 excludes $t = h - 1$. If $j \le t$, the proof of that lemma gives $p^{t+1-j} \mid T_X$: divide $\chi_Z(X)$ by p^{t-j}, then use its zero residue and total ramification of its own field at p. If $j > t$, integrality alone suffices. In either case $d_Z^2 T_X^2 \ge p^{2(t+1)} n^2$.

At an eligible height cross X with a u-free nonunit of P_s. The gcd is some $w = p^k$ with $1 \le k \le t$, and the normalized entry has residue w over s. Lemma 8.5 gives a positive integral s-word of weight w, since $(s - w)(\ell - 1) > w$. It is unique: a word shorter than s misses a position in every μ_s-coset, so its coefficients are the unique nonnegative coset representatives with minimum zero. Strictness gives a nonidentity term. The tame stabilizer acts freely on these terms and preserves multiplicities, hence has order $a \mid s - 1$ with $a \le w$. If a is even, its involution acts as conjugation on the pure word. The entry field is then real whereas K_X is nonreal by odd duality. Thus $[K_X : \mathbb{Q}] \ge b_s(t)$. Since $s - 1 > p^t$ (both s, p^t are odd), the entry ramifies at s. Separation gives $K_X \cap K_Z = \mathbb{Q}$, so the full orbit of X agrees on the p-active columns. The square action at p produces at least $(p-1)/2$ phases in the orbit of Z, each with a distinct dual pair, hence $[K_Z : \mathbb{Q}] \ge p - 1$. Finally $W(s, p^t)$ would forbid the word just obtained upon rescaling to full weight.

The X-orbit therefore costs at least $p^{2(t+1)} n^2 c_s(t)$ in (2.5) multiplied by d_Z^2: at eligible heights use the two field degrees over their rational intersection, and otherwise just the trace square. The q-free and r-free orbits are disjoint by degree. Counting also the unit yields $p^4 n^2 (A_q + A_r) < p^{2h} n^2$, as claimed.

If all N active labels have degree n, any s-free nonunit $X \in P_p$ hits all of them and has $1 \le t \le h - 2$ by the same height bounds. Such an X exists by Proposition 4.2. At an eligible height its orbit agrees on these columns, and each entry is a nonzero integer multiple of $p^t n$. The row averaged over the orbit has norm squared $D/[K_X : \mathbb{Q}]$ by orthogonality. At an ineligible height take the Galois mean of the squared row norm and use the lcm divisibility as in Lemma 20.4(2). In both cases $N p^{2t} b_s(t) < p^{2h}$, counting also the unit. In particular a permitted height exists on each axis. As $p^{2t} b_s(t) \ge p^2 A_s/(p-1)$ and $N \ge 2p^{h-1}(p-1)$ by that lemma, $A_s < p^{h-1}/2$ at each companion. Their sum is less than $p^{h-1} \le p^{2h-4}$. □

Theorem 20.6. *Let $p < q < r$ be odd primes, $h \ge 3$, $U = p^{h-1}$ and $H_t = \max\{a : a \mid r - 1,\ a \le p^t\}$. Suppose $r > U$, $W(r, 1 + \lfloor U/q \rfloor)$, and for every $1 \le t \le h - 2$ either*

$$W(r, p^t) \qquad \text{or} \qquad (p-1)(r-1) \ge p^{2(h-t-1)} H_t.$$

Then every braided fusion category of dimension dividing $p^{2h} q^B r^C d$ is weakly group-theoretical for all $B, C \ge 0$. In particular $r > U$ and $W(r, p^{h-2})$ suffice.

Proof. In a remaining divisor-minimal table take complementary nonunits of the principal q- and r-blocks, of degrees $p^i q^b, p^k r^c$. If their common weight g is at most U, then $r \ge g + 2$ and the two residues in Theorem 4.9 give a full r-word on at most $1 + \lfloor U/q \rfloor$ positions. Thus every such pair has weight p^h and their codegrees are p-units. Prime avoidance and defect-zero vanishing isolate p and force $D = p^{2h} n^2$, as in Theorem 20.1. This argument only uses $q < r$ and $r > U$ and works in either order of p, q.

In Theorem 20.5, all heights at r are eligible since $r > U \ge p^{t+1}$. Every retained height gives $p^{2(t-1)} c_r(t) \ge p^{2(t-1)}(p-1)(r-1)/H_t \ge p^{2h-4}$, a contradiction. The specialization uses $1 + \lfloor U/q \rfloor \le p^{h-2}$. □

Corollary 20.7. *For $h \ge 3$, the conclusion of Theorem 20.6 holds whenever*

$$r > \max\{p^{h-1},\ 33,\ (p^{h-2} - 1)^3\}.$$

In particular, $3^6 q^B r^C d$ is covered for all B, C and all odd primes $3 < q < r$ with $r \geq 11$. At $C = 4$ all primes $3 < q < r$ are covered.

Proof. The scalar test gives $W(s, j+1)$ for $s > \max(33, j^3)$ and $j \geq 1$. For $j = 1, 2, 3$ substitute directly. For $j \geq 4$, write $E = j(j+1)/2$ and use the $g(v)$ of Corollary 20.2. If $g(v) \leq j-2$ then $v \leq g(v)(g(v)+1)$; if $g(v) \geq j-1$, then $v - 1 \leq E - 1 \leq (j-1)^2$. Thus $(v-1)/g(v) \leq j-1$ and $2C(E) + 1 \leq 3 + j(j+1)(j-1) \leq j^3$. Also $3(1+2E)/2 < j^3$. Take $j = p^{h-2} - 1$. At $p = h = 3$ the exact test gives $W(r, 3)$ for every prime $r \geq 11$, and $r > 9$ throughout that range.

The quartic assertion follows also for the smaller primes from Corollary 20.11 below. □

Proposition 20.8. *For $h = 3$, the numerical alternative in Theorem 20.6 says that all divisors of $p^6 q^B r^C d$ are WGT dimensions whenever $r > p^2$, $W(r, 1 + \lfloor p^2/q \rfloor)$ and $(r-1)(p-1) \geq p^2 \max\{a : a \mid r-1,\ a \leq p\}$.*

20.3 Two traces at an isolated sextic prime

Let p, q, r be distinct odd primes, with $q < r$. Suppose p is isolated in an ordinary minimal table of dimension $p^6 n^2$, with n supported on q, r and divisible by both. For $s = q, r$ put

$$e_s = \frac{s-1}{\max_{a \mid s-1,\, a \leq p} a}, \qquad e_s^+ = \frac{s-1}{\max_{a \mid s-1,\, a \leq p} a/\gcd(a,2)}.$$

Set $c_s = (p-1)e_s^+$ if $s > p$ and $(s-p)(\min(p,q)-1) > p$, and $c_s = 1$ otherwise.

Proposition 20.9. *An isolated sextic as above is impossible if $c_q + c_r \geq p^2$. In particular, for $p < q$ it suffices that $(p-1)(e_q + e_r) \geq p^2$.*

Proof. At $h = 3$ the only height in Theorem 20.5 is $t = 1$, and $A_s \geq c_s$. When $p < q$ both primes are eligible because $s - p \geq 2$, and $e_s^+ \geq e_s$. □

Theorem 20.10. *Let $p < q < r$ be odd primes and d coprime and square-free. Put*

$$k = 1 + \left\lfloor \frac{\min(r, p^2)}{q} \right\rfloor.$$

If $W(r, k)$ holds, every divisor of $p^6 q^B r^4 d$ is a WGT dimension, with B arbitrary. If $r > p^2$, the exponent at r can also be arbitrary.

Proof. For $r < p^2$, one has $k < p$ since $(p-1)q > p^2$. The first alternative of Proposition 8.7 gives the forbidden word. The proof of Theorem 8.2 excludes its second alternative when $k < p$. For $r > p^2$, the two-residue argument of Theorem 20.6 excludes every complementary common p-weight $g \leq p^2$: since $r \geq g + 2$ it gives a full word in at most $1 + \lfloor g/q \rfloor$ positions. Thus all opposite witnesses have full p-height three, and their defect-zero rows isolate p at valuation six. This step works for arbitrary companion exponents. Now $e_r \geq p + 1$ because $r - 1 > p^2$, and $e_q \geq 1$, so $(p-1)(e_q + e_r) \geq (p-1)(p+2) \geq p^2$. Proposition 20.9 now excludes the isolated table. □

In particular all r are covered for the sextic–quartic conclusion if $p \geq 5$ and $q^3 \geq p^4$. Indeed, with $j = \lfloor \min(r, p^2)/q \rfloor$ we have $j^3 < r$: for $r \leq p^2$ use $r^2 \leq p^4 \leq q^3$ and $j < r/q$, and at $r > p^2$ use $(p^2/q)^3 \leq p^2 < r$. For $j \geq 4$ or $r > 33$, use the scalar obstruction in Corollary 20.7. At $j = 0, 1, 2$ strictness and the small scalar bounds suffice (at $j = 2$, $r > 2q$). At $j = 3$ and $r \leq 33$ we have $r > 3q$, so the only possible q is 7 and thus $p = 5$, incompatible with $q^3 \geq p^4$.

For $p = 5, q = 7$ and $r = 29, 31$, the four-position calculation in Corollary 8.9 gives $W(r, 4)$, so both companion exponents are unrestricted. For $p = 5, q = 7, r = 23$ the word bound $W(23, 4)$ remains undecided. Nevertheless $e_7 = 2$ and $e_{23} = 11$, so Proposition 20.9 excludes isolation at five for arbitrary companion exponents whenever the five-valuation is six.

Corollary 20.11. *Let $p < q < r$ be odd primes. For each line of the following table every divisor of $p^6 q^B r^C d$ is covered, for all $B, C \geq 0$ and all coprime square-free d:*

p	q	r
3	$q > 3$	$r \geq 11$
5	$q > 5$	$r \geq 29$
7	$q > 7$	$r \geq 53$.

For $C = 4$ the conclusion holds at $p = 3, 7$ for every q, r. At $p = 5$ the sole assignment left by these tests is $(q, r) = (7, 23)$.

Proof. In the three lines $r > p^2$, and $1 + \lfloor p^2/q \rfloor$ is at most $2, 4, 5$, respectively. At two positions use Theorem 19.5. For four and five positions use the exposed-support calculations of Corollary 8.9. At five positions these give the cases $53, 59, 61$; the scalar bound applies from 67 onward. For the quartic the only smaller triple at $p = 3$ is $(3, 5, 7)$, where $r < 2q$. At $p = 5$, $r < 29$ gives $r < 3q$ except for $(q, r) = (7, 23)$. Both ratio tests follow from Theorem 8.2. When $p = 7$ and $r < 53$ we have $r < 5q$, so Corollary 8.9 applies. □

For $r > 33$ the scalar bound alone has a useful consequence for the location of any surviving odd triple:

$$q^3 < p^4, \qquad q^{3/2} < r < (p^2/q)^3.$$

Indeed with $j = \lfloor \min(r, p^2)/q \rfloor$ a failure of the test forces $r \leq j^3$. Since $j < r/q$ and $j < p^2/q$, the two strict inequalities follow. For a fixed least prime p the possible companion primes in this interval are finite even when B is unbounded.

Applying the multiplier criterion. Let $p < q < r$ be odd primes, and d coprime and square-free. Take $m = \lfloor \min(r, p^2)/q \rfloor$. Write $E = m(m+1)/2$ and $H = (r-1)/2$. Under the exposure condition $1 + 2E < 2r/3$ in Section 8.10, if its scalar inequality and Proposition 8.38 exclude every $h \leq E$, then every divisor of $p^6 q^B r^4 d$ is a WGT dimension for all $B \geq 0$, by Theorem 20.10. The exponent at r is also arbitrary when $r > p^2$. Independently, if $m = \lfloor r/q \rfloor$ works and $m + 1 < p$, the same holds for $p^A q^B r^4 d$ with both A, B arbitrary (Proposition 8.7 and Theorem 8.2).

The first part of Proposition 8.38 gives $W(577, 10)$ and $W(1783, 14)$. At $(m, r) = (9, 577), (13, 1783)$ the only scalar survivors are $(h, k) = (42, 6), (90, 9)$ respectively, with equality and $h = k(k+1)$. In fact for every integer $k \geq 2$ with $r = 2k^3 + 4k^2 + 1$ prime, the same argument applies whenever $k(k+1) \leq E$ and $E(E-1) < (H-1)(k+1)$: for each lower j the largest scalar quotient $h(h-1)/j$ at $h = j(j+1)$ is increasing in j, and beyond $k(k+1)$ the new denominator is at least $k+1$. The first example applies for either middle prime $59, 61$ and every prime $10 < p < q$ at arbitrary first and middle exponents. In the second example one may take $q = 131$ or 137 and $14 < p < q$.

The second part gives $W(180547, 64)$, with $E = 2016$. The sole scalar survivor is $h = 2016$, $k = 45$ (compare the preceding endpoint $44 \cdot 45$ by the same monotonicity). Here $H = 90273$, $h - k = 1971 = 3^3 \cdot 73$, and $3^{51} \equiv -1 \pmod{30091}$ with $30091 \mid H$. The odd valuation at three excludes the mask. The scalar inequality and the order-two quotient condition both allow these three examples: H is odd in the last two and $h - k = 36$ is a square in the first.

20.4 Two unbounded exponents below the least-prime square

For odd primes $p < q < r$ put $g = p^2$. We give a criterion for every divisor of $p^6 q^B r^C d$, with both B, C unbounded, when $r < g$. Here and below d is coprime and square-free.

Lemma 20.12. *Let $q \geq 7$ and $s > q$ be primes and $s < g < 2s - 2$ an integer prime to q. Suppose that z is a cyclotomic integer expressed as a positive rational sum of roots of unity of*

weight g, with no phase prime below five, and that $z \equiv g$ at every prime above q and s. Every such expression has the same phase multiplicities. They are integers. Its q-parts consist of $t = g - s$ copies of 1 and s copies of one q-power root η. All other parts are s-primary, and the multiplicities at s-positions are $g\delta_{x,1}$ modulo q.

Proof. In the integral two-prime signed configuration give $+$ cost one and $-$ cost two as in Proposition 19.8. Consider a word of cost at most g with the two prescribed residues. On erasing q-parts its signed coefficients $m_x - g\delta_{x,1}$ are constant modulo q on each μ_s-coset. At most one such constant can be nonzero, since two would require at least $2s - 1 > g$ occupied positions.

Erase the s-parts instead and write v for the resulting integer in the q-power field. Set $\alpha = (v - t)/s$, which is integral because this field is unramified at s. Its signed rational word length is at most $(g + t)/s < 3$: subtract t from the word for v and divide all coefficients by s. On a single prime-power configuration the minimal signed length of an integral value is attained integrally. Indeed its coefficient vectors differ by constants on the prime-order cosets, and subtracting an integral median on each coset minimizes the sum of absolute values. Thus α is the sum of at most two signed q-power roots, the empty word being permitted. Let $M \in \{-2, -1, 0, 1, 2\}$ be the sum of their signs. Reducing $v = t + s\alpha$ above q shows that the signed augmentation is $t + sM$ modulo q. Since $g = t + s$, the sum of the coset constants of the preceding paragraph is $M - 1$ modulo q. For $M = -2, -1, 0$ the sole nonzero constant would be $-3, -2, -1$, respectively. At every one of at least $s - 1$ positions this requires signed cost at least two, impossible. For $M = 2$ the element v is a positive expression of mass $t + 2s$ on at most three q-positions. These are exposed among signed roots by the coset functional that is one at the $m \leq 3$ marked positions and $-m/(q - m)$ elsewhere; its absolute value on unmarked positions is strictly below one as $q \geq 7$. The cost of the expression for v would thus exceed g. Consequently $M = 1$ and α is a single positive q-root η. Exposing $1, \eta$ forces precisely the announced unsigned word of length g. Since $M = 1$, all coset constants vanish.

For clarity, the passage to rational physical expressions is unchanged from Proposition 19.8. Independent integral functionals on the complementary factors fix 1, send a uniformly chosen nonidentity prime-order root to -1, and vanish at the other nontrivial roots. Any term with complementary support has expected signed cost at most $2/(5 - 1) < 1$. The loss contradicts the signed integral minimum just proved, by two-prime unimodularity. All optimal integral expressions have at most $1 + \lfloor g/q \rfloor$ primary s-positions including 1. Two such supports together contain at most $1 + 2\lfloor g/q \rfloor < s$ positions, independent over the q-power field. At each of their positions at most three distinct q-roots appear across the two expressions; these too are independent. Consequently the optimum is unique, and every rational optimum, being a convex combination of integral optima, has these multiplicities. □

In odd ordinary data the difference-set argument applies simultaneously to any number of primary coordinates. Suppose a full positive word, up to a scalar, uses at most k_s positions in each coordinate s of a finite set of odd primes, with constant complementary part. Write $E_s = k_s(k_s - 1)/2$, $H_s = (s - 1)/2$ and $u(h) = \min\{j \geq 1 : j(j + 1) \geq h\}$. Such a word is impossible if, in every nonconstant coordinate, the following conditions hold with $(s, k, E, H) = (s, k_s, E_s, H_s)$:

$$2(2E + 1) < s, \qquad (H - 1)u(h) > h(h - 1) \quad (1 \leq h \leq E). \tag{20.1}$$

Indeed, at the fixed row choose a full word minimizing the sum of its numbers h_s of opposite nonzero differences, subject to $h_s \leq E_s$ and to the initially constant coordinates staying constant. Its adjoint support lies in the product of the coordinate difference sets. The coset functionals in Lemma 4.15 have absolute value strictly less than one off the marks under the first inequality. Their product, with normalized trace on the complement, therefore exposes this box. In each nonconstant coordinate the nonidentity tame multipliers modulo sign have average overlap at most $h_s(h_s - 1)/(H_s - 1)$ by freeness. Choose independently one with overlap $j_s < u(h_s)$, using the second inequality. Lemma 4.15 supplies a common nonunit whose phases belong to the

intersection of the two boxes. In each such coordinate its opposite differences number at most $j_s(j_s+1) < h_s$, as in Lemma 4.16. This contradicts minimality; a word constant everywhere already contradicts strictness.

For a rectangle of at most k positions at r and two at $q \geq 7$, the q-condition is automatic ($E_q \leq 1$). Thus only (20.1) at r, k need be checked.

Theorem 20.13. *Suppose $p \geq 5$ and $(p^2+2)/2 < r < p^2$. Let $k = 1 + \lfloor p^2/q \rfloor$ and*

$$e_s^+ = \frac{s-1}{\max_{a|(s-1),\ a\leq p} a/\gcd(a,2)}, \qquad s = q, r.$$

If (20.1) holds at $s = r, k$, and $(p-1)(e_q^+ + e_r^+) \geq p^2$, then all divisors of $p^6 q^B r^C d$ are WGT dimensions for all B, C.

Proof. In a remaining minimal table opposite r-free and q-free nonunits of P_q and P_r have common degree factor $p^i \geq q$ (Lemma 4.10). As $i \leq 3$ and $q > p$, the sole alternative to full p-height is $i = 2$. Their normalized crossing would then have the two residues and weight $g = p^2$ of Lemma 20.12. Its r-projection has at most k positions, excluded by the mixed support obstruction. Thus every such witness has full p-height three. The opposite principal hits separate p from both q and r as in Theorem 20.10, without a hypothesis on their exponents.

Here both companion primes are eligible. Proposition 20.9 excludes this isolated sextic by the stated orbit inequality. □

For $p = 5$, this covers every $r \in \{17, 19, 23\}$ with $11 \leq q < r$ and arbitrary B, C. Here $k \leq 3$, the three values of e_r^+ are $8, 6, 22$, and (20.1) is immediate with $E = 3$. For $p = 7$, it covers every $r \in \{29, 31, 37, 41, 43, 47\}$ with $17 \leq q < r$. Moreover at $r \geq 37$ it includes $q = 13$. For $r = 29, 31$ and $q \geq 17$ we have $E \leq 3$. For the other four primes use $E \leq 6$ and $s \geq 37$. The values of e_r^+ in order are $4, 6, 12, 8, 6, 46$. For $r = 29$ the possible middle primes are $17, 19, 23$, with $e_q^+ = 8, 6, 22$. For $r = 31, 43$ all admissible middle primes satisfy $e_q^+ \geq 4$: an index-two odd stabilizer of order at most seven would require $q \leq 15$ and $q \equiv 3 \pmod 4$, and an even stabilizer of index one is impossible since $q - 1 > 7$. The numerical criterion holds in every case.

Corollary 20.14. *For odd primes $p < q < r$ with $p \geq 5$ and $2q > p^2$, all divisors of $p^6 q^B r^C d$ are WGT dimensions for all B, C.*

Proof. Here $k \leq 2$. For $r > p^2$ apply Theorem 20.10. Otherwise $2r - 2 > p^2$ since $r \geq q + 2$, and (20.1) holds. Put $h = (p+1)/2$. For either $s = q, r$ we have $s - 1 \geq (p-1)h > p(h-1)$, so $e_s^+ \geq h$. Equality would require an effective stabilizer $a/\gcd(a,2) \geq p - 1$ for a divisor $a \leq p$ of $s - 1$. Its only possible value is p, so equality forces $s = ph + 1$. The two distinct primes cannot both have this value; hence $e_q^+ + e_r^+ \geq 2h + 1 = p + 2$, as required. □

Proposition 20.14a (Two positive residues at arbitrary ratio)**.** *Let $q < s$ be primes, g a positive integer with $q \nmid g$, and set $m = \lfloor g/s \rfloor$, $t = g - ms$. Assume $m \geq 1$ and $q \geq 5(m+2)$. Let z be a cyclotomic integer admitting an expression*

$$z = \sum_{\omega \in \mu_n} c_\omega \omega, \qquad c_\omega \in \mathbb{Q}_{\geq 0}, \qquad \sum_\omega c_\omega = g,$$

where every prime divisor of n is at least five. We call g the weight of this expression. Assume $z \equiv g$ at every prime ideal over q and over s in $\mathbb{Z}[\zeta_n]$. Then the multiplicities c_ω are integers and are the same for every expression satisfying these weight and phase-order conditions, allowing the level n to vary.

Every occurring phase is a product of a q-power root and an s-power root. Counting repetitions, its q-parts form t copies of 1 and s copies of each member of a multiset of m q-power roots. The multiplicity at each s-position x is $g\delta_{x,1} \pmod q$. In particular the two projections have at most $m + 1$ and $1 + \lfloor g/q \rfloor$ positions, respectively.

Proof. We may enlarge the level to a common multiple of qs and any two levels being compared. First consider an integral signed word on the q- and s-primary roots whose value has the stated residues. Give a positive letter cost one and a negative letter cost two. Write P, N for their numbers and suppose $P + 2N \le g$.

Erase the q-parts and let h_x be the resulting signed coefficients on the s-positions. Modulo every prime above q erasure fixes the value. Since q is unramified in the s-power field, the erased value is g modulo q in its ring of integers. In that ring the relations among the roots are generated by the sums over the μ_s-cosets (omit one position per coset to get an integral basis). Thus $h_x - g\delta_{x,1}$ is constant modulo q on each such coset C. Choose the representative a_C of the constant minimizing $w(a) = \max(a, -2a)$. At all positions except possibly 1 the cost is at least $w(a_C)$, giving $\sum_C w(a_C) \le g/(s-1) < m + 2$.

Erase the s-parts instead. The resulting q-primary value is $v = t + s\alpha$ with α integral, since $v \equiv g \equiv t$ at every prime over s and s is unramified here. Subtracting t and dividing by s gives a rational signed expression for α of absolute length at most $(g + t)/s < m + 2$. Among the coefficient vectors of this integral value, choose an integral vector and subtract a median on each μ_q-coset. All rational representations differ from this vector by a constant on each coset, so an integral vector achieves the minimum absolute length. We thus have a signed integral word of length at most $m + 1$ for α. Write L for the sum of its signs. Counting the h_x modulo q, and reducing v above q, gives

$$P - N \equiv g + s\sum_C a_C \equiv t + sL \pmod{q}.$$

Hence $\sum_C a_C = L - m$ as integers: their absolute values are at most $m + 1$ and $2m + 1$, and $q > 3m + 2$.

Mark 1 and every q-root in the chosen word for α. On a μ_q-coset containing k marks prescribe a functional equal to 1 there and $-k/(q-k)$ on the unmarked positions. Its coset sums are zero, so these prescriptions give a rational-linear functional. Its values lie in $[-1/4, 1]$ because $k \le m + 2 \le q/5$. Applied to the two expressions for v it gives

$$t + sL \le P + N/4 \le g - 7N/4.$$

Put $d = m - L \ge 0$. If some a_C is negative, write $a_C = -b_C$ on these cosets, put $B = \sum b_C$ and $b = \max b_C$. Then $B \ge d$ and $2b \le m + 1$. At each of at least $s - 1$ positions of such a coset there are at least b_C negative letters, or there are at least $q - b_C$ positive letters. The number of positions where the first bound fails is at most $g/(q-b)$. Consequently

$$N \ge B(s-1) - \frac{bg}{q-b} > Bs\left(1 - \frac{1}{s} - \frac{m+1}{q-b}\right) > \frac{4}{7}Bs,$$

using $g < (m+1)s$, $b \le B$, $1/s < 1/15$ and $(m+1)/(q-b) < 2/9$. This contradicts $N \le 4sd/7$. Hence every $a_C \ge 0$. Their sum is $-d \le 0$, so all vanish, $d = 0$ and $N = 0$. The word for α has signed sum m and length at most $m + 1$; by parity it consists of m positive letters. The functional now takes value g on v, so $P = g$ and every q-part is marked. The marks are linearly independent, since they miss a position in each μ_q-coset. This gives the stated q-multiplicities, and $a_C = 0$ gives the congruence on the s-multiplicities.

To pass to the original rational expression, split off all primary factors other than q, s as in Proposition 19.8. In each such u-power factor use the integral functional which fixes 1, takes value -1 on one uniformly chosen nonidentity uth root, and vanishes on the remaining roots. Tensor these independent functionals with the identity on the retained factors. The values on each μ_u-coset sum to zero. The contraction fixes g and preserves the two radical congruences: the radical at either retained prime is generated by its primary cyclotomic uniformizer. A term with a nontrivial complementary u-part has expected signed cost at most $2/(u-1) \le 1/2$. Its presence would give some integral contraction target with a rational signed expression of cost

less than g. The two-prime cyclotomic configuration is unimodular [4, Theorems 3 and 13]. In an integral basis of its column lattice, every nonsingular maximal minor has determinant ± 1; this property persists on adjoining the negative columns. A basic optimum for an integral target therefore has integral coefficients, contrary to the preceding calculation.

Only the q, s-factors remain. The cost-g face of the polyhedron of signed representations of z is a bounded optimal face. By the same unimodularity its vertices are integral, so the rational expression is a convex combination of integral optima. In each optimum the nonidentity s-positions have multiplicity at least q and the identity occurs since $q \nmid g$. Two optima together use at most $1 + 2\lfloor g/q \rfloor < s$ such positions in total, since $g < (m+1)s$ and $q \geq 5(m+2)$. Their union is independent over the q-power field by the coset basis and coprimality of the conductors. The sums of q-roots therefore agree at each s-position. The union of the q-positions contains at most $1 + 2m < q$ roots and is likewise independent. Thus every joint multiplicity agrees, proving uniqueness. □

For $g = p^2$ and $s = r < p^2$, Proposition 20.14a gives the two projection sizes in the product obstruction (20.1). The isolation and trace argument of Theorem 20.13 applies whenever $q \geq 5(\lfloor p^2/r \rfloor + 2)$, both projection tests hold and the isolated-orbit inequality is satisfied.

The small least prime. For $p = 3$ the conclusion of Corollary 20.14 follows from Corollary 20.11 except at $(3, 5, 7)$. In any minimal obstruction in this remaining unbounded-exponent family a complementary crossing of common weight nine is necessary. Moreover, *every* such crossing has, up to the choices of primitive roots $\eta \in \mu_5$, $\xi \in \mu_7$, the unique normalized spectrum

$$z = 1 + \xi + 3\eta + 4\eta\xi.$$

Indeed the common 3-weight of a 7-free nonunit of P_5 and a 5-free nonunit of P_7 is nine or 27 by Lemma 4.10. If all pairs have weight 27, full 3-height isolates a sextic three as before. The isolated-orbit argument applies with $e_5^+ = 4$, $e_7^+ = 2$ and $2(4+2) > 9$: the height-one auxiliary pure words are available since $(s-3)(3-1) > 3$ for $s = 5, 7$.

At weight nine repeat the argument of Lemma 20.12 with negative cost $7/4$. This keeps the contraction at the spectator prime three strict. The signed residue $-3, -2$ or -1 modulo five costs at least $7/4$ at each of six positions, already more than nine. For $M = 2$ the functional on at most three marked positions in a five-coset has negative magnitude at most $3/2 < 7/4$, so this case also fails. The rest of that proof gives two 5-parts of mass $2, 7$ and two 7-parts of mass $4, 5$, with 1 the first position in each coordinate. A singleton projection would give an excluded one- or two-phase word (Theorem 19.5). Write the nonzero exponents of the two projections as a, b, respectively.

If either relative order is wild, its adjoint projection $\{0, \pm a\}$ or $\{0, \pm b\}$ has at most one mark per prime-order coset. In the product coset functional the largest possible single negative magnitude is $3/2$ and the product of the two negative magnitudes is at most $(3/2)/4 < 1$. Tensoring with the normalized spectator trace (at least $-1/2$) exposes the adjoint rectangle. Independent units moving the nonzero pair off itself in each coordinate then contradict the common-nonunit argument. Thus both relative orders are tame. If an initial corner is missing, after scaling the exponents and choosing signs, the adjoint marks are the 3×3 square of positions $0, \pm 1$ in each coordinate with $(1, 1), (-1, -1)$ omitted. The following functional exposes these seven marks. In that square put 1 on the marks and 0 on the omitted corners. In the other four columns put $-3/4$ on row zero and $-1/2$ on rows ± 1. In each of the other two rows put $-3/2, -1, -1$ on columns $0, 1, -1$ and $7/8$ on the four remaining columns. Every row and every column sums to zero, so the array is a linear functional on the cyclotomic configuration. Extend by zero on other product cosets and tensor with the normalized spectator trace. As all its values are greater than -2, equality at one is possible only on the seven marks. The independent dilations again yield a contradiction. With all corners occupied the margins force exactly the four

stated coefficients. In particular the value generates $\mathbb{Q}(\zeta_{35})$ by uniqueness of the two projections, and its tame trace at five is $4(1+\xi)-(3+4\xi)=1$. These are necessary conditions. The $(3,5,7)$ assignment with two unrestricted companion exponents stays open; its quartic case follows from Section 20.2.

Twist-one orbits at the least prime three. For a minimal table in the unbounded-exponent case $(3,5,7)$ of Section 20.4, the weight-nine pair may be chosen in the pure five- and seven-twist fibers. Indeed Proposition 4.2 gives each hit in those fibers; if all their complementary pairs had full three-height, the vacuum-block argument of Section 20.4 would again isolate three. Take faithful characters at five and seven of orders four and six. At $1+\xi+3\eta+4\eta\xi$ their mixed Fourier coefficient is four times a product of two nonzero Gauss sums. On $(\mathbb{Z}/35)^\times$ their product has kernel generated by simultaneous negation: the two images intersect in $\{1,-1\}$ and the characters are faithful. Inverting either factor preserves this kernel. Therefore Proposition 19.20 supplies twist-one labels in P_5, P_7 in the same full three-block with fields containing $\mathbb{Q}(\zeta_{35})^+$. Each orbit has at least 24 members, since a nonunit in odd dimension has nonreal field and this real subfield has degree 12. More precisely, use characters with local orders $(4,6),(4,3),(2,6)$. All three mixed coefficients are nonzero by the same product of Gauss sums. With fixed generators a,b of the order-four and order-six groups, their kernels on $(\mathbb{Z}/35)^\times$ are respectively $\langle(a^2,b^3)\rangle$, $\langle(1,b^3)\rangle$, and $\langle(a,b^3)\rangle$. Inverting either component preserves each kernel. The kernel intersections are pairwise trivial, so any two of the three fields generate $\mathbb{Q}(\zeta_{35})$. In each of P_5, P_7 therefore either a twist-one row contains $\mathbb{Q}(\zeta_{35})$, or at least $24+12+6=42$ nonunit twist-one labels occur in three distinct orbits. The common three-block can depend on the character.

20.5 The sextic exponent at either smaller prime

Let P,Q,R be distinct odd primes with $Q<R$ and $R>P^2$.

Theorem 20.15. *If $W(R,1+\lfloor P^2/Q\rfloor)$ holds, then every divisor of $P^6Q^BR^Cd$ is a WGT dimension for arbitrary B,C and coprime square-free d. The exponent six can thus occur at the middle prime: there is no ordering hypothesis on P and Q.*

Proof. For the two principal blocks at Q,R, the complementary-pair argument in Theorem 20.6 at $h=3$ isolates P at valuation six, since $R>P^2$ and $W(R,1+\lfloor P^2/Q\rfloor)$ holds. The repeated odd primes permit ordinary data by Proposition 4.7. In Proposition 20.9, $R>2P$ is eligible and $e_R^+\ge e_R\ge P+1$, since $R-1>P^2$. The other companion costs at least one. Thus $c_R+c_Q\ge(P-1)(P+1)+1=P^2$. □

For example, for any fixed distinct odd P,Q, the theorem gives $P^6Q^BR^C$ for all sufficiently large primes R, independently of both B and C. An explicit sufficient threshold is $R>\max\{P^2,33,\lfloor P^2/Q\rfloor^3\}$ by the scalar word bound in Section 20.3 and Corollary 20.7.

A cubic scalar bound. For integer $j\ge1$ put $E=j(j+1)/2$. For $1\le h\le E$ one has

$$\frac{h}{k(h)}\le\sqrt{h+1/4}+1/2\le(j+1)/\sqrt{2}+1/2,\qquad k(h)=\min\{v:v(v+1)\ge h\}.$$

Consequently $3+2h(h-1)/k(h)<(j+1)^3/\sqrt{2}$. To verify the strict estimate, replace $2(h-1)$ by $j(j+1)-2$. The difference between the proposed upper bound and the resulting expression is $(j+1)(j+3)/\sqrt{2}-(j^2+j)/2-2>0$. Also $3(1+2E)/2<(j+1)^3/\sqrt{2}$ for $j\ge1$. Thus $r\ge(j+1)^3/\sqrt{2}$ forces $W(r,j+1)$ by Lemma 4.16.

An all-r inequality. Let $p < q < r$ be odd primes and $J = \lfloor p^2/q \rfloor$. If

$$(J+2)^2 \le \sqrt{2}\, q,$$

every divisor of $p^6 q^B r^4 d$ is a WGT dimension, for arbitrary B and coprime square-free d. For $r > p^2$ both companion exponents are arbitrary. In particular the all-r statement holds whenever $\sqrt{2}\, q^3 \ge (p^2+2q)^2$; for every fixed $\alpha > 1/\sqrt{2}$ it also holds throughout $q^3 \ge \alpha p^4$ once q is sufficiently large in terms of α.

Indeed put $j = \lfloor \min(r, p^2)/q \rfloor \le J$. The case $j = 0$ is strictness and $j = 1$ follows from the two-position theorem since $r \ge 7$. For $j \ge 2$, $(j+1)^3/j < (j+2)^2 \le (J+2)^2 \le \sqrt{2}\, q$, and $jq < r$. Thus the preceding cubic bound gives $W(r, j+1)$. Apply Theorem 20.10. Finally $q^3 \ge \alpha p^4$ gives $(p^2/q+2)/\sqrt{q} \le 1/\sqrt{\alpha} + 2/\sqrt{q}$, which is at most $2^{1/4}$ for the indicated range. For instance $p = 31, q = 97$ passes the integer-floor test while $q^3 < p^4$.

20.6 A quotient of the tame multiplier group

Let $W(r, m+1)$ denote the exclusion of a full positive r-primary word on at most $m+1$ positions for a nonunit label at a nonunit row in odd ordinary minimal data. A common scalar root is allowed. The following test keeps track of where differences lie in the group of tame multipliers.

Proposition 20.16. *Let r be an odd prime, $E = m(m+1)/2$, $H = (r-1)/2$, and assume $1 + 2E < 2r/3$. If $W(r, m+1)$ fails, there is an integer $1 \le h \le E$ with*

$$k = \min\{j \ge 1 : j(j+1) \ge h\}, \qquad \delta = h(h-1) - (H-1)k \ge 0.$$

If $\delta < 2(h-1)$, the following further condition holds for every divisor b of H. There are nonnegative integers x_i indexed by $i \in \mathbb{Z}/b$ with $\sum_i x_i = h$, and integers $e_j \ge 0$ with $\sum_j e_j = \delta$, such that

$$\sum_i x_i x_{i+j} = \frac{H}{b} k + (h-k)\mathbf{1}_{j=0} + e_j.$$

In particular, when H is even, the interval $[h-k-\delta, h-k+\delta]$ contains the square of an integer congruent to h modulo two.

Proof. By Lemma 4.16, a minimizing word at the fixed row has $h \le E$ opposite difference pairs A and every nonidentity tame multiplier satisfies $|A \cap uA| \ge k$.

If the populations of A on the free tame orbits are $h_1, \dots, h_v > 0$, then summing the overlaps over $u \ne 1$ gives $\sum_a h_a(h_a - 1)$. Two or more nonempty orbits decrease this sum from $h(h-1)$ by at least $2(h-1)$. This proves $\delta \ge 0$ and, under the strict bound on δ, identifies A with an h-element subset of a single cyclic orbit. Identify that orbit with $\mathbb{Z}/H$. Under reduction modulo b let x_i count the members of A in residue class i. The left side of the displayed equation counts all ordered pairs with difference j modulo b. Each of the H/b group differences in that class occurs at least k times; difference zero in the full group occurs h times. Summing gives the asserted equations and total excess. At $b = 2$ subtraction of the odd equation from the even gives $(x_0 - x_1)^2 = h - k + e_0 - e_1$. The two populations sum to h. □

For arbitrary odd primes $p < q < r$ put $m = \lfloor r/q \rfloor$. If $m + 1 < p$ and Proposition 20.16 has no feasible h at (r, m), every divisor of $p^A q^B r^4 d$ is a WGT dimension for all $A, B \ge 0$ and every coprime square-free d. Indeed its conclusion excludes the first alternative of Proposition 8.7, and $m + 1 < p$ excludes the second as in Theorem 8.2. For the sextic superfamily one can instead use $m = \lfloor \min(r, p^2)/q \rfloor$ without the hypothesis $m + 1 < p$, by Theorem 20.10; if $r > p^2$ the exponent at r may also be arbitrary.

Here are instances where the scalar inequality alone leaves a feasible h but the order-two quotient excludes it. Every column uses the same rule. The last column brackets the interval in

the preceding column by consecutive squares of the required parity.

m	r	h	k	δ	$[h-k-\delta, h-k+\delta]$	bracketing squares
17	3877	153	12	12	$[129, 153]$	$11^2, 13^2$
29	17981	435	21	21	$[393, 435]$	$19^2, 21^2$
33	26449	552	23	23	$[506, 552]$	$22^2, 24^2$
34	29453	595	24	30	$[541, 601]$	$23^2, 25^2$
45	66889	1035	32	14	$[989, 1017]$	$31^2, 33^2$

To check uniqueness of h without enumerating supports, on the interval $k(k-1) < h \le k(k+1)$ the scalar quotient $h(h-1)/k$ increases. Its value at a complete right endpoint is $(k+1)(k(k+1)-1)$, increasing with k. It suffices to compare the last two positions of the last complete interval and of the terminal interval, and the endpoint of the preceding complete interval. For example the first line gives $W(3877, 18)$. It applies at middle prime 223 or 227 for every prime $18 < p < q$ and arbitrary first and middle exponents. At $p = 67$, $q = 223$, Theorem 20.10 asks for this same word obstruction, with $h = 153$ left by the scalar test.

A uniform range at the sextic prime. For odd primes $5 \le p < q$ the condition

$$\left(\frac{p^2}{q} + 1\right)^3 \le \sqrt{2}\, p^2$$

suffices for every divisor of $p^6 q^B r^4 d$, for every prime $r > q$ and every B. In particular, for any $c > 1/\sqrt{2}$ and all sufficiently large p, it is enough that $q^3 \ge cp^4$; the choice $c = 4/5$ works for $p \ge 125$. This uniform estimate is already implicit in the exact scalar test, whereas the preceding quotient test eliminates witnesses that satisfy that test.

If $q \ge p^2$ then $j = \lfloor \min(r, p^2)/q \rfloor = 0$ and strictness suffices. Otherwise set $x = \min(r, p^2) > q$. The function $(x/q+1)^3/x$ is increasing for $x \ge q$. The displayed prime condition gives $(j+1)^3 \le (x/q+1)^3 \le \sqrt{2}\, x \le \sqrt{2}\, r$. Now apply Theorem 20.10. The asserted asymptotic form follows by dividing the cube root of the prime condition by $p^{2/3}$. For the explicit constant, $(5/4)^{1/3} < 27/25$ and $28/25 < 2^{1/6}$.

20.7 A small intersection at the largest quartic prime

Proposition 20.17. *At every prime $s \ge 173$, a full positive s-primary word of a nonunit label at a nonunit row cannot have at most seven positions, in either type of minimal table.*

Proof. Write $H = (s-1)/2$, $g(v) = \min\{t \ge 1 : t(t+1) \ge v\}$. Among all full positive s-primary words of nonunit labels on the given row, take one minimizing the number h of nonzero opposite pairs in its difference set D. We allow overall root-of-unity factors. The initial word gives $h \le 21$. Since $43 < s/2$, the argument of Lemma 4.16, applies. Every intersection of D with a unit multiple of D contains at least $g(h)$ opposite pairs: a common nonunit of the two adjoints has its balancing support in that intersection. Counting the nonidentity tame multipliers gives

$$(H-1)g(h) \le h(h-1).$$

The maximum of $h(h-1)/g(h)$ for $1 \le h \le 21$ is 95, attained at $h = 20$. Thus $s > 193$ is impossible. At $s = 173$ the inequality allows $h = 19, 20$, and at $179, 181, 191, 193$ only $h = 20$. In all cases $g(h) = 4$.

Let l be the number of nonzero opposite pairs in D of exact order s. For a uniform random unit at the chosen s-power level, the expected overlap on an orbit of n opposite pairs containing b pairs of D is b^2/n. The order-s orbit has H pairs, and every higher-order orbit has at least sH. Summing gives

$$4H \le l^2 + (h-l)^2/s.$$

If $l \le 18$, the right side is at most $18^2 + 20^2/173 < 344 \le 4H$. Thus at most one pair has higher order. No nonidentity tame multiplier modulo sign can match a higher-order pair with itself, so every overlap for such a multiplier lies in $\mathbb{F}_s$. The mean nonidentity overlap there is $l(l-1)/(H-1) \le 380/85 < 5$. Some intersection therefore equals $T = \{0, \pm x_1, \dots, \pm x_4\}$. A common nonunit has support contained in T. Its differences lie in $T - T$, which has at most $4 \cdot 5 = 20$ pairs and at least h. These actual differences are exposed. Comparing their dilates as above shows that $T - T$ meets each of its dilates in at least four pairs.

The following short finite check is on differences of four residues and is independent of the dimension. Scale one pair to ± 1 and choose the other three from the $H - 1$ remaining pairs. Form the mask of $x_i, 2x_i, x_i \pm x_j$ modulo sign. Retain masks with at least nineteen pairs at $s = 173$ and twenty elsewhere, and test their intersections with every nonidentity multiple. The columns count normalized choices of T, including choices with the same mask. The loops in the residue enumeration reproduce the counts using only integer arithmetic.

s	choices	at least the required size	all overlaps at least four
173	98770	52012	0
179	109736	50376	0
181	113564	52658	0
191	134044	65108	0
193	138415	67813	0

This excludes every possible T. □

Corollary 20.18. *For arbitrary nonnegative A, B, all divisors of $p^A q^B r^4 d$ are WGT dimensions whenever $11 \le p < q < r < 7q$ are primes and $q \ge 29$.*

Proof. For $r < 6q$, use the exposed-support calculation of Corollary 8.9. In the remainder, $r > 6q \ge 174$ gives $r \ge 179$, and $1 + \lfloor r/q \rfloor = 7 < p$. The four-pair calculation in Proposition 20.17, together with its scalar bound, excludes the first alternative of Proposition 8.7. The second is excluded by $7 < p$, as in the proof of Theorem 8.2. □

20.7.1 Extension to smaller middle primes

The finite check in Proposition 20.17 becomes larger if the scalar inequality allows smaller values of h. The following extension uses only the same common-constituent argument.

Proposition 20.19. *For every prime $s \ge 127$, the seven-position obstruction of Proposition 20.17 holds in both types of minimal table. Consequently every divisor of $p^A q^B r^4 d$ is a WGT dimension for $11 \le p < q < r < 7q$ and $q \ge 19$, with A, B arbitrary.*

Proof. Only the primes in the table below need checking. Let h_0 be the least $h \le 21$ satisfying $(H-1)g(h) \le h(h-1)$. Every word on this row has at least h_0 differences modulo sign, by the minimizing argument above. Here $g(h_0) = 4$. The initial seven positions lie in one coset of μ_s. Indeed, if split across cosets, at most $\binom{6}{2} = 15$ of their nonzero opposite differences have exact order s, by summing the binomial coefficients of the coset populations. All difference spectra in this initial comparison are exposed since $43 < s/2$. Averaging the overlaps over all units as above gives $4H \le 15^2 + 21^2/s < 252 \le 4H$. Translate the initial positions to $\mathbb{F}_s$. We can now minimize over full words supported there up to translation: every common nonunit obtained from them is again of this kind.

We describe the finite test explicitly. A forced mask D means that on the fixed row some nonunit has adjoint phases in $\{0\} \cup \{\pm d : d \in D\}$, with one entry per opposite pair. Use such a mask only if $2(1 + 2|D|) < s$. The starting mask consists of all differences of a candidate support, modulo sign. For a lower bound L on every true word's difference count, any two forced masks, including a mask with itself, give for each unit u a common nonunit with phases

in $T = \{0\} \cup \pm(D \cap uF)$. Consequently reject the seed if $|D \cap uF| < g(L)$ or $T - T$ has fewer than L nonzero opposite differences. Otherwise its nonzero differences form a further forced mask whenever exposed. This implication holds equally for masks that contain the true adjoint support strictly. It follows from Lemma 4.15, by exposing both adjoints and applying strictness to their common constituents.

At 127 only, first suppose that the true minimum is 17 or 18. The average nonidentity overlap of a minimizing word is $h(h-1)/62 \le 306/62 < 5$. Some intersection therefore contains exactly four opposite pairs. Enumerate all $\binom{62}{3} = 37820$ choices after scaling one pair to ± 1, and start the mask test from $T - T$ with $L = 17$. Exactly 1024 pass its comparison with its own dilates; none passes subsequent comparisons. Thus $L = 19$ is valid at 127. Elsewhere take $L = h_0$.

Pad the initial support to seven points. Among all ordered triples of distinct points in the padded support choose (a, b, c) minimizing the least positive residue of $(c-a)/(b-a)$; use a, b as the affine coordinates $0, 1$. The smallest remaining point is this minimum. Generate supports in increasing order, skipping any prefix with a smaller normalized third point on one of its triples. Also skip a prefix of length j with v existing difference pairs if $v + 21 - \binom{j}{2} < L$. Both omissions preserve at least one allowed representative of each possible support. Counts are

s	L	seven-point seeds	pass own dilates	survive all tests
127	19	7648354	181274	0
131	17	14348304	541163	0
137	17	17850526	284626	0
139	17	19150174	237257	0
149	18	23066734	6567	0
151	18	24674574	3705	0
157	19	24343632	5	0
163	19	29683165	0	0
167	19	33726047	0	0

The normalized-support enumeration gives the exact loops. It restricts forward comparisons to intersections with at most six pairs and at most 41 processed masks per seed. Reaching that limit counts as survival. Thus each zero comes from contradictions alone. The dimension statement follows as in Corollary 20.18: at $q = 19$ or 23 any prime r with $6q < r < 7q$ is at least 127. □

20.8 The remaining family with exponents six, six and four

Choose a braided counterexample of least dimension among the divisors of $p^6q^6r^4d$. All three primes must remain repeated by Dong's theorem. Lowering a sextic exponent gives the unordered box $(7, 5, 4)$ of Theorem 8.1; lowering the quartic exponent gives a cubic, covered by Theorem 4.12. No square-free prime survives: the table dimension Δ is square-full by Lemma 2.5; if a singleton two were removed in passing to a fermion quotient, its table would be odd, contrary to Proposition 4.7. The three exponents in the source are therefore exactly $6, 6, 4$. The quartic prime is the largest by Theorem 7.3. If the category is slightly degenerate then two occurs among the three primes, and is sextic. Theorem 21.3 excludes this possibility. The surviving category must be ordinary, integral, modular, perfect and fusion-simple.

For an even dimension the dyadic sextic theorem (Theorem 21.2) leaves only the odd pair $(3, 5)$, so the dimension is exactly 29160000. At this dimension 3 and 5 occur together in a twist order, and 2 occurs with at least one of them (Section 21.3 and Lemma 21.1). For odd primes the inequalities of Theorems 20.13 and 20.15 exclude whole families with two arbitrary companion exponents under the stated prime conditions. Some odd assignments still require separate twist configurations. For example, at $5^6 7^6 23^4$ both $5 \cdot 23$ and $7 \cdot 23$ must divide simple twist orders (Theorem 22.2); the witnesses can be different. These restrictions leave the unrestricted family open.

21 The dyadic sextic obstruction

21.1 Isolation at dyadic valuation six

Lemma 21.1. *Let an ordinary minimal table have precisely three dimension primes $2 < q < r$ and $v_2(D) = 6$. Some twist order is divisible by $2q$ or by $2r$.*

Proof. Suppose two is isolated, meaning that every label with nontrivial dyadic twist has trivial odd twist. There is at least one such label by categorical Cauchy. By Lemma 3.3, each has pure dyadic codegree, hence $D = 64n^2$ with n odd, and the active degrees are among $n, 2n, 4n$. (Codegree one would exhaust the norm at the unit.) Write P_2 for the principal two-block. By Proposition 6.2, applied to $\{2\}, \{q, r\}$, every degree of dyadic height less than three is in P_2. The integral column $u = 8W_2e_0$ then satisfies

$$\|u\|^2 = 64, \qquad v_2(u_X) = v_2(d_X) \quad (8 \nmid d_X).$$

A field ramified at r is unramified at two by the same proposition.

Every row in P_2 hits every odd degree. Their number, including the unit, is a positive multiple of four by the dimension equation modulo four. Every nonunit row also hits distinct even degrees omitting q and r, respectively, by Lemma 2.5. Indeed a nonunit with two primes missing would have prime-power degree. Each nonzero character entry costs at least one on average over its Galois conjugates. These observations already exclude an active row of codegree four.

We record an orbit estimate. If an r-free label X has degree $2q^j$, cross it with a q-free witness of degree $2^h r^l$ in P_r (Lemma 4.10). The lcm-normalized entry has weight two, strict house less than two and residue two at every prime above r. By Theorem 4.9 it has a unique unsigned expression as a sum of two r-power roots, since $r \geq 5$. The roots are distinct by strictness. The tame stabilizer acts freely on the nonidentity roots, so its order is at most two. Thus the r-inertia orbit of X has length $t \geq (r-1)/2$.

Suppose first that there is an active Z of codegree sixteen, so $d_Z = 2n$. Every supported X lies in P_2 by Theorem 2.9. If $i = v_2(d_X) \geq 1$, then $b = \chi_Z(X)/2^{i-1}$ is integral by lcm divisibility, and has zero residue at two. Indeed it is a dyadic unit times $\chi_X(Z)$, which has the residue of $2n$. Each such nonzero orbit of the entries $\chi_Z(X)$ costs at least 4^i by Theorem 5.3; the assertion about all primes above two follows by Galois covariance within P_2. The vacuum therefore excludes $i \geq 2$. At an r-free even hit X the preceding orbit estimate now applies. Its field ramifies at r and is unramified at two, whereas the field of Z is contained in a dyadic cyclotomic field (Corollary 2.8). Thus $\chi_Z(X)$ is a nonzero even rational integer, equal on its r-inertia orbit. The t columns cost at least $4t$. The other even prime-omitting orbit costs at least four. Including the odd degrees gives

$$16 \geq 4 + 4t + 4 \geq 4 + 2(r-1) + 4.$$

Consequently $r = 5$, $q = 3$, and equality holds throughout. There are precisely four odd labels, one the unit, and every even hit of this row omits three or five. This applies to every active codegree-sixteen row.

If any odd label is dyadically active, let 2^s be the greatest order of its dyadic twist among odd labels, and let L count odd labels of that order. All such labels have degree n. Choose m which is 2^{s-1} on the dyadic part of the twist level and zero on the odd part. In the dyadic cyclotomic field

$$n^{-2} \sum_X d_X^2 \theta_X^m = 64 - 2L + 4 \sum_{Y \text{ active},\ d_Y = 2n} (\theta_Y^m - 1).$$

The regular-object indicator $D^{-1} |\sum_X d_X^2 \theta_X^m|^2$ is integral over $\mathbb{Z}$ by (3.2). Complex conjugation preserves the unique prime above two; with $v(2) = 1$ the displayed number thus has valuation at least three. Every summand in parentheses has positive valuation or is zero. It follows that $v(2L) > 2$, hence $4 \mid L$. There are only three odd nonunit labels, a contradiction.

All active labels now have degree $2n$, so the saturation of the row norm just proved gives $S_{ZZ'} = 0$ for any two of them. Let A be their coordinate span. The odd local factors act trivially, up to scalars, on A by isolation (as in the proof of Proposition 6.2). Thus $W_2 A = sA \perp A$. The actual dyadic twist is the identity on $A^{\perp}$, and its W_2-conjugate is the identity on $(W_2 A)^{\perp}$. In particular both preserve A. The corresponding upper and lower unipotents differ from these operators by scalars and generate the dyadic group. This would make $W_2 A = A$, contradicting orthogonality.

It remains that every active degree is n. Let N count them. The same highest-order indicator calculation, now without its last sum, gives $N \geq 4$. Take an r-free even $X \in P_2$ by Proposition 4.2. Its degree $2^i q^j$ has $i = 1$ or 2 by Corollary 4.1. It hits all N active labels. On each of them the r-inertia conjugates of the X-row agree, since all active fields have dyadic conductor. Averaging those t orthogonal S-rows gives squared norm D/t. On the active columns the lcm scale is $2^i n$; averaging also over all Galois automorphisms gives squared mass at least $N 4^i n^2$, in addition to the vacuum. Thus $4^i t N < 64$. First $i = 1$, then $t \geq (r-1)/2 \geq 2$ by the orbit estimate, giving $N \leq 7$. By Lemma 6.1 the modulus of the vacuum coordinate of u is

$$|u_0| = \tfrac{1}{8}\Big|64 - \sum_{Z \text{ active}} (1 - \theta_{Z,2})\Big| \geq (64 - 2N)/8 > 6.$$

Hence $u_0^2 \geq 49$. The N odd active coordinates cost at least four in total. The $t \geq 2$ even r-free conjugates cost at least eight, and a q-free even member of P_2 costs at least four, by its height one or two. These disjoint contributions exceed 64. □

Theorem 21.2. *Let $q < r$ be odd primes other than the pair $(3, 5)$, and let e be square-free and coprime to $2qr$. Every weakly integral braided fusion category of dimension dividing $2^6 q^a r^b e$ is weakly group-theoretical, for all $a, b \geq 0$. At $(3, 5)$ an ordinary minimal table of dyadic valuation six requires a mixed dyadic twist and a complementary principal crossing of weight four.*

Proof. It suffices to consider three-prime minimal tables. Pair an r-free nonunit of P_q with a q-free nonunit of P_r. Their degrees are $2^i q^j, 2^h r^l$, and $g = 2^{\min(i,h)} \geq q$ by Lemma 4.10. Square divisibility gives $g \leq 8$ in ordinary data and $g \leq 4$ in the positive fermion quotient. If $g \leq 4$, then $g = 4$, $q = 3$. For $r \geq 7$ the two principal residues and Theorem 4.9 give $1 + 3\eta$, with η a nontrivial r-power root. The two positions have at most three differences, so Lemma 4.15 excludes the crossing: its single nonzero opposite pair is moved by a tame unit and $3 < r/2$. Thus for $(q, r) \neq (3, 5)$ only ordinary data remain and every such pair has $g = 8$. Each witness then has odd codegree. A q-active column hits an r-free nonunit of P_q by Proposition 4.2, so cannot be dyadically active by (3.1). Interchanging q, r isolates two, contrary to Lemma 21.1; the presence of height three forces $v_2(D) = 6$. The same argument shows that in the ordinary $(3, 5)$ case at valuation six at least one crossing has weight four. □

21.2 Eliminating the slightly degenerate dyadic sextic

Theorem 21.3. *A divisor-minimal slightly degenerate obstruction cannot have dimension $2^6 q^b r^c$ for distinct odd primes $q < r$ and arbitrary b, c. Consequently the slightly degenerate branch of the unordered $(6, 6, 4)$ divisor family is empty.*

Proof. The positive fermion quotient has dimension $\Delta = 2^5 q^b r^c$. Take an r-free nonunit of the principal q-block P_q and a q-free nonunit of P_r. The gcd of their degrees is a power of two at least q by Lemma 4.10 and at most four by square divisibility in the positive quotient. Thus $q = 3$ and the gcd is four. If $r > 5$, Theorem 4.9 gives $1 + 3\xi$ at the normalized crossing, where ξ has nontrivial r-power order. Theorem 19.5 excludes the resulting two-position character. Hence $r = 5$. All complementary witnesses have degree $4 \cdot 3^i$ or $4 \cdot 5^j$ and positive dyadic defect one.

Take the integral minimal modular extension of Lemma 4.4, with fermion g and matrices $S, s = S/\sqrt{4\Delta}$. Put $\chi_X(Y) = S_{XY}/d_X$. Even and odd refer to monodromy with g. Let A, B be

the symmetric and antisymmetric even spaces under tensoring by g, and C, D the corresponding odd spaces. The full Weyl interchanges this permutation and the monodromy sign, while odd local factors commute with both (Section 3). Thus the dyadic Weyl W preserves A, D and interchanges B, C. The dyadic upper unipotent $U = \lambda T_2$ interchanges A, B and preserves C, D since even mates have opposite twists and odd mates equal twists. Write $K = W|_A$, W_o for the product of odd Weyls and $\widehat{s} = s|_A = KW_o|_A$. The genuine congruence representation of Section 3 has $\lambda^8 = 1$ and trivial odd normalizing scalars. The dyadic group preserves full two-blocks and K preserves positive two-blocks by lifting their projectors (Lemmas 3.2 and 7.13).

Positive twists. The first assertion of Lemma 7.13 applies to each complementary witness, since $v_2(4\Delta) = 7$. Thus its row sees only the two squared dyadic phases $+1, -1$. A pair of any other squared phase cannot be active at three or five, as Proposition 4.2 modulo the opposite prime would force a complementary hit. On each such extra phase fiber the odd local groups act trivially: they preserve the fiber, their upper unipotents act identically there, and conjugate unipotents generate. Their Weyls in particular fix both full lines of its even pairs (the actual dyadic twists are opposite). By Lemma 3.3 such a pair V has pure dyadic positive codegree; hence both b, c are even. Set $n = 3^{b/2}5^{c/2}$ in this case. The column $Ke_V = \widehat{s}e_V$ has nonzero vacuum coefficient, so $V \in P_2$. It vanishes modulo two on even degrees. The estimate $f_V \geq [\mathbb{Q}(\chi_V) : \mathbb{Q}] + 8$ of Corollary 5.6 leaves only $f_V = 32$ among $32, 8, 2$, giving $d_V = n$.

Dimension orthogonality on such a row modulo five gives a five-free nonunit hit X, necessarily of degree $2 \cdot 3^h$ since $X \in P_2$ by Theorem 2.9, Corollary 4.1, Lemma 2.5. Crossing with a complementary $4 \cdot 5^j$ witness gives a weight-two entry of strict house below two and residue two at five. Its unique unsigned five-primary word is a nonscalar sum of two roots (Theorem 4.9), so the tame stabilizer has order at most two. The five-inertia orbit of X has size $t \geq 2$. Every extra pair, being of degree n odd, is hit by every member of this orbit by principality. All extra pairs are five-unramified. The averaged norm of Lemma 2.4, retaining the vacuum and all L extra pairs and using $\operatorname{lcm}(n, d_X) = 2n$, gives $4tL < 32$.

Thus $L \leq 3$. If these phases occurred, take the nonempty set H of pairs whose squared phase has highest order 2^m, $m \geq 2$. The operator $J = (T_2^2)^{2^{m-1}}$ on A is $1 - 2P_H$ with P_H their coordinate projection. The trace of $JKJ^{-1}K^{-1}$ is an algebraic integer since the scalar of the powered unipotent cancels and the resulting group element has finite order. Its expansion is

$$\dim A - 4|H| + \tfrac{1}{8} \sum_{V,Y \in H} |\chi_V(Y)|^2,$$

using $Ke_V = \widehat{s}e_V$. The sum is rational by Galois covariance and stability of H, hence belongs to $8\mathbb{Z}$. Its terms are congruent to one above two by principality; thus $|H| = 2$. If both labels are individually fixed, the four entries are odd-integer squares. Otherwise their fields are the same quadratic dyadic field. The squared moduli form a matrix with diagonal y and off-diagonal σy, where σ interchanges the labels: use symmetry and Galois covariance in either argument. For rational y the sum is four modulo eight by oddness. Otherwise y belongs to the real quadratic dyadic field and is $a + b'\sqrt{2}$ with a odd; the sum is $4a$, again impossible. Hence all even dyadic twists have order dividing four.

Lemma 7.13 now gives full dyadic level dividing eight and two equal-degree positive labels of opposite squared twists in each complementary block. It also gives Gaussian integer values on even objects for every purely dyadic odd character.

Odd spins. Put $Z = T_2^2|_A$, $F = KZK^{-1}$. These involutions anticommute, and U^4 is scalar on C, by Lemma 7.14. Over each complementary block the two odd orbits are free, and their actual dyadic twists have ratio $\pm i$. Their degrees satisfy $u^2 + v^2 = 2d^2$ and $v_2(u) = v_2(v) = 2$. For $d = 4 \cdot 3^i$, descent modulo three gives $u = v = d$; for $d = 4 \cdot 5^j$, reduction modulo three gives $3 \nmid uv$.

At its odd prime p each complementary witness has a full principal lift. Indeed, if Y is an odd label of p-unit degree, its square is even, so its character value is congruent to $\pm d_Y$. Tensoring

the witness by g chooses the positive sign. For every odd V, evaluating the even product YV then gives the dimension residue on V. Such a Y is supplied by the opposite odd mates just found. This lift hits both odd mates of every opposite block by principality. As W preserves its full two-block and W_o preserves odd dyadic spin fibers, its row sees only the two odd spins of its own block. Thus the same pair $\delta, i\delta$ occurs in every complementary block. At each odd prime the full principal block has only even labels (evaluate at g) and excludes g (evaluate at an odd mate of unit degree). Therefore every odd label active at three or five hits a complementary witness by Proposition 4.2 modulo the other prime. All such labels have one of these two spins.

Some other odd spin occurs, since the two-spin conclusion of Lemma 7.14 would contradict perfectness of R_+.

By scalar U^4 all other odd orbits have spin $-\delta$ or $-i\delta$. Call their set E. The odd local groups fix their full lines, by the same fiber argument as above, so their full codegrees are dyadic by Lemma 3.3. This forces b, c even. Write $n = 3^{b/2}5^{c/2}$. For $V \in E$, $We_V = se_V$ has nonzero vacuum coefficient, hence V is full two-principal. Its degree is $2^j n$, $j = 0, 1, 2$, by Corollary 4.1 at valuation seven.

The signed norm. At least eight odd-degree positive pairs lie in P_2. Its rank is even since Z, F restrict to anticommuting involutions. Its sum of squared degrees is divisible by 32 by Theorem 2.9 on the unit row. Write k_0, k_1 for the numbers of odd and even degrees. The even degrees have height one by Corollary 4.1. Modulo four, $4 \mid k_0$, so k_1 is even; modulo eight $k_0 + 4k_1 \equiv 0$. The vacuum gives $k_0 \geq 8$. Each of these k_0 values at $V \in E$ is nonzero by principality and has squared modulus at least one since it is Gaussian.

Compare the signed character of V with that of an odd mate of a $4 \cdot 5^j$ witness. These rows differ, since they have different spins. In their restricted orthogonality (Lemma 4.5), the vacuum survives modulo three and all three-bearing degrees contribute zero by lcm-integrality on the three-free mate. This gives a nonunit three-free hit X_3. A $4 \cdot 3^i$ mate gives a five-free hit X_5. Both belong to the full and positive principal two-block by Theorem 2.9, Lemma 4.5, with degrees $2 \cdot 5^l$ and $2 \cdot 3^h$, respectively, $l, h > 0$. Each $\chi_V(X)$ is a nonzero Gaussian integer vanishing above two. When its squared modulus is at least four, its Galois orbit costs at least four per pair. When it is two, $\chi_V(X)^2 = \pm 2i$ and $\chi_X(V^2) = (d_V/d_X)^2 \chi_V(X)^2$. Since V^2 is even, i belongs to the positive character field of $[X]$. Five-inertia fixes i, so another coset of its orbit is disjoint from the first. Every conjugate there also has signed absolute square two, by Galois covariance, using full lifts and ignoring the fermion sign. The five-inertia orbit of $[X_5]$ has size at least two by the weight-two crossing above; for $[X_3]$ use the trivial lower bound one. The disjoint supports cost at least eight and four, respectively. Thus the signed codegree is at least $8 + 8 + 4 = 20$. By (4.1) it equals $32/4^j$ for a free orbit and $64/4^j$ for a fixed one. Hence $j = 0$. A fixed label would have even d_V^2, since VV^* would be g-invariant and all even simples occur in free pairs. Every $V \in E$ is therefore free of odd degree n.

Strict multiplicity. Take $X = X_5$ as above. Every $V \in E$ hits X, since X is two-principal and d_V is odd. Lcm-integrality makes $\chi_V(X)$ a nonzero multiple of two in $\mathbb{Z}[i]$, of squared modulus at least four. The five-inertia orbit of $[X]$ has size $t \geq 2$. Its full orbit also has size t: the entries at the V are fixed by five-inertia by symmetry, equal degrees along the orbit, and pure dyadic codegree at V, whereas passage to gX negates them. All chosen representatives have the same dyadic twist.

Put $a_X = (e_X + e_{gX})/\sqrt{2}$, $b_X = (e_X - e_{gX})/\sqrt{2}$, and similarly c_V, d_V^- for symmetric and antisymmetric odd pair vectors. For $v = \sum_X b_X$ over the t representatives, $\|v\|^2 = t$. The coefficient of sb_X at c_V is $2S_{XV}/(\sqrt{128}\, n)$, independent of the representative and of squared modulus $|\chi_V(X)|^2/32 \geq 1/8$. Hence $t|E| \leq 8$. If $|E| = 4$, then $t = 2$ and equality gives $sv \in \langle c_V : V \in E \rangle$. The odd Weyls fix this space pointwise and commute with s, so $W_o v = v$. Applying U and using equal dyadic twist of the representatives gives $W_o a = a$ for $a = \sum_X a_X$. Thus $sa = Ka$ is supported on the positive P_2. A $4 \cdot 5^j$ witness Y lies outside P_2 by the dyadic height bound. Yet the Y-coordinate of sa is a nonzero common factor times $\sum_X \chi_Y(X) \equiv 2d_X \neq 0$

modulo a prime over five by five-principality. We have proved $1 \le |E| \le 3$.

Trace on the odd antisymmetric space. On D put $x = \lambda\delta$, $T = U/x$. Its eigenvalues lie in $\{1, i, -1, -i\}$. For the projection P_E onto $\langle d_V^- : V \in E\rangle$ one has

$$J = 1 - 2P_E = \tfrac{1}{2}\big((1-i)T + (1+i)T^{-1}\big).$$

We claim $\mathrm{Tr}_D(JWJW^{-1}) \in \mathbb{Z}$. Unitarity and cyclicity give

$$\mathrm{Tr}_D(JWJW^{-1}) = \Im\,\mathrm{Tr}_D(TWTW^{-1}) + \Re\,\mathrm{Tr}_D(TWT^{-1}W^{-1}).$$

The corresponding words in U, W in $\mathrm{SL}_2(\mathbb{Z}/8)$ have matrices

$$h = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \qquad k = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Both have order dividing six: $h^3 = -I$ and $k^3 = 5I$ modulo eight. Both are conjugate to their inverses in this group: the Weyl inverts k, and the coordinate swap inverts h while its product with $I + 2h$ has determinant one modulo eight. Their traces on D are consequently real algebraic integers in $\mathbb{Q}(\zeta_6)$, hence integers. Replacing U by T multiplies the first trace by $x^{-2} \in \mu_4$ and leaves the second unchanged, proving the claim. Expanding the projector gives

$$\mathrm{Tr}_D(JWJW^{-1}) = \dim D - 4|E| + 4 \sum_{V,Y \in E} |W_D(V,Y)|^2.$$

The odd Weyls fix each of these columns, so $W_D(V,Y)$ is the full s-entry in antisymmetric pair bases, $2S_{VY}/(\sqrt{128}\,n)$. It follows that

$$\sum_{V,Y \in E} |\chi_V(Y)|^2 \in 8\mathbb{Z}. \tag{21.1}$$

We prove an orbit congruence valid for any size of this packet. Its entries $\chi_V(Y)$ are dyadic units by principality and belong to $L = \mathbb{Q}(\zeta_8)$ by the pure codegrees and the level. The group $G = \mathrm{Gal}(L/\mathbb{Q})$ acts on the pairs: odd inertia fixes each full character, and the Galois square fixes its dyadic spin at level eight. Equal degrees and symmetry imply that conjugation in either index conjugates the entry, up to the sign from changing a fermion representative. Thus $y_{VY} = |\chi_V(Y)|^2$ is well defined on pairs and satisfies $y_{\sigma V, \tau Y} = \sigma\tau(y_{VY})$.

Partition E into G-orbits of sizes $n_i \in \{1, 2, 4\}$. On a rectangle $O_i \times O_j$ the sum of the y_{VY} is $n_i n_j m_{ij}$, where m_{ij} is their normalized absolute trace. Indeed averaging independently over two group elements averages their product uniformly over G. For an entry written $\alpha = \sum_{k=0}^{3} a_k \zeta_8^k$ with $a_k \in \mathbb{Z}$, the normalized trace of $|\alpha|^2$ is $\sum_k a_k^2$. It is odd since α is a dyadic unit and ζ_8 reduces to one. When either orbit is a singleton, conjugating that index acts on α only by sign. The rational character lines in L are $\mathbb{Q}, i\mathbb{Q}, \sqrt{2}\mathbb{Q}, \sqrt{-2}\mathbb{Q}$; an integral dyadic unit on these lines is an odd integer times one or i. Then $m_{ij} \equiv 1 \bmod 8$. In every other diagonal rectangle $4 \mid n_i^2$, so oddness of m_{ii} gives $n_i^2 m_{ii} \equiv n_i^2 \bmod 8$. The remaining off-diagonal rectangles occur in transpose pairs of equal sum with $8 \mid 2n_i n_j$. Altogether

$$\sum_{V,Y \in E} |\chi_V(Y)|^2 \equiv \Big(\sum_i n_i\Big)^2 = |E|^2 \pmod 8.$$

Consequently (21.1) forces $4 \mid |E|$, contradicting $1 \le |E| \le 3$.

For the unordered box, an odd-dimensional obstruction is ordinary. When the dyadic valuation is at most five, Theorem 8.3 covers the reduced source, with the lower faces in Section 4. Every remaining even source has dyadic valuation six, just excluded. □

21.3 Restrictions on an ordinary dyadic obstruction

Write $d_i = \dim i$, $f_i = D/d_i^2$, $\chi_i(j) = S_{ij}/d_i$, P_u for the full principal u-block of ordinary integral modular data, and K_i for the cyclotomic field generated by χ_i. A label is u-active when its twist has nontrivial u-part. We work with a perfect fusion-simple category and the positive spherical structure.

Trace transfer. Let s, t be distinct primes. Choose $E = P_t$ or its intersection with the fiber of trivial prime-to-t twist. Suppose $X \in P_s$ is t-inactive, H is a finite group of cyclotomic automorphisms preserving its t-inactivity, and g is a positive integer. For each s-free nonunit $Y \in E$, suppose that $\gcd(d_X, d_Y) = g$ and that

$$\sum_{\sigma \in H} \sigma(S_{XY}/\operatorname{lcm}(d_X, d_Y)) = \tau$$

for a common rational integer τ. If $v_s(g|H| - \tau) = 1$, every t-active J has $s \mid d_J$. More precisely, at every chosen prime over s there is an s-bearing $Y \in E$ with $v_s(\chi_J(Y)) < 1$, normalizing $v_s(s) = 1$. Consequently $1 \le v_s(d_Y) \le v_s(d_J)$. If J also lies in P_s, then K_X and K_J both ramify at s.

Indeed set $F(Y) = g\sum_{\sigma \in H} \chi_{X^\sigma}(Y) - \tau d_Y$. The characters transform by ordinary Galois action, with $d_{X^\sigma} = d_X$. Then $F(1) = g|H| - \tau$ and $F(Y) = 0$ at every s-free nonunit of E. Principality gives $F(Y) \equiv (g|H| - \tau)d_Y \equiv 0$ at every prime over s. Indeed $\sigma\chi_X(Y) = \chi_X(Y^\sigma)$ by symmetry, Galois covariance and invariance of integral dimensions. Thus the defining principal congruences are stable at a fixed prime under every cyclotomic automorphism (see also the proof of Theorem 2.9). This also gives the congruence at every conjugate prime. Restricted orthogonality gives

$$0 = \sum_{Y \in E} \chi_J(Y)\overline{F(Y)}.$$

For the full block this is Theorem 3.1. For its indicated fiber, both projectors commute with the t-local group and their product still does; conjugation by the global Weyl yields an operator commuting with the t-twist, exactly as in Proposition 4.2. The vacuum term has valuation one. Therefore, at some s-bearing Y, the product has valuation at most one. Since $\overline{F(Y)}$ has positive valuation, this forces the first assertion. The lcm divisibility of S_{JY} implies the height inequality. If $J \in P_s$, its character also has positive valuation at such Y. Unramifiedness of either field would make that factor divisible by s, since $\overline{F(Y)}$ lies in the abelian field K_X. This is impossible.

The ordinary sixth dyadic valuation. Assume $D = 2^6 3^b 5^c$. If two mixes with three, every five-active label has degree divisible by three. If two mixes with five, every three-active label has degree divisible by five. More precisely the preceding transfer applies with $(s, t) = (3, 5)$ and $(5, 3)$, respectively. For a column J mixing two and s, apply Proposition 4.2 at s, inside its trivial complementary fiber and avoiding t. It produces a nonunit t-free $X \in P_s$ with $S_{XJ} \neq 0$ and $\theta_{X,t} = 1$. By Lemmas 3.3 and 4.10, $v_2(d_X) = 2$: it is at least two, and at height three the row misses J. Every s-free nonunit $Y \in P_t$ has dyadic height at least two by the same height bound, hence the gcd is four. For the normalized crossing, principality gives residue four above three and five and strict house less than four. Theorem 4.9 leaves $1 + 3\xi$, for a nonidentity five-power root, or $-\sum_{j=1}^{4} \eta_j \zeta_5^j$, in which the η_j are one copy of one plus one coset of the cube roots of unity. In the first case ξ has order five by the two-position relative-order result (Theorem 19.5), applied to the full-mass expression $(d_Y/4)(1 + 3\xi)$ for $\chi_X(Y)$. Take H to be the order-four tame subgroup of five-inertia in a common cyclotomic level, fixing the other primary factors. In the first case its trace is $4 - 3 = 1$; in the second its trace is $\sum_j \eta_j = 1$. Thus $g|H| - \tau = 15$ in both orientations, using *the same* five-inertia group in each.

A spectrally pure product. If all constituents of XY have the same dyadic twist δ and no dyadically active row sees both X and Y, then $\delta = 1$. In particular the conclusion holds when either factor has odd codegree. Indeed the vector $v = [XY]$, written in the fusion basis, is an eigenvector of the dyadic upper unipotent with eigenvalue $a\delta$, where a is its common normalization scalar. On the other hand, $(Sv)_J = d_J\chi_J(X)\chi_J(Y)$ for the unnormalized S. Its support has trivial dyadic twist, so the conjugate generator $S^{-1}T_2S$ has eigenvalue a on v. These two conjugate unipotents generate the dyadic modular group, so that their actions on the common eigenline coincide. The odd-codegree specialization is Lemma 3.3. The two ordinary weight-four phase profiles therefore have the following consequence. When both factors belong to their trivial complementary fibers and one has full dyadic height, the negative profile is excluded. Without the full-height hypothesis that profile forces a dyadically active row meeting both factors.

Complementary height under separation. Assume further $c = 4$ and that three and five never mix. Since an isolated quartic prime is excluded (Lemma 7.2), two mixes with five. Choose the resulting $Y \in P_5$ of dyadic height two and three-free degree as above. For every five-free nonunit $X \in P_3$ the gcd is four. All five-active labels have full three-height by Lemma 3.3; thus b is even, Y and X are five-inactive (Lemma 4.10 bounds X below full three-height). The negative profile is impossible: at most one of its four distinct five-parts permits a five-inactive constituent, and at most two of its four three-parts permit a three-inactive constituent. The positive profile puts total dimension $3\,\mathrm{lcm}(d_X, d_Y)$ on its nontrivial five-phase. It follows that $3^{b/2} \mid 3\,\mathrm{lcm}(d_X, d_Y)$ and hence $v_3(d_X) = b/2 - 1$, again by Lemma 4.10. In particular the height is exactly two at $(6, 6, 4)$.

Lemma 21.4 (Separated twists and inertia orbits)**.** *Let an ordinary perfect fusion-simple integral modular table have square dimension n^2, where $n = As^e r^c$, $e, c \geq 1$, s, r are distinct primes, s is odd, and $\gcd(A, sr) = 1$. Suppose s and r never divide the same twist order. Each s-active Z has degree $zs^h r^c$, with $z \mid A$ and $h < e$, by Lemma 3.3 and defect-zero vanishing. Suppose each such Z has a nonzero entry at a principal row $X \in P_s$ satisfying*

$$d_X = xs^{e-1}, \quad x \mid A, \quad x \geq M, \qquad K_X \text{ unramified at } s, \qquad |I_r X| \geq m,$$

where I_r is cyclotomic inertia at r. If every s-active Z with $h > 0$ has at least b Galois conjugates and $mbM^2 \geq A^2$, then $h = 0$ for all s-active labels and

$$\sum_{Z \text{ active at } s} z^2 < A^2 s^2/m, \qquad m\,\mathrm{rad}(A)s^e < 2As^2.$$

Proof. Since $r \nmid f_Z$, inertia at r fixes K_Z by Corollary 2.8. Average m distinct I_r-conjugate unnormalized S-rows of X. Their average has squared norm n^2/m by orthogonality. At every conjugate Z' of Z its entry is $S_{XZ'}$, since the m entries coincide there. The vacuum contributes positively. At a nonzero entry divisible by an integer L in the algebraic integers, the mean squared modulus over conjugations is at least L^2, by the norm. These estimates still hold after conjugation since the Galois action permutes the rows and preserves degrees.

If $h > 0$, principality makes $\chi_X(Z)$ vanish at every prime above s; unramifiedness gives $s \mid \chi_X(Z)$. This and lcm divisibility make S_{XZ} a nonzero algebraic-integer multiple of $\mathrm{lcm}(x, z)s^e r^c$. Counting the Z-orbit and the vacuum therefore gives $mbM^2 < A^2$, a contradiction. Now fix one of the rows X. Since every s-active degree is prime to s, principality makes X meet all these columns. At each of them S_{XZ} is divisible by $zs^{e-1}r^c$. Counting all of them in the same averaged row gives the first bound.

In Lemma 6.1 select the s-part, killing the dyadic anomaly. Its vacuum entry gives the rational integer

$$L = \sum_{Z \text{ active at } s} z^2(1 - \theta_{Z,s}) = A^2 s^{2e}(1 - (W_s)_{00}).$$

It is positive since s is active by categorical Cauchy and each nontrivial phase has real part below one. Thus $L < 2A^2s^2/m$. At the primes of A the same lemma gives $n(1-(W_s)_{00}) \in \mathrm{rad}(A)\mathbb{Z}$. Consequently $\mathrm{rad}(A)As^e \mid r^cL$, hence divides L, proving the second bound. □

Ruling out separation of three and five. Suppose three and five are separated at $D = 2^6 3^{2e} 5^4$, $e \geq 2$. For every three-active Z, Proposition 4.2 gives a five-free nonunit $X \in P_3$ hitting it. By the preceding complementary-height calculation $d_X = 2^i 3^{e-1}$ with $i = 2, 3$. Its weight-four crossing with the chosen degree-20 member of P_5 puts $\mathbb{Q}(\zeta_5)$ in K_X; thus $|I_5 X| \geq 4$ and K_X is unramified at three by Proposition 6.2. Lemma 21.4 with $(A, s, r, M, m, b) = (8, 3, 5, 4, 4, 1)$ gives $3^e < 18$, impossible for $e \geq 3$, and $\sum_Z 2^{2k} < 144$ for the three-active degrees $25 \cdot 2^k$.

If three is isolated, Lemma 3.3 gives $k = 3$ on every three-active label. The preceding sum allows at most two such labels. Proposition 6.2, applied to the partition $\{3\}, \{2, 5\}$, gives $3^e(W_3)_{00} \in \mathbb{Z}$. With $k = 3$ the vacuum identity becomes

$$3^{2e}(1-(W_3)_{00}) = \sum_{Z \text{ three-active}} (1-\theta_{Z,3}).$$

Its right-hand side is positive and at most four, whereas its left-hand side is a multiple of 3^e. This excludes isolation also for $e = 2$.

When two and five are separated. Assume $e \geq 3$ and that two and five never mix. Two must mix with three by the exclusion of an isolated dyadic prime (Lemma 21.1). Choose a five-free dyadic-height-two witness $X \in P_3$ for a column mixing two and three. Its twist has trivial five-part by the choice of complementary fiber. For any three-free nonunit $Y \in P_5$, five is inactive by trace transfer, and $v_5(d_Y) = 1$ by the principal bound. Neither factor is dyadically active, as a dyadically active label has full five-height by Lemma 3.3. The negative crossing profile would place all constituents in the nontrivial dyadic twist -1 while some have nontrivial five-part, contrary to separation. The unsigned profile has a nontrivial five-phase of total dimension $3\,\mathrm{lcm}(d_X, d_Y)$. Each constituent at this phase has full dyadic height. Since $v_2(d_X) = 2$, it follows that every such Y has $v_2(d_Y) = 3$ and hence $d_Y = 40$.

21.4 Principal witnesses and the first even core

A lower bound in the principal block at the larger prime. Let a perfect fusion-simple ordinary integral table have just three dimension primes p, q, r, with $q < r$ odd. Every p-free nonunit $Y \in P_r$ has $q^2 \mid d_Y$. Indeed, Lemma 4.10 gives $d_Y = q^j r^h$ with $j, h \geq 1$ and supplies an r-free nonunit $X \in P_q$ of degree $p^a q^k$, $a, k \geq 1$. If $j = 1$, put $z = S_{XY}/\mathrm{lcm}(d_X, d_Y)$. Then $\chi_X(Y) = r^h z$ and $\chi_Y(X) = p^a q^{k-1} z$. The principal congruences give $z \equiv q$ at every prime over q and r, respectively. Balancing gives weight at most q, while fusion-simplicity and perfectness give strict house below q (see the proof of Lemma 4.10). This contradicts Theorem 4.9 at $g = q < r$.

Suppose now $v_q(D) = 6$, $v_r(D) = 4$ and a label J is active at both q and r. At each prime over p, Proposition 4.2, applied to J in the principal r-block and its fiber of trivial prime-to-r twist, gives a nonunit Y with $d_Y \chi_J(Y)$ a unit there. Thus $p \nmid d_Y$ and $S_{YJ} \neq 0$. Its degree is $q^j r$ by Lemma 4.10. If $j = 3$, square divisibility gives $q \nmid f_Y$, so (3.1) in Theorem 3.1 forces $S_{YJ} = 0$ because J is q-active. Hence $j \leq 2$, and the preceding lower bound gives $d_Y = q^2 r$. In particular, at the ordinary sextic core each jointly three- and five-active column is met in the indicated principal five-block fiber by an odd label of degree 45, with a unit character value at the chosen prime over two.

The same choice may be made with K_Y at most tamely ramified at p, hence unramified when $p = 2$. More generally this sharpening holds in both cases of Proposition 4.2 for ordinary integral data. In its restricted orthogonality or slice identity group the summands by the action of wild p-inertia in a common cyclotomic splitting field. This p-group preserves the principal block (Galois permutation of the argument in the defining congruences), the trivial complementary

fiber, and the orders of twists. Within each orbit the dimensions agree and $\chi_J(Y^\sigma) = \sigma\chi_J(Y)$, so all the summands have the same residue at the chosen prime. Every nonsingleton orbit contributes zero there. The vacuum term is a unit, so at least one nonunit singleton contributes a unit. Its character field is fixed by wild p-inertia.

One can impose an additional field condition on the same hit. Namely, put $F = \mathbb{Q}(\chi_J(Y))$. Grouping the same identity instead by full Galois orbits gives orbit contributions $c_Y d_Y[K_Y : F]\,\mathrm{Tr}_{F/\mathbb{Q}}(\chi_J(Y))$, with c_Y the integer slice coefficient (one in orthogonality). A nonzero contribution modulo p requires both factors $[K_Y : F]$ and $\mathrm{Tr}_{F/\mathbb{Q}}(\chi_J(Y))$ to be p-units. In particular the maximal subextension of $K_Y/\mathbb{Q}$ of p-power degree lies in $F \subset K_J$. The same orbit is tame at p: if its wild inertia were nontrivial, decomposing this orbit into wild inertia orbits as above would make its contribution zero. Select a conjugate in this orbit where the character value has nonzero residue at the specified prime.

Apply this refinement at the avoided prime two to a three-active J in the ordinary core $D = 2^6 3^6 5^4$, now using P_3. The selected odd Y has degree $3^j 5^h$, $1 \le j, h \le 2$. If $h = 1$, cross it with any three-free nonunit in P_5, of degree $2^i 5$ by Lemma 4.10. The weight-five result above gives tame five-inertia of order divisible by four on K_Y. Because $[K_Y : F]$ is odd and $F \subset K_J$, its order-four tame inertia image survives in K_J. If J is also five-active, $h = 2$ is impossible by defect-zero vanishing, so every jointly three- and five-active J has the full quartic tame five-inertia image. In particular its orbit under five-inertia has at least $4 \cdot 5^{a-1}$ labels when the five-part of its twist has order 5^a: twist covariance already makes wild inertia have at least 5^{a-1} images on that orbit. Conversely, if J is three-active and lacks this quartic field inertia, the selected odd P_3 witness necessarily has degree 75 or 225. This applies, for example, to every jointly two- and three-active J when two and five are separated, since then $5 \nmid f_J$.

Lemma 21.5 (A common hit in a principal block)**.** *Let B be a principal t-block in integral ordinary modular data with a unique invertible label, and let $p \neq t$ be prime. Suppose J, K have distinct t-twists and that for each nonunit p-free $Y \in B$ at least one of S_{JY}, S_{KY} vanishes. Then p divides both d_J and d_K, and neither J nor K is p-principal. In fact there is a common hit $Y \in B$ with $p \mid d_Y$, $v_p(d_Y) \le \min(v_p(d_J), v_p(d_K))$, at which both characters are units at the chosen prime above p (with conjugation on the second).*

Proof. Indeed the restricted orthogonality sum $\sum_{Y \in B} \chi_J(Y)\overline{\chi_K(Y)} = 0$ has vacuum term one. After reduction some p-bearing label must make a nonzero contribution. The height bound follows from lcm integrality. If either row were p-principal, its value on every p-bearing degree would vanish at the reduction prime. Principality holds above every prime over p by the Galois action on the argument; this also treats conjugation. □

For $D = 2^6 3^{2e} 5^4$, $e \ge 3$, suppose that two and five are separated. Take a two-active J and a five-active K. Then J is five-inactive. Every three-free nonunit of P_5 has degree 40 by the degree-forty argument in Section 21.3. It has odd codegree, and its row vanishes at J. The preceding transfer gives $3 \mid d_J, d_K$ and $J, K \notin P_3$. Thus every three-free degree and also every member of the principal three-block has pure ternary twist. In particular a dyadically active label has degree $25 \cdot 2^i 3^j$ with $j \ge 1$, and a five-active one has degree $8 \cdot 3^k 5^l$ with $k \ge 1$. This statement applies to both single and mixed active twists.

A simultaneous Gauss congruence. Let $D = n^2$ be an ordinary integral modular dimension, and T a nonempty set of its prime divisors. Set $c_t(Y) = 0$ when the t-twist of Y is trivial, $c_t(Y) = t/(t-1)$ when its order is t, and $c_t(Y) = 1$ at higher order. Then

$$n \prod_{\substack{u \mid n,\ u \notin T \\ u \text{ prime}}} u \quad \Big| \quad L_T = \sum_Y d_Y^2 \prod_{t \in T} c_t(Y).$$

In particular L_T is a nonnegative rational integer, positive precisely when a single twist is active at all primes in T. Indeed expand $\prod_{t\in T}(1-\theta_{Y,t})$ by inclusion and exclusion. For $J\subseteq T$ choose m_J equal to one at the normalized local levels in J and zero on their complement; include the order of the dyadic anomaly scalar in the dyadic congruence. By Lemma 6.1, the Jth weighted sum divided by n equals $n(W_J)_{00}\xi^{m_J}$, where $n(W_J)_{00}$ is a rational integer divisible by every listed outside prime (for $J=\varnothing$, use n). The normalized trace of a dyadic root of unity is 0 or ±1. On the other side the normalized trace factorizes over the coprime primary orders; the mean of $1-\theta_{Y,t}$ is exactly $c_t(Y)$.

For example, if both dyadic edges occur at $2^6 3^{2e} 5^4$, every simultaneously three- and five-active degree is $2^i 3^j 5$, with $1\le j<e$. Indeed the two degree transfers give the two odd prime divisors, and activity cuts off their full heights. Write $a_3=2c_3$, $a_5=4c_5$ on this nonempty joint sector. On cancelling its common factors, the congruence becomes

$$128\cdot 3^{e-2}\ \Big|\ \sum_{Y\ \text{joint}} 4^i 9^{j-1} a_3(Y)a_5(Y).$$

At $D=5400^2$ the positive integer on the right is therefore a multiple of 384.

There is also an exact balance. For an odd prime t choose a primitive tth root ζ_t. For a label of exact t-primary twist order t write $\epsilon_t(Y)=(a/t)$, the Legendre symbol for $\theta_{Y,t}=\zeta_t^a$. For any nonempty set T of odd primes,

$$\sum_{\substack{Y:\ \operatorname{ord}(\theta_{Y,t})=t\\ \text{for all } t\in T}} d_Y^2 \prod_{t\in T}\epsilon_t(Y)=0.$$

Indeed the partial Gauss sum selecting T is rational by the same vacuum identity, as its exponent kills the dyadic normalization. Average its Galois conjugates with the product of the quadratic characters on the factors in T. The rational sum has average zero. For a primary root of order above t the average vanishes on the subgroup congruent to one modulo t, and a trivial root is killed by the quadratic character. At order exactly t the average is the sign $\epsilon_t(Y)$ times a fixed nonzero quadratic Gauss sum divided by $t-1$. Independence of the primary factors gives the formula. Thus in the ordinary triangle the two signs $\epsilon_3\epsilon_5$ carry equal squared-dimension mass among the labels with primary orders exactly three and five, regardless of their dyadic twist.

The outside divisor can be increased if every label active at a prime of T has $v_u(f_Y)\le 2$, for a given $u\notin T$. In that case replace the factor u in the divisor by $u^{v_u(n)}$. By Theorem 6.9 each W_t, $t\in T$, has entries integral at every prime above u. The same holds for their product W_J, $J\subseteq T$. Hence the rational integer $n(W_J)_{00}$ is divisible by $u^{v_u(n)}$ in the inclusion–exclusion sum.

For example, if two and five mix at $2^6 3^{2e} 5^4$, every three-active label has degree $2^i 3^j 5^l$ with $l\ge 1$ by the sextic trace transfer. The sharpened singleton congruence gives

$$800\cdot 3^e\ \Big|\ \sum_{Y\ \text{three-active}} 4^i 9^j 25^{l-1}\, 2c_3(Y).$$

Proposition 21.6 (Minors on deficient rows and columns)**.** *Let an ordinary integral modular table have square dimension n^2 on three primes p,q,t, where p,q never divide the same twist order. Call x deficient if $k_x=v_p(n/d_x)>0$. For any deficient rows A and columns B, the residue rank at p of $(\chi_x(y))_{A,B}$ bounds below the number of intermediate labels in either factor order $W_tV_p=V_pW_t$ between A and B, where $V_j=\gamma_j^{-3}W_j$. Only the full blocks and complementary twist fibers preserved by each factor need be counted.*

In particular, every deficient Z has a $Y\in P_p$ in its full t-block with the same prime-to-t twist and $(W_t)_{ZY}\neq 0$. If $t\nmid f_Z$, one may take $Y=Z$. Hence a t-full label deficient at both p and q belongs to both principal blocks.

Proof. The p-local translates of the nontrivial p-phase spaces span an invariant subspace on which the q-upper unipotent acts by its normalization scalar: separation gives this on each phase space,

and the q-action commutes with its translates. The generating conjugates also act by scalars there, so $V_q = I$ there and $V_p = I$ on the orthogonal complement, as in Proposition 6.2. Thus $(V_p - I)(V_q - I) = 0$ and, up to a scalar root of unity, $S/n = W_t(V_p + V_q - I)$; a dyadic anomaly outside n acts by scalars. The q-active columns have p-unit codegrees, so V_q is p-integral by the lower-unipotent expansion in the preceding Gauss congruence. The single-column estimate in Theorem 6.9 bounds the valuation of column y of W_t below by $1 - k_y$ and, by unitarity applied to W_t^{-1}, that of row x by $1 - k_x$. Commutativity gives

$$v_p((W_t(V_q - I))_{xy}) \geq 1 - \min(k_x, k_y).$$

After scaling row x by n/d_x, the character minor therefore equals the scaled $W_t V_p$ minor in the residue field, up to a common unit. A nonzero residue minor is nonzero in characteristic zero; the factorization in either order gives the claimed count. For $A = \{Z\}$, $B = \{0\}$ the entry $\chi_Z(0) = 1$ is a unit. The intermediate label for $W_t V_p$ lies in P_p by the vacuum-column lemma and in the required block and fiber by W_t. The t-block is a singleton if $t \nmid f_Z$ by Theorem 3.1. □

A trace bound for several principal residues. Let Z be a nonunit in a perfect fusion-simple symmetric integral table with exactly three dimension primes a, b, t. Write H for any subset of these primes at which Z is principal. Then

$$f_Z \;\geq\; [K_Z : \mathbb{Q}] + \sum_{q \in \{a,b,t\}} \left(\prod_{p \in H \setminus \{q\}} p \right)^2 .$$

Indeed for each q, the dimension identity (2.6) in Lemma 2.4 has vacuum term one; reduction modulo q forces a nonzero absolute trace T_X at some nonunit q-free degree. That degree has both other prime factors by Lemma 2.5, so the three column orbits are distinct. Every conjugate of $\chi_Z(X)$ vanishes above each $p \in H$ dividing d_X, by principality and Galois covariance. Their absolute trace is a rational integer therefore divisible by the product of these primes. The orbit norm identity (Lemma 2.4), (2.5) counts at least T_X^2 for each of these three orbits in addition to the vacuum contribution.

Consequently when a, b are separated, every t-full degree misses at least two powers of a or at least two powers of b relative to n. Otherwise let H be the set of deficient primes. The preceding Weyl argument puts Z in P_p for every $p \in H$, while $f_Z = (\prod_{p \in H} p)^2$. The term $q = t$ alone equals this value, a contradiction after counting the vacuum. Codegree one is impossible by the ordinary row norm as soon as a nonunit is present. This excludes in particular $4 \cdot 3^e \cdot 5$ when two and five are separated and $4 \cdot 3^{e-1} \cdot 25$ when two and three are separated.

21.5 Congruences and quartic fields at the dyadic sextic

Proposition 19.16 determines the full height at a missing prime. At $D = 2^6 3^{2e} 5^4$, in the absence of a 30-twist, the joint $3, 5$ degrees are divisible by eight, the joint $2, 5$ degrees by 3^e, and the joint $2, 3$ degrees by 25. If all three pairs occur, Section 21.3 makes every joint $3, 5$ degree equal to $8 \cdot 3^j 5$ with $1 \leq j < e$. At $e = 3$ the possibilities are 120 and 360.

At $2^6 3^6 5^4$ there is an additional incompatibility. A dyadically active five-free P_3 witness and a three-active three-free P_5 witness cannot coexist without a twist divisible by 30. The same is true with their two activity conditions interchanged. Indeed their common weight is four: all relevant degrees have dyadic height at least two (Lemma 4.10), while a dyadically active degree cannot have height three. Neither witness can mix two and three in the absence of the triple, since such a label has full five-height, whereas the indicated five-heights are zero and one. Thus their dyadic and ternary twist products are both nontrivial. By Corollary 19.18 every constituent has these same two components, and the two different five-positions force some constituent also active at five.

More strongly, when both dyadic edges occur there are witnesses of dyadic height two on both sides, by Section 21.3. Every such label has trivial three-twist. Otherwise use $p = 2, T = \{5\}$ on its nontrivial three-fiber and pair it with any dyadic-height-two witness on the other side. Its codegree is even and the opposite degree divided by four is odd, whereas $3(\xi - \sigma_5\xi)$ is a two-unit for a nontrivial five-unit acting on ξ. In particular each jointly two- and three-active column in this triangle requires a five-free nonunit of degree 12 or 36 in P_3 with completely trivial twist: use Proposition 4.2 in the trivial complementary fiber.

In the weight-five case with $d_X = 20$ the prefactor η equals one even when X has trivial dyadic twist. Use $p = 2, T = \{3, 5\}$ at X; the opposite degree divided by five is odd. If $\eta \neq 1$, the mixed difference of z equals $3(\eta - \sigma_3\eta)(\xi - \sigma_5\xi)$, a two-unit for appropriate local inertia. This contradicts the congruence.

By the block congruence in Section 19.8, at the even core any three-free degree-20 member X of P_5 lies in one of four distinct full two-blocks, each entirely three-inactive if the triple twist is absent. Choose an odd $Y \in P_3$ of degree 15 or 45 from a joint three- and five-active column. The preceding result gives $\chi_X(Y) \equiv \xi \pmod 2$, with ξ a primitive fifth root, because $d_Y/5$ is odd. The four tame five-conjugates have distinct residues in characteristic two.

Applications at the even core. Suppose $D = 2^6 3^{2e} 5^4$ with $e \geq 3$. If two and five are separated, every three-active J of dyadic height below three has an odd member Y of P_3 with full five-height in its trivial prime-to-three twist fiber, such that $d_Y \chi_J(Y)$ is a unit at the prescribed prime above two. Indeed in the restricted orthogonality of Proposition 4.2 even degrees contribute zero there. The other nonunit degrees under consideration are $3^j 5$ and $3^j 25$, $1 \leq j \leq e - 1$. Each of the former has a full tame five-inertia orbit of four by its weight-five crossing with a three-free member of P_5 (Section 19.5). In this orbit the value $\chi_J(Y)$ is constant modulo two along this orbit by Proposition 19.17 applied to $\chi_J(Y)$, because $2 \mid f_J$ and the corresponding differences in the argument equal Galois differences of $\chi_J(Y)$ by symmetry and rationality of the degrees. Its orbit contributes zero. The vacuum then forces the required full-height hit. At $e = 3$ its degree is 75 or 225.

Theorem 21.7. *In a perfect fusion-simple ordinary minimal table suppose $D = 2^6 3^{2e} 5^4$, with $e \geq 3$, and some column is active at two and at one of the odd primes. Suppose no twist is divisible by* 30. *In each of P_3 and P_5 there is a simple Z with*

$$\theta_Z = 1, \qquad \mathbb{Q}(\zeta_5) \subseteq K_Z.$$

The two witnesses can be chosen in the same full two-block. Each block thus contains the unit and at least four nonunit simples of twist one, including a non-self-dual pair.

Proof. Use Proposition 4.2 in the pure twist part of the principal block of the odd prime at the mixed column, avoiding the other odd prime. The resulting nonunit has dyadic height two: the lower bound follows from Lemma 4.10, and height three would miss the dyadically active column. Take an opposite principal witness in its pure twist part, avoiding the first odd prime, by the same proposition applied to an active column at the other odd prime, whose existence follows from categorical Cauchy (Section 3). Their degrees are $2^i 3^j$ and $2^k 5$, their gcd is four, and their twists have 3-power and 5-power order, respectively. By Sections 19.3 and 21.3, their lcm-normalized crossing has two possibilities. The four negative phases of the signed possibility have different five-parts and ternary parts consisting of a singleton and a complete cubic coset. Both factors have trivial dyadic twist. Among the four negative packets at most one cancels the five-part of the input twists and at most two cancel the three-part. At least one constituent would have twist divisible by 30. The entry is therefore $1 + 3\xi$ for a primitive fifth root ξ. For a faithful character ψ of $(\mathbb{Z}/5)^\times$, the Fourier coefficient of this crossing is proportional to $3 \sum_{a=1}^{4} \psi(a)\xi^a$, a nonzero Gauss sum. Proposition 19.20, with $u = 3, v = 5, w = 2$, gives the two witnesses. The field $\mathbb{Q}(\zeta_5)$ forces an orbit of at least four in each principal block. The vacuum has rational field, and complex conjugation moves these witnesses to their duals. □

Which quartic field occurs. Let ζ be primitive of order five and ω primitive of order three. In the weight-four and weight-five crossings of Section 19.3 and Proposition 8.6 respectively, the field of the normalized entry z contains at least one of the explicit cyclic quartic fields

$$\mathbb{Q}(\zeta), \qquad \mathbb{Q}(\zeta_{15})^{+}.$$

In the unsigned alternative it contains the first. In fact the unsigned weight-four word is $1+3\zeta^{a}$, $a \neq 0$. At weight five the tame five-trace of $\eta(2+3\zeta^{a})$ equals 5η. The field of z is abelian and hence stable under these automorphisms, so one can divide out the ternary factor η. In the signed case one can first remove η at weight five by the same tame trace. Up to a rational constant and a rational sign the remaining number is

$$x = \sum_{i=1}^{4} \eta_i \zeta^{i},$$

where the η_i form one copy of one and one coset of μ_3 in the three-power roots. If the coset is not μ_3 itself, averaging over three-inertia kills precisely those three terms and leaves ζ^{a}. Otherwise let i, j be the two indices with coefficient one and k, l those with coefficients ω, ω^{2}. The same average gives $\frac{1}{2} + \frac{3}{2}(\zeta^{i} + \zeta^{j})$. If $j \neq -i$ modulo five, its stabilizer under five-inertia is trivial by independence of the four nonidentity fifth roots. If $j = -i$, then $l = -k$, and x is fixed by simultaneous complex conjugation at three and five. Its stabilizer is exactly this subgroup of order two: the averaged word only allows the identity or negation at five, and conjugating at three alone changes the coefficients at k, l. Thus $\mathbb{Q}(x) = \mathbb{Q}(\zeta_{15})^{+}$.

Consequently every jointly three- and five-active J at the sextic core contains one of these quartic fields in K_J. Indeed, choose the odd $Y \in P_3$ in the refined two-avoidance statement of Section 21.4. Since J is five-active, $d_Y = 3^{j}5$. Cross it with a three-free member of P_5 in the trivial complementary fiber. The field of this weight-five crossing lies in K_Y. The avoided-prime transfer puts every subfield of K_Y of two-power degree in K_J. If no twist has order divisible by thirty, the signed case is excluded, so $\mathbb{Q}(\zeta) \subseteq K_J$. In particular J cannot be self-dual, because a self-dual row has real character values.

At $2^{6}3^{6}5^{4}$ Proposition 19.21 shows that every joint three- and five-active label requires a second full tame five-orbit in its own five-block with exactly its prime-to-five twist. Indeed the first orbit is full by Section 21.4. This applies also if the joint label is dyadically active. For a second application, in the unsigned weight-five crossing of that section write $z = \eta(2+3\xi)$, ξ of order five. If the ternary root η is nontrivial, $\mathbb{Q}(z)$ contains $\mathbb{Q}(\zeta_3, \zeta_5)$: the uniqueness of the two phases, of different multiplicities, forces any stabilizer to fix η and ξ. In a dyadically trivial fiber containing either witness, the $(3,5)$-torus thus has at least four distinct orbits affording the product of the two quadratic characters, by Proposition 19.21.

The outside divisor for joint Gauss sums. In ordinary square dimension let T be a nonempty set of odd dimension primes, and let $u \notin T$ be another dimension prime. If no twist is active at all primes of $T \cup \{u\}$, then the simultaneous Gauss integer of Section 21.4 satisfies

$$u^{2v_u(n)} \mid L_T.$$

Here $L_T = \sum_X d_X^{2} \prod_{t \in T} c_t(X)$, where $c_t = 0, t/(t-1), 1$ at primary orders $1, t, > t$, respectively. Indeed the raw value $\sum_X d_X^{2} \prod_{t \in T}(1 - \theta_{X,t})$ is rational by the vacuum Weyl formula and inclusion–exclusion: all the CRT exponents kill the dyadic normalization. It is therefore equal to its normalized trace L_T. Every surviving term is divisible by $u^{2v_u(n)}$ in the cyclotomic integer ring because its label has full u-height (Proposition 19.16). At the even core the combined divisors give $43200 \mid L_{\{3,5\}}$. Equivalently, writing $d_J = 8 \cdot 3^{j}5$, $a_3 = 2c_3$, $a_5 = 4c_5$ on the joint sector,

$$24 \mid \sum_{J} 9^{j-1} a_3(J) a_5(J).$$

When two is included, the factors c_2, c_3 have denominators prime to five and c_2, c_5 have denominators prime to three. Therefore the omitted prime's full degree square divides the rational simultaneous sum directly in these two cases. Together with the outside divisor in Section 21.4, the three divisibilities in this triangle are

$$135000 \mid L_{\{2,3\}}, \qquad 145800 \mid L_{\{2,5\}}, \qquad 43200 \mid L_{\{3,5\}}.$$

21.6 A quartic field at twist one without a separation hypothesis

Theorem 21.8. *At $D = 29160000$ the ordinary minimal table has twist-one labels $V_3 \in P_3$ and $V_5 \in P_5$ in the same full two-block whose fields both contain one of*

$$\mathbb{Q}(\zeta_5), \qquad \mathbb{Q}(\zeta_{15})^+.$$

If no twist has order exactly 30, the witnesses can be chosen with the first field. In particular each principal block contains at least four nonunit simples of twist one. The two chosen labels may coincide.

Proof. Choose complementary pure-twist principal witnesses $X \in P_3$, $U \in P_5$, of degrees $2^i 3^j$, $2^k 5$. Here pure allows twist one. Use Proposition 4.2 at the odd endpoint of a dyadic edge and avoid the other odd prime. This gives dyadic height two at that witness by Section 21.3. Use the same proposition at an active column for the other odd prime. Both dyadic heights are at least two, so the common degree factor is four. Put $z = S_{XU}/\operatorname{lcm}(d_X, d_U)$. By Section 19.3 its unsigned form is $1 + 3\xi$ with ξ a primitive fifth root. Its signed form is

$$-\sum_{a=1}^{4} \eta_a \zeta_5^a, \qquad \{\eta_a\} = \{1, 1, \omega, \omega^2\}, \quad \operatorname{ord}(\omega) = 3.$$

The exclusion of a wild cubic coset in this formula is precisely Proposition 19.7.

A nonzero Fourier coefficient for a Galois character whose kernel is preserved upon inversion of either the three- or five-component puts the corresponding cyclic field into twist-one witnesses in the same full two-block (Proposition 19.20). In the unsigned case use a faithful quartic character ψ at five. Its coefficient is a nonzero multiple of the primitive Gauss sum of ψ. In the signed case let i, j be the two exponents carrying ω, ω^2. Take the product of ψ and the nontrivial character at three. Antisymmetrizing at three leaves a nonzero scalar times $\zeta_5^i - \zeta_5^j$. The five Fourier coefficient is a nonzero Gauss sum times $\overline{\psi}(i) - \overline{\psi}(j) \neq 0$. On $(\mathbb{Z}/15)^\times$ the product character has kernel generated by simultaneous complex conjugation, unchanged by either local inversion. Its fixed field is $\mathbb{Q}(\zeta_{15})^+$.

It remains to see which orders the signed case produces. The four signed terms are actual monodromy phases with equal total dimension $\operatorname{lcm}(d_X, d_U)$ at each phase (Proposition 19.6). Both inputs have trivial dyadic twist. At most one of the four terms cancels the five-twist product, and at most two cancel the three-twist product. Thus some constituent is active at all three primes. Both dyadic edges now occur. By the two degree transfers (Section 21.3), every three-active degree is five-bearing and every five-active degree three-bearing. Hence the five-free pure ternary X and the three-free pure quinary U both have twist one. The two marked phases with $\eta_a \neq 1$ are therefore actual twists of order exactly 30, and the other two have order 10. More generally replace one input by each of its Galois conjugates; it still has twist one, and the normalized entry is conjugated by the same automorphism. The uniqueness of the physical monodromy packets now realizes all eight primitive thirtieth roots and all four primitive tenth roots. Finally the orbit of a label whose field contains a quartic field has size at least four, and the rational vacuum is outside it. □

If an order divisible by 30 occurs, both dyadic edges occur. One can then take X and U at the two dyadic edges so that $d_X = 12$ or 36 and $d_U = 20$. Both have twist one by the same

degree transfers. Their fields already contain $\mathbb{Q}(z)$, which in the signed case is $\mathbb{Q}(\zeta_{15})^+$ or $\mathbb{Q}(\zeta_{15})$ according as the positions carrying the repeated ternary phase are opposite or nonopposite (Section 19.3). The two distinct degrees give two distinct nonunit orbits. In particular the triangle, with or without a triple twist, has at least nine twist-one simples, rising to seventeen for a nonopposite signed crossing.

Thirteen twist-one simples without a triple. Suppose all three edges occur and no twist order is divisible by 30. There are nonunit twist-one labels of each of the following degree types:

$$12 \text{ or } 36, \qquad 20, \qquad 15 \text{ or } 45.$$

Each can be chosen with character field containing $\mathbb{Q}(\zeta_5)$. For the first two use X, U in the preceding proof. Their unsigned crossing $1 + 3\xi$ gives the field on both sides. For the third, use Section 21.4 at a jointly three- and five-active column to obtain an odd $Y \in P_3$ in the pure three-twist fiber, of degree $3^j 5$ with $1 \le j \le 2$. By Section 21.5 its crossing with U has normalized word $2 + 3\xi'$ for a primitive fifth root. Its constituent packet at ξ' has dimension $3\,\mathrm{lcm}(20, d_Y)$. If Y had nontrivial ternary twist, every simple in this packet would be three- and five-active, since $\theta_U = 1$ and Y is pure ternary. All those degrees are divisible by eight in the absence of the triple (Proposition 19.16). But the packet dimension has dyadic valuation two. Thus $\theta_Y = 1$, and $\mathbb{Q}(\zeta_5) \subseteq K_Y$. The three degree types distinguish three Galois orbits, giving at least twelve nonunit twist-one simples in addition to the unit.

The quartic field throughout a dyadic residue block. There is also a block restriction valid when a twist is divisible by 30. More generally, take labels J, Y in ordinary integral data, put $x = \chi_J(Y)$ and $F = \mathbb{Q}(x)$, and suppose that $\ell \nmid \mathrm{Tr}_{F/\mathbb{Q}}(x)$. Let $F_{(\ell)}$ be the maximal subfield of F of ℓ-power degree. Every label in the full ℓ-block of J has character field containing $F_{(\ell)}$, and the orbit of that block under the cyclotomic Galois group has size divisible by $[F_{(\ell)} : \mathbb{Q}]$.

Indeed this Galois group acts on the full blocks at a fixed prime over ℓ: equivalence of two characters can be tested after permuting their arguments. If h stabilizes the block of J, then $gx \equiv ghx$ at that prime for every Galois element g, by using Y^g as the test argument. If the image of the block stabilizer on F had order divisible by ℓ, it would contain an element of order ℓ. Grouping the absolute trace into the orbits of this element would make it zero in the residue field. Thus that image has order prime to ℓ. It fixes $F_{(\ell)}$, since $F/\mathbb{Q}$ is abelian. The stabilizer of a label in the block is contained in the block stabilizer, proving both assertions.

For a jointly three- and five-active J take $Y \in P_3$ of odd degree 15 or 45 by the refined prime avoidance of Section 21.4. It ensures that $[K_Y : F]$ and $\mathrm{Tr}_{F/\mathbb{Q}}(x)$ are odd. A weight-five crossing puts one of $\mathbb{Q}(\zeta_5)$, $\mathbb{Q}(\zeta_{15})^+$ in K_Y (Section 21.5), so that field lies in $F_{(2)}$. The whole two-block of J has that field and the four tame five-conjugates of the block are distinct: five-inertia has image of order four on either quartic field. In particular none of these blocks is principal, and every label in them has five-height at most one, since a five-unit codegree would have five-unramified character field (Corollary 2.8).

Retaining the quartic field on removing the five-twist. Let J be any jointly three- and five-active column at the ordinary even sextic core. Its full five-block contains a five-inactive label Z with exactly its dyadic and ternary twists and with

$$\mathbb{Q}(\zeta_5) \subseteq K_Z \qquad \text{or} \qquad \mathbb{Q}(\zeta_{15})^+ \subseteq K_Z.$$

Without an order divisible by 30, the first inclusion holds.

Take the odd $Y \in P_3$ in its pure ternary fiber supplied by the avoided-two trace selection in Section 21.4. For $x = \chi_J(Y)$ and $F = \mathbb{Q}(x)$, both $[K_Y : F]$ and $\mathrm{Tr}_{F/\mathbb{Q}}(x)$ are odd. The degree of Y is 15 or 45, and the quartic-field calculation of Section 21.5 puts one of the displayed cyclic

quartic fields K in K_Y, the first in the absence of a 30-multiple. Then $K \subset F$, by the odd index. The algebraic integer $\mathrm{Tr}_{F/K}(x)$ cannot lie in the quadratic subfield of K, since its absolute trace is odd. Consequently x has nonzero Galois Fourier coefficient for a faithful character of $\mathrm{Gal}(K/\mathbb{Q})$ inflated to the splitting field. Equivalently, for the corresponding character orbit vector v on the conjugates of Y we have $(sv)_J \neq 0$ (replacing the character by its inverse if required). All its coordinates are five-inactive.

Let E project onto the full five-block and complementary twists of J. Since $s = W_5 W_{5'}$ and E commutes with W_5, the vector $W_{5'}v$ has a nonzero coordinate in E. It is still five-inactive and still affords the same character: the local Weyls at two and three invert only a trivial or quadratic component. The stabilizer of a nonzero coordinate therefore fixes K, proving the claim.

In the absence of a 30-multiple, the same Z can moreover be chosen in P_3. The full dyadic height of J makes its two-block a singleton, so its row of W_2 is scalar on the J-line. From $(W_2 W_5 W_3 v)_J \neq 0$ we now get $EW_3 v \neq 0$. The initial orbit lies in P_3, which W_3 preserves. As before, $W_3 v$ is five-inactive and affords the faithful conductor-five character. In the no-30 triangle the resulting Z is even by the exclusion of ternary-active odd degrees $15, 45$ in P_3 proved by the packet argument above (which also applies if the odd label has nontrivial five-twist: dilating the primitive fifth root on the opposite witness makes the three-coefficient packet five-active): three-activity forces five-height at least one, ramification at five bounds it by one, and an odd principal degree would be 15 or 45. Thus each such joint label requires in its own full five-block four even non-self-dual, five-inactive principal ternary conjugates at its exact ternary phase.

21.7 Residue ranks on a separated path

We apply Proposition 21.6 with $(p, q, t) = (2, 3, 5)$; write $z_{xy} = S_{xy}/\operatorname{lcm}(d_x, d_y)$.

Now take $D = 2^6 3^{2e} 5^4$, $e \geq 3$, and suppose two and three are separated. Choose a pure five-primary $U \in P_5$ of degree 20 from a joint two-five column. For a joint three-five column choose an odd $V \in P_3$ of degree $3^j 5$ in its pure ternary fiber. In fact V has twist one, since a three-active degree is divisible by eight. Let H be tame five-inertia, taken of order four. The weight-five calculation in Section 21.5 gives $z_{UV} = 2 + 3\xi$ for a primitive fifth root ξ. Index the four H-conjugates of either label by the nonzero elements of $\mathbb{F}_5$, and append the vacuum with index zero. The residue character matrix between these two sets of five labels at a fixed prime above two is

$$(\chi_{U_a}(V_b) \bmod \mathfrak{p})_{a,b \in \mathbb{F}_5} = (\bar{\xi}^{ab})_{a,b \in \mathbb{F}_5}.$$

For nonzero a, b this follows from $\chi_{U_a}(V_b) = 3^j(2 + 3\xi^{ab})$; the vacuum entries are odd. The matrix is invertible by the Vandermonde determinant.

All five rows belong to P_5 and have pure five-primary twists. For every full two-block C containing b_C of the five columns, the forward factor order thus gives at least b_C twist-one labels in $P_5 \cap C$. In particular if $V \in P_2$ there are at least five twist-one labels in $P_2 \cap P_5$, since P_2 is Galois-stable. For the reverse factor order let B be the five-block of V. Five-inertia fixes B. The four blocks C_a of the U_a, $a \neq 0$, are distinct and nonprincipal by their residues at V. The reverse factor order at column V places a label in each $C_a \cap B$ with pure five-primary twist $\theta_{U_a,5}$: V_p preserves the complementary phases of the row U_a, and W_t preserves those of the column V. The vacuum row similarly gives a twist-one label in $P_2 \cap B$. Thus each such five-block meets all five of these specified two-blocks.

Every label Z of C_a has $\mathbb{Q}(\xi) \subseteq K_Z$. For any Galois automorphism h fixing Z, the two values $\chi_{U_a}(V^h)$ and $\chi_{U_a}(V)$ agree modulo the same prime above two, since one may substitute Z for U_a. The displayed crossing and prime-to-two injectivity of reduction give $h(\xi) = \xi$. In particular all four intersections $C_a \cap B$ contain non-self-dual labels with this precise subfield. This applies to each distinct five-block of an odd pure-ternary P_3 witness of five-height one.

There is also a second hit for the same joint J. In $E = P_3 \cap \{\theta_{3'} = 1\}$ set $F(Y) = \sum_{h \in H} \chi_{U^h}(Y) - d_Y$. All four rows and the unit are three-inactive, whereas J is three-active.

Restricted orthogonality gives $\sum_{Y\in E} F(Y)\chi_J(Y) = 0$, using the dual of J in the conjugated row. Here $F(1) = 3$. At every odd nonunit of five-height one in E the crossing above gives $F(Y) = 0$. Other odd nonunits have full five-height and miss J. At $8 \mid d_Y$, lcm divisibility gives $F(Y) \in 2\mathcal{O}$. It follows, at every prescribed prime above two, that some even $Y \in E$ has both $F(Y)$ and $\chi_J(Y)$ nonzero in the residue field. Its two-height is one or two, its five-height is at most one, and Y itself has twist one by separation. Its tame five-inertia orbit has size four: otherwise the trace defining $F(Y)$ counts each value an even number of times. This supplies distinct even and odd twist-one P_3 orbits of full tame five-inertia meeting every joint J.

The rank conclusions apply to this second orbit as well. Put $r_h = \overline{\chi_{U^h}(Y)}$ in the residue field at two, where here the bar denotes reduction. Its sum is nonzero by the choice of Y. The four-by-four matrix $(r_{ab})_{a,b\in H}$ is invertible: after inverting the column indices it is the regular multiplication matrix of $\sum r_h[h]$ in the group algebra of the cyclic group H of order four. Its augmentation ideal in characteristic two is nilpotent. Thus the forward count in each two-block holds separately for this even orbit. Its five-block also meets each C_a at the pure five-phase of U_a, using any nonzero entry in the corresponding row, and meets P_2 at twist one: the vacuum crossing with Y is nonzero by the two-sided valuation estimate after scaling by n/d_Y. If $v_2(d_Y) = 2$, the four conjugates of Y occupy four different full two-blocks, since $\chi_{Y^b}(U_a) = (20/d_Y)\chi_{U_a}(Y^b)$ differs by a two-unit from the residue matrix just shown invertible.

A height bound from the exceptional fifth-root adjoint. Suppose at nonunits X, Y the two monodromy packets have primitive fifth-root ratio, and their larger dimension is $\lambda d_X d_Y$, $1/2 \le \lambda < 1$. Suppose also $\theta_{Y,2} = 1$. Among the adjoints of X and of a tame five-conjugate X', there is a nonunit constituent Z and there is a constituent T of YZ for which the dyadic monodromy is exactly -1. Furthermore

$$d_Z \le d_X^2 - 1, \qquad d_T \le \lambda d_X^2 d_Y.$$

Indeed the adjoint character word at row Y has the full mass on $A = \{1, \xi, \xi^{-1}\}$, where ξ is the relative fifth root. On the identity coset of the five-primary positions put 1 on A and $-3/2$ on its two vacant positions, and put zero on every other coset. Tensor with normalized trace on the prime-to-five factor. This rational functional has value one only on A, and is less than one everywhere else except at the negatives of the two vacant roots, where its value is $3/2$. Indeed a complementary root other than -1 has normalized trace at least $-1/2$. Apply also the dilated functional to the conjugate with multiplier two at five, for which $A \cap A^2 = \{1\}$. If the two balancing expansions of the adjoints avoided their respective exceptional negative phases, equality at the full mass would force all phases of their common nonunit constituent into $\{1\}$. Strictness and the common-nonunit lemma in the proof of Theorem 19.5 prohibit this. The unit has no negative phase. For either adjoint every such T occurs in $(YX)X^*$ or its counterpart with X'. The first product YX splits into the two indicated packets, of which each simple has degree at most $\lambda d_X d_Y$. The same holds after moving just the X-label by tame inertia: the resulting two positive fifth positions are exposed by the two-mark functional in Section 19.3, so their weights persist.

Apply this to $X = U$ of degree 20 and any odd $Y \in P_3$ of degree $3^j 5$ in the pure ternary fiber. Here $\lambda = 3/5$ and Y has twist one. At least one of Z, T is dyadically active and therefore has degree at least $2 \cdot 3^e$, since it has full three-height and no nonunit has prime-power degree. As $399 < 1200 \cdot 3^j$, the two bounds imply $3^{e-j} \le 600$, hence $j \ge e - 5$. If $e \ge 5$, every Z in this argument is dyadically inactive since $2 \cdot 3^e > 399$. The actual dyadic component of T is then -1. Both the lower height bound and this prescribed phase hold uniformly without an upper bound on e.

The five-free P_3 degrees satisfy a similar bound. Write one of them $d_Y = 2^i 3^j$ with $i = 2$ or 3 and $j < e$ by the complementary-height and principality bounds. It is dyadically inactive by separation. The crossing with U is $1 + 3\xi$ by the unsigned weight-four result; here $\lambda = 3/4$. It follows that $2 \cdot 3^e \le 300 \cdot 2^i 3^j$, hence $e - j \le 5$ if $i = 2$ and $e - j \le 6$ if $i = 3$.

In particular the number of three-active labels on this path is bounded independently of e. Every such label meets a five-free member of P_3 by restricted orthogonality modulo five. These witnesses have degree at least $8 \cdot 3^{e-6}$ by the two height bounds, so their number is at most $625 \cdot 3^{12}$. Every three-active degree is divisible by 40 by separation and the five-bearing transfer. For a fixed witness and any such label it meets, the lcm of their degrees is at least $40 \cdot 3^{e-6}$. A nonzero cyclotomic integer has mean squared conjugate modulus at least one. Averaging the ordinary row norm over embeddings thus bounds the number of such hits per witness by $25 \cdot 3^{12}$. Altogether there are at most $15625 \cdot 3^{24}$ three-active labels. A ternary phase of order 3^k yields 3^{k-1} different phases by the Galois square action, forcing $k \le 33$.

21.8 The upper ternary height when two and five are separated

Suppose $D = n^2 = 2^6 3^{2e} 5^4$, $e \ge 3$, in an ordinary perfect fusion-simple minimal table, with two and five separated and both joined to three. Fix the principal blocks P_3, P_5. Put

$$A = \{X \in P_5 : 3 \nmid d_X,\ X \ne 1\}, \qquad B_j = \{Y \in P_3 : v_5(d_Y) = 2,\ v_3(d_Y) = j\}.$$

Every $X \in A$ has degree 40 and $\mathbb{Q}(\zeta_5) \subseteq K_X$. Indeed the degree is the complementary degree theorem of Section 21.3, and the no-triple weight-five crossing of Corollary 19.18 with an odd degree-$3^j 5$ member of P_3 has the form $\eta(2 + 3\xi)$ with ξ primitive of order five. Its five-inertia trace recovers 5η. The odd member exists at a joint three–five column by prime avoidance. Also A contains a member X_0 tame at three, by wild-inertia avoidance at a five-active column in Section 21.4. Both sets above are Galois-stable.

Proposition 21.9. *Let C_j consist of the odd degree-$3^j 5$ members of P_3. Then*

$$4 \sum_j 9^j |B_j| + \tfrac{1}{4} \sum_j 9^j |C_j| < 9^e.$$

Proof. Take four conjugates of one $X \in A$ under the tame order-four subgroup of five-inertia. The average of their S-rows has squared norm $n^2/4$, including a positive vacuum contribution. At $Y \in B_j$ the entries coincide, since $5 \nmid f_Y$, and are nonzero by three-principality of Y and $3 \nmid d_X$. They are divisible by $200 \cdot 3^j$ by lcm integrality. At $V \in C_j$ the lcm is $40 \cdot 3^j$ and the normalized crossing $\eta(2 + 3\xi)$ above has tame average $5\eta/4$. Counting both sets in the Galois mean of the norm proves the inequality. □

An arithmetic form of the fusion moment test. For a rational row Y, all the eigenvalues $\lambda_X = d_Y \chi_Y(X)/d_X$ of multiplication by $[Y]$ are rational integers. For a prime ℓ with $a = v_\ell(D) > 0$, a residue $c \in \{0, \dots, \ell - 1\}$, and $0 \le t < a$, one has

$$\ell^{a-t} \ \Big|\ \sum_{\lambda_X \equiv c \pmod{\ell}} d_X^2 \big((\lambda_X - c)/\ell\big)^t.$$

In fact these conditions over all ℓ, c, t are equivalent to integrality of all vacuum moments $D^{-1} \sum_X d_X^2 \lambda_X^k$, $k \ge 0$. For necessity, put $m = \varphi(\ell^a)$ (or any multiple at least a). The polynomial $1 - (T - c)^m$ equals the indicator of the residue class c modulo ℓ^a on integers. Multiply it by $(T - c)^t$ and use integrality of the vacuum coefficient of every integer polynomial in $[Y]$. Conversely expand any monomial at each residue c using the binomial theorem. Terms of degree at least a in $T - c$ already vanish modulo ℓ^a, and the displayed conditions handle the others.

21.9 The principal three-block

Take an ordinary minimal table of dimension $2^6 3^{2e} 5^4$, $e \ge 3$. Separation of two and five forces both other edges by Section 21.3 and Lemma 21.1.

Theorem 21.10. *Suppose two and five are separated in an ordinary minimal table of dimension* $2^6 3^{2e} 5^4$, $e \geq 3$.

1. *No* $Y \in P_3$ *with* $v_5(d_Y) = 2$ *has* $v_3(d_Y) = e - 1$.
2. *Every member of* P_3 *of degree* $4 \cdot 3^j$ *or* $5 \cdot 3^j$ *has* $j \leq e - 2$. *Both types occur.*
3. *At* $e = 3$ *the odd nonunit degrees in* P_3 *are exactly* 15 *and* 75. *Each jointly* $2, 3$*-active column meets a degree-*75 *principal row, and each jointly* $3, 5$*-active column meets a degree-*15 *principal row.*

We prove part (1) in this section and part (2) in Section 21.10. For part (3), a nonunit cannot have prime-power degree, and the principal block bound gives $v_3(d_Y) \leq e - 1$. Each odd nonunit in P_3 therefore has degree $3^j 5$ or $3^j 25$. Here a column meets a row if their S-entry is nonzero. A jointly $3, 5$-active column meets an odd principal row by prime avoidance and misses full five-height rows by defect zero (Section 3, Proposition 4.2). For a jointly $2, 3$-active column the full five-height odd hit follows from Section 21.5; the parity identity below also proves this directly.

Proof of part (1). Write $B = B_{e-1}$ and use A, X_0 from Section 21.8. Suppose $B \neq \varnothing$. For $Y \in B$ put $i = v_2(d_Y)$, $L = \operatorname{lcm}(40, d_Y) = n/3$ and $z = S_{X_0 Y}/L$. By Corollary 2.8, $F = \mathbb{Q}(z)$ is unramified at two and five by the codegrees of X_0 and Y, respectively, and at most tame at three by the choice of X_0. Thus $F \subseteq \mathbb{Q}(\omega)$, $\omega = \zeta_3$. If $z = a + b\omega$ is nonrational, the nontrivial orbit Fourier coefficient in Proposition 19.19 gives

$$|b|\sqrt{3}\sqrt{[K_{X_0} : F][K_Y : F]} \leq n/L = 3.$$

But K_{X_0} contains $\mathbb{Q}(\zeta_5)$, disjoint from F, so the first index is at least four. Hence $z \in \mathbb{Z}$. Principality gives $(8/2^i)z = \chi_Y(X_0) \equiv 40 \pmod 3$, while the trivial Fourier coefficient gives $|z|\sqrt{[K_{X_0} : \mathbb{Q}][K_Y : \mathbb{Q}]} \leq 3$. Thus $z = (-1)^{i+1}$ and $[K_Y : \mathbb{Q}] \leq 2$. For any $X \in A$ the crossing field is again unramified at two and five and has degree at most two inside K_Y, so the same argument applies. In particular $s_{XY} = (-1)^{i+1}/3$ for the normalized matrix $s = S/n$. The purely ternary twist of Y lies in μ_3: order at least nine would give at least three square-Galois conjugates. The codegree at $i = 3$ is nine, impossible by Corollary 5.6.

The constant unit vector on A contributes $|A|/9$ on each $Y \in B$ to the squared norm of its s-image, besides the vacuum. Thus $|A|\,|B| < 9$, and $4 \mid |A|$ by the quartic field. If $|A| = 8$, $B = \{Y\}$ and the Y-row is rational. Reduced dimension orthogonality at two and five requires distinct nonunit hits of odd and five-free degree, respectively. Both are three-bearing, since nonunit degrees are not prime powers. Each costs at least nine by three-principality. For $i = 1, 2$ this overspends the norm: $1 + 8(8/2^i)^2 + 18 > (24/2^i)^2$. For $i = 0$, lcm integrality makes the character even at even degrees. Its squared norm 576 modulo four requires at least three odd nonunit values besides the vacuum; each is divisible by three. The five-free hit has even degree and value divisible by six. These hits, the vacuum and A already cost $3 \cdot 9 + 36 + 1 + 8 \cdot 64 = 576$. At a two-active column three-avoidance gives a further three-free nonunit in P_2 outside A (its dyadic height is at most two); its entry is nonzero by three-principality. This contradiction gives $|A| = 4$. It is a single Galois orbit. Proposition 4.2 provides a member with pure five-twist by avoiding three, and all three-free labels have pure three-twist by Section 21.4; thus all four have twist one.

Put $a = \frac{1}{2}\sum_{X \in A} e_X$ and $w = sa$. The orbit is stable under duality, so w is real. On B every coordinate has absolute value $2/3$. The modular relation $sTs = \xi T^{-1} s T^{-1}$ and $Ta = a$ give

$$\left| \sum_Z w_Z^2 \theta_Z \right| = |\langle a, w \rangle|.$$

If there were two labels in B, their phases would have sum of modulus at least one since both lie in μ_3. Their combined contribution on the left has modulus at least $4/9$. The remaining mass is $1/9$, giving $|\langle a, w\rangle| \geq 1/3$. But $w_0 = 80/n > 0$ lies outside $A \cup B$, and hence $|\langle a, w\rangle| \leq \|w|_A\| < 1/3$. Thus $B = \{Y\}$ and Galois invariance gives $K_Y = \mathbb{Q}$. In particular Y is self-dual.

Let $b_U = \chi_Y(U) \in \mathbb{Z}$, $i = v_2(d_Y)$, and $q = n/d_Y$. Thus $i = 0, 1, 2$, $q = 24/2^i$, and $|b_U| = 8/2^i$ on A. Let U be a five-free nonunit in P_3. Its degree is $2^k 3^j$ with $k = 2$ or 3, $j \geq 1$ (Lemma 4.10). If $b_U \neq 0$, then $3 \cdot 2^{k-i} \mid b_U$ by three-principality and lcm divisibility. For $k = 2$ its crossing with any member of A is $1 + 3\zeta$ after lcm normalization, with ζ a primitive fifth root: the signed weight-four crossing would produce a joint $2, 5$ twist (Section 21.3). Thus the Galois orbit of U has at least four members, all with the same b_U since Y is rational. For $k = 3$ a single label suffices. Either orbit costs at least $(24/2^i)^2 = q^2$ in $\sum_U b_U^2 = q^2$. This leaves no room for the unit. Hence $b_U = 0$ for every five-free nonunit U in P_3.

All members of P_3 have purely ternary twist (Section 21.4). If Y were three-active, prime avoidance at Y with avoided prime five (Proposition 4.2) would contradict the vanishing just proved. Thus $\theta_Y = 1$. Moreover every three-active J must have five-bearing degree. Indeed block orthogonality for distinct ternary twists gives

$$\sum_{U \in P_3} \overline{\chi_Y(U)} \chi_J(U) = 0.$$

The only five-free contribution is the unit. If $5 \nmid d_J$, every five-bearing U contributes zero modulo a prime over five by lcm divisibility, a contradiction.

The diagonal modular relation and $\theta_Y = 1$ give $|\sum_Z b_Z^2 \theta_Z| = q|b_Y|$. Since the row norm is q^2, its squared mass on twist one is at most $(q^2 + q|b_Y|)/2$.

At a dyadically active column use Proposition 4.2 on the principal two-block in the pure dyadic twist fiber, avoiding three. The resulting nonunit W is three-free, hence also has purely ternary twist and therefore twist one. It is outside A since its dyadic height is below three. Its degree is even, and $b_W \neq 0$ by three-principality of Y. By symmetry $\chi_W(Y) = d_Y b_W / d_W$ is an integer, congruent to d_Y modulo two by two-principality. Thus $|b_W| \geq 2$ for $i = 0, 1$, and $|b_W| \geq 1$ for $i = 2$.

By dimension orthogonality reduced modulo five there is at least one five-free nonunit Z with $b_Z \neq 0$. Every such Z has pure five-twist because it can be neither two-active (that requires full five-height) nor three-active (as just proved), and is five-inactive by defect-zero vanishing in the row of Y, since five does not divide q^2. Hence $\theta_Z = 1$. Write $d_Z = 2^k 3^j$ with $k, j \geq 1$, using the exclusion of nonunit prime-power degrees. Lcm divisibility and three-principality give

$$3 \cdot 2^{\max(k-i,0)} \mid b_Z.$$

The unit, A, Y and W give distinct twist-one contributions outside these positions. Subtracting them from the twist-one budget leaves at most

$$\frac{q^2 + q|b_Y|}{2} - 1 - 4(8/2^i)^2 - b_Y^2 - b_W^2 = \frac{68}{4^i} - 1 - b_W^2 - (|b_Y| - 6/2^i)^2.$$

For $i = 0, 1, 2$ this is at most $63, 12, 9/4$, respectively. Every five-free hit costs at least $36, 9, 9$, respectively. This excludes $i = 2$ outright. For $i = 0, 1$ exactly one hit remains and it has $k = 1$, since $k \geq 2$ costs at least $144, 36$, respectively. Uniqueness and the rationality of Y make this Z Galois-fixed, hence rational.

Now cross Z with any degree-40 label $X \in A \subset P_5$. The lcm-normalized entry $z = S_{XZ}/\operatorname{lcm}(d_X, d_Z)$ is rational since Z is rational, and integral by lcm divisibility. Since $\gcd(d_X, d_Z) = 2$ and $5 \nmid d_Z$, the congruence $\chi_X(Z) = (d_Z/2)z \equiv d_Z \pmod 5$ gives $z \equiv 2 \pmod 5$. Strictness gives $|z| < 2$, an impossibility.

At the first core the surviving degrees satisfy a common residue equation. More generally, take any jointly two- and three-active J at $D = 2^6 3^{2e} 5^4$ on the path separating two and five. For

every prime above two,

$$\sum_{\substack{U\in P_3,\ d_U=25\cdot 3^j\\ 1\le j\le e-2}} \chi_J(U)\equiv 1 \pmod{\mathfrak{p}}, \qquad \mathfrak{p}\mid 2.$$

Indeed block orthogonality gives $\sum_{U\in P_3} d_U\chi_J(U)=0$. Only odd degrees survive reduction. Every odd nonunit degree is 3^j5^t, with $1\le j\le e-1$ and $t=1,2$, by Corollary 4.1 and the exclusion of prime-power degrees. At $t=1$ the weight-five crossing gives an orbit of length four under the order-four subgroup of tame five-inertia (Section 21.8). That subgroup fixes J, since two-activity requires full five-height. The character is constant on the four arguments, so the contribution vanishes modulo two. At $t=2$ the upper ternary height has just been excluded. This proves the congruence. At $e=3$ only degree 75 occurs in the sum; in particular every such J meets at least one degree-75 row. If d_J is odd, the same argument works modulo four after retaining the factors d_U: lcm divisibility makes $d_U\chi_J(U)$ a multiple of four at every even degree. Thus at $e=3$ one has $\sum_{d_U=75,\ U\in P_3}\chi_J(U)\equiv 1 \pmod 4$ for such J, since $75\equiv -1 \pmod 4$.

The degree-75 witness at $e=3$ can be chosen with character field contained in $\mathbb{Q}(\zeta_{27})$. Indeed group the last residue sum by dyadic inertia. It is a two-group, and conjugate summands in an inertia orbit have the same residue. A nonzero contribution therefore comes from a fixed label. Its field is unramified at two, and at five by its codegree. It is consequently a ternary cyclotomic subfield. The weighted inequality for the sets B_j in Proposition 21.9 gives $4\cdot 9|B_1|<729$, so its entire orbit has size at most twenty. A ternary field of conductor at least 81 has inertia order divisible by 27. This proves the claim.

21.10 The local Weyl bound for a nontrivial tame character

Let $s=S/n$ be the integral square-dimensional normalization, $D=n^2$. For an odd prime p, take a character χ of the group of cyclotomic tori whose local factor ψ at p is inflated from $\mathbb{F}_p^\times$ and is nontrivial. In particular the diagonal torus is unsigned. Write E_0 for the projector onto p-inactive labels. On the χ-isotypic space,

$$\|E_0sE_0\|\le p^{-1/2}.$$

More precisely, let v be a p-inactive vector of torus type χ and E fix the complementary twist components, optionally inside one full p-block. When $\psi^2\neq 1$, in Esv the sums of squared absolute coordinates at the p individual phases $1,\zeta_p,\dots,\zeta_p^{p-1}$ are all equal. Higher p-power phases may also occur.

Indeed put $w=EW_{p'}v$, using $s=W_pW_{p'}$. The projector E commutes with the local group, so w is upper-unipotent-fixed and has local torus type ψ. At local level p^a, let Q_1 average over the kernel of reduction mod p. The lower–diagonal–upper factorization of this kernel as in Section 19.11 gives $Q_1w=\operatorname{avg}_{p|t}L(t)w$: the torus character is trivial on units congruent to one modulo p. Hence W_pQ_1w is precisely the restriction of $W_pw=Esv$ to phases in μ_p. Write $u=Q_1w$ in the level-one quotient and B for its upper Borel. If $t\neq 0$ in $\mathbb{F}_p$, $W_p^{-1}U(t)W_p\in BW_pB$. In the explicit Bruhat decomposition the two outer upper unipotents fix u, and the intervening torus factor has argument t or its inverse up to fixed sign. Thus the coefficient at u is a constant times $\psi(t)$ or $\psi(t)^{-1}$. Its sum over $t\neq 0$ is zero. Finite additive Fourier inversion at the zero frequency gives squared mass $\|u\|^2/p$ at the trivial phase. If $\psi^2\neq 1$ the coefficient vanishes individually since W_p sends the torus type ψ to the distinct type ψ^{-1}. Each of the p masses is then $\|u\|^2/p$. Taking $E=1$ proves the norm bound. The same proof in commuting local factors gives the norm bound $\prod_{p\in T}p^{-1/2}$ if input and output are inactive at all these primes and each local factor of χ is tame and nontrivial: after moving the other Weyls past the projectors, the local compressed operators preserve the other local character types.

In particular, for two lists of p-inactive orbits the Fourier matrix of Proposition 19.19 in this mode has norm at most $n/\sqrt{p}$. Its (X,Y) entry remains the usual product of $\operatorname{lcm}(d_X,d_Y)$, the

square root of $[K_X : F][K_Y : F]$, and $\sum_{g\in\mathrm{Gal}(F/\mathbb{Q})} \chi(g)g(z)$, where $z = S_{XY}/\operatorname{lcm}(d_X, d_Y)$ and $F = \mathbb{Q}(z)$, provided the character descends to F.

Two missing principal degrees on the separated path. Take $D = 2^6 3^{2e} 5^4$, $e \geq 3$, with two and five separated and both joined to three. Every member of P_3 and every three-free simple is five-inactive by Section 21.4. Choose a three-free nonunit $U \in P_5$ tame at three by Section 21.4 at a five-active column, avoiding three. Its degree is 40 by Section 21.3. Let R range through Galois orbit representatives of members of P_3 of degrees $4 \cdot 3^{j_R}$ or $5 \cdot 3^{j_R}$. Each type occurs: apply prime avoidance at a column mixing two and three to omit five, and at a column mixing three and five to omit two, respectively (Section 21.3 and Section 21.4). Here the odd hit has five-height one because a full-five label misses a five-active column. In the even case the hit has dyadic height two because it sees a two-active column.

At the even type the normalized entry is $1 + 3\xi$, with ξ a primitive fifth root. At the odd type it is $\eta(2 + 3\xi)$ for a ternary root η (Corollary 19.18). Its tame five-inertia trace is 5η, hence $\eta \in K_U$. Tameness at three gives $\eta \in \mu_3$. The entry field F_R is $\mathbb{Q}(\zeta_5)$ for $\eta = 1$ and $\mathbb{Q}(\zeta_{15})$ otherwise, interpreting $\eta = 1$ for the even type. For a faithful quartic character at five alone the absolute Fourier coefficient on F_R is $3\sqrt{5}$ in every case: when η is primitive cubic its two conjugates sum to -1. The lcm is always $40 \cdot 3^{j_R}$. Dividing the improved column bound by $5 \cdot 9 \cdot 40^2$ gives

$$\sum_R 9^{j_R}[K_U : F_R][K_R : F_R] \leq 9^{e-1}.$$

A single exponent $e-1$ would exhaust this budget, although the opposite degree type is compulsory. Thus both types satisfy $j_R \leq e - 2$, with no hypothesis on the high full-five degrees. At $e = 3$ each of the two types exists, all such degrees are respectively $12, 15$, and the displayed sum of relative index products without the weights is at most nine. In particular the chosen degree-40 witness has field $\mathbb{Q}(\zeta_5)$ or $\mathbb{Q}(\zeta_{15})$ at the first dimension. Wild five-inertia would contribute a factor five to each of the two compulsory index products, exceeding nine. This field is already unramified at two and at most tame at three.

Pure fifth phases on the triangle. At the core with all three edges, take the twist-one $X \in P_3$, $U \in P_5$ of degrees 12 or 36, and 20 from Section 21.6. Their crossing has a nonzero Fourier coefficient for a character cutting out $K = \mathbb{Q}(\zeta_5)$ in the unsigned case and $K = \mathbb{Q}(\zeta_{15})^+$ in the signed case. Both characters restrict at five to a faithful quartic. Apply the equal-norm identity to the χ-orbit vector on X in the full principal five-block with trivial complementary phases. Its coordinate at U after applying s is nonzero. By equality every primitive fifth phase has a nonzero coordinate there. This fiber is globally Galois-stable, and s inverts the torus character, so the field of each such label contains K. Tame five-inertia inversion acts nontrivially on K and preserves each fifth phase by the square action. Hence at least two such labels occur at each of the four phases in P_5.

For the nonopposite signed word, both quartic characters have a nonzero coefficient. For the conductor-five character, the trace over three assigns coefficient 2 to the repeated positions and -1 to the others, up to global sign. The resulting nonzero coefficient has magnitude $3\sqrt{10}$ because the two repeated positions are nonopposite; the character twisted by the quadratic character at three was treated in Section 21.6. Each phase therefore has witnesses for both quartic fields. If the witnesses are in distinct Galois orbits each contributes at least two. If they are in the same orbit its field contains their compositum $\mathbb{Q}(\zeta_{15})$, and the orbit has at least eight labels at precisely two fifth phases with equal multiplicities. Thus each primitive fifth phase occurs at least four times in this case.

If the triangle has no triple, let $Z \in P_3$ be the third twist-one witness of Section 21.6. Write $d_X = 4 \cdot 3^j$, $d_Z = 5 \cdot 3^k$ with $j, k \in \{1, 2\}$, and $m_T = [K_T : \mathbb{Q}(\zeta_5)]$ for these three labels. Their two unsigned crossings against U have faithful-quartic coefficient of magnitude $3\sqrt{5}$. The compressed

column norm gives

$$m_U(9^{j-1}m_X + 9^{k-1}m_Z) \le 36.$$

Consequently a degree-36 or degree-45 witness has relative-index product with U at most three. If both occur, $m_U(m_X + m_Z) \le 4$.

21.11 Which principal rows meet a joint three–five column?

Consider $D = 2^6 3^{2e} 5^4$, $e \ge 3$, on the ordinary path where two and three are separated. Let P_3 be the full principal three-block. A joint three–five column is a label whose twist order is divisible by 15. Every such column meets an odd member of P_3 of five-height one and an even member of five-height at most one (Section 21.7). We determine which even hits have five-height one and bound the number of distinct character values there.

Choose a joint two–five column. Apply the wild-inertia refinement of Section 21.4 to the principal five-block at this column, avoiding three. The resulting three-free label U has degree 20 by Section 21.3 and its field K_U is tame at three. Write $h = [K_U : \mathbb{Q}]$ and τ for normalized field trace to $\mathbb{Q}$. Its normalized orbit sum and eigenvalues are

$$B = \frac{8}{h} \sum_{U' \in \mathrm{Gal}\cdot U} [U'], \qquad b_Y = 8\tau(\chi_Y(U)), \qquad b_1 = 160.$$

The normalized-trace argument gives B in the three-local fusion ring and $b_Y \in \mathbb{Z}$. Indeed $3 \nmid h$ by tameness and the prime support $\{2, 3, 5\}$, and normalized traces of cyclotomic integers in this support lie in $\frac{1}{8}\mathbb{Z}$. For a nonunit Y one has $-160 < b_Y < 160$ by strictness. For $Y \in P_3$ one has $b_Y \equiv 160 \pmod 3$ by applying principality to each U'.

Proposition 21.11. *For every joint three–five J put $j = v_3(d_J)$. Among even $Y \in P_3$ with $v_5(d_Y) = 1$ and $S_{JY} \neq 0$, retain the distinct values $b = b_Y$ different from 10 and 40, forming L_J. Then*

$$\sum_{b \in L_J} v_3(160 - b) \;\ge\; 2e - j - 2.$$

In particular there are at least $\lceil (2e - j - 2)/5 \rceil$ distinct such values. At $e = 3$ either there are two distinct values or one is congruent to 160 modulo 9.

Proof. Let E be the idempotent for P_3, integral in the fusion ring over the local coefficient ring at three (Section 2). Let T be the set of distinct b_Y on nonunits of P_3 with $S_{JY} \neq 0$. Then

$$E[J] \prod_{b \in T} (B - b[1]) = d_J \prod_{b \in T} (160 - b)\, e_1,$$

as is seen on each character. The vacuum coefficient of the dimension idempotent e_1 is $1/D$. Its coefficient on the left is integral at three, so $2e \le j + \sum_{b \in T} v_3(160 - b)$.

Every five-free nonunit $Y \in P_3$ has dyadic height at least two, and the lcm-normalized crossing with U equals $1 + 3\xi$ for a primitive fifth root ξ (Corollary 19.18). Hence $\chi_Y(U) = 5(1 + 3\xi)$ and $b_Y = 10$. At every odd member of five-height one, the degree is $3^k 5$ and the crossing is $2 + 3\xi$ (Section 21.5). Thus $\chi_Y(U) = 4(2 + 3\xi)$ and $b_Y = 40$. Labels of full five-height vanish on J by the defect-zero criterion (Lemma 3.3). The two discarded values contribute at most $v_3(150) + v_3(120) = 2$ in total. Finally $1 \le 160 - b \le 319$ and $j < e$ by three-activity, proving the counts. □

One can prescribe a second nonzero crossing. Fix any joint two–five column K. For the subset of L_J realized by the same Y with $S_{KY} \neq 0$, the valuation sum is at least $e - j - 2$. Indeed, use $E[J][K]$ in the proof and $v_3(d_K) = e$ by separation. Every resulting witness has dyadic height one or two by the defect-zero criterion at K. Its three-height is deficient by principality. Its dyadic and ternary twists are trivial by separation and the deficient heights.

21.12 The exponent at the even frontier

Consider an ordinary minimal table of dimension $D = 2^6 3^{2e} 5^4$, $e \geq 3$. A prime is active on a label if it divides the order of its twist. By Section 21.3 and Section 21.9, some twist order is divisible by 15 and at least one is divisible by 6 or 10.

Proposition 21.12. *A minimal obstruction has $e \leq 35$. If some twist order is divisible by 10, it has $e \leq 18$.*

Proof. Choose a five-active column, also two-active when possible. By the wild-inertia version of prime avoidance in Section 21.4, it meets a three-free $U \in P_5$ for which K_U is tame at three and $d_U = 2^i 5 \leq 40$. If the column is two-active then $i \leq 2$ by defect zero, so $d_U \leq 20$. Our functional at three has $t = 2$. Theorem 16.5 now gives $2e \leq v_3(79!) = 36$ whenever $d_U \leq 20$.

It remains to consider $d_U = 40$ with no twist divisible by ten. Choose a jointly two- and three-active J. Every Y with $S_{JY} \neq 0$ has $v_2(d_Y) \leq 2$ by defect zero. Thus $\chi_Y(U)$ is twice an algebraic integer by lcm divisibility, and $b_Y = F(\chi_Y(U))$ is even. Its vacuum value is 80. At nonunits strictness gives gaps $80 - b_Y = 2k$, $1 \leq k \leq 79$. In the three-local fusion ring,

$$[J] \prod_{k=1}^{79} (B - (80 - 2k)[0]) = d_J \, 2^{79} 79! \, e_0,$$

where e_0 is the dimension-character idempotent. Inactive eigenvalues of $[J]$ vanish and every remaining nonvacuum eigenvalue is killed by the product. The vacuum coefficient gives $2e \leq v_3(d_J) + 36$. Three-activity implies $v_3(d_J) \leq e - 1$, proving $e \leq 35$. □

21.13 Odd ramification on the path separating two and three

Suppose an ordinary perfect, fusion-simple integral modular category has dimension $D = 2^6 3^{2e} 5^4$, with $e \geq 3$, and no twist order is divisible by six. Write P_ℓ for the full principal ℓ-block and $\chi_Y(U) = S_{YU}/d_Y$. By Section 21.11, a twist order divisible by ten occurs. Choose at such a column its degree-20 witness $U \in P_5$ whose character field K_U is tame at three. Use this same U for every J below.

Proposition 21.13. *Let $15 \mid \mathrm{ord}(\theta_J)$ and put $j = v_3(d_J)$. There is an even-degree $Y \in P_3$ with $v_5(d_Y) = 1$ and $S_{JY} \neq 0$ such that $F_Y = \mathbb{Q}(\chi_Y(U))$ ramifies at five, or $4 \mid d_Y$ and F_Y ramifies at three. If $2e - j - 2 > 5$, one can require ramification at five. In particular the first assertion applies at $e = 3$, and the second applies to every such J at $e \geq 7$.*

Proof. For every even-degree $Y \in P_3$ with $v_5(d_Y) = 1$ and $S_{JY} \neq 0$, put $x = \chi_Y(U)$ and $b = 8\tau(x)$, where τ denotes normalized field trace to $\mathbb{Q}$. By Proposition 21.11, the distinct b other than $10, 40$ satisfy

$$\sum_b v_3(160 - b) \;\geq\; 2e - j - 2.$$

All conjugates of x have modulus strictly below 20 and are congruent to 20 above three by $Y \in P_3$. They are also congruent to 20 above five, since $U \in P_5$ gives $\chi_U(Y) = (d_Y/20)x \equiv d_Y$ and $d_Y/20$ is a five-adic unit. Galois covariance gives these congruences at every conjugate prime.

Suppose F_Y is unramified at five. Then $z = x/5$ is an algebraic integer of conductor supported on two and three, by the character-field conductor bound. Hence $k = 2\tau(z) \in \mathbb{Z}$. Indeed the normalized trace of a dyadic root of unity is integral, that of a root of three-power order is half-integral, and the traces multiply at coprime orders. Roots of unity span the cyclotomic integer ring. Moreover $[F_Y : \mathbb{Q}]$ is prime to three since $F_Y \subseteq K_U$, the latter is tame at three, and the cyclotomic degrees at two and five are prime to three. Thus the congruences for the conjugates of z give $k \equiv 8 \pmod 3$. The bound $|k| < 8$ leaves $k = -7, -4, -1, 2, 5$, with $b = 20k$.

The value $k = 2$ is discarded. The others contribute at most $1+1+2+1 = 5$ to the sum, proving the five-ramification assertion.

If F_Y is unramified also at three, its conductor is dyadic. Then $\tau(z)$ is integral, so k is even. Alternatively, if $v_2(d_Y) = 1$, lcm divisibility gives $x \in 2\mathcal{O}_{F_Y}$, hence $z \in 2\mathcal{O}_{F_Y}$ and again k is even. In both cases only $k = -4, 2$ remain. Their values $b = -80, 40$ contribute at most $v_3(240) = 1$. The three-activity of J gives $j \le e-1$, so the required sum is at least $e - 1 \ge 2$. This proves the first assertion. □

22 The surviving odd twist path

22.1 The remaining ordered triple at least prime five

Write P_t for the principal t-block. We use the following consequence of Proposition 19.8. At the ordinary $5, 7, 23$ table, a complementary P_7, P_{23} crossing of common weight 25 has four distinct 23-positions and trivial 7-parts if its entry is unramified at seven. Its actual masses are $4\ell, 7\ell, 7\ell, 7\ell$, where ℓ is the lcm of the degrees. Indeed Proposition 19.8 gives a unique expression in which the seven-parts are two copies of 1 and 23 copies of η. If the entry is unramified at seven, its unique seven-projection is invariant under seven-inertia, so $\eta = 1$. The word obstruction $W(23, 3)$ then forces four positions, and the congruence of the multiplicities modulo seven gives the stated masses. The tame 23-inertia orbit has length 22 by Corollary 19.9.

If a column is active at both five and 23, principal support (Proposition 4.2) at 23 modulo seven supplies a seven-free nonunit $C \in P_{23}$ hitting it. At $5^6 7^{2e} 23^4$ its degree is $25 \cdot 23$: its 23-height is one by quartic principality, its five-height is at least two by the complementary bound in Section 21.4, and it cannot be three by defect-zero vanishing on the chosen column.

Proposition 22.1. *In the frontier dimension $5^6 7^6 23^4$ the following constraints hold on the two possible separations involving five.*

1. *If five and 23 are separated, every seven-free nonunit $C \in P_{23}$ has degree $125 \cdot 23$ and a tame 23-inertia orbit of length 22.*

2. *If five and seven are separated, there is a seven-free $C \in P_{23}$ of degree $25 \cdot 23$. Its crossings with each five-free $B \in P_7$ of 23-height one and with each 23-free $X \in P_7$ of five-height two have only 23-primary monodromy phases. Both resulting 23-words have four positions.*

Proof. Five cannot be isolated, by Proposition 20.9. If five and 23 are separated, a five- and seven-active column therefore exists. A 23-free $X \in P_7$ hitting it has five-height exactly two: the complementary bound gives at least two and defect-zero vanishing rules out three. Each seven-free nonunit $C \in P_{23}$ has 23-height one and five-height at least two, as above. If its five-height were two, neither X nor C could be 23-active, since 23-activity forces full five-height. At their weight-25 crossing, Proposition 19.8 and Corollary 19.9 give a nonidentity 23-position with multiplicity $m \in \{7, 14, 21\}$. Its summands are 23-active, so $125 \mid m \operatorname{lcm}(d_X, d_C)$, impossible at common five-height two. Thus $d_C = 125 \cdot 23$, and the same crossing still has common weight 25 and a tame orbit of length 22.

Suppose instead that five and seven are separated. Five must mix with 23. Use the degree-$25 \cdot 23$ witness C above for that mixed column. Seven-active degrees have full five-height, so C and both indicated partners are seven-inactive. The weight-23 crossing with B has one common seven-part η by Proposition 8.7. If $\eta \ne 1$, all constituents would be seven-active, contrary to $125 \nmid d_B d_C$. At the weight-25 crossing with X, a nontrivial η in Proposition 19.8 would put total mass $23 \operatorname{lcm}(d_X, d_C)$ on seven-active constituents. This again is not divisible by 125. The words at both crossings are therefore pure 23-words on at most four positions, and $W(23, 3)$ forces all four. □

The middle-prime height of an outer witness. For any odd $p < q < r$, if $W(q, 1 + \lfloor q/p \rfloor)$ holds, then every r-free nonunit $U \in P_p$ has $q^2 \mid d_U$, independently of all three exponent caps. Choose a p-free nonunit $B \in P_q$. By Lemma 4.10 their degrees have the forms $p^i q^a, q^k r^b$, with all four powers positive. Suppose $a = 1$ and set $z = S_{BU}/\operatorname{lcm}(d_B, d_U)$. Since $\chi_B(U) = p^i z$, its residue at q is zero. Similarly $\chi_U(B) = q^{k-1} r^b z$ gives $z \equiv q$ above p. Its weight is q, so Lemma 8.4 and rescaling give the forbidden full q-word. In particular at $p = 5, q = 7$ every such U has seven-height at least two, since $W(7, 2)$ holds by Lemma 4.16. At $v_7(D) = 6$ this height is either two or three; in a nonzero hit on a seven-active label only two is possible by defect-zero vanishing.

22.2 The last odd assignment at least prime five

Theorem 22.2. *Suppose that an ordinary perfect fusion-simple minimal table has dimension $D = 5^6 7^B 23^4$. Some twist is divisible by $7 \cdot 23$, for every B. At $B = 6$ some twist is also divisible by $5 \cdot 23$.*

Suppose seven and 23 are separated. Then $B = 2e$, $e \geq 3$, by Lemma 3.3 and Theorem 7.3. The quartic prime 23 is not isolated (Lemma 7.2), so it mixes with five and the degree-$25 \cdot 23$ witness $C \in P_{23}$ of Section 22.1 exists. Every 23-free nonunit $X \in P_7$ has degree $5^i 7^j$, where $i = 2, 3$ and $j < e$, by the complementary bounds. Its weight-25 crossing with C has 23-inertia orbit of length 22 by Corollary 19.9, so K_X is unramified at seven by Proposition 6.2. Neither factor is 23-active, as neither has full seven-height. Each nonidentity 23-phase has mass $7 \operatorname{lcm}(d_X, d_C)$ by Section 22.1, all on constituents of full seven-height; hence $j = e - 1$. Proposition 4.2 supplies such a row X hitting each seven-active Z. Odd-dimensional duality gives Z and Z^* as distinct Galois conjugates. Lemma 21.4 with $(A, s, r, M, m, b) = (125, 7, 23, 25, 22, 2)$ would give $22 \cdot 7^{e-2} < 50$, impossible for $e \geq 3$.

The separated endpoints five and 23. Now let $B = 6$ and suppose five and 23 are separated. Every seven-free nonunit of P_{23} has degree $125 \cdot 23$ by Proposition 22.1. Its row therefore vanishes on every five-active column. A five-active J and a 23-active K have distinct 23-twists. Lemma 21.5 on P_{23} at seven gives $7 \mid d_J, d_K$ and $J, K \notin P_7$. Thus the seven-free labels and all of P_7 have seven-primary twists.

There is an additional uniform restriction on crossings in this path. Choose a 23-free $X \in P_7$ meeting a joint five- and seven-active column; such a column exists since five is not isolated. Its degree is $25 \cdot 7^j$, $j = 1, 2$. Its crossing with a seven-free member of P_{23} has weight 25, so Corollary 19.9 makes K_X ramified at 23 and hence unramified at five. For every seven-free nonunit $Y \in P_5$, write $d_Y = 5^u 23^v$, $u, v \in \{1, 2\}$. Put $g = 5^u$, $L = \operatorname{lcm}(d_X, d_Y)$, and $z = S_{XY}/L$. Principality at seven gives $z \equiv g$ there. When $u = 2$, $\chi_Y(X) = 7^j z$ has zero residue above five, while $z \in K_X$ is unramified at five. Thus $w = 5z/g$ is integral in both cases. It has a positive expression of weight five, strict house below five, and residue five at seven. By Lemma 8.5 that expression uses only seven-power roots and w is an unsigned integral sum of five such roots. Uniqueness at length less than seven and strictness give a nonzero number, at most five, of nonidentity terms. The tame stabilizer acts freely on them, so tame inertia at seven has orbit size at least two on w. In particular every such Y has a seven-ramified field, and K_X ramifies at both seven and 23.

A signed-length test for ramification. In an odd ordinary minimal table on three distinct primes s, p, q, $q \geq 5$, suppose a nonunit crossing z of weight $g = ms + d$ has positive residues g at s, p, where $m \geq 1$ and $0 \leq d < s$. Put $L = \lfloor (g + d)/s \rfloor \leq m + 1$. If $p > m + L$ and z is unramified at s, then

$$z = d + s \sum_{j=1}^{m} \eta_j$$

for p-power roots η_j. Indeed $(z-d)/s$ is integral in the complementary two-prime field; normalized s-trace and two-prime unimodularity give a signed integral word of length at most L. Erase its p-parts. The resulting q-primary word equals m modulo every prime above p, and p is unramified there. Its difference from m has house at most $m + L < p$, hence is zero. Every signed q-letter other than $+1$ has normalized trace at most $1/(q-1)$. Two such letters would give trace less than m, since $L \le m+1$. A single exception must be rational and would require length at least $m+2$. This proves the formula.

Strictness ensures a nonidentity position, so uniqueness of the word of length $m < p$ and the free tame action there give orbit size at least $(p-1)/\max\{a \mid p-1 : 1 \le a \le m\}$. In particular the test holds for all d if $p > 2m+1$. When $g = 2s+d$, $0 < d < s/2$, one has $L = 2$ and $p \ge 5$ suffices. The two roots in this case are distinct and nonidentity: otherwise the displayed full-mass word, rescaled to the character value, contradicts strictness or the odd two-phase theorem.

Which outer prime ramifies. In fact every 23-free nonunit $U \in P_5$ and every label $E \in P_{23}$ of degree $49 \cdot 23$ ramify at 23 and are unramified at five. The assertion about E is nonvacuous: a joint seven- and 23-active column has such a hit on P_{23} by Section 21.4, applied modulo five. Write $d_U = 5^i 7^a$, where $i = 1, 2$ and $a = 2, 3$ by Section 22.1. Their lcm-normalized crossing z has weight 49, strict house below 49, and positive residue 49 at five and 23. If z is unramified at 23, apply the preceding test at $49 = 2 \cdot 23 + 3$ to get $z = 3 + 23(\eta_1 + \eta_2)$ for distinct nonidentity five-power roots. The tame stabilizer at five has order at most two by uniqueness. Thus the two witness fields ramify at a common outer prime, and separation prevents them ramifying at both. Since every such U can be paired with every such E, the prime is the same for all of them.

Suppose it were five. Fix a joint seven- and 23-active J. Then $d_J = 125\, 7^k 23^h$, $k = 1, 2$ and $h = 0, 1$, by separation and the preceding transfer. Its field is unramified at five and its orbit has at least 66 labels: the independent square actions on the seven- and 23-parts give at least $3 \cdot 11$ twist pairs of values, and duality pairs distinct labels in each fiber. Choose an E as above hitting J. If $h = 1$, E-principality and unramifiedness at 23 give $23 \mid \chi_E(J)$, and lcm divisibility makes S_{EJ} a nonzero multiple of $125 \cdot 49 \cdot 23^2$. Its Galois-mean cost on the J-orbit exceeds D because $66 > 49$. All joints therefore have $h = 0$ and every such E hits each one by principality. Fix E and set $L = 125 \cdot 49 \cdot 23$. Write $b = S_{EJ}/L$ and let $c = 7$ or 3 according as $k = 1$ or 2. Since $\chi_E(J) = 125b$, principality gives $b \equiv c \pmod{23}$ in the unramified field. At least two five-inertia conjugates of E agree on the J-orbit. Taking their average gives $M(b) < 161^2/(2 \cdot 66) < 197$, where M is the Galois mean square. If $b \ne c$, the integral element $(b-c)/23$ has mean square at least one; the triangle inequality then gives $M(b) \ge (23-c)^2 \ge 256$. Therefore $b = c$. At $k = 1$ this makes $\chi_E(J) = d_J$, contrary to strictness. All joints have degree $125 \cdot 49$ and $b = 3$. If their number is N, their total cost gives $18N < 161^2$.

More precisely, if the primary twist orders of a joint are $7^u, 23^v$, its orbit length is a multiple of $66 \cdot 7^{u-1} 23^{v-1}$ by the same square action and duality. Our bound thus gives $v = 1$ and $u = 1$ or 2. Write the counts in these two sectors $66A$ and $462B'$; then $1 \le A + 7B' \le 21$. The simultaneous Gauss congruence of Section 21.4 for $T = \{7, 23\}$ makes $5\sqrt{D}$ divide the positive integer

$$(125 \cdot 49)^2 \left(66A \frac{7}{6} \frac{23}{22} + 462B' \frac{23}{22} \right) = (125 \cdot 49)^2 \frac{7 \cdot 23}{2} (A + 6B').$$

In particular $23 \mid A + 6B'$, a contradiction. This eliminates the five-ramified alternative. In the surviving alternative z is unramified at five. Apply the converse test at $49 = 9 \cdot 5 + 4$ to get the unique word $z = 4 + 5 \sum_{j=1}^{9} \xi_j$ in 23-power roots. The 23-tame orbit length of every such U, E is at least eleven.

In particular any five- and seven-active Z whose degree is divisible by five has a 23-free hit of degree exactly $5 \cdot 49$ in P_5. Indeed take a hit U by prime avoidance. Its seven-height is two: it is at least two as above, and full height would miss Z. Write $d_Z = 5^h 7^k 23^2$ and $d_U = 5^i 49$. As K_U is unramified at five, $5 \mid \chi_U(Z)$, and S_{UZ} is a nonzero integral multiple of $5^{\max(h, i+1)} 49 \cdot 23^2$.

The Z-orbit has at least twelve members, by the square twist action and odd duality. At least eleven inertia-conjugate U rows agree there. If $i = 2$, their norm bound would give $11 \cdot 12 < 49$. Hence $i = 1$ as claimed.

We exclude the second alternative. The preceding unsigned word on at most ten 23-positions is exposed among all phases: on a μ_{23}-coset with $b \le 10$ marks use value one on the marks and $-b/(23-b)$ at unmarked positions, tensoring with normalized trace at complementary primes (Lemma 4.15). It is also linearly independent on its support by the cyclotomic coset relations. Thus every actual nonidentity balancing packet of U, E has dimension $5m \operatorname{lcm}(d_U, d_E)$ for a positive m, with the sum of these m at most nine. Both factors are 23-inactive since neither has full five-height, and every 23-active constituent has dimension divisible by 125. If $i = 1$, then $5 \mid m$ for every such packet. There would be at most one nonidentity position, contradicting strictness or Theorem 19.5. Hence all 23-free nonunits of P_5 have $i = 2$. The degree-$5 \cdot 49$ conclusion just proved rules out every five-bearing joint five- and seven-active label.

The remaining joint labels have degree $7^k 23^2$, $k = 1, 2$, by the seven-divisibility transfer and activity. Their set is nonempty since five is not isolated. Fix a 23-free $U \in P_5$ hitting one of them. Its degree must now be $25 \cdot 49$, and it hits every joint by principality. At such a joint J, put

$$L = 25 \cdot 49 \cdot 23^2, \qquad b = S_{UJ}/L.$$

Since K_U is unramified at five and K_J at 23, b lies in a seven-power field. The congruence $\chi_U(J) = 23^2 b \equiv 7^k 23^2$ above five therefore gives $b \equiv 7^k \pmod 5$. Counting at least eleven U-rows agreeing on the entire J-orbit as before, we get

$$M(b) < \frac{35^2}{11 \cdot 12} < 9.29.$$

For $k = 2$ this forces $b = -1$, since $(b+1)/5$ is integral and a nonzero value would give $M(b) \ge (5-1)^2$. For $k = 1$ let $w = (b-2)/5$. A nonzero integer in a seven-power field has mean square at least $5/3$ unless it is a signed seven-power root (Proposition 19.8). The reverse triangle inequality would then give $M(b) \ge (5\sqrt{5/3} - 2)^2 > 9.29$. For a signed seven-power root $M(b) = 29 + 20 \operatorname{Tr}_{\mathrm{mean}}(w)$, and this is at most 9.29 only at $w = -1$. Thus at $k = 1$ we have $b = 2$ or -3.

All these entries are rational. The whole Galois orbit of U now agrees on every joint column. Its length is divisible by eleven by the tame 23-bound (the only possibilities are $11, 22$), and even by odd-dimensional duality. If N_k counts the joint labels of each degree, the averaged row gives $4N_1 + N_2 < 35^2/22 < 56$. Each joint orbit length is divisible by $12 \cdot 5^{u-1} 7^{v-1}$, where its actual primary twist orders are $5^u, 7^v$. Thus $u = v = 1$ throughout and, writing $N_1 = 12A$, $N_2 = 12B'$, we have $4A + B' \le 4$ and $A + B' > 0$. The joint Gauss sum is now

$$L_{5,7} = \frac{5}{4}\frac{7}{6}\, 23^4 (7^2 N_1 + 7^4 N_2).$$

It is divisible by $23\sqrt{D}$, hence $25 \mid A + 49B'$. Consequently $A \equiv B' \pmod{25}$, impossible in the indicated range. We have proved that a twist divisible by $5 \cdot 23$ is compulsory. Together with the first result of this section both edges to the quartic prime are compulsory; only separation of five and seven, or a complete three-edge graph, remains.

A bound on two crossing fields. In the ordinary $5^6 7^{2e} 23^4$ family, apply this to either of the two crossings of Proposition 19.10, at weights 23 and 25. Suppose $5^3 7^{e-1} 23 \mid L_{XY}$. By Lemma 19.11, the seven-part is trivial. In particular the entry field is $F = \mathbb{Q}(\zeta_{23})$. Thus for $n_X = [K_X : F]$ and $n_Y = [K_Y : F]$ we obtain

$$n_X n_Y \le 3.$$

Indeed the three nonzero positions all have weight seven. At each of the four origins in the affine support classified by the enumeration in Proposition 19.10, Section 19.4 provides a nontrivial character at 23 sending those positions, after rotation, into an arc spanning at most four steps modulo 22, of length $4\pi/11$. A multiplicative Gauss sum of modulus $\sqrt{23}$ consequently gives a Fourier coefficient of modulus at least $21\sqrt{23}\cos(2\pi/11)$. On the other hand $\sqrt{D}/L_{XY} \le 161$. Since $\cos(2\pi/11) > 5/6$ and $35\sqrt{23} > 161$, the single-entry bound in Proposition 19.19 excludes $n_X n_Y \ge 4$.

In particular one of the two character fields equals $\mathbb{Q}(\zeta_{23})$ and the other has relative degree at most three. Neither label admits wild inertia at $5, 7$ or 23. At most one of the labels can have a twist divisible by seven, since a label with seven-active twist has at least three distinct seven-inertia conjugates by the square action, all fixing F. If such a label occurs, its seven-part has order exactly seven, its field has relative degree three, and the other label has field F.

More explicitly, the relative field of nontrivial degree can only be obtained by adjoining to F one of

$$\sqrt{5}, \quad \sqrt{-7}, \quad \sqrt{-35}, \quad \zeta_7 + \zeta_7^{-1}.$$

Indeed after excluding wild inertia the ambient field is $\mathbb{Q}(\zeta_5, \zeta_7, \zeta_{23})$. A subfield containing the last factor splits as its compositum with a subfield of $\mathbb{Q}(\zeta_5, \zeta_7)$, by the product description of the Galois group. The trivial, quadratic and cubic subfields of this latter field give the list. The square Galois action now shows that both original twists divide $5 \cdot 7 \cdot 23$, that at least one has order dividing 23, and that neither of them has twist order divisible by 35. In fact any five- or seven-active twist forces an extra inertia image of order at least two or three, respectively, and activity at both forces their relatively prime orbit sizes to divide the same relative degree.

If five and seven are separated. Each of P_5 and P_7 contains a twist-one orbit of length at least 22 outside the vacuum. In P_5 its row field contains $\mathbb{Q}(\zeta_{23})^+$; in P_7 it contains $\mathbb{Q}(\zeta_{23})$. In fact take a seven-free $U \in P_5$ in the pure five-twist fiber hitting a joint five- and 23-active column, and a five-free $V \in P_7$ in the pure seven-twist fiber hitting a joint seven- and 23-active column, by Proposition 4.2. Their degrees are $5^i 23, 7^j 23$, respectively, $i, j \in \{1, 2\}$, by Lemma 4.10 and defect-zero vanishing at 23. Since U lacks full seven-height it cannot be five-active; likewise V cannot be seven-active. Thus both twists equal one.

Take $E \in P_{23}$ of degree $49 \cdot 23$ and $C \in P_{23}$ of degree $25 \cdot 23$ meeting the corresponding joint columns as above. The normalized crossing of U, E has weight 23, positive residue 23 at five by U-principality ($\chi_U(E) = 49z$), and residue zero at 23 by E-principality ($\chi_E(U) = 5^i z$). The odd prime-weight result in Lemma 8.4 gives a common five-part η on the actual monodromy, and an r-word $h + 5\sum_{j=1}^{m} \xi_j$ after its removal, where $r = 23$, $h + 5m = 23$, and $1 \le m \le 4$ counts nonidentity positions with multiplicity in the sum. Both inputs are five-inactive by separation (their seven-heights are zero and two). If $\eta \ne 1$ then every constituent is five-active and has degree divisible by 7^3, whereas $7^3 \nmid d_U d_E$. Thus $\eta = 1$. Uniqueness in the prime-weight normal form shows that the order of the tame stabilizer divides m, so the orbit size is 11 or 22. Since the value is in a pure 23-power field, it generates a field containing the unique degree-eleven subfield $\mathbb{Q}(\zeta_{23})^+$. The orbit of U has even size by duality.

At the crossing of V, C, the same calculation uses $\chi_V(C) = 25z$, $\chi_C(V) = 7^j z$ with primes seven and 23. Both inputs are seven-inactive by separation, and $125 \nmid d_V d_C$ excludes any nontrivial common seven-part in Lemma 8.4. The resulting pure 23-word is $2 + 7(\xi_1 + \xi_2 + \xi_3)$: at most three nonidentity terms occur and $W(23, 3)$ requires four distinct positions including the identity. Its tame stabilizer order divides three and 22, so is one. This pure cyclotomic value therefore generates a field containing $\mathbb{Q}(\zeta_{23})$.

In fact all three principal blocks have large twist-one orbits here. At V, C, Proposition 19.10 puts the three distinct nonidentity positions in μ_{23}. The row C has pure 23-twist by separation and its deficient five- and seven-heights. For a faithful character at 23, the Fourier coefficient of this crossing is a nonzero Gauss sum times the sum of three 22nd roots of unity. The latter cannot

vanish: three unit vectors with sum zero differ by cube roots, and $3 \nmid 22$. Proposition 19.20 gives twist-one members of P_7, P_{23} in a common full five-block, both with character field containing $\mathbb{Q}(\zeta_{23})$. Any additional dyadic factor acts by scalars since the actual twists have odd order. Similarly the pure 23-word at U, E is moved by the order-eleven part of tame inertia. At least one of its nonzero Fourier components has order divisible by eleven, and its character has 23-power conductor. Here E has pure 23-twist by its deficient five- and seven-heights. The same proposition gives members of P_5, P_{23} in a common full seven-block with twist one and fields containing $\mathbb{Q}(\zeta_{23})^+$. The two choices of member in P_{23} may differ.

22.3 Residues of mixed words at five, seven and twenty-three

Apply Proposition 19.17 at the odd assignment $(5, 7, 23)$ without assuming that a particular edge is missing. Suppose no twist is active at all three primes. At a weight-25 crossing of a degree $25 \cdot 23$ label $C \in P_{23}$ and a degree $25 \cdot 7^j$ label $X \in P_7$, both five-heights are two. At a weight-23 crossing of the same C with a five-free $B \in P_7$ of degree $7^j 23$, the argument is identical. In both cases take $p = 5, T = \{7, 23\}$ at C and note that the opposite degree divided by the common weight is a five-unit. Write the 23-positions as the identity and three nonidentity positions as in Proposition 19.10. If the seven-phase η is nontrivial, apply seven-inversion and any 23-dilation carrying one occupied nonzero position to a vacant one. At weight 23 the coefficient of η at the occupied position is seven. At weight 25, at most two of the letters have seven-phase one, so at least one of the three nonidentity positions has all seven letters of phase η. Use that position. The mixed difference has zero constant coefficient, and at the image position its coefficient is, up to sign, $7(\eta - \eta^{-1})$, a unit modulo five. It cannot be zero modulo $5\mathcal{O}$: the only relation on the 23 positions over the seven-primary integers, also after reduction modulo five, is the constant coefficient vector. Thus $\eta = 1$ in both cases. At the weight-25 crossing X is seven-inactive as well: if it were seven-active the singleton test with $p = 5, T = \{23\}$ at X would make all 23-dilations of that pure word congruent modulo five, which the same occupied and vacant positions disprove.

At $(5, 7, 23)$ the row C of degree $25 \cdot 23$ in either crossing just considered forces 22 distinct full five-blocks, each entirely seven-inactive in the absence of a joint twist. Its reduced character at the opposite label is a nonzero scalar modulo five times a pure four-position 23-word. The three marked nonidentity exponents have common coefficient $7 \not\equiv 0 \pmod 5$. The polynomial of a primitive 23rd root is irreducible modulo five since 5 has multiplicative order 22 modulo 23. A dilation stabilizing the reduction must preserve the three nonzero marked positions, by the constant-vector relation and the unchanged identity coefficient. Its order divides both three and 22, hence it is the identity. All 22 reduced values are distinct.

22.4 An orientation for two separated odd primes

Write P_s for the principal s-block. A positive expression of weight g is a sum of roots of unity with nonnegative rational coefficients adding to g. The balancing expression for $z = S_{UV}/\operatorname{lcm}(d_U, d_V)$ has weight $g = \gcd(d_U, d_V)$, and all conjugates satisfy $|\sigma z| < g$ at a pair of nonunits in a minimal table. Congruences above a prime are understood at all primes above it.

Lemma 22.3. *Suppose a cyclotomic integer z has a positive expression of integer weight g in roots supported at primes $\geq \ell$, and all its conjugates have modulus strictly below g. If $z \equiv g$ above the two distinct primes s, t and $st > g\ell/(\ell - 1)$, then $\mathbb{Q}(z)$ ramifies at s or t.*

Proof. If both primes are unramified then $(z - g)/(st)$ is integral. Since $st > g$, the number $1 + (z - g)/(st)$ is a convex combination of 1 and z/g with both coefficients positive. Its house is strictly below one, so its norm forces it to vanish. This gives $z = g - st < -g/(\ell - 1)$. The normalized trace of a root of order n is $\mu(n)/\varphi(n) \geq -1/(\ell - 1)$ under the stated prime restriction. Averaging the positive expression gives a contradiction. □

Lemma 22.4. *Let s, t, r be distinct odd primes and suppose z is a cyclotomic integer with a positive expression of weight r in roots supported on these primes. Assume $z \equiv r$ above s, t and $\mathbb{Q}(z)$ is unramified at s. Write $r = sm + c$, $0 < c < s$, and $L = \lfloor (r + 2c)/s \rfloor$. Suppose*

$$L < t, \qquad L + 1 < r, \qquad L < 2(t - m), \qquad L < m + 3.$$

Then $z = c + s \sum_{j=1}^{m} \eta_j$ for t-power roots η_j. This expression gives the multiplicities in every positive expression of weight r on the three primes provided

$$m + 1 < t, \qquad \frac{m + 1}{t - m - 1} < \min(s - 1, r - 1).$$

Proof. In a cyclotomic field of sufficiently divisible conductor on s, t, r, the fixed field of s-inertia is the cyclotomic factor on t, r. Thus $\alpha = (z - c)/s$ is integral in that factor. Average the positive expression over s-inertia. An s-power root contributes 1, $-1/(s - 1)$ or 0, according as its order is $1, s$ or larger. Give positive letters cost one and negative letters cost two. Subtracting c and dividing by s expresses α with rational signed cost at most $(r + 2c)/s$. The signed two-prime cyclotomic configuration has integral cost minimizers for integral values (Theorem 4.9 and Proposition 19.8). Choose an integral word of cost at most L.

Erase the t-parts of its letters, obtaining an r-primary word v. Reduction above t gives $v \equiv \alpha \equiv m$. At ambient r-power order r^e ($e \geq 1$), the relations modulo t are generated by the full μ_r-coset sums. Indeed reduction of $\Phi_{r^e}(x) = \sum_{j=0}^{r-1} x^{jr^{e-1}}$ gives precisely these relations; here the congruence holds at all primes over t and t is unramified. The coefficient vector of $v - m$ has support at most $L + 1 < r$. Thus each coefficient vanishes modulo t. Away from 1 the absolute values are at most $L < t$, so those coefficients vanish in $\mathbb{Z}$. The integer v lies between $-L/2$ and L and is congruent to m modulo t. The hypotheses give $v = m$. Each negative letter would add three to the cost above the net augmentation m, whereas $L < m + 3$. Hence all m letters are positive and have trivial r-part.

For the last claim mark 1 and the η_j in the t-primary factor. In each μ_t-coset with b marks assign the value 1 at the marks and $-b/(t - b)$ elsewhere. The coset sums vanish, so this defines a rational-linear functional. Tensor it with normalized trace on the s, r factors. This trace on a nonidentity root is strictly below one, and its negative values have magnitude at most $1/\min(s-1, r-1)$. By the last inequality the product functional attains one only at marked pure t-roots. Its value at z is r, forcing every positive expression of weight r onto those marks. They are linearly independent, since no full μ_t-coset is marked. This identifies their multiplicities. □

Proposition 22.5. *Consider an ordinary minimal table of dimension $5^6 7^6 23^4$ with separated twist primes five and seven. Let U range over all seven-free nonunits of P_5 and V over all five-free nonunits of P_7. A single $t \in \{5, 7\}$ ramifies in every one of these row fields; the other prime s is unramified in all of them. If $t = 5$ every such V has seven-height two, and if $t = 7$ every such U has five-height two. Every s-active degree has 23-height at most one. There is a twist-one orbit of size at least 44 among the V; when $t = 5$ the same holds among the U.*

Proof. The degrees are $5^i 23^a$ and $7^j 23^b$, respectively, with $i, j, a, b \in \{1, 2\}$ by square divisibility, positive defect in the principal block and the exclusion of prime-power degrees. By Section 22.2 both sets contain twist-one witnesses at 23-height one, from the respective joint columns at five and 23, or seven and 23. Whenever $\min(a, b) = 1$, the common crossing weight is 23. Principality gives $z \equiv 23$ above five and seven, since the multipliers $d_V/23$ and $d_U/23$ in $\chi_U(V)$ and $\chi_V(U)$ are units at five and seven, respectively. Lemma 22.3 applies since $35 > 23 \cdot 5/4$. Each row field contains the entry field, and separation prevents a row field from ramifying at both endpoints (Proposition 6.2). Pair the two height-one witnesses, then pair arbitrary members of either set with the opposite height-one witness. This forces the common t, which ramifies also in every entry field with $\min(a, b) = 1$.

For $(s,t) = (7,5)$ the parameters (m,c,L) of Lemma 22.4 are $(3,2,3)$, and for $(s,t) = (5,7)$ they are $(4,3,5)$. Both satisfy its hypotheses, including exposure since $4 < 6$ and $5/2 < 4$, respectively. Strictness forces some $\eta_j \neq 1$. At a nonidentity phase the total dimension of the corresponding summands of $U \otimes V$ is $sm_x \operatorname{lcm}(d_U, d_V)$ for $1 \leq m_x \leq m < s$. Every t-active label has full s-height three by separation (Lemma 3.3). Both inputs have smaller s-height, so their t-twists are trivial. The packet therefore consists of t-active summands. Divisibility by s^3 proves $v_s(\operatorname{lcm}(d_U, d_V)) \geq 2$. Pairing again with the opposite height-one witness gives the degree assertion for each row.

Suppose an s-active H has 23-height two. Its degree is $t^3 s^k 23^2$, $0 \leq k \leq 2$, and its field lies in an s-power cyclotomic field by the codegree conductor bound. If $k > 0$, choose a t-free principal s-row Y hitting H by Proposition 4.2. Its field is unramified at s, so S_{YH}, lying in both fields, is rational. Principality gives $s \mid \chi_Y(H)$, and Y has s-height two. By lcm divisibility, S_{YH} is then a nonzero integer multiple of $5^3 7^3 23^2 = \sqrt{D}$, impossible in its row norm. If $k = 0$, choose the canonical 23-height-one Y in the same block. Principality makes it hit every conjugate of H, with rational entries divisible by $\sqrt{D}/s$. The orbit of Y has size at least 22 (Section 22.2). The orbit of H has size at least $s - 1$ by the square action on its s-twist and free duality on each twist fiber. The averaged Y-row has squared norm $D/|\operatorname{Gal}\cdot Y|$ and agrees with Y on this entire orbit of H. This would give $22(s-1) < s^2$, impossible for $s = 5, 7$.

For the two chosen twist-one witnesses Section 22.2 gives $\mathbb{Q}(\zeta_{23}) \subseteq K_V$ and $\mathbb{Q}(\zeta_{23})^+ \subseteq K_U$. Their common nontrivial entry field F is t-primary by Lemma 22.4, hence disjoint from $\mathbb{Q}(\zeta_{23})$. This gives $[K_V : \mathbb{Q}] \geq 44$. If $t = 5$ and F is real, then K_U contains the real compositum $F\mathbb{Q}(\zeta_{23})^+$, and free duality makes K_U nonreal, so its degree is at least 44. If $t = 5$ and F is nonreal, its degree is divisible by four: the involution in the cyclic group of order $4 \cdot 5^h$ acts nontrivially on F. The compositum then already has degree at least 44. □

For the chosen height-one U, V, let $F = \mathbb{Q}(z)$ as in the proof. The t-inertia conjugates of each label occupy at least $[F : \mathbb{Q}]$ distinct nonprincipal 23-blocks. Indeed the opposite character value is a 23-unit rational factor times z. If the tested row were principal at 23, its opposite value would reduce to zero. Since F is unramified there, $z/23$ would be integral and of strict house less than one, whereas $z \equiv 23$ above s. If two conjugates have the same 23-block, evaluate on the entire Galois orbit of the opposite label. All conjugates of the difference of their z-values vanish above 23, and that difference belongs to $23\mathcal{O}_F$. In the t-primary words of Lemma 22.4, the coefficient differences after cancelling s range between $-m$ and m. Their residues are constant on each μ_t-coset by the cyclotomic relation modulo 23. Since $2m < 23$, the integer differences themselves are constant there, giving equality of the two z-values. Finally t-inertia realizes every conjugate of F.

The complementary seven-blocks. On the separated five–seven path at $5^6 7^6 23^4$, Section 22.3 puts degree-$25 \cdot 23$ members of P_{23} in 22 distinct five-blocks, all seven-inactive. In the other direction, at least eleven distinct nonprincipal seven-blocks contain a degree-$49 \cdot 23$ member of P_{23}, and all their members are five-inactive.

Indeed Section 22.2 gives $E \in P_{23}$ of degree $49 \cdot 23$ and $U \in P_5$ of degree $5^i 23$, with $\chi_E(U) = 5^i(h + 5\sum_{j=1}^{m} \xi_j)$, where $1 \leq m \leq 4$ and the ξ_j are nonidentity 23-power roots, counted with multiplicity. If two tame 23-conjugates of E lie in the same seven-block, evaluate on all conjugates of U. All conjugates of the difference of the two m-term sums vanish above seven, so the difference belongs to $7\mathbb{Z}[\zeta_{23^a}]$ at a common conductor 23^a. By the formula for Φ_{23^a} its coefficient vector is constant modulo seven on each coset of μ_{23}. Each coset has a vacancy in the combined supports, and the coefficients have absolute value at most four. Thus the multisets coincide. Their tame stabilizer has order dividing both m and 22, giving at least eleven blocks. The principal seven-block is Galois-stable, so none is principal. Each contains a positive-seven-defect label whose residue character is moved by 23-inertia. Since separation forbids a joint $5, 7, 23$ twist, Section 19.8 makes the whole block five-inactive.

For the labels U, V of Proposition 22.5, their entire 23-blocks are s-inactive. At height two the block is a singleton, and the proposition applies directly. At height one pair with an opposite height-one witness. Some t-inertia conjugate changes the t-word; after removing the common factor s, the difference has house at most $2m < 23$. It therefore remains nonzero modulo 23 in its unramified field. The character multiplier at each endpoint is a 23-unit, so its residue character moves (evaluate on all conjugates of the opposite witness). The same full-block criterion applies.

22.5 The two active twists on the separated odd path

Let five and seven be separated in the ordinary minimal table of dimension $D = 5^6 7^6 23^4$. Use the orientation of Section 22.4, writing t for the prime ramifying in the complementary principal rows, s for the other of $5, 7$, and $r = 23$. Here a complementary principal s-row means a t-free nonunit in P_s.

Proposition 22.6. *Every label jointly active at s, r has degree divisible by s.*

Proof. A joint label H of s-free degree must have $d_H = t^3 r$: separation forces full t-height, activity precludes full r-height, and a nonunit cannot have prime-power degree. Fix the height-one witness $Y \in P_s$ of degree $s^2 r$ from the orientation. Its field is unramified at s and ramified at t, and its orbit has size at least 22 (Section 22.2 and Section 22.4). It hits every such H by principality. Write $S_{YH} = t^3 s^2 r z_H$. The cyclotomic integer z_H is unramified at s. Since $\chi_Y(H) = t^3 z_H \equiv t^3 r$ at every prime above s, $z_H - c \in s\mathcal{O}$ for the least absolute residue $c = \pm 2$ of 23 modulo s. If z_H is rational then $|z_H| \geq 2$. Otherwise $(z_H - c)/s$ is a nonzero algebraic integer, so the triangle inequality for the Galois mean-square norm gives $M(z_H) \geq (s-2)^2$.

All conjugate Y-rows agree on the rational entries, while at least two t-inertia conjugates agree on the remaining H since $t \nmid f_H$. Average the former set of rows on the rational coordinates and the latter on the other coordinates, and take Galois means. The two sets are Galois-stable. The number N_0 of these labels is consequently bounded by

$$N_0 < s^2 r^2 \left(\frac{1}{22 \cdot 4} + \frac{1}{2(s-2)^2} \right). \tag{22.1}$$

Here $D/(t^3 s^2 r)^2 = s^2 r^2$ and the vacuum gives strictness.

For a joint Galois orbit with primary orders s^u, r^v its size is $A(s-1)(r-1)s^{u-1}r^{v-1}/2$ with A a positive integer: the independent square actions are transitive on the possible twist pairs, and duality acts freely on each fiber. Set $k_p(1) = 1$ and $k_p(j) = (p-1)p^{j-2}$ for $j > 1$. The sum of $c_s(H)c_r(H)$ on that orbit, with c_p from the simultaneous Gauss congruence in Section 21.4, is $Asrk_s(u)k_r(v)/2$. Write B_0 for the sum of $Ak_s(u)k_r(v)$ on the s-free joint orbits. The congruence $s^3 \mid L_{s,r} = \sum_H d_H^2 c_s(H)c_r(H)$ applies since $s^3 \mid \sqrt{D}$. After multiplication by 2, each positive s-height orbit contributes a multiple of s^3. The remaining sum is $(t^3 r)^2 srB_0$, hence $s^2 \mid B_0$. Comparing the orbit size with $Ak_s(u)k_r(v)$ gives $N_0 \geq (s-1)(r-1)B_0/2$. A nonzero N_0 would therefore be at least 1100 at $s = 5$ or 3234 at $s = 7$; (22.1) gives respectively $N_0 < 886$ and $N_0 < 814$. □

Proposition 22.7. *Every label jointly active at s, r has twist of order exactly sr.*

Proof. By Proposition 22.6 write $d_J = t^3 s^h r^k$ for such a joint label, with $h = 1, 2$ and $k = 0, 1$. The degrees follow from separation and activity. By Proposition 4.2 choose a complementary principal s-row Y hitting J with pure s-primary twist. Its twist is one, because s-activity would require full t-height. It has $d_Y = s^2 r$: the s-height is two by Section 22.4, and full r-height would miss J. Put $n = \sqrt{D}$ and $S_{YJ} = (n/r)b$. Then b is a nonzero algebraic integer. Indeed, $\chi_Y(J)$ is divisible by s because $Y \in P_s$, $s \mid d_J$ and K_Y is unramified at s; lcm divisibility supplies the other factors. The field of J is unramified at t and that of Y at s, so b belongs to an r-power cyclotomic field.

Suppose the primary twists of J have orders s^u, r^v. Its orbit has at least $(s-1)(r-1)s^{u-1}r^{v-1}/2$ labels: use independent square actions on the two primary twists and the distinct duals in each fiber. At least two t-inertia conjugates of Y agree on this whole orbit. Their average has squared row norm at most $D/2$, with a positive vacuum contribution. Writing $M(b)$ for the mean of $|b|^2$ over embeddings, this gives

$$(s-1)(r-1)s^{u-1}r^{v-1}M(b) < r^2.$$

Since $M(b) \geq 1$, the only exception to $u = v = 1$ could be $(s, t, u, v) = (5, 7, 2, 1)$. In that case $|\operatorname{Orb}(J)| \geq 220$, and every nonzero b satisfies $M(b) < 529/440$.

An algebraic integer in an r-power field with this mean is a signed root of unity. To see this, write its integer coefficients in cosets of μ_r. Distinct cosets are orthogonal for the mean. For one coset with coefficients c_i the mean is $\sum_{i<j}(c_i - c_j)^2/(r-1)$. A nonconstant coset contributes at least one. If the range is at least two, the distances to a maximum and minimum give a contribution at least two. For range one, unless one value occurs just once, the contribution is at least $2(r-2)/(r-1)$. These bounds prove the assertion.

For every hit Y here K_Y contains $\mathbb{Q}(\zeta_{23})^+$: pair Y with the degree-$49 \cdot 23$ row of P_{23} in Section 22.2. The proof there applies to every such Y, since both inputs are five-inactive and the nontrivial five-part would force $7^3 \mid d_Y d_E$. The remaining pure 23-word has at most four nonidentity letters after removing its common factor five, so its tame orbit has size at least eleven. If b were rational, the entire Y-orbit would agree on the J-orbit, giving $11 \cdot 220 < 529$. Nor can b have order divisible by 23^2, since then $[K_J : \mathbb{Q}] \geq 22 \cdot 23$, whereas $2|\operatorname{Orb}(J)| < 529$. Hence b is a signed primitive 23rd root. In particular K_Y contains $\mathbb{Q}(\zeta_{23})$ and also a nontrivial seven-primary field by Section 22.4. Its orbit size is therefore a multiple of 22 and at least 44.

Choose a faithful tame character ψ of $\mathbb{F}_{23}^\times$ and form its normalized orbit vector v on $\operatorname{Orb}(Y)$, giving the same coefficient on the fibers of restriction to $\mathbb{Q}(\zeta_{23})$. This is a unit vector of 23-torus type ψ supported at twist one. Let m_Y, m_J be the two orbit sizes. For each conjugate of J the Fourier coefficient of the normalized Weyl image Sv/n has modulus

$$\frac{\sqrt{m_Y}}{22 \cdot 23}\left|\sum_{a \in \mathbb{F}_{23}^\times} \psi(a)\zeta_{23}^a\right| = \frac{\sqrt{m_Y}}{22\sqrt{23}},$$

since b is a signed primitive 23rd root. Thus the J-orbit alone carries squared mass $m_Y m_J/(22^2 \cdot 23) \geq 20/23$. On the other hand its 23-twists occupy exactly eleven primitive phases by the Galois square action. The local Weyl formula in Section 21.10, also used in Proposition 23.5, bounds the mass of Sv/n on each one by $1/23$ because v is 23-inactive and $\psi^2 \neq 1$. This gives at most $11/23$, contradicting the lower bound. □

Proposition 22.8. *Suppose $s = 7$ and that J in Proposition 22.7 exists. Its orbit has size 66 and $(v_7(d_J), v_{23}(d_J)) = (1, 1)$ or $(2, 0)$. Precisely one orbit of five-free pure-twist labels in P_7 meets J. Each member Y is also in P_5, has orbit size 44, field $\mathbb{Q}(\zeta_{23}, \sqrt{5})$, and $S_{YJ}/(n/23)$ is a signed primitive 23rd root. For this Y and every seven-free nonunit $U \in P_5$, the lcm-normalized crossing is $9 + 7(\eta + \eta^{-1})$ with η of order five.*

Proof. Retain the notation b, m_Y, m_J from above. Now m_J is a multiple of 66, by the independent square actions and duality. Also m_Y is a multiple of 22 and at least 44: the pure prime-weight crossing with degree $25 \cdot 23$ in P_{23} puts $\mathbb{Q}(\zeta_{23})$ in K_Y, by the argument in Section 22.2, and the orientation gives a nontrivial five-primary subfield. Here $m_J < 529/2$, so $23 \nmid m_J$. Thus b belongs to $\mathbb{Q}(\zeta_{23})$. Write $b = c_0 + \sum_{a \neq 0} c_a \zeta_{23}^a$ with integer coefficients. For every character ψ of $\mathbb{F}_{23}^\times$ with $\psi^2 \neq 1$, the same normalized orbit vector and local phase estimate just used give

$$m_Y m_J \left|\sum_{a \neq 0} c_a \psi(a)\right|^2 \leq 11 \cdot 22^2. \tag{22.2}$$

Indeed the orbit Fourier coefficient factors through the primitive Gauss sum, and all J-coordinates have the same modulus. This holds also for every Galois conjugate of the inner sum.

If $m_Y m_J > 11 \cdot 22^2$, all the indicated sums vanish by the algebraic norm. Then c_a is constant on squares and on nonsquares, so $b \in \mathbb{Q}(\sqrt{-23})$. A nonrational integer of that field has mean square at least six, whereas $M(b) < 529/132$. A rational nonzero b would give $m_Y m_J < 529$ by averaging the entire Y-orbit, also impossible. Thus $m_Y m_J \le 5324$, forcing $m_J = 66$ and $m_Y = 44$: the only further multiple of 22 in range is 66, and 3 cannot divide m_Y since K_Y is supported on $5, 23$. Its two required subfields now give the asserted compositum.

At these orbit sizes the right hand side of (22.2) divided by $m_Y m_J$ is $11/6 < 2$. Any nonzero inner sum w is therefore a root of unity: $1 - w\overline{w}$ has all conjugates of modulus less than one. To read this condition, write $\mathbb{F}_{23}^{\times} = \{\pm a^j : 0 \le j < 11\}$ with a of order eleven, and put $A_j = c_{a^j} + c_{-a^j}$, $B_j = c_{a^j} - c_{-a^j}$. The order-eleven Fourier coefficient of each vector is zero or a root of unity in $\mathbb{Q}(\zeta_{11})$. The only coefficient relation on the eleven positions is the constant relation. Thus each vector is constant apart from at most one change by ± 1. Since $A_j \equiv B_j \pmod 2$, any changes occur at the same index. Both constant would again place b in the quadratic field. Removing one signed term $\zeta_{23}^{\pm a^j}$ therefore leaves $\beta \in \mathcal{O}_{\mathbb{Q}(\sqrt{-23})}$.

Write $\beta = (x + y\sqrt{-23})/2$, $x \equiv y \pmod 2$. For a primitive 23rd root ζ, the quadratic Gauss sum gives

$$M(\beta \pm \zeta) \ge (x^2 + 23y^2)/4 + 1 - (|x| + 23|y|)/22.$$

For $y \neq 0$ this exceeds $529/132$ (use $|x| \ge 1$ when $|y| = 1$). Hence β is rational. The constant Fourier coefficient, by the two orbit norms, has absolute value at most $23/\sqrt{44 \cdot 66} < 1/2$. Since the mean of ζ is $-1/22$, this forces $\beta = 0$.

Choose the same faithful ψ for every such hit orbit. The resulting unit vectors are orthogonal, while their Weyl projections onto the J-orbit are collinear: that transitive orbit has at most one copy of their common full Galois character. Each projection has squared norm $44 \cdot 66/(22^2 \cdot 23) = 6/23$. For k' hit orbits a normalized linear combination therefore gives squared norm $6k'/23$ there, at most $11/23$ by the same phase estimate. Thus $k' = 1$.

Lemma 4.19 puts a member of the unique hit orbit in P_5 and hence puts all of it there. In its congruence the signed primitive roots over this orbit sum to ± 2. Since $d_Y = 7^2 23$ and $\chi_J(Y) = \pm 7^{3-h} 23^{1-k} \zeta$, it gives $2 \cdot 7^{5-h} 23^{2-k} \equiv \pm 1 \pmod 5$. Among $h = 1, 2$ and $k = 0, 1$ this leaves exactly the two asserted profiles.

Finally the pure five-word at Y, U in Section 22.4 is $2 + 7\sum_{j=1}^{3} \eta_j$. Its nontrivial field is contained in $\mathbb{Q}(\sqrt{5})$. The multiset of three roots is unique, since it has fewer than five positions in each defining coset. Every nonidentity orbit under the subgroup fixing $\sqrt{5}$ has size $2 \cdot 5^{e-1}$ at order 5^e. The three roots therefore consist of one identity and one inverse pair of order five. □

Corollary 22.9. *If $s = 7$ all jointly seven- and 23-active labels have the single degree $125 \cdot 7 \cdot 23$. Every seven-free nonunit of P_5 has degree $25 \cdot 23$, and every label of full 23-height has twist one.*

Proof. A seven-free nonunit $U \in P_5$ has degree $5^i 23^a$, $i, a \in \{1, 2\}$. Pair it with a 23-free nonunit $X \in P_7$ from Proposition 4.2. If $i = 1$, their lcm-normalized crossing z has weight five. Since $d_U/5$ is a seven-unit, X-principality gives $z \equiv 5$ above seven; K_U is unramified there by Section 22.4. The integral $1 + (z - 5)/7$ has strict house below one, so $z = -2$. The normalized trace of every root on $5, 7, 23$ is at least $-1/4$, giving a contradiction. Thus $i = 2$.

At least one joint exists by Section 22.2, hence so does its pure hit Y from Proposition 22.8. For every U above, the normalized entry there is $9 + 7(\eta + \eta^{-1})$ of Galois mean $11/2$. If $a = 2$, then $n/\operatorname{lcm}(d_U, d_Y) = 35$. Averaging the 44 conjugate Y-rows and retaining just U would give $44(11/2)^2 \le 35^2$, impossible. Thus every such U has degree $25 \cdot 23$. Fix the particular twist-one U of Section 22.2. Its orbit has at least 44 members by Section 22.4. On all the pure Y hit orbits described above, the averaged U-row has the same normalized entry $11/2$. Now $n/\operatorname{lcm}(d_U, d_Y) = 805$, so the number b of these Y-orbits satisfies $44^2 b(11/2)^2 < 805^2$, hence $b \le 11$.

Each such Y meets at most one joint orbit of each 23-twist square class: its faithful orbit vector puts $6/23$ squared mass on each joint orbit it meets, whereas the eleven possible phases of one square class carry at most $11/23$. Thus there are at most 22 joint orbits. All have size 66 and exact twist order $7 \cdot 23$. Let A_{hk} count those of $7,23$-degree heights (h,k). The only possibilities are $(1,1)$ and $(2,0)$. In $2L_{7,23}$ the two contributions together equal

$$7 \cdot 23 \cdot 5^6(7^2 23^2 A_{11} + 7^4 A_{20}).$$

Since $23^2 \mid L_{7,23}$ by Section 21.4, we have $23 \mid A_{20}$. The orbit bound forces $A_{20} = 0$.

Finally suppose a five-active H has full 23-height. Its degree is $7^3 5^h 23^2$, $h \le 2$, and its field is unramified at 23. Choose a seven-free hit $U \in P_5$ of the now determined degree. Its 23-inertia orbit contains at least eleven labels by the pure word against degree $49 \cdot 23$ (Section 22.2). They agree on at least four conjugates of H, by its five-twist square action and duality. Since $\operatorname{lcm}(d_H, d_U) = n/5$, their averaged row would give $44 < 25$. Thus no such H exists; Section 22.4 excludes seven-activity at full 23-height, and full height itself excludes 23-activity. □

22.6 The unramified endpoint at seven in dimension $5^6 7^6 23^4$

Consider the branch of Section 22.5 at $D = 5^6 7^6 23^4$, with five and seven separated and the complementary principal rows unramified at seven. Put $r = 23$, $n = \sqrt{D}$. Every seven-active label H then has

$$v_{23}(d_H) = 1.$$

Thus the 23-free possibility is excluded even for a pure seven-twist. At each such H, Lemma 4.19 supplies a twist-one 23-free hit in $P_7 \cap P_{23}$. Moreover W_7 has integral entries at every prime above 23.

Proof. Full 23-height was excluded in Section 22.5. Suppose $r \nmid d_H$. Separation and the absence of prime-power degrees give $d_H = 125 \cdot 7^j$, $j = 1, 2$. Take a five-free nonunit $Y \in P_7$ in the pure seven fiber meeting H; its twist is one, since seven-activity would require full five-height. Write $d_Y = 49r^a$, $a = 1, 2$, by the orientation result (Section 22.4). The lcm-normalized crossing z is unramified at seven in K_Y, and $\chi_Y(H) = 125z \equiv 0$ at every prime over seven, so $z \in 7\mathcal{O}$. At $j = 1$ its house is strictly below seven, impossible. At $a = 2$ the entry $S_{YH} = n(z/7)$ already costs at least n^2 in the Galois-averaged row norm. Thus $j = 2, a = 1$. Write $S_{YH} = (n/r)b$. Then b is a nonzero integer of an r-power cyclotomic field since K_H is unramified at five and K_Y at seven. Any such pure Y of degree $49r$ has $K_Y \supseteq \mathbb{Q}(\zeta_r)$ by its crossing with the seven-free degree-$25r$ row of P_r (Sections 22.2 and 22.5). Its field ramifies at five by the orientation in Section 22.4. Thus $m_Y = [K_Y : \mathbb{Q}]$ is a multiple of 22 and at least 44. Also $6 \mid m_H$ by the seven-twist square action and odd duality.

We claim $b \in \mathbb{Z}$. Otherwise let r^c be its conductor at r, and express $b = \sum_j d_j \zeta_{r^c}^j$, $d_j \in \mathbb{Z}$. If b lies outside the quadratic subfield there is a character ψ of exact conductor r^c, with ψ^2 of the same conductor, such that its Fourier coefficient is nonzero: this follows by Fourier inversion and minimality of c (at $c = 1$ omit the two characters of order at most two). This character descends to $\mathbb{Q}(b)$ and hence gives orbit vectors on both inputs. Their normalized s-matrix coefficient has absolute value

$$\frac{\sqrt{m_Y m_H}}{r\varphi(r^c)}\, r^{c/2} \left| \sum_{(j,r)=1} d_j \overline{\psi(j)} \right|.$$

This is the primitive Gauss-sum formula applied to the orbit matrix as in Proposition 19.19. Both inputs are r-inactive, by Section 22.5 for the r-free H (all jointly seven–r-active degrees there are $125 \cdot 7r$). Proposition 19.23 bounds the coefficient by $r^{-c/2}$. Thus $m_Y m_H |\sum d_j \overline{\psi(j)}|^2 \le (r-1)^2 = 484$, also for every conjugate character. The nonzero sum is an algebraic integer. But m_H is divisible by eleven if $c = 1$, and by 23 if $c > 1$, in addition to six, a contradiction. If

$b = (x+y\sqrt{-23})/2$ instead, the quadratic orbit coefficient has absolute value $\sqrt{m_Y m_H}\,|y|/(2\sqrt{23})$. The tame inactive compression (Section 21.10) applies also to the quadratic character, giving $m_Y m_H y^2 \le 4$. Thus b is rational.

Let v_Y be the unit vector constant on the orbit of Y and zero off it. The squared mass of sv_Y on the H-orbit is $m_Y m_H b^2/529 < 1$, since the vacuum also contributes. Thus $b = \pm 1$. We have $22 \mid m_Y$, $m_Y \ge 44$ and $6 \mid m_H$; moreover $3 \nmid m_Y$, since K_Y has conductor supported at five and 23. The possible pairs (m_Y, m_H) are $(44, 6), (44, 12), (88, 6)$.

To exclude the two pairs with product 528, retain also the disjoint orbit of the twist-one $U \in P_5$ from Corollary 22.9 and Proposition 22.5, of degree $25r$ and size $m_U \ge 44$. By Lemma 22.4 the lcm-normalized entry at U, Y has the unique weight-23 expression $z = 2 + 7\sum_{j=1}^{3} \eta_j$, with η_j of five-power order. Wild five-inertia fixes K_Y since $5 \nmid m_Y$. It permutes the three letters by uniqueness, so fixes each: its orbits on roots of order $5^a > 5$ have size 5^{a-1}. If t letters are trivial, the normalized Galois trace of z is $(35t - 13)/4$, of absolute value at least $13/4$. Since $n/\operatorname{lcm}(d_Y, d_U) = 805$, averaging the Y-orbit on the U-orbit gives squared mass at least $(44 \cdot 13/(4 \cdot 805))^2 > 1/529$ (use $44 \cdot 13/4 = 143 > 805/23$). This and $528/529$ exceed one. Consequently $m_H = 6$ and $m_Y = 44$. The seven-twist of H has order exactly seven. The two required subfields determine $K_Y = \mathbb{Q}(\zeta_{23}, \sqrt{5})$. Such a Y crosses each seven-free nonunit $U \in P_5$ in the same $9 + 7(\eta + \eta^{-1})$, up to Galois conjugation, as in Section 22.5: the proof uses the 5-primary word on three letters and the displayed quadratic five-part of K_Y. In particular for the fixed U-orbit of size at least 44 the mean normalized crossing is $11/2$. Here $d_U = 25r$ and $n/\operatorname{lcm}(d_U, d_Y) = 805$. The averaged U-row bounds the number B of these Y-orbits by $44^2 B(11/2)^2 < 805^2$, so $B \le 11$.

For each H-orbit at most two of these Y-orbits meet it, since three would cost $3 \cdot 44 \cdot 6 > 529$ in the averaged H-row. The double-principal residue at five in Section 22.5 counts only those lying in P_5. Each contributes $\pm 44 \cdot (49r) \cdot (7r) \equiv \pm 3 \pmod 5$, since $\chi_H(Y) = \pm 7r$. One orbit cannot give -1 modulo five. Thus precisely two meet H, both double-principal. Conversely, a fixed Y-orbit can meet at most two such H-orbits by the same norm bound. Hence there are at most eleven H-orbits.

Finally the singleton Gauss sum $L_7 = \sum_Z d_Z^2 c_7(Z)$ of Section 21.4 is divisible by 23. The terms at 23-bearing labels are zero modulo 23 (the denominator of c_7 divides six). Each orbit of 23-free seven-active labels would contribute exactly $7(125 \cdot 49)^2$ because its size is six and its seven-twist has order seven. Their number is therefore a multiple of 23, hence zero. We have proved $v_{23}(d_H) = 1$ for every seven-active H. This gives the divisibility hypothesis of Lemma 4.19 with $(p, q) = (7, 23)$. It also gives $v_{23}(f_H) = v_{23}(D) - 2v_{23}(d_H) = 4 - 2 = 2$. Apply the all-primes conclusion of Theorem 6.9 with $(q, p) = (7, 23)$. □

Theorem 22.10. *The separated five–seven path at $5^6 7^6 23^4$ cannot have the unramified endpoint $s = 7$. In any remaining separated table the complementary principal rows ramify at seven and are unramified at five.*

Proof. We first exclude seven-free seven-active labels. They have degree $125r$ by the preceding result and are pure seven-active by Section 22.5, since all joints have degree $125 \cdot 7r$. Fix the orbit of 44 labels Y of degree $49r$ from a joint column as in Section 22.5. Each hits every such H by seven-principality. Write $S_{YH} = nz/(7r)$. The lcm-normalized crossing z lies in $\mathbb{Q}(\zeta_r)$: $K_Y = \mathbb{Q}(\zeta_r, \sqrt{5})$ and K_H is unramified at five. It satisfies $z \equiv r \equiv 2$ modulo seven by the seven-principality of Y. Thus $z = 2 + 7y$ for a cyclotomic integer y. Every nontrivial r-Fourier coefficient of $z/(7r)$ equals the corresponding coefficient of y/r. Both labels are r-inactive. The same estimate for inactive orbit vectors gives $y \in \mathbb{Z}$: a nonquadratic y forces $66 \mid m_H$, making $m_Y m_H > 484$, and the quadratic coefficient would require $m_Y m_H \le 4$. Therefore z is a rational integer congruent to two modulo seven. Averaging the 44 rows on this set bounds its size by $N_0 < (7r)^2/(44 \cdot 4) < 148$.

For an orbit of exact seven-primary order 7^u write its size as $6 \cdot 7^{u-1} A$, $A \ge 1$, using the

independent square action and duality. The orbit sum of c_7 is $7Ak(u)$ where $k(1) = 1$ and $k(u) = 6 \cdot 7^{u-2}$ for $u > 1$. In the singleton congruence $7^3 \mid L_7$ all seven-bearing orbits disappear modulo 7^3. The remaining degrees equal $125r$, so 49 divides $\sum Ak(u)$ over the seven-free orbits. If nonempty this gives $N_0 \geq 6 \cdot 49 = 294$, a contradiction.

Now every seven-active degree is $125 \cdot 7^h r$, $h = 1, 2$. At each pure seven-active H repeat the five-free pure principal-hit analysis above. A hit Y of r-height two would again give S_{YH}/n nonzero integral at full lcm scale: the common crossing is divisible by seven because $Y \in P_7$ and $\chi_Y(H) = 125z$. Thus $d_Y = 49r$, and $b = S_{YH}/(n/r)$ is nonzero integral in the r-primary field. The same inactive compression makes b rational, and the two disjoint orbit estimates above again give $b = \pm 1$, $m_H = 6$ and $m_Y = 44$. Indeed $6 \mid m_H$ by seven-activity and duality, and the Y-orbit and the fixed U-orbit have the same degree and field constraints used there. Thus $K_Y = \mathbb{Q}(\zeta_r, \sqrt{5})$ and the seven-twist has exact order seven. In the double-principal residue modulo five each contributing Y-orbit now gives

$$\pm 44 \cdot 49r \cdot 7^{3-h} \equiv \begin{cases} \pm 2, & h = 1, \\ \pm 1, & h = 2. \end{cases}$$

At $h = 1$ at least two contributing orbits are needed and at $h = 2$ at least one.

Let A_1, A_2 count the pure seven-active orbits at the respective heights and B the joint orbits. Every joint has height one, orbit size 66, and a double-principal pure Y-orbit hitting it, with $S_{YJ}/(n/r)$ a signed primitive rth root by Section 22.5. All Y-orbits used for the two kinds of columns together number at most eleven: every one has the field and degree giving the common mean $11/2$ against the fixed U-orbit, so the same 805^2 bound applies to their union. For each Y there are at most two incident pure orbits by the row norm and at most two joint orbits by the primitive r-phase bound (Section 22.5). If two pure orbits are incident, the unit orbit vector constant on Y has, after s, squared mass $2 \cdot 44 \cdot 6/529 = 528/529$ on them. Each incident joint would use an additional $44 \cdot 66/(22^2 \cdot 529) = 6/529$, since the signed primitive 23rd root has Galois mean $\mp 1/22$. Therefore the number of incident pure orbits plus half the incident joint count is at most two for each Y. Counting these incidences gives

$$2A_1 + A_2 + B/2 \leq 22.$$

All seven-active twists have seven-component of exact order seven. Each of the two quadratic classes of primitive seventh phases occurs, by the quadratic Gauss balance of Section 21.4. At a fixed primitive phase α let P_α be its projector. Its squared-degree mass $M_\alpha = \sum_{\theta_{Y,7}=\alpha} d_Y^2$ is a rational integer. Since $se_0 = (d_Y/n)_Y$ and the complementary Weyl is unitary and commutes with P_α, we have

$$\frac{M_\alpha}{D} = \|P_\alpha W_7 e_0\|^2 = \sum_{\theta_{Y,7}=\alpha} (W_7)_{Y0}\, \overline{(W_7)_{Y0}}.$$

Every summand is integral at all primes above $r = 23$ by the preceding all-primes bound and conjugation. The rational number M_α/D therefore belongs to $\mathbb{Z}_{(23)}$. Since $v_{23}(D) = 4$, we obtain $23^4 \mid M_\alpha$. Write $A_{h,\pm}, B_\pm$ for the orbit counts in the corresponding square class at seven. A pure orbit gives two labels over each of its three seventh phases, and a joint gives twenty-two over each of its three phases. Dividing the squared degree sum by $2(125 \cdot 7r)^2$ gives

$$529 \mid A_{1,\pm} + 49A_{2,\pm} + 11B_\pm.$$

Both quantities are positive, whence $A_1 + 49A_2 + 11B \geq 1058$. The joint quadratic balance at $7, 23$ from Section 21.4 gives equal mass at both values of the sign product. Every joint orbit has the same size and degree and one sign product, so the nonempty joint count B is even. But the incidence inequality gives

$$A_1 + 49A_2 + 11B \leq 1078 - 97A_1 - \tfrac{27}{2}B \leq 1051.$$

This contradiction proves the assertion. □

The opposite ramification orientation at this odd dimension, and the ordinary modular case at 5400^2, remain open.

23 Degree and support restrictions at the first open dimension

23.1 Four disjoint orbits of witnesses

Say that a pair p, q occurs if pq divides some twist order. The pair $3, 5$ occurs and at least one of $2, 3$ and $2, 5$ occurs. We recall the following four disjoint sets, each of four labels of one degree.

- Complementary witnesses in P_3, P_5, of pure three- and five-primary twist, respectively, have degrees $a \in \{12, 24, 36, 72\}$ and $b \in \{20, 40\}$. Their weight-four crossing gives four distinct five-inertia conjugates of each (Theorem 21.8).
- At a jointly $3, 5$-active column, an odd witness in P_3 of pure ternary twist has degree $c = 15$ or 45. Its weight-five crossing gives four distinct five-inertia conjugates. The joint column itself has four such conjugates by Sections 19.5 and 21.4. Its degree d divides $360 = 2^3 3^2 5$, since both active primes are deficient in its degree (Lemma 3.3). The first three sets have pure twists and distinct degree types, so all four sets are disjoint.

23.2 Formal codegrees and the largest simple degree

Write $n = 5400$, $d_U = \mathrm{FPdim}(U)$, $b_U = S_{ZU}/d_Z$ for the character row at Z, and $f = f_Z = D/d_Z^2$. Recall that $d_U \mid n$, that every nonunit degree has at least two distinct prime factors, and that $S_{ZU}/\operatorname{lcm}(d_Z, d_U)$ is an algebraic integer (Section 2). Each nonunit row satisfies

$$b_0 = 1, \qquad \sum_U d_U b_U = 0, \qquad \sum_U |b_U|^2 = f, \qquad n = \left| \sum_U d_U b_U \theta_U \right|.$$

The last equality is the vacuum–Z entry of the modular relation $STS = p^+ T^{-1} S T^{-1}$, using $|p^+| = n$. Reducing the first orthogonality sum at each of $2, 3, 5$ gives a nonzero nonunit entry of degree prime to that prime. These three kinds of degrees are disjoint.

We give two details of the estimates on a row. Cyclotomic Galois conjugation of its entries permutes the argument U and preserves d_U. In an orbit O with $b_U \neq 0$, the mean of $|b_U|^2$ is at least one by the arithmetic–geometric mean inequality applied to its absolute norm. Thus $\sum_O |b_U| \leq \sum_O |b_U|^2$ by Cauchy–Schwarz, and the support has at most f labels. If a nonzero algebraic-integer value c in an orbit reduces to zero at a prime over p, its rational norm is divisible by p. Writing $t = [\mathbb{Q}(c) : \mathbb{Q}]$, its distinct conjugates have squared mass at least $tp^{2/t}$. For $p = 2$ this is at least 4, and for $p = 3$ at least 6, as follows by checking $t = 1, 2, 3$ and monotonicity thereafter. For f a power of p, the row is supported in the principal p-block P_p (Theorem 2.9). On this block $v_p(d_U) \leq v_p(n) - 1$ (Corollary 4.1) and $S_{UZ}/d_U \equiv d_Z$ at the chosen prime above p.

Codegree four is excluded by Lemma 2.6.

Codegree nine. Here $d_Z = 1800$ and the support lies in P_3. Any supported degree exceeding 600 must have three-height exactly two and contain all three primes: the other possible heights give degrees at most $8 \cdot 3 \cdot 25 = 600$, and the prime-omitting bounds are at most $225, 200, 72$ in P_3. At such a position, $b_U = (d_U/d_Z)\chi_U(Z)$ vanishes modulo the prime over three, since d_U/d_Z is a local unit there and $U \in P_3$. Its orbit costs at least six in squared mass in addition to the unit and the three disjoint prime-omitting orbits. This exceeds nine. All supported degrees are therefore at most 600. Since the unit has degree one, the modular relation gives $5400 < 600 \cdot 9$ if the row exists, a contradiction.

Codegree sixteen. Here $d_Z = 1350$, and the support lies in P_2. For an even supported degree of two-height k, the number $b_U/2^{k-1}$ is integral by lcm divisibility. It has zero residue at two by two-principality of U: up to a dyadic unit it equals $\chi_U(Z)$. Its orbit costs at least $4 \cdot 4^{k-1}$. Consequently $k = 1$ at every even hit. In particular the three-free and five-free nonunit hits have degrees at most 50 and 54, respectively, and cost at least four each.

No two rows of degree 1350 have a nonzero mutual entry, including a row paired with itself. Otherwise symmetry and prime-power support put both in P_2. A row in P_2 meets every odd degree by principality. There are at least eight odd-degree labels: the unit and four odd conjugates described in Section 23.1 already give five, and their total number is divisible by four by reduction of $\sum d_U^2 = D$ modulo four. All odd positions therefore cost at least eight in total. The compulsory three-free and five-free nonunit orbits are even. Their additional cost exhausts sixteen before the proposed degree-1350 hit.

All remaining support degrees are at most 675. Indeed $k \le 1$, and the divisors of $2 \cdot 3^3 \cdot 5^2$ bigger than 675 consist only of 1350. In $\sum d_U|b_U|$, replace the sum of absolute values on each orbit by its squared mass, then reserve the unit and the two compulsory even orbits. This gives

$$n \le 675 \cdot 16 - (675 - 1) - 4(675 - 50) - 4(675 - 54) = 5142.$$

This contradiction excludes codegree sixteen. Codegree 25 is excluded by Proposition 5.7 at the quartic prime five. The degree 5400 would alone exhaust D. The only divisors of 5400 above 900 are $1080, 1350, 1800, 2700, 5400$, completing the degree bound.

23.3 Excluding a simple object of degree six

Theorem 23.1. *Neither an ordinary divisor-minimal table nor its positive reduced counterpart has a nonunit label of degree six. In particular every nonunit degree in either table is at least ten.*

Proof. Suppose $d_X = 6$ and write Δ for the table dimension, equal to the category dimension in the ordinary case and half of it in the reduced case. By Corollary 4.18 its prime support is $\{2, 3, 5\}$. By Lemma 4.10 a three-free nonunit in P_5 has five-height $j \ge 1$ with $2j + 2 \le v_5(\Delta)$. Thus in both tables $v_5(\Delta) \ge 4$.

Work in a common cyclotomic splitting field with Galois group G. Use prime avoidance at any five-active column in the full principal five-block, avoiding three. The wild-inertia argument of Section 21.4 gives a three-free nonunit U with K_U tame at three. Indeed at a five-active J the orthogonality relation $\sum_{Y \in P_5} d_Y \chi_J(Y) = 0$ holds in either table (Proposition 4.2). Wild ternary inertia preserves P_5 and permutes its arguments without changing their residues at three. Nonsingleton orbits contribute zero there. The vacuum term forces a singleton nonunit orbit with nonzero contribution. Its degree is $2^i 5^j$ with $i, j \ge 1$. The entry $z = S_{XU}/\operatorname{lcm}(6, d_U)$ has weight two and residue two above five, since $\chi_U(X) = 3z$. By Theorem 4.9, it is a sum of two five-power roots. They are distinct by strictness, and their ratio has order five by Theorem 19.5, applied to the full-weight word $3z$. Consequently $|\chi_U(X)|^2 = 9(2 + t)$, where $t = \zeta + \zeta^{-1}$ for a primitive fifth root ζ.

The unit occurs exactly once in XX^*, also in the positive quotient: for a lift X the fermion f does not occur in XX^* since $fX \ne X$. The other constituents have total degree 35. Each of these degrees is supported on $2, 3, 5$ and has at least two prime divisors. The only possible odd degree at most 35 is 15, occurring once. The remaining mass 20 can only be one degree 20 or two degrees 10. Write $XX^* = 1 + V + B$, with $d_V = 15$ and $d_B = 20$. Neither $v = \chi_U(V)$ nor $b = \chi_U(B)$ is rational. Indeed, if v is rational, $v/3$ is a rational integer of absolute value below five by lcm integrality. Five-principality gives $v \equiv 0$ above five, whence $v = 0$. Now $b = 17 + 9t$ has a conjugate greater than 20. If b is rational, then $b \in \mathbb{Z}$, $|b| < 20$ and $b \equiv 0 \pmod 5$. Since $v = 17 + 9t - b \in 3\mathcal{O}$, where $\mathcal{O}$ is the ring of algebraic integers, $b \equiv 2 \pmod 3$. The possibilities $b = -10, 5$ both give a conjugate of v greater than 15.

For every $\sigma \in G$, the adjoints of X and X^σ share a nonunit (Theorem 19.5). If B is one simple or twice one simple, every σ fixes V or B. Their stabilizers are proper because v, b are irrational, and a group is never the union of two proper subgroups. Hence $B = W + W'$ with W, W' distinct of degree ten. No degree-ten simple can be fixed by G. Otherwise its character eigenvalues would be integers, the vacuum value 10 and all others strictly between -10 and 10. Applying $\prod_{k=-9}^{9}(T - k)$ to this simple in the fusion ring gives 19! times the vacuum idempotent, whose unit coefficient is $19!/\Delta$. This is impossible since $v_5(19!) = 3 < v_5(\Delta)$.

Suppose first that W, W' lie in different G-orbits. Then every σ fixes at least one of V, W, W'. If a finite abelian group is the union of three proper subgroups, they are the three subgroups of index two over their common intersection. To see this, each subgroup has an element in neither of the others. Adding such an element of the third subgroup to an element in the first two but outside the third would leave all three, so the three pairwise intersections coincide. Quotient by that intersection. Translation by a nonzero element of one subgroup embeds the nonzero part of another into that of the third. Consequently all have the same order n, and $n \mid 3n - 2$, so $n = 2$. Thus $K_V, K_W, K_{W'}$ are the distinct quadratic subfields of a biquadratic field. Their nonrational components span three independent rational lines. Since

$$v + \chi_U(W) + \chi_U(W') = 17 + 9t,$$

$\mathbb{Q}(\sqrt{5})$ is one of those subfields. If it is K_V, then b is rational; otherwise v is rational. Both alternatives were excluded.

It remains that W, W' lie in the same orbit. Identify the orbit with the additive group $A = \mathrm{Gal}(K_W/\mathbb{Q})$, taking W at 0 and W' at $s \neq 0$. Let $\pi : G \to A$ be restriction and H the stabilizer of V. For each $\sigma \notin H$, the translate by $\pi(\sigma)$ of $\{0, s\}$ meets $\{0, s\}$. Thus $\pi(G \setminus H) \subseteq \{0, s, -s\}$. If $\ker \pi \not\subset H$, every fiber has a point outside H, giving $|A| \leq 3$. Otherwise $H = \pi^{-1}(J)$ for a proper subgroup J and $A \setminus J \subseteq \{s, -s\}$, giving $|A| \leq 4$.

At $|A| = 2$ the pair sum is Galois-fixed. At $|A| = 3$ its character values at U are rational: K_W has conductor supported on $2, 3, 5$ by Corollary 2.8, and any such cubic field ramifies wildly at three. Indeed the dyadic and quinary cyclotomic factors have degrees prime to three, as does the tame ternary factor. Since K_U is tame at three, $K_W \cap K_U = \mathbb{Q}$. Both cases make b rational. Finally at $|A| = 4$, the complement of J must be the two different elements $s, -s$. Thus A is cyclic with generator s, and $F = K_V$ is its quadratic subfield. The displayed identity puts $t \in K_W$, hence $F = \mathbb{Q}(\sqrt{5})$ and $b \in F$. Write $\alpha = \chi_U(W)$ and γ for the generator corresponding to s. Galois covariance gives $b = \alpha + \gamma\alpha$. Since $\gamma^2 b = b$, the element $y = \alpha - \gamma^2\alpha$ satisfies $\gamma y = -y$ and $\gamma^2 y = -y$. These imply $y = 0$. Thus $\alpha \in F$ and $b = \mathrm{Tr}_{F/\mathbb{Q}}(\alpha)$ is rational, the final contradiction. Every integer between two and nine other than six is a prime power, already excluded in Section 2. □

Rank at the even core. At $D = 5400^2$ the rank is at most 46774. Write A, B, C for the sets of nonunits whose degrees omit two, three, five, respectively. They are disjoint, since a nonunit degree cannot be a prime power. For $X \in P_p$ the nonzero entries on all p-free nonunits cost at least $(d_Y / \gcd(d_X, d_Y))^2$ per label in the row norm D/d_X^2, after averaging on Galois orbits: division by this scale gives a nonzero algebraic integer of mean squared conjugate modulus at least one. The vacuum costs one separately.

Avoid five at a two-active column to obtain a five-free $X \in P_2$. Its degree is $2^i 3^j \geq 12$ by Theorem 23.1. The scale on A is at least five. If $d_X = 12$, the known degree-45 member of P_5 has scale 15 and costs an extra 200 above the baseline 25. Hence $25|A| + 200 \leq D/12^2 - 1$, giving $|A| \leq 8091$. If $d_X > 12$, then $d_X \geq 18$ and $|A| \leq 3599$. Since all degrees in A are at least 15, the degree-45 witness gives

$$\sum_{Y \in A} (900 - d_Y^2) \leq 675 \cdot 8091 - 1800.$$

In P_3 a five-free witness has degree at least 12, and gives scale at least five on B, so $|B| \le 8099$. All its degrees are at least 10, and the complementary P_5 quartet has degree 20 or 40. Thus the analogous sum over B is at most $800 \cdot 8099 - 4 \cdot 300$.

Use both the degree-45 row and a row of the complementary quartet of common degree $b = 20$ or 40 in P_5 (Section 23.1). Put $s_m(d) = d/\gcd(d, m)$. For every five-free nonunit degree d,

$$35(900 - d^2) \le 1557 s_{45}(d)^2 + 172 s_b(d)^2.$$

Indeed, the only such degrees below 30 are $12, 18, 24$. The pairs of squared scales are $(16, 9), (4, 81), (64, s_b(24)^2)$, respectively. The coefficients give equality in the first two cases, and the first term alone suffices in the third. Summing the two row-norm bounds gives

$$\sum_{Y \in C} (900 - d_Y^2) \le \frac{1557(14400 - 1) + 172(72900 - 1)}{35} = \frac{34957871}{35}.$$

All remaining nonunits have degree at least 30. Hence

$$\text{rank} \le 1 + \left\lfloor \frac{D - 1 + 5459625 + 6478000 + 34957871/35}{900} \right\rfloor = 46774.$$

The degree-45 witness is supplied by Section 21.4.

23.4 The largest five-free principal degree on the path missing six

Suppose $D = n^2 = 2^6 3^{2e} 5^4$, $e \ge 3$, in an ordinary minimal table, and no twist order is divisible by six. The two–five and three–five pairs both occur. Every three-active degree is divisible by 40, and every two-active degree has full three-height. These are the separation and degree-transfer results of Section 21.3.

Proposition 23.2. *Suppose P_3 contains a five-inactive X of degree $8 \cdot 3^{e-1}$ (72 at the first core). All such labels form a single Galois orbit, with $K_X = \mathbb{Q}(\zeta_5)$. Every degree-20 label of P_5 has the same field and twist of order exactly five. There are between two and four such orbits. Every jointly two–five-active K has five-component of order exactly five and meets a degree-20 label of P_5 at each of the four primitive fifth phases.*

Proof. A jointly two–five-active column supplies $U \in P_5$ of degree 20; fix any such degree-20 label. Its twist is purely quinary by the two degree transfers above. At a joint three–five column choose an odd $V \in P_3$ in the pure ternary fiber of degree $5 \cdot 3^k$, $1 \le k < e$. It has twist one since its degree is odd. Its crossing with U, after division by the lcm of the two degrees, is $2 + 3\eta$; that of X with U is $1 + 3\xi$. Here η, ξ are primitive fifth roots (Section 21.7). In particular all three fields contain $L = \mathbb{Q}(\zeta_5)$. Write $m_A = [K_A : L]$.

Use unit vectors on the Galois orbits of X, V in a faithful quartic torus character at five, and the inverse mode on the orbit of U. For either input $R = X, V$ the squared matrix coefficient of $s = S/n$ is

$$\frac{45}{n^2} \operatorname{lcm}(d_R, 20)^2 m_R m_U.$$

Indeed the Fourier coefficient of each displayed crossing is three times a primitive quartic Gauss sum (Section 21.10). For $R = X$ this number is $m_X m_U/5$. The contribution from V is positive. The sum over the two inputs is at most one by unitarity. If the twist of U has order at least 25, then $5 \mid m_U$ by square Galois covariance, a contradiction. When its twist is trivial the sum is at most $1/5$, by the five-inactive compression in Section 21.10, also impossible. In the remaining case the orbit of U occupies two of the four primitive fifth phases. On the five-inactive input space of this torus character the squared norm of the projection onto any one tame phase after s is at most $1/5$, by the equal-mass identity in the same section. Thus the sum is at most $2/5$.

This forces $m_X = m_U = 1$ and excludes a second orbit of such X, since V contributes as well. In the reverse count, fix X. An orbit of degree-20 labels in either square class of fifth phases uses $1/5$ of the norm of its unit orbit vector after s. That class has budget $2/5$, so there are at most two orbits per class.

To show that both square classes occur, take any jointly two–five-active K and let $\alpha = \theta_{K,5}$. Let E project onto P_5 in the fiber with trivial complementary twists. In the local modular representation write $s = W_5W'$, $v = W'e_0$, and $w = W'e_K$. At five, v is fixed by the upper unipotents and the torus, and w has upper-unipotent phase α. The projector E commutes with this local group. For every unit u modulo the five-part of the level we have

$$H(u) := \sum_{Y \in E} d_Y \chi_K(Y) \theta_{Y,5}^u = C\alpha^{-u^{-1}}.$$

In fact $H(u)$ is D/d_K times $\langle v, EW_5^{-1}U(u)W_5 w\rangle$. In the lower-unipotent Bruhat decomposition the left upper-triangular factor fixes v, and the right upper unipotent has parameter $-u^{-1}$. This gives the formula with a constant C independent of u; in particular $C = \alpha H(1)$ is integral. Also $H(0) = 0$ by restricted block orthogonality.

Reduce at a prime above three. Every three-free nonunit in P_5 has degree 20 or 40 by Section 21.3; degree 40 misses K by full dyadic height. Hence $\overline{H}$ is the discrete Fourier transform of masses on μ_5, with mass one at the trivial phase (only the vacuum contributes there). It is periodic modulo five and nonzero by Fourier inversion. Since $H(0) = 0$, the constant C has nonzero residue. Comparing $u = 1$ and $u = 6$ gives $\alpha^5 = 1$, since reduction is injective on five-power roots. Thus α is primitive of order five. Fourier inversion at phase one gives $\overline{C} = -5$. At every other fifth phase δ, the mass is consequently

$$-\sum_{u \in \mathbb{F}_5^\times} \overline{\alpha^{-u^{-1}}\delta^{-u}}.$$

This sum is nonzero: Φ_5 is irreducible over $\mathbb{F}_3$, so a vanishing sum of four fifth roots would have all five multiplicities congruent modulo three. Their sum is four, so each would be at least two. Thus K meets degree-20 labels at each primitive phase. Each orbit occupies just two phases, proving the asserted lower bound. □

Two further consequences concern the whole principal block and the opposite end of the degree range.

Corollary 23.3. *Under the same upper-degree hypothesis, every member of P_3 has purely ternary twist. The orbit of X misses every three-bearing label of full five-height. There is at most one three-free label of full five-height in the entire table.*

Proof. Retain E, K, α and the three-adic reduction from the preceding proof. For $A \in P_3$ write $\beta_A = \theta_{A,5}$ and replace d_Y in $H(u)$ by $\overline{\chi_A(Y)}$, giving $H_A(u)$. Principality gives $\overline{H_A(u)} = \overline{H(u)}$ in the residue field (here the outer bar denotes reduction). Let $h(t)$ denote the local diagonal torus $\mathrm{diag}(t, t^{-1})$, which permutes labels without signs. The same Bruhat calculation now gives

$$H_A(u) = (\alpha\beta_A)^{-u^{-1}} C_{h(-u)A}, \qquad C_B = \frac{D}{d_B d_K}\langle W'e_B, EW_5W'e_K\rangle.$$

Indeed the left upper unipotent also has parameter $-u^{-1}$, and the intervening torus is $h(-u^{-1})$. Each C_B is integral by the formula at $u = 1$. Since $C_0 = C$ has nonzero residue, reduction gives $C_{h(-u)A} \equiv C_0 \beta_A^{u^{-1}}$. Apply the same formula at parameter one to $h(u)A \in P_3$. Its five-phase is $\beta_A^{u^{-2}}$ by torus conjugation, so $\beta_A^{u^{-1}} = \beta_A^{u^{-2}}$ by injectivity of reduction on five-power roots. Taking $u = 2$ proves five-inactivity. Three-principality bounds the three-height below e, which excludes dyadic activity by separation.

If $25 \mid d_Y$, its codegree is a five-unit, so K_Y is unramified at five. Therefore $x = \chi_X(Y)$ lies in $K_X \cap K_Y = \mathbb{Q}$ and lcm divisibility gives $25 \mid x$. Since $s_{XY} = x/75$, three-principality at X makes this an integer whenever $3 \mid d_Y$, hence zero by unitarity. Otherwise $d_Y = 25 \cdot 2^i$, and $x/25$ is congruent to 2^i modulo three. The entries $s_{X'Y} = x/75$ on the four conjugates of X coincide. Column unitarity gives $|x/25| = 1$. For $a = \frac{1}{2}\sum_{X' \sim X} e_{X'}$, the real unit vector $w = sa$ therefore has $|w_Y| = 2/3$ at every such Y. There are at most two.

If there were two, their Galois orbits would have size at most two. Their twists are dyadically inactive by separation and five-inactive by their codegrees. Their ternary twists consequently lie in μ_3, since an order divisible by nine gives at least three Galois conjugates. Now $Ta = a$ and the modular relation gives $|\sum_Z w_Z^2 \theta_Z| = |\langle a, w\rangle|$. Two third roots sum to a number of modulus at least one. The two indicated coordinates therefore contribute modulus at least $4/9$ on the left, and the remaining mass is $1/9$. But $|\langle a, w\rangle| < 1/3$, because the complement of these two coordinates has norm $1/3$ and contains the vacuum with $w_0 > 0$, outside the support of a. This excludes the second label. □

23.5 Character fields inside a signed fusion product

Suppose an ordinary minimal table at $n = 5400$ has all three twist pairs. Take the twist-one principal witnesses $X \in P_3$, $U \in P_5$ of degrees $4 \cdot 3^j$, 20, with $j = 1, 2$, as in Section 21.6. Consider a signed crossing

$$z = \frac{S_{XU}}{L} = -\sum_{a=1}^{4} \eta_a \zeta^a, \qquad L = 20 \cdot 3^j, \qquad \{\eta_a\} = \{1, 1, \omega, \omega^2\},$$

where ζ, ω have orders five and three. Write k, l for the two positions with coefficients ω, ω^2 respectively. For a label A let K_A be its character field. Put $F = \mathbb{Q}(z)$ and $h_A = [K_A : F]$ for $A = X, U$.

Proposition 23.4. *If $k = -l \pmod 5$, then $9^{j-1} h_X h_U \le 36$; otherwise $9^{j-1} h_X h_U \le 18$. In particular, for $d_X = 36$ both fields are tame at five. In the nonopposite case their compositum is at most quadratic over $\mathbb{Q}(\zeta_{15})$ and tame at three as well.*

Proof. Take the product χ of a faithful quartic character ψ at five and the quadratic character at three. The sum of $\chi(g)g(z)$ over $g \in \mathrm{Gal}(\mathbb{Q}(\zeta_{15})/\mathbb{Q})$ has squared modulus

$$3 \cdot 5\, |\overline{\psi}(k) - \overline{\psi}(l)|^2.$$

Indeed ternary antisymmetrization gives $-(\omega - \omega^2)(\zeta^k - \zeta^l)$, and the remaining sum is a quartic Gauss sum times the difference of the character values. This gives 60 at opposite positions and 30 at nonopposite positions. In the former case $F = \mathbb{Q}(\zeta_{15})^+$ and in the latter $F = \mathbb{Q}(\zeta_{15})$ (Section 21.6). The character is trivial on simultaneous complex conjugation, so the sum over $\mathrm{Gal}(F/\mathbb{Q})$ has squared modulus 15 and 30, respectively. Since X, U are both inactive at three and five, the two-prime local Weyl bound in Section 21.10 bounds $L^2 h_X h_U$ times this squared modulus by $n^2/15$. Substituting L yields the inequalities. At $j = 2$ the relative-index products are at most four and two respectively, giving the field assertions. □

Here the Fourier estimate can also control the constituents of the tensor product. We give the underlying orbit identity for arbitrary primes. For simples A, B in ordinary square-dimensional integral modular data, let E be an abelian field in the splitting field containing K_A, K_B and a twist value t. If the total degree of the constituents of AB of twist t is M_t, then

$$M_t = \sum_V N_{AB}^V d_V [EK_V : E],$$

taking one representative per orbit under $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ among those constituents. In fact in integral square dimension the Galois permutation is a based ring automorphism: applying σ to each character identity permutes the arguments without signs and preserves degrees. Automorphisms fixing the inputs therefore preserve all multiplicities. The square Galois action preserves the indicated twist when t belongs to E (in the normalized lift the anomaly correction is trivial, since the two-primary scalar has order dividing eight and every odd square equals one modulo eight). The orbit length is the stated field index. In particular, if all degrees in the packet are multiples of g and a prime p does not divide M_t/g, some constituent has $p \nmid (d_V/g)[EK_V : E]$.

For the signed crossing, each of its four phases has $M_t = L$ by uniqueness of the balancing packets, and conjugating one input realizes every primitive thirtieth root (Section 21.6). These conjugate products have the same input fields. At order 30 all constituents have degrees $2^i 3^a 5$ with $0 \le i \le 2$, $1 \le a \le 2$, by the two principal degree transfers. Consequently:

- If $d_X = 12$, each of the eight primitive thirtieth roots occurs on a simple of degree 15, 30 or 60. Indeed its packet degree divided by 15 is four, and at least one quotient degree is prime to three. The tame fifth-inertia involution moves each jointly three- and five-active label and preserves its twist and degree (Section 21.4). Thus each phase has at least two labels of one of these three degrees.

- If $d_X = 36$, each primitive thirtieth root occurs on a simple whose character field is tame at five. Take $E = K_X K_U \mathbb{Q}(\zeta_{15})$, which is tame at five by the proposition, and divide the orbit identity by 15. The resulting packet mass is 12, a five-unit. Moreover any constituent of degree at least 45 is tame at five since $[EK_V : E] \le 180/d_V < 5$. In the nonopposite case all constituents of degree at least 90 are likewise tame at three, as their indices over E are at most two. A degree-180 constituent has its entire field inside E.

23.6 Five-free hits and degree-forty-five traces

At $n = 5400$ assume that every pair of dimension primes occurs in a twist. At a jointly two- and three-active column pick $X \in P_3$ by five-avoidance in the pure ternary fiber, with K_X tame at five (Section 21.4). It has twist one and degree $4 \cdot 3^j$, $j = 1, 2$ (Section 21.3 and Section 21.6). Write τ for the normalized trace to $\mathbb{Q}$. The five-local fusion-ring element

$$B = \frac{8}{[K_X : \mathbb{Q}]} \sum_{X' \sim X} [X']$$

has integer eigenvalues $b_Y = 8\tau(\chi_Y(X))$ with $b_0 = 32 \cdot 3^j$. Indeed the conductor primes are two, three, five, and tameness at five makes the orbit size a five-unit. The normalized traces of primary roots have denominators dividing $1, 2, 4$ respectively. Principality gives $b_Y \equiv b_0 \pmod 5$ on P_5. Put $u_Y = (b_0 - b_Y)/5$ there.

Five-free witnesses. Let J be jointly three- and five-active. For every prime above five, either some five-free nonunit $Y \in P_5$ with $\chi_J(Y)$ a unit there has $b_Y \equiv b_0 \pmod{25}$, or such hits attain at least three distinct nonzero residues $u_Y \bmod 5$. The possible degrees of these hits are $12, 18, 24, 36, 72$: perfectness rules out prime-power degrees, three-activity rules out full three-height and Theorem 23.1 excludes degree six.

For the proof, the vacuum coefficient of $E_{P_5}[J](b_0 - B)^a$ is five-integral. Dividing the resulting numerator by $d_J 5^a$ and using $d_Y^2 \chi_Y(J) = d_J d_Y \chi_J(Y)$ gives

$$\sum_{Y \in P_5} d_Y \chi_J(Y) u_Y^a \equiv 0 \pmod{\mathfrak{p}}, \qquad 0 \le a \le 2, \quad \mathfrak{p} \mid 5,$$

since $v_5(d_J) = 1$ and $v_5(n^2) = 4$. A polynomial of degree at most two therefore cannot vanish on all contributing nonvacuum residues and be nonzero at zero. Five-bearing degrees drop out

after reduction. More generally the same argument gives at least $v_p(D) - v_p(d_J)$ distinct nonzero residues, unless zero is realized at a nonunit p-free hit, for any p-local element with rational integral eigenvalues congruent to the vacuum on P_p.

In particular, if all these five-free unit hits have rational $x = \chi_Y(X)$, at least two distinct ones have ternary height two. Indeed write $t = 3^{j-1}$, so $d_X = 12t$. For $j = 2$ the integer x is divisible by three by lcm integrality and three-principality at X: if $v_3(d_Y) = 1$, the identity $\chi_X(Y) = (d_Y/d_X)x \equiv d_Y$ even gives $9 \mid x$, and if $v_3(d_Y) = 2$ it gives $3 \mid x$. Always x/t is an integer, is congruent to 12 modulo five and has absolute value below 12. Hence $x/t \in \{-8, -3, 2, 7\}$, giving four nonzero residues for $u_Y = 8(12t - x)/5$ modulo five. At $v_3(d_Y) = 1$ three-principality gives $3 \mid x/t$ also when $j = 1$, leaving only $x/t = -3$. At least two of the three prescribed residues must therefore occur at the higher ternary height.

The rest of this section determines the possible rational traces on the known odd degree-45 rows of P_5 when no twist is divisible by 30. This is a constraint on the spectrum even when many five-free rows meet J.

The first prime below twice the weight. Suppose ordinary square modular data have prime support $\{2, q, r\}$ with q, r distinct odd primes and $r \geq 5$. For two nonunits A, B in a perfect fusion-simple table set $L = \operatorname{lcm}(d_A, d_B)$ and $z = S_{AB}/L$. Assume

$$g = \gcd(d_A, d_B) = (r + 1)/2, \qquad z \equiv g \pmod{\mathfrak{r}} \quad (\mathfrak{r} \mid r),$$

and that z is tame at r. If $q \geq 5$, then $v_r(g - \tau(z)) = 1$. The same holds for $q = 3$ when no twist is divisible by $6r$, $v_2(L) < v_2(n)$ and $v_r(L) < v_r(n)$.

To prove this, average at two and over wild q-inertia. These averages preserve integrality on cyclotomic integers: on primary roots, the dyadic trace is $0, 1, -1$, while the wild trace kills precisely the roots of order greater than q. Their denominators are prime to r, so the residue is preserved. By tameness at r the resulting z_0 lies in $\mathbb{Q}(\zeta_q, \zeta_r)$. Equivalently one can also apply the wild r-trace to a positive balancing word, since it fixes z. This produces a rational signed expression of length at most g for z_0 and preserves its normalized trace. Two-prime unimodularity in the proof of Theorem 4.9 gives an integral signed expression of length at most g. On erasing r-parts its value w still has residue g and is unramified at r. The integer $1 + (w - g)/r$ has all conjugates in a disk tangent to the unit circle solely at one. The norm gives $w = g$ or $g - r = -(g - 1)$.

In the first case all g terms are positive rth roots. In the second, erase r and write the signed coefficient vector on μ_q as $(k - (g - 1), k, \ldots, k)$. Its absolute sum is at most g, and opposite pairs cost two. Thus all terms are $g - 1$ negative rth roots except possibly at $q = 3$. There one may have $g - 2$ negatives and one each of ω, ω^2, all multiplied by rth roots. Equal r-positions for the two positive terms contract them to a single negative term.

Suppose the two positive positions b, c differ. This last representation has minimum length exactly g: a shorter word must be pure-r by the preceding classification, while ternary inversion changes this word by $(\omega - \omega^2)(\zeta^b - \zeta^c) \neq 0$. It is the unique minimal signed word. Indeed the vectors of ternary coefficients at the r positions in two such words differ by one common constant. If they have no common vacant position, their supports overlap in at most one position since $r = 2g - 1$. When the first support has g positions they all contain single letters, and off the possible overlap at least two of the original $-1, \omega, \omega^2$ coefficients differ. When it has $g - 1$ positions there is no overlap, and the coefficients at b, c have distinct imaginary parts. Smaller support leaves a common vacancy automatically. A nonzero constant is therefore impossible. At each position the remaining signed ternary vector is unique: its form is $(-m, 0, 0)$, $(-m, 1, 0)$ or $(-m, 0, 1)$, $m \geq 0$, and adding a nonzero constant to all three entries strictly increases the absolute sum. Unimodularity gives integral vertices for the minimum signed-weight polytope, hence the rational minimum is unique as well.

The positive balancing expression has full weight g. Equality after the projections and contractions forces every physical phase to survive and the projected coefficients to equal this unique signed word. Thus $z = z_0$, the two positive packets have degree L each and each negative packet has degree mL, $1 \le m \le g-2$. The field of z ramifies at three and at r: ternary inversion changes it as above, and the nonzero ternary difference cannot be fixed by all r-inertia (its r-trace is zero when b, c are both nonzero and nonrationality is immediate if either is zero). Both input fields contain this field. Hence both inputs have trivial dyadic twist in the absence of a triple (Theorem 19.12). If their twist product is ternary-active, the negative packets require either a triple or degrees of full r-height (Proposition 19.16); $r \nmid m$ excludes the latter. Otherwise at least one positive packet is jointly three- and r-active, since $b \neq c$. Its degrees would all have full dyadic height, which cannot sum to L. This excludes the mixed case.

It remains to take traces of the g positive or $g-1$ negative rth roots. If t positions are trivial, then $(r-1)(g-\tau(z))$ equals $r(g-t)$ or $r(g-1+t)$ respectively. The multiplier belongs to $\{1, \dots, r-1\}$; strictness excludes $t = g$ in the positive case. This proves the valuation.

Return to degree twelve. In the no-triple triangle suppose $d_X = 12$. Let Y be any degree-45 member of P_5. Such rows are forced at every jointly three- and five-active column by Section 21.4. For $z = S_{XY}/180$ the principal-five congruence is $z \equiv 3$ above five and z is tame at five since $z \in K_X$. The previous argument gives $v_5(3-\tau(z)) = 1$ and hence $v_5(96 - b_Y) = 1$. More explicitly $b_Y = 32\tau(z)$ belongs to $\{-64, -24, 16, 56\}$.

A bound on the number of letters carrying wild ramification. Let q, r be different odd primes, $r \ge 5$, and let k be the cyclic degree-q subfield of $\mathbb{Q}(\zeta_{q^2})$. Suppose an integer $z \in k\mathbb{Q}(\zeta_r)$ has a signed word of length at most g in $q^2 r$-th roots and rational integer residue at every prime above r. Let u generate wild q-inertia and v generate r-inertia. If

$$(1-u)(1-v)z \in 2\mathcal{O}$$

and z ramifies at q, its normalized q-trace has a signed word of length at most $g - 4(q-1)$ in rth roots.

Indeed the q cosets of μ_q inside μ_{q^2} span independent modules over the r-cyclotomic integers. Within each coset the unique relation is the sum of its letters. Tame q-inertia fixes z and permutes the $q-1$ nonidentity cosets transitively. Their components therefore have equal minimum length $m > 0$. Their q-traces vanish. The identity-coset component A is fixed by the tame action, hence belongs to $\mathbb{Q}(\zeta_r)$. Averaging its signed expression at the tame q-factor and using the one-prime integer-word minimum gives a pure r-word for A, of length at most $g - (q-1)m$.

Consider one nonidentity coset. There is nothing to prove if $m \ge r$. Otherwise write h_a for the signed sum of its q-letters in position $a \in \mathbb{F}_r$ of a minimum word. On erasing r-parts the coset has zero residue modulo r: the rational congruence holds above every prime, and erasure puts the congruence in the q-cyclotomic ring, unramified at r, where the different cosets remain independent. The q integer coefficients of $\sum_a h_a$ on this coset are equal modulo r. Their pairwise differences have absolute value at most $m < r$, so $\sum_a h_a = 0$ in characteristic zero. On this coset modulo two the element u cyclically permutes q letters modulo their zero-sum relation. It has no invariants since q is odd. Consequently the mixed congruence makes this component v-invariant modulo two. Comparing coefficients in the r positions allows a common shift by the cyclotomic relation, but the shift is zero since position zero is fixed. Thus all h_a for $a \neq 0$ are congruent modulo two. If this residue is nonzero each of the $r-1$ positions costs a letter. Otherwise h_0 is even too, and at least two nonzero positions occur since their sum is zero. Each nonzero even position costs at least two letters. Hence $m \ge 4$, as asserted.

Return to degree thirty-six. For the same degree-45 row Y and $d_X = 36$ put $z = S_{XY}/180$. Its weight is nine, and $b = 32\tau(z)$. Three-principality at X gives $5z \equiv 45$ above three, while

five-principality at Y gives $z \equiv 9$ above five. The no-triple congruence in Section 19.8 at the dyadically deficient X gives $(1-\sigma_3)(1-\sigma_5)z \in 2\mathcal{O}$ for arbitrary inertia elements at the two odd primes, since $\chi_X(Y) = 5z$.

If the crossing field is unramified at three, $z/3$ is a cyclotomic integer of conductor supported on two and five. Thus $4\tau(z/3) \in \mathbb{Z}$ and $24 \mid b$. Otherwise K_X ramifies at three and contains $\mathbb{Q}(\zeta_5)$ by its unsigned weight-four crossing. The absence of the triple forces it to be unramified at two (Corollary 19.14). If z is tame at three, it belongs to $\mathbb{Q}(\zeta_{15})$ by tameness at five. Its trace $8\tau(z)$ over this field is divisible by three by the residue congruence. It is also even: modulo two the trace operator is $(1-\sigma_3)(1-\sigma_5)(1-\sigma_5^2)$ for generators of the two inertia groups. Hence $24 \mid b$ once again. The inequalities $|b| < 288$ and $b \equiv 288 \pmod 5$ now give $b = 168, 48, -72, -192$.

Finally suppose that z is wild at three. Write $m_T = [K_T : \mathbb{Q}(\zeta_5)]$. The no-triple estimate of Section 21.10 gives $m_U(9m_X + 9^{k-1}m_Z) \le 36$ with a compulsory third twist-one witness Z of degree $5 \cdot 3^k$, whence $m_X \le 3$. Wild ternary ramification forces $m_X = 3$. The conductor of K_X divides twice the formal codegree (Corollary 2.8), of ternary valuation two at $d_X = 36$. Tameness at five puts K_X inside $\mathbb{Q}(\zeta_{45})$, so $K_X = \mathbb{Q}(\zeta_5)\mathbb{Q}(\zeta_9)^+$. Relative trace to $\mathbb{Q}(\zeta_{45})$ and two-prime unimodularity give an integral signed word of length at most nine. Indeed the relative trace contracts dyadic roots to signs or zero and kills higher odd primary orders. Apply the preceding bound with $(q, r, g) = (3, 5, 9)$. The normalized ternary trace has a pure fifth signed word of length at most one, giving $b \in \{0, 8, -8, 32, -32\}$. Only $8, -32$ have the prescribed residue at five.

Thus at degree thirty-six all six possible odd traces have five-adic gap exactly one.

23.7 Counting active twists by a local torus character

Work in ordinary integral modular data of square dimension with unsigned cyclotomic tori. Let E be a coordinate subspace invariant under the local modular group G_p, and let H_p be its diagonal torus. Examples include the whole table and a full p-block restricted to specified complementary twist phases: the upper unipotent preserves the coordinates and the local Weyl preserves both conditions (Lemma 3.2 and Section 21.6).

Proposition 23.5. *Let $p \ge 5$ and let $L \subset E$ be an m-dimensional space supported on p-inactive labels, of tame torus type ψ with $\psi^2 \ne 1$, $\psi(-1) = -1$. If the projector to local level at most p has rank r on L, then E contains at least $2(p-1)r$ labels of twist order exactly p at p, and at least $p(p-1)(m-r)$ of higher p-power order. They may be selected from contributing torus orbits, each of size divisible by the order of ψ. In particular, at five, m disjoint inactive orbits of size divisible by four force $8m$ active labels in equal-degree quartets within E.*

Proof. Write Q_1 for averaging over the kernel of reduction modulo p. For $v \in L$ the squared norm of the α-phase projection of $W_p v$ is $\|Q_1 v\|^2/p$ for every $\alpha \in \mu_p$ (Section 21.10). Each such projection has rank r. The unsigned involution $H_p(-1)$ fixes each phase and acts by minus one on the image, requiring two labels per independent vector. Sum over the $p-1$ primitive phases.

The Weyl image of the $(m-r)$-dimensional kernel is supported at higher orders. Torus type ψ^{-1} restricts to a space of dimension at most one on each orbit, with stabilizer in $\ker\psi$ whenever the restriction is nonzero. Its support thus meets at least $m-r$ orbits. At order p^a, the square action gives $p^{a-1}(p-1)/2$ phases, each containing at least two positions by oddness. This proves the second bound. At five each specified inactive orbit affords the same faithful tame quartic character. The contributing active orbits have constant degree and size divisible by four, giving the stated selection. □

Compulsory active orbits. Consider the four disjoint quartets in Section 23.1. The complementary rows $X \in P_3, U \in P_5$ have degrees $a \in \{12, 24, 36, 72\}$ and $b \in \{20, 40\}$, respectively. An odd $V \in P_3$ has degree $c \in \{15, 45\}$, and a joint three–five label J has degree dividing 360. Each has a local five-inertia orbit of size divisible by four, by Section 21.6 and Sections 19.5

and 21.4. The selected X, V have purely ternary twist and U purely quinary twist. For each five-inactive orbit of this kind the permutation representation of the cyclic group H_5 contains the same tame quartic character, since its orbit size is divisible by four. Distinct orbits thus count independently in Proposition 23.5.

Every five-active degree divides 1080 and is at most 540 by Section 23.2. We also use the five-inactive label Z of Section 21.6 in the full five-block of J, with the same dyadic and ternary twists. Its field contains $\mathbb{Q}(\zeta_5)$ or $\mathbb{Q}(\zeta_{15})^+$, giving a fourth-order five-inertia action and five-height at most one. If no triple occurs, one can take $Z \in P_3$; then its ternary height is at most two. If a triple occurs, take J triple-active and its ternary activity again bounds that height by two. In all cases $d_Z \mid 360$.

In the triangle X, U have twist one (Section 21.6). In the absence of a triple, that argument also supplies the odd V of twist one; Z is three-active and so distinct from all three quartets. With a triple, Z is both two- and three-active, while V has pure ternary twist. Thus X, U, V, Z give four distinct inactive quartets and at least 32 five-active labels. Their J quartet has degree at most 360.

If ten is missing, $(a, b, c) = (12, 40, 15)$ and the chosen U has twist one (Sections 21.9 and 21.10). Again there are at least 24 five-active labels. All have full two-height and three-bearing degree (Section 21.4); their degrees divide 1080, and the largest such degree below 1080 with full two-height is 360. There are also five-inactive odd labels of degrees $45, 75$ by Section 21.4 and Theorem 21.10. Counting the unit, these two and the four degree-15 labels gives seven odd labels; their total number is divisible by four, so there is another, of degree at most 675. At a two-active column obtain a three-free nonunit $W \in P_2$ by prime avoidance; its degree divides 100 and is even, and it is five-inactive since it has deficient two-height. It is distinct from the degree-40 quartet by that same deficiency.

It remains that six is missing. Here $b = 20$. Every three-active label has full two-height by separation and is five-bearing because two and five occur together (Section 21.3). Hence the pure ternary X, V have twist one. The three-active Z is distinct from them and from U, and has full two-height and five-height one. In fact every degree-20 row of P_5 has pure quinary twist (its two- and three-heights are deficient) and full quartic five-inertia by its crossing with V. If even one is five-inactive, there are four inactive quartets and hence at least 32 active labels. Otherwise there are at least eight active degree-20 labels. Indeed a five-phase of order at least 25 gives an orbit of size divisible by 20, by the square action and the quartic inertia. If all their phases have order exactly five, use the terminal Fourier argument in the proof of Proposition 23.2: at a jointly two–five-active column the degree-40 rows miss by full two-height, so the only three-free terms are the vacuum and degree-20 rows at primitive fifth phases. That argument gives a hit at each primitive phase; its inertia involution supplies a second label of the same degree. This use of the Fourier identity needs only the stated phases. Thus the three inactive quartets of X, V, Z force at least twenty-four five-active labels. A three-free witness $W \in P_2$ in the pure dyadic fiber has twist one: every two-active degree has full three-height by separation. Its degree distinguishes it from X, V, and its twist distinguishes it from Z. Take a joint two–five label K. Its full five-block contains a label T with the same dyadic twist and trivial five-twist, by removal of the primary twist (Proposition 19.16). Its ternary twist is trivial by separation. It has full three-height, deficient dyadic height, and five-height at most one: a full five-height label would be a singleton five-block and could not share that block with K. Thus $d_T \le 540$. Its nontrivial pure dyadic twist distinguishes it from the four witness orbits.

Now consider the full five-block of J at its exact ternary phase, denoted by E. Each label there has degree $40 \cdot 3^j$, $j = 0, 1, 2$: three-activity gives full two-height and a factor of five, and full five-height would make its block a singleton. If n_j count these three degrees then

$$n_0 + 9n_1 + 6n_2 \equiv 0 \pmod{25}.$$

Indeed E commutes as a coordinate projector with W_5, so $\sum_{Y \in E} d_Y^2 = D\|EW_2W_3e_0\|^2$. On this nontrivial ternary phase W_2 is scalar: the dyadic upper and its conjugates are scalar there

by separation. The operator W_3 is integral at every prime above five, by the enhanced Weyl estimate (Section 21.4), since every three-active degree has five-height at least one. Hence the sum is divisible by 5^4 and division by 40^2 gives the congruence.

Section 24.2 uses this congruence to give the size of each such fiber.

Intersections of principal blocks on the path missing ten. In the ordinary minimal table at $D = 5400^2$, suppose two and five are separated. Each five-active column has a degree-40 twist-one hit in $P_5 \cap P_3$, and each two-active column has a three-free twist-one hit in $P_2 \cap P_3$ of degree $10, 20, 50$ or 100. Indeed, in this graph three divides every two-active and every five-active degree (Section 21.4). Apply Lemma 4.19 with $q = 3$ and $p = 5, 2$, respectively. The three-free nonunits in P_5 have degree 40 by Section 21.3; the positive two-height in P_2 is at most two (Corollary 4.1), and a nonunit degree has at least two prime divisors. Both hits can be chosen tame at three: in the lemma's principal-block congruence group the terms by wild three-inertia. Every nonsingleton orbit contributes zero to the residue (Section 21.4). In particular the chosen degree-40 label has field $\mathbb{Q}(\zeta_5)$ or $\mathbb{Q}(\zeta_{15})$ (Section 21.10).

For each jointly three- and five-active column J, the same lemma at $(p, q) = (5, 2)$ gives a degree-45 twist-one hit in $P_5 \cap P_2$ whose field is unramified at two. In fact any odd nonunit $X \in P_5$ meeting this J must have degree $3^j 5$: it has positive ternary and five-height by the exclusion of prime-power degrees, and the five-height in P_5 is at most one (Corollary 4.1). The odd-prime-weight bound in Section 21.4 gives $j \geq 2$, while three-activity of J excludes $j = 3$ by defect-zero vanishing. Hence $j = 2$. Wild two-inertia fixes some contributing row in the resulting intersection by the same orbit argument (and the tame quotient at two is trivial).

For a jointly two- and three-active column the five-free hit from $(p, q) = (2, 5)$ lies in $P_2 \cap P_5$, has twist one, and has degree $12, 18$ or 36: the two-height and three-height are each at most two, and degree six is excluded by Section 23.3.

Four further principal labels when six is missing. At $D = 5400^2$ suppose two and three are separated. At any jointly three–five-active J, Lemma 4.19 gives an odd twist-one hit $V \in P_2 \cap P_3$. Its degree is $5 \cdot 3^j$, $j = 1, 2$: it is deficient at three by principality and at five since it meets J. At any jointly two–five-active column the reversed lemma gives a three-free twist-one $W \in P_2 \cap P_3$ of degree $2^i 5$, $i = 1, 2$, by the corresponding dyadic and quinary height bounds.

In fact $V \notin P_5$. Otherwise $z = S_{WV}/\operatorname{lcm}(d_W, d_V)$ has weight five and residue five above each of $2, 3, 5$. At two use $\chi_W(V) = 3^j z$ and $W \in P_2$; at three and five use $\chi_V(W) = 2^i z$ and $V \in P_3 \cap P_5$. The congruences hold at every conjugate prime. By Proposition 8.6, the residues at three and five and strictness leave just

$$z = \eta(2 + 3\xi) \quad \text{or} \quad z = \eta\left(1 - \sum_{t=1}^{4} \eta_t \zeta^t\right),$$

where η is ternary, ξ is a nonidentity five-power root, ζ is a primitive fifth root and $\{\eta_t\} = \{1, 1, \omega, \omega^2\}$ for a primitive cubic root ω. The first expression gives $\eta\xi = 1$ on reduction at two, impossible by injectivity on odd-order roots. For the second expression work at a common odd conductor. Two is unramified, so the congruences at every prime over two are equality modulo 2. The four powers ζ^t, $1 \leq t \leq 4$, are an integral basis over the ternary cyclotomic integer ring (coprime conductors), also after reduction. Substituting $1 = -\sum_{t=1}^{4} \zeta^t$ in the congruence shows that all η_t are equal modulo two, again impossible.

Consequently $P_2 \cap P_5$ contains at least four nonunit twist-one labels distinct from the V quartet. Indeed, Section 21.7 uses the four tame five-conjugates of V together with the vacuum as columns, and the analogous four degree-20 labels $U \in P_5$ together with the vacuum as rows. Their residue matrix at two is the invertible five-by-five Fourier matrix. All five columns now belong to P_2. The forward factorization in that section therefore requires five distinct twist-one labels in $P_2 \cap P_5$, including the vacuum. The X quartet is outside P_2 since $\chi_X(U) = 5(1 + 3\xi') \not\equiv d_U = 20$

above two, with ξ' primitive of order five. No three-free nonunit in P_5 belongs to P_2: its degree is 20 or 40, the latter has full dyadic height, and in degree 20 its value at V reduces to a primitive fifth root instead of the odd dimension residue. This separates the four new labels from W and U as well.

23.8 The smallest simple degree and wild ramification

Assume $D = 5400^2$. We write P_5 for the full principal five-block and K_X for the character field of a simple X. A degree-ten simple X with K_X unramified at five requires the triangle, and has $\mathbb{Q}(\sqrt{-3}) \subset K_X$, by Section 19.12. We sharpen the count and the twists of its required witnesses.

Proposition 23.6. *If $d_X = 10$ and K_X is unramified at five, then P_5 contains at least ten five-free nonunits whose fields are tame at five. If there is no twist of order divisible by thirty, it contains at least eleven such labels. In the latter case θ_X has order dividing eight, and there is a five-free $Y \in P_5$ for which*

$$\chi_Y(X) = 5\epsilon(1-\omega), \qquad \theta_X\theta_Y = -\epsilon, \qquad \epsilon \in \{1,-1\}, \quad \operatorname{ord}(\omega) = 3.$$

The product XY has two twist values $-1, \omega$, each carrying total dimension $5d_Y$. Both Y, Y^* lie outside $P_2 \cup P_3$, are five-inactive, and have degree $12, 18, 24, 36$ or 72.

Proof. Take the contracted eigenvalues $z = \beta_Y$ of Section 19.12. In P_5 they are Eisenstein integers, the vacuum value is 2, and the nonunits give zero, sixth roots, or Eisenstein integers of norm three. Their squared-degree masses obey the mixed moments of degree at most three modulo five. Put $x = z + \bar{z}$, $t = z\bar{z}$. Group the nonvacuum points into their complex-conjugacy orbits in this order:

$$(x,t) = (0,0), (-2,1), (2,1), (-1,1), (1,1), (0,3), (-3,3), (3,3).$$

Their sizes are $m = (1,1,1,2,2,2,2,2)$. Let v_j be the mass at *one* point of the indicated orbit, reduced modulo five. The two conjugate points have equal masses. The equations for $1, x, x^2, t, x^3, xt$ give

$$(v_j) = (2-a-2b,\ 2+a+b,\ a+b,\ -2-2a-b,\ 1-2a-b,\ a,\ 1+b,\ b).$$

Indeed substituting the eight points and vacuum $(4,4)$ into the six equations determines coordinates $1,2,3,4,5,7$ from $a = v_6, b = v_8$. Wild five-inertia fixes the contracted point z and the degree. Its nonsingleton orbits have zero mass modulo five, so only its fixed labels need be counted. As every five-free squared degree is 1 or -1 modulo five, at least $\sum_j m_j \|v_j\|_5$ distinct nonunits are necessary, where $\|h\|_5 = \min_{k\in\mathbb{Z}} |h - 5k|$. The row minima over $b = 0, \dots, 4$, for $a = 0, \dots, 4$ in order, are $11, 12, 15, 13, 10$. This proves the first count, and $v_7 - v_8 = 1$ supplies a nonreal hit with $x = \pm 3$, hence $z = \epsilon(1-\omega)$ up to conjugation. More generally each norm-three hit is a sum of two adjacent sixth roots, and the exposure argument of Section 19.12 gives two physical packets of mass $5d_Y$. Their relative twist order is six. Since their normalized crossing has norm three, it contains $\mathbb{Q}(\sqrt{-3})$ and is nonzero modulo two but zero above three. This excludes $Y \in P_2, P_3$ by $d_X = 10$, prime-to-three and even. The ternary ramification implies $v_3(d_Y) \le 2$. The five-free degree then has the asserted possibilities by the elementary degree restrictions.

Suppose thirty is missing. If both packets have nontrivial five-phase, the one active at two would require full three-height by Proposition 19.16. Their masses have insufficient three-height. Since the relative twist has order six, both are five-inactive. A packet active at two and three would require full five-height, again impossible for its mass. Thus one twist is exactly -1 and the other a primitive cube root. This applies also to XY^*, since the twist of Y^* equals that of Y. A purely imaginary norm-three z changes sign under duality, giving opposite sets of packet twists,

impossible. This proves $a = 0$, hence the second count. For $z = \epsilon(1 - \omega)$, write $c = \epsilon\theta_X\theta_Y$. The twists are $c, -c\omega$ at Y and $c, -c\omega^{-1}$ at Y^*. Their required forms force $c = -1$. Since the three pairs occur, every five-active degree is three-bearing and every three-active degree is five-bearing (Section 21.3). Thus X has trivial five-component and Y trivial ternary component. Their product now shows that both twists are dyadic. Replacing X by any Galois conjugate changes at most ω to its inverse in the displayed value, hence gives the same twist for X. The Galois square action bounds its order by eight. □

The tame case with a varying prime. There is also a prime-uniform alternative which permits tame ramification itself. Let an ordinary minimal table have dimension $2^A 3^B p^C$, $p \geq 5$, $C \geq 4$, and a label X of degree $2p$ tame at p. For $p \geq 7$ one must have $C = 4$, all three twist pairs must occur, and some p-free $Y \in P_p$ satisfies

$$\chi_Y(X) = p\eta\epsilon(1 - \omega), \qquad \eta \in \mu_p, \quad \epsilon \in \{1, -1\}.$$

In particular the character fields at X, Y contain $\mathbb{Q}(\sqrt{-3})$, and X is not self-dual. At $p = 5$ a second possible form is

$$\chi_Y(X) = 5\gamma(\xi + \xi'), \qquad \gamma \in \mu_6, \quad \xi, \xi' \in \mu_5, \quad \xi \neq \xi'.$$

Here the alternative is compulsory on both separated paths at 5400^2. It forces $\sqrt{5} \in K_X \cap K_Y$ and two twist packets of equal mass $5d_Y$ and relative order five. The witness may be chosen tame at five and lies outside $P_2 \cup P_3$. On the path missing ten it has degree $24, 72$ or 216. If the five-inertia of K_X has order two then $\xi' = \xi^{-1}$. Thus besides the wild case, the two paths retain the tame case with this explicit crossing.

To prove the assertion put $L = K_X\mathbb{Q}(\zeta_3)$. Tameness at p implies that $[L : \mathbb{Q}(\zeta_3)]$ is prime to p: apart from the tame p-component the conductor has only two and three as prime factors. Average the fusion label over this group. Its eigenvalues on P_p, divided by p, are p-integral elements b_Y of $\mathbb{Q}(\zeta_3)$ by principality, since this last field is p-unramified. Consequently the mixed moments $\sum_{Y \in P_p} d_Y^2 P(b_Y, \bar{b}_Y)$ vanish modulo p for $\deg P < C$ by the integral block idempotent as in Section 19.12. The vacuum value is 2. Only p-free nonunits contribute on reduction, and the coefficients b_Y are invariant under wild p-inertia, so only its singleton orbits need be retained.

For such a nonunit Y, $\alpha = \chi_Y(X)/p$ is an algebraic integer by lcm divisibility. Project α by normalized relative trace onto $\mathbb{Q}(\zeta_{3p})$, in a common cyclotomic field. This contraction fixes the roots in μ_{6p} and kills all other roots: the omitted primary levels are only higher ternary, dyadic and p-power levels. It therefore preserves integrality on the full ring of integers and gives the image w a rational signed word of weight at most two. Two-prime unimodularity (the proof of Theorem 4.9) gives an integral signed word of length at most two. The residue of b_Y is precisely its image under erasing p-parts, since the further normalized trace gives $-1/(p-1) \equiv 1$ on nonidentity pth roots. This statement only uses residues of b_Y; the normalized trace at tame p need not preserve global integrality.

Apply the cubic $(b + \bar{b})(b\bar{b} - 1)$, whose vacuum value 12 is a p-unit. At a contributing nonunit, the erased signed word has norm three or four and nonzero real trace modulo p; the possibilities for integral sums of at most two sixth roots are norms $0, 1, 3, 4$. Its two sixth-root positions therefore are adjacent or identical. No shorter integral word gives this residue. By the same unimodularity its minimum rational weight is two. Thus no positive mass from the original physical word was killed by the contraction onto $\mathbb{Q}(\zeta_{3p})$, and $w = \alpha$ with full weight two. The two-phase theorem (Theorem 19.5), rescaled to full degree $2p$, rules out a relative order exceeding six. In the adjacent case the p-components are therefore equal, giving the first displayed form. Raising the entry to the pth power shows that both fields contain ω. The two physical packets have dimension pd_Y by the exposed hexagon edge (Section 19.12), after rotation by the common root. The separation contradictions of that section use only the relative order six, the packet masses and $\omega \in K_Y$, so all three twist pairs occur. In the identical case the distinct

p-components have ratio of order p (an identical full phase contradicts strictness). This requires $p = 5$. Conversely for $p \geq 7$ no nonunit p-free position can have erased norm four, by the same identical-phase argument. The degree-four polynomial of Section 19.12 then vanishes on every nonunit residue, while the vacuum value 72 is a p-unit. Hence $C = 4$.

For the five-case the positive quinary pair is exposed: assign value one to the two selected fifth roots and $-2/3$ to the other three. Multiply by a functional at the sixth roots which takes value one uniquely at γ, minus one at its opposite and absolute value less than one elsewhere (use the real part after rotation). Extend by the same higher-level traces. This exposes exactly the displayed two phases and gives equal masses. Their squared modulus contains $\sqrt{5}$. A row in P_2 is impossible since reduction at two preserves distinct fifth roots, whereas the degree ten is even. A row in P_3 would require $\xi + \xi' \equiv \pm 2$ at three after reducing γ, impossible by irreducibility of Φ_5 modulo three: the coefficient difference from ± 2 has a vacancy and a nonzero nonidentity coefficient. If ten is missing at 5400^2, one of the two packets is five-active and all such degrees have full two-height. Its mass forces $8 \mid d_Y$, giving the degree list by $5 \nmid d_Y$. At $d_Y = 216$ the formal codegree is 625, so the tame choice of Y has field $\mathbb{Q}(\sqrt{5})$ or $\mathbb{Q}(\zeta_5)$. Finally five-inertia of order two fixes the crossing under inversion of fifth roots with sixth roots fixed; equality of these two-term sums forces the multisets to coincide, hence the pair is inverse. At $D = 5400^2$ Proposition 19.23 yields the following restrictions on every five-inactive label, including those outside the principal blocks.

- If its five-conductor is at least 25 and it is nonselfdual or its tame five-inertia has order four, there are at least 48 five-active labels, at least 40 of five-primary twist order ≥ 25.
- If its five-conductor is at least 125, there are at least 124 five-active labels, including at least 100 whose degree is five-free. The rank is at least 174, or at least 300 if the label is nonselfdual or the tame inertia order is four.

Indeed Theorem 3.5 at odd primes gives $\operatorname{ord}(\theta_{Z,5}) \mid f_Z$. At five-height ≥ 1, $v_5(f_Z) \leq 2$, hence degrees at twists of five-order ≥ 125 are five-free and at most 216. Other five-active degrees are at most 540. In every case Section 23.7 supplies the vacuum and three disjoint five-inactive quartets with degree bounds $72, 45, 360$, respectively, and in the triangle a fourth with bound 40 (interchanging the last two when necessary). Four jointly three–five-active labels have degree at most 360. If ten is missing the three inactive bounds are $12, 15, 40$ and every five-active degree is at most 360. Substitute these counts in rank $\geq D/900^2 + \sum(1 - d_Z^2/900^2)$. Also count the orbit of the wild input itself. Its field is five-ramified, so its degree divides 1080 and is at most 540. At conductor at least 25, full tame order four gives an orbit of at least twenty, meeting at most one of the specified inactive quartets since they come from different Galois orbits. Thus at least sixteen are additional. Otherwise the orbit misses every such quartet, and nonselfduality doubles its minimum size of five. At conductor at least 125 the same argument gives at least twenty-five additional labels in all, and at least fifty in the doubled case (at full tame order there are at least one hundred before removing any overlap). Each additional label saves at least $1 - 540^2/900^2$.

24 Twist fibers and rank at the first open core

At $D = 5400^2 = 29160000$, every nonunit degree divides 5400, lies between 10 and 900 and has at least two distinct prime factors (Section 23.3). A joint label is active at the indicated primes. Write m_z for the number of simples of degree z, R for rank, τ for normalized field trace and $\mathsf{M}(b) = \tau(b\bar{b})$.

Theorem 24.1. *A perfect fusion-simple integral modular category of dimension* 5400^2 *has rank* $101 \le R \le 46774$. *The last column gives the pages where the bounds for each twist configuration are proved.*

Twist orders	*Min. rank*	*Min. five-active*	*Rank without* 900	*Proof, p.*
None divisible by 6	141	60	≥ 141	*213*
None divisible by 10	101	40	≥ 127	*215*, *216*
All pairs, none divisible by 30	106	56	≥ 130	*214*
Some divisible by 30	112	56	≥ 142	*214*

If $R = 101$, *at least* 21 *labels have degree* 900, *in at least four distinct Galois orbits and at least two twists. At least* 18 *of their rows meet a label active at two and three. At every rank the field degree at a simple of degree* 900 *is at most six. At degree* 675 *the field is rational or quadratic and* $m_{675} \le 7$. *At degree* 600 *the field is contained in* $\mathbb{Q}(\zeta_9)$, *and its degree is at most three if no twist order is divisible by six.*

24.1 The three largest possible degrees

For the remainder of this section let $\mathcal{C}$ be a perfect fusion-simple integral modular category with $D = 5400^2$. By Section 23.2 every simple degree divides 5400 and every nonunit degree is at most 900.

Corollary 24.2. *If no twist order is divisible by six at* 5400^2, *every simple degree is at most* 675.

Proof. A degree-900 row in Proposition 6.12 has $p = 2$, $q = 3$, $a = 2$, $b = 3$ and $f = 36$. A two-active degree is divisible by 27 and at most 900. Among divisors of 5400 with these constraints, the largest is 675. But $2 \cdot 675 \cdot 35 = 47250 < 48600 = 5400 \cdot 3^2$. The next possible degree after 900 is 675. □

Corollary 24.3. *At* 5400^2, *with at most three exceptions among all degree-*900 *simples, every such row has a nonzero entry on a label whose twist is jointly two- and three-active and five-inactive. Any exception has twist one and belongs to* $P_2 \cap P_3$. *It requires* $\xi = \pm 1$, *and the possible pairs* $(k_{2,Z}, k_{3,Z})$ *are* $(3, -2\xi)$ *and* $(-3, 4\xi)$. *In particular every orbit of at least four degree-*900 *simples meets the indicated joint sector at every member.*

Proof. A degree-900 row misses all five-active labels. If it also misses the indicated joint sector, the separated-sum equation $H_2 + H_3 = 5400\xi\theta_Z^{-1}$ holds for this row even without global separation. By Theorem 6.11, $3 \mid k_{2,Z}$ and $2 \mid k_{3,Z}$ since $f_Z = 36$. Neither integer can vanish, as the other would then have modulus one. Local Weyl vacuum columns belong to their respective principal blocks and have trivial twist at all complementary primes, by local commutation. This puts Z in both principal blocks at twist one. Each such label accounts for at least $9/36$ of the norm squared of $W_2 e_0$. Four labels would exhaust this norm on nonunit twist-one labels, impossible by the strict norm estimate following Proposition 6.12. Likewise $|k_{2,Z}| = 6$ is impossible as it would exhaust that norm on Z alone. Thus $k_{2,Z} = \pm 3$. Since $\gamma_3 = 1$, $\gamma_2^3 = \xi$ and $\theta_Z = 1$, the separated-sum equality becomes $k_{3,Z} = \xi(1 - k_{2,Z})$. This proves the assertions; the exceptional property is preserved by Galois. □

For a nonunit Z put $K = K_Z$, $h = [K : \mathbb{Q}]$, $f = D/d_Z^2$, and $b_Y = \chi_Z(Y)$. In a cyclotomic field L containing K denote the normalized trace to $\mathbb{Q}$ by τ. For $\beta \in L$ put $a_Y = \tau(\beta b_Y)$ and $B = \mathrm{Tr}_{L/K}(\beta)/[L : K]$. Orthogonality of the h distinct conjugate rows gives

$$\sum_Y a_Y^2 = \frac{f}{h}\tau(|B|^2) \le \frac{f}{h}\tau(|\beta|^2), \qquad \sum_Y d_Y a_Y = 0. \tag{24.1}$$

Indeed a is the sum of those rows with coefficients $\sigma(B)/h$. Equality in the inequality holds precisely when $B = \beta$. If $\tau(\beta\mathcal{O}_L) \subset \mathbb{Z}$, then

$$a_Y \in \frac{d_Y}{\gcd(d_Z, d_Y)}\mathbb{Z}$$

by lcm divisibility. When $a_0 = 1$, reducing the last equality of (24.1) modulo $2, 3, 5$ requires a nonzero nonunit coefficient at a degree omitting each prime. These are three distinct degree types, since no nonunit has prime-power degree.

We will use the same second-indicator calculation on two Galois orbits. Put $s = S/n$, $T = \operatorname{diag}(\theta_Y)$, and write e_0 for the vacuum vector.

Lemma 24.4. *Let $\mathcal{O} = GZ$ be a Galois orbit in an integral modular category of square dimension n^2, where $G = \operatorname{Gal}(K_Z/\mathbb{Q})$. For $\rho \in \widehat{G}$ let $u_\rho(gZ) = \rho(g)$, extended by zero off $\mathcal{O}$. Suppose $\lambda(g) = \theta_Z/\theta_{gZ}$ is a character. If su_δ is supported at twist one, then $su_{\delta\lambda}$ is a unit multiple of $u_{(\delta\lambda)^{-1}}$ and*

$$\left|\sum_{g\in G} (\delta\lambda)(g)\, g(\chi_Z(Z))\right| = n/d_Z.$$

Suppose moreover that λ has order three. Let a Galois permutation fixing the vacuum and commuting with s, T have fixed space on $\mathcal{O}$ spanned by the modes

$$u_{\delta\lambda^j}, \qquad 0 \le j < 3,$$

where δ runs over distinct characters of order at most two, including the trivial character. If every su_δ is supported at twist one, then

$$|(sT^2se_0)_Y| = d_Z/n \qquad (Y \in \mathcal{O}).$$

Proof. Since $Tsu_\delta = su_\delta$, the modular relation $sTs = \xi T^{-1}sT^{-1}$ gives $sT^{-1}u_\delta = \xi^{-1}Ts^2u_\delta$, supported on $\mathcal{O}$. Since $T^{-1}u_\delta = \theta_Z^{-1}u_{\delta\lambda}$, its s-image is supported on that orbit. For a Galois permutation P, the unsigned covariance $sP = P^{-1}s$ gives the inverse character there. The scalar has modulus one by unitarity. Evaluating at Z gives the sum, since $s_{Z,gZ} = (d_Z/n)g(\chi_Z(Z))$. In the cubic case $\delta^2 = 1$ and s^2 acts by duality, so s exchanges the two lines $u_{\delta\lambda^{\pm1}}$. Hence

$$H_\delta = \operatorname{span}\{u_\delta, u_{\delta\lambda}, u_{\delta\lambda^{-1}}, su_\delta\}$$

is invariant under s, T. Indeed T cycles the three orbit modes up to scalar, and no label of $\mathcal{O}$ has twist one, by square Galois covariance and nontriviality of λ. The displayed vectors over all δ are therefore orthogonal, each of squared norm $|G|$. Since $(su_\delta)_0 = (d_Z/n)\sum_g \delta(g)$, the projection of e_0 onto their sum is $(d_Z/n)su_1$. Its invariant orthogonal complement in the fixed space vanishes on $\mathcal{O}$. Thus there $sT^2se_0 = (d_Z/n)sT^2u_1$. As T^2u_1 is a unit multiple of u_λ, the claim follows. □

Theorem 24.5. *In the integral modular category at $D = 5400^2$, a degree-900 character field is contained in $\mathbb{Q}(\zeta_{36})$ and has degree at most six. A degree-675 character field is rational or quadratic. Its twist order divides eight unless its field is $\mathbb{Q}(\sqrt{2})$, in which case it divides sixteen.*

Proof. At degree 900 the formal codegree is 36. By Theorem 6.13 the field lies in $L = \mathbb{Q}(\zeta_{36})$ and the dyadic part of the twist divides eight. As $[L : \mathbb{Q}] = 12$, it suffices to exclude $K = L$. Write $\omega = \zeta_3$ and take $\beta = 2(1 - \omega^{-1})/3$. On third roots, the traces $\tau(\beta\omega^j)$ are $1, -1, 0$ in some order with value 1 at $j = 0$. On primitive ninth roots the trace is zero. Normalized traces of dyadic roots belong to $\{0, 1, -1\}$, and the traces factor across the two conductors. Thus a is integral, $a_0 = 1$, and $\sum a_Y^2 \le 48/h$.

Assume $h = 12$. The vacuum and the three mandatory nonunit positions exhaust the norm, with coefficients ± 1. Automorphisms fixing ω preserve a and fix each of these positions by uniqueness of its degree type. Conjugating β replaces a by its dual. By row orthogonality their inner product is $(36/12)\tau(\beta^2) = 2$. Thus exactly one of the three nonunit positions is self-dual, since each common position is fixed and contributes one.

All four positions have twist one. Indeed the twist of Z divides $8 \cdot 9$ by the individual twist bounds. If it divides 24, its conjugate twists agree by square covariance. Averaging the vacuum entry of $STS = p^+T^{-1}ST^{-1}$ with weights $g(\beta)/12$ would give $5400 = |\sum_Y d_Y a_Y \theta_Y| \leq 1 + 3 \cdot 900$. The ternary component is therefore primitive of order nine. Now the same average is zero: the three units $1, 4, 7$ modulo nine fix β and the dyadic part, and give three equally spaced inverse twists. Subtract $\sum_Y d_Y a_Y = 0$. Modulo five only the difference at the five-free nonunit survives and modulo three only the one at the three-free nonunit survives. Each corresponding root of unity differs from one by a multiple of 5 or 3, of house at most two, hence coincides with one. The remaining twist is then one as well.

The square action makes $\lambda(g) = \theta_Z/\theta_{gZ}$ a cubic character: for every unit lift t it equals $\theta_Z^{1-t^2}$, and $9 \mid (t^2-1)(u^2-1)$ for any two units. Dyadic inversion extended trivially at the odd primes commutes with s, T and fixes e_0. Its fixed modes on $\mathcal{O}$ are exactly $u_{\delta\lambda^j}$ with $\delta = 1, \epsilon$, where ϵ is the ternary quadratic character. The vectors with coordinates $g(\beta)$ and $g(\bar{\beta})$ span u_1, u_ϵ. Their s-images are proportional to the trace vectors a and its dual, supported at twist one. Lemma 24.4 therefore gives $|(sT^2se_0)_Y| = 1/6$ on $\mathcal{O}$.

Put $A = \sum_Y a_Y[Y]$. By character orthogonality $\chi_Y(A^*) = 0$ off the orbit, with the twelve values $3g(\beta)$ on it up to reindexing. The indicator formula in the proof of Theorem 6.11 gives $\nu_2(A) = 12 \cdot 3\tau(\beta)/36 = 1$. The vacuum and the unique nonunit self-dual position give 1 ± 1, a contradiction.

At 675, $f = 64$ and $K \subset L = \mathbb{Q}(\zeta_{64})$ by Theorem 6.13. Use $\beta = 1$. Normalized dyadic trace is integral, so $\sum_Y a_Y^2 = 64/h$. The three-free and five-free nonunit hits have even degree, hence even coefficients. Thus $64/h \geq 1 + 1 + 4 + 4 = 10$. Since h is a power of two, $h \leq 4$. The fourth powers in $(\mathbb{Z}/64)^\times$ contain all units congruent to one modulo 16, so $K \subset \mathbb{Q}(\zeta_{16})$. Square covariance bounds the twist order by 32, and the codegree excludes all odd twist primes.

At $h = 4$ the total squared norm is 16. Every even-degree hit has two-height one and coefficient ± 2; a coefficient divisible by four alone would exhaust the norm. Its degree is 18 or 54 in the five-free class, and 10 or 50 in the three-free class. For the first class, a crossing with a three-free $U \in P_5$ of degree 20 or 40 (Section 23.1) has gcd two and is a sum of two distinct five-power phases by Theorem 4.9. The squared modulus of their sum is $2 + \eta + \eta^{-1}$ for a nontrivial five-power root η. Its real cyclotomic field contains $\mathbb{Q}(\sqrt{5})$, so K_Y contains this quadratic field. The coefficients a_Y are constant on entire orbits. The five-free class costs at least eight in squared mass, leaving room for only one three-free label, rational of degree 50: a rational degree-ten row is excluded by the 19! vacuum idempotent argument (Section 23.3). Thus the five-free orbit has exactly two labels and field $\mathbb{Q}(\sqrt{5})$. Including the vacuum leaves three units of squared mass, all at odd nonunits with coefficients ± 1. If θ_Z has order at most eight all its conjugate twists coincide, forcing $5400 \leq 1 + 4 \cdot 54 + 2 \cdot 50 + 3 \cdot 675$, a contradiction. Hence its order is 16 or 32. Write B, B' for the two five-free labels of common degree e, and A for the rational degree-50 label. Modulo five, this gives $1 \pm 4e \equiv 0$, forcing $e = 54$ with negative sign. Modulo three the coefficient of A is negative as well. The three signed odd degrees divided by fifteen therefore sum to 21. The absolute values belong to $\{1, 3, 5, 9, 15, 45\}$, by degree divisibility and the absence of nonunit prime-power degrees. In decreasing order of absolute value the possible triples are

$$(15, 5, 1), \quad (15, 3, 3), \quad (9, 9, 3), \quad (15, 9, -3), \quad (15, 15, -9), \quad (45, -15, -9).$$

Indeed if 45 occurs the other two give -24, and otherwise one need only write $21 - 15 = 6$ or $21 - 9 = 12$ as a signed sum of two allowed terms.

All the support labels have twist one. To prove this, average the vacuum entry of the modular relation over the orbit of Z. We get $\sum d_Y a_Y \theta_Y = 0$ since the square orbit of either a primitive sixteenth or a primitive thirty-second root sums to zero. Modulo three the sole terms are the vacuum and $-100\theta_A$, hence $\theta_A - 1 \in 3\mathcal{O}$, and its house forces $\theta_A = 1$. Modulo five one obtains $\theta_B + \theta_{B'} - 2 \in 5\mathcal{O}$. Its house is at most four. Therefore both phases are one. For a positive triple of odd terms equality in the triangle inequality forces the three phases to be one. For $(45, -15, -9)$ move the two negatives to the other side and use the same equality. For $(15, 15, -9)$ divide by three and reduce modulo five to see that the negative term has twist one; the two positives then do too. In $(15, 9, -3)$ the three degrees are distinct, so their labels are rational. Their twists have order dividing 24 by square covariance. For a nontrivial root of order dividing 24 the rational normalized trace of $1 - \theta$ is at least $1/2$. The negative term can compensate a deficit of at most six, so the weight-15 twist is one. The resulting identity $9\theta_2 = 6 + 3\theta_3$ again gives equality in the triangle inequality.

With u_1 the orbit indicator, su_1 is proportional to a. The twist ratio $\lambda(g) = \theta_Z^{1-t^2}$ for a unit lift t is a character since $64 \mid (t^2-1)(v^2-1)$ for all odd t, v. Lemma 24.4 gives $|\sum_g \lambda(g) g(b_Z)| = 8$. In particular $b_Z \neq 0$. At order sixteen λ is the quadratic character cutting out $\mathbb{Q}(\sqrt{2})$. The sum belongs to its anti-invariant rational line $\mathbb{Q}\sqrt{2}$ and cannot have absolute value eight.

At order thirty-two, square covariance puts the real field $\mathbb{Q}(\zeta_{16})^+$ of degree four in K_Z, so they coincide. In this field the normalized trace of an algebraic integer has the same residue as the integer at the dyadic prime: an integral basis is 1 and $\zeta_{16}^j + \zeta_{16}^{-j}$ for $j = 1, 2, 3$; the last three have trace zero and residue zero. But $a_Z = 0$, since none of the three nonunit odd trace positions can contain the four-element orbit of Z. On the other hand the nonzero b_Z puts Z in the full principal two-block by prime-power codegree support (Theorem 2.9), whence $b_Z \equiv 675 \equiv 1$ at the dyadic prime. This contradiction excludes $h = 4$. The asserted twist bound for $h \leq 2$ follows from the same square covariance and the dyadic codegree. □

The subgroup lattice of $\mathrm{Gal}(\mathbb{Q}(\zeta_{36})/\mathbb{Q}) = C_2 \times C_6$ now gives the possible degrees $1, 2, 3, 4, 6$ at 900. The unique quartic field is $\mathbb{Q}(\zeta_{12})$. A sextic field is $\mathbb{Q}(\zeta_9)^+$ times one of $\mathbb{Q}(i), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{3})$.

Corollary 24.6. *Every degree-900 label with orbit size six has twist order divisible by nine.*

Proof. The individual twist bounds, including Theorem 6.13 give $\mathrm{ord}(\theta_Z) \mid 8 \cdot 9$. If it divides 24, every conjugate row has the same twist by square covariance. Take the integral trace selector $a_Y = \tau(\beta\chi_Z(Y))$ of Section 24.1 in $\mathbb{Q}(\zeta_{36})$, whether or not β lies in K_Z. Its vacuum value is one, and $\sum_Y a_Y^2 \leq 48/6 = 8$. Its signed dimension sum vanishes, so it has three distinct nonunit hits omitting respectively two, three and five. The degree bounds there are $675, 200, 216$. The vacuum entry of the modular relation would give

$$5400 \leq \sum_Y d_Y |a_Y| \leq 8 \cdot 900 - (900 - 1) - (900 - 675) - (900 - 200) - (900 - 216) = 4692,$$

using $|a_Y| \leq a_Y^2$ at integral positions. Hence nine divides the twist order. □

The two ramified sextic possibilities. Suppose $h = 6$ and $K_Z = \mathbb{Q}(\zeta_9)^+ E$ with $E = \mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{3})$. Then Z misses all degrees divisible by eight. Moreover the rational trace $a_Y = \tau(b_Y)$ has exactly six nonzero coefficients, all ± 1, and its support includes two different Galois orbits of nontrivial twists with opposite coefficient signs.

Indeed $\zeta_3 \notin K_Z$, so projection of β to K_Z is one and the integral selector in Corollary 24.6 is a. Its squared norm is six. Its mandatory four distinct positions, including the vacuum, rule out a coefficient of magnitude two. Thus $A = \sum a_Y [Y] = 6e_{\mathcal{O}}$, where $e_{\mathcal{O}}$ is the character idempotent of the orbit. The diagonal of its orthogonal projection at Y is $\mathsf{M}(b_Y)/6 < 1$: the vacuum line is outside this projection and has a nonzero Y-coordinate. If $8 \mid d_Y$, the field of Y is unramified at two, so $b_Y = 2z$ with z in the integer ring of $\mathbb{Q}(\zeta_9)^+$. Normalized trace is integral on this real

ring, since it is generated additively by 1 and the real root sums $\zeta_9^j + \zeta_9^{-j}$. Hence $\mathsf{M}(z)$ is an integer smaller than $6/4$. A nonzero z would have mean square one, and so would be a real root of unity, giving $a_Y = \pm 2$.

If all six support twists were one, $su_1 = a$ would be supported there. The twist-ratio character λ is cubic by Corollary 24.6. Dyadic inversion fixes precisely the modes $u_1, u_\lambda, u_{\lambda^2}$ on this orbit. Lemma 24.4 and the indicator formula would give $\nu_2(A) = 6 \cdot 6/36 = 1$. Yet its six signed positions are stable under duality, so an even number are self-dual and $\nu_2(A)$ is even. Finally $\sum_Y d_Y a_Y \theta_Y = 0$ since the ninth components of the orbit twists average to zero. Subtract the untwisted dimension sum and take real parts. The nontrivially twisted positions must occur with both signs, on different orbits.

Corollary 24.7. *At any prescribed twist there are at most thirteen degree-900 simples, on every admissible twist graph. Twelve is an upper bound whenever $R \le 7035$.*

Proof. Let m be their number at the prescribed twist; then $M_t \ge 900^2 m$. All other nonunit degrees are at least ten (Section 23.3). Thus

$$B \le 900m + 1 + (D - 900^2 m - 1)/10.$$

For $m = 14$ this is less than 1794601, whereas the bound of Proposition 6.14 would require $\sqrt{B} \ge (9 \cdot 14 \cdot 900^2 - 3D - 4n)/(2n) = 1348$. The upper bound on B decreases with m and the required lower bound increases. For $m \ge 13$ the proposition similarly requires $\sqrt{B} \ge 673$. Since $B \le n\sqrt{R}$, this implies $R \ge 673^4/5400^2 > 7035$. □

Proposition 24.8. *At 5400^2 write k for the total number of degree-675 labels. Then $k \le 7$. If any member of the full principal two-block has degree divisible by four, then $k \le 3$. Consequently, when $k \ge 4$ every even member of that block has dyadic height one.*

Proof. For $W \in P_2$ put $i = v_2(d_W)$. Since every row of degree 675 hits W by two-principality and $d_W / \gcd(d_W, 675) = 2^i$, Lemma 6.15 gives $ek4^i < 64$ for its five-inertia orbit size e. There exists a five-free nonunit $W \in P_2$ by Proposition 4.2. Its degree involves two and three. If $i \ge 2$ the bound gives $k \le 3$. Otherwise $d_W = 18$ or 54, since degree six is excluded. A three-free $U \in P_5$ has degree 20 or 40 (Section 23.1). Their crossing, of weight two, is a sum of two distinct five-power roots up to a scalar (Theorem 4.9). Its squared absolute value generates a field containing $\mathbb{Q}(\sqrt{5})$, so $e \ge 2$ and $k \le 7$. □

Where all odd degrees must lie if a quadratic field occurs at 675. Suppose a degree-675 label Z has quadratic field $\mathbb{Q}(\sqrt{\delta})$, where $\delta = -1, 2$ or -2. Then every odd-degree label belongs to P_2. In particular every jointly three- and five-active label is even by the nonprincipal two-block result (Section 21.6). There are at least four degree-675 labels and every even degree in P_2 has dyadic height one.

Indeed write $b_Y = x_Y + \sqrt{\delta}\, y_Y$ with rational integers x_Y, y_Y. Each is divisible by $2^{v_2(d_Y)}$ by lcm integrality. Orthogonality of the two conjugate rows gives

$$\sum x_Y^2 = 32, \qquad \sum y_Y^2 = 32/|\delta|.$$

At twist order at most eight the twists of Z and its conjugate coincide. Adding their vacuum modular identities gives $5400 \le \sum d_Y |x_Y|$. For nonzero x_Y the ratio $d_Y/|x_Y|$ is at most 225 unless $d_Y = 675$ and $|x_Y| \le 2$: the odd part divides 675 and any proper divisor is at most 225. Multiples $2 \cdot 675$ are excluded by the degree bound. Let m_1, m_2 count degrees 675 with $|x_Y| = 1, 2$. Their excess over $225 x_Y^2$ is 450 in either case. The vacuum, three-free nonunits and five-free nonunits have respective ratios at most $1, 25, 27$ and compulsory squared masses at least $1, 4, 8$. For the last bound, a five-free position has dyadic height at least one, and height one

entails degree 18 or 54 (degree six excluded), with at least a conjugate pair by the weight-two crossing with $U \in P_5$ recalled above. Thus

$$5400 \le 4592 + 450(m_1 + m_2).$$

In particular some 675 hits a 675. At twist order sixteen we have $\delta = 2$ and the two conjugate twists are opposite. Subtracting the two modular identities gives $5400/\sqrt{2} \le \sum d_Y |y_Y|$. We have $\sum y_Y^2 = 16$, so if all y_Y vanished on degree 675 the right side would be at most $225 \cdot 16 = 3600$. This again gives a mutual hit. In fact multiplying the difference identity by $\sqrt{2}\,\xi^{-1}\theta_Z$ and taking normalized trace gives

$$5400 = \sum_Y d_Y y_Y \, \tau(\sqrt{2}\,\xi^{-1}\theta_Z\theta_Y).$$

Here the difference was taken with the factor $2\sqrt{2}$ removed, so that its right side before multiplication is $5400\xi\theta_Z^{-1}/\sqrt{2}$. Each displayed trace has modulus at most one: $\sqrt{2} = \zeta_8 + \zeta_8^{-1}$, and for any dyadic root at most one of the two resulting terms has nonzero normalized dyadic trace. Traces factor across the odd conductor. The trace vanishes unless θ_Y has dyadic order exactly sixteen, as ξ has order dividing eight. Each degree 675 with $|y_Y| = 1, 2$ can improve the allowance $225y_Y^2$ by 450. Thus there must already be at least $(5400 - 225 \cdot 16)/450 = 4$ degree-675 labels in this case.

Every degree-675 row has support exactly P_2: its codegree is a two-power, and each principal position gives $\chi_Y(Z) \equiv 675 \not\equiv 0$ above two. By symmetry a mutual hit puts a degree-675 label in P_2, then every one of them in P_2. Testing any odd degree on such a principal row shows that all odd labels are in its support. At a joint three–five column pick the odd $V \in P_3$ of degree 15 or 45 with field unramified at two (Section 21.4); its five-inertia orbit has at least four labels. On these four b_Y is a rational odd integer by principality and the intersection of the two fields. Their contribution saves at least $4(225 - 45) = 720$ against $225x_Y^2$, disjoint from the reserved positions. The bound 4592 at order at most eight can consequently be replaced by 3872, proving $m_1 + m_2 \ge 4$. If $W \in P_2$ has dyadic height k, it hits every degree 675 with lcm-scale $675/\gcd(675, d_W) = 675 \cdot 2^k/d_W$. Averaging its row norm over the Galois group and reserving the vacuum gives $4(675 \cdot 2^k/d_W)^2 < D/d_W^2$. Hence $k \le 1$.

Proposition 24.9. *At $D = 5400^2$ any simple of degree 600 has character field contained in $\mathbb{Q}(\zeta_9)$ and twist order dividing nine.*

Proof. Here $f_Z = 81$, so $K_Z \subset \mathbb{Q}(\zeta_{81})$ and its degree h divides 54. Put $b_Y = \chi_Z(Y)$ and take $a_Y = \tau(\beta b_Y)$ with $\beta = 2(1 - \omega^{-1})/3$, $\omega = \zeta_3$. The normalized trace takes integral values on β times every ternary cyclotomic integer: on powers of ω its values are $1, -1, 0$, and on roots of higher ternary order they vanish. We have $a_0 = 1$, $\sum_Y d_Y a_Y = 0$ and $\sum_Y a_Y^2 \le 108/h$ by row orthogonality, by character orthogonality and $\tau(|\beta|^2) = 4/3$.

For every three-bearing degree d the sum of the a_Y at degree d is a multiple of three by the degree-sum obstruction. Reduction of $\sum d_Y a_Y = 0$ modulo two, respectively five, requires a nonzero such sum at an odd, respectively a five-free, nonunit degree. These disjoint types are both three-bearing since a nonunit degree has at least two prime factors. Each type costs at least three in squared norm. The vacuum and the necessary three-free nonunit add two, showing $h \le 108/8$. If $h > 6$, the only possibility is $h = 9$, so the field is the real subfield of $\mathbb{Q}(\zeta_{27})$ and $a_Y = \tau(b_Y)$. Now $\sum a_Y^2 = 81/9 = 9$. In particular no nonzero coefficient occurs at ternary height two or higher, since lcm divisibility would make it a multiple of three.

The five-free entries thus have degree 12 or 24 (degree six was excluded). Their allowance is at most four in squared norm after reserving the vacuum, the odd type and the three-free type. A nonzero signed sum in a single one of these degrees must therefore come from exactly three equal signs at that degree. A fourth nonzero entry could neither preserve the multiple of three at that degree nor give a multiple of three at the other degree. Modulo five the equation is $1 \pm 3d \equiv 0$, so $d = 12$ and all three signs are negative. However every degree-12 member of

P_3 has tame five-inertia orbit of size divisible by four. To see this, cross it with a three-free $U \in P_5$ of degree 20 or 40 (Section 23.1). The common weight is four. In the unsigned case of Section 19.3 the crossing is $1 + 3\xi$ with ξ primitive of order five. In the signed case its four nonidentity fifth positions have coefficients given, up to a common sign, by one copy of one and a coset of the three cube roots. The square of a tame generator pairs the four positions and cannot preserve these coefficients since some multiplicity is odd. The sole relation on the fifth roots is the constant vector over the ternary field, and the position zero stays fixed. This proves the orbit assertion in both cases. Since a is an ordinary normalized trace, it is constant on that orbit. This contradicts the three labels and gives $h \le 6$. The cyclic Galois group at 81 then puts K_Z inside $\mathbb{Q}(\zeta_9)$. Any higher ternary twist would require at least nine square conjugates, and the codegree excludes every other prime. □

The cubic conductor at degree 600 **on the path missing six.** In fact the degree-600 character field has degree at most three on that path. To exclude $h = 6$, let $K_Z = \mathbb{Q}(\zeta_9)$ and use the two conjugate selectors $a_Y = \tau(\beta b_Y)$, $a_{Y^*} = \tau(\bar{\beta} b_Y)$ defined in the proof for degree 600. Then $\sum a_Y^2 = 18$. A three-free $U \in P_5$ of degree 20 exists on this path. Each five-free Y with nonzero a_Y belongs to P_3. Its ternary height is at most two by lcm divisibility. At height two the possibilities $18, 36, 72$ have $3 \mid a_Y$; their five-inertia orbit sizes are multiples of two, four and four respectively, by their weight-two or weight-four crossings with U by the weight-two and weight-four classifications. Even two such positions would exhaust the norm before the vacuum. The remaining five-free degrees are $12, 24$. Their crossings with U have weight four and hence five-inertia orbit sizes divisible by four. Both signed degree sums $\sum_{d_Y = d} a_Y$ are therefore multiples of twelve by the degree-sum obstruction at three. Reserving the odd type, the three-free type and the unit leaves at most 13 in squared mass. Modulo five the only possibility is twelve positive degree-12 coefficients: $1 + 12 \cdot 12 \equiv 0$, whereas neither choice of sign at 24 works. A thirteenth position of that type cannot preserve the signed sum.

There are five further units of norm after the vacuum and the twelve. A three-free position is needed. The odd hits are of degree 15 or 75 as a higher ternary height would cost nine. Their signed sums are multiples of three, and at 15 also of four by the weight-five crossing with U (Section 19.8). Thus precisely three odd positions of degree 75 have equal sign. The other two positions are three-free: a three-bearing signed sum at a new degree cannot be obtained from a single spare unit, and neither known type admits one additional position. The possible three-free degrees are $10, 20, 40, 50, 100, 200$. The signed dimension equation $1 + 144 \pm 225 + e_1 + e_2 = 0$ now gives negative sign and $(e_1, e_2) = (40, 40)$ or $(100, -20)$.

No hit can be active at two or five, since the degree of Z has full height at both. A three-active degree has full dyadic height on this path, so the only possible three-active trace positions are the two of degree 40. If the twist of Z has order at most three, the sum of $a_Y d_Y \theta_Y$ has modulus 5400, impossible on the listed support. Thus θ_Z has order nine. Averaging over wild ternary inertia makes this sum zero, since β is fixed and the three conjugate twists are equally spaced. In the two-40 alternative this gives $40(\theta_1 + \theta_2 - 2) = 0$. Consequently every trace position has twist one.

Let $G = \mathrm{Gal}(K_Z/\mathbb{Q})$ and put $u_\chi(gZ) = \chi(g)$ on the orbit and zero elsewhere. Put $v(g) = g(\beta)$ on the orbit. The s-image of v is supported at twist one. As in the proof at 675, $sT^{-1}v$ is supported on the orbit. Here v has both trivial and quadratic components, so $T^{-1}v$ has the two components $\lambda, \lambda\epsilon$, where $\lambda(g) = \theta_Z/\theta_{gZ}$ is cubic and ϵ is the quadratic character. Torus equivariance and s^2 equal to duality now put the s-images of these two modes and their two inverses on this orbit. Each of the four internal Fourier coefficients $\sum_g \chi(g) g(b_Z)$ has modulus $5400/600 = 9$. Write c_Y for the mean of b_Y over the three wild ternary conjugates; then $\sum |c_Y|^2 = 81/3 = 27$ by row orthogonality, and its orthogonal complement has norm squared 54. The four characters are all wild-nontrivial. Parseval shows that their contribution on the orbit of Z alone is $4 \cdot 9^2/6 = 54$. Thus $b_Y = c_Y \in \mathbb{Q}(\omega)$ off the orbit.

At a positive degree-12 position write $b_Y = x + y\omega$ with $x, y \in \mathbb{Z}$. The two selectors are $x - y$ and x. Their sum $2x - y$ is zero modulo three since b_Y has zero residue at three (principality and $v_3(12) = v_3(600)$). Yet both selectors at degree 12 are zero or one, and at this position one is positive. This final contradiction excludes the sextic field.

A restriction on the central charge. On every graph, the existence of a degree-600 simple forces $\xi = \pm 1$. Indeed a hit of this simple has trivial dyadic and quinary twist components by full height of 600 at two and five. Its character field is ternary and θ_Z is a ninth root. Thus $5400\xi\theta_Z^{-1} = \sum_Y d_Y \chi_Z(Y)\theta_Y$ belongs to a cyclotomic field of odd ternary conductor. The dyadic root of unity ξ in this field is real.

24.2 Local Weyl transitions in joint twist fibers

Write E for a full five-block restricted to fixed dyadic and ternary twists. Proposition 23.5 applies inside this fiber.

Two distinct ternary phases. Let J be jointly three- and five-active at 5400^2. There are two different nontrivial ternary phases with the *same* dyadic and quinary components as J at which a joint label occurs. Indeed at local order three the two primitive cubic phases have equal positive squared mass when the complementary phases are fixed (Section 19.11). At order $3^a \geq 9$, the ternary square action alone supplies 3^{a-1} phases. In each such phase choose a joint label, and take its full five-block and exact complementary phases. The resulting fibers E are disjoint. Each contains a five-inactive quartic orbit by Section 21.6. Consequently each contains at least eight five-active labels. Every active label there is jointly three- and five-active, so the local five-inertia involution moves it (Sections 21.4 and 21.6); in particular all active labels occur in quartets.

Equal mass at tame fifth phases. In a tame fifth fiber E as above suppose the active degrees are $d_0 b$ with b odd. Section 19.11 gives equal squared mass at each of the four primitive phases. After division by d_0^2, reduction modulo eight shows that the four label counts agree modulo eight. If all active orbits are quartets, each contributes twice its squared degree at each phase of its square class. Cancelling this factor two before reduction shows that the numbers of quartets in the two classes agree modulo eight. In particular, five quartets are impossible and four split two to each class. If in addition the two-term sums of the available b^2 are distinct up to order, the two classes have the same multiset of quartet degrees.

At least twenty-four labels in each joint fiber. Every such E contains at least 24 labels, including at least 16 active at five. All degrees in it are at most 360, since the two odd heights are deficient: the ternary phase is nontrivial, and full five-height makes a five-block a singleton. When the dyadic phase is nontrivial too, the upper degree bound drops to 180.

Write A for the number of five-active labels in E, all of them jointly three- and five-active. These labels occur in local five-inertia orbits of size divisible by four. If a phase has five-primary order $5^a \geq 25$, its orbit contains at least $2 \cdot 5^{a-1}$ distinct phases by the square action, with at least two labels over every phase by the inertia involution. Then $A \geq 20$. When $A < 20$ all active twists are tame and each active orbit is one quartet: the local group action factors modulo five because $U(5)$ acts trivially, and its conjugates generate the kernel of reduction modulo five. At any successive depth the conjugates of an upper elementary matrix span the trace-zero quotient over $\mathbb{F}_5$, which proves the normal generation by induction. The parity theorem (Section 19.10) says that the number of local five-orbits of tame order four in E is even. If $A = 8$, its two active quartets therefore require an even number of inactive quartets. There is at least one, so at least two, forcing $A \geq 16$ by the local Weyl count.

If $A = 12$, the three active quartets cover both square classes since the masses at all four individual fifth phases are equal by Section 19.11. Each quartet contributes twice its squared degree to each of the two individual phases in its class. The two quartets on one side and the single quartet on the other would therefore have degrees satisfying $d_1^2 + d_2^2 = d_3^2$. When two and three are separated or when two and five are separated, these degrees all have full dyadic height by separation. This also holds in the triangle without a triple (Proposition 19.16). After removing their common dyadic part the equation is impossible modulo four. For a triangle containing a triple, we choose the starting J triple-active, so all active labels in these fibers have five-height exactly one: positivity of the five-height is the ternary degree transfer and the upper bound is activity (Section 21.3). After dividing by five, every degree is supported on two and three. In a primitive Pythagorean triple the odd hypotenuse must have a prime divisor equal to one modulo four, since each of its prime divisors admits a square root of -1 modulo that prime. Thus the required equality is again impossible. The same height-one argument works for a dyadically trivial fiber whenever the ternary degree transfer is available.

It follows that $A \geq 16$. If $A = 16$, its four tame active orbits force an even number of inactive quartic orbits by the same parity theorem. Hence there are at least eight inactive labels in this case; if $A \geq 20$ we use the four already proved. This gives $|E| \geq 24$ as asserted.

On the path missing six every degree in E is $40, 120$ or 360 (Section 23.7). In fact $|E| \geq 25$. If equality held in the bound 24 with twenty active labels, the four inactive labels would form a single orbit. The orbit filtration (Section 19.11) then gives only orders dividing five on E. With sixteen active labels the same tameness is automatic. The normalized squares $1, 9, 81$ have distinct two-term sums by their base-nine digits. The tame count above excludes twenty active labels; at sixteen the degree multisets in the two square classes agree. The eight inactive labels are two quartets. The fiber mass divided by 40^2 is divisible by 25 (Section 23.7). Removing its common factor four, its residue would be $s + 2t$ with $s, t \in \{2, 7, 10, 12, 15, 18\}$, the sums of two residues from $1, 9, 6$. None is the negative of twice another. This excludes size twenty-four.

Two nontrivial ternary phases therefore supply at least 50 labels of these three degrees, including at least 32 jointly active at three and five. The twist-one quartets of X, V supply another 16 purely five-active labels by the local estimate. In addition, a jointly two- and five-active phase gives at least four jointly two- and five-active labels: at quintic order five use equal mass at the four primitive phases with the dyadic twist fixed, and a higher order already has at least ten square phases. With a ternary phase of order at least nine at the joint J, the corresponding three–five counts rise to 75 and 48.

When all three pairs occur. In the absence of a triple the other three inactive quartets X, U, V have twist one, giving at least 24 pure five-active labels outside the two fibers. In the presence of a triple, choose J triple-active, so the fibers share a nontrivial dyadic phase. The three inactive quartets again give 24 active labels outside those fibers: all three have trivial dyadic phase, even when the pure ternary V is active at three. There are consequently at least 56 five-active labels in the triangle. If a joint label at the chosen dyadic phase has ternary order at least nine, the corresponding argument gives at least 72.

Five inactive orbits on the path separating two and five. In an ordinary minimal table of dimension $2^6 3^{2e} 5^4$, $e \geq 3$, suppose the ternary prime occurs jointly with each of two and five, which are separated. There are at least five distinct five-inactive local five-inertia orbits of size divisible by four. Consequently there are at least forty five-active labels in equal-degree quartets. At $e = 3$ one can choose all five inactive orbits with ternary defect.

Let H_5 be local five-inertia. Fix a tame character ψ of order four at five. For each five-inactive orbit $O_Y = H_5 Y$ whose size is divisible by four, put $\ell_Y = |O_Y| = 4m_Y$. Its stabilizer lies in $\ker \psi$

since the group of units at a power of five is cyclic. The input and output unit vectors are

$$v_Y = \frac{1}{\sqrt{\ell_Y}} \sum_{g \in H_5/\mathrm{Stab}(Y)} \overline{\psi(g)}\, e_{gY}, \qquad v_Y^- = \frac{1}{\sqrt{\ell_Y}} \sum_{g \in H_5/\mathrm{Stab}(Y)} \psi(g)\, e_{gY}.$$

We use the Hermitian inner product linear in its second variable. They form orthonormal bases of the five-inactive spaces L_ψ and $L_{\psi^{-1}}$ of the indicated torus types. Each space has one line per orbit. Every such orbit is dyadically inactive, since two-activity forces full five-height. The dyadic group acts scalarly: its diagonal upper generator is scalar on this invariant space and its conjugates generate the group. The ternary group acts on L_ψ. Section 21.10 gives three compulsory orbits U, X, V with $d_U = 40$, $d_X = 4 \cdot 3^j$, $d_V = 5 \cdot 3^k$, $1 \le j, k \le e-2$. Their fields contain $\mathbb{Q}(\zeta_5)$; U is neutral and tame at three. For $Y = X, V$ the full-weight normalized entries are $1 + 3\zeta$ and $\eta(2 + 3\zeta')$, respectively, with primitive fifth roots ζ, ζ' and a ternary scalar η. The local five-orbit fixes η. In the double sum for $\langle v_Y^-, sv_U \rangle$, the entry at (gY, tU) depends on the tame product gt mod 5. Each of its four values occurs $4m_Y m_U$ times with the same character weight. Division by $\sqrt{\ell_Y \ell_U}$ leaves the factor $\sqrt{m_Y m_U}$. The constant term has zero quartic Fourier coefficient and the remaining Gauss sum has squared modulus $3^2 \cdot 5$. Since $n = 200 \cdot 3^e$ and $\mathrm{lcm}(d_U, d_Y) = 40 \cdot 3^{v_3(d_Y)}$, this gives

$$5|\langle v_Y^-, sv_U \rangle|^2 = 5^a / 9^h, \qquad 5^a = m_U m_Y, \quad h = e - v_3(d_Y) - 1 \ge 1.$$

The local five-block at each of two distinct nontrivial ternary phases contains an inactive quartic orbit. For $e = 3$ this was proved above. For varying e, the avoided-two trace selection of Section 21.4 in the full principal three-block of a joint label gives an odd hit of degree $5 \cdot 3^l$ with five-height one by defect-zero vanishing. Its crossing with U puts $\mathbb{Q}(\zeta_5)$ in its field. The selected value has odd relative field index and odd absolute trace, so a faithful quartic Fourier coefficient is nonzero (Section 21.6). Applying the complementary Weyls gives the required inactive orbit at the specified phase.

At $e = 3$ first set aside the coordinate orbits of full ternary height in each mode space. Each label then has ternary-unit codegree, hence a singleton three-block and trivial ternary twist. Its coordinate line affords the trivial ternary representation since the upper unipotent acts trivially and its conjugates generate the local group. These lines contribute no transitions between different ternary phases. For $e > 3$ retain all orbits.

Suppose the retained space has dimension at most four. It contains the three compulsory orbits at distinct degrees d_U, d_X, d_V. A ternary phase of order at least nine has at least three square conjugates at the same degree, on different five-orbits, forcing at least five orbits in total. Thus the local upper unipotent has cube one. Its representation factors through $\mathrm{SL}_2(\mathbb{F}_3)$. Indeed, in each congruence quotient of depths $j, j+1$ with $j \ge 1$, the image of the representation kernel contains the conjugates of E_{12}, from the upper unipotent at depth j. These span $\mathfrak{sl}_2(\mathbb{F}_3)$: interchanging the coordinate axes gives $-E_{21}$, and conjugation by a lower unipotent gives a nonzero diagonal component. Descending induction kills the entire first congruence subgroup.

Write $z = -I$ in this quotient. As an unsigned torus permutation it preserves degree and permutes the five-orbit lines. If a compulsory orbit is fixed setwise, the action on its labels commutes with H_5 and therefore is translation by an element of $H_5/\mathrm{Stab}(Y)$. Because K_Y contains $\mathbb{Q}(\zeta_5)$ and z fixes the fifth roots, this translation is wild, hence of odd order. The involution fixes that orbit pointwise. At most one additional orbit is available. If z exchanges it with a compulsory orbit the exchanged pair has a single negative eigenline, and otherwise only the extra line can be negative. On the negative eigenspace z has determinant one. Indeed the Weyl matrix and $\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ anticommute over $\mathbb{F}_3$, so their commutator is z; the eigenspace is stable under both. Its negative multiplicity is therefore even, hence zero. This works for either quartic mode. Both retained representations descend to A_4, with upper generator u a 3-cycle and Weyl element w a nonidentity element of the Klein four-group.

Let E_0 project onto the five-inactive coordinates, let $B = E_0 W_5|_{L_\psi} : L_\psi \to L_{\psi^{-1}}$, and let Q project onto local level at most five. Both operators here are defined on the full mode spaces, including any trivial lines set aside at $e = 3$. The five-phase estimate of Section 21.10, polarized, gives $5B^*B = Q$ on L_ψ. Both B and Q intertwine the ternary action. The compression of s between the two mode spaces equals BW_3 up to the dyadic unit scalar. At least one of X, V is nonneutral since both primitive cubic phases occur and at most four orbit lines are available. There is thus a nonzero transition from U to a different u-phase. Character summands cannot give this transition: w acts trivially on them and B intertwines u. Lemma 6.10 therefore gives a unique tetrahedral summand T on each side. The projection Q acts identically on the input T, since it commutes with the group and B is nonzero on T. Thus $\sqrt{5}B$ identifies the two tetrahedral summands unitarily. Any coordinate lines set aside at $e = 3$ were trivial constituents and do not affect this identification.

A transition from a simple u-phase to a different simple phase has squared modulus $4/9$ for $\sqrt{5}BW_3$, by the lemma. On either compulsory transition the actual value $5^a/9^h$ is one modulo four, where all denominators are odd. If the neutral phase were simple, X, V would therefore have to be the two orbit lines at one doubled nonneutral phase. The sum of their two squared transitions would be $4/9$, the squared norm of the projection onto that phase from the neutral tetrahedral line. The sum is two modulo four. This forces the neutral phase to be doubled, with basis $v_U, v_{U'}$ in the input; the two nonneutral phases are simple and the fourth constituent is trivial. For a nonneutral $Y \in \{X, V\}$ we now have

$$\tfrac{4}{9}x = 5|\langle v_Y^-, sv_U\rangle|^2 = 5^a/9^h, \qquad x = \|P_T v_U\|^2.$$

Applying the last assertion of Lemma 6.10 in the input space gives

$$|\langle v_{U'}, W_3 v_U\rangle|^2 = \tfrac{16}{9}x(1-x) = \frac{5^a(4\cdot 9^{h-1} - 5^a)}{9^{2h-1}}.$$

The ternary torus fixes every label of O_U. Its wild part fixes U by tameness and commutes with H_5. Its involution preserves O_U because it acts identically on the retained mode space, and the translation argument above applies. The Galois formula of Lemma 6.1 sends W_3 to W_3 times a ternary torus element on the right. The columns of W_3 indexed by O_U are consequently rational. In particular, writing $G_Z = H_5/\mathrm{Stab}(Z)$, the two normalized input modes give

$$\sqrt{\ell_{U'}\ell_U}\,\langle v_{U'}, W_3 v_U\rangle = \sum_{g\in G_{U'},\ t\in G_U} \psi(g)\overline{\psi(t)}(W_3)_{gU',tU} \ \in \mathbb{Q}(i).$$

Its squared modulus is a Gaussian norm. So is $\ell_{U'}\ell_U = 16 m_{U'} m_U$, since $5 = |2+i|^2$. Hence the squared transition has even valuation at every prime congruent to three modulo four. But $4\cdot 9^{h-1} - 5^a$ is positive (it is nonnegative since $x \le 1$, and odd) and is three modulo four, so it has such a prime to odd valuation. Neither 5^a nor the square denominator changes this parity. This contradiction proves the five-orbit assertion. Proposition 23.5 gives the forty five-active labels.

24.3 Rank from compulsory simple degrees

For rank R, let m_d count simples of degree d. Suppose distinct positions have upper degree bounds $t_1, \dots, t_j$. For any cutoff b,

$$b^2 R + \sum_{d>b}(d^2 - b^2)m_d \ \ge\ D + \sum_{i=1}^{j}(b^2 - \min\{t_i, b\}^2). \tag{24.2}$$

Indeed the left side minus D counts $b^2 - \min\{d, b\}^2$ at every position. We use this with $b = 600$ or 675.

The path separating two and three. Here $m_{900} = 0$ by Corollary 24.2. Put $g = 600^2$, $C(d) = g - d^2$ and $k = m_{675}$. By the compulsory active orbits and the double-principal-block argument, the following labels are distinct: the vacuum, neutral X, V quartets with degrees at most $72, 45$, a neutral three-free $W \in P_2 \cap P_3$ of degree 10 or 20, and four further neutral members of $P_2 \cap P_5$. There is also a pure two-active member of the five-block of a joint two–five label, of degree at most 540, and at least four jointly two–five-active labels with the same upper bound. The two joint ternary fibers have different ternary twists. The fiber count in this separated case gives at least 25 labels of degrees $40, 120, 360$ in each.

The four neutral members of $P_2 \cap P_5$ can be taken in a single tame five-inertia orbit Q. Indeed, let H be the order-four tame subgroup and adjoin the unit both to HU, for the pure five-primary $U \in P_5$ of degree 20, and to HV for the neutral $V \in P_2$. Section 21.7 gives a minor of $W_5 V_2$ between their five-dimensional coordinate spaces. After multiplying row Y by n/d_Y, its reduction at two is the Fourier matrix $(\bar{\zeta}^{ij})_{i,j \in \mathbb{F}_5}$, up to a scalar unit, for a primitive fifth root ζ. The complex minor is invertible. Its factorization through V_2 and W_5 only uses intermediate labels of twist one in $P_2 \cap P_5$, by block and complementary phase preservation. Both coordinate projections commute with H; V_2 commutes with H and W_5 intertwines its action with inversion. A faithful quartic eigenline in the domain survives in the intermediate space, giving the asserted orbit. The block distinctions separate Q from X, V, U and W.

The three neutral quartets X, V, Q supply at least 24 pure five-active positions, selectable in equal-degree quartets by the local Weyl count. The same conclusion holds if wild inertia enlarges one of these orbits: take its trivial-wild, faithful-tame character. The central involution acts negatively on the relevant Weyl images at both tame and wild phases. If a degree-20 row in P_5 is neutral, it gives a fourth neutral quartet, and we get 32 pure five-active positions. In the other case at least eight pure five-active labels have degree 20, in full quartets (Section 23.7); choose them among the 24. Since the joint ternary fibers supply 32 five-active labels and the two–five edge supplies four others, there are at least 60 five-active labels in either case.

For $Y \in Q$ let $i = v_2(d_Y)$. The inertia-orbit estimate of Lemma 6.15, with $z = 675$ and $p = 5$, gives $k4^i < 16$: every degree-675 row hits Y by two-principality. Hence the common degree of Q is bounded by $v = 540, 270, 135$ for $k = 0$, $1 \le k \le 3$, $4 \le k \le 7$, respectively. In the last case it is odd, of degree $15, 45$ or 135.

To count the two other five-free labels required by the dimension identity, put $x = d_X$. For $x = 12, 24, 36, 72$ the residues $1 + 4x^2$ modulo 25 lie in $\{2, 5, 10, 12\}$, none zero or the negative of a unit square. A five-free position outside Q and the X quartet contributes at least $540^2 - 216^2 = 244944$ beyond its previous allowance, whether filling a 540 slot or newly reserved. The joint ternary fibers, V, W and all degree-20 labels are five-bearing. If Q itself is five-free, its four degrees are at most 108 at $k = 0$ and at most 54 at $k > 0$. Accordingly the additional five-free contribution is at least $2 \cdot 244944$, $4(270^2 - 54^2)$, $2 \cdot 244944$ in the three ranges of k. In the middle range this is the cheaper of the two possibilities for Q.

For the branch with a neutral degree-20 quartet, the resulting lower bound on $gR + 95625k$ is the sum of D, these five-free contributions, and

$$C(1) + 4C(72) + 4C(45) + C(20) + 4C(v) + 5C(540) + 50C(360) + 4C(20) + 32C(540).$$

In the other branch replace the last two terms by $8C(20) + 16C(540)$, increasing the bound by 344000. For the first branch the baseline and its remaining allowance at $R = 140$ are

k	baseline	$140g + 95625k -$ baseline
0	48983451	1416549
1, 2, 3	49648299	847326, 942951, 1038576
4, 5, 6, 7	50076951	705549, 801174, 896799, 992424.

In any such fiber, if no inactive label has the minimum ternary height, at least 36 labels attain it. Indeed $\sum_E d_Y^2 = \sum_{Y \text{ inactive}} d_Y^2 k_{\{2,3\},Y}^2$. Divide by the square of the smallest degree in E

and reduce modulo nine: the number attaining it is a multiple of nine, and it is also a positive multiple of four since all such labels occur in equal-degree quartets. The two fibers E may have more than fifty labels or degrees below 360. Let $t = \sum_E (|E| - 25)$, and let a, b count their labels of degree $40, 120$ respectively. Since each fiber mass is divisible by $25 \cdot 40^2$ (Section 23.7), their extra contribution obeys

$$230400t + 128000a + 115200b, \qquad 6t - 5a + 3b \equiv 0 \pmod{25}.$$

Outside the fibers the previously reserved three-free positions are the unit, W of degree $w = 10, 20$, and four, respectively eight, positions of degree 20. Denote the number of further three-free positions outside by c. Each contributes at least 290000: a pure five-active three-free degree is at most 40 instead of its cap 540; an unreserved three-free degree is at most 200. No dyadically active label here is three-free, and none of the neutral nonunits in $P_2 \cap P_5$ is three-free (see the four further principal labels on the path missing six). None of these three-free savings coincide with a five-free one, by prime-power exclusion. In the first branch $a + c$ is a positive multiple of three: divisibility follows modulo three; if zero, modulo nine gives $1 + 4 \cdot 20^2 + w^2 = 0 \bmod 9$, hence $w = 10$. The unique degree-ten label would be rational, contrary to Section 19.12. In the second branch $a + c \equiv 2 \bmod 3$.

The extra cost is at least 870000, including the 344000 branch correction where applicable. If $a = 0$ use $c \geq 3, 2$, respectively. If $a > 0$, the only fiber cost below 870000 has $(t, a, b) = (0, 5, 0)$ and costs 640000. Indeed $t \leq 3$, $b \equiv -2t \bmod 5$ and $5a \equiv 6t + 3b \bmod 25$. At $t = 0$ test $b = 0, 5$; $b = 5$ forces $a \geq 3$. At $t = 1, 2$ the least b is $3, 1$, respectively, forcing $a \geq 3$ at these minima; at $t = 3$, already $b \geq 4$. In the surviving case the first branch needs $c \geq 1$; the second incurs the branch correction. This excludes $k = 1, 4, 5$.

The odd-degree count is divisible by four. At $k = 0$ three more odd labels are needed beyond the unit and V quartet. Each saves at least $540^2 - 135^2 = 273375$ in a capped slot or $C(225)$ outside. An odd degree distinct from 675 properly divides 675 and is at most 225; it is three- and five-bearing. At $k = 2$ at least one more odd is required. If Q is odd, its improved cap and the external five-free allowance already add $4(270^2 - 135^2) + (489888 - 279936) > 273375$; otherwise use the same odd saving outside Q. At $k = 6$ the unit, V, Q and the six degree-675 labels give fifteen odd labels, so one more is needed. These savings exclude $k = 0, 2, 6$. At $k = 3$, five-bearing Q increases the five-free allowance by $489888 - 279936$. Five-free Q has degree $d = 18$ or 54 and $1 + 4x^2 + 4d^2 \neq 0 \bmod 25$ for the four values of x, costing another 244944 outside the two quartets. Both exceed the allowance.

Finally $k = 7$ forces $w = 10$ by Proposition 24.8. Five-ramification supplies another neutral degree-ten conjugate outside the reserved positions. Its cost 359900 leaves less than 640000, so $a = 0$. We have $c \leq 3$; as the selected pure five-active positions are grouped in full quartets, none of these c positions lies there. Each besides the new degree-ten label therefore costs at least $C(200) = 320000$. This gives at least $359900 + 2 \cdot 320000$ in the first branch or $344000 + 359900 + 320000$ in the second. Both exceed 992424. This establishes $R \geq 141$.

All three pairs occur. Select 85 distinct positions: the vacuum, the three inactive quartets X, U, V of degrees at most $72, 40, 45$, twenty-four five-active positions outside the joint fibers capped at 540, and forty-eight labels in the two joint fibers. The complementary twist conditions separates these fibers from the other five-active positions. These last degrees are at most $u = 360$ if there is no triple, or $u = 180$ if a triple is used to define the fibers. At least two additional five-free labels besides the X quartet are required by the same residue argument. All degrees in the two chosen fibers are five-bearing, by the ternary degree transfer, and so are U, V. Each additional five-free label gains $540^2 - 216^2$ even if already counted among the twenty-four five-active labels. There is also a further three-free label: the vacuum and the U quartet give five, whereas their total number is a multiple of three. It differs from every five-free nonunit. If in a joint fiber, its degree divides 40 by the five-defect; the generic five-active positions outside have

three-bearing degrees by the degree transfer, and an unreserved three-free degree divides 200. This adds at least $u^2 - 40^2$. With a baseline G equal to 900^2 or 600^2, put

$$B(G,u) = D+(G-1)+4\sum_{t\in\{72,40,45\}}(G-t^2)+24(G-540^2)+48(G-u^2)+2(540^2-216^2)+(u^2-40^2).$$

Thus $900^2R \geq B(900^2,u)$, giving $R \geq 106,112$, respectively. In the absence of degree 900, (24.2) gives $gR + 95625k \geq B(g,u)$, whose right hand sides are $47123451, 51691851$. With $k \leq 7$ this proves $R \geq 130,142$, respectively.

The path separating two and five. At $n = 5400$, a five-inertia quartet of degree $12, 15, 20, 36$ or 45 meets at most eight degree-900 labels. A quartet of degree $24, 40$ or 72 meets at most two. Thus the three mandatory quartets of degrees $12, 15, 40$ on the path missing ten collectively meet at most eighteen such labels. Every other degree-900 row vanishes on all twelve columns.

There is a stronger bound on that path. Let $4a$ count all degree-12 labels in P_3. We have $1 \leq a \leq 8$: in the weighted sum of Section 21.10 a degree-12 orbit costs at least one quarter of its size, and the opposite degree-15 type costs at least one of the available nine units. At any jointly two- and three-active J, orthogonality on P_3 gives

$$1 + 12\sum_{d_X=12,\ X\in P_3}\chi_J(X) \equiv 0 \pmod{\mathfrak{p}}, \qquad \mathfrak{p} \mid 5.$$

Indeed a five-free nonunit of P_3 has degree 2^i3^j with $i = 2, 3$ (Lemma 4.10); full dyadic height misses J, and the other type has degree 12 by Section 21.10. The degree of J has full five-height by separation, so its row agrees on the four tame five-conjugates in each quartet. Write b_J for one quarter of the sum in this congruence. It is a rational integer: the full set of X is Galois-stable, and the sum is four times an algebraic integer. Thus $b_J \equiv -2 \pmod 5$. Applying s to the normalized indicator of the $4a$ labels gives

$$\sum_{J \text{ jointly two-, three-active}} d_J^2 b_J^2 < aD/4.$$

Hence the joint sector has squared mass strictly below $aD/16$. In particular fewer than $9a/4$ degree-900 simples can be joint, so there are at most 17 in any rank. At $a = 1$ there are at most two, and their ternary twist has order three by square Galois covariance.

Let $G = 675^2$, $g = 900^2$, and $m = \#\{Z : d_Z = 900\}$. The two joint ternary fibers from Section 24.2 contain at least 48 labels, all of degrees at most 360. The five-orbit argument gives five five-inactive quartets, including X, V, U of degrees $12, 15, 40$. A neutral quartet here means one with trivial ternary phase; all five have trivial dyadic phase. In particular U is neutral.

If at least two of the five quartets are neutral, their two orbits supply at least sixteen pure five-active labels in addition to the eight neutral labels and the 48 members of the joint fibers. The compulsory quartets have ternary defect by the five-orbit argument, so all these 72 degrees are at most 360. Reserving the vacuum, and the much smaller degrees of X, V, U whether inside this count or outside, gives squared mass at most

$$M = 1 + 4(12^2 + 15^2 + 40^2) + 60 \cdot 360^2$$

on 73 compulsory positions (if necessary replace unspecified positions by X, V, U). A further label of degree 75 lies outside: it has full five-height, so it is neither five-active, nor in a ramified quartet, nor in a five-block containing an active simple. Thus (24.2) with $b = 675$ gives $GR + (g - G)m \geq D + 74G - M - 75^2$. In particular $R \geq 101$.

It remains to treat the case where just U among the five is neutral. Select 40 five-active labels in ten equal-degree quartets: sixteen in each of the two joint fibers and eight in the pure five-fiber of U. All have degree $24, 72, 120, 216$ or 360: five-activity gives full dyadic height and

positive ternary height (Section 21.4). Together with the unit and the five inactive quartets this gives 61 labels. Write b, c for the degrees of the other two quartets besides X, V, U; they divide 360. The following five further labels are distinct and outside those 61:

$$45, \qquad 75, \qquad w \in \{10, 20, 50, 100\}, \qquad y \in \{12, 18, 36\}, \qquad o \in \{15, 45, 75, 135, 225, 675\}.$$

Indeed use the twist-one 45 and w from the double-block argument, and at a joint dyadic–ternary column the five-free twist-one witness $y \in P_2 \cap P_5$ from the same argument. All are distinct from the neutral 40 and the four nonneutral quartets. The witness of degree $75 \in P_3$ is five-inactive with full five-height, hence cannot lie in a five-ramified quartet. Finally the total odd count is divisible by four, and in the 61 selected labels the odd count is one modulo four. In addition to the $45, 75$ just retained a further odd o is therefore required. The multiplicity of degree 18 is even, as is that of 54: a degree-40 row in P_5 gives at each such label a sum of two distinct five-power phases after lcm normalization (Theorem 4.9). Its squared modulus puts $\mathbb{Q}(\sqrt{5})$ in the character field.

The resulting 66 labels have total squared mass at most

$$M_0 = 1 + 4(12^2 + 15^2 + 40^2 + 2 \cdot 360^2) + 40 \cdot 360^2 + 45^2 + 75^2 + 100^2 + 36^2 + 675^2 = 6703248.$$

In particular $GR + (g - G)m \geq D + 66G - M_0 = 52528002$.

At $R \leq 94$ the excess above D is at most 223248. Thus every remaining degree must be 900, $b = c = 360$, and $o = 675$: decreasing one of these upper bounds costs at least 354375, $4(360^2 - 180^2)$ or $675^2 - 225^2$, respectively. The value $y = 18$ is impossible by its even multiplicity. Modulo nine, the only three-free labels give $1 + 4 \cdot 40^2 + w^2 \equiv 0$, so $w = 50$. Modulo sixteen, the seven fixed odd squares give 7, the extra odd square gives 9, and w^2 gives 4, with all other contributions zero. This rules out $R \leq 94$.

With $b = 600$, the same 74 or 66 positions in (24.2) give respectively $D + 74b^2 - M - 75^2$ and $D + 66b^2 - M_0 + (675^2 - b^2)$: in the latter the cap on the reserved odd label becomes 600. Thus

$$360000R + 450000m + 95625k \;\geq\; 48010498 \quad \text{or} \quad 360000R + 450000m + 95625k \;\geq\; 46312377.$$

Here $k = m_{675} \leq 7$. For $m = 0$ they give $R \geq 132$ and $R \geq 127$ respectively. If $R \leq 105$ there are at least 18 degree-900 labels; if $R = 101$ there are at least 21. Corollaries 24.3 and 24.7, and Theorem 24.5 give the orbit, twist and incidence assertions of Theorem 24.1.

24.4 The exact rank calculation

When two and five are separated, only the one-neutral case at ranks 95 through 100 remains. The integer recursions below enumerate the candidates. For each of the five survivors at rank 100, the certificates give both disjoint fibers and their norm witnesses. A separate calculation checks each pair. The argument on tame phases excludes all five.

Let $A = \{24, 72, 120, 216, 360\}$, $J = A \setminus \{216\}$, $g = 900^2$, and let $\mathcal{D}$ be the divisors of 5400 between 10 and 675 with at least two prime factors. The $R - 66$ additional degrees are 900 apart from a multiset L in $\mathcal{D}$. The ten selected five-active quartets contribute $2304S$, where S is a sum of ten squares from $\{1, 9, 25, 81, 225\}$. For the initial enumeration take $b \leq c$ from $\mathcal{D}$ dividing 360, $o \in \{15, 45, 75, 135, 225, 675\}$, $w \in \{10, 20, 50, 100\}$, and $y \in \{12, 18, 36\}$. Retain precisely the tuples satisfying

$$\begin{aligned} M_0 + (R - 66)g - D = {} & 4(2 \cdot 360^2 - b^2 - c^2) + (675^2 - o^2) \\ & + (100^2 - w^2) + (36^2 - y^2) + 2304(2250 - S) + \sum_{d \in L} (g - d^2). \end{aligned}$$

Here $|L| \leq R - 66$ and the multiplicities of $18, 54$ in $L \cup \{y\}$ must be even. An explicit enumeration subtracts $g - d^2$ from the remaining nonnegative budget, adding $d \in \mathcal{D}$ in decreasing order with

repetitions. Include the empty multiset. The least decrement is 354375. For the admissible S, start with $\{0\}$ and take ten successive sumsets with $\{1, 9, 25, 81, 225\}$. A tuple records S only once regardless of its number of representations.

R	95	96	97	98	99	100
initial tuples	0	1	43	384	3659	30100
$\max \#\{d \in L : d \in A\}$	–	0	1	2	3	5
after the two exact fiber norms	0	0	0	0	0	5

The maximum counts occurrences with multiplicity. Thus there are at most 45 five-active labels overall, with eight already reserved at U. Three distinct joint fibers would require at least 48 joint labels. All joint labels consequently lie in the two selected fibers, and their ternary phases have order three, since order nine or higher would generate a third fiber. Each fiber has 16 or 20 five-active labels, since its joint labels occur in quartets. The tame count in Section 24.2 excludes a fiber with twenty active labels, since every possible active degree divided by 24 is odd. A wild phase requires twenty label positions and, at that very fifth phase, the other nontrivial cubic phase also has positive mass. This would use at least $20 + 20 + 8 > 45$ five-active labels. Thus both fibers contain sixteen tame active labels.

The normalized squares $1, 9, 25, 225$ available in a joint fiber (activity excludes 216) have distinct two-term sums. Thus its quartet degrees are (u, u, v, v) by the same tame count. For a fixed fifth phase the two primitive cubic phases also have equal squared mass, so the very same u, v occur in both fibers. The ten selected quartet degrees altogether consist of four u, four v, and a pair $h, i \in A$ at U. We initially allow this last pair to be unequal.

Here is the exact second computation underlying the last line of the table. Take every $u \le v$ in J and $h \le i$ in A with

$$576S = 4u^2 + 4v^2 + h^2 + i^2.$$

Form a list of possible five-inactive positions in the two fibers. Include the four quartets $12, 15, b, c$ as groups of multiplicity four. From $L \cup \{o\}$ keep the degrees dividing 360. If $y = 18$ first remove its twist-one conjugate of degree 18 from L. Group the remaining 18's and 54's in pairs, and use multiplicity one for the others. This pairing is allowed because each five-inertia orbit at these two degrees has even size and the complementary phases and full five-block are invariant. The already reserved quartets and conjugate pair are themselves stable under tame five-inertia.

Take disjoint sublists for the two fibers. For each sublist I of pairs (m, d) require $\sum_I m \ge 8$, and put

$$F = 8(u^2 + v^2) + \sum_I m d^2, \qquad F = 25 \sum_I m d^2 x_{m,d}^2, \quad x_{m,d} \in \mathbb{Z}, \tag{24.3}$$

with a separate variable for each group, including repeated pairs. This identity follows by commuting the fiber projector through W_5 in the squared norm of the vacuum column: only five-inactive positions contribute on the $W_{\{2,3\}} e_0$ side. Theorem 6.11 gives the factor 25 because every degree in the fiber has positive five-defect. The normalized coordinate is constant on each selected five-inertia group. To test the equality, start with the single remainder $F/25$, discarding nonintegers. At each (m, d) replace each remainder t by the set of $t - md^2x^2$ for integers $0 \le x \le \sqrt{t/(md^2)}$. Keep the sublist if zero occurs after its last group. Two successful sublists must be disjoint, and unused positions are permitted. These recursions give the following complete output, all at $R = 100$, $w = 50$, and $u = v = 360$.

(b, c)	o	y	L	S	(h, i)
$(15, 18)$	135	12	$360, 72, 12, 12$	2034	$72, 360$
$(15, 36)$	135	12	$360, 36, 12, 12$	2034	$72, 360$
$(18, 180)$	135	12	$72, 30, 12, 12$	2034	$72, 360$
$(24, 360)$	15	36	$600, 600, 270, 24, 12$	1882	$24, 216$
$(36, 180)$	135	12	$36, 30, 12, 12$	2034	$72, 360$

Eliminating the five remaining profiles. In each row at most two entries of L could be five-active. Hence the pure five-fiber at U contains at most ten active labels. The two selected quartets cannot come from a wild orbit, which would need twenty labels if it contained the faithful quartic mode. An additional wild orbit would itself require at least ten more. By the tame-phase count the four phase counts agree modulo eight. All are positive with sum at most ten, hence equal. There are just eight active labels, two quartets in opposite square classes, of equal degree. This contradicts every displayed pair (h, i) and proves the bound.

Proof of Theorem 24.1. The three row-norm estimates on p. 189 give the upper bound. The character-field assertions follow from Theorem 24.5, Proposition 24.9 and its separated-case refinement; Proposition 24.8 bounds m_{675}.

For the lower bounds, the two disjoint ternary fibers each contain at least 24 labels. If 10 is missing, the five inactive orbits give 40 five-active labels. The degree budget and congruences in the second row's proof give $R \geq 95$ and $R \geq 127$ if $m_{900} = 0$. In Section 24.4, the rank enumeration tests the degree multisets and the two disjoint fiber norms (24.3). It leaves five profiles, all of rank 100. Their incompatible tame phases exclude all five. At $R = 101$ the same budget gives $m_{900} \geq 21$; Theorem 24.5 bounds each orbit by six, Corollary 24.7 bounds the multiplicity at one twist, and Corollary 24.3 allows at most three rows missing the joint sector. □

25 The unbraided boundary in dimension 5400

Let $\mathcal{A}$ be a non-weakly-group-theoretical fusion category of dimension $n = 5400$. Its center is a counterexample in dimension n^2. The minimal reduction at that dimension therefore applies to the center itself. In particular it is integral and fusion-simple, its only invertible simple is the unit, and its nonunit degrees belong to

$$\mathcal{D} = \{a : 10 \leq a \leq 900,\ a \mid n,\ a \text{ has at least two prime divisors}\}.$$

Dominance of the forgetful functor $F : \mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ implies that $\mathcal{A}$ is integral as well: a positive sum of square roots of integers is integral only when every summand is integral, by independence of distinct squarefree radicals.

Theorem 25.1. *Every proper fusion subcategory of $\mathcal{A}$ has dimension at most* 216. *Every formal codegree of $K_0(\mathcal{A})$ other than that of the dimension character lies between* 6 *and* 540. *The rank satisfies the following bounds.*

	$K_0(\mathcal{A})$ *commutative*	$K_0(\mathcal{A})$ *noncommutative*
arbitrary $\mathcal{A}$	$10 \leq \operatorname{rank} \mathcal{A} \leq 1512$	$12 \leq \operatorname{rank} \mathcal{A} \leq 1512$
$\mathcal{A}$ *simple and perfect*	$11 \leq \operatorname{rank} \mathcal{A} \leq 462$	$13 \leq \operatorname{rank} \mathcal{A} \leq 490$

25.1 The canonical induction algebra and character moments

Write I for the right adjoint of F. For each irreducible representation E of $K_0(\mathcal{A})_{\mathbb{C}}$, let $m_E = \dim E$ and f_E be its formal codegree. Ostrik's induction theorem identifies E with a distinct simple summand of $I(\mathbf{1})$, of degree $a_E = n/f_E$ and multiplicity m_E [48, Theorem 2.13]. In particular

$$\sum_E m_E a_E = n, \qquad \sum_E m_E^2 = \operatorname{rank} \mathcal{A}, \qquad n\tau = \sum_E a_E \chi_E, \tag{25.1}$$

where τ takes the coefficient of the unit. The dimension character has $(m_E, a_E) = (1, 1)$, and all other a_E belong to $\mathcal{D}$. This proves the codegree assertion.

Indices and the length of induction. A subcategory of index e corresponds to a connected étale subalgebra of $I(\mathbf{1})$ of degree e [19, Theorem 4.10]. Hence $e \mid n$ and $e-1$ is a sum from $\mathcal{D}$. Below 24 the only odd such sum is 15, whereas $16 \nmid n$. For the odd divisors $1 < e < 25$, namely $3, 5, 9, 15$, none of $e-1$ is such a sum. Thus $e \geq 25$ for a proper subcategory. In particular the pointed subcategory has dimension $g \leq 216$, since a pointed category is weakly group-theoretical. All other degrees are at least two, so $\operatorname{rank} \mathcal{A} \leq g + (5400 - g)/4 \leq 1512$.

We next show that $L = \sum_E m_E \geq 10$. In the sum of the nonunit degrees of $I(\mathbf{1})$, equal to 5399, three different terms must omit two, three and five, respectively. Their upper bounds are $675, 200, 216$. Seven terms can contribute at most 4691. For eight terms the upper sum is 5591, leaving a deficit of 192. The other five terms must therefore equal 900 and the odd term 675. The two remaining terms sum to 224. The three-free term divides 200 and is 2 modulo three, hence is $20, 50$ or 200; only 200 leaves a divisor of 216. The three degrees $24, 200, 675$ occur once, giving three integer-valued linear characters h, u, v: each is unique at its codegree and thus Galois invariant. The sum of the remaining characters other than FPdim is also integer-valued, by the same Galois invariance and integrality of character values. For a simple X write $d = d_X$ and use h, u, v also for their values there. Applying (25.1) to X and XX^* gives, modulo 900,

$$d + 24h + 200u + 675v \equiv 0, \qquad d^2 + 24h^2 + 200u^2 + 675v^2 \equiv 0.$$

Modulo 25 the first gives $h \equiv d$. Modulo nine, eliminating u gives $3(d-h)^2 \equiv 0$, so $75 \mid d - h$. As $|h| \leq d$, every simple of degree at most 37 satisfies $h(X) = d_X$. The labels with this equality span a proper fusion subcategory: duality preserves it and equality in the triangle inequality forces it on every constituent of a product. Its inverse image under F is proper by dominance and contains the central simple of degree 24, contradicting fusion-simplicity of the center. This proves $L \geq 10$. By (25.1), rank is at least L, or $L+2$ when some $m_E \geq 2$. If the center has no degree-900 simple, even nine nonunit terms sum to at most $7 \cdot 675 + 200 + 216 = 5141$. In this case $L \geq 11$ already without a simplicity assumption. This applies in particular when the center has no twist order divisible by six.

How many degree-three simples can occur. Suppose now that $\mathcal{A}$ is simple and perfect and let n_j count its degree-j simples. A degree-two simple would produce a fusion subcategory of dimension 60 by the categorical Nichols–Richmond theorem [23, Theorem 3.4]; the other alternatives there require invertibles. Hence $n_2 = 0$. The following inequality holds in every simple perfect integral fusion category with $n_2 = 0$ and $n_3 \geq 2$:

$$n_3 \leq 2K, \qquad K = \min(4n_4, 5n_5) + 6n_6 + 9n_9. \tag{25.2}$$

Indeed put $P_X = XX^* - \mathbf{1}$ for $d_X = 3$. Its degree partition is $(8), (3,5)$ or $(4,4)$. For $X \neq Y$ of degree three the product X^*Y is either a sum of degree-three simples or has partition $(4,5), (3,6)$ or (9). In the first case its squared norm in the simple basis is at least three; in every other case it is at most two. Frobenius reciprocity gives $\|X^*Y\|^2 = 1 + \langle P_X, P_Y \rangle$. Call two partitions compatible if this last inner product is at least two, using actual simple labels. At most K partners $Y \neq X$ are incompatible with a fixed X, since XB contains at most d_B degree-three simples.

Suppose $n_3 > 2K$. A singleton (8) cannot occur since every other label is incompatible with it. Every remaining partition is compatible with itself, and any two occurring partitions must have a common compatible partner. A $(3,5)$ partition is compatible only with itself; if one occurs, all P_X equal $U + V$ with $d_U = 3, d_V = 5$. Here $U = U^*$ and $U^2 = \mathbf{1} + U + V$. Left multiplication by U preserves the nonempty set of degree-three labels other than U: for $Y \neq U$, the product UY has only degree-three constituents and $[UY : U] = [U^2 : Y] = 0$. This is impossible, since U generates the category and Y^*Y contains the unit.

For $(4,4)$ the only compatible pairs of different types are aa, ab. Two distinct repeated pairs, or two distinct pairs with unequal entries, are incompatible. A repeated pair aa and a pair bc

with $a \notin \{b, c\}$ have no common compatible partner. Thus there is a pair compatible with all, and unique among occurring pairs of its kind (repeated or distinct): two repeated pairs aa, bb force the distinct type to be ab, and two different distinct pairs force a unique repeated type. If only one kind occurs, all pairs coincide. Choose X with this pair. Since $\|XX^*\|^2 = \|X^*X\|^2$, the type at X^* is the same and $P_{X^*} = P_X$. For $Y \neq X$ of degree three, X^*Y has only degree-three constituents and omits X^* since $[XX^* : Y] = 0$. Multiplying these constituents by X again gives only degree-three labels; it gives no X since $[XX^*Y : X] = \langle X^*Y, X^*X\rangle = 0$. Thus the nonunit constituents of XX^* and their duals preserve the degree-three labels other than X, contradicting generation as before. This proves (25.2).

Upper ranks. Set $H = \sum_{j\geq 4} j^2 n_j$ and $E = \sum_{j\geq 4}(j^2 - 9)n_j$. If $n_3 \leq 1$ the degree-square sum gives rank at most 338. Otherwise

$$5399 = 9n_3 + H \leq 18K + H \leq 60E/11.$$

The last inequality holds coefficientwise except at $4, 5$ where it is $198 \min(4n_4, 5n_5) \leq 244n_4 + 685n_5$. Since $E = 5399 - 9(\operatorname{rank}\mathcal{A} - 1) \equiv 8 \pmod 9$, this yields $E \geq 998$ and rank at most 490.

If the fusion ring is commutative, reduce the last identity of (25.1) at a prime above two. Distinct algebra characters over the residue field are linearly independent, so some $\chi \neq \mathrm{FPdim}$ with a_χ odd reduces to FPdim. In particular $a_\chi \geq 15$, $f_\chi \leq 360$. At each odd-degree label the character is nonzero. Its squared norm, and that of each Galois conjugate, is f_χ; a nonzero algebraic integer has mean squared conjugate modulus at least one. Thus $O = \sum_{j\geq 3,\ j \text{ odd}} n_j \leq 359$. From (25.2) we have $n_3 \leq 10n_5 + 12n_6 + 18n_9$. If $n_3 \geq 2$, comparison of the coefficients at $3, 4, 5, 6, 9$ gives

$$\operatorname{rank}\mathcal{A} - 1 \leq \frac{5399}{16} + \frac{61O}{176} + \frac{n_3 - 10n_5 - 12n_6 - 18n_9}{11} \leq \frac{5399}{16} + \frac{61 \cdot 359}{176} < 462.$$

The two upper rank bounds are attained by degree multisets satisfying these numerical inequalities. In the general and commutative cases, respectively, they are

$$5400 = 1 + 401 \cdot 3^2 + 48 \cdot 4^2 + 38 \cdot 5^2 + 2 \cdot 6^2,$$
$$5400 = 1 + 328 \cdot 3^2 + 100 \cdot 4^2 + 31 \cdot 5^2 + 2 \cdot 6^2.$$

Their ranks are 490 and 462, and the second has $328 + 31 = 359$ odd-degree nonunits. These are arithmetic profiles. Any improvement of the upper bounds must use information beyond the inequalities they satisfy.

The two endpoint ranks. For rank ten, or for noncommutative rank twelve, the lower bound $L \geq 10$ and (25.1) force $L = 10$, respectively with all $m_E = 1$ or with exactly one $m_E = 2$. In fact for commutative rank ten the assertion is immediate; in the other case $\sum m_E(m_E - 1) \geq 2$ and equality is necessary. There are nine nonunit summands counted with multiplicity. Group the non-dimension representations by their common $a = a_E$, and put $k_a = \sum m_E$, $b_a = \sum m_E^2$, $f_a = n/a$ within each group. For a nonunit simple X put $d = d_X$, $s_a = \sum \chi_E(X)$ and $t_a = \sum \chi_E(XX^*)$. These are integers by Galois invariance; t_a is a sum of squared Hilbert–Schmidt norms in unitary realizations of the based-ring representations. The necessary row conditions are

$$\sum as_a = -d, \quad \sum at_a = n - d^2, \quad t_a \geq 0, \quad s_a^2 \leq k_a t_a, \quad |s_a| < k_a d. \tag{25.3}$$

For strictness, $\|\rho_E(X)\| \leq d$ by the spectral radius of XX^*. Equality in the trace bound would force $\rho_E(X)$ to act as a scalar of modulus d for each E in the block. Then every constituent of XX^* acts by its positive dimension, by equality in the trace bound for XX^*. A nonunit constituent and its dual generate the category, forcing E to be the dimension character. When $k_a = 1$, also $t_a = s_a^2 < f_a$ by character orthogonality and the vacuum contribution.

Including the unit row $(1, (k_a, k_a)_a)$, character orthogonality gives

$$\sum_X d^2 = n, \quad \sum_X d s_a = 0, \quad \sum_X t_a = b_a f_a. \tag{25.4}$$

Indeed the dimension vector is orthogonal to every non-dimension character, and $\sum_X XX^*$ acts on E by $m_E f_E$. Dual labels have the same row. A self-dual label has $s_a^2 - t_a \in 2\mathbb{Z}$, by taking the second elementary symmetric function of the eigenvalues in the block. Thus a row violating any such parity occurs an even number of times.

For reproducibility, enumerate nonincreasing lists of nine terms from $\mathcal{D}$ summing to 5399; there are 26. In the noncommutative case choose the block containing the two-dimensional representation, which gives 36 choices of a list and a block. For each, generate the rows (25.3) for $3 \le d \le 73$, and subtract the unit row's contribution from the target in (25.4). For a candidate row let v be its vector of moments in that equation and put $g = 2$ if the parity fails, $g = 1$ otherwise. Starting with the unit monomial, multiply the truncated geometric series $(1 - z^g \boldsymbol{x}^{gv})^{-1}$ over the row types. The coefficient at $z^{r-1}\boldsymbol{x}^{\text{target}}$ is zero in all 26 and 36 instances. Equivalently, recursively subtract g slots and gv in nondecreasing row order until the slots are zero. This integer calculation gives the two lower bounds in Theorem 25.1.

26 Decomposition over prime divisors

A modular category is *prime* if every braided Deligne decomposition has a trivial factor. For *tensor-prime*, allow all tensor decompositions. Write $\pi(D)$ for the prime divisors of D.

26.1 Splitting the group and its action

For an integral WGT modular category, a maximal Tannakian subcategory $\mathrm{Rep}(G)$ gives a faithful braided G-crossed extension of a pointed metric core $\mathcal{C}(A, \eta)$ whose equivariantization is the category [26, Proposition 4.56(i)–(ii), Definition 5.8 and Corollary 5.19], [42, Theorem 1.1]. Conversely a pointed core implies WGT by closure under extensions and equivariantizations. Thus

$$D = |G|^2 |A|, \qquad A = \perp_{\ell \| A|} A_\ell, \qquad \rho_\ell : G \longrightarrow O(A_\ell, \eta_\ell).$$

There is no nonzero G-invariant isotropic subgroup in A: its equivariantization would enlarge $\mathrm{Rep}(G)$ inside its centralizer [26, Proposition 5.18]. The action may have kernel; a twisted double can have trivial core.

For a braided G-crossed extension of a pointed category with metric group (A, η), Etingof and Galindo [28, Lemma 4.2] show that each simple object in the g-component has Frobenius–Perron dimension $\sqrt{|(1 - \rho(g))A|}$. Their proof uses the stabilizer description of Davydov and Nikshych [20, Corollary 5.6 and Remark 5.7]. Equivariantization then gives the degrees and codegrees below.

Proposition 26.1 (Degrees and codegrees of a gauging)**.** *Let Y be simple in the g-component of a braided G-crossed extension of $\mathcal{C}(A, \eta)$. Write $H = G_Y$ for its inertia subgroup. The simple objects above the G-orbit of Y are indexed by the irreducible projective H-representations π of its equivariance cocycle. Their degrees and codegrees are*

$$d_{Y,\pi} = [G : H] \dim(\pi) \sqrt{|(1 - \rho(g))A|}, \qquad f_{Y,\pi} = |A^g| \left(\frac{|H|}{\dim \pi} \right)^2 .$$

In particular the gauging is integral exactly when $|(1 - \rho(g))A|$ is a square for every g.

Proof. The A-orbits on the simples of $\mathcal{F}_g$ are module summands; indecomposability gives one orbit, say A/B. Its $|A/B|$ simples have equal degree, whose square is $|B|$ because the component

has dimension $|A|$. To identify B, let σ be the polar form on A and β the stabilizer bicharacter. For all $a \in A$, $b \in B$,

$$(\rho(g) - 1)a \in B, \qquad \beta(b, (\rho(g) - 1)a) = \sigma(b, a)^{-1}$$

by [20, Corollary 5.6 and Remark 5.7]. The right side exhausts $\widehat{B}$ as a varies, by nondegeneracy and restriction of characters. Hence $B = (1 - \rho(g))A$ and the component has $|A^g|$ simples. The degree formula now follows from equivariantization [13, Corollary 2.13], and $f = D/d^2$ gives the second formula. □

Lemma 26.2 (Coprime descent)**.** *Let $G = G_1 \times G_2$ have coprime factor orders. Let $\mathcal{B}_i$ be nondegenerate braided categories and $A_i = \mathrm{Inv}(\mathcal{B}_i)$. Suppose $\gcd(|G_1|, |A_2|) = \gcd(|G_2|, |A_1|) = 1$. If the action on $\mathcal{B}_1 \boxtimes \mathcal{B}_2$ in a faithful braided G-crossed extension is the product of homomorphisms $G_i \to \mathrm{Aut}_{\mathrm{br}}(\mathcal{B}_i)$ on isomorphism classes, then the extension and its equivariantization are external products of the corresponding constructions on the two factors.*

Proof. The opposite group has no positive cohomology on A_i, since its order acts invertibly. Inflation therefore gives $H^n(G, A_1 \oplus A_2) = \bigoplus_i H^n(G_i, A_i)$. For trivial scalar coefficients and $n > 0$ one also has $H^n(G, \mathbb{C}^\times) = \bigoplus_i H^n(G_i, \mathbb{C}^\times)$: positive integral homology is killed by the group order, so the mixed tensor and Tor terms in Künneth vanish, and $\mathbb{C}^\times$ is divisible.

In the Picard classifying space the next two homotopy groups are $A_1 \oplus A_2$ and $\mathbb{C}^\times$; external product includes the individual groups A_i and multiplies the scalar coordinates [31, Theorem 5.2, Proposition 7.3 and Theorem 7.12]. By naturality the two factor $H^3(G_i, A_i)$ obstructions are the components of the vanishing obstruction for the given extension. Correct the two lower lifts by the components of their $H^2(G, A_1 \oplus A_2)$ difference from the prescribed lift. Their product now has vanishing scalar fourth obstruction. Upon restriction to G_i the other lift is constant and the inclusion of the ith factor is the identity on scalars, so each factor fourth obstruction vanishes. The remaining difference of the two extensions is in $H^3(G, \mathbb{C}^\times)$; its two inflated components adjust the factor associators. Equivariantization by the product group then gives the assertion. □

We use the Hall-factor criterion from the order graph of Erdős and Pálfy [27]. Partition $\pi(|G|)$ into two parts. If across the cut $s \nmid \ell^j - 1$ for $1 \le j \le v_\ell(|G|)$ in either direction, G is the product of its two Hall factors. Indeed, for even order every odd prime joins two. For odd order use solvability [33, p. 775] and induct on $|G|$. Choose a minimal normal subgroup $M \cong \mathbb{F}_\ell^e$. In G/M the opposite Hall factor is normal; its preimage is $M \rtimes H$ by Schur–Zassenhaus. The cut makes $|H|$ coprime to $|\mathrm{GL}_e(\ell)|$, so this preimage is $M \times H$ and its unique opposite Hall subgroup H is normal in G. A complementary Hall subgroup P commutes with H modulo M and $[P, H] \subseteq M \cap H = 1$.

Theorem 26.3 (Arithmetic and action components)**.** *For $D = \prod_\ell \ell^{a_\ell}$ form a graph $\Gamma(D)$ on the primes of D. Join distinct s, ℓ if in either direction*

$$s \mid \ell^{2j} - 1 \qquad (1 \le j \le \lfloor a_\ell/2 \rfloor).$$

An integral WGT modular category of dimension D is a braided product over the connected components Π of $\Gamma(D)$; the factor for Π has dimension $\prod_{\ell \in \Pi} \ell^{a_\ell}$. This conclusion holds for every integral modular category when all repeated primes lie in one component, including the square-free case. In particular it holds whenever $2 \mid D$ or $3 \mid D$.

For any modular category with pointed core, the same decomposition holds after adjoining the edges 2–ℓ for the odd primes with $a_\ell = 1$ and $v_\ell(d_X^2)$ odd for some simple X. When the action is known, one may instead use the following finer graph on $\pi(|G||A|)$: join s, ℓ if in either direction $s \mid |\rho_\ell(G)|$, or if $s, \ell \mid |G|$ and $s \mid \ell^j - 1$ for some $1 \le j \le v_\ell(|G|)$.

Proof. Across a cut in the finer graph, G splits by the Hall-factor criterion. Each factor acts trivially on the opposite primary summands of A. Lemma 26.2 thus splits the gauging into factors of dimensions $|G_i|^2|A_i|$. It remains to bound the edges of this graph.

The group edges are in $\Gamma(D)$ since $2v_\ell(|G|) + v_\ell(|A|) = a_\ell$. For an action edge choose an isometry of A_ℓ of prime order $s \neq \ell$. If s is odd, a primitive sth root occurs among its eigenvalues on $A_\ell/\ell A_\ell$: the kernel of reduction is an ℓ-group. Put $f = \mathrm{ord}_s(\ell)$ and $m = v_\ell(|A|)$. The composition factors of A_ℓ as a $\mathbb{Z}[C_s]$-module have total $\mathbb{F}_\ell$-dimension m. Their eigenvalues over $\overline{\mathbb{F}}_\ell$ are stable under Frobenius, and under inversion because the polar pairing identifies A_ℓ with its character dual as a C_s-module. If f is even the Frobenius orbit already has size f; if f is odd its inverse orbit is disjoint and adds another f. Thus $s \mid \ell^{2j} - 1$ with $j = f/\gcd(2, f) \leq \lfloor m/2 \rfloor$. This works also at $\ell = 2$ and retains the rank bound $f \leq \dim_{\mathbb{F}_\ell}(A_\ell/\ell A_\ell)$.

If $s = 2$ and $a_\ell \geq 2$, the edge belongs to $\Gamma(D)$. If $a_\ell = 1$, then $A_\ell = C_\ell$ has isometry group $\{1, -1\}$. The action is nontrivial exactly when some $(1 - \rho(g))A$ has ℓ-valuation one; this is exactly the odd-valuation test on d_X^2 by Proposition 26.1. In an integral gauging A_ℓ is therefore fixed whenever $|A_\ell| = \ell$ is odd, also when $\ell \mid |G|$, by the same moved-subgroup formula. More generally a pointwise fixed A_ℓ with $\ell \nmid |G|$ splits off by Lemma 26.2.

Finally, even without WGT all exponent-one primes in an integral modular category split off as pointed modular primary factors [14, Corollary 1.22]. If the repeated primes are in one component, every other component is an isolated exponent-one prime, giving the required decomposition. If 2 or 3 divides D, it joins each other repeated prime through $\ell^2 - 1$. □

Primes seen by simple dimensions. For integral modular $\mathcal{C}$, let $S_0 = \{p : p \mid d_X \text{ for some } X\}$. If $p \mid D$ and $p \notin S_0$, Burciu's categorical Itô–Michler theorem gives $p^{v_p(D)} \mid |U(\mathcal{C})|$ and splits off a pointed modular factor of dimension $p^{v_p(D)}$ [10, Proposition 1.3]. The divisibility also implies $p \notin S_0$ [12, Corollary 1.7]. Since $D = |U(\mathcal{C})| \,\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})$, it follows that $S_0 = \pi(\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}))$. Burciu extends this splitting to weakly integral braided categories using conjugacy-class dimensions [11, Theorem 1.1]. For WGT categories the graph of the remaining centralizer is $\Gamma(D)[S_0]$; removing only the exponent-one factors gives $\Gamma(Q(D))$. An exponent-one vertex may have joined two repeated primes. In particular, if all d_X are odd and $D = 2^\alpha M$ with M odd, the remaining odd-dimensional centralizer has dimension M. WGT for it gives solvability of $\mathcal{C}$, for instance when $M = p^a q^b c$ with c square-free and coprime to pq [41, Theorem 7.4].

In the Frobenius–Perron prime graph [43, Definitions 2.1 and 4.1], which we call the simple-dimension graph, one joins distinct primes p, q if $pq \mid d_X$ for some X. In a product over arithmetic components Π, it is the join of the induced graphs on the nonempty $S_0 \cap \Pi$: witnesses from different factors can be tensored. Thus each connected component of its complement lies in one arithmetic factor, and with two nonempty factor supports the simple-dimension graph has diameter at most two. In the integral modular case, the class dimensions are the d_X^2, so the simple-dimension graph is also Burciu's class graph; if it is disconnected then the category is solvable [11, Theorem 1.2 and Lemma 6.2(1)]. A nonpointed prime integral modular category has $S_0 = \pi(D)$. Natale and Pacheco Rodríguez treat Frobenius groups with abelian kernel and complement in [43, Example 8.6], illustrating the calculation with $C_{341} \rtimes C_{10}$. Here take $G = C_{341} \rtimes C_5$, with faithful actions on C_{11} and C_{31}. The nonidentity fluxes have abelian centralizers of index 5 or 341, and the pure charges have degrees 1 or 5. Thus the simple dimensions of $\mathcal{Z}(\mathrm{Vec}_G)$ are $1, 5, 341$. The simple-dimension graph has edge 11–31; the arithmetic graph has edges 5–11 and 5–31.

26.2 Multiplicities and prime-power codegrees

For a weakly integral modular category let n_λ count the simples of degree λ, $N = \operatorname{ord}(T)$ and $K = \mathbb{Q}(S)$ for the unnormalized matrix. Then

$$\prod_{\substack{\ell \mid D \\ \ell \text{ odd}}} \frac{\ell - 1}{2} \;\Big|\; \frac{\varphi(N)}{|(\mathbb{Z}/N\mathbb{Z})^\times[2]|} \;\Big|\; [K : \mathbb{Q}] \;\Big|\; \prod_\lambda n_\lambda!.$$

By [8, Theorems 2.3 and 2.5(a), Lemma 2.6], $K \subseteq \mathbb{Q}(\zeta_N) \subseteq \mathbb{Q}(\zeta_n)$, where n is the order of a normalized twist matrix, and $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/K)$ has exponent at most two. The same holds for its quotient $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/K)$. Galois permutes the characters S_{ij}/d_j faithfully, since their values generate K (the unit column contains the d_i), and preserves the positive simple dimensions [8, Lemma 3.9]. The categorical Cauchy theorem [9, Theorem 3.9] identifies the primes of N and D. For instance $7 \mid D$ forces a repeated dimension of multiplicity at least three. If K is multiquadratic, then $N \mid 240$ by [8, Corollary 3.14]. Pointed core alone does not imply this: a nondegenerate cyclic metric group of order seven has $K = \mathbb{Q}(\zeta_7)$.

Proposition 26.4 (The smallest codegrees at prime-power dimension)**.** *An integral modular category of dimension p^n, $n \geq 4$, has no simple of formal codegree p^2.*

Proof. Such a category is nilpotent by successive cyclic gradings [30, Theorem 8.28], hence WGT, so use the pointed-core description above. For a simple above $Y \in \mathcal{F}_g$ with projective stabilizer representation π, the dimension formula gives

$$f = |A^g|\,|G_Y|^2/(\dim \pi)^2 \;\geq\; |A^g|\,|G_Y|.$$

Here $(\dim \pi)^2 \leq |G_Y|$ by the dimension of the twisted group algebra, and $g \in G_Y$: the crossed braiding identifies $Y \otimes Y^*$ with ${}^g(Y^*) \otimes Y$, which therefore contains the unit.

Suppose $f = p^2$. Then $p^n = f d_X^2$ is square, hence so is $|A| = p^n/|G|^2$. If $A \neq 0$, a p-group element fixes at least p elements of A by the orbit equation. Since $|(1-g)A|$ is square by integrality, $|A^g|$ is square as well and is at least p^2. The inequality forces $G_Y = 1$, hence $g = 1$ and $|A| = p^2$. Every $h \in G$ then fixes all of A by the same argument. The neutral label $Y \in A$ has stabilizer G, giving $G = 1$, contrary to $n \geq 4$.

If $A = 0$ the crossed category has one invertible in each component, so the gauging is the twisted double of a p-group G of order at least p^2. Now $G_Y = C_G(g)$ has order at least p^2: for noncentral g it contains $Z(G)\langle g\rangle \supsetneq Z(G)$, and for central g it is G. Equality $f = p^2$ would force the twisted group algebra of $C_G(g)$ to be a single matrix algebra. For $g = 1$ its cocycle is trivial. For $g \neq 1$ its basis element at g is central: the centralizer cocycle is $\alpha_g(x,y) = \omega(g,x,y)\omega(x,y,g)/\omega(x,g,y)$ for $x, y \in C_G(g)$ [43, (8.1)], so $\alpha_g(g,x) = \alpha_g(x,g) = \omega(g,x,g)$. Either way the center has dimension greater than one. □

Corollary 26.5 (Nilpotent modular categories and prime-power doubles)**.** *In an integral nilpotent modular category of dimension D, $v_p(D) \geq 4$ implies $v_p(f_X) \neq 2$. In particular every simple of $\mathcal{Z}(\mathrm{Vec}_P^\omega)$ has codegree divisible by p^4 when P is a p-group of order at least p^2.*

Proof. Use the prime-power decomposition [25, Theorem 1.1] and multiplicativity of codegrees. In a double of dimension $|P|^2$ all codegrees are even powers of p greater than one. □

The nilpotence hypothesis is necessary. The center of a Tambara–Yamagami category on an abelian group of order p^2, p odd, is integral modular of dimension $4p^4$. Its degree-p simples $Z_{\rho,\Delta}$ have codegree $4p^2$ and lie outside the principal p-block: their character values on $Y_{a,b}$, $a \neq b$, are zero rather than two modulo p [35, Proposition 4.1 and the S-matrix on p. 977]. Gelaki, Naidu and Nikshych credit Izumi with an earlier computation of these modular data [35, Remark 4.3].

Lemma 26.6 (Factors of a cyclic gauging)**.** *Let a cyclic group G act faithfully on a finite metric group A of coprime order. Join $s \mid |G|$ to $t \mid |A|$ when the Sylow subgroup G_s acts nontrivially on A_t. Suppose that each target has a neighbor and $A_t^{G_s} = 0$ on each edge. The action admits a modular gauging, and the connected components give the prime braided factors of every gauging.*

Proof. Coprimality gives $H^3(G, A) = 0$ and cyclicity gives $H^4(G, \mathbb{C}^\times) = 0$, so the two obstructions to a braided crossed extension vanish [31, Proposition 7.3 and Theorem 7.12]. Its gauging is modular [26, Proposition 4.56(ii)], and Lemma 26.2 splits it over the components. Conversely, its entire pointed subcategory is $\mathrm{Rep}(G)$: Proposition 26.1 excludes invertibles over $g \neq 1$ by faithfulness, and in the neutral component only $0 \in A$ is fixed by G. Any braided factorization therefore splits $\mathrm{Rep}(G)$ as the product of the pointed parts of its factors. This splits G as $G_1 \times G_2$ of coprime orders, since G is cyclic. De-equivariantizing factor by factor splits $A = A_1 \perp A_2$, with G_i acting trivially on A_{3-i}. If $G_s \leq G_i$ acts on A_t, the opposite summand of A_t lies in $A_t^{G_s} = 0$. Thus both ends of every edge belong to one factor. Since the factor dimensions are $|G_i|^2|A_i|$, a connected component admits no further splitting. □

26.3 Two exact criteria on a square-free set of primes

For a bipartition $S \sqcup T$ of a set of primes and a sign $\epsilon_t \in \{1, -1\}$ at each $t \in T$, draw the allowed edges

$$s \longrightarrow t \quad (s \in S,\ t \in T,\ s \mid t - \epsilon_t).$$

Connectivity refers to the underlying undirected graph. The signs belong to targets, not to individual edges.

Theorem 26.7. *Let L be composite and square-free.*

1. *A tensor-prime integral WGT fusion category of dimension L exists if and only if some bipartition has connected allowed graph with all signs $+1$. Moreover for any integral WGT category of this dimension, tensor-primeness is equivalent to primeness of its center.*

2. *A prime integral WGT modular category of dimension L^2 exists if and only if some bipartition and choice of target signs has connected allowed graph.*

Proof. Groups of square-free order. Every group H of square-free order has the form $F \rtimes E$, where $F = [H, H]$ and E are cyclic of coprime orders [49, Theorem 10.1.10]. Draw the actual nontrivial actions from the prime factors of E to those of F. Every edge $s \to t$ has $s \mid t - 1$. Subgroups belonging to distinct components commute and have coprime orders, so H is their direct product. Conversely a direct product must separate the prime factors without cutting an edge. Every connected all-positive allowed graph is realizable: let each C_s act nontrivially on every incident C_t through its cyclic automorphism group.

By [32, Proposition 9.1], an integral WGT category of dimension L has a group-theoretical presentation $\mathcal{C}(H, \omega, K, \alpha)$ with $|H| = L$. A coprime product $H = H_1 \times H_2$ splits this presentation. Indeed $K = K_1 \times K_2$ by taking powers to isolate each coordinate, and both ω and its trivialization on K can be chosen as products by the scalar cohomology calculation in Lemma 26.2. The twisted algebra and its category of bimodules then split. If H is directly indecomposable, its center is trivial: a nontrivial center is a central Hall subgroup and splits off by Schur–Zassenhaus, while a cyclic group of composite square-free order is decomposable. Its twisted double, the center of every presentation above, is prime by [39, Proposition 6.7(ii), p. 4215]: a decomposition would give commuting normal subgroups with product H and intersection in $Z(H) = 1$. The center of a tensor product is a braided product. This proves (i), using Vec_H for existence.

Modular categories of square-free root. For even L the all-positive star with source two is connected, and (i) supplies a prime center. Now let L be odd, and write $L^2 = |G|^2|A|$ as above. The order of G is square-free and coprime to $|A|$. Each primary summand A_t is weakly

anisotropic, since an isotropic subgroup invariant under all its isometries would be G-invariant. As t is odd and $|A_t| = t^2$, A_t is a metric plane over $\mathbb{F}_t$ by [26, Theorem A.24(i),(iv)]. The odd image of G acts on it through the cyclic rotation group. The plane is anisotropic, since rotations of a split plane preserve its isotropic lines, contrary to maximality of $\mathrm{Rep}(G)$. Thus for $G = F \rtimes E$ as above, F acts trivially on the planes. The prime factors of E act on the cyclic factors F_t through groups of order $t-1$ and on A_t through rotations of order dividing $t+1$. These are the positive and negative targets of a single signed graph. Across any cut G splits by its actual group actions, and its factors fix the opposite planes. Lemma 26.2 splits the category, so primeness implies connectivity.

Conversely, given a connected signed graph, take $G = \prod_{s \in S} C_s$ and one metric plane A_t for each target, split for sign $+1$ and anisotropic for sign -1. Its rotation group is cyclic of order $t - \epsilon_t$; choose the image of G there to have order the product of the incident primes. The action is faithful and every nonidentity rotation has no nonzero fixed vector. Lemma 26.6 gives a prime modular gauging; it is WGT, integral since each moved plane contributes t^2 to $|(1-\rho(g))A|$, and has dimension L^2. □

Both tests apply without a WGT hypothesis when L has at most five prime divisors by Theorem 15.2, using the center in (i). For three odd primes $p < q < r$, a connected graph is a star. All positive signs give $p \mid r-1$ and either $p \mid q-1$ or $q \mid r-1$. With arbitrary signs the criterion is

$$(p \mid q^2-1 \text{ and } p \mid r^2-1) \quad \text{or} \quad pq \mid r-1 \quad \text{or} \quad pq \mid r+1.$$

The first alternative has source p; the last two have target r. The hypothesis that L is composite matters in (i): Vec_{C_p} is tensor-prime for an odd prime p, but its center is the split metric plane.

Sharpness of the exponents and of integrality. Let C_s have prime order and act on a nonzero metric group A of coprime order with $A^{C_s} = 0$. The graph is a star, so the gauging is prime by Lemma 26.6. Its degrees and multiplicities are

$$(1, s), \qquad (s, (|A|-1)/s), \qquad (\sqrt{|A|}, s(s-1)).$$

The neutral contribution comes from the unit and the free orbits in A. Each nonidentity component has a single simple of dimension $\sqrt{|A|}$ and s equivariant structures, since $H^2(C_s, \mathbb{C}^\times) = 0$.

For distinct odd primes s, ℓ and $n = \ell^h$, use the norm on $\mathbb{F}_n \times \mathbb{F}_n$ or $\mathbb{F}_{n^2}$, followed by the trace to $\mathbb{F}_\ell$. Its polar form is nondegenerate by the trace pairing. The cyclic norm-one torus has order $n-1$ or $n+1$ and acts freely off zero. Either divisibility $s \mid n \pm 1$ therefore realizes the displayed type at $|A| = n^2$. For $(s, \ell, h) = (5, 7, 2)$ the first edge occurs at $\ell^4 - 1$; compare the anisotropic plane construction and its simple degrees in [34, Definition 3.1 and Proposition 4.2].

For $s = 2$, inversion on a cyclic odd metric group of order $M > 1$ gives the metaplectic categories [3, Theorem 3.2]. At $M = 35$ the two extra edges of Theorem 26.3 connect the isolated vertices of $\Gamma(140)$ through two. In dimension $4qr$ with q, r distinct odd primes, integral modular categories are pointed [6, Theorem 3.1]. Every fusion category of this dimension is WGT by applying Theorem 4.12 to its center.

Code, data and certificates

The verification programs, their inputs and the certificates are available at
`https://github.com/sebastienpalcoux/WGT-fusion-categories`.

To reproduce a calculation, look up its theorem or subsection in the verification manifest. The dimension-check instructions give the command and expected output for the four bounds in Theorem 17.1. The printed locations in the manifest refer to the repository edition. The source labels identify the corresponding results in the present text. Each entry gives the hypotheses,

the command to run from the `code/` directory, and the expected output. The `data/` directory contains the rank-15 data of Dong and Palcoux. The witness file records the seven terminal trace obstructions of Theorem 9.15 and the five pairs of disjoint fibers at rank 100 in Section 24.4.

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