A proof of a conjecture of Cazenave on graph eigenvalues

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Abstract

We prove Cazenave's conjecture $\mu_2^2 \geq 4\lambda_2 + 1$ for connected graphs, where λ_2 and μ_2 are the second largest adjacency and Laplacian eigenvalues. We also characterize equality in this inequality and in the linear bound $\mu_2 \geq \lambda_2 + 1$.

Let G be a finite simple connected graph on $n \geq 2$ vertices, with adjacency matrix A and Laplacian $L = D - A$. Write λ_2 and μ_2 for their second largest eigenvalues. Stars include K_2. Cazenave [2, Theorem 1] proved the first inequality below and conjectured the second [2, Conjecture 6(ii)].

Theorem 1. *Every such graph satisfies*

$$\mu_2 \geq \lambda_2 + 1 \qquad \text{and} \qquad \mu_2^2 \geq 4\lambda_2 + 1.$$

Equality holds exactly for stars in the first bound, and for stars on at least three vertices in the second.

Let T be $K_{1,3}$ with one edge subdivided, and let γ be the middle root of $x^3 - 7x^2 + 13x - 5$. Direct calculation gives

$$\mu_2(T) = \gamma \in (9/4, 7/3), \qquad \mu_2(P_5) = \frac{3+\sqrt{5}}{2}, \qquad \mu_2(K_3) = 3.$$

We use Laplacian subgraph monotonicity [1, Proposition 3.2.1(ii)], which does not require induced subgraphs. The following consequence of [3, Observation 3.3 and Theorem 3.4] has a short direct proof.

Lemma 2. *Unless G is a star, P_4 or C_4, it satisfies $\mu_2(G) \geq \gamma > \sqrt{5}$.*

Proof. We show that G contains K_3, P_5 or T. If its maximum degree is at most two, it is a path or cycle, and the claim follows from the exclusions. Otherwise choose a vertex h of degree at least three. Since G is not a star, connectivity gives a neighbour a of h with a neighbour $b \neq h$. If $b \sim h$, then hab is a triangle. Otherwise two further neighbours c, d of h give the copy ha, ab, hc, hd of T. Subgraph monotonicity and the displayed spectra prove the bound. □

The rounded bound below follows from Cazenave's signed-degree estimate [2, Theorem 1 and the following remark]; we additionally prove strictness when $\lambda_2 > 0$.

Lemma 3. *If $t = \lambda_2 > 0$, then $\mu_2 \geq \lceil t \rceil + 1$ and $\mu_2 > t + 1$.*

Proof. Set $k = \lceil t \rceil$ and let y be a t-eigenvector of A. By Perron–Frobenius, y has both signs. At a positive maximum and a negative minimum, the eigenvalue equation gives at least k neighbours of the same sign. Retain k such edges at each vertex, forming a spanning subgraph H consisting of two disjoint $K_{1,k}$'s and isolated vertices.

Choose nonzero z in the two-dimensional $(k+1)$-eigenspace of $L(H)$, orthogonal to a top eigenvector of $L(G)$. Then

$$(\mu_2 - k - 1)\|z\|^2 \geq z^{\mathsf{T}}(L(G) - L(H))z = \sum_{uv \in E(G)\setminus E(H)} (z_u - z_v)^2 \geq 0.$$

Thus $\mu_2 \geq k+1$. If $\mu_2 = t+1$, then $t = k$ and every omitted edge has equal z-values at its endpoints. The support of z is a nonempty union of whole stars (each star's top Laplacian eigenvector is proportional to $(k, -1, \ldots, -1)$), so no edge leaves it. Connectivity forces full support; hence y has exactly $k+1$ positive and $k+1$ negative coordinates.

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Put $S = \{v : y_v > 0\}$ and $d_S(v) = |N(v) \cap S|$. Summing $Ay = ky$ over S gives

$$k \sum_{v \in S} y_v = \sum_{v \in S} d_S(v) y_v + \sum_{v \notin S} d_S(v) y_v < k \sum_{v \in S} y_v,$$

since $d_S(v) \leq k$ on S, all coordinates outside S are negative, and connectivity supplies a crossing edge. This contradiction proves strictness. □

Proof of Theorem 1. The pairs (λ_2, μ_2) for K_2, stars of order at least three, P_4 and C_4 are, respectively,

$$(-1, 0), \qquad (0, 1), \qquad \left(\frac{\sqrt{5} - 1}{2}, 2 \right), \qquad (0, 2).$$

These give the stated equalities for stars and strict inequalities for P_4, C_4. Together with Lemma 2, they also give $\mu_2 \geq 2$ for every non-star. Thus the linear bound is strict for non-stars with $\lambda_2 \leq 0$; Lemma 3 handles $\lambda_2 > 0$.

For the quadratic bound, consider a non-star other than P_4, C_4. If $\lambda_2 \leq 1$, Lemma 2 gives $\mu_2^2 > 5 \geq 4\lambda_2 + 1$. Otherwise put $t = \lambda_2 > 1$ and $k = \lceil t \rceil \geq 2$. Then

$$\mu_2^2 \geq (k + 1)^2 \geq 4k + 1 \geq 4t + 1.$$

Equality requires $k = t = 2$ and $\mu_2 = t + 1$, contrary to Lemma 3. □

Corollary 4. *Put $\alpha = (\sqrt{5} - 1)/2$. Among connected graphs with $\lambda_2 > \alpha$, the minimum of μ_2 is γ, attained by T.*

Proof. Stars, P_4 and C_4 have $\lambda_2 \leq \alpha$, so Lemma 2 gives the lower bound. The graph T attains it, since $\lambda_2(T) = \sqrt{2 - \sqrt{2}} > \alpha$. The threshold is sharp: $\lambda_2(P_4) = \alpha$ and $\mu_2(P_4) = 2 < \gamma$. □

References

[1] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Universitext, Springer, New York, 2012. doi:10.1007/978-1-4614-1939-6.

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