hexagon:2611.00230v1

NONEXISTENCE OF DSRGS WITH TEN PARAMETER SETS: THE RANK OF $A-\theta I$ VERSUS A SUBMATRIX PRESCRIBED AT A VERTEX

IURII IVANOV

Abstract. For a directed strongly regular graph (DSRG), the parameters determine rank(A − θI) at nontrivial eigenvalues θ. For small λ or t = λ + 1 they almost determine a piece of this matrix at a vertex: a piece needing more rank excludes the graph, and a small margin leaves finitely many cases. We exclude (50, 11, 7, 0, 3), (68, 13, 7, 0, 3), (100, 16, 7, 0, 3), (108, 19, 9, 0, 4), (48, 9, 5, 0, 2), (55, 16, 12, 1, 6), (66, 10, 5, 4, 1) and (91, 12, 6, 5, 1), open in Hobart and Brouwer's table, and their complements. Counting twin classes reproves (96, 15, 10, 9, 1) and (28, 6, 3, 2, 1): for the first, rank replaces one counting bound; for the second, it is unused. Lean 4 with Mathlib proves nine exclusions; for (28, 6, 3, 2, 1) its kernel also checks a SAT certificate. (48, 9, 5, 0, 2) ends with a search by two separately written programs, the second certified by CaDiCaL unsatisfiability proofs checked by drat-trim.

Independent researcher, Pattaya, Thailand
Email address: nosdormas95@gmail.com

1. Introduction

A directed strongly regular graph (DSRG) with parameters (v, k, t, λ, μ) is a loopless directed graph on v vertices whose adjacency matrix A satisfies

$$AJ = JA = kJ, \qquad A^2 = tI + \lambda A + \mu(J - I - A),$$

where J is the all-ones matrix [1]. Every vertex has k outgoing and k incoming arcs, t of them reciprocal, and there are exactly λ paths of length 2 from x to $y \neq x$ if there is an arc $x \to y$, and exactly μ if there is none. For $t = k$ this is an undirected strongly regular graph, and throughout we assume $0 < t < k < v - 1$. Duval [1] introduced these graphs and found their spectrum and necessary conditions on the parameters. The matrix $J - I - A$ of the complement also defines a DSRG, with parameters $(v, v-k-1, v-2k+t-1, v-2k+\mu-2, v-2k+\lambda)$ [1], so parameter sets are considered in pairs: a set and its complement exist or fail to exist together.

Hobart and Brouwer maintain a table of feasible parameter sets with $v \leq 110$ [2]: for each set it gives a construction or a condition that rules the set out, or marks the set as open. In the version of the table dated 6 November 2026, the ten pairs that this paper is about are open:

Theorem A. *There are no directed strongly regular graphs with parameters*

$$(50, 11, 7, 0, 3),\ (68, 13, 7, 0, 3),\ (100, 16, 7, 0, 3),\ (108, 19, 9, 0, 4),\ (48, 9, 5, 0, 2),$$

2020 *Mathematics Subject Classification.* Primary 05C20; Secondary 05E30.
Key words and phrases. directed strongly regular graph, nonexistence, matrix rank, twin vertices, formalization in Lean, SAT certificate.
Corresponding author. ORCID: 0009-0009-4137-6788.

$$(55,16,12,1,6),\ (66,10,5,4,1),\ (91,12,6,5,1),\ (96,15,10,9,1),\ (28,6,3,2,1)$$

and none with the parameters of their complements.

The method of the paper is a comparison of two ranks. Let θ be a nontrivial eigenvalue of A, let θ' be the other one, and let m' be the multiplicity of θ'. The rank of the matrix $A-\theta I$ equals $1+m'$ (Lemma 3.1): it is determined by the parameters and is small when the multiplicity is small. On the other hand, for small λ and for $t=\lambda+1$ the parameters almost completely prescribe a piece of the same matrix at a single vertex: the principal submatrix on the neighbourhood, the rows of the vertex and its out-neighbours, coinciding rows. If the piece needs a rank greater than $1+m'$, the graph does not exist. If the rank suffices with a small margin, the piece determines the rest of the matrix up to a matrix of small rank, and the structure of the graph at the vertex reduces to finitely many cases. The principle by itself excludes nothing: for an existing graph the rank is exactly $1+m'$, and what excludes a set is only a piece forced by the parameters.

The exclusions of eight pairs of Theorem A rest on this comparison. For the four pairs with $\lambda=0$ it gives a contradiction at once. For $(48,9,5,0,2)$, $(55,16,12,1,6)$, $(66,10,5,4,1)$ and $(91,12,6,5,1)$ it reduces the structure at a vertex to finitely many cases, which are excluded by counting, and for $(48,9,5,0,2)$ by a search. Two pairs excluded earlier by other methods get short proofs from counting twin classes for $t=\lambda+1$, and these proofs do not need the rank: for $(96,15,10,9,1)$ path counting gives both bounds on a twin class, and the rank can replace one of them; the exclusion of $(28,6,3,2,1)$ also uses a certificate of SAT unsatisfiability.

What is known. The nonexistence of the pairs $(96,15,10,9,1)$ and $(28,6,3,2,1)$ has already been proved by other methods in [3] and [4]. For the first pair there is also a Lean theorem based on [3] in the repository of Chung and Wesley [5]. Nevertheless the table, in its version of 6 November 2026, keeps both pairs open. Two more pairs of the same family, $(40,9,6,5,1)$ and $(80,19,13,12,2)$ with their complements, are already marked in the table as nonexistent by note N13: the note cites the repository [5], while the argument itself is the platform paper [6]. Theorem 7.6 gives these two pairs another short proof, and we do not call these results new. For the other eight pairs of Theorem A we know of no other proofs.

Earlier papers. Our proofs of all ten nonexistence results appeared earlier, without peer review, in two papers [7] and [8] on rStation, an open platform where AI agents publish research papers: the first excluded the six sets with $\lambda\le 1$, the second the four sets with $t=\lambda+1$ together with $(40,9,6,5,1)$ and $(80,19,13,12,2)$. The present paper merges the two and reworks them (Section 10): eight of the ten nonexistence results are derived here from one principle and two from counting twin classes, all are compared with the literature, and all except $(48,9,5,0,2)$ are proved in Lean.

Further results. The comparison of ranks also yields conditions on the parameters, not only exclusions of individual sets: $2k-t\le m'$ for $\lambda=0$ and a root $\theta\ne 0$ with $\theta^3-t\theta-\mu(k-t)\ne 0$ (Theorem 4.3), and a bound on the number of components of the predecessor map for $\lambda=1$, $t=2\mu$ (Corollary 6.3). Counting twin classes gives the inequality

$$(\lambda+2-\mu)^2\le\mu(v-k-1)$$

TABLE 1. The ten pairs of Theorem A. The spectrum of A is written as eigenvalues with multiplicities. The notation of the column "Argument" is introduced in Sections 3 and 7; $K = k + 1$.

Set and complement	Spectrum of A	Argument	Where
$(50, 11, 7, 0, 3)$ $(50, 38, 34, 29, 28)$	$11^1, 1^{37}, (-4)^{12}$	the submatrix of $A - I$ on $N[x]$ of order 16 is nonsingular, while $\operatorname{rank}(A - I) = 13$	Theorem 4.5
$(68, 13, 7, 0, 3)$ $(68, 54, 48, 43, 42)$	$13^1, 1^{51}, (-4)^{16}$	order 20, rank 17	Theorem 4.5
$(100, 16, 7, 0, 3)$ $(100, 83, 74, 69, 68)$	$16^1, 1^{76}, (-4)^{23}$	order 26, rank 24	Theorem 4.5
$(108, 19, 9, 0, 4)$ $(108, 88, 78, 72, 70)$	$19^1, 1^{86}, (-5)^{21}$	order 30, rank 22	Theorem 4.5
$(48, 9, 5, 0, 2)$ $(48, 38, 34, 30, 30)$	$9^1, 1^{33}, (-3)^{14}$	order 14, rank 15: the Schur complement has rank 1, so the types of the outer vertices and the arcs of one pivot vertex determine the graph; the required cover does not exist	Theorem 5.3
$(55, 16, 12, 1, 6)$ $(55, 38, 34, 27, 24)$	$16^1, 1^{44}, (-6)^{10}$	the left kernel of the 17 rows on $\{x\} \cup \Gamma^+(x)$, of dimension at least 7, is incompatible with the integrality of the arc counts	Theorem 6.4
$(66, 10, 5, 4, 1)$ $(66, 55, 50, 45, 50)$	$10^1, 4^{11}, (-1)^{54}$	$\operatorname{rank}(A + I) = K + 1$: every vertex has a twin, all classes are pairs, and $\lvert N^+[x]\rvert = 11$ is odd	Corollary 7.10
$(91, 12, 6, 5, 1)$ $(91, 78, 72, 66, 72)$	$12^1, 5^{13}, (-1)^{77}$	the same, $\lvert N^+[x]\rvert = 13$	Corollary 7.10
$(96, 15, 10, 9, 1)$ $(96, 80, 75, 65, 75)$	$15^1, 9^8, (-1)^{87}$	a twin class has at least $K - f = 8$ and at most $K - q = 6$ vertices: both bounds come from counting (Proposition 7.5), the lower one also from the rank (Proposition 7.3)	Theorem 7.6
$(28, 6, 3, 2, 1)$ $(28, 21, 18, 15, 18)$	$6^1, 2^7, (-1)^{20}$	a vertex without a twin is excluded by a SAT formula whose certificate is checked by the Lean kernel; without such vertices all classes are pairs, but $\lvert N^+[x]\rvert = 7$	Theorem 7.12

for DSRGs with $t = \lambda + 1$ and $\mu < \lambda + 2$ (Theorem 7.6). The inequality does not change when the graph is blown up by cliques, so together with each set it excludes the whole line of its blow-ups.

How the results are checked. The nonexistence results for nine pairs out of ten and for their complements are theorems of Lean 4 [9] with the Mathlib library [10], with no hypotheses and no axioms beyond the three standard ones; for $(28, 6, 3, 2, 1)$ the Lean kernel also checks the proof of unsatisfiability of a SAT formula. For $(48, 9, 5, 0, 2)$ the reduction is proved in the text, and the last step is a search carried out by two programs that share no code; the unsatisfiability of the formulas of the second program is confirmed by checked proofs. Section 9 describes the verification, its Table 2 names the Lean theorems for each pair, and Table 3 in Appendix B names what each step rests on.

Outline. Section 2 places the method among the known conditions on the parameters. Section 3 proves the rank principle. Sections 4–7 are applications: the closed neighbourhood for $\lambda = 0$, its Schur complement, the left kernel for $\lambda = 1$, and twin classes for $t = \lambda + 1$; the comparison with known results stands next to

each statement. Section 8 says where the method is silent, and Section 9 is about the machine verification. Appendix A contains the reduction and the search for $(48, 9, 5, 0, 2)$, and Appendix B the table of the statements of the paper and the Lean theorems.

2. Related work

The definition of a DSRG, the spectrum, the multiplicity formulas and the complement are due to Duval [1]. The table [2] collects the sets with $v \leq 110$ and the conditions by which some of them are excluded. The conditions are of three kinds: spectral ones, namely the integrality of the multiplicities [1] and the absolute bound [11, Theorem 2], [12, Theorem 4.2]; rank conditions, namely descriptions of the graphs with $t = \mu$ in which the rank of A is small [11, 13, 14, 15]; and local ones, namely path counts at one vertex [11, Theorems 10–12] and principal submatrices of idempotents [16]. Our method combines the second kind with the third: it takes a matrix of known small rank and a piece that the parameters force at one vertex.

The rank of A for $t = \mu$. When 0 is an eigenvalue, the rank of A equals $1 + k/(\mu - \lambda)$ [13, Section 3], and when the rank is small, each row is a combination of a few rows. Arguments of this kind choose rows that span the whole row space, analyse the types of columns on them and obtain a short list of possible ratios k/v. In this way the DSRGs in which the rank of A is at most 4 were described [11, Theorem 3 and Lemma 7], it was proved that for each rank there are finitely many such ratios [13, Theorem 2], and the ranks 5 [14] and 6 [15] were analysed. Section 7 works with the same binary matrix of small rank: the rows of $A + I$ are equal exactly when the rows of the complement matrix $J - I - A$ are equal, and the complement has $t = \mu$. But it does not enumerate the column types of the whole matrix; it compares its rank with a piece at one vertex. Sections 4–6 carry the same comparison over to roots for which the matrix $A - \theta I$ is not binary.

Representations. Godsil, Hobart and Martin [12, Section 3] assign to a vertex i a pair of vectors $x_i, y_i \in \mathbb{R}^m$, where m is the multiplicity of an eigenvalue, such that the inner product $\langle x_i, y_j \rangle$ equals 1, α or β according to whether i and j coincide, there is an arc $i \to j$, or there is none. The matrix of these products is the projection onto the eigenspace multiplied by v/m. This is a directed version of the Euclidean representation of a strongly regular graph, but it has no positivity: $x_i \neq y_i$. From the representation the authors derive the absolute bound [12, Theorem 4.2], the same as [11, Theorem 2], and a description of the graphs with an eigenvalue of multiplicity 2 [12, Section 5]; according to them, this description was obtained independently and by another route by Jørgensen [11, Theorems 3–5]. We look at the same matrix differently: its rank equals the multiplicity, so each of its blocks has rank at most the multiplicity, and the lever is a block that the parameters prescribe at one vertex.

Undirected models and local comparisons. For strongly regular graphs the same techniques are known in a symmetric form, where positivity is added to rank: the Gram matrix of the vectors of the Euclidean representation on a set of vertices is determined by the induced subgraph and has rank at most the multiplicity of the eigenvalue [17, 18], and the graph is recovered from a star complement [19, 20]. How these works relate to Sections 4 and 5 is said after Theorem 4.5 and Proposition 5.2. The conditions of Hobart and Williford [16] are compared with

Theorem 4.3, Corollary 6.3 and Proposition 8.1 where these statements are proved, and the works on classes of vertices with equal rows in Section 7.

3. The rank principle

Notation. V is the vertex set, $\mathbf{1}$ is the all-ones vector, e_x is the unit vector of a vertex x, and $\mathbf{1}_S$ is the indicator vector of a set S. For a vertex x: $\Gamma^+(x)$ is the set of its out-neighbours, the heads of the arcs from x; $\Gamma^-(x)$ is the set of its in-neighbours, the tails of the arcs into x; $N^+[x] = \{x\} \cup \Gamma^+(x)$ and $N^-[x] = \{x\} \cup \Gamma^-(x)$ are the closed out-neighbourhood and in-neighbourhood; $N[x] = \{x\} \cup \Gamma^+(x) \cup \Gamma^-(x)$ is the closed neighbourhood. For a matrix X and sets R, C we denote by $X[R, C]$ its block on the rows R and the columns C, and by $X[\cdot, C]$ and $X[R, \cdot]$ the whole columns C and the whole rows R. For a statement P the number $[P]$ equals 1 if P holds, and 0 otherwise.

The nontrivial eigenvalues of a DSRG are the roots of the polynomial $x^2 - (\lambda - \mu)x - (t - \mu)$ [1]. Below $\mu \neq 0$, and the discriminant $\Delta = (\lambda - \mu)^2 + 4(t - \mu)$ is positive. For $0 < t < k < v - 1$ both conditions always hold: this was proved by Duval [1] (see [12, Lemma 2.1 and Theorem 2.2]). Sections 3 and 4 do not use the inequality $t < k$, and their statements hold also for $t = k$ provided $\mu \neq 0$ and $\Delta > 0$. We denote the roots by $r > s$, and their multiplicities as eigenvalues of A by f and g. Each of the two roots can play the role of θ; then θ' is the other root, m' is its multiplicity, $q = \theta' - \theta$ and $B_\theta = A - \theta I$. We write E_θ for $\ker(A - \theta I)$ over $\mathbb{R}$.

Lemma 3.1 (rank principle)**.** *Under these assumptions:*

(1) Identity. $B_\theta^2 = qB_\theta + \mu J$, $B_\theta \mathbf{1} = (k - \theta)\mathbf{1}$, $\mathbf{1}^\top B_\theta = (k - \theta)\mathbf{1}^\top$ *and* $k \neq \theta$. *The root* $\theta = 0$ *occurs exactly when* $t = \mu$, *and the root* $\theta = -1$ *exactly when* $t = \lambda + 1$; *in these cases* B_θ *is a matrix of zeros and ones:* A *or* $A + I$.

(2) Rank. $\mathbb{R}^v = \langle \mathbf{1} \rangle \oplus E_\theta \oplus E_{\theta'}$, $\ker B_\theta = E_\theta$, *the image of* B_θ *is* $\langle \mathbf{1} \rangle \oplus E_{\theta'}$, *and*

$$\operatorname{rank} B_\theta = 1 + m', \qquad m' = \frac{-k - (v-1)\theta}{\theta' - \theta}.$$

(3) Submatrix. *For any sets of rows* R *and columns* C *we have* $\operatorname{rank} B_\theta[R, C] \leq 1 + m'$. *In particular, a nonsingular principal submatrix* $B_\theta[R, R]$ *has order at most* $1 + m'$.

(4) Recovery. *If* $|R| = |C| = 1 + m'$ *and the block* $B_\theta[R, C]$ *is nonsingular, then* $B_\theta = B_\theta[\cdot, C]\, B_\theta[R, C]^{-1}\, B_\theta[R, \cdot]$.

(5) Schur complement. *If the principal submatrix* $B_\theta[R, R]$ *is nonsingular and* $\bar{R} = V \setminus R$, *then the matrix* $B_\theta[\bar{R}, \bar{R}] - B_\theta[\bar{R}, R]\, B_\theta[R, R]^{-1}\, B_\theta[R, \bar{R}]$ *has rank* $1 + m' - |R|$.

(6) Scaled projection. *The matrix* $\mathcal{K}_\theta = vB_\theta - (k - \theta)J$ *satisfies* $\mathcal{K}_\theta^2 = vq\mathcal{K}_\theta$, *has rank* m', *and its image is* $E_{\theta'}$.

Proof. (1) The roots satisfy $\theta + \theta' = \lambda - \mu$ and $\theta\theta' = \mu - t$, so $(A - \theta I)(A - \theta' I) = A^2 - (\lambda - \mu)A + (\mu - t)I = \mu J$ by definition. Since $A - \theta' I = B_\theta - qI$, this is exactly $B_\theta^2 = qB_\theta + \mu J$. The row and column sums of B_θ equal $k - \theta$. If $k = \theta$, we would get $B_\theta \mathbf{1} = 0$ and $\mu v \mathbf{1} = \mu J \mathbf{1} = (B_\theta - qI)B_\theta \mathbf{1} = 0$, contrary to $\mu \neq 0$. The number 0 is a root exactly when $t - \mu = 0$, and the number -1 exactly when $1 + (\lambda - \mu) - (t - \mu) = 0$, that is, when $t = \lambda + 1$. Off the diagonal B_θ coincides with A, and on the diagonal it has $-\theta$.

(2) From $\mathbf{1}^\top A = k\mathbf{1}^\top$ it follows that A maps $\mathbf{1}^\perp$ into itself. On $\mathbf{1}^\perp$ the matrix J acts as zero, so there $(A-\theta I)(A-\theta' I) = 0$, and a polynomial with distinct real roots annihilates the restriction of A. Hence A is diagonalizable on $\mathbf{1}^\perp$ with eigenvalues in $\{\theta, \theta'\}$. An eigenvector u for θ lies in $\mathbf{1}^\perp$, because $k\mathbf{1}^\top u = \mathbf{1}^\top A u = \theta \mathbf{1}^\top u$ and $k \neq \theta$; the same holds for θ'. This gives the decomposition. The matrix B_θ acts on the three summands as multiplication by $k - \theta \neq 0$, by 0 and by $q \neq 0$, which gives the kernel, the image and the rank. The formula for m' is the vanishing of the trace: $0 = \operatorname{tr} A = k + (v - 1 - m')\theta + m'\theta'$.

(3) The rank of a submatrix does not exceed the rank of the matrix.

(4) The rows of $B_\theta[R, \cdot]$ are linearly independent, because already their parts $B_\theta[R, C]$ are, and there are $1 + m' = \operatorname{rank} B_\theta$ of them. Hence they form a basis of the row space, and the row u equals $c_u B_\theta[R, \cdot]$ for some c_u. Comparing the columns C, we get $c_u = B_\theta[u, C]\, B_\theta[R, C]^{-1}$.

(5) Block elimination writes B_θ as a product of invertible block triangular matrices and of the block diagonal matrix formed by $B_\theta[R, R]$ and the Schur complement, so the rank of B_θ equals $|R|$ plus the rank of the complement; it remains to apply item 2.

(6) Both sides of the equality $\mathcal{K}_\theta = vq\Pi$, where Π is the projection onto $E_{\theta'}$ along $\langle \mathbf{1} \rangle \oplus E_\theta$, act in the same way on the three summands of item 2: on $\mathbf{1}$ as zero, because $J\mathbf{1} = v\mathbf{1}$; on E_θ as zero; on $E_{\theta'}$ as multiplication by vq. A projection of rank m' gives the rest. □

Items 1, 2 and 6 are known. Jørgensen [11, Section 2] derives from the definition the minimal polynomial of A and its diagonalizability, and writes the projections onto the eigenspaces as combinations of I, A and $J - I - A$; Godsil, Hobart and Martin [12, Propositions 3.1–3.3] do the same, and Hobart and Williford write the projection as such a combination in the proof of Theorem 3 of [16]. The matrix $\mathcal{K}_\theta$ is, up to a factor, the projection onto $E_{\theta'}$ and the matrix of inner products of the representation of [12] from Section 2. The equality $\operatorname{rank} B_\theta = 1 + m'$ for $\theta \in \{0, -1\}$ is noted in the introduction of [11]: for the root 0 the rank of A equals $1 + m'$, and for the root -1 the rank of the complement matrix $J - I - A$ equals $1 + m'$. Items 3–5 are standard linear algebra. The method lies not in the lemma itself but in the choice of the piece to which it is applied.

Corollary 3.2 (rank lever)**.** *If at some vertex the parameters force sets R, C with $\operatorname{rank} B_\theta[R, C] > 1 + m'$, then there is no graph with these parameters. If p rows of the matrix $\mathcal{K}_\theta$ are forced, their left kernel has dimension at least $p - m'$.*

Proof. The first statement is item 3 of Lemma 3.1, the second is item 6: p rows of a matrix of rank m' span a space of dimension at most m'. □

Complement and transpose. The matrix $A^\top$ is also the adjacency matrix of a DSRG, with the same parameters: the defining equalities are mapped to themselves under transposition. Its out-neighbours are the in-neighbours in A. The matrix $J - I - A$ is the adjacency matrix of a DSRG with the parameters of the complement [1]. Therefore every statement about the out-neighbours of a vertex also holds for its in-neighbours, and the nonexistence of a set implies the nonexistence of its complement.

How the applications are organized. In each of Sections 4–7 we choose a root θ and a vertex x and find a piece of B_θ whose structure is determined by the parameters. In Sections 4–6 the exclusion comes from comparing this piece

with Lemma 3.1. In Section 7 the rank enters two statements, Proposition 7.3 and Theorem 7.8, and the rest is done by counting paths of length 2 by item 1:

Section	Parameters	θ	Piece of B_θ	What it uses
4	$\lambda = 0$	r or s, $\theta \neq 0$	principal submatrix on $N[x]$, of order $2k - t + 1$	item 3
5	$\lambda = 0$	the same	Schur complement of this submatrix	items 1, 4, 5
6	$\lambda = 1$, $t = 2\mu$	1	rows of $\mathcal{K}_1$ on $\{x\} \cup \Gamma^+(x)$	item 6
7	$t = \lambda + 1$	-1	rows of $B_{-1} = A + I$ on $N^+[x]$ and on vertices outside $N^+[x]$	item 2 in Proposition 7.3 and Theorem 7.8, item 1 elsewhere

4. The closed neighbourhood for $\lambda = 0$

In this section $\lambda = 0$. Fix a vertex x and put $\Gamma^+ = \Gamma^+(x)$, $\Gamma^- = \Gamma^-(x)$, $M = \Gamma^+ \cap \Gamma^-$, $Q = \Gamma^+ \setminus \Gamma^-$ and $P = \Gamma^- \setminus \Gamma^+$. The vertices of M are joined to x in both directions, Q consists of the out-neighbours of x that are not in-neighbours, and P of the in-neighbours that are not out-neighbours.

Lemma 4.1. *Let $\lambda = 0$. Then:*

(1) *if $a \to b$ and $b \to c$, then there is no arc $a \to c$;*
(2) *the sets Γ^+ and Γ^- are independent, there are no arcs from Γ^- to Γ^+, and the only arcs inside $N[x]$ are the arcs from x to Γ^+, from Γ^- to x and from Q to P;*
(3) *every vertex of Q has exactly μ out-neighbours in P;*
(4) *$|M| = t$, $|Q| = |P| = k - t$ and $|N[x]| = 2k - t + 1$.*

Proof. (1) If $a \to b \to c$ and $a \to c$, then $a \neq c$, because there are no loops, and the path $a \to b \to c$ gives $A^2(a, c) \geq 1$, contrary to $A^2(a, c) = \lambda = 0$.

(2) An arc $a \to b$ inside Γ^+ together with $x \to a$ and $x \to b$ gives a triple forbidden by item 1; an arc inside Γ^- gives the triple $a \to b \to x$ with $a \to x$; an arc from $a \in \Gamma^-$ to $b \in \Gamma^+$ gives the triple $a \to x \to b$ with $a \to b$. Now let $a \to b$ be an arc inside $N[x]$ not incident with x. Its tail does not lie in Γ^-, because arcs from Γ^- lead neither to Γ^- nor to Γ^+; hence $a \in Q$. Its head does not lie in Γ^+, because Γ^+ is independent; hence $b \in P$.

(3) For $u \in Q$ there is no arc $u \to x$, so there are exactly μ paths of length 2 from u to x. These are the paths $u \to y \to x$ with $y \in \Gamma^-$, and by item 2 all such y lie in P.

(4) $|M| = A^2(x, x) = t$, hence $|\Gamma^+ \cup \Gamma^-| = 2k - t$ and $|Q| = |P| = k - t$, while $x \notin \Gamma^+ \cup \Gamma^-$. □

Proposition 4.2. *For $\lambda = 0$ and any root $\theta \neq 0$*

$$\det B_\theta[N[x], N[x]] = (-1)^{t+1}\, \theta^{2k-t-2}\big(\theta^3 - t\theta - \mu(k - t)\big).$$

The determinant depends neither on the graph nor on the vertex.

Proof. Order $N[x]$ as follows: x, then M, Q and P. By Lemma 4.1 the submatrix $L = B_\theta[N[x], N[x]]$ has the form

$$L = \begin{pmatrix} -\theta & \mathbf{1}^\top & \mathbf{1}^\top & 0 \\ \mathbf{1} & -\theta I & 0 & 0 \\ 0 & 0 & -\theta I & A_{QP} \\ \mathbf{1} & 0 & 0 & -\theta I \end{pmatrix},$$

where $A_{QP} = A[Q, P]$ is the matrix of arcs from Q to P; by item 3 of Lemma 4.1 its row sums equal μ. The block L' on $M \cup Q \cup P$ is block upper triangular, and $\det L' = (-\theta)^{2k-t}$. Let $b^\top = (\mathbf{1}^\top, \mathbf{1}^\top, 0)$ be the row of x off the diagonal, and $c = (\mathbf{1}, 0, \mathbf{1})$ the column of x. The solution y of the system $L'y = c$ is as follows: $y_M = y_P = -\theta^{-1}\mathbf{1}$, and $-\theta y_Q + A_{QP}y_P = 0$ gives $y_Q = -\mu\theta^{-2}\mathbf{1}$. Hence $b^\top y = -t\theta^{-1} - \mu(k-t)\theta^{-2}$, and by Schur's formula

$$\begin{aligned} \det L = \det L' \cdot \bigl(-\theta - b^\top y\bigr) &= (-\theta)^{2k-t}\bigl(-\theta + t\theta^{-1} + \mu(k-t)\theta^{-2}\bigr) \\ &= (-1)^{t+1}\,\theta^{2k-t-2}\bigl(\theta^3 - t\theta - \mu(k-t)\bigr), \end{aligned}$$

since $(-1)^{2k-t+1} = (-1)^{t+1}$. □

For example, for $(50, 11, 7, 0, 3)$ and $\theta = 1$ the neighbourhood $N[x]$ consists of 16 vertices, and the determinant equals $(-1)^8(1 - 7 - 12) = -18$ at every vertex of every graph with these parameters, while the rank of $A - I$ equals $1 + g = 13$.

Theorem 4.3. *Let $\lambda = 0$, and let a root θ satisfy $\theta \neq 0$ and $\theta^3 - t\theta - \mu(k-t) \neq 0$. Then $2k - t \leq m'$, where m' is the multiplicity of the other root.*

Proof. By Proposition 4.2 the principal submatrix of B_θ of order $|N[x]| = 2k-t+1$ is nonsingular, and by item 3 of Lemma 3.1 we get $2k - t + 1 \leq 1 + m'$. □

Corollary 4.4. *Let $\lambda = 0$, $\mu > 0$ and $t = 2\mu + 1$. Then $(2k-t)(\mu+2) \leq k + v - 1$.*

Proof. The polynomial $x^2 + \mu x - (\mu+1) = (x-1)(x+\mu+1)$ has the roots $r = 1$ and $s = -(\mu+1)$, and $\Delta = (\mu+2)^2 > 0$. The vanishing of the trace and the dimension count give $k + f - (\mu+1)g = 0$ and $f + g = v - 1$, whence $g = (k+v-1)/(\mu+2)$. Take $\theta = 1$. Since $t = |M| \leq k$, the number $1 - t - \mu(k-t)$ is at most $1 - t = -2\mu < 0$. Theorem 4.3 gives $2k - t \leq g$, and it remains to multiply by $\mu + 2$. □

Theorem 4.5. *There are no DSRGs* $(50, 11, 7, 0, 3)$, $(68, 13, 7, 0, 3)$, $(100, 16, 7, 0, 3)$, $(108, 19, 9, 0, 4)$, *and none with the complementary parameters* $(50, 38, 34, 29, 28)$, $(68, 54, 48, 43, 42)$, $(100, 83, 74, 69, 68)$, $(108, 88, 78, 72, 70)$.

Proof. All four sets have $\lambda = 0$ and $t = 2\mu + 1$, and the inequality of Corollary 4.4 fails: $15{\cdot}5 = 75 > 60$, $19{\cdot}5 = 95 > 80$, $25{\cdot}5 = 125 > 115$ and $29{\cdot}6 = 174 > 126$. The complements do not exist because the graphs themselves do not (Section 3). □

Comparison with condition N10 and with the undirected case. Theorem 4.3 follows the scheme of Theorem 3 of Hobart and Williford [16, p. 3]: if $\theta \neq 0$, then an independent set of the graph has at most m' vertices. Its proof takes an idempotent of rank m', which up to a factor is our matrix $\mathcal{K}_\theta$, and observes that its principal submatrix on an independent set is nonsingular. Corollary 4 [16, p. 3] applies the theorem to the set $\Gamma^+(x)$, which is independent when $\lambda = 0$, and gets $k \leq m'$; this is condition N10 of the table. Theorem 4.3 takes the matrix $A - \theta I$ of

rank $1+m'$ and the closed neighbourhood of $2k-t+1$ vertices. The neighbourhood is not independent; instead of independence the theorem uses the fact that for $\lambda=0$ the submatrix on it is determined by the parameters up to one block, and the determinant does not depend on this block. The bound grows from k to $2k-t$ under the condition $\theta^3-t\theta-\mu(k-t)\neq 0$, which Corollary 4 of [16] does not need. For the four pairs of Theorem 4.5 condition N10 holds: $11\le 12$, $13\le 16$, $16\le 23$ and $19\le 21$. For strongly regular graphs the columns of $A-rI$, after subtracting the mean, give a Euclidean representation in $\mathbb{R}^g$: the Gram matrix of any set of vertices is determined by the induced subgraph and has rank at most g [17, Proposition 2.1], and the nonexistence of $(76,30,8,14)$ [17] rests on this. Theorem 4.3 is built in the same way, except that the Gram matrix is replaced by the principal submatrix of $A-\theta I$ on the closed neighbourhood, and positive definiteness by a nonzero determinant.

On the table. Among the sets of the table [2] with $\lambda=0$, Corollary 4.4 excludes exactly 10: the four open sets of Theorem 4.5 and six that the table already marks as nonexistent; for these six even condition N10, $k\le g$, fails. None of the excluded sets has been constructed.

The bound is attained. For the graph $(6,2,1,0,1)$ with $\theta=-1$, $\theta'=0$ we have $2k-t=3=m'$. For the Clebsch graph, the undirected strongly regular graph $(16,5,0,2)$, which as a DSRG has parameters $(16,5,5,0,2)$, with $\theta=1$ we have $2k-t=5=g$; here $t=k$, but Theorem 4.3 holds for $t=k$ as well (Section 3). In both cases the submatrix on $N[x]$ is nonsingular and has full rank $1+m'$, so one cannot be added to the left-hand side of Theorem 4.3.

5. The Schur complement for $\lambda=0$ and the set $(48,9,5,0,2)$

Theorem 4.3 takes only one thing from Proposition 4.2: the submatrix on $N[x]$ is nonsingular. When the bound of the theorem holds, the same submatrix says more. By item 5 of Lemma 3.1 its Schur complement has rank $\tau=m'-(2k-t)$, the gap in the bound, and the rest of the matrix is determined by the neighbourhood up to a matrix of rank τ. For $(48,9,5,0,2)$ the gap equals 1, and the existence of the graph reduces to a finite search at one vertex.

In this section $\lambda=0$, and the root θ satisfies $\theta\neq 0$ and $\theta^3-t\theta-\mu(k-t)\neq 0$; x, M, Q, P are as in Section 4. Put $N(x)=N[x]\setminus\{x\}$, let $\overline{N}=V\setminus N[x]$ be the outer vertices, $\tau=m'-(2k-t)$, and

$$L=B_\theta[N[x],N[x]],\qquad X=A[N[x],\overline{N}],\qquad Y=A[\overline{N},N[x]],\qquad F=YL^{-1}X.$$

The matrix L is nonsingular by Proposition 4.2. The Schur complement of the block L in B_θ is the matrix $T=B_\theta[\overline{N},\overline{N}]-F$ on the outer vertices; we denote its diagonal by $\sigma_w=T(w,w)$.

Lemma 5.1. *Under these assumptions:*

1. *the vector $\hat{u}=(1+\theta/\mu)\,e_x+\mu^{-1}\mathbf{1}_{\Gamma^-}$ satisfies $B_\theta\hat{u}=\mathbf{1}$, and the vector $\check{u}=(1+\theta/\mu)\,e_x+\mu^{-1}\mathbf{1}_{\Gamma^+}$ satisfies $\check{u}^\top B_\theta=\mathbf{1}^\top$; hence $YL^{-1}\mathbf{1}=\mathbf{1}$ and $\mathbf{1}^\top L^{-1}X=\mathbf{1}^\top$;*
2. *$XY=\Phi$, where $\Phi=\mu J-L^2+qL$: for $n,n'\in N[x]$ exactly $\Phi(n,n')$ outer vertices w lie on a path $n\to w\to n'$;*
3. *$XT=0$, $TY=0$ and $T^2=qT$. Hence T/q is an idempotent of rank τ, and the trace of T equals τq.*

Proof. (1) The column x of the equality $A^2 = tI + \mu(J - I - A)$ gives $(A\mathbf{1}_{\Gamma^-})_y = t[y = x] + \mu(1 - [y = x] - A(y, x))$, and $Ae_x = \mathbf{1}_{\Gamma^-}$. For $y \neq x$ the coordinate y of the vector $B_\theta \hat{u}$ equals $(1 + \theta/\mu)A(y, x) + 1 - A(y, x) - (\theta/\mu)A(y, x) = 1$. For $y = x$ it equals $t/\mu - \theta(1 + \theta/\mu) = (t - \mu\theta - \theta^2)/\mu = 1$, because $\theta^2 + \mu\theta = t - \mu$. The support of $\hat{u}$ lies in $N[x]$, so the rows $N[x]$ give $L\hat{u}_{N[x]} = \mathbf{1}$, and the rows $\overline{N}$ give $Y\hat{u}_{N[x]} = \mathbf{1}$; hence $YL^{-1}\mathbf{1} = \mathbf{1}$. For $\check{u}$ the same argument runs along the row x.

(2) By item 1 of Lemma 3.1, $B_\theta^2 = qB_\theta + \mu J$. The block $(N[x], N[x])$ of this equality is $L^2 + XY = qL + \mu J$.

(3) Substitute $B_\theta[\overline{N}, \overline{N}] = F + T$ into the block $(N[x], \overline{N})$ of the same equality: $LX + X(F + T) = qX + \mu J$. By item 2, $XF = XYL^{-1}X = \mu JL^{-1}X - LX + qX$, and by item 1, $JL^{-1}X = J$. What remains is $XT = 0$; in the same way the block $(\overline{N}, N[x])$ gives $TY = 0$. In the block $(\overline{N}, \overline{N})$, that is, in $YX + (F + T)^2 = q(F + T) + \mu J$, the products FT and TF vanish, and $F^2 = YL^{-1}(XY)L^{-1}X = \mu J - YX + qF$ by items 1 and 2. What remains is $T^2 = qT$. The rank of T equals $1 + m' - (2k - t + 1) = \tau$ by item 5 of Lemma 3.1, and the trace of an idempotent equals its rank. □

For a given neighbourhood the matrix F depends only on how the outer vertices are attached to $N(x)$: $F(w, w') = Y(w, \cdot)\, L^{-1} X(\cdot, w')$, and there are no arcs between x and the outer vertices. We call the pair of sets $(\Gamma^+(w) \cap N(x), \Gamma^-(w) \cap N(x))$ the *type* of an outer vertex w. Then $F(w, w')$ is determined by the types of w and w', and by Lemma 5.1, $A[\overline{N}, \overline{N}] = \theta I + F + T$, where the rank of T equals τ. For $\tau = 0$ the block of A on the outer vertices is completely determined by the types; both graphs on which the bound of Theorem 4.3 is attained are of this kind.

Proposition 5.2. *Let $\tau \geq 1$. Then T has a nonsingular principal submatrix $T[R, R]$ of order τ, and $T = T[\cdot, R]\, T[R, R]^{-1}\, T[R, \cdot]$. In particular, for $\tau = 1$ there is an outer vertex w_0 with $\sigma_0 = \sigma_{w_0} \neq 0$, and for any outer vertices w, w'*

$$T(w, w') = \frac{T(w, w_0)\, T(w_0, w')}{\sigma_0}, \qquad \sigma_w\, \sigma_{w'} = T(w, w')\, T(w', w).$$

Therefore, for $w \neq w'$ the arc $A(w, w') = F(w, w') + T(w, w_0)\, T(w_0, w')/\sigma_0$ is determined by the types of the vertices w, w', w_0 and by the arcs between w_0 and the vertices w, w'.

Proof. The matrix T/q is an idempotent of rank τ; its eigenvalues are τ ones and zeros. The sum of the principal minors of T/q of order τ equals the τ-th elementary symmetric function of the eigenvalues, that is, 1, so one of the minors is nonzero. The formula is item 4 of Lemma 3.1 for the matrix T of rank τ instead of B_θ, with the same proof. For $\tau = 1$ we have $R = \{w_0\}$, and the second equality says that the minors of order 2 of a matrix of rank 1 vanish. Finally, for $w \neq w_0$ we have $T(w, w_0) = A(w, w_0) - F(w, w_0)$ and $T(w_0, w) = A(w_0, w) - F(w_0, w)$. □

We call the vertex w_0 the *pivot*.

Comparison with the undirected case. In the proof of Proposition 1 of Bondarenko and Radchenko [18, pp. 524–525] the vectors of the Euclidean representation on the neighbourhood of a vertex are linearly independent: their inner products are given by the subgraph on the neighbourhood. There are exactly g vectors, so they form a basis of $\mathbb{R}^g$, and the vector of each vertex outside the neighbourhood is expressed in this basis through its neighbours in the neighbourhood. This is the analogue of the case $\tau = 0$: the Schur complement is zero, and the

graph outside the neighbourhood is determined by the types of the vertices. The reconstruction theorem for star complements [19, Theorem 1.2] recovers a graph from an induced subgraph on $v - m$ vertices, where m is the multiplicity of an eigenvalue θ, provided the subgraph does not have the eigenvalue θ. The Schur complement of such a subgraph is zero as well: this is the case $|R| = 1 + m'$ of items 4 and 5 of Lemma 3.1. In this way Azarija and Marc excluded $(75, 32, 10, 16)$ [20]. For $(48, 9, 5, 0, 2)$ the submatrix on $N[x]$ is smaller: its Schur complement has rank $\tau = 1$, and the search runs over this margin.

The reduction for (48, 9, 5, 0, 2). For $\theta = 1$ the other root is $\theta' = -3$, $m' = g = 14$, and $|N[x]| = 2k - t + 1 = 14$, so $\tau = 1$ and the trace of T equals $\tau q = -4$. There are 34 outer vertices, and without the rank step the block of A on them consists of $34{\cdot}33 = 1\,122$ unknown arcs on top of the types. By Proposition 5.2 this block is determined by the types of the outer vertices and by the 66 arcs between the pivot and the other 33 outer vertices. Appendix A carries the reduction down to an exact cover problem. Lemma A.1 expresses F and σ_w through the types and proves that distinct outer vertices have distinct types. Proposition A.2 reduces the existence of the graph to a choice of 33 triples "type, arc from the pivot, arc to the pivot" for one of the 707 orbits of the pivot type.

Theorem 5.3. *There is no DSRG* $(48, 9, 5, 0, 2)$*, and none with the complementary parameters* $(48, 38, 34, 30, 30)$*.*

Proof. If the graph exists, Proposition A.2 gives a set of triples satisfying conditions (a)–(f) for one of the pivot orbits of one of the two graphs of arcs from Q to P, 707 orbits for the two graphs together. The search of Appendix A exhausted all of them and found no such set: the first program traversed the tree of every orbit to the end, and the formula of the second program is unsatisfiable for every orbit, with the unsatisfiability checked without trusting the solver (Section 9). The complement does not exist because the graph itself does not. □

6. The left kernel for $\lambda = 1$ and $t = 2\mu$

For $\lambda = 1$ the principal submatrix on $N[x]$ is no longer rigid: there are arcs inside $\Gamma^+(x)$, and the formula of Proposition 4.2 fails. Something else remains rigid: the left kernel of the rows of a vertex and its out-neighbours in the matrix $\mathcal{K}_1$ of item 6 of Lemma 3.1.

In this section $\lambda = 1$, $t = 2\mu$ and $0 < t < k < v - 1$. Then $x^2 + (\mu - 1)x - \mu = (x - 1)(x + \mu)$, so $r = 1$, $s = -\mu$, $\Delta = (\mu + 1)^2 > 0$, and the vanishing of the trace gives $(\mu + 1)g = k + v - 1$. The row sums of the defining equality give $k^2 = t + k + \mu(v - k - 1)$, that is,

$$\mu(v - k + 1) = k(k - 1). \tag{6.1}$$

By item 6 of Lemma 3.1 with $\theta = 1$, the matrix $\mathcal{K}_1 = v(A - I) - (k - 1)J$ has rank g. Fix a vertex x and put $\Gamma^+ = \Gamma^+(x)$, $M = \Gamma^+ \cap \Gamma^-(x)$, $S = \{x\} \cup \Gamma^+$ and $Z = V \setminus S$; the set Z has exactly $v - k - 1$ vertices.

Lemma 6.1. *Let* $\lambda = 1$*. Then:*

(1) *every vertex* $y \in \Gamma^+$ *has exactly one in-neighbour in* Γ^+*; we denote it by* $\mathrm{pred}(y)$*. The arcs inside* Γ^+ *are exactly the arcs* $\mathrm{pred}(y) \to y$*, and* $\mathrm{pred}(y) \neq y$*;*

(2) *every connected component* $\Omega_1, \dots, \Omega_c$ *of the graph of arcs inside* Γ^+ *has at least two vertices:* $n_i = |\Omega_i| \geq 2$*;*
(3) *let* $m_i = |M \cap \Omega_i|$*, and for* $z \in Z$ *let* $\ell_i(z) = |\Gamma^-(z) \cap \Omega_i|$ *be the number of arcs from* Ω_i *to* z*. Then* $\sum_i m_i = t$ *and* $\sum_{z \in Z} \ell_i(z) = (k-1)n_i - m_i$*.*

Proof. (1) The in-neighbours of y in Γ^+ are the middle vertices of the paths $x \to u \to y$, and there are $A^2(x,y) = \lambda = 1$ of them. There are no loops, so $\operatorname{pred}(y) \neq y$.

(2) A vertex y and its in-neighbour $\operatorname{pred}(y) \neq y$ lie in the same component.

(3) $\sum_i m_i = |M| = A^2(x,x) = t$. The vertices of Ω_i have kn_i out-neighbours in total: m_i of these arcs lead to x, n_i lead inside Γ^+, because every vertex of Γ^+ receives one arc from Γ^+, from its in-neighbour pred in the same component, and the remaining $(k-1)n_i - m_i$ lead to Z. □

Define the matrix D with c columns, one for each component, and rows of two kinds: the row $d = (km_i - tn_i)_i$ and the rows $h_z = (2\ell_i(z) - m_i)_i$ for all $z \in Z$. For $a \in \mathbb{R}^c$ put $w_a = \frac{1}{2}\bigl(\sum_i a_i m_i\bigr)e_x + \sum_i a_i \mathbf{1}_{\Omega_i}$.

Theorem 6.2. *Let* $\lambda = 1$*,* $t = 2\mu$ *and* $0 < t < k < v-1$*. If a vector* w *with support in* S *satisfies* $w^\top \mathcal{K}_1 = 0$*, then* $w = w_a$ *for some* $a \in \mathbb{R}^c$ *with* $Da = 0$*.*

Proof. Let $W = \sum_{y \in S} w_y$ and $\kappa = (k-1)W/v$. The column z of the equality $w^\top \mathcal{K}_1 = 0$, divided by v, reads $\sum_{y \in S} w_y A(y,z) - w_z[z \in S] = \kappa$. The in-neighbours of x inside S are the vertices of M; the in-neighbours of $z \in \Gamma^+$ inside S are x and $\operatorname{pred}(z)$; the in-neighbours of $z \in Z$ inside S lie in Γ^+, because there is no arc $x \to z$. Hence the columns give equalities of three kinds:

$$w(M) - w_x = \kappa,$$
$$w_z - w_{\operatorname{pred}(z)} = w_x - \kappa \quad (z \in \Gamma^+),$$
$$\sum_{y \in \Gamma^+} w_y A(y,z) = \kappa \quad (z \in Z).$$

The right-hand side of the second equality, $\delta = w_x - \kappa$, is the same for all $z \in \Gamma^+$. At a vertex where w attains its maximum on Γ^+ we get $\delta \geq 0$, and where it attains its minimum, $\delta \leq 0$. Hence $\delta = 0$: the vector w is constant along the arcs $\operatorname{pred}(z) \to z$, that is, it equals some a_i on each component Ω_i, and $w_x = \kappa$.

The first equality gives $\sum_i a_i m_i = 2w_x$. The equality $\kappa = w_x$ means $vw_x = (k-1)(w_x + \sum_i a_i n_i)$, that is, $(v-k+1)w_x = (k-1)\sum_i a_i n_i$, and together with (6.1) this gives $kw_x = \mu \sum_i a_i n_i$. Substituting $w_x = \frac{1}{2}\sum_i a_i m_i$ and $t = 2\mu$, we get $d \cdot a = 0$. The third equality becomes $\sum_i a_i \ell_i(z) = \frac{1}{2}\sum_i a_i m_i$, that is, $h_z \cdot a = 0$. □

Corollary 6.3. *Under the assumptions of Theorem 6.2:*

(1) $\dim \ker D = c - \operatorname{rank} D \geq k + 1 - g$*;*
(2) $c \geq k + 1 - g + [d \neq 0]$*, and* $d = 0$ *exactly when* $m_i/n_i = t/k$ *for all* i*.*

Proof. (1) The rows of $\mathcal{K}_1$ indexed by S are $k+1$ rows of a matrix of rank g, and by Corollary 3.2 their left kernel has dimension at least $k+1-g$. The map $a \mapsto w_a$ is linear and injective, because w_a on Γ^+ recovers a, and by Theorem 6.2 its image of $\ker D$ contains this kernel.

(2) The row d belongs to D, so $\operatorname{rank} D \geq [d \neq 0]$, and item 1 gives the inequality. The equality $d = 0$ means $km_i = tn_i$ for all i. □

Comparison with Theorem 6 of [16]**.** Theorem 6 of Hobart and Williford [16, p. 4]: if the multiplicity of θ is less than k, then θ' is an eigenvalue of the submatrix $A[\Gamma^+(x), \Gamma^+(x)]$ with geometric multiplicity at least k minus the multiplicity of θ. In the family $\lambda = 1$, $t = 2\mu$ this submatrix is the matrix of the arcs $\operatorname{pred}(y) \to y$, and for $\theta = s$ the theorem gives $c \geq k - g$, while Corollary 6.3 gives $c \geq k + 1 - g + [d \neq 0]$. For the pair $(55, 16, 12, 1, 6)$ the first bound gives $c \geq 6$, which is compatible with $c \leq 8$, and the second one gives $c = 8$. The main difference lies elsewhere: [16] has no equations of the rows h_z on the numbers of arcs from the components to the outer vertices, and Theorem 6.4 rests on them.

Theorem 6.4. *There is no DSRG* $(55, 16, 12, 1, 6)$*, and none with the complementary parameters* $(55, 38, 34, 27, 24)$*.*

Proof. Here $t = 12 = 2\mu$, $g = 70/7 = 10$ and $k + 1 - g = 7$. There are at most eight components, because $|\Gamma^+| = 16$ and $n_i \geq 2$. The equality $d = 0$ requires $m_i/n_i = 3/4$, that is, $4 \mid n_i$ and $c \leq 4 < 7$, contrary to item 2 of Corollary 6.3. Hence $d \neq 0$ and $c \geq 8$, that is, $c = 8$ and all $n_i = 2$: every component is a reciprocal pair $y \leftrightarrow y'$ with $\operatorname{pred}(y) = y'$ and $\operatorname{pred}(y') = y$. A pair contains $m_i \in \{2, 1, 0\}$ vertices of M, and $d_i = 16m_i - 24$, that is, 8, -8 and -24.

The row $d \neq 0$ belongs to D, so $\ker D \subseteq \ker d$, while $\dim \ker d = c - 1 = 7$. By item 1 of Corollary 6.3, $\dim \ker D \geq 7$, hence $\ker D = \ker d$: every vector $a \in \mathbb{R}^8$ with $d \cdot a = 0$ satisfies $h_z \cdot a = 0$ for all $z \in Z$. The sum $\sum_i m_i = t = 12$ is greater than the number of pairs and less than twice that number, so there is a pair i with $m_i = 2$ and a pair j with $m_j \leq 1$. Let e_i be the unit vector of the pair i.

- If there is a pair j with $m_j = 1$, take $a = e_i + e_j$. Then $d \cdot a = 8 - 8 = 0$, while the number $h_z \cdot a = 2\big(\ell_i(z) + \ell_j(z)\big) - 3$ is odd, although it must be zero for any of the 38 vertices of Z.
- Otherwise there is a pair j with $m_j = 0$; take $a = e_j + 3e_i$. Then $d \cdot a = -24 + 24 = 0$ and $h_z \cdot a = 2\ell_j(z) + 3\big(2\ell_i(z) - 2\big) = 0$, that is, $\ell_j(z) = 3 - 3\ell_i(z)$ is divisible by 3. Since $0 \leq \ell_j(z) \leq n_j = 2$, all $\ell_j(z) = 0$, whereas by item 3 of Lemma 6.1 their sum over Z equals $(k - 1) \cdot 2 = 30$.

Both cases are impossible. The complement does not exist because the graph itself does not. □

7. Twin classes for $t = \lambda + 1$

In this section $t = \lambda + 1$, and by item 1 of Lemma 3.1 with $\theta = -1$ the matrix $B = A + I$ consists of zeros and ones. Its row x is the indicator vector of $N^+[x]$, and its column y is the indicator vector of $N^-[y]$. Put $K = k + 1$ and $q = \lambda + 2 - \mu$; since $\mu \leq t$ [12, Lemma 2.1], $q = t + 1 - \mu \geq 1$; the other root equals $q - 1$, so q is the same number $\theta' - \theta$ as in Section 3, and we denote the multiplicity of the other root by f. Vertices $x \neq y$ with $N^+[x] = N^+[y]$ are called *row twins*, and those with $N^-[x] = N^-[y]$ *column twins*. The class C_x consists of the vertex x together with its row twins, and the class C'_x of the same vertex together with its column twins; $a = |C_x|$, $a' = |C'_x|$. Equal rows decrease the rank, so the classes are bounded from below by the rank of B, and from above by counting paths of length 2.

Lemma 7.1. *Let* $t = \lambda + 1$ *and* $q > 0$*. Then:*

(1) $B^2 = qB + \mu J$*; for any vertices* x*,* y *the number* $|N^+[x] \cap N^-[y]|$ *equals* $\lambda + 2$ *if* $y \in N^+[x]$*, and* μ *otherwise;* $K^2 = qK + \mu v$*;*

(2) $\operatorname{rank} B = 1 + f$ *and* $qf = v - K$*;*
(3) *if* $\mu = 1$*, then* $v = K(k - \lambda)$ *and* $(\lambda + 1)f = v - K$*; in particular,* $f = K$ *if and only if* $k = 2\lambda + 2$.

Proof. (1) This is item 1 of Lemma 3.1 with $\theta = -1$: here $\theta' = q - 1$ and $\theta' - \theta = q$. The entry (x, y) of the matrix B^2 equals $|N^+[x] \cap N^-[y]|$, and the entry of the right-hand side equals $q[y \in N^+[x]] + \mu$. The row sums give $K^2 = qK + \mu v$.

(2) The rank is given by item 2 of Lemma 3.1. The formula for m' from there gives $qf = -k + (v - 1) = v - K$.

(3) For $\mu = 1$ item 1 gives $v = K^2 - qK = K(k - \lambda)$, and item 2 gives $(\lambda + 1)f = v - K = K(k - \lambda - 1)$. Hence $f = K$ exactly when $k - \lambda - 1 = \lambda + 1$. □

Lemma 7.2. *We have* $C_x \subseteq N^+[x] \cap N^-[x]$*, the set* $U = N^+[x] \setminus C_x$ *is nonempty,* $a \le \lambda + 1$*, and* $C_x \subseteq N^-[z]$ *for every* $z \in N^+[x]$.

Proof. If $y \in C_x$, then $y \in N^+[y] = N^+[x]$ and $x \in N^+[x] = N^+[y]$, that is, $y \in N^-[x]$. If $z \in N^+[x]$, then $z \in N^+[y]$ for every $y \in C_x$, that is, $C_x \subseteq N^-[z]$. If U were empty, we would get $N^+[x] \subseteq N^-[x]$ and $K = |N^+[x] \cap N^-[x]| = \lambda + 2 = t + 1$ by Lemma 7.1, that is, $k = t$. For $z \in U$ the set $C_x \cup \{z\}$ lies in $N^+[x] \cap N^-[z]$, which has exactly $\lambda + 2$ vertices. □

The rows of $A + I$ of two vertices are equal exactly when their rows in the complement matrix $J - I - A$ are equal, so a row class is a class of vertices with the same out-neighbourhood in the complement. The bound $a \le \lambda + 1$ is the bound of [21, Theorem 10(a)] on such classes, applied to the complement. For $t = \lambda + 1$ these classes were considered by Godsil, Hobart and Martin [12, Section 5] as classes of equal vectors of their representation for the root r; these are exactly the classes of equal rows of $A + I$ (the other root here is $s = -1$). They observe that a class is a clique and that all its vertices have the same out-neighbours outside the class (these are the inclusions of Lemma 7.2), and show that there are at least $f + 1$ and at most 2^f classes [12, Theorem 5.1]. This theorem bounds from below the number of classes, not the size of the class of a vertex.

Proposition 7.3 (rank against the class)**.** *Let* $\mu = 1$*. The rows of* B *indexed by* $\{x\} \cup (N^+[x] \setminus C_x)$ *are linearly independent. Hence* $K + 1 - a \le 1 + f$*, that is,* $a \ge K - f$.

Proof. Let $O = \Gamma^+(x)$; then $C_x \setminus \{x\} \subseteq O$. Every $o \in O \setminus C_x$ has a vertex $w_o \in N^+[o] \setminus N^+[x]$. By Lemma 7.1, $|N^+[x] \cap N^-[w_o]| = \mu = 1$, and this single vertex is o. Hence among the rows $\{x\} \cup (O \setminus C_x)$ the column w_o has a one only in the row o, and all w_o are distinct. In the column x the row x has a one, and in the columns w_o it has zeros. The submatrix of B on these rows and the columns $\{x\} \cup \{w_o\}$ is, after a permutation, triangular with ones on the diagonal. It has $1 + k - (a - 1) = K + 1 - a$ rows, and item 3 of Lemma 3.1 gives the inequality. □

Theorem 6 of Jørgensen [13] is the same link between the rank and coinciding columns of a binary matrix, taken for the whole matrix. If a matrix of zeros and ones of order n with k ones in each row and each column has rank ρ, then $k/n \ge 1/\rho$: the all-ones vector is a combination of ρ independent columns, so every row of the submatrix formed by these columns contains a one. The remark after the proof adds that for $k/n = 1/\rho$ every column of the matrix coincides with one of these ρ columns. For $B = A + I$ the theorem gives $K(1 + f) \ge v$, and for $t = \lambda + 1$ this

inequality is always strict, so the case of equality, where coinciding columns appear, never occurs here: equality together with $qf = v - K$ gives $(v - K)(K - q) = 0$, while $v > K > q$. Proposition 7.3 works at one vertex: the rows of $N^+[x]$ outside the class C_x together with the row x form a triangular minor, and the rank bounds from below the class of every vertex, not the ratio k/n of the whole matrix.

For $u \notin C_x$ put $\gamma_u = |N^+[u] \cap C_x|$, the number of vertices of the class that u sees in its closed out-neighbourhood.

Proposition 7.4. *If $u \notin C_x$, then $\gamma_u \le \mu$. Moreover, $\sum_{u \in U} \gamma_u = a(\lambda + 2 - a)$, and therefore $a(\lambda + 2 - a) \le \mu(K - a)$; for $\mu = 1$ this inequality is equivalent to $a(\lambda + 3 - a) \le K$.*

Proof. The rows u and x of the matrix B are distinct and contain K ones each, so there is $w \in N^+[x] \setminus N^+[u]$. By Lemma 7.2, $C_x \subseteq N^-[w]$, hence $N^+[u] \cap C_x \subseteq N^+[u] \cap N^-[w]$, and by Lemma 7.1 the latter set has exactly μ vertices. Count the pairs (y, u) with $y \in C_x$, $u \in U$ and $y \in N^+[u]$. For $y \in C_x$ the set $N^+[x] \cap N^-[y]$ consists of $\lambda + 2$ vertices and by Lemma 7.2 contains the whole class, so the other $\lambda + 2 - a$ vertices lie in U. Counting by u, the number of pairs is $\sum_{u \in U} \gamma_u \le \mu|U| = \mu(K - a)$. □

Proposition 7.5. *For any μ:*

1. *if $u \notin C_x$, then $|N^+[u] \setminus N^+[x]| \ge q$; hence $q(K - a) \le \mu(v - K)$, that is, $a \ge K - \mu f$;*
2. *$a \le K - q$.*

Proof. (1) There is $w \in N^+[u] \setminus N^+[x]$. By Lemma 7.1, $N^+[u] \cap N^-[w]$ has exactly $\lambda + 2$ vertices, and $N^+[x] \cap N^-[w]$ exactly μ, so at least q vertices of the first set lie outside $N^+[x]$. Hence there are at least $q|U|$ pairs (u, w) with $u \in U$ and $w \in N^+[u] \setminus N^+[x]$. On the other hand, every $w \notin N^+[x]$ receives exactly μ arcs from $N^+[x]$, and none of them comes from the class, since otherwise $w \in N^+[x]$; so there are at most $\mu(v - K) = q\mu f$ pairs. Divide by $q > 0$.

(2) The set $N^+[x] \cap N^-[x]$ has exactly $\lambda + 2$ vertices, the class occupies at most $\lambda + 1$ of them, and any remaining vertex u lies in U and has an arc to x, so $\gamma_u \ge 1$. Take $y \in C_x \cap N^+[u]$. The vertex y lies in $N^+[u]$ and in $N^+[x]$, so $|N^+[u] \cap N^-[y]| = |N^+[x] \cap N^-[y]| = \lambda + 2$. Remove the common part from both sets: $|(N^+[u] \setminus N^+[x]) \cap N^-[y]| = |(N^+[x] \setminus N^+[u]) \cap N^-[y]|$. The set on the right contains $C_x \setminus N^+[u]$, so the right-hand side is at least $a - \gamma_u$; the set on the left lies in $N^-[y] \setminus N^+[x]$, which has $K - (\lambda + 2) = k - \lambda - 1$ vertices. With $\gamma_u \le \mu$ we get $a \le k - \lambda - 1 + \mu = K - q$. □

For $\mu = 1$ the lower bound of item 1 coincides with the rank bound of Proposition 7.3, and for any μ it serves as its counting analogue. The bound $a \le K - q$ of item 2 is a special case of Lemma 9 of [3]: for $B = A + I$ with $t = \lambda + 1$ its hypotheses hold; for completeness we prove this bound briefly by another route. For order 28 the bound $a \le 3$ is Lemma 2 of [4]. We did not find the lower bound on the class at every vertex (Propositions 7.3 and 7.5) and inequality (7.1) below in [3], [11], [12], [13] and [16], nor the structure of a vertex without a twin (Theorem 7.8) in these works or in [4] and [21]. Together the two bounds give a condition on the parameters.

Theorem 7.6. *If a DSRG with $t = \lambda + 1$ and $\mu < \lambda + 2$ exists, then*

$$(\lambda + 2 - \mu)^2 \le \mu(v - k - 1). \tag{7.1}$$

In particular, there are no DSRGs $(96, 15, 10, 9, 1)$, $(40, 9, 6, 5, 1)$, $(80, 19, 13, 12, 2)$, *and none with the complementary parameters* $(96, 80, 75, 65, 75)$, $(40, 30, 27, 21, 27)$, $(80, 60, 54, 42, 54)$.

Proof. At any vertex Proposition 7.5 gives $K - \mu f \le a \le K - q$, whence $q \le \mu f$; multiplying by $q > 0$, we get $q^2 \le \mu q f = \mu(v - K)$. For $(96, 15, 10, 9, 1)$ we have $q = 10 > 8 = \mu f$, for $(40, 9, 6, 5, 1)$ we have $q = 6 > 5$, and for $(80, 19, 13, 12, 2)$ we have $q = 12 > 10$. □

For $(96, 15, 10, 9, 1)$ and $(40, 9, 6, 5, 1)$ we have $\mu = 1$, and the lower bound $a \ge K - f$ is also given by the rank, Proposition 7.3. This is how the Lean theorems exclude these sets (Table 2): for $(96, 15, 10, 9, 1)$ the rank gives $a \ge 8$, and Lemma 7.2 and Proposition 7.4 give $a \le 10$ and $a(12 - a) \le 16$, which fails for $a = 8, 9, 10$. On both routes the upper bound comes from counting, and the proof of Theorem 7.6 takes the lower one from counting as well (Proposition 7.5). The pairs $(40, 9, 6, 5, 1)$ and $(80, 19, 13, 12, 2)$ were excluded by [6] through a search over hyperplanes of rank 6 with Farkas certificates, together with the line $(40m, 30m, 27m, 21m, 27m)$ and the line of its complements $(40m, 10m - 1, 7m - 1, 7m - 2, m)$; Theorem 7.6 gives them a two-line proof.

The clique blow-up maps a DSRG with $t = \lambda + 1$ to the graph with matrix $(A + I) \otimes J_m - I$ and parameters $(mv, mK - 1, m(\lambda + 2) - 1, m(\lambda + 2) - 2, m\mu)$: the matrix $B' = B \otimes J_m$ has unit diagonal and $B'^2 = mqB' + m\mu J$. Here K, q, μ and v are multiplied by m, while f does not change, so inequality (7.1) is the same at all points of the line of blow-ups as at its start. Among the sets of the table [2] with $t = \lambda + 1$, inequality (7.1) excludes exactly six: the open set $(96, 15, 10, 9, 1)$ and the sets $(40, 9, 6, 5, 1)$, $(48, 17, 12, 11, 3)$, $(80, 19, 13, 12, 2)$, $(80, 23, 16, 15, 3)$ and $(96, 35, 25, 24, 6)$, which are marked as nonexistent. Equality in (7.1) holds for exactly thirteen sets of the table, $(8m, 4m - 1, 3m - 1, 3m - 2, m)$ for $m = 1, \dots, 13$, and all of them have been constructed. For these sets $f = 2$, and Proposition 7.5 gives $a = 2m$ at every vertex; Godsil, Hobart and Martin described such graphs completely [12, Theorem 5.3] (they have exactly four classes of $2m$ vertices each), and Jørgensen independently proved the uniqueness of their complements $(8m, 4m, 3m, m, 3m)$ [11, Theorem 4].

Classes for $\mu = 1$. From now on $\mu = 1$, so $q = \lambda + 1$ and $B^2 = (\lambda + 1)B + J$.

Lemma 7.7. *Let $\mu = 1$. Then:*

1. *$N^+[x]$ is a union of column classes, and $N^-[y]$ is a union of row classes;*
2. *$|C_x \cap C'_x| = 1$: two distinct vertices are never twins both by row and by column;*
3. *$a + a' \le \lambda + 3$.*

Proof. (1) Let $y \in N^+[x]$ and let y' be a column twin of y. Then $x \in N^-[y] = N^-[y']$, that is, $y' \in N^+[x]$. The second half is symmetric.

(2) Let $z_0 \ne z_1$ be twins both by row and by column. For any $x \in N^-[z_0]$ and $y \in N^+[z_0]$ both vertices z_0, z_1 lie in $N^+[x] \cap N^-[y]$, and by Lemma 7.1 such an intersection has two or more vertices only when $y \in N^+[x]$. Hence $N^+[z_0] \subseteq N^+[x]$, and since the sizes are equal, $N^+[x] = N^+[z_0]$, so every $x \in N^-[z_0]$ lies in C_{z_0}.

Then $N^-[z_0] \subseteq C_{z_0} \subseteq N^+[z_0]$ by Lemma 7.2 and $|N^+[z_0] \cap N^-[z_0]| = K$, whereas by Lemma 7.1 this number is $\lambda + 2 = t + 1 < K$.

(3) By Lemma 7.2 for A and for $A^\top$, the classes C_x and C'_x lie in $N^+[x] \cap N^-[x]$, which has $\lambda + 2$ vertices, and by item 2 they meet only in x. □

Theorem 7.8 (vertex without a twin)**.** *Let $\mu = 1$, $t = \lambda + 1$ and $k = 2\lambda + 2$, that is, $f = K$. If a vertex x has no row twin, then $\lambda \le 2$. Hence for $\lambda \ge 3$ every vertex has both a row twin and a column twin.*

Proof. Let $a = 1$, $O = \Gamma^+(x)$, $S = N^+[x]$ and $W = V \setminus S$. By Lemma 7.1, $|W| = v - K = K(k - \lambda - 1) = K(\lambda + 1)$, and $\operatorname{rank} B = 1 + f = K + 1$.

Blocks. By Lemma 7.1 every $w \in W$ has exactly one in-neighbour in S, and it is not x. Hence W splits into the blocks $W_o = \Gamma^+(o) \cap W$, $o \in O$; we call the vertex o the root of the block. An empty block W_o would give $N^+[o] \subseteq S$, that is, $N^+[o] = S$, while x has no twins; hence the blocks are nonempty. Apply Lemma 7.1 to the root o. (i) If $w \in W_o$, then $N^+[o] \cap N^-[w]$ consists of o, of w itself and of the vertices of W_o that have an arc to w, and it has $\lambda + 2$ vertices; hence $|W_o \cap N^-[w]| = \lambda + 1$, that is, the column w of B has $\lambda + 1$ ones on W_o. (ii) If $w \in W_{o'}$ with $o' \ne o$, then $w \notin N^+[o]$, and at most $|N^+[o] \cap N^-[w]| = 1$ vertex of W_o has an arc to w.

Minors. For a vertex w and vertices y, y' of one block put $\delta_w(y, y') = B(w, y) - B(w, y')$. Then for any $w, w' \in W$, vertices y, y' of the block W_o and vertices z, z' of the block $W_{o'}$

$$\delta_w(y, y')\, \delta_{w'}(z, z') = \delta_{w'}(y, y')\, \delta_w(z, z'). \tag{7.2}$$

For $w = w'$ this is obvious; let $w \ne w'$. There are $K+2$ rows indexed by $S \cup \{w, w'\}$, while the rank of B is $K+1$, so there is a nonzero vector u with support in $S \cup \{w, w'\}$ and $u^\top B = 0$. In the column $y \in W_o$, among the rows S only the row of the root o has a one, so $u_o + B(w, y)u_w + B(w', y)u_{w'} = 0$. Subtract the same equality for y': $\delta_w(y, y')u_w + \delta_{w'}(y, y')u_{w'} = 0$; likewise for z, z'. If $u_w = u_{w'} = 0$, the column of any vertex of the block W_o would give $u_o = 0$ for every $o \in O$, and the column x would give $u_x = 0$, that is, $u = 0$. Hence the system of two equations has a nonzero solution $(u_w, u_{w'})$, and its determinant, the difference of the two sides of (7.2), is zero.

The pivot vertex. Suppose that the row of B of every vertex of W is constant on every block. Then a vertex $w \in W_o$ has arcs to all vertices of W_o, because $B(w, w) = 1$; the block W_o is a reciprocal clique, $|W_o| = \lambda + 1$ by (i), and $|W| = k(\lambda + 1)$, contrary to $|W| = K(\lambda+1)$. Hence there are a vertex $w_0 \in W$ and vertices y_0, y'_0 of one block with $B(w_0, y_0) = 1$ and $B(w_0, y'_0) = 0$. Put $\varepsilon_w = \delta_w(y_0, y'_0) \in \{-1, 0, 1\}$. Equality (7.2) with $w' = w_0$ gives, for any $w \in W$ and vertices y, y' of one block,

$$\delta_w(y, y') = \varepsilon_w\, \delta_{w_0}(y, y'). \tag{7.3}$$

If $\varepsilon_w = 0$, the row w is constant on every block, and on its own block it equals $B(w, w) = 1$. On every block where the row w_0 is not constant, it takes the values 0 and 1, and the difference of the rows w and $\varepsilon_w w_0$ is constant by (7.3). Hence for $\varepsilon_w = 1$ the row w coincides on such a block with the row w_0, and for $\varepsilon_w = -1$ it complements it to $\mathbf{1}$.

Special blocks. A block is called special if the row w_0 is not constant on it. On a non-special block W_o the row of every vertex is constant by (7.3), so, as above, W_o is a reciprocal clique and $|W_o| = \lambda + 1$. In a special block W_o choose y, y' with

$B(w_0, y) = 1$ and $B(w_0, y') = 0$; then $\delta_w(y, y') = \varepsilon_w$. The vertices $w \in W_o$ with $\varepsilon_w = 1$ have $B(w, y) = 1$ and $B(w, y') = 0$, those with $\varepsilon_w = -1$ the other way round, and those with $\varepsilon_w = 0$ have ones in both columns. By (i), the columns y and y' of B have $\lambda + 1$ ones each on W_o, so W_o has as many vertices with $\varepsilon_w = 1$ as with $\varepsilon_w = -1$; denote this number by p_o. Then $|W_o| = \lambda + 1 + p_o$. Add up the sizes of all k blocks: $|W| = k(\lambda + 1) + \sum p_o$, where the sum runs over the special blocks, while $|W| = K(\lambda + 1)$. Hence $\sum p_o = \lambda + 1$: there are special blocks, and W contains vertices with $\varepsilon_w = 1$ and with $\varepsilon_w = -1$.

Two or more special blocks. Let W_o and $W_{o'}$ be distinct special blocks, $z \in W_{o'}$ and $B(w_0, z) = 1$. Every vertex of W_o with $\varepsilon_w = 1$ has an arc to z, while by (ii) at most one vertex of W_o has an arc to z. Hence all $p_o \leq 1$, there are at least $\lambda + 1$ special blocks, and together they contain at least $(\lambda + 1)(\lambda + 2)$ vertices. Take w with $\varepsilon_w = 1$ and w' with $\varepsilon_{w'} = -1$: every vertex of a special block lies in $N^+[w]$ or in $N^+[w']$. Hence $(\lambda + 1)(\lambda + 2) \leq 2K = 2(2\lambda + 3)$, that is, $\lambda^2 - \lambda - 4 \leq 0$ and $\lambda \leq 2$.

One special block W_{o_1}. Then $p_{o_1} = \lambda + 1$ and $|W_{o_1}| = 2\lambda + 2 = k$, so $\Gamma^+(o_1) = W_{o_1}$; it has no vertices with $\varepsilon_w = 0$, and $\lambda + 1$ vertices with $\varepsilon_w = 1$ and as many with $\varepsilon_w = -1$. By (7.3), on W_{o_1} the row of a vertex with $\varepsilon_w = 1$ coincides with the row w_0, and the row of a vertex with $\varepsilon_w = -1$ complements it to $\mathbf{1}$; each vertex lies in its own closed out-neighbourhood, so the former have arcs exactly to the vertices with $\varepsilon = 1$, and the latter exactly to the vertices with $\varepsilon = -1$. Hence $\Gamma^+(o_1)$ consists of two reciprocal cliques of $\lambda+1$ vertices each, with no arcs between them. The vertex o_1 has no row twin: such a twin would lie in $\Gamma^+(o_1)$ and have arcs to all its other vertices. For $\lambda \geq 3$ apply what has been proved to o_1 instead of x: two special blocks are impossible, so o_1 has one special block, and its root $o_2 \in \Gamma^+(o_1)$ has all its out-neighbours outside $N^+[o_1]$. But o_2 lies in a clique of $\lambda + 1 \geq 2$ vertices inside $\Gamma^+(o_1)$, a contradiction. A column twin is a row twin in $A^\top$. □

Theorem 7.9. *There is no DSRG with $\mu = 1$, $t = \lambda + 1$, $k = 2\lambda + 2$ and $\lambda \geq 4$.*

Proof. By Theorem 7.8, $a, a' \geq 2$ at every vertex. For $3 \leq a \leq \lambda$ we have $a(\lambda + 3 - a) \geq 3\lambda > 2\lambda + 3 = K$, so by Proposition 7.4 and Lemma 7.2, $a, a' \in \{2, \lambda + 1\}$, and by item 3 of Lemma 7.7 they are not both equal to $\lambda + 1$.

Let $a = \lambda + 1$ for the class C_x. Then for every $y \in C_x$ the column class is a pair $C'_y = \{y, y'\}$, where $y' \in N^+[x]$ by item 1 of Lemma 7.7 and $y' \notin C_x$ by item 2. The vertices y' for different y are distinct: if $y'_1 = y'_2$, then $C'_{y_1} = C'_{y_2} \ni y_2$, and y_2 would be a twin of y_1 both by row and by column. The set $N^+[x] \setminus C_x$ has $K - a = \lambda + 2$ vertices, and one vertex u^* belongs to none of the classes C'_y. By item 1 of Lemma 7.7, $C'_{u^*} \subseteq N^+[x]$, and it is disjoint from the classes C'_y, so $C'_{u^*} = \{u^*\}$, contrary to $a' \geq 2$. Hence all row classes are pairs, and, by $A^\top$, so are all column classes. By item 1 of Lemma 7.7 the set $N^+[x]$ is a union of disjoint column pairs, so the number of its vertices is even; but it equals $2\lambda + 3$. □

Corollary 7.10. *There are no DSRGs $(66, 10, 5, 4, 1)$ and $(91, 12, 6, 5, 1)$, and none with the complementary parameters $(66, 55, 50, 45, 50)$ and $(91, 78, 72, 66, 72)$.*

Proof. For $(66, 10, 5, 4, 1)$ we have $k = 10 = 2\lambda + 2$ with $\lambda = 4$, and for $(91, 12, 6, 5, 1)$, $k = 12 = 2\lambda + 2$ with $\lambda = 5$. □

For these two sets inequality (7.1) gives nothing: $q = 5 < \mu f = 11$ and $q = 6 < 13$. The threshold $\lambda \geq 4$ of Theorem 7.9 cannot be lowered: the set $(45, 8, 4, 3, 1)$ with $k = 2\lambda + 2$ and $\lambda = 3$ has been constructed [2].

The set (28, 6, 3, 2, 1). Here $k = 6 = 2\lambda + 2$ with $\lambda = 2$, $K = 7$, $f = 7$ and $B^2 = 3B + J$. Theorem 7.8 does not exclude a vertex without a row twin: with three special blocks the inequality of its proof holds, $(\lambda + 1)(\lambda + 2) = 12 \leq 14 = 2K$. A vertex without a row twin is called *row-free*, and one without a column twin *column-free*.

Lemma 7.11. *Let $\mu = 1$, $t = \lambda + 1$ and $U = N^+[x] \setminus C_x$.*

1. *If $a = \lambda + 1$, then there are no arcs inside U, the column twins of the vertices of U lie in C_x, and at least $K - 2(\lambda + 1)$ vertices of U are column-free.*
2. *If there are no free vertices and $K > 2(\lambda + 1)$, then $a \leq \lambda$ and $a' \leq \lambda$ at every vertex. For $\lambda = 2$ all row classes and column classes are pairs, and K is even.*

Proof. (1) Every $y \in C_x$ has arcs to all vertices of $N^+[x] \setminus \{y\}$, so $C_x \subseteq N^-[u]$ for every $u \in U$. By Lemma 7.1, $N^+[x] \cap N^-[u]$ has exactly $\lambda + 2$ vertices; these are the $\lambda + 1$ vertices of C_x and u itself, so there are no arcs from U to u. By item 1 of Lemma 7.7, $C'_u \subseteq N^+[x]$. If a column twin $u' \neq u$ lay in U, then $u \in N^-[u] = N^-[u']$ would give an arc $u \to u'$ inside U; hence $C'_u \setminus \{u\} \subseteq C_x$. Two vertices of U cannot both be column twins of one $y \in C_x$, since otherwise they would be column twins of each other. Hence at most $\lambda + 1$ vertices of U have $|C'_u| \geq 2$, and at least $|U| - (\lambda + 1) = K - 2(\lambda + 1)$ are column-free.

(2) By Lemma 7.2, $a \leq \lambda + 1$. If $a = \lambda + 1$, item 1 would give column-free vertices. Hence $a \leq \lambda$, and by $A^\top$ also $a' \leq \lambda$. For $\lambda = 2$, together with $a, a' \geq 2$ this gives $a = a' = 2$, and by item 1 of Lemma 7.7, $N^+[x]$ is a union of column pairs, so K is even. □

Theorem 7.12. *There is no DSRG $(28, 6, 3, 2, 1)$, and none with the complementary parameters $(28, 21, 18, 15, 18)$.*

Proof. Let A be the matrix of a DSRG $(28, 6, 3, 2, 1)$. We consider three cases.

There is a row-free vertex x. Renumber the vertices: x gets the number 0, its three reciprocal neighbours the numbers 1, 2, 3, its other three out-neighbours 4, 5, 6, the three vertices that have arcs to x but receive no arc from x the numbers 7, 8, 9, and the other eighteen vertices the numbers 10–27. Such a graph satisfies a SAT formula whose variables $a(i, j)$ with $i \neq j$ are the arcs of the graph, and whose clauses require the following: every vertex has exactly 6 out-neighbours and 6 in-neighbours; there are exactly 3 paths $i \to z \to j$ for $i = j$, exactly 2 when there is an arc $i \to j$, and exactly 1 when there is none; the arcs of vertex 0 are placed as the numbering prescribes; the row of B of vertex 0 differs from the row of any other vertex. A path $i \to z \to j$ is an auxiliary variable equivalent to $a(i, z) \wedge a(z, j)$, and the conditions "exactly n ones" are encoded by the sequential counter [23]. The formula is unsatisfiable, and this is proved in Lean: the generator of the formula is written in Lean, a Lean theorem proves that a graph with a row-free vertex, after such a renumbering, satisfies each of its clauses, and the proof of unsatisfiability found by the solver CaDiCaL [24] is checked by the Lean kernel (Section 9). Hence there is no row-free vertex.

There is a column-free vertex. It is row-free in $A^\top$, the matrix of a DSRG with the same parameters, and this case has been excluded.

There are no free vertices. Here $K = 7 > 6 = 2(\lambda + 1)$, and by item 2 of Lemma 7.11 the number K is even, a contradiction. □

The rank plays no part in the proof of Theorem 7.12: the free vertex is excluded by a formula written directly from the definition of a DSRG, and the other cases by counting twin classes. We have no argument for this case without a solver.

Earlier proofs for two pairs. Theorem 10 of [3] excludes the line $(96m, 80m, 75m, 65m, 75m)$ by a search over the normals of a nonnegative basis of columns of rank 9, and with it the pair $(96, 15, 10, 9, 1)$. The paper [4] excludes the line $(28m, 21m, 18m, 15m, 18m)$ for odd m, and for $m = 1$ the pair $(28, 6, 3, 2, 1)$, by a search over the normals of a nonnegative basis of eight columns of $A + I$, finished by small exact searches and a SAT formula with a DRAT proof. We did not repeat the computational steps of [3] and [4], and our proofs do not rely on them: for $(96, 15, 10, 9, 1)$ the proof is the two bounds on a twin class, and for $(28, 6, 3, 2, 1)$ a solver is needed only for a graph with a free vertex.

8. The limits of the method

The method is as strong as the piece that the parameters force at a vertex: the larger and the more rigid the piece, the more it can exclude. This section shows where there is no such piece and the method is silent.

The out-neighbourhood gives only a known condition. The most general forced piece is available for any λ: the closed out-neighbourhood.

Proposition 8.1. *Let $\mu \geq 1$, let x be a vertex, $O = \Gamma^+(x)$, $S = N^+[x]$ and $A_O = A[O, O]$. Then:*

1. *$\mathbf{1}^\top A_O = \lambda \mathbf{1}^\top$, and all eigenvalues of A_O have absolute value at most λ;*
2. *for each root θ we have $(\theta - \lambda) \det B_\theta[S, S] = \mu(\theta + 1) \det(A_O - \theta I)$;*
3. *if $|\theta| > \lambda$ and $\theta \neq -1$, then $B_\theta[S, S]$ is nonsingular, and $k \leq m'$.*

Proof. (1) For $o \in O$ there are exactly λ paths $x \to z \to o$, and their middle vertices are the in-neighbours of o in O. The absolute value of an eigenvalue of a nonnegative matrix is at most the largest column sum of the matrix.

(2) Order S as x, O. Then $B_\theta[S, S] = \begin{pmatrix} -\theta & \mathbf{1}^\top \\ \mathbf{a} & E \end{pmatrix}$, where $E = A_O - \theta I$ and $\mathbf{a}$ is the indicator vector of $O \cap \Gamma^-(x)$. Expansion gives $\det B_\theta[S, S] = -\theta \det E - \mathbf{1}^\top \operatorname{adj}(E)\, \mathbf{a}$. From $\mathbf{1}^\top E = (\lambda - \theta)\mathbf{1}^\top$ it follows that $(\lambda - \theta)\mathbf{1}^\top \operatorname{adj}(E) = \det E \cdot \mathbf{1}^\top$. Multiply by $\lambda - \theta$ and use $\mathbf{1}^\top \mathbf{a} = t$: $(\lambda - \theta) \det B_\theta[S, S] = \det E\, (\theta^2 - \lambda\theta - t)$. A root satisfies $\theta^2 = (\lambda - \mu)\theta + t - \mu$, hence $\theta^2 - \lambda\theta - t = -\mu(\theta + 1)$.

(3) For $|\theta| > \lambda$ the number θ is not an eigenvalue of A_O by item 1, and $\theta \neq \lambda$. For $\theta \neq -1$ the right-hand side of item 2 is nonzero, so the block of order $k + 1$ is nonsingular, and by item 3 of Lemma 3.1 we get $k + 1 \leq 1 + m'$. □

The bound $k \leq m'$ for $|\theta| > \lambda$ is Corollary 7 of Hobart and Williford [16, p. 5] with θ and θ' interchanged, read as its contrapositive. In [16] it is derived from [16, Theorem 6] (Section 6): θ' is an eigenvalue of $A[\Gamma^+(x), \Gamma^+(x)]$, whence $|\theta'| \leq \lambda$ if the multiplicity of θ is less than k; this is condition N11, and the table already applies it. For $\theta = -1$, that is, for $\lambda = 0$ and $t = 1$, Corollary 7 of [16] is even stronger than item 3, because it has no exception $\theta = -1$. Thus the out-neighbourhood by itself gives no new exclusion. The exclusions of Sections 4–6 and of Theorem 7.9 rest on pieces larger or more rigid than it: on the neighbourhood $N[x]$ of $2k - t + 1$

vertices for $\lambda = 0$, on the left kernel of the rows $\{x\} \cup \Gamma^+(x)$ for $\lambda = 1$, $t = 2\mu$, and on the rows of $N^+[x]$ together with two outer vertices for $t = \lambda + 1$, $\mu = 1$, $f = K$. Such pieces are forced only for small λ and small μ, and beyond them the method in its present form is silent.

$\lambda = 0$. Of the seven open sets of the table [2] with $\lambda = 0$, Theorems 4.5 and 5.3 close five. For the two remaining ones the bound of Theorem 4.3 holds with a margin: for $(70, 11, 5, 0, 2)$ at $\theta = 1$ the gap τ equals 3, and for $(88, 9, 3, 0, 1)$ it is 17 at $\theta = 1$ and 40 at $\theta = -2$. For $\tau = 3$ Proposition 5.2 determines the block outside $N[x]$ by the types of the outer vertices and the arcs of three pivots, but a search from three pivots is beyond our means: by Knuth's estimate [25] with random descents, its tree for each of the four graphs of arcs from Q to P has from $4.6 \cdot 10^{15}$ to $1.5 \cdot 10^{17}$ nodes. This is an estimate, not a proof, and nothing in the paper rests on it.

$\lambda = 1$, $t = 2\mu$. Of the twelve open sets of this family in the table [2], seven have $k + 1 - g \le 0$, and Corollary 6.3 gives them nothing. For the other five the corollary bounds the number of components from below, $c \ge k + 1 - g + [d \ne 0]$, while $c \le k/2$ by Lemma 6.1. Only for $(55, 16, 12, 1, 6)$ do the two bounds come close, $7 \le c \le 8$, and Theorem 6.4 excludes it. For $(44, 12, 8, 1, 4)$, $(63, 16, 10, 1, 5)$, $(95, 20, 10, 1, 5)$ and $(98, 22, 12, 1, 6)$ the margin $k + 1 - g$ equals 2, 4, 2, 6 against $k/2 = 6$, 8, 10, 11, and the kernel bound leaves several structures of a vertex that we have not excluded.

$t = \lambda + 1$. Of the five open sets of the table with $\mu = 1$ and $t = \lambda + 1$, the paper closes four. For the fifth, $(70, 9, 3, 2, 1)$, we have $f = 20$, while $K = 10$. The structure of Theorem 7.8 rests on the equality $f = K$: for $f > K$ the rank of B is greater than $K + 1$, the rows of $N^+[x]$ together with two outer vertices may be independent, and the minors of order 2 from its proof need not vanish. Lemma 7.11 applies to this set, but $K = 10$ is even, and there is no contradiction. For $\mu \ge 2$ the lower bound $K - \mu f$ on a class from Proposition 7.5 is usually negative: every vertex outside $N^+[x]$ receives μ arcs from $N^+[x]$, and the count allows classes of a single vertex. Of the 19 open sets of the table with $t = \lambda + 1$, $\mu \ge 2$ and integral f that inequality (7.1) does not exclude, 18 have this bound at most zero, for example $17 - 34 = -17$ for $(102, 16, 6, 5, 2)$. For the nineteenth, $(81, 17, 10, 9, 2)$, it equals $18 - 14 = 4$; together with $a \le K - q = 9$ and Proposition 7.4 it leaves $a \in \{4, 9\}$, and there is no contradiction.

9. Machine verification

Lean. The nonexistence results for nine pairs of Theorem A and their complements are proved in Lean 4 [9] with the Mathlib library [10]: Lean version v4.35.0, Mathlib at the tag v4.35.0. A graph is given by the predicate `IsDSRG n A k t lam mu`. It requires the integer matrix A of order n to consist of zeros and ones, to have zero diagonal and row and column sums k, and to satisfy $A^2 = tI + \lambda A + \mu(J - I - A)$ entrywise; it has no other conditions. The nonexistence of a set (v, k, t, λ, μ) is a theorem `¬ IsDSRG v A k t lam mu` for any matrix A, without other hypotheses. The other theorems of the project are about properties of a graph and take the hypothesis `IsDSRG` with the parameters of the statement and the hypotheses of the statement itself; which of them are proved for any matrix, and how the Lean statements differ from the text, is named in Table 3 of Appendix B. Eigenspaces are taken over $\mathbb{R}$, and the multiplicity m' is the dimension of $\ker(A - \theta' I)$. For each theorem

the program prints, with the command `#print axioms`, the axioms it depends on: these are only `propext`, `Classical.choice` and `Quot.sound`. The proofs contain neither `sorry`, nor axioms of our own, nor native computation (`native_decide`).

TABLE 2. The nonexistence results of Theorem A in Lean. The modules belong to the project `DsrgRank` in the repository of Section 11.

Set and complement	Lean theorems	Module	Route of the proof
$(50, 11, 7, 0, 3)$, $(50, 38, 34, 29, 28)$	`ClosedNbhd_no_50_11_7_0_3`, `compl_no_50_38_34_29_28`	`ClosedNeighbourhood`, `Complement`	Corollary 4.4
$(68, 13, 7, 0, 3)$, $(68, 54, 48, 43, 42)$	`ClosedNbhd_no_68_13_7_0_3`, `compl_no_68_54_48_43_42`	the same	Corollary 4.4
$(100, 16, 7, 0, 3)$, $(100, 83, 74, 69, 68)$	`ClosedNbhd_no_100_16_7_0_3`, `compl_no_100_83_74_69_68`	the same	Corollary 4.4
$(108, 19, 9, 0, 4)$, $(108, 88, 78, 72, 70)$	`ClosedNbhd_no_108_19_9_0_4`, `compl_no_108_88_78_72_70`	the same	Corollary 4.4
$(48, 9, 5, 0, 2)$, $(48, 38, 34, 30, 30)$	—	—	the reduction in the text, the search by two programs
$(55, 16, 12, 1, 6)$, $(55, 38, 34, 27, 24)$	`LeftKer_no_55_16_12_1_6`, `compl_no_55_38_34_27_24`	`LeftKernelComponents`	Theorem 6.4 by the same route
$(66, 10, 5, 4, 1)$, $(66, 55, 50, 45, 50)$	`TwinMuOne_no_66_10_5_4_1`, `compl_no_66_55_50_45_50`	`TwinClassesMuOne`, `Complement`	Theorems 7.8 and 7.9
$(91, 12, 6, 5, 1)$, $(91, 78, 72, 66, 72)$	`TwinMuOne_no_91_12_6_5_1`, `compl_no_91_78_72_66_72`	the same	Theorems 7.8 and 7.9
$(96, 15, 10, 9, 1)$, $(96, 80, 75, 65, 75)$	`TwinMuOne_no_96_15_10_9_1`, `compl_no_96_80_75_65_75`	the same	Propositions 7.3 and 7.4, Lemma 7.2
$(28, 6, 3, 2, 1)$, $(28, 21, 18, 15, 18)$	`Case28_no_28_6_3_2_1`, `Case28_no_28_21_18_15_18`	`Case28`, `VertexFrame`, `Complement`, `Sat`	Theorem 7.12 with the SAT certificate in the kernel

The complements are derived from the nonexistence results by the general complement theorem `not_isDSRG_compl`. By the same route the project proves the nonexistence of $(40, 9, 6, 5, 1)$ and $(80, 19, 13, 12, 2)$ from Theorem 7.6 together with their complements, and also of six other sets with $\lambda = 0$ that Corollary 4.4 excludes.

The SAT certificate in the Lean kernel. The first case of Theorem 7.12 is closed by a SAT formula. It has 131,518 variables and 300,889 clauses. The solver CaDiCaL [24], version 1.9.5, from PySAT [22] proves it unsatisfiable in less than a second and writes a DRAT proof; drat-trim [26] checks it against the formula and extracts from it an LRAT proof [27] of 525 steps. This proof is checked by the Lean kernel. The generator of the formula is written in Lean, and a Lean theorem proves that the matrix of a DSRG with a row-free vertex, after the renumbering from the proof of Theorem 7.12, satisfies every clause of the generator. The LRAT steps refer to 1,627 clauses of the formula; the kernel checks by computation that each of them is a clause of the generator for the parameters $(28, 6, 3, 2, 1)$ and that the

525 steps derive the empty clause from them. The LRAT checker in Lean is built on the kernel of the package lrat-catcher [28]. Thus both the unsatisfiability of the formula and its correspondence to a graph with a free vertex are proved formally, and neither the solver nor the program that builds the formula needs to be trusted.

The search for (48, 9, 5, 0, 2). The last step of Theorem 5.3 is a computation on the reduction of Proposition A.2, which is proved only in the text. It was carried out by two programs of different design that share no code (Appendix A). The second program writes a SAT formula for each pivot orbit, and the unsatisfiability of each of the 707 formulas has been checked without trusting the solver. For 166 orbits the formula contains the empty clause: one of its cardinality constraints requires a cell to take more triples than the cell has candidates, or a negative number of triples. The remaining 541 formulas were solved by CaDiCaL [24], version 1.9.5; for it the cardinality constraints are encoded by the sequential counter [23]. For each of these formulas CaDiCaL wrote a DRAT proof, drat-trim accepted all 541 proofs, and all the LRAT proofs that drat-trim extracted from them were accepted by the LRAT checker of the repository: it checks them again against the same formula by unit propagation and accepts no RAT steps. The SHA-256 hashes of the formulas and proofs, the computing times and the commands that recreate them are given in the README of the repository (Section 11).

The checkers confirm the unsatisfiability of the formula they are given; whether it encodes the conditions of Proposition A.2, they do not check. This correspondence is not proved formally: it rests on the construction of the formula and on the controls of the programs on known graphs (Appendix A).

What is checked outside Lean. The nonexistence of $(48, 9, 5, 0, 2)$ remains outside Lean. Its reduction (item 3 of Lemma 5.1, the formula of Proposition 5.2, Lemma A.1 and Proposition A.2) is proved in the text, and the search is carried out by programs; for this set Lean proves only part of the reduction and does not give the nonexistence. The statements on which no nonexistence result rests are also proved only in the text: the determinant formula of Proposition 4.2, items 3–5 of Lemma 3.1 in general form (the applications use their particular forms), the identity $\sum_{u\in U}\gamma_u = a(\lambda+2-a)$ of Proposition 7.4, and Proposition 8.1.

Controls on known graphs. Every statement of Sections 4–7 has been checked by exact computation at every vertex of the known DSRGs that satisfy its hypotheses, complements included; the formula of Proposition 4.2 fails on graphs with $\lambda = 1$, as it should. These checks are not part of the proofs: they show that the statements say what we claim they say.

10. Use of AI

The arguments, the proofs, the checking programs and the text of the paper were produced by AI agents running on Claude models (Anthropic) under the direction of the author. The author chose the problem, directed the work and is responsible for the publication. The two rStation papers [7] and [8] that this paper merges and reworks were posted by an AI agent directed by the author.

No human mathematician has checked the proofs of the text line by line. The proofs of nine nonexistence results out of ten are checked by the Lean kernel (Section 9). The reduction for $(48, 9, 5, 0, 2)$ and the steps that are not in Lean (Table 3) were read by independent passes of AI reviewers.

11. Data availability

The Lean modules, the checking programs, the graphs on which the controls run, and the certificates needed to repeat the checks of the paper are kept in the public repository on GitHub `https://github.com/Nosdormas/melete-math` (release 1.1.0) and preserved in the Software Heritage archive as the snapshot `swh:1:snp:660b3b805773567b1a87f0f5764b21a3b580fc44`. For every check, the README of the repository names the command, the expected outcome and the computing time, and for the search of Theorem 5.3 and the certificate of Theorem 7.12 also the SHA-256 hashes of the formulas and proofs.

Appendix A. The reduction and the search for $(48, 9, 5, 0, 2)$

Here $(v, k, t, \lambda, \mu) = (48, 9, 5, 0, 2)$, $\theta = 1$, and the notation is that of Section 5.

Lemma A.1. *Let $(v, k, t, \lambda, \mu) = (48, 9, 5, 0, 2)$ and $\theta = 1$. Then $\theta' = -3$, $m' = g = 14$, $\theta^3 - t\theta - \mu(k - t) = -12$, $|M| = 5$, $|Q| = |P| = 4$, $|N(x)| = 13$, there are 34 outer vertices, $\tau = 1$, and the trace of T equals -4. Moreover:*

1. *the arcs from Q to P form a bipartite graph in which every vertex is incident with exactly two arcs: a cycle of length 8 or two cycles of length 4;*
2. *the row and the column x of the matrix Φ are zero, and on $N(x)$, in the order M, Q, P,*
$$\Phi[N(x), N(x)] = \begin{pmatrix} 3I + J & J & 2J \\ 2J & 3I + 2J & 2(J - A_{QP}) \\ J & J & 3I + 2J \end{pmatrix};$$
every vertex of M and of P has 8 out-neighbours among the outer vertices, and every vertex of Q has 7; every vertex of M and of Q has 8 in-neighbours among the outer vertices, and every vertex of P has 7;
3. *let $\alpha_w = |\Gamma^+(w) \cap Q|$ and $\beta_w = |\Gamma^-(w) \cap P|$, let $\nu_2(w, w')$ be the number of paths $w \to n \to w'$ with $n \in N(x)$, and let $\nu_3(w, w')$ be the number of paths $w \to u \to p \to w'$ with $u \in Q$, $p \in P$. Then for outer vertices w and w'*
$$F(w, w') = \frac{(1 + \alpha_w)(1 + \beta_{w'})}{3} - \nu_2(w, w') - \nu_3(w, w'),$$
$$\sigma_w = -1 - \frac{(1 + \alpha_w)(1 + \beta_w)}{3} + \nu_2(w, w) + \nu_3(w, w),$$
and the number $3\sigma_w$ is an integer;
4. *distinct outer vertices have distinct types.*

Proof. The roots of the polynomial $x^2 + 2x - 3$ are 1 and -3; by Lemma 3.1, $f + g = 47$ and $9 + f - 3g = 0$, whence $g = 14$, $f = 33$. Further, $1 - 5 - 8 = -12$, the sizes of the parts are given by Lemma 4.1, $\tau = 14 - 13 = 1$, and the trace of T equals $1 \cdot (-3 - 1) = -4$ by Lemma 5.1.

(1) By Lemma 4.1 every $u \in Q$ has exactly two out-neighbours in P. For $p \in P$ there is no arc $x \to p$, so there are exactly $\mu = 2$ paths $x \to o \to p$; their middle vertices lie in Γ^+, and by Lemma 4.1 in Q. A bipartite graph on $4 + 4$ vertices in which every vertex is incident with two arcs is a union of even cycles of length at least 4.

(2) By Lemma 5.1, $\Phi(n, n') = (XY)(n, n')$ is the value of $A^2(n, n')$ given by the definition minus the number of paths of length 2 inside $N[x]$, and the paths inside $N[x]$ are given by Lemma 4.1. For example, for $m \neq m'$ from M there is no arc,

$A^2(m,m') = \mu = 2$, inside $N[x]$ there is one path $m \to x \to m'$, and $\Phi(m,m') = 1$; for $u \in Q$ and $p \in P$ with an arc $u \to p$ we have $A^2(u,p) = 0$ and $\Phi(u,p) = 0$. There are no arcs from x to the outer vertices, so the row and the column x are zero. The degrees are $k = 9$ minus the numbers of arcs inside $N[x]$.

(3) Let $\xi = X(\cdot, w')$ and $z = L^{-1}\xi$. By Lemma 4.1 the only out-neighbour of $i \in \Gamma^-$ inside $N[x]$ is x, those of $u \in Q$ are two vertices of P, and those of x are the vertices of Γ^+. Hence the rows of $Lz = \xi$ give $z_i = z_x - \xi_i$ for $i \in \Gamma^-$, $z_u = \sum_{u \to p} z_p - \xi_u$ for $u \in Q$, and $\sum_{o \in \Gamma^+} z_o = z_x$, since there is no arc $x \to w'$. Substitute the first two equalities into the third. Every $p \in P$ has exactly two in-neighbours in Q, and w' has exactly $\mu = 2$ in-neighbours in Γ^+ and $\beta_{w'}$ in-neighbours in P, so $12 z_x = 2 + 2\beta_{w'}$. Finally, $F(w,w') = \sum_{n \in \Gamma^+(w) \cap N(x)} z_n$; the vertex w has exactly 2 out-neighbours in Γ^- and α_w out-neighbours in Q, and the subtracted terms ξ_n add up to $\nu_2(w,w') + \nu_3(w,w')$. This gives the first formula; the second follows from $\sigma_w = -1 - F(w,w)$.

(4) Let $w \neq w'$ have the same type. Then they have two common in-neighbours in Γ^+, and by item 1 of Lemma 4.1 there are no arcs between w and w'. The value $F(w,w')$ depends on the out-neighbours of w and the in-neighbours of w', and w' has the same in-neighbours as w; hence $F(w,w') = F(w,w)$ and $T(w,w') = -F(w,w) = 1 + \sigma_w$. Likewise $T(w',w) = 1 + \sigma_w$ and $\sigma_{w'} = \sigma_w$. The equality $\sigma_w \sigma_{w'} = T(w,w')T(w',w)$ of Proposition 5.2 gives $\sigma_w^2 = (1+\sigma_w)^2$, that is, $\sigma_w = -1/2$, while the number $3\sigma_w$ is an integer, a contradiction. □

Symmetries and types. Label Q and P so that the graph of arcs from Q to P coincides with one of the two graphs of item 1 of Lemma A.1. Permutations of M and pairs of permutations of Q and P that preserve the arcs from Q to P preserve L and Φ and map types to types without changing F and σ; we call them symmetries of the frame, that is, of the subgraph on $N[x]$ with the chosen arcs from Q to P. For the cycle there are $5! \cdot 8 = 960$ of them, for two cycles $5! \cdot 32 = 3840$. A type is called admissible if a vertex w of this type has exactly two in-neighbours in Γ^+ and exactly two out-neighbours in Γ^-, and the subgraph on $N[x] \cup \{w\}$ contains no transitive triangle; the type of every outer vertex is admissible. There are 57,920 admissible types for the cycle and 58,912 for two cycles; the symmetries of the frame split them into 415 and 418 orbits, and $\sigma \neq 0$ on 352 and 355 of them, 707 orbits in total. The *key* of an orbit is the smallest code, in a fixed order, of a type in it.

Proposition A.2. *Suppose that a graph* $(48, 9, 5, 0, 2)$ *exists, and let* x *be a vertex of it. Then, for a suitable labelling of* Q *and* P*, there is an outer vertex* w_0 *with* $\sigma_0 \neq 0$ *whose type is the fixed representative of its orbit, and the key of this orbit is at most the key of any outer vertex with* $\sigma \neq 0$*. Each of the other 33 outer vertices* w *is given by a triple: the type of* w*,* $a_w = A(w_0, w)$ *and* $b_w = A(w, w_0)$*. The triples satisfy the following conditions:*

(a) minor of the pair: $\sigma_0 \sigma_w = (a_w - F(w_0,w))(b_w - F(w,w_0))$*;*
(b) arcs to the pivot: *the arc* $w_0 \to w$ *does not close a transitive triangle with the third vertex in* $N[x]$*, and without this arc there are at most* μ *paths* $w_0 \to n \to w$ *with* $n \in N[x]$*; the same holds for* $w \to w_0$*;*
(c) types: *the types of all 34 outer vertices are distinct, and if* $\sigma_w \neq 0$*, the key of the orbit of the type of* w *is not smaller than the key of the pivot;*

(d) compatibility: *for any two triples the numbers $A(w, w')$ and $A(w', w)$ given by the formula of Proposition 5.2 equal 0 or 1, an arc between w and w' does not close a transitive triangle with the third vertex in $N[x] \cup \{w_0\}$, and without an arc there are at most μ paths of length 2 between them through $N[x] \cup \{w_0\}$;*

(e) cover: *for any $n, n' \in N(x)$ exactly $\Phi(n, n')$ of the 34 outer vertices lie on a path $n \to w \to n'$; the degrees of the vertices of $N(x)$ towards the outer vertices are as in item 2 of Lemma A.1; the vertex w_0 has exactly $7 - \alpha_{w_0}$ out-neighbours and $7 - \beta_{w_0}$ in-neighbours among the other outer vertices, and every w has exactly $7 - \alpha_w$ and $7 - \beta_w$;*

(f) reciprocal arcs of the pivot: *the number of triples with $a_w = b_w = 1$ equals t minus the number of vertices of $N(x)$ joined to w_0 by arcs in both directions.*

Proof. The trace of T equals -4, so $\sigma_w \neq 0$ for at least one outer vertex. Among such vertices take a vertex w_0 with the smallest orbit key and apply the symmetry of the frame that maps its type to the representative of the orbit. Condition (a) is the second equality of Proposition 5.2 with $w' = w_0$, and the arc formula in (d) is its first equality. The rest of (b) and (d) is the definition of a graph with $\lambda = 0$ and $\mu = 2$ on a subgraph: over an arc there are no paths of length 2, and without an arc there are exactly μ of them in the whole graph. Condition (c) is item 4 of Lemma A.1 and the choice of w_0. The first two parts of (e) are item 2 of Lemma 5.1 and item 2 of Lemma A.1; the last one holds because an outer vertex has 9 out-neighbours, of which 2 lie in Γ^- and α_w in Q, and 9 in-neighbours, of which 2 lie in Γ^+ and β_w in P. Condition (f) is the equality $A^2(w_0, w_0) = t$: the vertex w_0 has exactly t neighbours joined to it by arcs in both directions; x is not among them, those that lie in $N(x)$ are given by the type of w_0, and the rest are the outer vertices with $a_w = b_w = 1$. □

The first program. For each of the 707 pivot orbits it looks for 33 triples satisfying conditions (a)–(e). It also checks the last part of (e), the degrees of the other outer vertices among the outer vertices. It does not check condition (f): weakening the conditions only enlarges the search. First it builds the candidates, the triples satisfying (a)–(c), and their pairwise compatibility (d); everything is computed exactly, in integers, because the matrix $12L^{-1}$ is integral. Then it solves an exact cover problem with pairwise compatibility. There are 197 cells: 169 pairs (n, n') from $N(x)$, 13 cells each for the out-degrees and the in-degrees of the vertices of $N(x)$, and two cells for the degrees of the pivot. The depth-first search takes the cell with the fewest fitting candidates and branches over them; in the branch of the i-th candidate the candidates of the same cell with smaller indices are excluded, so every cover occurs in exactly one branch, and the search is complete. For the cycle, a cover is impossible already at the root for 38 of the 352 pivot orbits, and for two cycles, for 90 of 355. The search over the remaining 314 and 265 orbits visited 36,012 and 12,620 nodes and found no cover.

The second program is designed differently and shares no code with the first: it builds the frame, the types, the order of the orbits and the key with its own code, and writes one SAT formula for each pivot orbit. A variable of the formula is a candidate, that is, a triple satisfying (a)–(c). The clauses forbid taking two triples of the same type and two triples that are incompatible by the arc formula and the count of paths of length 2 in (d); triangles in which the arc between two triples is one of the two arcs of a path through the third vertex are not forbidden by the

formula, and this weakening, too, only enlarges the search. Cardinality constraints encode the same cells of the cover as in the first program, the number 33 of triples and condition (f); the last part of (e) is absent from the formula as well. The solver Minicard from PySAT [22] found the formulas of all 707 orbits unsatisfiable, but the result does not rest on it: each unsatisfiability has been checked separately, 166 formulas contain the empty clause, and for the other 541 CaDiCaL wrote proofs accepted by two checkers (Section 9). Both programs rest on the same reduction, Proposition A.2, which is proved only in the text, and each checks a part of its conditions. Therefore the second program catches an implementation error in the first one, but not an error in Lemma A.1 or in Proposition A.2.

Cuts beyond the conditions. Besides conditions (a)–(f), both programs discard candidates by direct consequences of the identity $A^2 = tI + \mu(J - I - A)$ and of the cover (e): there are at most μ paths of length 2 between vertices already chosen and none over an arc, a vertex has at most t two-way neighbours, and a path $n \to w \to n'$ is taken only while the number $\Phi(n, n')$ is not used up. Every graph with these parameters satisfies these cuts.

Controls of the programs. The case $\tau = 1$ is checked on the Petersen graph as a DSRG $(10, 3, 3, 0, 1)$ with $\theta = 1$. There the rank of T equals 1, every outer vertex has $\sigma_w = -1/2$, and the types of outer vertices repeat: this is exactly what item 4 of Lemma A.1 excludes for $(48, 9, 5, 0, 2)$ by the integrality of $3\sigma_w$. Therefore, for the Petersen graph the second program builds a formula of the same construction without the condition of distinct types. The positive control of the second program: for 190 pairs (vertex, θ) of known graphs with $\tau \in \{0, 2, 3\}$ and for all ten vertices of the Petersen graph at $\theta = 1$, where $\tau = 1$, the formula is satisfiable on the actual graph, and the first solution found is a DSRG. Three deliberate faults, namely a sign in the formula of Proposition 5.2, a shift of the matrix Φ and a strict order of the keys, lose the actual graph at all vertices of the Petersen graph for both roots. The first program is built for the frame of $(48, 9, 5, 0, 2)$ and was not run on the Petersen graph. Its exact cover engine has been checked against exhaustive search on random instances, and on the Clebsch graph at $\theta = 1$, where $\tau = 0$, it finds exactly one cover, which assembles into the graph itself. The first program has no control at $\tau = 1$, and it writes no checkable trace of its search.

Appendix B. Statements of the paper and Lean theorems

Table 3 names what each step of the proofs rests on: "Lean" is a theorem accepted by the Lean kernel (Section 9); "by hand" is a proof in the text. Theorems about properties of a graph take the hypothesis `IsDSRG` with the parameters of the statement; where "for any matrix" is said, the theorem has no hypothesis `IsDSRG`. The nonexistence results for the sets and their complements are listed in Table 2.

Table 3: Statements of the paper, what they rest on, and Lean theorems.

Statement	Rests on	Lean theorems	Difference in the Lean statement
Section 3: $\mu > 0$ and $\Delta > 0$ for $0 < t < k < v - 1$	Lean	`mu_pos`, `disc_pos`	—

Statement	Rests on	Lean theorems	Difference in the Lean statement
Lemma 3.1, items 1, 2	Lean	`sub_mul_sub_eq`, `k_sub_mul_k_sub`, `spectrum_counts`, `rank_sub_root`, `rank_sub_root_duval`	—
Lemma 3.1, items 3–5	by hand; standard linear algebra	—	the applications use particular forms, rows below
Lemma 3.1, item 6	Lean	`KerFrame_rank`, `KerFrame_range`	hypotheses $\theta \neq \theta'$ and $k \neq \theta'$
Complement and transpose	Lean	`isDSRG_compl`, `not_isDSRG_compl`, `isDSRG_transpose`	—
Lemma 4.1	Lean	`ClosedNbhd_no_tt`, `ClosedNbhd_arcs`, `ClosedNbhd_muZ`, `ClosedNbhd_card`	—
Proposition 4.2	nonsingularity in Lean, the determinant formula by hand	`ClosedNbhd_kernel`, `ClosedNbhd_rank`	a kernel vector with support in $N[x]$ is zero; no determinant formula
Theorem 4.3, Corollary 4.4, Theorem 4.5	Lean	`ClosedNbhd_bound`, `ClosedNbhd_r_one`; the nonexistence results in Table 2	—
Lemma 5.1, items 1, 2	Lean	`Schur_yhat`, `Recover_cover`	only the vector $\hat{u}$; item 2 in counting form
Lemma 5.1, item 3; the formula of Proposition 5.2	by hand	—	—
Proposition 5.2 for $\tau = 1$, the second equality, for $(48, 9, 5, 0, 2)$	Lean in its own form	`Schur_kernel_of_card`, `Schur_pair48`, `Recover_det_zero_of_rank_lt`, `Recover_det16`	for outer $w \neq w'$ there is a nonzero vector of the kernel of $A - I$ with support in $N[x] \cup \{w, w'\}$, and every submatrix of $A - I$ of order 16 is singular; the passage to the minor of T is not stated; `Schur_kernel_of_card` and `Recover_det_zero_of_rank_lt` are for any matrix
Lemma A.1: $g = 14$ and the rank of $A - I$	Lean	`Schur_g48`, `Schur_rank48`	rank 15 instead of $\tau = 1$
Lemma A.1, item 2: degrees towards the outer vertices	Lean in general form, the numbers by hand	`Recover_out_degree`, `Recover_in_degree`	the degree of any vertex towards the outer vertices equals k minus its degree inside $N[x]$; the numbers 8 and 7 are not stated

Statement	Rests on	Lean theorems	Difference in the Lean statement
Lemma A.1, items 1, 3, 4; Proposition A.2	by hand	—	—
The search in the proof of Theorem 5.3	two programs; the unsatisfiability of the formulas of the second one: checked proofs or the empty clause (Section 9)	—	—
Lemma 6.1	Lean	`LeftKer_pred`, `LeftKer_cls_two`, `LeftKer_comp_card`, `LeftKer_sum_back`, `ProfBound_out`	the components are the classes of the equivalence relation generated by the arcs inside Γ^+, without the map pred; item 1 in the form "y has an in-neighbour in Γ^+ other than y"; the count of item 3 for any function on Γ^+ constant along these arcs
Theorem 6.2	Lean	`LeftKer_eqs`, `LeftKer_prof_lift`, `LeftKer_prof_le`	instead of $\ker D$, functions on Γ^+ constant along the arcs and satisfying the equations of the rows d and h_z
Corollary 6.3	Lean	`LeftKer_dim`, `LeftKer_bound`, `LeftKer_g`, `LeftKer_comp_bound`	item 1 for the same space of functions; in addition $2c \le k$
Theorem 6.4	Lean	`LeftKer_55_16_12_1_6`; the nonexistence in Table 2	—
Lemma 7.1	Lean	`Twin_inter`, `TwinMuOne_inter`, `TwinMuOne_order`, `Twin_spec`, `TwinMuOne_spec`	—
Lemma 7.2, Propositions 7.4, 7.5	Lean, except the identity of Proposition 7.4	`Twin_sub`, `Twin_le`, `Twin_gamma`, `Twin_class`, `Twin_far`, `Twin_lower`, `Twin_upper`	inequalities only, without the identity $\sum_{u\in U}\gamma_u = a(\lambda+2-a)$; `Twin_sub` for any matrix
Proposition 7.3	Lean	`TwinMuOne_rank_ge`, `TwinMuOne_rank`, `TwinMuOne_admissible`	—
Theorem 7.6	Lean	`Twin_cond`; the nonexistence results in Table 2	for the two sets with $\mu = 1$ the lower bound comes from the rank, the upper one from Proposition 7.4 and Lemma 7.2
Lemma 7.7	Lean	`TwinMuOne_col_union`, `TwinMuOne_double`, `TwinMuOne_ad`	`TwinMuOne_col_union` for any matrix

Statement	Rests on	Lean theorems	Difference in the Lean statement
Theorem 7.8	Lean	`TwinMuOne_rank_free`, `TwinMuOne_det`, `TwinMuOne_free_twin`, `TwinMuOne_free_twin_col`	the contrapositive: for $\lambda \geq 3$ the row class and the column class of every vertex have at least 2 vertices; from one special block only what is needed further is derived: every vertex of $\Gamma^+(o_1)$ has an out-neighbour inside $\Gamma^+(o_1)$, and not every vertex of $N^+[o_1]$ is its out-neighbour
Theorem 7.9, Corollary 7.10	Lean	`TwinMuOne_family`, `TwinMuOne_no_family`; the nonexistence results in Table 2	—
Lemma 7.11	Lean	`VFrame_indep`, `VFrame_col_in_row`, `VFrame_colfree`, `VFrame_no_full`, `VFrame_even`, `VFrame_pairs`	`VFrame_even` for any matrix
Theorem 7.12	Lean	`VFrame_28`, `Case28_free_row`, `Case28_core_checked`; the nonexistence in Table 2	—
Proposition 8.1	by hand	—	—

References

[1] A. M. Duval. A directed graph version of strongly regular graphs. *Journal of Combinatorial Theory, Series A* 47 (1988), 71–100. `https://doi.org/10.1016/0097-3165(88)90043-X`

[2] S. A. Hobart, A. E. Brouwer. Parameters of directed strongly regular graphs. Table, version of 6 November 2026 (header Last-Modified: Fri, 06 Nov 2026 06:02:12 GMT). `https://homepages.cwi.nl/~aeb/math/dsrg/dsrg.html`

[3] mira_loom. Nonexistence of DSRGs (50,35,28,23,28), (72,21,7,4,7), (81,24,8,5,8), (81,63,56,47,56), (96,80,75,65,75), (100,35,14,9,14) and all positive multiples. rStation, problem strongly_regular_graphs, paper 7, version 1, 2026. `https://rstation.ai/problems/strongly_regular_graphs/papers/7/v1`

[4] mira_loom. Nonexistence of DSRGs (28m,21m,18m,15m,18m) for every odd m: (28,6,3,2,1), (28,21,18,15,18), (84,20,11,10,3), and (84,63,54,45,54). rStation, problem strongly_regular_graphs, paper 11, version 1, 2026. `https://rstation.ai/problems/strongly_regular_graphs/papers/11/v1`

[5] S. Chung, W. J. Wesley. AI research conducted on rStation, with results available on GitHub, October 2026. Repository `https://github.com/dualverse-ai/strong_regular_graph`, accessed on 6 November 2026; the Lean theorem on the nonexistence of $(96, 15, 10, 9, 1)$ is `DSRG.no_order96_complement` in the file `nonexistence/DSRG/Nonexistence96.lean`, based on [3]. The authors and the title are given as in the table [2].

[6] kirkman_loom. Nonexistence of 42 open DSRGs via rank-6 and rank-7 hyperplane duality, existence of DSRG(64,14,7,3,3) and DSRG(80,18,8,4,4), and closed-form abelian group-ring families. rStation, problem strongly_regular_graphs, paper 5, version 1, 2026. The title is not in the text of the paper; it is taken from citations in other papers of the platform. `https://rstation.ai/problems/strongly_regular_graphs/papers/5/v1`

[7] Nonexistence of directed strongly regular graphs (50, 11, 7, 0, 3), (68, 13, 7, 0, 3), (100, 16, 7, 0, 3), (108, 19, 9, 0, 4), (55, 16, 12, 1, 6) and (48, 9, 5, 0, 2): the rank of $A - \theta I$ against

the neighbourhood of a vertex. rStation, problem strongly_regular_graphs, paper 74, version 2, 2026-11-01. Merged into the present paper and reworked; posted by an AI agent directed by the author. https://rstation.ai/problems/strongly_regular_graphs/papers/74/v2

[8] Directed strongly regular graphs with t = λ + 1: twin classes of closed neighbourhoods and the nonexistence of (66, 10, 5, 4, 1) and (91, 12, 6, 5, 1). rStation, problem strongly_regular_graphs, paper 76, version 2, 2026-11-01. Merged into the present paper and reworked; posted by an AI agent directed by the author. Besides the two sets of the title, the text excludes $(40, 9, 6, 5, 1)$, $(80, 19, 13, 12, 2)$, $(96, 15, 10, 9, 1)$ and $(28, 6, 3, 2, 1)$ with their complements. https://rstation.ai/problems/strongly_regular_graphs/papers/76/v2

[9] L. de Moura, S. Ullrich. The Lean 4 theorem prover and programming language. *Automated Deduction — CADE 28*, LNCS 12699 (2021), 625–635. https://doi.org/10.1007/978-3-030-79876-5_37

[10] The mathlib Community. The Lean mathematical library. *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020)*, 367–381. https://doi.org/10.1145/3372885.3373824

[11] L. K. Jørgensen. Non-existence of directed strongly regular graphs. *Discrete Mathematics* 264 (2003), 111–126. https://doi.org/10.1016/S0012-365X(02)00555-1

[12] C. D. Godsil, S. A. Hobart, W. J. Martin. Representations of directed strongly regular graphs. *European Journal of Combinatorics* 28(7) (2007), 1980–1993. https://doi.org/10.1016/j.ejc.2006.08.008

[13] L. K. Jørgensen. Rank of adjacency matrices of directed (strongly) regular graphs. *Linear Algebra and its Applications* 407 (2005), 233–241. https://doi.org/10.1016/j.laa.2005.05.005

[14] L. K. Jørgensen. Directed strongly regular graphs with rank 5. *Linear Algebra and its Applications* 477 (2015), 102–111. https://doi.org/10.1016/j.laa.2015.03.019

[15] J. Ma, B. Zhang, J. Liu, J. Zhao. Directed strongly regular graphs with rank 6. *Discrete Mathematics* 340(6) (2017), 1367–1373. https://doi.org/10.1016/j.disc.2016.10.013

[16] S. A. Hobart, J. Williford. New feasibility conditions for directed strongly regular graphs. *Electronic Journal of Combinatorics* 24(1) (2017), #P1.16. https://doi.org/10.37236/5496

[17] A. V. Bondarenko, A. Prymak, D. Radchenko. Non-existence of (76, 30, 8, 14) strongly regular graph. *Linear Algebra and its Applications* 527 (2017), 53–72. https://doi.org/10.1016/j.laa.2017.03.033

[18] A. V. Bondarenko, D. V. Radchenko. On a family of strongly regular graphs with $\lambda = 1$. *Journal of Combinatorial Theory, Series B* 103(4) (2013), 521–531. https://doi.org/10.1016/j.jctb.2013.05.005

[19] P. Rowlinson, B. Tayfeh-Rezaie. Star complements in regular graphs: old and new results. *Linear Algebra and its Applications* 432(9) (2010), 2230–2242. https://doi.org/10.1016/j.laa.2009.04.022

[20] J. Azarija, T. Marc. There is no (75, 32, 10, 16) strongly regular graph. *Linear Algebra and its Applications* 557 (2018), 62–83. https://doi.org/10.1016/j.laa.2018.07.019

[21] Directed strongly regular graphs: general construction and nonexistence theorems from 38 papers of the rStation problem. rStation, problem strongly_regular_graphs, paper 43, version 1, 2026-10-25. Posted by an AI agent directed by the author. https://rstation.ai/problems/strongly_regular_graphs/papers/43/v1

[22] A. Ignatiev, A. Morgado, J. Marques-Silva. PySAT: A Python toolkit for prototyping with SAT oracles. *Theory and Applications of Satisfiability Testing (SAT 2018)*, LNCS 10929, 428–437. https://doi.org/10.1007/978-3-319-94144-8_26

[23] C. Sinz. Towards an optimal CNF encoding of Boolean cardinality constraints. *Principles and Practice of Constraint Programming (CP 2005)*, LNCS 3709, 827–831. https://doi.org/10.1007/11564751_73

[24] A. Biere, T. Faller, K. Fazekas, M. Fleury, N. Froleyks, F. Pollitt. CaDiCaL 2.0. *Computer Aided Verification (CAV 2024)*, LNCS 14681, 133–152. https://doi.org/10.1007/978-3-031-65627-9_7. The paper describes version 2.0; we used version 1.9.5 from PySAT [22].

[25] D. E. Knuth. Estimating the efficiency of backtrack programs. *Mathematics of Computation* 29 (1975), 121–136. https://doi.org/10.1090/S0025-5718-1975-0373371-6

[26] N. Wetzler, M. J. H. Heule, W. A. Hunt Jr. DRAT-trim: Efficient checking and trimming using expressive clausal proofs. *Theory and Applications of Satisfiability Testing (SAT 2014)*, LNCS 8561, 422–429. https://doi.org/10.1007/978-3-319-09284-3_31

[27] L. Cruz-Filipe, M. J. H. Heule, W. A. Hunt Jr., M. Kaufmann, P. Schneider-Kamp. Efficient certified RAT verification. *Automated Deduction — CADE 26*, LNCS 10395 (2017), 220–236. https://doi.org/10.1007/978-3-319-63046-5_14

[28] S. Szeider. lrat-catcher, version 0.2.1. A Lean 4 program, 2026. https://github.com/leansolving/lrat-catcher