ACTUAL IMAGES AND LOCAL FIBERS IN THE SECOND COMMUTATOR LAYER

SHENGYONG PAN

Abstract. Let K be an algebraically closed field of characteristic zero and let J be the commutator ideal of the free associative algebra. We prove that, for every proper multilinear polynomial $f \in J^2 \setminus J^3$ and every $n \geq 3$, the set of values of f on $M_n(K)$ is $\mathfrak{sl}_n(K)$ if $\operatorname{tr} f \equiv 0$, and $M_n(K)$ otherwise. In the first case the evaluation map is smooth at an input with nonzero rank-one square-zero value. In the second case it is flat at such an input, with a local complete-intersection fiber of the expected dimension. The theorem allows arbitrary proper multilinear corrections of the same degree in J^3. We use restrictions to two active variables with commuting passive inputs and detect the first nonzero trace term on the exact fiber of the traceless projection.

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1. Introduction

Throughout, K is algebraically closed of characteristic zero. We write $M_n = M_n(K)$ and $\mathfrak{sl}_n = \mathfrak{sl}_n(K)$. For a polynomial f in m original variables, let

$$\operatorname{Val}_n(f) = \{f(A_1, \dots, A_m) : A_i \in M_n\}$$

be its set of single values, and write $S_f = \operatorname{Val}_n(f)$. No linear span or Zariski closure is taken in this notation. An ordinary multilinear polynomial uses each of its m distinct original variables exactly once in every word. It is proper if substituting 1 for any variable gives zero in the free algebra, equivalently if it is invariant under adding a scalar to any input. Let J denote the two-sided commutator ideal of the unital free associative algebra.

The L'vov–Kaplansky conjecture predicts that the actual image of an ordinary multilinear polynomial on M_n is one of $\{0\}$, KI_n, $\mathfrak{sl}_n$, and M_n (see [2]). We studied the quartic case by repeated-input restrictions in [5]. Tang gives a quartic classification in [9, Theorem 1.1]. In [6, Theorem A], we classify the images in degrees four through six, starting with proper sextics.

Here we fix the second commutator layer and allow arbitrary degree. For $n \geq 3$ we determine the images and construct local fibers for proper multilinear polynomials in $J^2 \setminus J^3$.

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Theorem 1.1 (The proper second layer in every size at least three)**.** *Let f be an ordinary proper multilinear polynomial in m distinct original variables, and suppose $f \in J^2 \setminus J^3$. For every fixed $n \geq 3$ its actual image is*

$$\mathrm{Val}_n(f) = \begin{cases} \mathfrak{sl}_n(K), & \operatorname{tr} f \text{ is identically zero on } M_n(K)^m, \\ M_n(K), & \operatorname{tr} f \text{ is not identically zero on } M_n(K)^m. \end{cases}$$

Put $h = \operatorname{tr} f$ and $g = f - n^{-1}hI_n$. There is an input $a \in M_n^m$ with $f(a) = N$, $\operatorname{rank} N = 1$ and $N^2 = 0$, such that

$$\operatorname{rank}(dg_a) = n^2 - 1, \qquad h|_{g^{-1}(N)} \neq 0 \text{ in } \mathcal{O}_{g^{-1}(N),a} \quad \text{if } h \not\equiv 0.$$

At this input the map to $\mathfrak{sl}_n$ is smooth with local fiber dimension $mn^2 - n^2 + 1$ if $h \equiv 0$. If $h \not\equiv 0$, the map to M_n is flat with local complete-intersection fiber of dimension $mn^2 - n^2$.

These assertions hold for $f + R$ for every proper ordinary multilinear $R \in J^3$ in the same variables. The trace and the input a are taken for the complete polynomial $f + R$.

The proof uses complete restrictions to two active variables. Section 6 treats a nonzero restriction with one common passive matrix. If all such restrictions vanish, $\mathrm{Id}(UT_3) = J^3$ (see [3, Theorem 2.1]) gives a nonzero pair of symbols with independent diagonal passive matrices. Section 7 treats that pair in an étale chart of the original source. The alternating polynomial s_4 belongs to this second case.

The two local conditions in Theorem 1.1 hold at the same original input. If the restriction has nonzero trace, its full differential is surjective at a rank-one square-zero input. If $h \not\equiv 0$ and its restriction is zero, the first nonzero term in the transverse expansion of h gives a bilinear test on the two full active matrices. Theorems 6.2 and 7.2 give a nilpotent input at which this test is nonzero and the traceless differential along the restriction is surjective. Solving $g = N$ in those tangent directions leaves the transverse variables unchanged. The selected term therefore survives on the exact fiber $g^{-1}(N)$ (see Section 6.4 and Proposition 7.14). A correction in J^3 can change this term even though it vanishes on the restriction. The input is selected after the correction is included in h. The local-image criterion of Section 2 then gives the stated sets of values.

Section 3 gives coordinates for the proper second quotient and the vanishing lemma for restrictions with few active variables. Section 4 constructs the one-order operators. Section 5 gives the size-three two-symbol construction and its traceful extension to larger sizes. Sections 6 and 7 treat the common-background and independent-diagonal restrictions, respectively. Section 8 formulates the remaining local-fiber questions and relates them to the higher-degree results in [6, 7, 8].

2. Local fibers and actual images

2.1. **A single local condition for arbitrary degree.** For noncentral-valued f define the target vector space

$$V_f = \begin{cases} \mathfrak{sl}_n(K), & \operatorname{tr} f \equiv 0, \\ M_n(K), & \operatorname{tr} f \not\equiv 0. \end{cases}$$

Interior and closure in the next theorem use the Zariski topology on V_f.

Theorem 2.1 (Local characterization)**.** *Let f be noncentral-valued and multilinear. With $N = E_{12}$,*

$$\boxed{S_f = V_f \iff N \in \operatorname{Int}_{V_f}(S_f) \iff N \notin \overline{V_f \setminus S_f}^{V_f}.}$$

Consequently LK, for this field class and all degrees and sizes, is equivalent to the middle condition for every noncentral-valued f. Moreover, $N \in S_f$ already holds for every such f; the issue is coverage of a neighborhood, not merely the existence of a nilpotent value.

Proof. First note that S_f is invariant under conjugation and multiplication by nonzero scalars, and contains zero.

If $\operatorname{tr} f \not\equiv 0$, then $KI_n \subseteq S_f$. To see this, choose Weyl matrices U, V with $UV = qVU$, $U^n = V^n = I$, where q is a primitive nth root. The $U^a V^b$ form a basis, and only its degree-zero element has nonzero trace. Expand a nonzero-trace evaluation multilinearly in that basis. Some basis tuple has nonzero output trace. All words at this tuple have the same total $(\mathbb{Z}/n\mathbb{Z})^2$ degree, so their sum is a multiple of one basis element. It must be a nonzero multiple of I. Scaling one input gives all scalars. If $\operatorname{tr} f \equiv 0$, the only scalar in V_f is zero, already in S_f.

Here is the elementary degeneration underlying the result. Every nonscalar W has a similar representative $\left(\begin{smallmatrix} u & v \\ w & Z \end{smallmatrix}\right)$ with $v \neq 0$. For otherwise every row covector would be a left eigenvector, forcing W to be scalar. The polynomial curve

$$R(t) = \begin{pmatrix} tu & v \\ t^2 w & tZ \end{pmatrix} = t \begin{pmatrix} t^{-1} & 0 \\ 0 & I \end{pmatrix} W \begin{pmatrix} t & 0 \\ 0 & I \end{pmatrix} \quad (t \neq 0)$$

has nonzero rank-one square-zero limit $R(0)$, conjugate to N. If $W \in \mathfrak{sl}_n$, the whole curve lies in $\mathfrak{sl}_n$. Suppose an open neighborhood O of N in V_f lies in S_f. After a fixed conjugation the curve passes through N. Its inverse image of O is an open subset of $\mathbb{A}^1$ containing zero, hence contains some nonzero t. Conjugation and scalar invariance now give $W \in S_f$. The scalar discussion completes $S_f = V_f$. The reverse implication and the closure formulation are immediate from the definition of interior. Central-valued polynomials have image either zero or KI_n by multilinearity, so the asserted equivalence with LK follows.

Finally, a nonscalar value has, after conjugation, a nonzero off-diagonal entry. Expand its inputs in matrix units. Some matrix-unit tuple gives an output with a nonzero off-diagonal entry. Regard these units as directed edges. Any nonzero word is an Euler trail using the same edges. An open trail has uniquely determined initial and terminal vertices, so every nonzero word in this tuple is a multiple of the same off-diagonal unit. The output is therefore a nonzero multiple of E_{ij}, $i \neq j$. Scaling and conjugation give N itself. □

This argument is independent of degree, properness, and a chosen coefficient family. We use it as a characterization and make no priority claim for the local reformulation. The proof also applies to a restricted evaluation domain whenever its image has the stated conjugation and scalar invariance, and all the required scalar targets have separately been obtained.

Corollary 2.2 (Two sufficient certificates)**.** *Let $d = \dim V_f$ and $q = mn^2$. Either of the following suffices for $S_f = V_f$:*

(1) There is $a \in f^{-1}(N)$ with $\operatorname{rank}(df_a : M_n^m \to V_f) = d$.
(2) There is $a \in f^{-1}(N)$ with

$$\dim \mathcal{O}_{f^{-1}(N),a} = q - d.$$

In particular the required fiber dimension is $(m-1)n^2$ in the traceful case and $(m-1)n^2 + 1$ in the trace-zero case.

Proof. The first condition makes f smooth at a. Smooth morphisms are open [11]; restricting to the smooth locus produces the neighborhood required in Theorem 2.1. For the second, the local rings of source and target at these closed points are regular of dimensions q, d. The source is Cohen–Macaulay, so the displayed dimension equality implies flatness by miracle flatness [12]. The flat locus is open [13], and flat morphisms locally of finite presentation are open [14]. Apply Theorem 2.1 again. Over algebraically closed K, a nonempty fiber of a finite-type scheme over a K-point has a K-point, so this yields actual matrix evaluations. □

The first condition implies the second, but neither is claimed necessary for polynomial surjectivity. In particular, generic full rank only proves dominance; it does not locate an open image neighborhood at N. The distinction is real even for invariant constructible cones: $\{0\} \cup \{A : \det A \neq 0\}$ is dense, contains every scalar, and omits N. It is an illustration of the logical issue, not a claimed polynomial image.

The rank-one attainment used in Theorem 2.1 is already in Kanel-Belov–Malev–Rowen: see Lemma 5 on p. 6 of arXiv:1005.0191v3, the author version cited in [15], and its explicit restatement for arbitrary n in Lemma 2 on p. 2 of arXiv:1306.4389v3 [16]. These locators refer to the specified preprint versions.

2.2. **Higher-order trace on a smooth traceless fiber.**

Proposition 2.3 (Exact local trace criterion)**.** *Let $f : M_n^m \to M_n$ be multilinear, let $h = \operatorname{tr} f$ and $g = f - n^{-1}hI$. Suppose $f(a) = N$ is nonzero rank-one square-zero and $\operatorname{rank}(dg_a) = n^2 - 1$. Put $Y = g^{-1}(N)$ and $Q = mn^2$. The following conditions are equivalent:*

(1) $h|_Y \neq 0$ in $\mathcal{O}_{Y,a}$;
(2) $\dim \mathcal{O}_{f^{-1}(N),a} = Q - n^2$;
(3) f is flat at a.

Each implies $\operatorname{Val}_n(f) = M_n$. In formal local coordinates on Y, a nonzero leading term of any positive weighted degree suffices for (1). No upper bound on the order of the first nonzero term is asserted.

Proof. The rank assumption makes Y smooth at a, with regular local domain $R = \mathcal{O}_{Y,a}$ of dimension $Q - n^2 + 1$. Since $h(a) = 0$, its restriction lies in the maximal ideal. If it is nonzero, it is a nonzerodivisor and $\dim R/(h) = Q - n^2$. If it is zero, the dimension stays $Q - n^2 + 1$. Since $f^{-1}(N)$ is cut out in Y by $h = 0$, this proves (1)⇔(2). Miracle flatness proves (2)⇒(3), and the dimension formula for flat local maps gives the converse. Corollary 2.2 proves surjectivity. Completion is faithfully flat, so a nonzero formal leading term detects a nonzero element of R. □

This is the local-ring form of the higher-order mechanism. The compatible algebraic degeneration tools in Glazer–Hendel [10, Proposition 2.10 and Corollaries 2.12–2.15] provide a framework for proving flatness from weighted initial maps. In the present one-equation fiber problem, the preceding elementary argument suffices; no convolution or Waring decomposition is needed.

For clarity, the corrected higher terms can be defined recursively. Let $K_a = \ker dg_a$ and choose a linear right inverse $R_a : \mathfrak{sl}_n \to T_a(M_n^m)$ of dg_a. The formal implicit function construction gives unique homogeneous maps $v_r : K_a \to \mathfrak{sl}_n$ of degree $r \geq 2$ such that

$$\phi(u) = a + u + R_a \sum_{r \geq 2} v_r(u), \qquad g(\phi(u)) = N.$$

They are obtained by

$$v_r(u) = -[\text{degree } r] \left(g\left(a + u + R_a \sum_{2 \leq j < r} v_j(u) \right) - N \right).$$

The trace germ to test is $H(u) = h(\phi(u))$, not $h(a+u)$. If $dh_a = 0$, its first two possibly nonzero homogeneous terms are

$$H_2(u) = \tfrac{1}{2} d^2 h_a(u,u),$$
$$H_3(u) = \tfrac{1}{6} d^3 h_a(u,u,u) - \tfrac{1}{2} d^2 h_a\big(u, R_a d^2 g_a(u,u)\big).$$

Thus an uncorrected cubic derivative cannot in general certify the trace on the fiber. For positive weights on u, a nonzero initial form of H certifies Proposition 2.3. If the entire germ is zero, this particular seed is not flat; this does not rule out another seed or surjectivity by another construction.

3. Coordinates in the second layer

3.1. **Coordinates on the proper second layer.**

Let $A = K\langle x_1, \ldots, x_m \rangle$ be the unital free associative algebra, J its two-sided commutator ideal, and $\mathcal{P}_m$ its multilinear component. Write

$\mathcal{P}_m^{\mathrm{prop}}$ for the proper subspace, consisting of polynomials killed by every unit deletion. All quotients in this subsection are free-algebra quotients; no matrix-identity ideal is imposed. Put

$$\mathcal{Q}_m^{\mathrm{prop}} = \frac{J^2 \cap \mathcal{P}_m^{\mathrm{prop}}}{J^3 \cap \mathcal{P}_m^{\mathrm{prop}}}.$$

For a subset $S \subseteq [m]$ of size at least two, set $a = \min S$ and, for $j \in S \setminus \{a\}$, define the left-normed comb

$$b_{S,j} = [x_j, x_a, x_{i_1}, \dots, x_{i_{|S|-2}}], \qquad i_1 < \cdots < i_{|S|-2}, \quad \{i_1, \dots, i_{|S|-2}\} = S \setminus \{a, j\}.$$

Proposition 3.1. *For $m \geq 4$, the classes*

$$b_{S,j} b_{S^c,\ell}, \qquad |S|, |S^c| \geq 2, \quad j \in S \setminus \{\min S\}, \quad \ell \in S^c \setminus \{\min S^c\},$$

form a basis of $\mathcal{Q}_m^{\mathrm{prop}}$. In particular its ordered-bipartition components are independent, and

$$\begin{aligned} D_m := \dim \mathcal{Q}_m^{\mathrm{prop}} &= \sum_{p=2}^{m-2} \binom{m}{p} (p-1)(m-p-1) \\ &= 2^{m-2}(m^2 - 5m + 4) + 2m - 2. \end{aligned} \tag{1}$$

Proof. We first compute the dimension without assuming independence of the proposed products. Use the multigrading by the occurrences of the original letters, and put

$$s = t_1 + \cdots + t_m, \qquad C = \prod_{i=1}^{m} (1 - t_i)^{-1}, \qquad H_A = (1-s)^{-1}, \qquad B = 1 + (s-1)C.$$

Order the letters. For each word w whose proper prefixes are nondecreasing but which is not itself nondecreasing, let $g_w = w - \mathrm{sort}(w)$. Its leading word in degree-lexicographic order is w. These leading words are prefix-free. Reduction at the first nonordered prefix decreases the order and terminates at the sorted representative. Therefore the g_w generate J as a right ideal. Prefix-freeness makes their right multiples have distinct leading words, proving

$$J = \bigoplus_w g_w A, \qquad \sum_w \mathbf{t}^{\deg w} = H_J / H_A = B.$$

Because J^{r-1} is also a left ideal, the same decomposition gives $J^r = \bigoplus_w g_w J^{r-1}$. Consequently

$$H_{J^r} = B^r H_A, \qquad H_{J^r/J^{r+1}} = B^r C.$$

On a nonempty squarefree support of size p, the coefficient of B is $p-1$, and its constant coefficient is zero.

The PBW decomposition orders the original generators before a basis of the derived free Lie algebra. It expresses A as the vector-space tensor product of sorted commutative monomials and the proper algebra. This decomposition respects each J^r. Indeed, the derivations $\partial_i(x_j) = \delta_{ij}$ preserve J, hence every J^r; on a multilinear component they are unit deletions. Successive coefficient extraction uses these derivations and the projections $f \mapsto f - x_i \partial_i f$, as well as multiplication by original generators, all of which preserve the ideal filtration. Thus removing the sorted singleton factors divides the squarefree exponential generating series by e^x. Since the squarefree series of B is

$$B(x) = \sum_{p \geq 2} (p-1) \frac{x^p}{p!} = (x-1)e^x + 1,$$

the series for the proper second layer is $B(x)^2$. Taking its $x^m/m!$ coefficient gives both expressions in (1).

It remains to prove spanning. The proper algebra is generated by nontrivial free Lie commutators, each a linear combination of combs. Modulo J^2, Jacobi moves the least letter of a comb into its initial bracket. Later adjoint letters can be reordered: interchanging adjacent ones has an error of the form $[L, [x, y]]$, a difference of two products of nontrivial commutators. The resulting one-factor normal forms are exactly $b_{S,j}$. Their classes are independent modulo J^2: in UT_2, set $x_j = E_{12}$ and every other letter of S equal to $\mathrm{diag}(1, 0)$. The selected comb is a

nonzero multiple of E_{12}, and every other canonical comb begins with two equal diagonal inputs and vanishes. Since J^2 vanishes on UT_2, this isolates every coefficient.

Now express a proper element of J^2 as a sum of products of combs. After normalizing its one-factor terms, their canonical remainder must be zero by that independence. Modulo J^3, only two-factor terms remain. Normalize each factor modulo J^2; its error multiplied by the other factor lies in J^3. The displayed products therefore span. Their number is the first expression in (1), already proved to be the dimension, so they are a basis. □

Lemma 3.2 (Vanishing on common-background sources)**.** *If an ordinary multilinear polynomial lies in J^r, then its restriction with fewer than r active original slots and every other slot equal to one common matrix D is identically zero. The identity holds for every D, not just diagonal matrices, and all derivatives tangent to that linear source vanish.*

Proof. Expand a member of J^r as a sum of products of r commutator-ideal factors with possible word contexts and take its original multihomogeneous component. The original variable sets in each term are disjoint. Fewer than r active variables leave one ideal factor with only passive inputs. It is evaluated in the commutative algebra $K[D]$ and is zero. This proves an identity on the entire linear restriction, and differentiation within that restriction preserves zero. No assertion is made about transverse derivatives. □

4. Ordered symbols and smooth nilpotent values

Write $\mathcal{H}_n = \{A \in M_n : A_{ii} = 0 \text{ for every } i\}$ for the hollow subspace in the fixed basis. Every traceless matrix is similar to a hollow matrix, by the prescribed-diagonal theorem of Fillmore [1].

4.1. **Ordered words and mixed differential depths.**

The fixed-depth hypothesis can be removed when the complete specialization has one order of its two active letters. The following statement does not require homogeneity.

Theorem 4.1. *Let $0 \neq F(u,v,w) = \sum a_{rst} u^r v^s w^t \in K[u,v,w]$, and put*

$$H_F(D,A,B) = \sum a_{rst} D^r A D^s B D^t.$$

For $n \geq 3$, let $\mathcal{V}_F$ be the union of the conjugates of $H_F(D,A,B)$, where D is regular diagonal and $A, B \in \mathcal{H}_n$. Then

$$\mathcal{V}_F = \begin{cases} M_n(K), & F(u,v,u) \neq 0, \\ \mathfrak{sl}_n(K), & F(u,v,u) = 0. \end{cases}$$

Proof. Suppose first that $F(u,v,u) \neq 0$. Number the indices cyclically and put $A = \sum_i E_{i,i-1}$ and $D = \operatorname{diag}(d_1, \ldots, d_n)$. Then

$$(H_F(D,A,B))_{ij} = F(d_i, d_{i-1}, d_j) B_{i-1,j}. \tag{2}$$

Each $F(d_i, d_{i-1}, d_i)$ is a nonzero polynomial. Each $F(d_i, d_{i-1}, d_{i+1})$ is also nonzero, since $n \geq 3$ and its three indices are distinct. Choose D outside the zero set of the product of these polynomials and the differences $d_i - d_j$. The source coordinates $B_{i-1,i}$ and $B_{i-1,i+1}$ are distinct hollow coordinates. Equation (2) therefore realizes every matrix supported on the diagonal and the forward cyclic band, by one hollow B. Set the wrap entry to zero to obtain any upper Jordan form. This proves $\mathcal{V}_F = M_n$.

Now suppose that $F(u,v,u) = 0$. Write $F = (u-w)G$, where $G \neq 0$. The exact word identity

$$H_F(D,A,B) = [D, H_G(D,A,B)]$$

shows that every value is traceless. For fixed diagonal D, this commutator is hollow. We show that every hollow target occurs. For a column j, the off-diagonal part of H_G is a linear function of the $n-1$ entries B_{kj}, $k \neq j$. Its matrix, with row and column indices different from j, is

$$M^{(j)}_{ik} = \begin{cases} G(d_i, d_k, d_j) A_{ik}, & i \neq k, \\ 0, & i = k. \end{cases}$$

For pairwise distinct i,k,j, the coefficient $G(d_i,d_k,d_j)$ is nonzero. A derangement of the $n-1\geq 2$ indices contributes a nonzero monomial to $\det M^{(j)}$. Distinct derangements give distinct monomials in the independent entries A_{ik}, so they cannot cancel. Thus every $\det M^{(j)}$ is a nonzero polynomial in D and hollow A. Their product with the differences d_i-d_j is nonzero. Choose one pair (D,A) where this product does not vanish. All column maps are then invertible simultaneously. Given $T\in\mathcal{H}_n$, prescribe the off-diagonal entries of H_G to be $T_{ij}/(d_i-d_j)$ and solve these column systems for one hollow B. This gives $H_F(D,A,B)=T$. Every traceless matrix is similar to a hollow matrix, proving the second assertion. □

Corollary 4.2. *Let $0\neq P(s,t)=\sum p_{rs}s^rt^s\in K[s,t]$, and put*

$$H_P(D,A,B)=\sum p_{rs}(\mathrm{ad}_D^r A)(\mathrm{ad}_D^s B).$$

For $n\geq 3$, the union of the conjugates of its values with D regular diagonal and $A,B\in\mathcal{H}_n$ is M_n if $P(s,-s)\neq 0$, and is $\mathfrak{sl}_n$ if $P(s,-s)=0$. If P is homogeneous, these conditions are equivalent to $P(1,-1)\neq 0$ and $P(1,-1)=0$, respectively.

Proof. Apply Theorem 4.1 to $F(u,v,w)=P(u-v,v-w)$. This substitution is injective, and $F(u,v,u)=P(u-v,v-u)$. □

4.2. **An ordered symbol with nonzero trace specialization.**

Lemma 4.3 (An ordered symbol with nonzero trace specialization)**.** *Let $h(u,v,w)=\sum_{a,b,d}h_{abd}u^av^bw^d\in K[u,v,w]$ satisfy $h(u,v,u)\neq 0$. For $n\geq 3$, the morphism*

$$H_h: M_n(K)\times\mathfrak{sl}_n(K)\times\mathfrak{sl}_n(K)\longrightarrow M_n(K),\qquad H_h(D,A,B)=\sum_{a,b,d}h_{abd}D^aAD^bBD^d,$$

has a nonzero rank-one square-zero value at a point where its differential is onto $M_n(K)$.

Proof. For a diagonal $D=\mathrm{diag}(d_1,\dots,d_n)$, entrywise multiplication gives

$$(H_h(D,A,B))_{i\ell}=\sum_{j=1}^{n}A_{ij}B_{j\ell}h(d_i,d_j,d_\ell). \tag{3}$$

Let $\mathcal{D}\subseteq K^n$ be the open set defined by pairwise distinct d_i and

$$h(d_1,d_j,d_1)\neq 0\qquad (2\leq j\leq n).$$

It is nonempty: each displayed factor is a nonzero polynomial by $h(u,v,u)\neq 0$, and the product of these factors and the pairwise differences is nonzero in the polynomial ring $K[d_1,\dots,d_n]$. Put $\widehat{q}=(0,q_2,\dots,q_n)^{\mathsf{t}}\neq 0$, take A hollow, and set $B=\widehat{q}e_1^{\mathsf{t}}$. Both A and B are traceless, and (3) becomes

$$H_h(D,A,B)=te_1^{\mathsf{t}},\qquad t_i=\sum_{j>1}A_{ij}q_jh(d_i,d_j,d_1). \tag{4}$$

Consider the parameter space

$$\mathcal{Z}=\left\{(d,q,A): d\in\mathcal{D},\ q\in K^{n-1}\setminus\{0\},\ A_{ii}=0,\ \sum_{j>1}A_{1j}q_jh(d_1,d_j,d_1)=0\right\}.$$

The last equation is a nonzero linear functional of the hollow entries of A at every base point (d,q): some q_j is nonzero, and all its possible $h(d_1,d_j,d_1)$ factors are nonzero. Its kernel is therefore a vector bundle of hyperplanes over the irreducible base $\mathcal{D}\times(K^{n-1}\setminus\{0\})$. In particular, $\mathcal{Z}$ is irreducible. Every point of $\mathcal{Z}$ has $t_1=0$, so its output in (4) is square-zero.

Fix a column $\ell>1$. Varying the $n-1$ hollow entries $B_{j\ell}$, $j\neq\ell$, gives outputs confined to column ℓ, with column vectors equal to columns $j\neq\ell$ of

$$(M_\ell)_{ij}=A_{ij}h(d_i,d_j,d_\ell).$$

These are legitimate variations in the full B-space $\mathfrak{sl}_n$. Simultaneous conjugation in all three original inputs is also a legitimate source direction. For its generator $E_{1\ell}$, the output derivative is

$$[E_{1\ell}, te_1^{\mathsf{t}}] = t_\ell E_{11} - te_\ell^{\mathsf{t}}. \tag{5}$$

Modulo matrices supported in the first column, this supplies $-t$ in column ℓ. Let Δ_ℓ be the determinant of the matrix obtained from M_ℓ by replacing its ℓ-th column by t. If $\Delta_\ell \neq 0$, these n derivative directions span the entire ℓ-th output column modulo the first column.

We show that Δ_ℓ is not identically zero on $\mathcal{Z}$. Specialize $\widehat{q} = e_\ell$. Then the equation defining $\mathcal{Z}$ reduces to $A_{1\ell} = 0$, and

$$t_i = A_{i\ell} h(d_i, d_\ell, d_1).$$

Choose a fixed-point-free permutation σ of $[n]$ with $\sigma(1) \neq \ell$. Such a permutation exists because $n \geq 3$: choose an n-cycle whose successor of 1 is an index outside $\{1, \ell\}$. The corresponding determinant term has the monomial

$$\prod_{i=1}^{n} A_{i,\sigma(i)}.$$

It uses neither a diagonal entry of A nor the entry $A_{1\ell}$. Its coefficient, up to sign, is a product of nonzero polynomials in d. Indeed, in an ordinary column $j \neq \ell$ the factor is $h(d_i, d_j, d_\ell)$ with $i \neq j$. If $i = \ell$, this is the nonzero trace specialization $h(d_\ell, d_j, d_\ell)$. Otherwise i, j, ℓ are distinct, and the factor is a copy of the nonzero polynomial h in three independent variables. In the replacement column ℓ, the row i is neither ℓ nor 1, by the two conditions on σ. Its factor $h(d_i, d_\ell, d_1)$ is again a copy of h in three independent variables. Different permutations give different monomials in the entries of A, so this term cannot cancel. Restricting a nonzero coefficient polynomial to the nonempty open set $\mathcal{D}$ cannot make it identically zero. This proves the required nonvanishing of Δ_ℓ on $\mathcal{Z}$.

It remains to control the first output column. At fixed D, B, the hollow A-variation E_{ij} with $j > 1$ and $i \neq j$ has output

$$q_j h(d_i, d_j, d_1) E_{i1}. \tag{6}$$

For $i = 1$ choose $j = 2$. For each $i > 1$ choose $j(i) \notin \{1, i\}$, which is possible since $n \geq 3$. Each selected h factor is a nonzero polynomial: for $i = 1$ this is the defining trace specialization, and for $i > 1$ its three coordinates are distinct. Requiring these factors and the finitely many selected q_j to be nonzero gives a nonempty open subset of the base (d, q). The hyperplane equation only constrains A, so its inverse image in $\mathcal{Z}$ is nonempty and open.

This open set and all the nonempty opens $\{\Delta_\ell \neq 0\}$, $2 \leq \ell \leq n$, meet because $\mathcal{Z}$ is irreducible. At a point of their intersection, (6) fills the first column, and the B-variations together with (5) fill every other column modulo it. Thus the full differential of H_h is onto M_n. Moreover, $\Delta_\ell \neq 0$ implies $t \neq 0$, since its replacement column is t. The common value is therefore nonzero of rank one and square-zero. The variations of A need not preserve the hyperplane equation defining $\mathcal{Z}$; they are variations in the original full source, as required. □

5. Two-symbol nilpotent seeds

5.1. **The complete shared-background two-symbol map on M_3.** Let $P, Q \in K[s, t]$ be homogeneous of the same degree $d \geq 2$, each divisible by st; either may be zero. Define the actual three-input polynomial map

$$H_{P,Q}(D, X, Y) = \sum_{r+s=d} P_{rs}(\mathrm{ad}_D^r X)(\mathrm{ad}_D^s Y) + \sum_{r+s=d} Q_{rs}(\mathrm{ad}_D^r Y)(\mathrm{ad}_D^s X). \tag{7}$$

Put

$$p = P(1, -1), \quad q = Q(1, -1), \quad \epsilon = (-1)^d, \quad \tau = p + \epsilon q.$$

We retain this reverse-symbol convention throughout the proof. The next statement supplies smooth nilpotent points for every nonzero pair in size three.

Write $E = \mathfrak{sl}_3$ if $\tau = 0$ and $E = M_3$ if $\tau \neq 0$.

Theorem 5.1. *The actual image of* (7) *on* M_3^3 *is*

$$\begin{cases} \{0\}, & P = Q = 0, \\ \mathfrak{sl}_3, & (P,Q) \neq (0,0),\ \tau = 0, \\ M_3, & \tau \neq 0. \end{cases}$$

In either nonzero case, there is an actual original input (D, X, Y) *with a nonzero rank-one square-zero value at which the map into its stated target is smooth. The local fiber is a complete intersection of dimension* 19 *in the trace-zero case and* 18 *in the traceful case.*

Proof of Theorem 5.1. For regular diagonal $D = \operatorname{diag}(d_1, d_2, d_3)$, direct multiplication gives the exact trace identity

$$\operatorname{tr} H_{P,Q}(D, X, Y) = \tau \sum_{i,j} (d_i - d_j)^d X_{ij} Y_{ji}. \tag{8}$$

Equivalently the sum is $\operatorname{tr}((\operatorname{ad}_D^d X)Y)$, so density extends this identity to all D. It is a nonzero polynomial function, as seen at $D = \operatorname{diag}(1, 0, 0)$, $X = E_{12}$, $Y = E_{21}$. Moreover a nonzero symbol is detected on a path through three distinct indices: if $P \neq 0$, use $X = E_{ij}$ and $Y = E_{j\ell}$, so that the reversed product is zero and the output is $P(d_i - d_j, d_j - d_\ell)E_{i\ell}$. Its two differences are independent. The case $Q \neq 0$ is obtained by interchanging X, Y. Thus no nonzero pair defines an identity or central-valued map on M_3.

5.1.1. *A column-star chart for the two nonzero symbols.* Assume initially that P, Q are both nonzero. At a regular diagonal D take

$$X = uE_{23} + vE_{32}, \qquad Y = c_2E_{21} + c_3E_{31}, \qquad uvc_2c_3 \neq 0. \tag{9}$$

Write

$$F_{ij\ell} = P(d_i - d_j, d_j - d_\ell), \qquad K_{ij\ell} = Q(d_i - d_j, d_j - d_\ell).$$

The exact value of the complete map is

$$N = n_2E_{21} + n_3E_{31}, \qquad n_2 = uc_3F_{231}, \quad n_3 = vc_2F_{321}. \tag{10}$$

Choose D so that all triple-distinct symbol values needed below are nonzero. Each is a nonzero polynomial under an invertible linear change of the two diagonal differences, so these finitely many open conditions are simultaneously possible. In particular $n_2n_3 \neq 0$, and N is nonzero of rank one and square-zero.

Four variations of the original Y slot give pure off-block output directions:

δY	E_{12}	E_{13}	E_{21}	E_{31}
$dH(\delta Y)$	$uK_{123}E_{13}$	$vK_{132}E_{12}$	$vF_{321}E_{31}$	$uF_{231}E_{21}$.

Their multipliers are nonzero. We work modulo these four output coordinates when adding the remaining directions.

Suppose first that $p^2 \neq q^2$. The variations $\delta Y = E_{23}, E_{32}$ give a basis of the two lower diagonal coordinates: their determinant is

$$uv\epsilon(q^2 - p^2)(d_2 - d_3)^{2d} \neq 0.$$

The variations $\delta X = E_{12}, E_{13}$ give respectively

$$c_2F_{121}E_{11} + c_2K_{212}E_{22} + c_3K_{312}E_{32}, \qquad c_3F_{131}E_{11} + c_3K_{313}E_{33} + c_2K_{213}E_{23}.$$

Together with the previous six controls these have rank eight. With respect to the trace pairing, normalize their one-dimensional conormal by $C_{11} = 1$. Its only other possibly nonzero entries are

$$C_{23} = -\frac{c_2F_{121}}{c_3K_{312}}, \qquad C_{32} = -\frac{c_3F_{131}}{c_2K_{213}}.$$

Simultaneous conjugation of the full original tuple (D, X, Y) gives output tangent $[Z, N]$. If every such tangent were annihilated by C, then $[C, N] = 0$, which forces

$$C_{23}n_3 = n_2, \qquad C_{32}n_2 = n_3.$$

These equations cannot hold for every choice of the independent nonzero parameters u, v, c_2, c_3: for fixed diagonal data and c_2, c_3, the ratio n_2/n_3 varies with u/v. If $p = 0$, both displayed entries of C vanish and the contradiction is immediate. Choose parameters where one equation fails. An actual orbit tangent then supplies the ninth direction.

Suppose next that $p^2 = q^2$ and $\tau \neq 0$. Necessarily $p \neq 0$ and $q = \epsilon p$. Put

$$x = d_2 - d_1, \quad y = d_3 - d_1, \quad f = F_{231}, \quad g = F_{321},$$
$$h = K_{312} = Q(y, -x), \quad j = K_{213} = Q(x, -y), \quad A = (-x)^d, \quad B = (-y)^d.$$

The lower-diagonal variations now give $E_{22} + E_{33}$. In target coordinates $(11, 22, 33, 23, 32)$, the two X variations, that normalized lower-diagonal variation, and the simultaneous orbit directions generated by E_{12}, E_{13} have columns

$$\begin{pmatrix} pc_2A & pc_3B & 0 & n_2 & n_3 \\ pc_2A & 0 & 1 & -n_2 & 0 \\ 0 & pc_3B & 1 & 0 & -n_3 \\ 0 & c_2j & 0 & 0 & -n_2 \\ c_3h & 0 & 0 & -n_3 & 0 \end{pmatrix}. \tag{11}$$

Its determinant is

$$2fguvc_2^2c_3^2\big(p^2x^dy^d - Q(y, -x)Q(x, -y)\big). \tag{12}$$

If the final polynomial is nonzero, choose the regular diagonal outside its zero set as well as the preceding symbol zero sets. This proves rank nine at the same actual nilpotent value. Interchanging X, Y and P, Q gives the same conclusion when the corresponding expression for P is nonzero.

If both final polynomials vanish identically, unique factorization in $K[s, t]$ forces each symbol to be a monomial: a nonmonomial irreducible factor of either symbol would divide s^dt^d. The endpoint equalities $P(1, -1) = p$ and $q = \epsilon p$ then fix the coefficients. The pairs not yet covered are exactly

$$P(s, t) = p(-1)^{d-r}s^rt^{d-r}, \qquad Q(s, t) = p(-1)^\ell s^\ell t^{d-\ell}, \quad 1 \leq r, \ell \leq d - 1, \tag{13}$$

where $p \neq 0$. Conversely each such pair makes both normal polynomials zero. We now treat every one of these pairs uniformly.

5.1.2. *All residual monomial pairs.* For (13), set

$$k = r + \ell - d, \qquad \sigma = (-1)^k, \qquad c_0 = p(-1)^{d-r}.$$

At a fixed regular diagonal D, make the invertible linear changes on the hollow input spaces $U = \mathrm{ad}_D^r X$ and $V = \mathrm{ad}_D^{d-r} Y$. Then

$$c_0^{-1}H_{P,Q} = J_k(D, U, V) := UV + \sigma\,\mathrm{ad}_D^k(V)\,\mathrm{ad}_D^{-k}(U). \tag{14}$$

Any negative exponent in this formula denotes only the inverse of ad_D on its hollow root spaces at the chosen regular diagonal. It is not a negative exponent in the original polynomial, nor a claim about a rational map at singular D.

Take $D = \mathrm{diag}(0, x, y)$ with $xy(y - x) \neq 0$, and put

$$a = x^k, \qquad b = y^k, \qquad c = (y - x)^k.$$

These three scalars are nonzero for every integer k. Choose

$$V = E_{21} + E_{31}, \qquad U = \begin{pmatrix} 0 & 1 & -1 \\ S & 0 & a \\ T & b & 0 \end{pmatrix}. \tag{15}$$

The exact normalized value is independent of S, T:

$$J_k = N = \begin{pmatrix} 0 \\ a \\ b \end{pmatrix} \begin{pmatrix} 1 & 1/a & -1/b \end{pmatrix}. \tag{16}$$

Its row-column pairing is $1-1=0$, so it is nonzero of rank one and square-zero. Set $\Delta = Ta - Sb$ and choose $\Delta \neq 0$, for example $S = 0, T = 1$.

Here are entry formulas giving explicit tangent certificates without any bound on k. For arbitrary hollow $U = (u_{ij})$, $V = (v_{ij})$ at this fixed D, the entries of J_k are

$$\begin{aligned}
(J_k)_{11} &= u_{12}v_{21} + u_{13}v_{31} + v_{12}u_{21} + v_{13}u_{31},\\
(J_k)_{12} &= u_{13}v_{32} + (b/c)v_{13}u_{32},\\
(J_k)_{13} &= u_{12}v_{23} + \sigma(a/c)v_{12}u_{23},\\
(J_k)_{21} &= u_{23}v_{31} + (c/b)v_{23}u_{31},\\
(J_k)_{22} &= u_{21}v_{12} + u_{23}v_{32} + v_{21}u_{12} + v_{23}u_{32},\\
(J_k)_{23} &= u_{21}v_{13} + (a/b)v_{21}u_{13},\\
(J_k)_{31} &= u_{32}v_{21} + \sigma(c/a)v_{32}u_{21},\\
(J_k)_{32} &= u_{31}v_{12} + (b/a)v_{31}u_{12},\\
(J_k)_{33} &= u_{31}v_{13} + u_{32}v_{23} + v_{31}u_{13} + v_{32}u_{23}.
\end{aligned} \tag{17}$$

They follow by multiplying the six root multipliers of ad_D^k, and also verify (16) directly.

Use row-major target order $(11, 12, 13, 21, 22, 23, 31, 32, 33)$. When $\sigma = 1$, the nine hollow input columns in the order

$$u_{12}, u_{13}, u_{23}, u_{32}, v_{12}, v_{13}, v_{21}, v_{23}, v_{32}$$

have determinant $4ab\Delta/c$ at (15), by differentiating (17). When $\sigma = -1$, retain the first eight of those columns and add the output column $[E_{12}, N]$. Their determinant is $2ab\Delta/c$. For an additional direct check of this last normal direction, a trace-pairing conormal to all hollow U, V directions is

$$C = \begin{pmatrix} c\Delta & 0 & 0 \\ -2ac\Delta & -c\Delta - 2a^2b^2 & 2a^3b \\ 2bc\Delta & -2ab^3 & -c\Delta + 2a^2b^2 \end{pmatrix}.$$

The first eight columns have minor $2a^3b^2/c^2$ on target rows $(11, 12, 13, 21, 22, 23, 31, 33)$, and $[C, N]$ has entries $21 = 2ac\Delta$, $31 = 2bc\Delta$, with every other entry zero. Thus the orbit direction is genuinely outside the hollow tangent image.

Every hollow U or V direction is realized by an actual X or Y direction at fixed D through the invertible changes above. The last column in the odd case is obtained by simultaneously conjugating the actual original variables D, X, Y; equivariance of (7) gives precisely the corresponding commutator output. No differentiation of inverse powers at a singular diagonal is used. Restoring c_0 rescales all outputs by a nonzero scalar. These nonzero determinants prove the traceful smooth assertion for all residual pairs, including negative, zero, and positive k.

5.1.3. *The trace-zero branch and the one-order boundary.* Assume $P, Q \neq 0$ and $\tau = 0$. Use the column-star input (9), with the same nonzero symbol opens and notation f, g, h, j, A, B as above. The four pure off-block directions remain available. In the remaining traceless coordinates $(11, 22, 23, 32)$, with 33 determined by the trace, the X directions E_{12}, E_{13} and simultaneous orbit directions E_{12}, E_{13} give respectively

$$\begin{aligned}
R_2 &= (pc_2A, -pc_2A, 0, c_3h), & R_3 &= (pc_3B, 0, c_2j, 0),\\
O_2 &= (n_2, -n_2, 0, -n_3), & O_3 &= (n_3, 0, -n_2, 0).
\end{aligned}$$

Here $q = -\epsilon p$, so the repeated diagonal contribution has opposite sign from (11). If $p \neq 0$, a normalized lower-block Y variation gives $B_0 = (0, 1, 0, 0)$, representing $E_{22} - E_{33}$. Expansion gives

$$\det(R_2, R_3, B_0, O_2) = jc_2\big(Agpvc_2^2 + fhuc_3^2\big).$$

It is a nonzero polynomial in the independent nonzero source parameters. If $p = 0$, then $q = 0$, and instead

$$\det(R_2, R_3, O_2, O_3) = -fghjuvc_2^2c_3^2 \neq 0.$$

Choose one common input outside the required zero sets. Together with the four off-block directions these give rank eight. Since (8) vanishes identically, the original map is smooth into $\mathfrak{sl}_3$ there.

If exactly one symbol is nonzero and its trace specialization is nonzero, Lemma 4.3 applies to $h(u,v,w) = P(u-v, v-w)$, after interchanging X, Y if necessary. This is an actual smooth point of the same three-input map, since the hollow inputs used there are allowed here. If its trace specialization is zero, the proof of Theorem 4.1 fixes one regular diagonal D and one hollow first input for which the second-input linear map is onto the entire hollow subspace. Its column derangement determinants are simultaneously nonzero; this is a fixed-input assertion, not just surjectivity of a union. Choose the second input giving the connected column-star nilpotent $E_{21} + E_{31}$. The second-input variations fill all six hollow coordinates, while simultaneous conjugation gives both traceless diagonal directions, modulo those hollow coordinates. Hence the full original differential has rank eight. This proves the smooth assertion on the one-order trace-zero boundary as well.

5.1.4. *Actual image and the original fibers.* A smooth point gives an open image neighborhood of its value in the specified affine target. The image of (7) is conjugation invariant and is a cone by scaling X. The rank-one contraction in the proof of Theorem 2.1 therefore supplies every nonscalar target in that target space. If $\tau = 0$, this gives all nonzero traceless matrices; zero is obtained by $X = 0$.

For $\tau \neq 0$, scalar attainment must also be checked for this repeated-input map. Take a regular diagonal D, index cyclically modulo three, and put

$$X_{i,i-1} = 1, \qquad Y_{i-1,i} = (d_i - d_{i-1})^{-d},$$

with all other entries zero. The forward product contributes p to every diagonal entry and the reversed product contributes ϵq; all off-diagonal entries vanish. Thus the actual value is τI_3. Scaling X supplies every scalar, proving image M_3. This step does not apply ordinary multilinear scalar extraction to a map with repeated D. Finally the Jacobian criterion gives smooth fibers of dimension $27 - \dim E$; in the regular source local ring their $\dim E$ defining equations form a regular sequence. This completes the proof of Theorem 5.1. □

5.2. **Traceful pairs in every size.** Use the map $H_{P,Q}$ in (7), with homogeneous degree $d \geq 2$ symbols divisible by st, and put $p = P(1,-1)$, $q = Q(1,-1)$, $\epsilon = (-1)^d$ and $\tau = p + \epsilon q$.

Theorem 5.2 (Traceful two-symbol pairs)**.** *Suppose $\tau \neq 0$. For every $n \geq 3$, the map $H_{P,Q} : M_n^3 \to M_n$ has actual image M_n and a smooth original three-input point over a nonzero rank-one square-zero matrix. Its local fiber there is a complete intersection of dimension $2n^2$.*

We prove the theorem by producing compatible seeds for an exact-fiber extension. Surjectivity in a smaller matrix size alone is insufficient.

Lemma 5.3. *Suppose $P, Q \neq 0$, $\tau \neq 0$, and at size $h \geq 3$ there is a regular diagonal D_0 and hollow X_0, Y_0 with a nonzero rank-one square-zero value N_0. Assume the derivative of*

$$(D, X, Y, g) \longmapsto g H_{P,Q}(D, X, Y) g^{-1}, \qquad D \text{ regular diagonal}, \quad X, Y \text{ hollow}, \quad g \in GL_h,$$

is onto M_h at this input, and $\det(\mathrm{ad}_{D_0}^j X_0) \neq 0$ for $0 \leq j \leq d$. Then there is a seed with these same properties in size $h+1$.

Proof. We use an exact singleton extension with a rank-one target and allow one endpoint coefficient to be zero. Write $N_0 = u_0 v_0^{\mathrm{t}}$, where $v_0^{\mathrm{t}} u_0 = 0$. Put $D = \mathrm{diag}(D_0, t)$, $E = D_0 - tI$, and initially $X = \mathrm{diag}(X_0, 0)$. If the upper and lower new Y rectangles are v, w, their output rectangles are

$$A_t v, \quad w B_t, \qquad A_t = \sum_{i+j=d} P_{ij} (\mathrm{ad}_{D_0}^i X_0) E^j, \quad B_t = \sum_{i+j=d} Q_{ij} (-1)^i E^i (\mathrm{ad}_{D_0}^j X_0). \tag{18}$$

The highest power of t in each sum has an invertible coefficient: homogeneity makes its other exponent unique, and the corresponding symbol coefficient is nonzero. Choose t so that D is regular and A_t, B_t are invertible.

If $q \neq 0$, take the exact target $N_1 = \left(\begin{smallmatrix} N_0 & 0 \\ v_0^{\mathrm{t}} & 0 \end{smallmatrix}\right)$. It has rank one and square zero. Solve (18) with $v = 0$ and $w \neq 0$. If $q = 0$, then $p \neq 0$; instead take $N_1 = \left(\begin{smallmatrix} N_0 & u_0 \\ 0 & 0 \end{smallmatrix}\right)$, solving with $v \neq 0$ and $w = 0$. The old output block is unchanged and the new bottom-right entry is zero, so these are exact values at single actual inputs.

For the new upper and lower X components u, z, the bottom-right derivative is

$$L(u, z) = (-1)^d\big(pzE^d v + qwE^d u\big). \tag{19}$$

It is nonzero by the chosen rectangle. Old input and old similarity variations have zero bottom-right derivative. Their old outputs are onto M_h, and their rectangular contributions can be removed using (18). Hence, together with L, the derivative is onto M_{h+1}. Every $(u, z) \in \ker L$ extends to a tangent to the exact target fiber, without changing these raw cross components of X.

The provisional X has a zero last row and column. Along such a kernel tangent, the first possible determinant coefficient is

$$(-1)^{j+1} \det(\mathrm{ad}_{D_0}^j X_0) \cdot zE^j (\mathrm{ad}_{D_0}^j X_0)^{-1} E^j u \quad (0 \leq j \leq d), \tag{20}$$

in order two. The last diagonal entry remains identically zero on the hollow source, so no second-order diagonal correction enters this coefficient. If $q \neq 0$, choose a nonzero u in the hyperplane $wE^d u = 0$, then choose z outside the finitely many annihilating hyperplanes in (20). If $q = 0$, make the dual choice, first fixing $z \neq 0$ in $zE^d v = 0$. The hyperplanes have a nonzero vector since $h \geq 3$, and all matrices in these Schur forms are invertible.

The auxiliary source is smooth and its exact target fiber is smooth at the provisional point. The selected tangent therefore lifts to a formal arc in that fiber. All the determinant coefficients (20) are nonzero on it; the surjectivity minor and regular-diagonal conditions remain units. Thus the finite-type exact fiber localized at these finitely many functions has a $K((t'))$-point and is nonempty. The Nullstellensatz supplies an actual K-point. This uses nonemptiness of a localized algebraic fiber, not algebraization of an arbitrary formal arc. Removing the similarity parameter only conjugates the rank-one target and preserves the raw determinant conditions. Surjectivity of the auxiliary map implies surjectivity for the actual three-input map, since all its directions are induced by actual D, X, Y variations. □

5.2.1. *Nonresonant pairs.* Assume $P, Q \neq 0$ and $p^2 \neq q^2$. Use the smooth column-star point (9) from the proof of Theorem 5.1. Its value is $N = n_2 E_{21} + n_3 E_{31}$ with $n_2 n_3 \neq 0$. The six hollow Y columns span the coordinates $12, 13, 21, 31, 22, 33$. In the remaining quotient coordinates $(11, 23, 32)$, the two raw upper-row X columns are R_2, R_3, and the actual orbit columns generated by E_{12}, E_{13} are

$$O_2 = (n_2, 0, -n_3), \qquad O_3 = (n_3, -n_2, 0).$$

The matrix $[R_2\ R_3\ O_2\ O_3]$ has rank three by that proof. The orbit columns are independent, so a nonzero vector in its kernel has a nonzero raw upper-row X component. Cancel its remaining output using the six Y columns. Raw variations in X_{21}, X_{31} have zero output, since they and Y have only a lower-left column, so the raw lower column can be chosen freely.

At this point every $\mathrm{ad}_D^j X$ has an invertible lower two-by-two block A_j and zero first row and column. A kernel tangent with nonzero upper row u and lower column z gives Schur determinant coefficient $-\det(A_j) u_j A_j^{-1} z_j$, where u_j, z_j are obtained from u, z by invertible root multipliers. For fixed $u \neq 0$ this is a nonzero linear functional of z. Choose z outside the finitely many kernels for $0 \leq j \leq d$. As in the proof of Lemma 5.3, lift in this same smooth exact target fiber and localize. We obtain a genuine size-three seed with all determinant conditions, while its rank-one nilpotent value and surjectivity are retained. The lemma now gives such a point in every size. This argument establishes determinant nonvanishing on the actual nilpotent fiber, rather than intersecting unrelated open sets in the unrestricted source.

5.2.2. *The entire traceful resonance.* Suppose $p^2 = q^2$ and $\tau \neq 0$. Then $q = \epsilon p \neq 0$. If both symbols are monomials, use (15) with $S = 0, T = 1$. Its smoothness was proved in Section 5. The first original input X has exactly one nonzero determinant cycle, $X_{12} X_{23} X_{31}$. Every power ad_D^j

multiplies this cycle by nonzero root differences, so all determinant conditions of Lemma 5.3 hold at this same point. This extends the residual monomial cases without importing any unproved all-size smoothness implication.

Otherwise interchange the actual X, Y roles, and hence P, Q, if necessary, so that Q is not a monomial. The resonance remains $q = \epsilon p \neq 0$. Define

$$\Phi_1(x, y, z) = qx^d Q(y, z) - Q(y, -x)Q(x, z),$$
$$\Phi_2(x, y, z) = qy^d Q(x, z) - Q(x, -y)Q(y, z).$$

These are both nonzero. Indeed, if $\Phi_1 = 0$, setting $x = 1$ gives $qQ(y, z) = Q(y, -1)Q(1, z)$. The two factors are nonzero, and their Cartesian-product monomial support has constant total degree only when each consists of one monomial. Thus Q itself is a monomial. Conversely a monomial satisfies that identity. Finally $\Phi_2(x, y, z) = \Phi_1(y, x, z)$.

Choose $D = \operatorname{diag}(0, x, y, -z)$ regular and set

$$a = Q(x, z), \quad b = Q(y, z), \quad f = P(x - y, y), \quad g = P(y - x, x), \quad u = a/f, \quad v = b/g.$$

Require these four symbol values, Φ_1, Φ_2, and every P, Q value on a distinct-three-vertex path to be nonzero. For any such path, its two differences are independent linear forms in x, y, z. Each required factor is therefore a nonzero polynomial. Their finite product with the regularity factors is nonzero, so one common choice exists. With a further parameter $\rho \neq 0$, put

$$X = E_{14} + \rho E_{41} + uE_{23} + vE_{32}, \qquad Y = E_{21} + E_{31}. \tag{21}$$

Direct multiplication of the complete map gives

$$H_{P,Q}(D, X, Y) = N = (0, a, b, 0)^{\mathsf{t}}(1, 0, 0, 1).$$

The forward paths $2 \to 3 \to 1$, $3 \to 2 \to 1$ and the reverse paths $2 \to 1 \to 4$, $3 \to 1 \to 4$ are the only surviving paths. Thus the value is independent of ρ, nonzero of rank one, and square-zero. The two two-cycles supporting X show that

$$\det(\operatorname{ad}_D^j X) = \rho uv\big((d_1 - d_4)(d_2 - d_3)\big)^{2j} \neq 0 \quad (j \geq 0).$$

It remains to prove smoothness at a single such input. Put $A = \{1, 4\}$, $B = \{2, 3\}$, and let σ exchange $1, 4$ and $2, 3$. Write $h_i = X_{i,\sigma(i)}$. The eight cross-block hollow Y directions split into four pairs. For $a' \in A, b' \in B$, the columns $Y_{a'b'}, Y_{\sigma(a')\sigma(b')}$ and the rows $H_{\sigma(a')b'}, H_{a'\sigma(b')}$ give the matrix

$$\begin{pmatrix} h_{\sigma(a')} P_{\sigma(a'),a',b'} & h_{\sigma(b')} Q_{\sigma(a'),\sigma(b'),b'} \\ h_{b'} Q_{a',b',\sigma(b')} & h_{a'} P_{a',\sigma(a'),\sigma(b')} \end{pmatrix},$$

where $P_{ijk} = P(d_i - d_j, d_j - d_k)$ and similarly for Q; the analogous pairs with A, B exchanged supply the other block. Each determinant is the difference of a nonzero multiple of ρ and a nonzero multiple of uv. A single $\rho \neq 0$ avoids their finitely many roots. The eight cross-block outputs are then independent. The within-pair Y directions supply $E_{11} + E_{44}$ and $E_{22} + E_{33}$, by $q = \epsilon p \neq 0$.

Let C annihilate these directions and all remaining hollow X and actual orbit directions under the trace pairing. The preceding ten outputs force

$$C_A = \begin{pmatrix} c & e \\ h & -c \end{pmatrix}, \qquad C_B = \begin{pmatrix} k & \ell \\ m & -k \end{pmatrix},$$

with no cross blocks. The raw X_{42} direction gives $P_{421}E_{41}$, so $e = 0$. Actual simultaneous conjugation of D, X, Y gives $[Z, N]$ and hence $[C, N] = 0$. Comparing the two rank-one factors of N gives

$$h = -2c, \qquad \ell = -(c + k)a/b, \qquad m = -(c - k)b/a.$$

The X_{12} direction has diagonal part $qx^d(E_{11} + E_{22})$ and remaining entry $Q(y, -x)E_{32}$. Pairing with C yields $(c + k)\Phi_1/b = 0$. Likewise X_{13} yields $(c - k)\Phi_2/a = 0$. Thus $c = k = 0$, and then $C = 0$. These actual hollow and orbit directions span all M_4, including its trace direction. Lemma 5.3 now extends this guarded point to every $n \geq 4$; size three is separately covered by Theorem 5.1.

5.2.3. *Actual images and local fibers.* If one complete symbol is zero, the traceful assertion follows from Lemma 4.3. Otherwise the nonresonant and resonant cases exhaust $\tau \neq 0$. At each smooth point, openness and the cone/conjugation argument in Theorem 2.1 give all nonscalar values. For scalars in this repeated-input map, choose regular diagonal D, use cyclic indices modulo n, and set

$$X_{i,i-1} = 1, \qquad Y_{i-1,i} = (d_i - d_{i-1})^{-d}.$$

All other entries are zero. Direct multiplication gives $H_{P,Q} = \tau I_n$; scaling X gives every scalar. This is not ordinary-multilinear scalar extraction with the repeated D treated as independent. The Jacobian criterion gives the asserted three-input fiber dimension and regular sequence. This proves Theorem 5.2.

6. Complete two-active coefficient fibers in every size at least three

We retain the standing algebraically closed, characteristic-zero field K. The results concern the unchanged original ordinary multilinear polynomial. We first treat a nonzero complete two-active restriction with one common passive matrix. Section 7 then treats the genuine second-layer portion of the common kernel of all these tests.

Theorem 6.1 (The complete original coefficient fiber)**.** *Let f be ordinary multilinear in $m = d + 2$ original variables, $d \geq 2$. Suppose that for some two original positions its* complete *restriction, with X, Y in those positions and D in all others, is $H_{P,Q}$ of (7), where P, Q are homogeneous of degree d, st $\mid P, Q$, and $(P, Q) \neq (0, 0)$. For each $n \geq 3$,*

$$\mathrm{Val}_n(f) = \begin{cases} \mathfrak{sl}_n, & \operatorname{tr} f \text{ is an identity on } M_n, \\ M_n, & \operatorname{tr} f \text{ is not an identity on } M_n. \end{cases}$$

In the first case $f : M_n^m \to \mathfrak{sl}_n$ has a smooth original input over a nonzero rank-one square-zero matrix; its local fiber dimension is $mn^2 - n^2 + 1$. In the second case $f : M_n^m \to M_n$ has a flat original input over such a matrix, and its local fiber is a complete intersection of dimension $mn^2 - n^2$.

The restriction identity may be an actual identity on the fixed M_n; a free identity suffices to apply the statement in every size. Arbitrary same-degree ordinary corrections with zero complete selected restriction are permitted, including the entire proper J^3 and all higher powers, with every crossing partition and order. The target is always chosen from the actual size-n trace of the corrected polynomial*, and the nilpotent input may change with that correction.*

Theorem 5.2 already proves the smooth full-matrix conclusion when the trace of the restriction is nonzero. The remaining problem is not merely to find a trace-zero restriction value. One must select a nilpotent input of that *same restriction* where the first surviving trace term of the complete original map does not disappear on its exact traceless fiber. The following theorem provides the needed compatibility.

Put $V_n = \{A \in M_n : A_{ii} = 0\}$, the hollow subspace, and keep $p = P(1, -1)$, $q = Q(1, -1)$, $\epsilon = (-1)^d$ and $\tau = p + \epsilon q$. At a fixed regular diagonal D, write

$$H_D(X, Y) = H_{P,Q}(D, X, Y), \qquad \Gamma_D(X, Y, g) = gH_D(X, Y)g^{-1}, \quad X, Y \in V_n, \quad g \in GL_n.$$

For $\tau = 0$, let $\mathcal{G}_D$ consist of hollow pairs whose raw value is nonzero rank-one square-zero and for which $d\Gamma_D$ is onto $\mathfrak{sl}_n$ at (X, Y, I). Let $\mathcal{G}_D^{\mathrm{guard}}$ be the subset satisfying

$$\det(\mathrm{ad}_D^j X) \neq 0 \qquad (0 \leq j \leq d). \tag{22}$$

The simultaneous conjugation in Γ_D is realized in actual inputs by

$$(D, X, Y) \longmapsto (gDg^{-1}, gXg^{-1}, gYg^{-1}).$$

In the original polynomial it conjugates *every* passive D slot as well. Raw tests and determinant conditions below refer to X, Y before this auxiliary conjugation.

Theorem 6.2 (Compatible nilpotent bilinear detection)**.** *Assume $(P,Q)\neq(0,0)$ and $\tau=0$. For every $n\geq 3$ there is a nonempty Zariski-open set Ω_n of regular diagonal matrices such that, for each fixed $D\in\Omega_n$, the tensors*

$$\{X\otimes Y:(X,Y)\in\mathcal{G}_D\}\quad \text{span } V_n\otimes V_n.$$

For $n\geq 4$ one may require (22); if $p\neq 0$, these guards can also be required in size three. After allowing arbitrary diagonal parts in the original X,Y, every nonzero bilinear form on $M_n\times M_n$ is nonzero at one actual smooth nilpotent seed of this same $H_{P,Q}$ and this same D.

The span in this statement is a linear-functional detector, not an assertion that a polynomial image can be added. Given a nonzero bilinear form, it supplies *one* compatible original input. The fixed open Ω_n is independent of that form. In particular, a form whose coefficients depend on D is first specialized at a suitable $D\in\Omega_n$ and then detected at that same D.

The proof splits exhaustively into $p\neq 0$ and $p=q=0$. In the latter case the cubic cycle norm separates invertible hollow operators from a two-dimensional cycle kernel. The former case needs one four-vertex Laurent calculation. Once a guarded spanning set is available in size four, a tensor uses at most four vertices, and the remaining size induction is uniform. We first record those common mechanisms, so that they are not repeated in each coefficient family.

6.1. **Diagonal enrichment and exact guarded extension.**

Lemma 6.3. *For fixed regular diagonal D, hollow tensor detection by $\mathcal{G}_D$ implies full $M_n\times M_n$ bilinear detection by actual smooth nilpotent seeds of $H_{P,Q}$.*

Proof. All active adjoint exponents are positive. Adding arbitrary diagonal matrices D_X,D_Y to the original X,Y therefore leaves the value unchanged. The hollow controls and simultaneous conjugation, including these added parts, retain the same smooth certificate. Suppose a bilinear form vanishes on every enriched pair. Its coefficient bilinear in D_X,D_Y is zero. Its coefficients linear in either added part vanish on the projections of the hollow seed set. Each projection spans V_n: otherwise a nonzero functional vanishing on that projection, tensored with any nonzero functional on the other factor, would contradict the hollow tensor span. Thus both mixed parts vanish. The remaining hollow-by-hollow part vanishes by hypothesis. □

Lemma 6.4 (Trace-zero guarded singleton extension)**.** *Suppose $P,Q\neq 0$, $\tau=0$, and $(X_0,Y_0)\in\mathcal{G}_{D_0}^{\mathrm{guard}}$ in size $h\geq 3$. For a generic new diagonal entry z the same symbol pair has a guarded smooth rank-one square-zero seed in size $h+1$ at $D=\operatorname{diag}(D_0,z)$. Moreover, any raw function nonzero at the provisional input below can be retained while restoring all guards on its exact saturated fiber.*

Proof. Write the raw value as $N_0=u_0v_0^{\mathsf{t}}$, with $v_0^{\mathsf{t}}u_0=0$, and set $E=D_0-zI$ and $X=\operatorname{diag}(X_0,0)$. The full new upper and lower Y rectangle maps are

$$F(z)y,\quad wG(z),\qquad \begin{aligned} F(z)&=\sum_{r+s=d}P_{rs}(\operatorname{ad}_{D_0}^r X_0)E^s,\\ G(z)&=\sum_{r+s=d}Q_{rs}(-1)^rE^r(\operatorname{ad}_{D_0}^s X_0).\end{aligned}\tag{23}$$

The largest s in the first sum and largest r in the second have invertible leading coefficients by (22). Homogeneity makes the other exponent unique, so there is no leading cancellation. Choose one z making D regular and both matrices invertible. For $\lambda\neq 0$ take $y=0$ and $w=\lambda v_0^{\mathsf{t}}G(z)^{-1}$. The complete raw value is

$$N_1=\begin{pmatrix}N_0 & 0\\ \lambda v_0^{\mathsf{t}} & 0\end{pmatrix}=\begin{pmatrix}u_0\\ \lambda\end{pmatrix}(v_0^{\mathsf{t}},0),\qquad N_1^2=0,\quad \operatorname{rank}N_1=1.\tag{24}$$

Old controls give the old $\mathfrak{sl}_h$ block after correcting any new rectangles; (23) fills both rectangles. A genuine new orbit direction $Z=\left(\begin{smallmatrix}0 & e\\ 0 & 0\end{smallmatrix}\right)$ gives

$$[Z,N_1]=\begin{pmatrix}\lambda ev_0^{\mathsf{t}} & -N_0e\\ 0 & -\lambda v_0^{\mathsf{t}}e\end{pmatrix}.$$

Choose $v_0^{\mathsf{t}} e \neq 0$. Its bottom-right entry supplies the missing scalar direction; its old-block trace is the opposite scalar, and the remaining old traceless part and rectangles can be corrected. Thus $d\Gamma_D$ is onto $\mathfrak{sl}_{h+1}$. This is an actual orbit normal and does not require either endpoint to be nonzero.

Prescribe any raw new X upper column a and lower row b. Their complete derivative is traceless and can be canceled by the old, Y-rectangle and orbit controls just described. These corrections do not change the specified raw X cross components. Hence (a, b) extends to a kernel tangent of the *exact* $\Gamma_D = N_1$ fiber. For a formal arc parameter t, the first determinant coefficient is

$$\det(\mathrm{ad}_D^j X(t)) = (-1)^{j+1} t^2 \det(\mathrm{ad}_{D_0}^j X_0) b E^j (\mathrm{ad}_{D_0}^j X_0)^{-1} E^j a + O(t^3). \tag{25}$$

Choose $a \neq 0$ and then b outside the finitely many annihilating hyperplanes. All coefficients in (25) are nonzero. The exact fiber is smooth at the provisional point, so the kernel tangent lifts to a formal arc in it. A surjectivity minor, and any raw function nonzero at the provisional point, are units on this arc. The finite-type localization inverting these functions and all guards is nonempty, and has an actual point over the algebraically closed base field. This uses nonemptiness of a localization, not an assertion that an arbitrary formal arc algebraizes. Finally discard the auxiliary g; the raw value is conjugate to N_1, and raw guards and tests have been retained. No global irreducibility of the exact fiber is needed. □

The rectangle and Schur mechanisms also appear in Lemma 5.3. Here the trace-zero orbit normal permits arbitrary raw cross tangents and preserves raw tests in the same exact nilpotent fiber, including when $p = q = 0$.

Lemma 6.5 (Four vertices suffice for the size induction)**.** *Suppose the same nonzero trace-zero pair has guarded hollow tensor spanning in size four at generic diagonal parameters. Then it has that spanning property in every size $n \geq 4$.*

Proof. Work over the algebraic closure of $K(d_1, \ldots, d_n)$. On an old $h = n-1$ vertex set choose a finite guarded spanning collection defined over the algebraic closure of its old diagonal field. The new diagonal entry is transcendental over these data, so Lemma 6.4 applies. Spanning survives scalar extension to the full diagonal field.

If a bilinear T vanishes at all guarded good size-n pairs, its value at each provisional extension must be zero: a nonzero value would be retained by the exact-fiber guard restoration. That value is

$$T_h(X_0, Y_0) + \lambda T\left(\mathrm{diag}(X_0, 0), \begin{pmatrix} 0 & 0 \\ v_0^{\mathsf{t}} G(z)^{-1} & 0 \end{pmatrix}\right).$$

It vanishes for every $\lambda \neq 0$, hence its constant term vanishes. The old spanning collection forces $T_h = 0$. This works after removing any vertex. For $n \geq 5$, a pair of hollow matrix units uses at most four vertices and lies in some $(n-1)$-vertex subset. All coefficients of T are therefore zero, proving the induction. □

Lemma 6.6 (One diagonal open independent of the test)**.** *A hollow tensor-spanning assertion proved at the generic diagonal point by actual smooth nilpotent seeds descends to one nonempty open Ω_n of regular diagonal tuples, with the same fixed-D assertion at every $D \in \Omega_n$. The guards may be included.*

Proof. Select finitely many generic seed tensors forming a basis. Place D, their raw inputs, auxiliary invertible matrices, raw rank-one square-zero equations, a nonzero entry in each value, chosen surjectivity minors, the tensor-basis determinant, and the required nonzero guard determinants in one finite-type incidence scheme. Its projection to diagonal D is dominant. By constructibility its image contains a nonempty open. Every nonempty finite-type fiber over a K-point has an actual K-point, since K is algebraically closed. This gives the same diagonal open for all tests simultaneously.

When a construction uses Laurent series, a nonzero raw test yields an extension-field point of this incidence, hence its nonemptiness. One does not descend a fixed fiber whose target itself

has Laurent coordinates. In applications below the Laurent parameter changes raw X, Y, g while D, and any D-dependent coefficients of the test, are fixed. □

6.2. **The nonzero-endpoint trace-zero stratum.** Throughout this subsection, the complete symbols satisfy

$$p = P(1,-1) \neq 0, \qquad q = Q(1,-1) = -\epsilon p, \qquad \epsilon = (-1)^d. \tag{26}$$

In particular, both symbols are nonzero. There is no restriction on their other coefficients. We write $\mathcal{G}_D^{\mathrm{guard}}$ for the hollow pairs (X,Y) such that $H_D(X,Y)$ is nonzero of rank one and square zero, the differential of Γ_D is onto $\mathfrak{sl}_n$ at (X,Y,I), and

$$\det(\mathrm{ad}_D^j X) \neq 0 \qquad (0 \leq j \leq d).$$

The guards and all tensor evaluations in this subsection concern the *raw* matrices X, Y, before the auxiliary conjugation in Γ_D.

Theorem 6.7. *Assume (26). For every $n \geq 3$ there is a nonempty open set Ω_n of regular diagonal matrices such that, at each fixed $D \in \Omega_n$,*

$$\mathrm{span}\{X \otimes Y : (X,Y) \in \mathcal{G}_D^{\mathrm{guard}}\} = V_n \otimes V_n.$$

Consequently, after arbitrary diagonal parts are allowed in the two original active inputs, the actual smooth nonzero rank-one square-zero seeds detect every nonzero bilinear form on $M_n \times M_n$.

We prove the tensor assertion first over the algebraic closure of the field of generic diagonal parameters. This keeps D fixed throughout each fiber argument. In particular, the coefficients of a bilinear form under examination are allowed to belong to this field, but they are held fixed during every source deformation. The final passage to one common open set of diagonal parameters is Lemma 6.6.

6.2.1. *Guarded detection in size three.*

Lemma 6.8. *Under (26), at a generic regular diagonal D of size three the tensors of $\mathcal{G}_D^{\mathrm{guard}}$ span $V_3 \otimes V_3$. Moreover, the exact local fiber at a generic column-star seed has a nonempty open subset on which all the raw adjoint-power guards hold. One may impose simultaneously any prescribed finite collection of additional guards of the form $\det(W \circ X) \neq 0$, where $W \circ X = (w_{ij}X_{ij})$ and all six off-diagonal w_{ij} are nonzero constants in the fixed coefficient field.*

Proof. Choose $D = \mathrm{diag}(d_1, d_2, d_3)$ with all distinct-three-vertex path values of P and Q nonzero. These conditions hold at the generic diagonal: on any such path the two differences are independent linear forms, and neither symbol is zero. Use the column-star chart (9)–(10), namely

$$X = uE_{23} + vE_{32}, \qquad Y = c_2E_{21} + c_3E_{31}, \qquad uvc_2c_3 \neq 0. \tag{27}$$

For clarity, put

$$\begin{gathered} f = P(d_2 - d_3, d_3 - d_1), \qquad g = P(d_3 - d_2, d_2 - d_1), \\ h = Q(d_3 - d_1, d_1 - d_2), \qquad j = Q(d_2 - d_1, d_1 - d_3), \\ A = (d_1 - d_2)^d, \qquad B = (d_1 - d_3)^d, \qquad C = (d_2 - d_3)^d, \\ n_2 = uc_3f, \qquad n_3 = vc_2g, \qquad \eta = n_2/n_3. \end{gathered}$$

The exact value is $N = n_2E_{21} + n_3E_{31}$. The trace-zero column calculation in the proof of Theorem 5.1, following (11), gives the remaining four-column minor

$$jc_2\big(Agpvc_2^2 + fhuc_3^2\big). \tag{28}$$

Together with the four pure off-block Y columns already displayed there, it proves that Γ_D is smooth onto $\mathfrak{sl}_3$ at a nonempty open set of these star parameters. All the columns use hollow input variations or true simultaneous-conjugation directions; no variation of the diagonal parameters is needed.

We first identify a complete kernel tangent, since both the guard restoration and the tensor calculation depend on its full output. The hollow Y tangent image is spanned by the four

off-block coordinates $12, 13, 21, 31$ and the line $E_{22} - E_{33}$. Modulo this image, use coordinates $(11, 23, 32)$. The raw X directions E_{12}, E_{13} and the orbit directions generated by E_{12}, E_{13} have columns

$$R_2 = (pAc_2, 0, hc_3), \quad R_3 = (pBc_3, jc_2, 0), \quad O_2 = (n_2, 0, -n_3), \quad O_3 = (n_3, -n_2, 0).$$

Set

$$L_2 = pAc_2 + \eta hc_3, \qquad L_3 = pBc_3 + \eta^{-1} jc_2. \tag{29}$$

The following are variations of the actual auxiliary source at (X, Y, I):

$$\begin{aligned} \delta X &= L_3 E_{12} - L_2 E_{13}, \\ \delta g &= \frac{hc_3 L_3}{n_3} E_{12} - \frac{jc_2 L_2}{n_2} E_{13}, \\ \delta Y &= \frac{L_2 L_3}{pCu} E_{32}. \end{aligned} \tag{30}$$

The first two lines have zero output in the displayed quotient. Their remaining output is $-L_2 L_3(E_{22} - E_{33})$. The last line cancels it, because $dH_D(E_{32} \text{ in } Y) = pCu(E_{22} - E_{33})$. Thus *the complete differential* of Γ_D on (30) is zero. In particular, the Y_{32} correction cannot be omitted.

The two scalars L_2, L_3 are nonzero for generic star parameters. Independently add $z_2 E_{21} + z_3 E_{31}$ to δX; it has zero output, since every product of this lower column with the current Y is zero. For a fixed entry-weight matrix W as in the lemma, the determinant along any arc with this tangent has coefficient

$$[t^2] \det(W \circ X(t)) = w_{12} w_{23} w_{31} L_3 u z_3 - w_{13} w_{32} w_{21} L_2 v z_2. \tag{31}$$

Indeed, a hollow 3×3 determinant is the sum of its two oriented three-cycle products. Each such product here contains two entries vanishing at $t = 0$, so second-order corrections to the arc cannot affect this coefficient. The right side is a nonzero linear form in (z_2, z_3). This applies to each $w_{ab} = (d_a - d_b)^j$, $0 \le j \le d$, and to every additional W under consideration. Choose (z_2, z_3) outside the finitely many kernels.

Let $\mathcal{F}$ be the exact fiber $\Gamma_D^{-1}(N)$ and let x_* be the chosen star point with auxiliary parameter I. The source and the map are smooth at x_*, so $\mathcal{F}$ is smooth there, its local ring is a regular local domain, and the selected kernel tangent lifts to a formal arc in this same fiber. Formula (31) makes every required guard a nonzero element of that local domain. A chosen surjectivity minor is a unit. Their product therefore defines a nonempty open subset of the local fiber. More concretely, its formal point proves that the corresponding finite-type localization is nonempty, and over an algebraically closed coefficient field it has an actual point. Discarding the auxiliary g only conjugates the value; it does not change the raw guards or the saturated differential rank.

This argument also gives the precise continuation principle needed below. If a raw bilinear form T vanishes on all guarded good pairs, then its germ on $\mathcal{F}$ at x_* is zero. Otherwise its product with the nonzero guard product would be nonzero in the local domain, and localizing also at T would give a guarded good pair with $T \neq 0$. Consequently both the value of T and every kernel derivative of T vanish at each generic column star. We do not assert that the unguarded star itself satisfies the determinant guards.

The same continuation principle holds at row stars. Write $R^{\mathrm{op}}(s, t) = R(t, s)$. Direct transposition gives

$$H_{P,Q}(D, X^{\mathsf{t}}, Y^{\mathsf{t}}) = \epsilon\, H_{Q^{\mathrm{op}}, P^{\mathrm{op}}}(D, X, Y)^{\mathsf{t}}. \tag{32}$$

The transformed endpoints are $-p$ and ϵp, so the transformed pair again satisfies (26). Moreover

$$\mathrm{ad}_D^j(X^{\mathsf{t}}) = (-1)^j (\mathrm{ad}_D^j X)^{\mathsf{t}}.$$

Thus transposition preserves the guards on the *first* raw slot as well as the rank-one, square-zero and saturated smoothness conditions. Applying the column argument to the transformed symbols proves the row assertion for the original symbols.

We now compute the annihilator. Vanishing at the column stars and row stars, with each isolated vertex in turn, kills every coefficient $T(E_{ab}, E_{cd})$ for which $\{a, b\} \neq \{c, d\}$. To see the coverage explicitly, two different edges in a three-vertex set share exactly one endpoint. A column

star covers either orientation of the complementary X edge with a Y edge ending at the isolated vertex, and a row star covers the orientations with that Y edge starting there. Independence of u, v, c_2, c_3 then kills the individual coefficients.

At (27), consider the raw variation

$$\delta Y = uE_{23} + \epsilon v E_{32}. \tag{33}$$

It preserves the exact value: the new lower diagonal contributions are multiples of $\epsilon p + q = 0$. Its tangent therefore belongs to the exact local fiber above. Since the germ of T on that fiber is zero, its derivative on this tangent is zero. Expanding in u, v and repeating with each isolated vertex gives

$$T(E_{ab}, E_{ab}) = 0, \qquad T(E_{ba}, E_{ab}) = -\epsilon T(E_{ab}, E_{ba}) \quad (a < b).$$

Only this derivative of the germ is used; no guard condition on the points of the shifted-star family is being presumed. The remaining form has the shape

$$T(X, Y) = \sum_{a<b} c_{ab}(X_{ab}Y_{ba} - \epsilon X_{ba}Y_{ab}). \tag{34}$$

Apply its derivative to the *complete* kernel tangent (30). The result is

$$0 = c_{12}c_2L_3 - c_{13}c_3L_2 + \frac{c_{23}}{pC}L_2L_3. \tag{35}$$

For fixed nonzero c_2, c_3, varying u/v makes η vary through a nonempty open subset of the multiplicative line; the smoothness restriction (28) removes only a proper closed subset. Hence (35) is a Laurent-polynomial identity in η. With $\zeta = c_{23}/(pC)$, its coefficients of η^{-1} and η respectively give

$$c_{12} = -\zeta pA, \qquad c_{13} = \zeta pB, \qquad \zeta c_2c_3(hj - p^2AB) = 0. \tag{36}$$

The final equation is the constant coefficient after the first two are substituted.

Suppose first that Q is not a monomial. Put $x = d_2 - d_1$, $y = d_3 - d_1$. If the norm difference in (36) vanished identically, it would give

$$Q(y, -x)Q(x, -y) = p^2x^dy^d = q^2x^dy^d.$$

Unique factorization forces both factors on the left to be monomials, and the invertible substitutions force Q itself to be a monomial, a contradiction. Thus $hj - p^2AB \neq 0$ at the generic diagonal, so $\zeta = 0$ and $T = 0$. If Q is a monomial but P is not, apply the same argument to the transformed pair in (32) and the form $\widetilde{T}(X, Y) = T(X^{\mathsf{t}}, Y^{\mathsf{t}})$. Its reverse symbol P^{op} is not a monomial. This transposition, rather than an exchange of the two active inputs, retains the guard on the required raw X slot.

It remains to treat two monomial symbols. Write

$$P = \alpha_0 s^r t^{d-r}, \qquad Q = \beta_0 s^\ell t^{d-\ell}, \qquad k = r + \ell - d, \qquad \sigma = (-1)^k.$$

The endpoint condition says $\beta_0/\alpha_0 = -\sigma$. As in (14), make the invertible hollow root-space changes $U = \mathrm{ad}_D^r X$, $V = \mathrm{ad}_D^{d-r} Y$. The present trace-zero normalization is

$$\alpha_0^{-1} H_D(X, Y) = J^-(U, V) := UV - \sigma\, \mathrm{ad}_D^k(V)\, \mathrm{ad}_D^{-k}(U). \tag{37}$$

Negative powers here mean only inverse root multipliers at the fixed regular diagonal. They do not change the original polynomial. Put $a = x^k$, $b = y^k$, $c = (y - x)^k$, all nonzero, and take the same support as (15):

$$U = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & a \\ \theta & b & 0 \end{pmatrix}, \qquad V = E_{21} + E_{31}, \qquad \theta \neq 0. \tag{38}$$

Its exact normalized value is

$$J^-(U, V) = \begin{pmatrix} 0 \\ a \\ b \end{pmatrix} \begin{pmatrix} 1 & -1/a & 1/b \end{pmatrix},$$

which is nonzero of rank one and square zero.

For completeness, the trace-zero tangent minor is recorded explicitly. Use the eight columns

$$u_{12}, u_{13}, u_{23}, u_{32}, v_{12}, v_{13}, v_{21}, v_{23}$$

and output rows $(11, 12, 13, 21, 22, 23, 31, 32)$, with the last diagonal entry determined by trace zero. The entry formulas (17) apply with the sign of every reverse-order summand changed. Their indicated derivative matrix is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -\theta & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b^2/c & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sigma a^2/c & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -c\theta/b \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & -b \\ 0 & -a/b & 0 & 0 & 0 & 0 & a/b & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & b & 0 \\ -b/a & 0 & 0 & 0 & \theta & 0 & 0 & 0 \end{pmatrix}.$$

Its determinant is

$$\begin{cases} -ab(\theta c + 2ab^2)/c^2, & \sigma = 1, \\ ab(-\theta c + 2ab^2)/c^2, & \sigma = -1. \end{cases} \tag{39}$$

Choose $\theta \neq 0$ away from its one root. These hollow directions already give the required rank eight. Furthermore, the raw X has exactly one nonzero determinant cycle, $X_{12}X_{23}X_{31}$. Every ad_D^j multiplies its factors by nonzero root differences, so all the guards hold at this very seed.

Equations (36) leave at most the form

$$T = \zeta p\big[-A(X_{12}Y_{21} - \epsilon X_{21}Y_{12}) + B(X_{13}Y_{31} - \epsilon X_{31}Y_{13}) + C(X_{23}Y_{32} - \epsilon X_{32}Y_{23})\big].$$

At (38), undoing the root-space changes gives

$$T(X, Y) = -2\zeta\alpha_0.$$

Indeed, only Y_{21}, Y_{31} are nonzero, and the two contributions inside the brackets both equal $-(-1)^{d-r}$, while $p = \alpha_0(-1)^{d-r}$. Thus $\zeta = 0$ also in the last case. This proves the lemma. □

6.2.2. *The four-vertex tensor coefficients.*

Proof of Theorem 6.7. Size three is Lemma 6.8. By the guarded extension and support argument of Lemmas 6.4 and 6.5, size four is the only further tensor calculation needed. We give that calculation at one fixed generic $D = \mathrm{diag}(d_1, d_2, d_3, d_4)$.

Let T be a bilinear form on $V_4 \times V_4$ vanishing on all guarded good pairs. Restriction to each three-vertex set is zero. Here the use of the extension lemma is essential: if $H_{D_0}(X_0, Y_0) = u_0 v_0^{\mathsf{t}}$ is an old guarded seed, its provisional extension has new lower Y row $\lambda v_0^{\mathsf{t}} G^{-1}$ and $X = \mathrm{diag}(X_0, 0)$. A nonzero value of T there would survive restoration of the guards in its exact fiber. Thus the provisional value is zero for every $\lambda \neq 0$. Its constant term in λ is the value of the restriction of T at the old seed, so the size-three tensor span makes that restriction zero. Consequently we may write

$$T(X, Y) = \sum_{a,b,c,e \text{ pairwise distinct}} C_{ab,ce} X_{ab} Y_{ce}. \tag{40}$$

Designate vertex 4 as new, and isolate vertex 1 in the old three-vertex block. Write $z = d_4$, $D_0 = \mathrm{diag}(d_1, d_2, d_3)$. Start with the generic column star (27), whose value is $N_0 = ne_1^{\mathsf{t}}$, where $n = (0, n_2, n_3)^{\mathsf{t}}$. The two rectangle matrices of Lemma 6.4, in the convention of (18), are

$$\begin{aligned} F(z)_{ab} &= P(d_a - d_b, d_b - z) X_{ab}, \\ G(z)_{ab} &= Q(z - d_a, d_a - d_b) X_{ab} \qquad (a \neq b), \end{aligned} \qquad F(z)_{aa} = G(z)_{aa} = 0. \tag{41}$$

At the generic four-tuple each of these six weights is nonzero: the two displayed arguments are independent diagonal differences. They are fixed constants during the following deformation.

Use the exact-fiber tangent and formal arc from the proof of Lemma 6.8, imposing also $\det F(z) \neq 0$ and $\det G(z) \neq 0$ as its additional entry-weight guards. Formula (31) permits all these choices simultaneously. Thus there is an arc

$$(X(t), Y(t), g(t)) \in \Gamma_{D_0}^{-1}(N_0), \qquad g(t) = I + O(t),$$

on which every required determinant has nonzero quadratic leading term. Only the raw X, Y and the auxiliary g vary with t. The diagonal entries d_1, d_2, d_3, z, and the coefficients of T, do not vary. The surjectivity minor at the initial star remains a unit. The raw old value factors as

$$\begin{aligned} H_{D_0}(X(t), Y(t)) &= g(t)^{-1} N_0 g(t) = u(t) v(t)^{\mathsf{t}}, \\ u(t) &= g(t)^{-1} n, \\ v(t)^{\mathsf{t}} &= e_1^{\mathsf{t}} g(t) = e_1^{\mathsf{t}} + O(t). \end{aligned} \tag{42}$$

The change of the right factor will be included in the inverse calculation; it is not replaced by the constant row e_1^{t}.

Split $G(z)$ into vertex 1 and the block $B = \{2, 3\}$:

$$G(z, t) = \begin{pmatrix} 0 & \rho(t) \\ \kappa(t) & B(t) \end{pmatrix}.$$

At $t = 0$, $B(0)$ is the invertible hollow two-by-two block whose entries are the nonzero weights in (41) times u, v. Both $\rho(t)$ and $\kappa(t)$ are $O(t)$. Its scalar Schur complement satisfies

$$s(t) = -\rho(t) B(t)^{-1} \kappa(t) = \gamma t^2 + O(t^3), \qquad \gamma \neq 0, \tag{43}$$

because $\det G(z, t) = \det B(t)\, s(t)$ and its quadratic coefficient was required to be nonzero. Write $v(t)^{\mathsf{t}} = (v_1(t), v_B(t))$. The block inverse gives exactly

$$\begin{aligned} (v(t)^{\mathsf{t}} G(z, t)^{-1})_1 &= (v_1 - v_B B^{-1} \kappa) s^{-1}, \\ (v(t)^{\mathsf{t}} G(z, t)^{-1})_B &= -(v_1 - v_B B^{-1} \kappa) s^{-1} \rho B^{-1} + v_B B^{-1}. \end{aligned}$$

Here $v_1 = 1 + O(t)$ and $v_B = O(t)$, so the Schur correction $v_B B^{-1} \kappa$ is $O(t^2)$. In particular,

$$(v(t)^{\mathsf{t}} G(z, t)^{-1})_1 = \gamma^{-1} t^{-2} + O(t^{-1}), \qquad (v(t)^{\mathsf{t}} G(z, t)^{-1})_2, (v(t)^{\mathsf{t}} G(z, t)^{-1})_3 = O(t^{-1}). \tag{44}$$

Thus the moving auxiliary conjugation does not cancel the leading pole.

Over the Laurent extension field form the provisional size-four input

$$\widehat{X} = \begin{pmatrix} X(t) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \widehat{Y} = \begin{pmatrix} Y(t) & 0 \\ v(t)^{\mathsf{t}} G(z, t)^{-1} & 0 \end{pmatrix}. \tag{45}$$

The complete output, not merely its leading term, is

$$H_D(\widehat{X}, \widehat{Y}) = \begin{pmatrix} u(t) v(t)^{\mathsf{t}} & 0 \\ v(t)^{\mathsf{t}} & 0 \end{pmatrix} = \begin{pmatrix} u(t) \\ 1 \end{pmatrix} \begin{pmatrix} v(t)^{\mathsf{t}} & 0 \end{pmatrix}.$$

It is nonzero of rank one and square zero. The rectangle matrices are invertible, so Lemma 6.4 gives saturated smoothness at this provisional point as well as restoration of all size-four raw X guards while retaining any raw function nonzero there.

By (40), terms supported on three old vertices are absent. The raw X rectangles involving the new vertex are zero. Equations (27) and (44) therefore give

$$[t^{-2}]\, T(\widehat{X}, \widehat{Y}) = \gamma^{-1} \big(C_{23,41} u + C_{32,41} v \big). \tag{46}$$

Indeed, terms with Y_{42} or Y_{43} must use an old X edge incident to vertex 1. Such X entries are $O(t)$, whereas those two Y entries are only $O(t^{-1})$, so they cannot contribute to the displayed coefficient. The only possible order t^{-2} terms use Y_{41} and $X_{23} = u + O(t)$ or $X_{32} = v + O(t)$.

If either of the two coefficients on the right of (46) were nonzero, choose the independent generic star parameters u, v so that their displayed linear combination is nonzero, retaining the finitely many smoothness and guard conditions. Then T is nonzero at the provisional Laurent input. The guard-restoration part of Lemma 6.4 supplies, over an algebraically closed extension

field, a guarded good raw pair with $T \neq 0$. To turn this into the required contradiction at the fixed generic D, use the finite-type incidence defined by the conditions

$$\operatorname{rank} H_D(X,Y) = 1, \quad H_D(X,Y)^2 = 0, \quad H_D(X,Y) \neq 0,$$

$$d\Gamma_D \text{ onto } \mathfrak{sl}_4, \quad \prod_{j=0}^{d} \det(\operatorname{ad}_D^j X) \neq 0, \quad T(X,Y) \neq 0.$$

Rank one and the differential condition are imposed on the appropriate open minor charts. The construction gives a point of this incidence over an extension of the fixed diagonal coefficient field, so it is nonempty. The algebraically closed coefficient field therefore supplies an actual point, contradicting the hypothesis on T. It is the incidence of *raw* rank-one square-zero values, smoothness, guards and $T \neq 0$ that is descended here. We do not attempt to descend an exact target fiber whose target coordinates have themselves become Laurent series.

It follows that $C_{23,41} = C_{32,41} = 0$. The same argument with each isolated old vertex kills all disjoint-edge coefficients whose Y edge points from the designated new vertex. Transposition (32), applied to the transformed symbol pair and transformed bilinear form, gives the identical lower-row argument for coefficients whose Y edge points to that vertex. It preserves all the raw X guards and does not invoke the size-four conclusion being proved. Finally the designation of the new vertex is free: for any tested Y edge one may designate either of its endpoints as the new vertex and use the other three vertices as the old block. This relabeling includes terms whose X edge contains the original label 4; permuting only the old three labels would not suffice. Every coefficient in (40) is zero.

This proves guarded size-four spanning. Lemma 6.5 now proves it in every size $n \geq 5$: a pair of hollow matrix units uses at most four vertices and is contained in an $(n-1)$-vertex set. The induction is carried out over the generic diagonal field. For each old vertex subset, its finite spanning collection may first be chosen over the algebraic closure of its own diagonal field; the added eigenvalue is transcendental over those old seed data. The rectangle inverses in Lemma 6.4 therefore exist by their invertible leading coefficients. Scalar extension preserves the old span, even when the coefficients of T depend on all diagonal entries. Thus no incompatible numerical diagonal choices are made at different stages. Lemma 6.6 then supplies one nonempty open Ω_n and actual spanning collections at every $D \in \Omega_n$.

The full-matrix detection statement now follows from Lemma 6.3. Its hypothesis holds because st divides both complete symbols, so adding arbitrary diagonal parts to the two original active inputs does not change the value or the available hollow and simultaneous-conjugation smoothness controls. □

6.3. **The complete zero-endpoint seed detector.** We retain the notation H_D, V_n, and Γ_D and the adjoint-power guards introduced above. Throughout this subsection

$$P(1,-1) = Q(1,-1) = 0, \qquad (P,Q) \neq (0,0).$$

Since st divides both symbols, every nonzero symbol is divisible by $st(s+t)$, and in particular $d \geq 3$. For path indices put

$$P_{ikj} = P(d_i - d_k, d_k - d_j), \qquad Q_{ikj} = Q(d_i - d_k, d_k - d_j).$$

At diagonal D the complete map, including both orders, satisfies

$$(H_D(X,Y))_{ij} = \sum_k \big(P_{ikj} X_{ik} Y_{kj} + Q_{ikj} Y_{ik} X_{kj}\big), \qquad (H_D(X,Y))_{ii} = 0. \tag{47}$$

The terms with $k = i$ or $k = j$ vanish. Thus H_D maps $V_n \times V_n$ into V_n.

Theorem 6.9 (Zero-endpoint detection in every size)**.** *For every $n \geq 3$ there is a nonempty Zariski-open set Ω_n of regular diagonal matrices such that, at each fixed $D \in \Omega_n$, the tensors $X \otimes Y$, $(X,Y) \in \mathcal{G}_D$, span $V_n \otimes V_n$. Thus these are actual nonzero rank-one square-zero values, and $d\Gamma_D$ is onto $\mathfrak{sl}_n$ at (X,Y,I). For $n \geq 4$ the same spanning assertion holds for $\mathcal{G}_D^{\mathrm{guard}}$, with all adjoint-power guards on the raw first input X. Allowing arbitrary diagonal additions to the original X and Y gives tensors spanning $M_n \otimes M_n$.*

The one-order fixed-input operators are given in Theorem 4.1 and Corollary 4.2. The size-three star differential is also contained in the proof of Theorem 5.1. We write the cycle operators in the convention (7) and prove the tensor-detection and exact-fiber assertions below.

6.3.1. *The common-zero-row detector in size three.*

Lemma 6.10. *The unguarded hollow assertion of Theorem 6.9 holds in size three, including either one-order boundary.*

Proof. Choose one regular diagonal D for which every triple-distinct evaluation of every nonzero symbol is nonzero. These are finitely many nonzero polynomial conditions: for a fixed triple the two successive differences are independent linear coordinates. Their simultaneous nonvanishing therefore defines a nonempty open set.

Fix an index i, and let $R_i \subset V_3$ be the linear space of hollow matrices with row i zero. For $X, Y \in R_i$ their complementary two-by-two hollow blocks have products supported on the diagonal. Those diagonal contributions vanish in (47). Consequently $H_D(X, Y)$ is supported on column i, off its diagonal. Every nonzero value on $R_i \times R_i$ is therefore rank one and square-zero. We show that a nonempty open subset of this entire irreducible pair space consists of smooth seeds.

Relabel $i = 1$. If both symbols are nonzero, take

$$X = uE_{23} + vE_{32}, \qquad Y = c_2E_{21} + c_3E_{31}, \qquad uvc_2c_3 \neq 0.$$

The value is

$$N = n_2E_{21} + n_3E_{31}, \qquad n_2 = P_{231}uc_3, \quad n_3 = P_{321}vc_2,$$

with $n_2n_3 \neq 0$. Four actual Y variations and two actual X variations give the following pure hollow directions:

variation	δY_{31}	δY_{21}	δY_{13}	δY_{12}	δX_{13}	δX_{12}
output	$P_{231}uE_{21}$	$P_{321}vE_{31}$	$Q_{132}vE_{12}$	$Q_{123}uE_{13}$	$Q_{213}c_2E_{23}$	$Q_{312}c_3E_{32}$.

Here each variation denotes a unit change of the indicated raw entry. All multipliers are nonzero; any potential diagonal contribution is zero by the endpoint hypothesis. The fixed-D derivative therefore fills V_3. The orbit directions generated by E_{12}, E_{13} have diagonal projections $n_2(E_{11} - E_{22})$ and $n_3(E_{11} - E_{33})$. Thus $d\Gamma_D$ has rank eight.

If $Q = 0$ and $P \neq 0$, instead take both X and Y in R_1 with all four entries $21, 31, 23, 32$ nonzero. The output again has both lower column entries nonzero. The Y variations E_{21}, E_{31} give the two pure lower-column directions, with multipliers $P_{321}X_{32}$ and $P_{231}X_{23}$. The X variations E_{12}, E_{13} give the two pure upper-row directions, with multipliers $P_{123}Y_{23}$ and $P_{132}Y_{32}$. Finally the Y variations E_{12}, E_{13} give the two lower off-diagonal directions, with multipliers $P_{312}X_{31}$ and $P_{213}X_{21}$. Thus the fixed-D hollow derivative is again onto, and the same two orbit directions complete it to $\mathfrak{sl}_3$. If $P = 0$, exchange the actual input slots. This proves the required nonempty openness for every $R_i \times R_i$.

Let a bilinear form T vanish on all smooth nilpotent seeds. Its restriction to $R_i \times R_i$ vanishes on a nonempty open subset, hence vanishes identically. Given any two hollow units E_{ab} and E_{cd}, there is an index i different from both row indices a, c. Both units belong to R_i, so $T(E_{ab}, E_{cd}) = 0$. All hollow tensor coefficients vanish. This is a size-three argument: the complementary block has size two, and no common-zero-row assertion in larger size is being used. □

6.3.2. *A nonzero cubic cycle norm.* Set $u = -s - t$ and define the homogeneous polynomial

$$\Delta_{P,Q}(s, t) = P(s, t)P(t, u)P(u, s) + Q(s, t)Q(t, u)Q(u, s). \tag{48}$$

Its coefficients are cubic polynomials in the symbol coefficients. A nonzero one-order symbol has nonzero cycle norm, since all three linear substitutions in its product are invertible.

Lemma 6.11. *If $\Delta_{P,Q} \neq 0$, then for every $n \geq 3$ the determinant of*

$$L_{D,X} : V_n \longrightarrow V_n, \qquad Y \longmapsto H_D(X, Y),$$

is a nonzero polynomial in diagonal D and hollow X. Consequently, for D in a nonempty regular-diagonal open set there is a dense open set of hollow X on which $L_{D,X}$ is invertible, even after imposing all adjoint-power guards on X.

Proof. First take $n = 3$. Use input order

$$(Y_{13}, Y_{21}, Y_{32}; Y_{12}, Y_{23}, Y_{31})$$

and output order

$$(H_{12}, H_{23}, H_{31}; H_{13}, H_{21}, H_{32}).$$

In these orders the full matrix of $L_{D,X}$ is block diagonal with blocks

$$\begin{pmatrix} Q_{132}X_{32} & 0 & P_{132}X_{13} \\ P_{213}X_{21} & Q_{213}X_{13} & 0 \\ 0 & P_{321}X_{32} & Q_{321}X_{21} \end{pmatrix}, \qquad \begin{pmatrix} Q_{123}X_{23} & P_{123}X_{12} & 0 \\ 0 & Q_{231}X_{31} & P_{231}X_{23} \\ P_{312}X_{31} & 0 & Q_{312}X_{12} \end{pmatrix}. \tag{49}$$

Their determinants are respectively

$$X_{13}X_{21}X_{32}(P_{132}P_{213}P_{321} + Q_{132}Q_{213}Q_{321}),$$

$$X_{12}X_{23}X_{31}(P_{123}P_{231}P_{312} + Q_{123}Q_{231}Q_{312}).$$

For $s = d_1 - d_2$, $t = d_2 - d_3$, and $u = d_3 - d_1$, the second parenthesis is $\Delta_{P,Q}(s,t)$ and the first is $(-1)^{3d}\Delta_{P,Q}(t,s)$. Hence, with precisely the displayed input and output orders,

$$\det L_{D,X} = (-1)^{3d} \left(\prod_{i \neq j} X_{ij} \right) \Delta_{P,Q}(s,t)\Delta_{P,Q}(t,s). \tag{50}$$

A different pair of coordinate orders changes only the overall sign. When the cycle norm is nonzero, a regular D making both factors nonzero exists. This gives the size-three assertion.

For the induction with both symbols nonzero, suppose that, at one fixed regular diagonal D_0 of size h, the determinant of $L_{D_0,X}$ is a nonzero polynomial in $X \in V_h$. For each $0 \le j \le d$, the polynomial $\det(\mathrm{ad}_{D_0}^j X)$ is also nonzero: a matrix supported on a Hamiltonian directed cycle is invertible and remains invertible after its edges are multiplied by the corresponding nonzero jth powers of root differences. The intersection of these finitely many nonempty opens in the full affine space V_h is nonempty. Choose X_0 in it. This intersection takes place before any nilpotent-value condition is imposed.

Put

$$D = \mathrm{diag}(D_0, z), \quad E = D_0 - zI_h, \quad X = \begin{pmatrix} X_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} B & v \\ w & 0 \end{pmatrix}, \qquad B \in V_h.$$

The complete hollow operator splits into three independent blocks:

$$H_D(X,Y) = \begin{pmatrix} L_{D_0,X_0}(B) & F(z)v \\ wG(z) & 0 \end{pmatrix}, \quad \begin{aligned} F(z) &= \sum_{r+s=d} P_{rs}(\mathrm{ad}_{D_0}^r X_0)E^s, \\ G(z) &= \sum_{r+s=d} Q_{rs}(-1)^r E^r(\mathrm{ad}_{D_0}^s X_0). \end{aligned} \tag{51}$$

In particular these rectangles use the full P, Q coefficients. Let $s_* = \max\{s : P_{d-s,s} \neq 0\}$ and $r_* = \max\{r : Q_{r,d-r} \neq 0\}$. Their leading coefficients as matrix polynomials in z are

$$[z^{s_*}]F = P_{d-s_*,s_*}(-1)^{s_*}\,\mathrm{ad}_{D_0}^{d-s_*} X_0, \qquad [z^{r_*}]G = Q_{r_*,d-r_*}\,\mathrm{ad}_{D_0}^{d-r_*} X_0. \tag{52}$$

They are invertible. Homogeneity leaves exactly one symbol monomial at each stated largest exponent, so there is no possible leading-term cancellation. Thus both rectangle determinants are nonzero polynomials in z. Choose z avoiding their roots and the eigenvalues of D_0. The complete operator in size $h+1$ is invertible. In the new full hollow X-space its invertibility open again intersects all adjoint-power opens, preparing the next induction step. No intersection with a constrained nilpotent fiber is asserted here.

For completeness, the one-order cases do not use an extension with an absent rectangle. If $Q = 0$, choose regular D with all triple-distinct $P_{ikj} \neq 0$. In column j the operator is the $(n-1)$-square hollow matrix

$$M_j(X)_{ik} = P_{ikj} X_{ik} \quad (i, k \neq j,\ i \neq k).$$

A derangement σ of the indices different from j contributes

$$\operatorname{sgn}(\sigma) \prod_{i \neq j} P_{i,\sigma(i),j} X_{i,\sigma(i)}$$

to its determinant. This coefficient is nonzero, and distinct permutations give distinct monomials in the original entries of X. Thus each $\det M_j$ is nonzero and their product is nonzero. This is the fixed-input operator from Theorem 4.1, not merely an assertion about a union of images. If $P = 0$, write $Q^{\mathrm{sw}}(s,t) = Q(t,s)$ and use the exact identity

$$H_{0,Q}(D, X, Y)^t = (-1)^d H_{Q^{\mathrm{sw}},0}(D, X^t, Y^t). \tag{53}$$

It transfers the preceding invertibility while keeping the first input fixed; replacing a fixed-Y assertion by a fixed-X assertion without this transpose would not suffice. The same unconstrained intersection with the adjoint-power opens applies to both one-order cases.

Finally, a nonzero coefficient in X of the polynomial $\det L_{D,X}$ remains nonzero on a nonempty diagonal open set. Intersecting with the regular locus gives the asserted common open in D; for every D there, the required opens in X are dense. □

Lemma 6.12. *Under the hypothesis of Lemma 6.11, the tensors of guarded smooth nilpotent seeds span $V_n \otimes V_n$ for every $n \geq 3$.*

Proof. Fix an allowed D, and let $U_D \subset V_n$ be the dense open set where $L_{D,X}$ and all the guard matrices are invertible. For $X \in U_D$, a column index j, and a column w with $w_j = 0$ and $w_i \neq 0$ for every $i \neq j$, set

$$N = w e_j^t, \qquad Y = L_{D,X}^{-1}(N). \tag{54}$$

This Y is the actual preimage under the complete operator. With both symbols present it need not itself have column support. The value $H_D(X, Y) = N$ is nonzero rank one and square-zero. Its actual Y derivative fills V_n. Moreover

$$\operatorname{diag}[E_{ji}, N] = w_i (E_{jj} - E_{ii}) \quad (i \neq j),$$

so the actual orbit directions complete the derivative to $\mathfrak{sl}_n$.

Suppose a bilinear form T annihilates every guarded smooth seed. For fixed $X \in U_D$, the linear functional

$$N \longmapsto T\big(X, L_{D,X}^{-1} N\big)$$

vanishes on the full-support open subset of each hollow column space. It therefore vanishes on that entire column space. All these column spaces span V_n, so $T(X, Y) = 0$ for every hollow Y. This holds for a dense open set of X, and consequently $T = 0$. This argument uses all column-target preimages of one actual operator. It does not add values of H_D or assume a column shape for its second input. □

6.3.3. *The zero cycle norm: exact lower kernels.* It remains to treat $\Delta_{P,Q} = 0$. Both symbols are then nonzero, since a nonzero one-order symbol has nonzero cycle norm. In size four choose a regular diagonal tuple such that every triple-distinct P_{ikj} and Q_{ikj} is nonzero. Their finite product is a nonzero polynomial, so these choices form a nonempty open set. We first use three lower vertices a, b, c, reserving a fourth vertex 0.

Lemma 6.13. *Let $A \in V_3$ have all six entries nonzero. The complete lower operator $L_A : B \mapsto H_{D_{\mathrm{low}}}(A, B)$ has rank four. Its kernel has a basis supported on the two opposite directed three-cycles, with each nonzero coordinate a fixed-D nonzero scalar times the corresponding entry of A. Its conormal for the hollow trace pairing also has a basis of two invertible matrices supported on opposite directed three-cycles.*

Proof. In input coordinates (B_{ac}, B_{ba}, B_{cb}) and output coordinates (H_{ab}, H_{bc}, H_{ca}) one block is

$$\begin{pmatrix} Q_{acb}A_{cb} & 0 & P_{acb}A_{ac} \\ P_{bac}A_{ba} & Q_{bac}A_{ac} & 0 \\ 0 & P_{cba}A_{cb} & Q_{cba}A_{ba} \end{pmatrix}. \tag{55}$$

The opposite cycle gives the other block, exactly as in (49). Each determinant is zero by (48), while its upper-left two-by-two minor is nonzero. Each block has rank exactly two.

For the displayed block an explicit right-kernel matrix $B^{(1)}$ is

$$B^{(1)}_{ac} = A_{ac}, \qquad B^{(1)}_{ba} = -\frac{P_{bac}}{Q_{bac}}A_{ba}, \qquad B^{(1)}_{cb} = -\frac{Q_{acb}}{P_{acb}}A_{cb}, \tag{56}$$

with its other entries zero. The first two equations are immediate; the third is precisely $P_{acb}P_{bac}P_{cba} + Q_{acb}Q_{bac}Q_{cba} = 0$. Interchanging b, c gives $B^{(2)}$ on the opposite cycle. In particular the six entries of A remain independent variables; the coefficients in these two kernel vectors depend only on D.

For an equally explicit conormal, a left null vector of (55) is

$$\left(1, -\frac{Q_{acb}A_{cb}}{P_{bac}A_{ba}}, \frac{Q_{acb}Q_{bac}A_{ac}}{P_{bac}P_{cba}A_{ba}}\right).$$

Its three coordinates are nonzero. Transposing its output positions for the trace pairing gives

$$C_1 = E_{ba} - \frac{Q_{acb}A_{cb}}{P_{bac}A_{ba}}E_{cb} + \frac{Q_{acb}Q_{bac}A_{ac}}{P_{bac}P_{cba}A_{ba}}E_{ac}. \tag{57}$$

It satisfies $\operatorname{tr}(C_1 L_A(B)) = 0$ for every hollow B, and is an invertible directed-cycle matrix. Interchanging b, c gives C_2. Their supports are disjoint and opposite. They form a basis of the two-dimensional conormal of the full rank-four lower operator. □

6.3.4. *A full size-four star differential.*

Lemma 6.14. *For every D in the size-four open just chosen, a nonempty open subset of the entire column-star parameter space*

$$X = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \qquad Y = ve_0^t, \qquad (A, v) \in V_3 \times K^3,$$

consists of nonzero rank-one square-zero seeds for which the fixed-D derivative dH_D is onto V_4. The same assertion holds for row stars. The lower block A can simultaneously satisfy all its adjoint-power guards.

Proof. At the displayed input,

$$H_D(X, Y) = N = we_0^t, \qquad w = Fv, \qquad F_{ij} = P_{ij0}A_{ij} \quad (i, j \in \{a, b, c\}), \tag{58}$$

where the diagonal of F is zero. The upper-row Y derivative is $r \mapsto rG$, with

$$G_{ij} = Q_{0ij}A_{ij} \quad (i, j \in \{a, b, c\}), \qquad G_{ii} = 0.$$

The determinants of F and G each have the two distinct cycle monomials $A_{ab}A_{bc}A_{ca}$ and $A_{ac}A_{cb}A_{ba}$ with nonzero coefficients. Each determinant is therefore a nonzero polynomial in the same six original entries of A. The determinants $\det(\mathrm{ad}^j_{D_{\mathrm{low}}} A)$ are likewise nonzero polynomials, as is seen on a single directed-cycle support. Choose one A where all six entries, both rectangle determinants, and all those lower guards are nonzero. This is an intersection of opens in the full irreducible space V_3.

The complete fixed-D hollow Y derivative is the direct sum of these two invertible three-dimensional rectangles and the rank-four operator L_A. Its rank is $3 + 3 + 4 = 10$. Its two-dimensional hollow conormal is precisely the span of C_1, C_2 from Lemma 6.13, embedded in the lower block.

Now vary the three entries of the raw upper row of X by $a = (a_j)_{j \in \{a,b,c\}}$. Its only possibly nonzero output is the lower hollow block

$$(dH_D)_{ij} = Q_{i0j}v_i a_j \quad (i \neq j). \tag{59}$$

The diagonal entries are zero by the endpoint hypothesis. For a lower conormal C, its pairing with this output is

$$\operatorname{tr}(C\,dH_D) = \sum_j a_j (M_C v)_j, \qquad (M_C)_{ji} = C_{ji} Q_{i0j}, \quad (M_C)_{jj} = 0.$$

Both M_{C_1} and M_{C_2} are still invertible directed-cycle matrices: every multiplier on their supports is nonzero. Their opposite disjoint supports prevent proportionality. It follows that $M_{C_1}v, M_{C_2}v$ are independent on a nonempty open set of v. Indeed, dependence for every v would make every vector an eigenvector of $M_{C_1}^{-1}M_{C_2}$, forcing that operator to be scalar, contrary to the supports.

Choose v in that open with both v and Fv of full support. These conditions are simultaneously possible because F is invertible and the vector space is irreducible. The three actual upper-row X directions then fill the two-dimensional quotient of V_4 by the Y image. Thus the full fixed-D hollow derivative has rank 12. The value N in (58) is nonzero rank one and square-zero, with all three complementary column entries nonzero. The orbit directions E_{0i} have diagonal projections $w_i(E_{00} - E_{ii})$, giving the remaining three independent directions and rank 15 for $d\Gamma_D$.

All the conditions used are open in the whole star parameter space, and we have shown their simultaneous nonemptiness. To obtain row stars for the same original symbols use

$$H_{P,Q}(D, X^t, Y^t) = (-1)^d H_{Q^{\mathrm{sw}},P^{\mathrm{sw}}}(D, X, Y)^t, \qquad R^{\mathrm{sw}}(s,t) = R(t,s). \tag{60}$$

The transformed pair has zero endpoints and zero cycle norm, since its norm is $\Delta_{P,Q}(t,s)$. Its triple-distinct weights are nonzero under our same conditions on D, by reversal of the path and homogeneity. The preceding column proof therefore applies to it. Transpose preserves all ranks and the adjoint-power guards, since $\operatorname{ad}_D^j(X^t) = (-1)^j(\operatorname{ad}_D^j X)^t$. □

6.3.5. *Restoring guards in the exact raw fiber.*

Lemma 6.15. *At any column star in the nonempty open constructed in Lemma 6.14, the exact raw fiber $H_D = N$ is smooth, and its local ring contains a nonzero product of all raw X guard determinants. Every function whose germ on this fiber is nonzero can be retained at an actual guarded smooth nilpotent seed in this same fiber. The corresponding assertion holds at row stars.*

Proof. The surjectivity of the fixed-D hollow differential makes

$$\mathcal{F}_N = \{(X', Y') \in V_4 \times V_4 : H_D(X', Y') = N\}$$

smooth at the star p. Its local ring $R = \mathcal{O}_{\mathcal{F}_N,p}$ is a regular local domain. The map from the three raw upper-row X variations to the two-dimensional quotient by the Y image has rank two. Choose a nonzero row a in its one-dimensional kernel. Its complete output lies in the Y image and can therefore be cancelled by an actual hollow Y variation. Independently, every raw lower-column X variation z has zero output: both products with the current $Y = ve_0^t$ vanish. Consequently, for every z, the raw variation

$$\dot{X} = \begin{pmatrix} 0 & a \\ z & 0 \end{pmatrix}$$

can be completed by a hollow $\dot{Y}$ to a tangent vector of the exact fiber $\mathcal{F}_N$. No output conjugation is involved.

Put $E = D_{\mathrm{low}} - d_0 I_3$ and $A_j = \operatorname{ad}_{D_{\mathrm{low}}}^j A$. All E, A_j are invertible. Smoothness lifts any such tangent to a formal arc in $\mathcal{F}_N$. Along it the raw X remains hollow, its upper and lower cross blocks start with ta and tz, and its lower block starts with A. The Schur complement gives, for every guard exponent j,

$$\det(\operatorname{ad}_D^j X(t)) = (-1)^{j+1} t^2 \det(A_j)\, aE^j A_j^{-1} E^j z + O(t^3). \tag{61}$$

In particular second-order changes in the lower block or the cross blocks cannot alter the displayed coefficient; the upper-left raw entry is identically zero. For each j the row $aE^jA_j^{-1}E^j$ is nonzero. Choose one z outside the finitely many annihilating hyperplanes. Every determinant in (61) is then nonzero on the formal arc. Their product is consequently a nonzero element of R. The fixed-D and saturated smoothness minors already nonzero at p are units in R.

Let $b \in R$ be any nonzero function germ. The product of b, the guard product, and these minors is nonzero because R is a domain. On a sufficiently small finite-type neighborhood in the component through p, its nonvanishing defines a nonempty open set. As K is algebraically closed, this open set has an actual K-point. At that point the raw value is still exactly N, all guards hold, the seed is smooth, and b is nonzero. This proves the stated retention property, not merely a formal existence claim. The transpose identity (60) transports the entire raw-fiber argument to row stars. □

Corollary 6.16. *If a bilinear form T on $V_4 \times V_4$ vanishes at every guarded smooth nilpotent seed, its entire germ on the exact raw fiber at every star in Lemma 6.14 is zero.*

Proof. A nonzero germ could be retained at a guarded smooth seed in the same raw fiber by Lemma 6.15, contradicting the hypothesis on T. □

6.3.6. *Six independent monomials and the guarded base.*

Lemma 6.17. *For each D in the above size-four diagonal open, the tensors of guarded smooth nonzero rank-one square-zero seeds span $V_4 \otimes V_4$.*

Proof. Suppose a bilinear form T annihilates all such tensors. By Corollary 6.16 it vanishes at every generic column or row star. Polynomial continuation on the whole irreducible star parameter space makes its restriction identically zero there.

First consider two hollow units E_{ab}, E_{cd} with $\{a,b\} \neq \{c,d\}$. The second edge has an endpoint outside $\{a,b\}$. If d is such an endpoint, use the column star isolated at d: E_{ab} belongs to its lower block and E_{cd} to its column. If c is such an endpoint, use the row star isolated at c instead. Thus

$$T(E_{ab}, E_{cd}) = 0 \qquad \text{if } \{a,b\} \neq \{c,d\}. \tag{62}$$

This eliminates four-distinct-vertex coefficients as well, without an embedding assertion from size three. No relation between the remaining same-edge and reversed-edge coefficients has been assumed.

Fix now an isolated vertex 0 and a generic lower block A. Choose a v making its column star one of the smooth points above. For either of the explicit lower kernel matrices $B^{(\nu)}$ in (56), the line

$$(X, Y_t) = \left(\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, ve_0^t + t \begin{pmatrix} 0 & 0 \\ 0 & B^{(\nu)} \end{pmatrix} \right) \tag{63}$$

lies in the exact same raw fiber: the additional full output is $tL_A(B^{(\nu)}) = 0$. The zero germ from Corollary 6.16, restricted to this line, gives

$$T\left(\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & B^{(\nu)} \end{pmatrix} \right) = 0.$$

Let $\mathcal{C}$ be the directed three-cycle supporting one such kernel vector, and write $B_e = \gamma_e A_e$ on it. Each $\gamma_e \in K^*$ depends only on D, and the cycle contains exactly one orientation of each unordered lower edge. Using (62), the preceding identity is

$$\sum_{e \in \mathcal{C}} \gamma_e \left(T(E_e, E_e) A_e^2 + T(E_{\bar{e}}, E_e) A_{\bar{e}} A_e \right) = 0, \tag{64}$$

where $\bar{e}$ denotes the reverse edge. This identity holds for a nonempty open set of the six independent entries of A, hence is a polynomial identity in them. Its three square monomials and three unordered-edge product monomials are six distinct monomials. Every coefficient displayed in (64) therefore vanishes. The opposite kernel cycle kills the other three square coefficients and the opposite three ordered reverse-edge coefficients on this lower triangle.

Varying the isolated vertex covers every unordered edge of the four-vertex set. All coefficients left after (62) consequently vanish, and $T = 0$. The two kernel shifts were taken in a fiber of the full operator, at a common original input, rather than in fibers of its summands. □

6.3.7. *Completion in all sizes and diagonal enrichment.*

Proof of Theorem 6.9. If the cycle norm is nonzero, Lemmas 6.11 and 6.12 give even guarded hollow detection in every size at least three. If the cycle norm is zero, Lemma 6.10 gives the required unguarded size-three result, while Lemma 6.17 supplies the guarded size-four base for both nonzero symbols.

For this latter case apply the exact guarded singleton extension 6.4 and the support induction 6.5. Their hypotheses concern the complete trace-zero map and guarded smooth nilpotent seeds, all of which have just been verified; they do not require a nonzero endpoint. They give full guarded hollow tensor span in every size $n \geq 5$. The generic fixed-diagonal descent in 6.6 then gives a nonempty numerical diagonal open Ω_n supporting a full spanning collection at each one of its points. In particular D is not selected separately for different tensor coefficients. The size-three branch needs no claim that its raw first input can satisfy the guards.

The common diagonal-enrichment lemma 6.3 upgrades these hollow assertions to full $M_n \otimes M_n$ detection. Its hypotheses hold because st divides both symbols, and the smoothness controls used here are actual hollow input directions and simultaneous conjugation of the entire original tuple. □

The conclusion is a detector for the complete zero-endpoint restriction with arbitrary coefficient dependence between P and Q. Together with the original trace transfer in Theorem 6.1, it applies to every ordinary coefficient fiber having this restriction, including corrections whose complete chosen-pair restriction is zero. No standalone outer-adjoint or corank-one hypothesis is needed, and no size-two or all-pairs-zero assertion is part of Theorem 6.9.

6.4. **The complete original trace on the same nilpotent fiber.** We now prove Theorem 6.1. This step uses the actual matrix-entry polynomial $\operatorname{tr} f$ at the specified size, not a formal free trace coefficient and not a selected low-activity detector. In particular, all fixed-size polynomial identities are included before the first nonzero trace term is selected.

Proof of Theorems 6.2 and 6.1. For the seed theorem, $\tau = 0$ implies $q = -\epsilon p$. The case $p \neq 0$ is Theorem 6.7; the case $p = 0$ is Theorem 6.9. In the latter proof the cycle norm is either nonzero or identically zero. If only one symbol is nonzero its norm cannot vanish, since it is a product of three nonzero polynomials under invertible substitutions. Thus these cases exhaust every nonzero pair, without imposing a product or fixed-depth hypothesis. Lemmas 6.6 and 6.3 give the stated common diagonal open set and the full, rather than hollow, bilinear detector.

For the original map, if $\tau \neq 0$, the smooth nilpotent input of Theorem 5.2 is an input of the complete restriction of f. Its tangent directions are original-source directions, so the full original differential is onto M_n. Writing $h = \operatorname{tr} f$ and $g = f - n^{-1} h I_n$, we obtain $\operatorname{rank}(dg_a) = n^2 - 1$ and $dh_a|_{\ker(dg_a)} \neq 0$ at this input a. Thus $h|_{g^{-1}(N)}$ has a nonzero linear term at a. Theorem 2.1 gives the actual image and smoothness gives the fiber assertion. Hence assume $\tau = 0$ for the rest of the proof, and put

$$\pi(A) = A - \frac{\operatorname{tr} A}{n} I_n, \qquad g = \pi f, \qquad h = \operatorname{tr} f.$$

Choose one passive original variable as an anchor D. In each of the other $m - 3$ passive positions put $D + Z_i$; retain the original active variables X, Y. These are invertible linear coordinates on M_n^m. The linear subspace $S = \{Z_1 = \cdots = Z_{m-3} = 0\}$ is the complete restriction source. On it $h = 0$ and $g = H_{P,Q}$. If h is an actual identity, any seed in Theorem 6.2 gives an original smooth input into $\mathfrak{sl}_n$. Theorem 2.1 then gives $\operatorname{Val}_n(f) = \mathfrak{sl}_n$ and the local fiber dimension $mn^2 - n^2 + 1$.

Suppose instead that h is a nonzero polynomial in actual matrix entries. Expand it by total degree in the normal variables:

$$h(D, X, Y, Z) = \sum_{j=v}^{m-3} H_j(D, X, Y, Z), \qquad H_v \neq 0, \quad 1 \leq v \leq m - 3. \tag{65}$$

Every H_j is homogeneous of degree j in Z and bilinear in the *full* original matrices X, Y. Indeed, every word of f uses each original slot once. Lower components in (65) vanish as actual size-n polynomials, including cancellations arising from trace identities. Each component is separately invariant under simultaneous conjugation: apply that invariance to $h(D, X, Y, tZ)$ and compare powers of the scalar t.

The conjugation-saturation of the nonempty diagonal open set Ω_n in Theorem 6.2 is dense in M_n. Indeed, if a polynomial vanishes on that saturation, then for each fixed conjugating matrix it vanishes on the corresponding conjugate of the full diagonal space by density of Ω_n in that space. It therefore vanishes on the dense regular semisimple locus. Invariance of H_v now shows that there are $D_0 \in \Omega_n$ and fixed normal matrices Z^0 for which

$$T(X, Y) = H_v(D_0, X, Y, Z^0)$$

is a nonzero bilinear form on $M_n \times M_n$. For if it vanished for every such D_0, Z^0, X, Y, conjugation and density would force $H_v = 0$. The seed detector chooses original X_0, Y_0 at this *same* D_0 such that $T(X_0, Y_0) \neq 0$, the value $N = H_{P,Q}(D_0, X_0, Y_0)$ is nonzero rank-one square-zero, and the restriction differential onto $\mathfrak{sl}_n$ is surjective. Let $a = (D_0, X_0, Y_0, 0)$ in the original linear coordinates.

At a the map g is smooth, already using directions in T_aS. Choose an $(n^2 - 1)$-dimensional control space $L \subset T_aS$ on which dg_a is an isomorphism onto $\mathfrak{sl}_n$. Complete it to linear source coordinates (ℓ, w, Z) centered at a. The formal implicit function theorem solves the *exact* partial fiber $g = N$ uniquely as $\ell = \phi(w, Z)$ in a formal neighborhood, with $\phi(0, 0) = 0$. Since $L \subset T_aS$, this solution does not change any normal coordinate Z_i.

The total lowest Taylor term of h at a is precisely

$$H_v(D_0, X_0, Y_0, Z). \tag{66}$$

It is nonzero by the chosen Z^0. A variation of any coefficient variable D, X, Y in H_v adds at least one to its total degree; all H_j with $j > v$ have larger total degree already. Substitution of the exact control correction $\phi(w, Z)$ therefore leaves (66) unchanged. This is a statement about the complete lowest Taylor term in actual original coordinates, not just a normal symbol before solving the fiber.

The local ring of $g^{-1}(N)$ at a is regular of dimension $mn^2 - n^2 + 1$ and is a domain. Its completion is the formal ring just used, so h is a nonzero element of that local ring. It vanishes at a and is a non-zero-divisor. Adding the trace equation gives the full fiber $f^{-1}(N)$ as a local complete intersection of dimension $mn^2 - n^2$. The same local trace-germ and miracle-flatness argument as Proposition 2.3 proves that f is flat at this original input. Corollary 2.2 and Theorem 2.1, including their ordinary multilinear scalar extraction, give the actual image M_n. No scalar value is supplied merely by taking a closure. If $v = 1$, the surviving linear trace term supplies the missing normal and the point is smooth into M_n. If $v \geq 2$, its Jacobian rank is exactly $n^2 - 1$: it is flat but not smooth. These alternatives are compatible with the asserted local complete-intersection fiber. □

Corollary 6.18 (Full higher-ideal coefficient fibers)**.** *Let F satisfy the complete restriction hypothesis of Theorem 6.1. Every same-degree ordinary polynomial $f = F + R$ with zero complete selected restriction for R has the image and local-fiber classification in that theorem. In particular, R may range over the entire proper multilinear J^3 component and all higher powers. The target is determined separately by* $\operatorname{tr}(F + R)$ *on the fixed M_n.*

Proof. The complete selected restriction is unchanged. The vanishing of proper J^3 on the whole two-active source is Lemma 3.2. We use that vanishing only to identify the restriction. Transverse derivatives of R are not set to zero: they are included in h and in every coefficient of (65). The theorem reselects the seed for the corrected polynomial, rather than asserting that an old nilpotent input remains suitable. □

6.5. **Coefficient scope, examples, and the remaining boundary.** For a fixed ordered pair of original active positions, the free proper degree-m J^2 coefficient space maps onto the full

symbol space

$$\{(P,Q) : \deg P = \deg Q = m-2,\ st \mid P,Q\}, \qquad \dim = 2(m-3).$$

To see surjectivity without a dimension heuristic, partition the $m-2$ passive original leaves into sets of sizes $r, s \geq 1$ and form one iterated commutator on the first active leaf and its r passive leaves, and another on the second active leaf and its s passive leaves. Their ordered product restricts, up to a nonzero sign, to $(\mathrm{ad}_D^r X)(\mathrm{ad}_D^s Y)$. Reversing the product gives the other order. All passive leaves remain distinct. As $r+s = m-2$ varies these are exactly the two monomial bases. Thus the kernel of this *fixed pair* map has codimension $2(m-3)$. This is not a codimension formula for the common kernel over all pairs and is not a quotient by size-n polynomial identities. Every nonzero fiber of this map is covered, with the actual trace deciding its target.

Remark 6.19 (A high-activity fixed-size trace identity does not obstruct the transfer). Take twenty-eight distinct original variables, arranged into fourteen pairs, and write $A_i = [a_i, a_i']$, $B_i = [b_i, b_i']$, $1 \leq i \leq 7$. Set

$$R = \sum_{\sigma \in S_7} \mathrm{sgn}(\sigma) A_{\sigma(1)} B_1 A_{\sigma(2)} B_2 \cdots A_{\sigma(7)} B_7.$$

This is proper, ordinary multilinear, and belongs to J^{14}. On M_3, every common-D restriction with at most fourteen active original slots is zero. With fewer than fourteen, one basic commutator has both entries D. With fourteen, either that still happens or each factor has one active entry. In the latter case all seven A_i belong to $\mathrm{im}\,\mathrm{ad}_D$. For regular diagonal D this space has dimension six, so alternation kills the value. Conjugation and polynomial density extend the identity to every D. This covers every choice of active original slots.

Nevertheless $\mathrm{tr}\, R$ is not an identity. Use

$$(A_1, \ldots, A_7) = (E_{12}, E_{13}, E_{21}, E_{23}, E_{11} - E_{22}, E_{31}, E_{32}),$$

$$(B_1, \ldots, B_7) = (E_{21}, E_{32}, E_{12}, E_{31}, E_{13}, E_{13}, E_{21}).$$

Here is a direct certificate requiring no enumeration of 7! terms. Let (u_i, v_i) be the following ordered coordinate pairs:

$$(1,2), (1,3), (2,1), (2,3), (1,1), (3,1), (3,2).$$

Then $B_i = E_{v_i, u_{i+1}}$ cyclically and

$$\mathrm{tr}(C_1 B_1 \cdots C_7 B_7) = \prod_{i=1}^{7} (C_i)_{u_i v_i}.$$

The coordinate-evaluation matrix of the seven displayed A_i is I_7. Hence the alternating trace is its determinant, namely one. All values have independent original lifts: $E_{ij} = [E_{ii}, E_{ij}]$ and $E_{11} - E_{22} = [E_{12}, E_{21}]$. Embedding these inputs in a three-by-three block proves actual trace nonvanishing for every $n \geq 3$. With $D = \mathrm{diag}(0,1,2)$, the thirteen off-diagonal factors can be written $[D, E_{ij}/(d_i - d_j)]$, using one active original slot each; the diagonal factor uses two. Thus fifteen active slots already detect the trace. The example is not invisible to every common-D template.

Label the same twenty-eight original leaves $x_1, \ldots, x_{28}$, and let C_{14} denote a left-normed commutator on fourteen distinct leaves. Put

$$F = [C_{14}(x_1, \ldots, x_{14}), C_{14}(x_{15}, \ldots, x_{28})].$$

The complete restriction with active x_1, x_{15} is $[\mathrm{ad}_D^{13} X, \mathrm{ad}_D^{13} Y]$; the two signs from the left-normed combs cancel. Thus $F + R$ has actual image M_n for every $n \geq 3$ by Theorem 6.1. In size three its entire trace is still zero on every common-D source with at most fourteen active slots. The theorem uses the first surviving actual normal term, not the false inference that a low-activity detector must see all traceful higher-ideal coefficients.

The common zero restriction region remains outside this theorem. It contains proper J^3 itself and also genuine second-layer examples such as s_4, whose two passive equal inputs make every chosen-pair restriction zero by alternation. Individual members may have other classifications. In particular, $\tau = 0$ alone is not the hypothesis: some *complete* pair (P, Q) must be nonzero. The fixed-size actual trace is still required to determine the target. Size two is not covered by the present seed theorem. No resolution of general LK is asserted.

Here the zero-restriction limitation refers to Theorem 6.1. The independent-diagonal Theorem 1.1 covers every genuine proper second-layer member of that region, including s_4, while retaining the exclusions of pure J^3, size two, and general LK.

The actual-image criterion, scalar extraction, local trace-germ criterion, one-order column operators, cubic cycle determinants, and highest-power rectangle arguments are established earlier in this manuscript. The proofs here reuse those mechanisms and supply the compatible full tensor detectors, exact raw-fiber controls and whole original trace transfer needed for this complete coefficient region. They do not claim historical priority for those ingredients.

7. The entire proper second commutator layer

The common-background theorem of Section 6 retains only the sums of the independent passive coefficients at each fixed depth. We now keep each passive diagonal independently. This recovers every proper ordinary polynomial in the second commutator layer, including the kernel of all common-background pair tests. The field remains algebraically closed of characteristic zero. The ideal J is the two-sided commutator ideal of the unital free associative algebra, as in Proposition 3.1.

The common-background branch is already Theorem 6.1. The additional proof has two parts: a compatible nilpotent tensor span for the complete independent-diagonal restriction, and an étale chart of the entire original source that retains its first nonzero trace normal. In particular, a triangular value alone is not used as an ambient flatness certificate.

7.1. **The exact original two-slot restriction.** Choose two original positions X, Y and label the other positions $D_1, \dots, D_d$, where $d = m - 2 \geq 2$. For diagonal passive inputs write $\boldsymbol{v}_a = ((D_i)_{aa})_{i=1}^d$. For a subset $S \subseteq [d]$ and a vector $a = (a_i)$ put $a_S = \prod_{i \in S} a_i$, with empty product 1.

Proposition 7.1 (Independent passive coefficients)**.** *For each original pair of a proper multilinear $f \in J^2$ there are unique arrays $(c_S), (e_S)$, indexed by $\varnothing \neq S \subsetneq [d]$, for which*

$$P(a,b) = \sum_{\varnothing \neq S \subsetneq [d]} c_S a_S b_{S^c}, \qquad Q(a,b) = \sum_{\varnothing \neq S \subsetneq [d]} e_S a_S b_{S^c}, \tag{67}$$

and the complete actual restriction is

$$\begin{aligned} H_D(X,Y)_{ac} &= f(X, Y; D_1, \dots, D_d)_{ac} \\ &= \sum_b \big[P(\boldsymbol{v}_a - \boldsymbol{v}_b, \boldsymbol{v}_b - \boldsymbol{v}_c) X_{ab} Y_{bc} \\ &\qquad + Q(\boldsymbol{v}_a - \boldsymbol{v}_b, \boldsymbol{v}_b - \boldsymbol{v}_c) Y_{ab} X_{bc} \big]. \end{aligned} \tag{68}$$

The positions on the first line are the original positions, regardless of the order used to display their names. At least one pair has $(P, Q) \neq (0, 0)$ when $f \notin J^3$.

Proof. Expand each ordinary word according to the order of X, Y and the three passive gaps before, between and after them. This gives two path coefficients separately linear in each triple $((D_i)_{aa}, (D_i)_{bb}, (D_i)_{cc})$. Properness is scalar-translation invariance in every passive variable, so each triple occurs through one of its two successive differences. Thus each coefficient has a unique expansion in the 2^d products $a_S b_{S^c}$ before the two extreme subsets are removed.

A one-active evaluation with diagonal passive inputs vanishes. An off-diagonal entry is an evaluation of f on a two-dimensional upper triangular block, where J^2 vanishes; diagonal entries also vanish. Make X diagonal, and let Y have a single off-diagonal entry. The two independent endpoint entries of X give $P(0, b) = Q(a, 0) = 0$. Interchanging the roles of X, Y gives $P(a, 0) =$

$Q(0,b)=0$. Hence both extreme coefficients vanish separately. This proves (67) and (68) for the unchanged original polynomial.

There is also exact finite extraction. On three vertices put

$$D_i^{(S)}=\begin{cases}E_{11}, & i\in S,\\ -E_{33}, & i\notin S.\end{cases}$$

Then

$$c_S=f(E_{12},E_{23};D^{(S)})_{13},\qquad e_S=f(E_{23},E_{12};D^{(S)})_{13}.$$

Every UT_3 evaluation expands into diagonal and strictly upper triangular inputs. Terms with at most one radical input vanish as above, and a word with three radical inputs vanishes because the radical has cube zero. The remaining terms have exactly two original active positions and are precisely these path tests. Thus vanishing of all the pair symbols would make f an identity of UT_3. The classical triangular identity theorem $\mathrm{Id}(UT_r)=J^r$, used in the commutator-degree framework of [3, Theorem 2.1], gives the final assertion for $r=3$. □

The last implication uses the triangular identity theorem, not a claim that a nonzero free polynomial is nonzero on every fixed matrix algebra. Here the fixed-size issue can be settled directly: for $n\geq 3$ the map

$$UT_3\longrightarrow M_n,\qquad T\longmapsto \mathrm{diag}(T,t_{11}I_{n-3})$$

is a unital algebra embedding, since $T\mapsto t_{11}$ is a character. Consequently

$$\mathrm{Id}(M_n)\subseteq\mathrm{Id}(UT_3)=J^3\qquad(n\geq 3).\tag{69}$$

In particular the theorem's polynomial is not an identity in any of its asserted sizes. No such inclusion is used for $n=2$.

If some common-D restriction is nonzero, the ordinary homogeneous symbols obtained by setting $a=s\mathbf{1}$, $b=t\mathbf{1}$ satisfy the hypotheses of Theorem 6.1. We therefore assume for the new branch that all original pair restrictions on a common D vanish, and choose a nonzero independent pair from Proposition 7.1. Then

$$\sum_{|S|=r}c_S=\sum_{|S|=r}e_S=0\qquad(1\leq r\leq d-1).$$

It follows as a polynomial identity in all d coordinates that

$$P(a,-a)=a_{[d]}\sum_S(-1)^{d-|S|}c_S=0,\qquad Q(a,-a)=0.\tag{70}$$

Thus H_D is hollow for every diagonal tuple and all full X,Y. It ignores diagonal parts of the active inputs, since both S and S^c are nonempty. The seed argument below only requires (67), a nonzero pair, and (70); it does not require the stronger grouped coefficient sums once the endpoints have been established.

7.2. **Independent-vector nilpotent tensor detection.** Let $V_n=\{X\in M_n : X_{aa}=0\}$ and, for a fixed diagonal tuple $D=(D_1,\dots,D_d)$, define

$$\Gamma_D(X,Y,g)=gH_D(X,Y)g^{-1},\qquad g\in GL_n.$$

Its orbit directions are actual directions of the original polynomial: one simultaneously conjugates every D_i,X,Y. All tests and guards below concern the raw X,Y, before this conjugation. For $S\subseteq[d]$ define the composition of commuting adjoint operators

$$K_S(X)=\left(\prod_{i\in S}\mathrm{ad}_{D_i}\right)(X),\qquad K_\varnothing(X)=X.\tag{71}$$

Thus $(K_S(X))_{ab}=X_{ab}\prod_{i\in S}((D_i)_{aa}-(D_i)_{bb})$; (71) is not a product of the matrices $\mathrm{ad}_{D_i}(X)$.

Theorem 7.2 (One fixed diagonal open for all bilinear tests)**.** *Assume (67), (70) and $(P,Q)\neq(0,0)$. For every $n\geq 3$ there is a nonempty open set Ω_n of diagonal tuples, with each D_i regular, such that at every fixed $D\in\Omega_n$ the tensors $X\otimes Y$ of hollow pairs satisfying*

$$N=H_D(X,Y)\neq 0,\qquad \operatorname{rank}N=1,\qquad N^2=0,\qquad \operatorname{rank}d\Gamma_D|_{(X,Y,I)}=n^2-1$$

span $V_n\otimes V_n$. For $n\geq 4$ the same assertion holds after imposing all 2^d guards

$$\det K_S(X)\neq 0\qquad (S\subseteq[d]),\tag{72}$$

including $S=\varnothing$. Allowing arbitrary diagonal parts of X,Y then supplies one such original nilpotent seed where any prescribed nonzero bilinear form on full $M_n\times M_n$ is nonzero.

The span is only a bilinear detector. It selects one input compatible with a subsequently specified nonzero test and does not add actual polynomial values. Both the fixed-D quantifier and the full active matrix spaces will be needed for the trace normal.

For notation write

$$P_{ikj}=P(\boldsymbol{v}_i-\boldsymbol{v}_k,\boldsymbol{v}_k-\boldsymbol{v}_j),\qquad Q_{ikj}=Q(\boldsymbol{v}_i-\boldsymbol{v}_k,\boldsymbol{v}_k-\boldsymbol{v}_j).$$

For every distinct triple i,k,j each nonzero symbol gives a nonzero polynomial in the diagonal parameters: the two vector differences can be prescribed independently. The product of all such nonzero path weights and all coordinatewise eigenvalue differences is therefore nonzero. We may impose their simultaneous nonvanishing on one diagonal open. At any such D, each guard determinant is a nonzero polynomial on the full affine space V_h, $h\geq 3$: take X supported on a directed Hamiltonian cycle, whose nonzero edges remain nonzero after every subset rescaling. This observation alone does not put the guards on a nilpotent fiber; that step is proved below.

Lemma 7.3 (The common zero row in size three)**.** *The unguarded assertion of Theorem 7.2 holds in size three on the simultaneous path-weight open just described.*

Proof. Let $R_i\subset V_3$ be the matrices with row i zero. For $X,Y\in R_i$, the complementary two-by-two hollow block can contribute only diagonal products, and these vanish by (70). Every nonzero output is consequently a column star supported in column i, of rank one and square zero.

Use $i=1$ first. If both symbols are nonzero, put

$$X=uE_{23}+vE_{32},\qquad Y=c_2E_{21}+c_3E_{31},\qquad uvc_2c_3\neq 0.$$

The value has entries $N_{21}=P_{231}uc_3$ and $N_{31}=P_{321}vc_2$, both nonzero. The Y variations E_{21},E_{31} control respectively H_{31},H_{21}; the Y variations E_{12},E_{13} control H_{13},H_{12} through Q; the X variations E_{12},E_{13} control H_{32},H_{23} through Q. The indicated weights and parameters are all nonzero, so these are six independent hollow controls. The actual orbit directions E_{12},E_{13} have diagonal projections $N_{21}(E_{11}-E_{22})$ and $N_{31}(E_{11}-E_{33})$. Thus the saturated differential has rank eight.

If $Q=0$, take every entry $21,31,23,32$ of both X,Y nonzero. The Y_{21},Y_{31} variations give the two lower-column coordinates; X_{12},X_{13} give the upper row using Y_{23},Y_{32}; and Y_{12},Y_{13} give the lower off-diagonal coordinates using X_{31},X_{21}. Additional lower-column components, if present, are removed with the first two controls. Again the hollow rank is six and the orbit rank is eight. If $P=0$, exchange the actual active inputs. Relabeling vertices gives the construction on each $R_i\times R_i$.

The good conditions are a nonempty open in the whole irreducible space $R_i\times R_i$. A bilinear form vanishing on all good seeds therefore vanishes identically on each such space. Any two hollow matrix units omit a common row in size three; hence every tensor coefficient is zero. This proves the assertion without imposing a one-variable homogeneity condition or a size-three guard. □

7.2.1. *Cycle determinants with vector differences.* Put $a = \boldsymbol{v}_1 - \boldsymbol{v}_2$, $b = \boldsymbol{v}_2 - \boldsymbol{v}_3$, and $c = -a - b$. Define

$$\Delta(a, b) = P(a, b)P(b, c)P(c, a) + Q(a, b)Q(b, c)Q(c, a). \tag{73}$$

The norm is a polynomial in $2d$ independent coordinates. In size three, use the input and output orders

$$(Y_{13}, Y_{21}, Y_{32}; Y_{12}, Y_{23}, Y_{31}), \qquad (H_{12}, H_{23}, H_{31}; H_{13}, H_{21}, H_{32}).$$

The operator $L_{D,X} : Y \mapsto H_D(X, Y)$ then has the two cyclic blocks

$$\begin{pmatrix} Q_{132}X_{32} & 0 & P_{132}X_{13} \\ P_{213}X_{21} & Q_{213}X_{13} & 0 \\ 0 & P_{321}X_{32} & Q_{321}X_{21} \end{pmatrix}, \qquad \begin{pmatrix} Q_{123}X_{23} & P_{123}X_{12} & 0 \\ 0 & Q_{231}X_{31} & P_{231}X_{23} \\ P_{312}X_{31} & 0 & Q_{312}X_{12} \end{pmatrix}. \tag{74}$$

Their determinants are respectively

$$(-1)^{3d} X_{13}X_{21}X_{32}\Delta(b, a), \qquad X_{12}X_{23}X_{31}\Delta(a, b).$$

The sign follows from simultaneous reversal of every difference: $P(-b, -a) = (-1)^d P(b, a)$, and similarly for Q. Only total degree d is used, not homogeneity in one scalar difference. If Δ is nonzero, both oriented factors are simultaneously nonzero on an open. If $\Delta = 0$, both P, Q are nonzero: when only one is present, the three invertible linear substitutions in its product cannot make a nonzero polynomial vanish in a domain.

Lemma 7.4 (Independent diagonal rectangle operators)**.** *Suppose $P, Q \neq 0$ and every $K_S(X_0)$ is invertible in size h. Adjoin new independent diagonal entries $z = (z_1, \dots, z_d)$ and set $E_i = D_i^{\mathrm{old}} - z_i I_h$, $E_S = \prod_{i \in S} E_i$. At $X = \mathrm{diag}(X_0, 0)$ the complete two new Y rectangles are $F(z)y$ and $wG(z)$, where*

$$F(z) = \sum_S c_S K_S(X_0) E_{S^c}, \qquad G(z) = \sum_S e_S (-1)^{|S|} E_S K_{S^c}(X_0). \tag{75}$$

Both determinant polynomials in the independent z_i are nonzero.

Proof. The block formula follows directly from (68). In particular the P and Q sums act on separate rectangles. The old hollow operator is unchanged, and diagonal outputs vanish identically.

Assign weights $w_i = 2^{i-1}$ to the z_i. All subset weight sums are distinct, and every proper submonomial of a squarefree subset monomial has smaller weight. Choose S_* with $c_{S_*} \neq 0$ maximizing $\sum_{i \in S_*^c} w_i$. The unique top-weight term of F is

$$(-1)^{|S_*^c|} c_{S_*} K_{S_*}(X_0) z_{S_*^c}.$$

Every other monomial in every entry has strictly smaller weight. Thus in the determinant the unique top-weight monomial has coefficient

$$\det\big((-1)^{|S_*^c|} c_{S_*} K_{S_*}(X_0)\big) \neq 0$$

and monomial $z_{S_*^c}^h$. Any determinant product using a lower-weight term has smaller total weight and cannot cancel it. For G select S_* maximizing its subset weight among $e_S \neq 0$. The signs from $(-1)^{|S|} E_S$ cancel in the top coefficient, which is $e_{S_*} K_{S_*^c}(X_0)$, again invertible. This proves actual matrix determinant nonvanishing in every fixed size; it makes no assumption that a formal noncommutative leading word avoids matrix identities. □

Lemma 7.5 (Invertible hollow operators and bilinear detection)**.** *If $\Delta \neq 0$, the guarded hollow assertion of Theorem 7.2 holds in every $n \geq 3$.*

Proof. When both symbols are present, the cyclic determinants give the invertible $L_{D,X}$ locus in size three. Intersect it with all subset guards in the full affine hollow X-space. Given such a size-h point, the old operator and the two rectangles of (75) are independent blocks of the operator at $X = \mathrm{diag}(X_0, 0)$. Lemma 7.4 permits one choice of z avoiding both determinant zero sets and every coordinatewise eigenvalue collision. The new operator is invertible. Its nonempty open locus in the entire new V_{h+1} intersects every new subset guard, preparing the next step. No constrained nilpotent fiber is used in this open intersection.

If $Q = 0$, a different proof avoids an absent rectangle. For each output column j, the operator on vertices other than j is the hollow $(n-1)$-square matrix

$$M_j(X)_{ik} = P_{ikj}X_{ik} \quad (i,k \neq j,\ i \neq k).$$

A derangement of these $n-1 \geq 2$ vertices contributes a determinant monomial with nonzero path-weight coefficient. Distinct permutations give distinct monomials in the independent entries of X, so $\det M_j$ is nonzero. The product over j, and with all the subset guards, is nonzero. For $P = 0$ the row version follows by transposition of the full map, with the two difference arguments interchanged. Thus it also produces an invertible fixed-X operator, rather than merely an invertible operator with the other input fixed.

A nonzero coefficient of $\det L_{D,X}$ remains nonzero on a nonempty diagonal open. At each such fixed D let $U_D \subset V_n$ be its dense invertibility-and-guards open. For $X \in U_D$, choose any full-support hollow column star $N = we_j^t$ and set $Y = L_{D,X}^{-1}N$. This is an actual input; with both orders present Y need not itself be a column star. The Y derivative fills V_n, and the orbit directions have diagonal projections $w_i(E_{jj} - E_{ii})$, $i \neq j$. Hence this is a smooth nilpotent seed.

If T vanishes at all such seeds, for fixed $X \in U_D$ the linear functional $N \mapsto T(X, L_{D,X}^{-1}N)$ vanishes on a dense open of every hollow column space. It vanishes on all those spaces and therefore on their span V_n. Thus $T(X,Y) = 0$ for all hollow Y; density in X gives $T = 0$. No addition of actual polynomial values occurs in this linear-functional argument. □

7.2.2. *The zero cycle norm on four vertices.* It remains to treat $\Delta = 0$. Both symbols are then nonzero, since a nonzero one-order symbol has nonzero cycle norm. In size four choose a diagonal tuple with every D_i regular such that every triple-distinct P_{ikj} and Q_{ikj} is nonzero. Their finite product is a nonzero polynomial, so these choices form a nonempty open set. We first use three lower vertices a, b, c, reserving a fourth vertex 0.

Lemma 7.6. *Let $A \in V_3$ have all six entries nonzero. The complete lower operator $L_A : B \mapsto H_{D_{\mathrm{low}}}(A,B)$ has rank four. Its kernel has a basis supported on the two opposite directed three-cycles, with each nonzero coordinate a fixed-D nonzero scalar times the corresponding entry of A. Its conormal for the hollow trace pairing also has a basis of two invertible matrices supported on opposite directed three-cycles.*

Proof. In input coordinates (B_{ac}, B_{ba}, B_{cb}) and output coordinates (H_{ab}, H_{bc}, H_{ca}) one block is

$$\begin{pmatrix} Q_{acb}A_{cb} & 0 & P_{acb}A_{ac} \\ P_{bac}A_{ba} & Q_{bac}A_{ac} & 0 \\ 0 & P_{cba}A_{cb} & Q_{cba}A_{ba} \end{pmatrix}. \tag{76}$$

The opposite cycle gives the other block, exactly as in (74). Each determinant is zero by (73), while its upper-left two-by-two minor is nonzero. Each block has rank exactly two.

For the displayed block an explicit right-kernel matrix $B^{(1)}$ is

$$B^{(1)}_{ac} = A_{ac}, \qquad B^{(1)}_{ba} = -\frac{P_{bac}}{Q_{bac}}A_{ba}, \qquad B^{(1)}_{cb} = -\frac{Q_{acb}}{P_{acb}}A_{cb}, \tag{77}$$

with its other entries zero. The first two equations are immediate; the third is precisely $P_{acb}P_{bac}P_{cba} + Q_{acb}Q_{bac}Q_{cba} = 0$. Interchanging b, c gives $B^{(2)}$ on the opposite cycle. In particular the six entries of A remain independent variables; the coefficients in these two kernel vectors depend only on D.

For an equally explicit conormal, a left null vector of (76) is

$$\left(1, -\frac{Q_{acb}A_{cb}}{P_{bac}A_{ba}}, \frac{Q_{acb}Q_{bac}A_{ac}}{P_{bac}P_{cba}A_{ba}}\right).$$

Its three coordinates are nonzero. Transposing its output positions for the trace pairing gives

$$C_1 = E_{ba} - \frac{Q_{acb}A_{cb}}{P_{bac}A_{ba}}E_{cb} + \frac{Q_{acb}Q_{bac}A_{ac}}{P_{bac}P_{cba}A_{ba}}E_{ac}. \tag{78}$$

It satisfies $\operatorname{tr}(C_1 L_A(B)) = 0$ for every hollow B, and is an invertible directed-cycle matrix. Interchanging b, c gives C_2. Their supports are disjoint and opposite. They form a basis of the two-dimensional conormal of the full rank-four lower operator. □

7.2.3. *A full size-four star differential.*

Lemma 7.7. *For every D in the size-four open just chosen, a nonempty open subset of the entire column-star parameter space*

$$X = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \qquad Y = ve_0^t, \qquad (A, v) \in V_3 \times K^3,$$

consists of nonzero rank-one square-zero seeds for which the fixed-D derivative dH_D is onto V_4. The same assertion holds for row stars. The lower block A can simultaneously satisfy all its subset adjoint guards.

Proof. At the displayed input,

$$H_D(X, Y) = N = we_0^t, \qquad w = Fv, \qquad F_{ij} = P_{ij0} A_{ij} \quad (i, j \in \{a, b, c\}), \tag{79}$$

where the diagonal of F is zero. The upper-row Y derivative is $r \mapsto rG$, with

$$G_{ij} = Q_{0ij} A_{ij} \quad (i, j \in \{a, b, c\}), \qquad G_{ii} = 0.$$

The determinants of F and G each have the two distinct cycle monomials $A_{ab}A_{bc}A_{ca}$ and $A_{ac}A_{cb}A_{ba}$ with nonzero coefficients. Each determinant is therefore a nonzero polynomial in the same six original entries of A. The determinants $\det K_S(A)$ for all $S \subseteq [d]$ are likewise nonzero polynomials, as is seen on a single directed-cycle support. Choose one A where all six entries, both rectangle determinants, and all those lower guards are nonzero. This is an intersection of opens in the full irreducible space V_3.

The complete fixed-D hollow Y derivative is the direct sum of these two invertible three-dimensional rectangles and the rank-four operator L_A. Its rank is $3 + 3 + 4 = 10$. Its two-dimensional hollow conormal is precisely the span of C_1, C_2 from Lemma 7.6, embedded in the lower block.

Now vary the three entries of the raw upper row of X by $a = (a_j)_{j \in \{a,b,c\}}$. Its only possibly nonzero output is the lower hollow block

$$(dH_D)_{ij} = Q_{i0j} v_i a_j \quad (i \neq j). \tag{80}$$

The diagonal entries are zero by the endpoint hypothesis. For a lower conormal C, its pairing with this output is

$$\operatorname{tr}(C\, dH_D) = \sum_j a_j (M_C v)_j, \qquad (M_C)_{ji} = C_{ji} Q_{i0j}, \quad (M_C)_{jj} = 0.$$

Both M_{C_1} and M_{C_2} are still invertible directed-cycle matrices: every multiplier on their supports is nonzero. Their opposite disjoint supports prevent proportionality. It follows that $M_{C_1} v, M_{C_2} v$ are independent on a nonempty open set of v. Indeed, dependence for every v would make every vector an eigenvector of $M_{C_1}^{-1} M_{C_2}$, forcing that operator to be scalar, contrary to the supports.

Choose v in that open with both v and Fv of full support. These conditions are simultaneously possible because F is invertible and the vector space is irreducible. The three actual upper-row X directions then fill the two-dimensional quotient of V_4 by the Y image. Thus the full fixed-D hollow derivative has rank 12. The value N in (79) is nonzero rank one and square-zero, with all three complementary column entries nonzero. The orbit directions E_{0i} have diagonal projections $w_i(E_{00} - E_{ii})$, giving the remaining three independent directions and rank 15 for $d\Gamma_D$.

All the conditions used are open in the whole star parameter space, and we have shown their simultaneous nonemptiness. To obtain row stars for the same original symbols use

$$H_{P,Q}(D, X^t, Y^t) = (-1)^d H_{Q^{\mathrm{sw}}, P^{\mathrm{sw}}}(D, X, Y)^t, \qquad R^{\mathrm{sw}}(a, b) = R(b, a). \tag{81}$$

The transformed pair has zero endpoints and zero cycle norm, since its norm is $\Delta(b, a)$. Its triple-distinct weights are nonzero under our same conditions on D, by reversal of the path and homogeneity. The preceding column proof therefore applies to it. Transpose preserves all ranks and the subset adjoint guards, since $K_S(X^t) = (-1)^{|S|} K_S(X)^t$. □

7.2.4. *Restoring guards in the exact raw fiber.*

Lemma 7.8. *At any column star in the nonempty open constructed in Lemma 7.7, the exact raw fiber $H_D = N$ is smooth, and its local ring contains a nonzero product of all raw X guard determinants. Every function whose germ on this fiber is nonzero can be retained at an actual guarded smooth nilpotent seed in this same fiber. The corresponding assertion holds at row stars.*

Proof. The surjectivity of the fixed-D hollow differential makes

$$\mathcal{F}_N = \{(X', Y') \in V_4 \times V_4 : H_D(X', Y') = N\}$$

smooth at the star p. Its local ring $R = \mathcal{O}_{\mathcal{F}_N, p}$ is a regular local domain. The map from the three raw upper-row X variations to the two-dimensional quotient by the Y image has rank two. Choose a nonzero row a in its one-dimensional kernel. Its complete output lies in the Y image and can therefore be cancelled by an actual hollow Y variation. Independently, every raw lower-column X variation z has zero output: both products with the current $Y = ve_0^t$ vanish. Consequently, for every z, the raw variation

$$\dot{X} = \begin{pmatrix} 0 & a \\ z & 0 \end{pmatrix}$$

can be completed by a hollow $\dot{Y}$ to a tangent vector of the exact fiber $\mathcal{F}_N$. No output conjugation is involved.

Put $E_i = D_i^{\mathrm{low}} - (D_i)_{00} I_3$, $E_S = \prod_{i \in S} E_i$ and $A_S = K_S(A)$, with the latter computed using the lower diagonal tuple. All E_i, A_S are invertible. Smoothness lifts any such tangent to a formal arc in $\mathcal{F}_N$. Along it the raw X remains hollow, its upper and lower cross blocks start with ta and tz, and its lower block starts with A. The Schur complement gives, for every subset $S \subseteq [d]$,

$$\det K_S(X(t)) = (-1)^{|S|+1} t^2 \det(A_S)\, a E_S A_S^{-1} E_S z + O(t^3). \tag{82}$$

In particular second-order changes in the lower block or the cross blocks cannot alter the displayed coefficient; the upper-left raw entry is identically zero, even for $S = \varnothing$. For each S the row $a E_S A_S^{-1} E_S$ is nonzero. Choose one z outside the finitely many annihilating hyperplanes. Every determinant in (82) is then nonzero on the formal arc. Their product is consequently a nonzero element of R. The fixed-D and saturated smoothness minors already nonzero at p are units in R.

Let $b \in R$ be any nonzero function germ. The product of b, the guard product, and these minors is nonzero because R is a domain. On a sufficiently small finite-type neighborhood in the component through p, its nonvanishing defines a nonempty open set. As K is algebraically closed, this open set has an actual K-point. At that point the raw value is still exactly N, all guards hold, the seed is smooth, and b is nonzero. This proves the stated retention property, not merely a formal existence claim. The transpose identity (81) transports the entire raw-fiber argument to row stars. □

Corollary 7.9. *If a bilinear form T on $V_4 \times V_4$ vanishes at every guarded smooth nilpotent seed, its entire germ on the exact raw fiber at every star in Lemma 7.7 is zero.*

Proof. A nonzero germ could be retained at a guarded smooth seed in the same raw fiber by Lemma 7.8, contradicting the hypothesis on T. □

7.2.5. *Six independent monomials and the guarded base.*

Lemma 7.10. *For each D in the above size-four diagonal open, the tensors of guarded smooth nonzero rank-one square-zero seeds span $V_4 \otimes V_4$.*

Proof. Suppose a bilinear form T annihilates all such tensors. By Corollary 7.9 it vanishes at every generic column or row star. Polynomial continuation on the whole irreducible star parameter space makes its restriction identically zero there.

First consider two hollow units E_{ab}, E_{cd} with $\{a, b\} \neq \{c, d\}$. The second edge has an endpoint outside $\{a, b\}$. If d is such an endpoint, use the column star isolated at d: E_{ab} belongs to its

lower block and E_{cd} to its column. If c is such an endpoint, use the row star isolated at c instead. Thus

$$T(E_{ab}, E_{cd}) = 0 \qquad \text{if } \{a,b\} \neq \{c,d\}. \tag{83}$$

This eliminates four-distinct-vertex coefficients as well, without an embedding assertion from size three. No relation between the remaining same-edge and reversed-edge coefficients has been assumed.

Fix now an isolated vertex 0 and a generic lower block A. Choose a v making its column star one of the smooth points above. For either of the explicit lower kernel matrices $B^{(\nu)}$ in (77), the line

$$(X, Y_t) = \left(\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, ve_0^t + t \begin{pmatrix} 0 & 0 \\ 0 & B^{(\nu)} \end{pmatrix} \right) \tag{84}$$

lies in the exact same raw fiber: the additional full output is $tL_A(B^{(\nu)}) = 0$. The zero germ from Corollary 7.9, restricted to this line, gives

$$T\left(\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & B^{(\nu)} \end{pmatrix} \right) = 0.$$

Let $\mathcal{C}$ be the directed three-cycle supporting one such kernel vector, and write $B_e = \gamma_e A_e$ on it. Each $\gamma_e \in K^*$ depends only on D, and the cycle contains exactly one orientation of each unordered lower edge. Using (83), the preceding identity is

$$\sum_{e \in \mathcal{C}} \gamma_e \left(T(E_e, E_e) A_e^2 + T(E_{\bar{e}}, E_e) A_{\bar{e}} A_e \right) = 0, \tag{85}$$

where $\bar{e}$ denotes the reverse edge. This identity holds for a nonempty open set of the six independent entries of A, hence is a polynomial identity in them. Its three square monomials and three unordered-edge product monomials are six distinct monomials. Every coefficient displayed in (85) therefore vanishes. The opposite kernel cycle kills the other three square coefficients and the opposite three ordered reverse-edge coefficients on this lower triangle.

Varying the isolated vertex covers every unordered edge of the four-vertex set. All coefficients left after (83) consequently vanish, and $T = 0$. The two kernel shifts were taken in a fiber of the full operator, at a common original input, rather than in fibers of its summands. □

7.3. **Exact guarded extension and one common diagonal open.**

Lemma 7.11 (Singleton extension in the exact saturated fiber)**.** *Suppose $P, Q \neq 0$ and a hollow smooth nilpotent seed in size $h \geq 4$ satisfies every subset guard. For a generic new diagonal vector $z = (z_1, \ldots, z_d)$ it extends to a guarded smooth nilpotent seed in size $h+1$. Any raw function nonzero at the provisional input in the proof can be retained while restoring the guards.*

Proof. Write $H_{D^{\mathrm{old}}}(X_0, Y_0) = N_0 = u_0 v_0^t$, with $v_0^t u_0 = 0$, and provisionally take $X = \mathrm{diag}(X_0, 0)$. By Lemma 7.4, both $F(z), G(z)$ are invertible for a generic z, simultaneously with every E_i invertible and all diagonal entries distinct in each passive coordinate. Put

$$Y = \begin{pmatrix} Y_0 & 0 \\ \lambda v_0^t G(z)^{-1} & 0 \end{pmatrix}, \qquad \lambda \neq 0. \tag{86}$$

The complete raw value, with no discarded cross terms, is

$$N_1 = \begin{pmatrix} u_0 v_0^t & 0 \\ \lambda v_0^t & 0 \end{pmatrix} = \begin{pmatrix} u_0 \\ \lambda \end{pmatrix} (v_0^t, 0), \qquad \operatorname{rank} N_1 = 1, \quad N_1^2 = 0. \tag{87}$$

The old saturated controls produce the old $\mathfrak{sl}_h$ block. Any extra cross components they create are corrected with the two invertible Y rectangles, without changing that old block. These rectangles themselves supply the two full cross blocks. A genuine orbit direction $Z = \left(\begin{smallmatrix} 0 & e \\ 0 & 0 \end{smallmatrix}\right)$ has

$$[Z, N_1] = \begin{pmatrix} \lambda e v_0^t & -N_0 e \\ 0 & -\lambda v_0^t e \end{pmatrix}.$$

Choosing $v_0^t e \neq 0$ gives the last trace-balanced diagonal direction. The saturated derivative is therefore onto $\mathfrak{sl}_{h+1}$.

These controls can be taken in the old raw variables, the new Y rectangles and the auxiliary orbit variables. None changes the new raw cross blocks of X. Prescribe an arbitrary column a and row b for those two cross variations. Their output can be cancelled by the preceding controls, producing a tangent of the exact saturated fiber $\Gamma_D = N_1$ with these prescribed raw X components.

That fiber is smooth at the provisional point. Lift the tangent to a formal arc and keep raw X hollow. The new diagonal raw entry is identically zero, including for the empty-subset guard. With $A_S = K_S(X_0)$ and $E_S = \prod_{i\in S}(D_i^{\mathrm{old}} - z_i I_h)$, the Schur complement gives

$$\det K_S(X(t)) = (-1)^{|S|+1} t^2 \det(A_S)\, b E_S A_S^{-1} E_S a + O(t^3). \tag{88}$$

The coefficient is a nonzero bilinear form in (a, b), since E_S, A_S are invertible. Fix any $a \neq 0$ and choose b outside the finitely many annihilating hyperplanes. All coefficients are then nonzero. Unknown second-order corrections cannot cancel them: the upper-left old block begins at A_S, both cross blocks begin in order one, and the new scalar diagonal is identically zero.

The smooth local fiber ring is a regular local domain. The arc shows that its product of guards is nonzero. Multiply it by a surjectivity minor and any raw function nonzero at the provisional point; those latter functions are units locally. A finite-type localization of the component through the point is nonempty and has an actual point over the algebraically closed field. This is not an assertion that a chosen formal arc algebraizes. Discarding the auxiliary conjugator leaves a raw value conjugate to N_1, so rank one and square zero persist, while the raw guards and test have been retained. In contrast to the size-four restoration, this argument fixes the saturated value, not necessarily the raw value. □

Lemma 7.12 (Tensor induction and fixed-diagonal descent)**.** *Guarded hollow tensor spanning in size four at generic independent diagonal parameters implies the same assertion in every $n \geq 4$, on one nonempty open of diagonal tuples independent of the test.*

Proof. Work first over an algebraic closure of the field of all independent diagonal entries. On an old $h = n-1$ vertex set choose finitely many guarded seed tensors spanning $V_h \otimes V_h$ over the algebraic closure of the old diagonal function field. The new d entries z_i remain algebraically independent over that field and over the finitely many algebraic coordinates of these seeds. Hence the two nonzero determinant polynomials from Lemma 7.4 stay nonzero, and Lemma 7.11 applies to each seed. The spanning property is preserved by scalar extension.

Let T vanish on all guarded good pairs in the new size. It must vanish at each provisional input: otherwise exact-fiber guard restoration retains a nonzero value of T. At (86) its value is

$$T_h(X_0, Y_0) + \lambda T\left(\operatorname{diag}(X_0, 0), \begin{pmatrix} 0 & 0 \\ v_0^t G(z)^{-1} & 0 \end{pmatrix}\right).$$

It vanishes for every nonzero λ, so its constant term vanishes. The old spanning collection gives $T_h = 0$. This works on every $(n-1)$-vertex subset. For $n \geq 5$, any two hollow matrix units use at most four vertices and fit in such a subset. Every coefficient of T is therefore zero, closing the induction from the guarded size-four base.

To obtain the claimed numerical open, choose finitely many generic seed tensors forming a basis and put their coordinates into one finite-type incidence scheme. Its conditions consist of the raw rank-one square-zero equations, a nonzero raw entry for each value, the chosen saturated surjectivity minors, the tensor-basis determinant, and all subset guards. The projection to the independent diagonal tuples is dominant, because the construction has a point over the algebraic closure of their function field. Its constructible image contains a nonempty open. Intersect with the coordinatewise regular-diagonal open. Every nonempty finite-type fiber over a K-point has a K-point, since K is algebraically closed. Thus every D in this one open supports a spanning collection simultaneously. The diagonal tuple is not reselected separately for different bilinear forms or tensor coordinates. □

Proof of Theorem 7.2. Lemma 7.3 handles size three. If $\Delta \neq 0$, Lemma 7.5 gives all sizes, including the guards. If $\Delta = 0$, both symbols are present. The full four-vertex construction in Lemma 7.10 gives the guarded base, and Lemma 7.12 gives every larger size and the one

common diagonal open. These cases exhaust every nonzero symbol pair satisfying the zero-endpoint identities.

It remains to pass from hollow to full bilinear tests. Arbitrary independent diagonal parts can be added to X, Y without changing H_D, by the nonempty proper subset conditions in (67). The hollow differential and its actual simultaneous-conjugation controls persist; the added diagonal parts are conjugated along with the other original inputs. Suppose a bilinear form on $M_n \times M_n$ vanishes at every such enriched seed. Expanding in the two independent added diagonal matrices first kills its diagonal-by-diagonal part. Its two mixed parts vanish on the projections of the hollow seed set. Each projection spans V_n: otherwise a nonzero linear functional annihilating that projection, tensored with any nonzero functional on the other factor, would contradict hollow tensor spanning. Thus both mixed parts vanish. The hollow-by-hollow part vanishes by the already established tensor span. The full form is zero, proving the final assertion. □

If one also requires the full enriched input to retain the guards, the nonempty-subset guards are unchanged. The empty-subset guard $\det X$ may change after a diagonal addition; restrict those additions to the open set where it stays nonzero. This open contains the zero addition at a guarded hollow seed, so the same polynomial-continuation argument applies. The trace transfer below only needs the enriched smooth nilpotent seed, not its guards.

7.4. **An original-input chart along the commuting-passive locus.** The independent passive matrices cannot be replaced by the linear common-D coordinates used in Theorem 6.1. In particular, a nonzero polynomial on the original source need not remain nonzero on the locus of commuting passive tuples. We retain all transverse directions by the following chart. Write $\mathcal{D}_n$ for the vector space of diagonal matrices and V_n for the hollow matrices. As above, $d = m - 2$.

Lemma 7.13 (A full original-input chart)**.** *Let $D = (D_1, \dots, D_d) \in \mathcal{D}_n^d$, with D_1 having pairwise distinct diagonal entries. Take*

$$A \in V_n, \qquad Z_i \in V_n \quad (2 \le i \le d), \qquad X, Y \in M_n,$$

and restrict to the open set where $g = I_n + A$ is invertible. The map

$$\begin{aligned}\Psi(D, A, X, Y, Z) = \big(&gD_1g^{-1},\, g(D_2 + Z_2)g^{-1}, \dots, g(D_d + Z_d)g^{-1},\\ &gXg^{-1},\, gYg^{-1}\big)\end{aligned} \tag{89}$$

puts these entries in their specified original positions. Its source has dimension mn^2, and Ψ is étale at every point with $A = 0$ and $Z = 0$. In particular, it is dominant onto M_n^m.

The subspace $S = \{Z = 0\}$ gives a local étale chart on the simultaneously conjugated commuting-passive locus with regular semisimple first passive input. Its dimension and codimension are

$$\dim S = 3n^2 + (d-1)n, \qquad \operatorname{codim} S = (d-1)(n^2 - n).$$

If $d\Gamma_D$ is onto $\mathfrak{sl}_n$ at (X, Y, I_n), the restriction of $d(f \circ \Psi)$ to $T_{(D,0,X,Y,0)}S$ is also onto $\mathfrak{sl}_n$.

Proof. The number of coordinates in the source of (89) is

$$dn + (n^2 - n) + (d-1)(n^2 - n) + 2n^2 = (d+2)n^2 = mn^2.$$

At a point $(D, 0, X, Y, 0)$ its derivative is

$$\begin{aligned}\delta B_1 &= \delta D_1 + [\delta A, D_1],\\ \delta B_i &= \delta D_i + \delta Z_i + [\delta A, D_i] \quad (2 \le i \le d),\\ \delta B_X &= \delta X + [\delta A, X], \qquad \delta B_Y = \delta Y + [\delta A, Y].\end{aligned} \tag{90}$$

Given an arbitrary tangent vector $(\delta B_1, \dots, \delta B_d, \delta B_X, \delta B_Y)$ to the original source, the diagonal of δB_1 determines δD_1. For $a \ne b$,

$$[\delta A, D_1]_{ab} = \big((D_1)_{bb} - (D_1)_{aa}\big)\delta A_{ab}.$$

The nonzero eigenvalue differences therefore determine the hollow matrix δA uniquely. For each $i \ge 2$, subtract $[\delta A, D_i]$ from δB_i and take its diagonal and hollow parts; these determine δD_i and δZ_i uniquely. Finally, the last line of (90) determines the unrestricted $\delta X, \delta Y$. Thus the

derivative is an isomorphism between smooth spaces of equal dimension, proving étaleness at the stated point.

The source is a nonempty open subset of an affine space, hence is irreducible. The étale locus is open and contains the stated points, and an étale map is open. Consequently the image of Ψ contains a nonempty open subset of the irreducible original source. This proves dominance, and in particular injectivity of pullback on its ring of polynomial functions.

For completeness, the commuting-passive locus used here is smooth. A regular semisimple matrix B_1 has centralizer $K[B_1]$, of dimension n. Every matrix commuting with it has a unique expression

$$B_i = \sum_{r=0}^{n-1} b_{ir} B_1^r.$$

These expressions identify its centralizer bundle with the trivial rank-n bundle over the regular semisimple locus: the displayed powers are linearly independent in every fiber, and the inverse coefficients are regular locally, by an invertible matrix-entry minor. Choosing $B_2, \dots, B_d$ in this bundle already makes them commute with one another. The passive locus thus has dimension $n^2 + (d-1)n$; the two full active matrices add $2n^2$ dimensions. The image of S lies in this locus. Its tangent map is injective by (90), and its dimension equals that of the locus. This proves the asserted local chart statement and the codimension formula.

It remains to compare the allowed conjugation directions. By the actual restriction formula (68), on S one has

$$(f \circ \Psi)(D, A, X, Y, 0) = g H_D(X, Y) g^{-1}.$$

The coordinate A provides every hollow infinitesimal conjugator. If B is diagonal, then $[B, D_i] = 0$ for every i, and simultaneous conjugation by $I_n + tB$ to first order has exactly the same input tangent as

$$\delta X = [B, X], \qquad \delta Y = [B, Y], \qquad \delta A = 0, \quad \delta D_i = 0, \quad \delta Z_i = 0.$$

Both active variations are allowed because X, Y are full matrices. Equivariance gives the output derivative $[B, H_D(X, Y)]$. Decomposing an arbitrary infinitesimal conjugator into its hollow and diagonal parts proves that every direction supplied by $d\Gamma_D$ is supplied in TS. No diagonal-torus direction has been lost by taking A hollow. □

7.5. **The first actual trace term on the same exact fiber.** The next argument uses the complete matrix-entry trace polynomial at the fixed size n. It does not substitute a free cyclic class for that polynomial, and does not require a trace to appear at a prescribed small order.

Proposition 7.14 (Trace transfer in the independent chart)**.** *Suppose the complete selected restriction of the original ordinary multilinear polynomial f is (68), and its nonzero symbol pair satisfies the zero-endpoint hypotheses of Theorem 7.2. Fix $n \geq 3$ and put $h = \operatorname{tr} f$ on M_n^m. If $h = 0$, the original map into $\mathfrak{sl}_n$ has a smooth input over a nonzero rank-one square-zero matrix, with local fiber dimension $mn^2 - n^2 + 1$, and its actual image is $\mathfrak{sl}_n$.*

If $h \neq 0$, the original map into M_n has a flat input over such a matrix, with local complete-intersection fiber of dimension $mn^2 - n^2$, and its actual image is M_n. More precisely, the construction has a first nonzero normal trace order $1 \leq v \leq d-1$. Its differential has rank n^2 when $v = 1$ and rank exactly $n^2 - 1$ when $v \geq 2$. Thus in the latter case the constructed point is flat but not smooth.

Proof. Let $\Omega_n \subset \mathcal{D}_n^d$ be the single nonempty open set supplied by Theorem 7.2; it already requires regular diagonal passive matrices. Put

$$\pi(B) = B - \frac{\operatorname{tr} B}{n} I_n, \qquad F = f \circ \Psi, \qquad G = \pi \circ F.$$

Trace invariance cancels g from the pulled-back trace:

$$(h \circ \Psi)(D, A, X, Y, Z) = \widetilde{h}(D, X, Y, Z) = \operatorname{tr} f(D_1, D_2 + Z_2, \dots, D_d + Z_d, X, Y), \tag{91}$$

with the same original slot order as in (89). In particular, this is a polynomial in D, X, Y, Z and is independent of A. The zero endpoints in (68) make $H_D(X, Y)$ hollow for all full X, Y. Consequently

$$\widetilde{h}(D, X, Y, 0) = 0, \qquad F|_S = gH_D(X, Y)g^{-1}.$$

If $h = 0$ on the original source, choose any full seed from Theorem 7.2. For example, its assertion applied to any nonzero bilinear form ensures that the seed set is nonempty. At the corresponding point $p = (D, 0, X, Y, 0)$, the value $N = F(p)$ is nonzero rank-one square-zero. Lemma 7.13 makes $d(F|_S)_p$ onto $\mathfrak{sl}_n$. Since Ψ is étale at p, the original differential is onto $\mathfrak{sl}_n$ as well. The original map is therefore smooth there, and its fiber has dimension $mn^2 - (n^2 - 1)$. Smoothness gives an open image neighborhood of N; Theorem 2.1 gives the asserted actual image.

Now assume that the actual polynomial h is nonzero. Dominance of Ψ, proved in Lemma 7.13, implies $\widetilde{h} \neq 0$. Expand by total degree in the independent hollow coordinates $Z_2, \dots, Z_d$:

$$\widetilde{h}(D, X, Y, Z) = \sum_{j=v}^{d-1} H_j(D, X, Y, Z), \qquad H_v \neq 0, \qquad 1 \leq v \leq d - 1. \tag{92}$$

Every H_j is homogeneous of degree j in the Z coordinates and bilinear in the *full* active matrices X, Y. Indeed, each original passive position other than the first contributes either its D_i or its Z_i, at most once, and each active position occurs exactly once. The term of degree zero vanishes by the preceding hollow restriction. The bound is $d-1 = m-3$ because there are precisely that many transverse passive positions. All coefficients below order v vanish as polynomials in actual size-n entries. This includes every fixed-size trace identity and every cancellation involving higher-ideal terms of f.

There are $D^* \in \Omega_n$ and fixed hollow matrices $Z^0 = (Z_2^0, \dots, Z_d^0)$ such that

$$T(X, Y) = H_v(D^*, X, Y, Z^0) \tag{93}$$

is a nonzero bilinear form on $M_n \times M_n$. Here is a coefficient-level justification of this simultaneous choice. Expand H_v in the matrix-entry variables of X, Y, Z. At least one coefficient is a nonzero polynomial in the independent diagonal coordinates of D. Its nonvanishing open set meets the nonempty open Ω_n in the irreducible affine space $\mathcal{D}_n^d$. At such a D^* the polynomial in X, Y, Z remains nonzero. Some coefficient of an X-entry times a Y-entry is then a nonzero polynomial in Z. Since K is infinite, choose Z^0 where this coefficient does not vanish. This proves the assertion without a density claim for commuting tuples in the original source.

Apply Theorem 7.2 to this T at this *same* D^*. It gives full original matrices X_0, Y_0 for which

$$T(X_0, Y_0) \neq 0, \qquad N = H_{D^*}(X_0, Y_0), \qquad \operatorname{rank} N = 1, \qquad N^2 = 0,$$

and the saturated differential is onto $\mathfrak{sl}_n$. Set $p = (D^*, 0, X_0, Y_0, 0)$ in the chart and $a = \Psi(p)$ in the original source. The two properties required for the trace argument now hold at one input: $G(p) = N$, and

$$d(G|_S)_p : T_pS \longrightarrow \mathfrak{sl}_n \quad \text{is onto}, \qquad H_v(D^*, X_0, Y_0, Z) \neq 0.$$

The second assertion follows by evaluation at Z^0; the first is Lemma 7.13. Nothing here adds values from different input tuples.

We next solve the exact traceless equations without changing the normal coordinates. Write $q = mn^2$ and $r = n^2 - 1$. Choose an r-dimensional linear subspace $L \subset T_pS$ on which dG_p is an isomorphism onto $\mathfrak{sl}_n$. In the affine coordinates of the chart centered at p, choose coordinates ℓ along L and complementary coordinates w within S, leaving every coordinate of Z untouched. Thus

$$(\ell, w, Z) \quad \text{are local coordinates}, \qquad \#\ell = r, \qquad \#(w, Z) = q - r, \qquad S = \{Z = 0\}.$$

Choose any linear coordinates on $\mathfrak{sl}_n$. The Jacobian of the r functions $G - N$ with respect to ℓ is invertible at p. The formal implicit function theorem therefore gives a unique formal solution

$$\ell = \phi(w, Z), \qquad \phi(0, 0) = 0, \qquad G(\phi(w, Z), w, Z) = N.$$

In particular, for the exact partial fiber $\mathcal{Y} = G^{-1}(N)$,

$$\widehat{\mathcal{O}}_{\mathcal{Y},p} \simeq K[[w, Z]]. \tag{94}$$

The correction ϕ may have a linear term. That causes no problem: its entries have zero constant term, and no Z coordinate is replaced or eliminated.

We verify the initial trace term after this correction explicitly. Let $\mathfrak{m}$ be the maximal ideal in the completed chart at p. Equation (92) gives

$$\widetilde{h} \equiv H_v(D^*, X_0, Y_0, Z) \pmod{\mathfrak{m}^{v+1}}, \qquad \widetilde{h} \in \mathfrak{m}^v. \tag{95}$$

To prove the congruence, every monomial in H_v already has Z-degree v. Replacing a coefficient variable among D, X, Y by its value at p changes that coefficient by an element of $\mathfrak{m}$, raising the total order by at least one. The terms H_j with $j > v$ have total order at least $v + 1$ before any such replacement. The polynomial is independent of A, so that coordinate introduces no further term. Upon substituting $\ell = \phi(w, Z)$, every centered coefficient variable remains in (w, Z) and the normal coordinates stay exactly Z. Hence (95) becomes

$$\widetilde{h}|_{\mathcal{Y}} \equiv H_v(D^*, X_0, Y_0, Z) \pmod{(w, Z)^{v+1}} \quad \text{in } K[[w, Z]]. \tag{96}$$

Its right-hand side is nonzero. Thus the first surviving normal trace term is also the full lowest total Taylor term on the *exact* partial fiber; an implicit correction cannot cancel it.

The surjective differential of G makes $R = \mathcal{O}_{\mathcal{Y},p}$ a regular local ring of dimension $q - r = mn^2 - n^2 + 1$. In particular, R is a domain. The completion map is injective (equivalently here, completion is faithfully flat and the maximal-ideal filtration is separated). Equation (96) shows that $\widetilde{h}|_{\mathcal{Y}}$ is a nonzero element of R. It lies in the maximal ideal because $F(p) = N$ has trace zero, and, since R is a domain, it is a non-zero-divisor. The full fiber over N is obtained by adding this one trace equation to $G = N$, so

$$\dim \mathcal{O}_{F^{-1}(N),p} = (q - r) - 1 = mn^2 - n^2.$$

Moreover, the r traceless equations form a regular sequence in the regular local ring of the chart: their independent linear parts can be completed to a regular system of parameters. Appending the non-zero-divisor $\widetilde{h}$ gives a regular sequence of length n^2. The full fiber is therefore a local complete intersection at p.

The target M_n and the chart source are regular, and the latter is Cohen–Macaulay. The displayed fiber dimension is the difference of their dimensions, so miracle flatness proves that F is flat at p. Since Ψ is étale at p, the original fiber at $a = \Psi(p)$ has the same local dimension and is a local complete intersection. Flatness descends through this local faithfully flat change of source; equivalently, miracle flatness applies directly to the original regular local source of dimension mn^2 and its fiber of dimension $mn^2 - n^2$. Thus the asserted point belongs to the original map f, rather than just to a restriction or a quotient representative.

Flatness for morphisms of finite presentation is open on the source, and a flat morphism locally of finite presentation is open. It follows that the actual image contains an open neighborhood of N in M_n. Indeed, each K-point in this open set has a nonempty finite-type fiber, which has a K-point because K is algebraically closed. The degeneration in Theorem 2.1 now yields every nonscalar target. The scalar part of that theorem applies to the *original ordinary multilinear* polynomial f; we spell it out to keep its scope explicit.

Choose Weyl matrices $U, V \in M_n$ with $UV = \zeta VU$ and $U^n = V^n = I_n$, where ζ is a primitive nth root of unity. The matrices $U^a V^b$, $0 \le a, b < n$, form a basis, and their traces are zero except for $U^0 V^0 = I_n$. Since $\operatorname{tr} f \neq 0$, some original input tuple has nonzero output trace. Expanding that tuple multilinearly in the Weyl basis shows that a tuple of basis matrices has nonzero output trace. Every word of f uses exactly the same original inputs once, so at this basis tuple every word is a scalar multiple of the same Weyl basis matrix, of degree $\sum_i (a_i, b_i)$ in $(\mathbb{Z}/n\mathbb{Z})^2$. Its nonzero trace forces that degree to be zero. The output is therefore a nonzero scalar matrix. Scaling one original input gives every scalar matrix. Together with the open-image argument this proves $\mathrm{Val}_n(f) = M_n$, as a set of single evaluations.

Finally, if $v = 1$, (96) is a nonzero linear form on the tangent space of $\mathcal{Y}$. Thus $d\widetilde{h}_p$ is nonzero on $\ker dG_p$, and the combined map $(G, \widetilde{h})$ has rank $r+1 = n^2$. This is equivalent to dF_p having full rank. If $v \geq 2$, (95) gives $d\widetilde{h}_p = 0$, whereas dG_p still has rank r. Hence dF_p has rank exactly $n^2 - 1$. The étale chart preserves these ranks for df_a. The first case is smooth, and the second is flat but not smooth. □

Completion of the proof of Theorem 1.1. If some ordered pair of original positions has a nonzero complete common-D restriction, Theorem 6.1 gives both the actual-image classification and the required original local fiber. For the remaining case every such complete restriction is zero. Proposition 7.1 supplies a nonzero independent pair, and (70) gives its zero endpoints. Apply Theorem 7.2 and Proposition 7.14 at the specified size n. These two cases exhaust proper ordinary $J^2 \setminus J^3$. Throughout, the test deciding the target is whether the *actual* polynomial $\operatorname{tr} f$ on this M_n^m is zero. No conclusion about that test is inferred merely from its value on the commuting-passive slice, and its answer is allowed to depend on n. □

7.6. **Higher-ideal corrections and the exact second-layer boundary.**

Lemma 7.15 (Vanishing on the whole commuting-passive slice)**.** *Let R be ordinary multilinear and belong to J^r. If fewer than r original positions are active and all other original inputs commute with one another, then R vanishes for arbitrary matrices in the active positions. In particular, every ordinary multilinear $R \in J^3$ vanishes on the entire two-active, commuting-passive source in every matrix size.*

Proof. Write R as a sum of products of r commutator-ideal factors, allowing arbitrary word contexts, and take its component of degree one in each original variable. In every surviving summand the variable sets of the factors and contexts are disjoint. With fewer than r active positions, at least one of the r ideal factors uses only passive inputs. Those matrices lie in a commutative subalgebra, on which the commutator ideal evaluates to zero. That factor, and therefore the whole summand including its surrounding words, vanishes.

One can also see that no cancellation or choice of presentation is being assumed by multiplying the active inputs by independent central parameters $t_1, \ldots, t_k$, where $k < r$. Modulo the ideal $(t_1, \ldots, t_k)$ the evaluation takes values in a commutative algebra, so it sends J^r into $(t_1, \ldots, t_k)^r$. Ordinary multilinearity makes the evaluated R equal to $t_1 \cdots t_k$ times its unscaled value, of total parameter degree $k < r$. Its membership in that ideal forces the unscaled value to be zero. The identity holds on the entire slice and hence also along its simultaneous conjugates. □

Corollary 7.16 (Same-degree corrections with no lost trace terms)**.** *Let F be proper ordinary multilinear of degree m with $F \in J^2 \setminus J^3$. For every same-degree proper ordinary multilinear $R \in J^3$, the corrected polynomial $f = F + R$ has the conclusions of Theorem 1.1 in each size $n \geq 3$. Its complete independent symbol pair is unchanged for every selected active pair. Its target and its compatible nilpotent input are determined using the actual corrected polynomial.*

Proof. The class of $F + R$ in J^2/J^3 is the nonzero class of F. Lemma 7.15 shows that the entire selected commuting-passive restriction, not just one triangular value, is unchanged. Thus its complete independent symbols are unchanged. Apply Theorem 1.1.

The restriction identity is the only use made of the vanishing of R on this slice. Its transverse derivatives need not vanish. Every one of them and every higher transverse trace coefficient is retained in (91) and (92) for $f = F + R$. The first surviving order and the bilinear form (93) may therefore change, and the seed is reselected for that form. The assertion does not say that a previously chosen triangular witness stays flat for every correction. Cross-partition terms outside J^3 are likewise not removed: their contributions are already in the full coefficient arrays of (67). □

Remark 7.17 (The alternating quartic lies in the independent branch). Use the standard convention

$$s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} \operatorname{sgn}(\sigma) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}.$$

Its commutator expansion is

$$\begin{aligned} s_4 = {} & [x_1,x_2][x_3,x_4] + [x_3,x_4][x_1,x_2] \\ & - [x_1,x_3][x_2,x_4] - [x_2,x_4][x_1,x_3] \\ & + [x_1,x_4][x_2,x_3] + [x_2,x_3][x_1,x_4]. \end{aligned}$$

Set $x_1 = D_1$, $x_2 = D_2$, $x_3 = X$, $x_4 = Y$, with D_1, D_2 diagonal. The first two products vanish, and the complete symbols in the convention of (68) are exactly

$$P(a,b) = a_2b_1 - a_1b_2, \qquad Q(a,b) = a_1b_2 - a_2b_1 = -P(a,b). \tag{97}$$

Indeed, the XY-ordered terms contribute $-a_1b_2 + a_2b_1$, and the YX-ordered terms contribute its negative. Both common-D specializations are zero, both endpoints $P(a,-a), Q(a,-a)$ are zero, and P is not zero. Writing $c = -a - b$, its triangle norm is

$$P(a,b)P(b,c)P(c,a) + Q(a,b)Q(b,c)Q(c,a) = 0.$$

Thus this example uses the size-three zero-row and size-four cycle-kernel branch of Theorem 7.2; it is not covered by a nonzero common-D restriction. In fact every common-D chosen-pair restriction of s_4 vanishes, because two of its original inputs coincide.

The polynomial is proper: inserting I in one position makes all its commutator products vanish. The expansion places it in J^2. Its nonzero independent restriction, together with Lemma 7.15, proves $s_4 \notin J^3$. Cyclically moving the first letter of a four-letter word to the last position preserves its trace and changes the alternating sign by $(-1)^3 = -1$. Summing in s_4 gives $\operatorname{tr} s_4 = -\operatorname{tr} s_4$, hence $\operatorname{tr} s_4 = 0$ in characteristic zero. Theorem 1.1 consequently gives actual image $\mathfrak{sl}_n$ and an original smooth nonzero rank-one square-zero input for every $n \geq 3$; its local fiber dimension is $3n^2 + 1$.

For a small exact instance, take

$$D_1 = \operatorname{diag}(0,1,2), \quad D_2 = \operatorname{diag}(0,1,4), \quad X = E_{23} + E_{32}, \quad Y = E_{21} + E_{31}.$$

Formula (97) gives the actual value $N = -2E_{21} + 2E_{31}$, which has rank one and square zero. The all-size smoothness assertion above follows from the proved seed and trace arguments, not from extrapolation of this one value or from a numerical rank calculation.

8. Further questions

8.1. **The size-two boundary.** The image classification on M_2 is due to Kanel-Belov, Malev and Rowen [15, Theorem 2]. The local statement of Theorem 1.1 has a separate size-two boundary. By the Amitsur–Levitzki theorem [4], s_4 is an identity on M_2, although $s_4 \in J^2 \setminus J^3$. Moreover, with zero endpoints in (68), every surviving term requires three distinct vertices. Thus the independent-diagonal restriction used in that branch vanishes on M_2. For noncentral polynomials on M_2, it remains to determine the scope of the smooth and flat nilpotent-fiber conclusions.

8.2. **The third commutator layer.** The classification in degrees four through six is given in [6, Theorem A]. The higher-arity study in [7] includes local-fiber results for the entire six-slot third layer, for classes arising from s_6 with same-degree J^4 corrections in sizes $n \geq 4$, and for specified families of higher arity. The general proper third layer J^3/J^4 in higher arity remains open. The fixed-size identity and central cases must be distinguished from the noncentral case. On M_3, higher-arity third-layer classes can have nonzero scalar image [7, section *Central weighted third classes in size three*].

For an ordinary multilinear $f \in J^3 \setminus J^4$ and $n \geq 4$, some restriction with three active original variables and independently chosen diagonal passive inputs is nonzero. Indeed, $\operatorname{Id}(UT_4) = J^4$ by [3, Theorem 2.1]. In a nonzero evaluation on UT_4, expand each input into its diagonal part and its strictly upper triangular part. Terms with fewer than three strictly upper triangular inputs vanish by Lemma 7.15. Terms with at least four vanish because the fourth power of the strictly upper triangular ideal of UT_4 is zero. A nonzero term therefore has exactly three active inputs, and it embeds into M_n. By the same lemma, every same-degree multilinear correction in J^4 vanishes on each three-active restriction.

In size three, the three-active restrictions can all vanish on a nonidentity in $J^3 \setminus J^4$. An example in seven variables is

$$\sum_{i=1}^{5}(-1)^{i-1}s_4(x_1,\dots,\widehat{x_i},\dots,x_5)[[x_6,x_7],x_i],$$

where the hat indicates omission [7, section *Higher third symbols and original correction trace fibers*]. In this case one needs a construction using a larger set of original input directions.

For a noncentral polynomial f in the third layer, write $h = \operatorname{tr} f$ and $g = f - n^{-1}hI_n$ on the full original source M_n^m. The following question specifies a sufficient extension of the method of this paper.

Problem (Original fibers in the third layer)**.** Let $n \geq 3$ and let f be a proper ordinary multilinear polynomial in $J^3 \setminus J^4$ with at least one nonscalar value on M_n. Is there an original input a with $f(a) = N$, $\operatorname{rank} N = 1$, and $N^2 = 0$ such that

$$\operatorname{rank}(dg_a) = n^2 - 1, \qquad h|_{g^{-1}(N)} \neq 0 \text{ in } \mathcal{O}_{g^{-1}(N),a} \quad \text{if } h \not\equiv 0?$$

For $h \equiv 0$, the differential condition gives the smooth traceless fiber. For $h \not\equiv 0$, Proposition 2.3 gives the flat full-matrix fiber. Thus the rank condition and the trace condition must hold at the same original input. Nonvanishing on UT_4 supplies a restriction, and the conditions in the problem concern the local ring of the original fiber.

Suppose $h \not\equiv 0$ and h vanishes on a three-active slice. In an original-input chart, the first nonzero transverse trace coefficient is trilinear in the three active matrices. A sufficient statement for extending Proposition 7.14 is that, for every passive tuple in a nonempty regular-diagonal open set, the active tensors $X \otimes Y \otimes Z$ span $M_n^{\otimes 3}$ as (X, Y, Z) ranges over inputs with nonzero rank-one square-zero value at which the slice differential, with simultaneous conjugation, surjects onto $\mathfrak{sl}_n$. This would permit the choice of an input on which the first nonzero trace coefficient is nonzero. The exact traceless equations could then be solved along the slice with the transverse parameters fixed. Degree eight is the first degree in which a nonzero J^4 correction can occur, so it is the first case in which this trace condition must allow a nonzero correction with the same original variables. Polynomials lying wholly in J^4 require a separate treatment.

8.3. **Beyond the proper second layer.** For nonproper polynomials in higher degree, a unit restriction can supply a surjective traceless differential along the retained variables at a rank-one nilpotent value. After restoring the deleted inputs, one must control the trace on the exact traceless fiber of the original map, as in the bounded-degree argument of [6]. The constructions for traceless containment in [8] address the strengthened Mesyan conjecture in higher degrees. The local-fiber question here also requires full-matrix coverage for the traceful noncentral branch. The arbitrary-degree L'vov–Kaplansky conjecture remains open.

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School of Mathematics and Statistics, Beijing Jiaotong University, Beijing 100044, China
Email address: shypan@bjtu.edu.cn