hexagon:2611.00227v1

An analytic lower bound for the cubic-lattice dimer constant

Yair Lavi

6 October 2026

Abstract

We prove that the close-packed dimer entropy per site ℓ_3 of the simple cubic lattice satisfies

$$\ell_3 \geq \frac{1}{2}\log\frac{3125}{1296} + \frac{47}{10000} = 0.444775842629\ldots.$$

This exceeds the previous best published lower bound, $0.441075842629\ldots$, obtained by Csikvári. The proof uses finite matching inequalities and elementary integration, without numerical quadrature or a computational certificate.

1 Introduction and result

For an even positive integer L, let G_L be the graph with vertex set $\{0,\ldots,L-1\}^3$, where two vertices are adjacent precisely when they differ by one in one coordinate and agree in the other two. A matching is a set of edges with no shared vertices; it is perfect if every vertex belongs to one of those edges. A perfect matching describes a close-packed dimer covering: each edge represents a dimer covering its two endpoint sites. Write $\mathrm{PM}(H)$ for the number of perfect matchings of a finite graph H. The close-packed dimer entropy per site of the simple cubic lattice, also called the cubic-lattice dimer constant, is

$$\ell_3 = \lim_{\substack{L\to\infty \\ L \text{ even}}} L^{-3}\log \mathrm{PM}(G_L). \tag{1}$$

For even $L \geq 4$, let F_L be the periodic counterpart of G_L: it has the same vertices and edges, together with the edges between vertices that agree in two coordinates and have 0 and $L-1$ in the third. All logarithms are natural, and entropy is normalized per vertex. Existence of the limit (1) and its equality with the periodic limit $\lim_{\substack{L\to\infty \\ L \text{ even}}} L^{-3}\log \mathrm{PM}(F_L)$ follow from Friedland and Peled [4, Theorem 4.1 and (4.7)], using Hammersley's density-limit theorem [5].

Known bounds and approximations

Every vertex of F_L has six neighbors, and the parity of the sum of its coordinates divides the vertices into two classes with every edge joining different classes. Schrijver's theorem [1] states that every k-regular bipartite graph on $2n$ vertices, for $k \geq 2$, has at least $\big((k-1)^{k-1}/k^{k-2}\big)^n$ perfect matchings. Applied to F_L with $k = 6$ and $2n = L^3$, and combined with the Friedland–Peled equality of free-box and periodic limits stated above, it gives the lower bound

$$B_0 := \frac{1}{2}\log\frac{3125}{1296} = 0.440075842629\ldots.$$

Csikvári [2, Remark 3.4] improves this to $\ell_3 \geq B_0 + 1/1000 = 0.441075842629\ldots$. For comparison, a tensor-network calculation of Vanderstraeten, Vanhecke and Verstraete [6] estimates ℓ_3 as 0.44988452, while He's computer-assisted theorem [7] proves $\ell_3 \leq 0.452130$. The tensor-network value is a numerical estimate without rigorous error control; Csikvári's bound, He's bound and our Theorem 1.1 below are proved inequalities.

Theorem 1.1. *For the limit in* (1)*,*

$$\ell_3 \geq B_0 + \frac{47}{10000} = 0.444775842629\ldots. \tag{2}$$

The gain over Csikvári's displayed bound is exactly 37/10000. The proof follows Csikvári's entropy-gap framework [2, Theorem 1.4]: a finite Heilmann–Lieb identity [2,3] represents the relevant defect as a sum of nonnegative path terms, which controls the derivative of the entropy gap. Our new ingredient is a reflection bound, uniform in the dimer activity, for the probability that two opposite edges of a coordinate unit square are simultaneously occupied. It gives a sharper lower bound for the four-cycle terms in the path sum. Reflection methods for dimer systems have established precedents; see, for example, Starr and Williams [8]. We give the needed argument directly. Retaining a quadratic term in the elementary logarithm estimate gives an integrable gain over B_0; the final comparison with 47/10000 is a rational identity.

2 Finite matching inequalities

Fix an even integer $L \geq 4$, and put $F = F_L$ and $N = L^3$, with coordinates interpreted modulo L. As observed in the introduction, F is six-regular and bipartite. Pairing consecutive vertices in the first coordinate gives a perfect matching.

For any finite graph H and $t > 0$, define

$$Z_H(t) = \sum_{M \text{ a matching of } H} t^{|M|}.$$

Here $|M|$ is the number of edges in M, and uncovered vertices are called monomers. This is the monomer–dimer partition function of H [2,3]. The parameter t is known as the dimer activity. It specifies the probability $\mathbb{P}_t(M) = t^{|M|}/Z_F(t)$ on matchings of F. Write $\mathbb{E}_t$ for expectation with respect to this probability. For a vertex set A, $H - A$ means the induced graph after deleting A; we also write $H - u - v$ for $H - \{u, v\}$. Set

$$p = p(t) = \frac{2\mathbb{E}_t|M|}{N} = \frac{2tZ_F'(t)}{NZ_F(t)}, \qquad q = \frac{p}{6}. \tag{3}$$

Translations, coordinate permutations and coordinate sign changes make all vertices equivalent and all edges equivalent. Conditioning on $|M| = k$ preserves these symmetries. A matching of size k covers exactly $2k$ of the N vertices and occupies exactly k of the $6N/2 = 3N$ edges. Hence, for any vertex v, any edge e, and $k = 0, \ldots, N/2$,

$$\mathbb{P}_t(v \text{ is covered} \mid |M| = k) = \frac{2k}{N}, \qquad \mathbb{P}_t(e \in M \mid |M| = k) = \frac{k}{3N}.$$

The law of total probability therefore gives

$$\mathbb{P}_t(v \text{ is covered}) = \sum_{k=0}^{N/2} \frac{2k}{N} \mathbb{P}_t(|M| = k) = \frac{2\mathbb{E}_t|M|}{N} = p,$$

$$\mathbb{P}_t(e \in M) = \sum_{k=0}^{N/2} \frac{k}{3N} \mathbb{P}_t(|M| = k) = \frac{\mathbb{E}_t|M|}{3N} = \frac{p}{6} = q.$$

Fix an edge uv. Removing uv from a matching containing it gives a matching of $F-u-v$; conversely, every matching of $F-u-v$ extends uniquely by adding uv. Adding this edge multiplies the weight by t, so the total weight of matchings containing uv is tZ_{F-u-v}. A matching leaves u uncovered precisely when it uses no edge incident to u. Such matchings are exactly those of $F-u$, with their weights unchanged, so their total weight is Z_{F-u}. Applying the same argument to v and dividing by Z_F gives

$$q = t\frac{Z_{F-u-v}}{Z_F}, \qquad 1-p = \frac{Z_{F-u}}{Z_F} = \frac{Z_{F-v}}{Z_F} \tag{4}$$

for every edge uv. The common argument t is suppressed in partition functions when no confusion is possible. At finite positive t, $0 < p < 1$ and $0 < q < 1/6$.

Let e_i, e_j be two distinct coordinate unit vectors and $x \in (\mathbb{Z}/L\mathbb{Z})^3$. A coordinate plaquette C is the four-vertex cycle with vertices, in cyclic order,

$$v_1 = x, \qquad v_2 = x + e_i, \qquad v_3 = x + e_i + e_j, \qquad v_4 = x + e_j,$$

where coordinates are interpreted modulo L. Its two pairs of opposite edges are

$$D_1 = \{v_1v_2, v_3v_4\}, \qquad D_2 = \{v_2v_3, v_4v_1\}.$$

Define $a := \mathbb{P}_t(D_1 \subseteq M)$, the probability that both edges of the first pair are occupied.

For either $k = 1$ or $k = 2$, removing the two edges of D_k from a matching containing them leaves a matching of $F - C$, where $F - C$ denotes the graph obtained by deleting all four vertices of C. Conversely, adding D_k to any matching M' of $F - C$ yields a unique matching containing that pair, with weight $t^{|M'|+2}$. Therefore

$$\mathbb{P}_t(D_k \subseteq M) = \frac{\sum_{M' \text{ a matching of } F-C} t^{|M'|+2}}{Z_F} = t^2\frac{Z_{F-C}}{Z_F}.$$

In particular, both pairs have the same probability a, and

$$a = t^2\frac{Z_{F-C}}{Z_F}. \tag{5}$$

The two pair events are mutually exclusive, so the probability that either pair is occupied is $2a$. Translations, coordinate permutations and coordinate sign changes carry any coordinate plaquette to any other while preserving matching weights. Hence a is independent of the plaquette chosen.

The preceding definitions and identities are standard in the monomer–dimer literature. Our first new ingredient is the following quantitative inequality for the joint occupancy of opposite plaquette edges.

Lemma 2.1. *For every $t > 0$,*

$$a(1 - 2q + a) \geq q^2. \tag{6}$$

Proof. We split the torus along two cuts into halves exchanged by a reflection. Once the occupied crossing edges are fixed, the remaining matchings in the two halves are independent and have identical distributions after reflection. This lets us compare the occupancy probabilities of an edge and its reflected copy using the Cauchy–Schwarz inequality.

By the symmetry noted after (5), we may take C to be the plaquette with $x = (0,0,0)$, $e_i = (1,0,0)$ and $e_j = (0,1,0)$, so that $v_1 = (0,0,0)$, $v_2 = (1,0,0)$, $v_3 = (1,1,0)$ and $v_4 = (0,1,0)$. Put $e = v_1v_2$ and $e' = v_4v_3$; these are the two edges of D_1, and they lie in the layers $y = 0$ and $y = 1$ respectively. The two edges $b_1 = v_1v_4$ and $b_2 = v_2v_3$ of D_2 join these layers.

Reflect vertices by $\theta(x, y, z) = (x, 1-y, z)$ modulo L; this is an automorphism of F with $\theta e = e'$. Let H be the set of vertices whose y coordinate lies in $\{0, L/2+1, L/2+2, \dots, L-1\}$; then θH consists of the vertices with $y \in \{1, 2, \dots, L/2\}$, and H, θH partition the vertex set. The edge e lies inside H and e' inside θH. Let B be the set of edges with one endpoint in H and the other in θH. These are exactly the y-direction edges across the two cuts of the layer cycle, between layers $0, 1$ and between layers $L/2, L/2+1$. Each edge of B joins a vertex to its reflection, so θ fixes every edge of B; the edges of B are pairwise vertex-disjoint because $L \geq 4$; and b_1, b_2 are the only edges of B at the endpoints of e. Every other edge of F lies inside H or inside θH. Write $F[H]$ for the subgraph induced by H; then θ maps $F[H]$ isomorphically onto $F[\theta H]$.

Condition on $S = M \cap B$, the set of occupied crossing edges. Let $V_H(S)$ and $V_{\theta H}(S)$ denote the endpoints of S in the two halves. Given S, the rest of M consists of a matching of $F[H] - V_H(S)$ and a matching of $F[\theta H] - V_{\theta H}(S)$; conversely, every such pair of matchings together with S is a matching of F. Since θ fixes S and maps $F[H] - V_H(S)$ isomorphically onto $F[\theta H] - V_{\theta H}(S)$, both residual graphs have the same partition function, which we call A_S. Therefore

$$Z_F = \sum_S t^{|S|} A_S^2, \qquad w_S := \mathbb{P}_t(M \cap B = S) = \frac{t^{|S|} A_S^2}{Z_F},$$

and, given S, the two residual matchings are independent, each distributed with weights $t^{|\cdot|}$ on its residual graph. Let f_S be the conditional probability, given $M \cap B = S$, that e is occupied; by the same isomorphism, f_S is also the conditional probability that e' is occupied. If S covers an endpoint of e, then $f_S = 0$. Conditional independence gives

$$q = \sum_S w_S f_S, \qquad a = \sum_S w_S f_S^2.$$

Let U be the set of S containing neither b_1 nor b_2; these are exactly the S covering no endpoint of e. Each of b_1, b_2 is occupied with probability q, and both are occupied with probability a, so inclusion–exclusion gives $\sum_{S \in U} w_S = 1 - 2q + a$. Since $f_S = 0$ for $S \notin U$, the Cauchy–Schwarz inequality gives

$$q = \sum_{S \in U} w_S f_S \leq \Big(\sum_{S \in U} w_S \Big)^{1/2} \Big(\sum_{S \in U} w_S f_S^2 \Big)^{1/2} \leq \big((1 - 2q + a)\, a \big)^{1/2},$$

and squaring yields (6). □

Put $Q = q(1-q)$ and $r = Q - t(1-p)^2$. Here t is the dimer activity and $p = p(t)$ is the mean fraction of covered vertices. The quantity r measures departure from the relation $t(1-p)^2 = q(1-q)$, which holds on the infinite six-regular tree—the connected graph with no cycles and six neighbors at every vertex [2, Proposition 6.5]. The path identity (8) below, the edge-transitive specialization of [2, Proposition 2.5(c)], expresses this departure through nonnegative contributions from cycles. The following lemma retains four plaquette terms of that sum and bounds them through the plaquette probability a. In Section 3 the resulting lower bound $r \geq 4a^2$ improves the entropy lower bound.

Lemma 2.2. *For every $t > 0$,*

$$4a^2 \leq r < Q. \tag{7}$$

Proof. For adjacent u, v, let $P : u \rightsquigarrow v$ denote a simple path, that is, a path with no repeated vertex. Its vertex set is $V(P)$, and $|V(P)|$ counts vertices, not edges. The finite Heilmann–Lieb

identity, in the bipartite form of [2, Lemma 2.4(c)], states that

$$Z_F Z_{F-u-v} - Z_{F-u} Z_{F-v} = \sum_{P:u\rightsquigarrow v} t^{|V(P)|-1} Z^2_{F-V(P)}.$$

All terms have the displayed positive sign because a path between the two parity classes has an odd number of edges. Multiply by t/Z_F^2 and subtract the term for the single-edge path uv. By (4),

$$r = \sum_{\substack{P:u\rightsquigarrow v \\ P\neq(uv)}} t^{|V(P)|} \left(\frac{Z_{F-V(P)}}{Z_F}\right)^2. \tag{8}$$

The edge uv belongs to four coordinate plaquettes: two choices of the other coordinate and two choices of its sign; since $L \geq 4$, these four plaquettes are distinct. Going around each plaquette gives a distinct three-edge path, whose vertex set is the plaquette and whose contribution is $t^4(Z_{F-C}/Z_F)^2 = a^2$ by (5). Retaining these four terms and dropping all other paths, which contribute nonnegative terms (for $L = 4$ these include the straight three-edge path around the torus), proves the lower inequality. The upper inequality follows from $t > 0$ and $p < 1$. □

Solving (6) as a quadratic inequality in a gives $a \geq a_*(q)$, where

$$a_*(q) = \frac{\sqrt{(1-2q)^2 + 4q^2} - (1-2q)}{2}.$$

We use the polynomial lower bound below to make the final integration elementary. The coefficient $29/10$ is a convenient rational choice: the bound holds throughout the required density interval, and its integrated gain exceeds $47/10000$.

Lemma 2.3. *For every $t > 0$, let $h(q) = 1 + 2q + \frac{29}{10}q^2$. Then*

$$a \geq q^2 h(q). \tag{9}$$

Proof. For $0 < q \leq 1/6$, the function $f_q(z) = z^2 + (1-2q)z - q^2$ is strictly increasing on $z \geq 0$. Writing $s = q^2 h(q)$, direct expansion gives

$$f_q(s) = q^4 T(q), \qquad T(q) = -\frac{1}{10} - \frac{9}{5}q + \frac{49}{5}q^2 + \frac{58}{5}q^3 + \frac{841}{100}q^4.$$

Since

$$T''(q) = \frac{98}{5} + \frac{348}{5}q + \frac{2523}{25}q^2 > 0, \quad T(0) = -\frac{1}{10}, \quad T(1/6) = -\frac{8759}{129600},$$

convexity puts T below its negative endpoint chord. Consequently $f_q(s) < 0 \leq f_q(a)$ for $q > 0$, by (6), so $a > s$. In particular, since $h(q) \geq 1$, we have $a \geq q^2$. □

3 Entropy integration and proof of the theorem

The finite-volume entropy identity turns the estimate $r \geq 4a^2$ from Lemma 2.2 into an integrable gain over B_0. For a random variable X, write $\mathrm{Var}_t(X) = \mathbb{E}_t(X^2) - (\mathbb{E}_t X)^2$. Differentiating $\mathbb{E}_t|M| = tZ_F'(t)/Z_F(t)$ gives $(tZ_F'/Z_F)' = \mathbb{E}_t(|M|^2)/t - (\mathbb{E}_t|M|)^2/t$, so the density in (3) obeys

$$p'(t) = \frac{2}{Nt}\,\mathrm{Var}_t(|M|) > 0.$$

The variance is strictly positive because the matching with no edges and any one-edge matching both have positive probability, so $|M|$ is not constant.

The constant term of $Z_F(t)$ is 1, and its highest-degree term is $\mathrm{PM}(F)t^{N/2}$. Consequently, the density identity (3) gives $p(t) \to 0$ as $t \downarrow 0$ and $p(t) \to 1$ as $t \to \infty$.

Thus p is a strictly increasing bijection from $(0,\infty)$ onto $(0,1)$; let $t_F(p)$ be its inverse, which is continuously differentiable on $(0,1)$. Following Csikvári [2, Sections 2–3], we define

$$\lambda_F(p) = N^{-1} \log Z_F(t_F(p)) - \frac{p}{2} \log t_F(p), \tag{10}$$

$$G_6(p) = \frac{1}{2}\left[p \log \frac{6}{p} + (6-p)\log(1-p/6) - 2(1-p)\log(1-p)\right], \tag{11}$$

$$g_F(p) = \lambda_F(p) - G_6(p).$$

Here $G_6(p)$ is the matching entropy per vertex of that infinite tree at covered-vertex density p [2, Section 3]. Thus $g_F(p)$ compares the entropy on the periodic cubic graph with this tree benchmark. The bound $r \geq 4a^2$ turns the plaquette occupancy estimate into a lower bound on the derivative of this entropy gap. Use continuous endpoint values for products $x \log x$. For $t = t_F(p)$, the matching probabilities satisfy $\log \mathbb{P}_t(M) = |M| \log t - \log Z_F(t)$. Averaging this identity and using $\mathbb{E}_t|M| = Np/2$ gives

$$-\frac{1}{N}\sum_M \mathbb{P}_t(M) \log \mathbb{P}_t(M) = \frac{1}{N}\log Z_F(t) - \frac{p}{2}\log t = \lambda_F(p).$$

Thus $\lambda_F(p)$ is the entropy of the matching distribution, normalized per vertex.

As $p \to 0$, the distribution concentrates on the single empty matching, whose entropy is zero. As $p \to 1$, it becomes uniform on the $\mathrm{PM}(F)$ perfect matchings, since all have the same weight $t^{N/2}$; this distribution has entropy $\log \mathrm{PM}(F)$. Together with $G_6(0) = 0$ and $G_6(1) = B_0$, these continuous endpoint values give

$$g_F(0) = 0, \qquad g_F(1) = N^{-1} \log \mathrm{PM}(F) - B_0. \tag{12}$$

For $0 < p < 1$, differentiating (10) by the chain rule yields

$$\lambda_F'(p) = \left[\frac{Z_F'(t)}{N Z_F(t)} - \frac{p}{2t}\right] t_F'(p) - \frac{1}{2}\log t, \qquad t = t_F(p).$$

The bracket vanishes by the density identity (3), leaving $\lambda_F'(p) = -\frac{1}{2}\log t_F(p)$. Direct differentiation of (11) gives

$$G_6'(p) = \frac{1}{2}\log \frac{(1-p)^2}{q(1-q)}, \qquad q = p/6.$$

Subtracting these derivatives therefore gives

$$g_F'(p) = \frac{1}{2}\log \frac{q(1-q)}{t_F(p)(1-p)^2} = -\frac{1}{2}\log\left(1 - \frac{r}{Q}\right). \tag{13}$$

The last equality uses $Q = q(1-q)$ and $t_F(p)(1-p)^2 = Q - r$, with r, Q evaluated at $t = t_F(p)$. By Lemma 2.2, $x := r/Q$ lies in $[0,1)$. The inequality $-\log(1-x) \geq x + x^2/2$ for $0 \leq x < 1$ follows by integrating $(1-u)^{-1} \geq 1 + u$ over $0 \leq u \leq x$. Applying it in (13) and then using Lemmas 2.2 and 2.3 gives

$$\begin{aligned} g_F'(p) &\geq \frac{r}{2Q} + \frac{r^2}{4Q^2} \\ &\geq \frac{2q^3 h(q)^2}{1-q} + \frac{4q^6}{(1-q)^2} \geq \frac{2q^3 h(q)^2}{1-q} + 4q^6. \end{aligned} \tag{14}$$

Csikvári uses the linear estimate $g_F'(p) \geq r/(2Q)$; here we retain the additional quadratic term. The first contribution in (14) uses $r \geq 4a^2$ and $a \geq q^2 h(q)$; the second uses $r \geq 4a^2$ and $a \geq q^2$. The function g_F is continuous on $[0,1]$ with the endpoint values (12) and continuously differentiable on $(0,1)$. Integrating (14) over $[\varepsilon, 1-\varepsilon]$, letting $\varepsilon \downarrow 0$, and substituting $p = 6q$ gives

$$N^{-1} \log \mathrm{PM}(F) \geq B_0 + J, \qquad J := 12 \int_0^{1/6} \frac{q^3 h(q)^2}{1-q}\, dq + 24 \int_0^{1/6} q^6\, dq. \tag{15}$$

To evaluate J, expand $h(q)^2 = 1 + 4q + \frac{49}{5}q^2 + \frac{58}{5}q^3 + \frac{841}{100}q^4$ and use

$$\int_0^{1/6} \frac{q^m}{1-q}\, dq = \log \frac{6}{5} - \sum_{j=1}^{m} \frac{1}{j 6^j} \qquad (m \geq 1).$$

Substitution in (15) gives

$$J = \frac{10443}{25} \log \frac{6}{5} - \frac{3108938251}{40824000} + \frac{1}{81648}. \tag{16}$$

Integrating the geometric series for $2/(1-u^2)$ from 0 to $1/z$ gives, for real $z > 1$,

$$\log \frac{z+1}{z-1} = 2 \sum_{k=0}^{\infty} \frac{1}{(2k+1) z^{2k+1}}.$$

At $z = 11$, the first four positive terms give the strict bound

$$\log \frac{6}{5} > \frac{373057792}{2046152955}.$$

Substituting this bound into (16) gives

$$J - \frac{47}{10000} > \frac{10443}{25} \frac{373057792}{2046152955} - \frac{3108938251}{40824000} + \frac{1}{81648} - \frac{47}{10000} = \frac{11298741211}{3977721344520000} > 0. \tag{17}$$

For orientation, $J \approx 0.00470288014$, exceeding $47/10000 = 0.0047$ by approximately 2.88×10^{-6}. These decimal values illustrate the margin; the strict comparison is proved by the rational bound in (17).

Proof of Theorem 1.1. Equations (15)–(17) give the same lower bound on every even periodic cubic graph F_L, $L \geq 4$. Friedland–Peled [4, Theorem 4.1 and (4.7)] identify the periodic perfect-matching entropy with the free-box limit whenever the volume is even and every side tends to infinity. Our boxes have three even sides $L \to \infty$, and their free and periodic matchings are exactly the dimer tilings in that theorem. Hence

$$\ell_3 = \lim_{\substack{L \to \infty \\ L \text{ even}}} L^{-3} \log \mathrm{PM}(F_L) \geq B_0 + J > B_0 + \frac{47}{10000}.$$

This proves (2). □

Acknowledgments

The proof of this lower bound was carried out by GPT-5.6-sol, GPT-6 Astra, Claude Fable 5, and Claude Fable 5.1, under the guidance of the author. The author has reviewed the resulting proof arguments. Responsibility for the final text rests with the author.

References

[1] A. Schrijver, "Counting 1-factors in regular bipartite graphs," *Journal of Combinatorial Theory, Series B* **72** (1998), 122–135. doi:10.1006/jctb.1997.1798.

[2] P. Csikvári, "Matchings in vertex-transitive bipartite graphs," *Israel Journal of Mathematics* **215** (2016), 99–134. doi:10.1007/s11856-016-1375-9. The cited numbering refers to the author's V5 version, dated 31 March 2015.

[3] O. J. Heilmann and E. H. Lieb, "Theory of monomer-dimer systems," *Communications in Mathematical Physics* **25** (1972), 190–232. doi:10.1007/BF01877590.

[4] S. Friedland and U. N. Peled, "Theory of computation of multidimensional entropy with an application to the monomer-dimer problem," *Advances in Applied Mathematics* **34** (2005), 486–522. doi:10.1016/j.aam.2004.08.005. The cited numbering refers to arXiv:math/0402009v1 (2004).

[5] J. M. Hammersley, "Existence theorems and Monte Carlo methods for the monomer-dimer problem," in *Research Papers in Statistics: Festschrift for J. Neyman*, Wiley, 1966, 125–146.

[6] L. Vanderstraeten, B. Vanhecke and F. Verstraete, "Residual entropies for three-dimensional frustrated spin systems with tensor networks," *Physical Review E* **98** (2018), 042145. doi:10.1103/PhysRevE.98.042145.

[7] Q. He, "A new upper bound on the dimer constant of $\mathbb{Z}^3$," arXiv:2607.28810v2 (2026).

[8] S. Starr and S. Williams, "Rough bounds for emptiness formation probability in the 2d dimer model using reflection positivity," arXiv:1810.08846 (2018).