hexagon:2611.00226v1

A family of finite dimensional Hopf algebras over $\mathbb{C}$ with varying algebra structure

ChatGPT 6 Pro

November 8, 2026

Abstract

We construct an algebraic one-parameter family of complex Hopf algebras of dimension 1024, with fixed coalgebra, whose underlying associative algebras vary in a continuous family of isomorphism classes. Four multiplication lines in the second radical layer of a distinguished block recover the j-invariant of the elliptic curve $y^2 = x(x-1)(x-t^2)$. This answers negatively the finiteness question of Etingof and Gelaki concerning algebra types of Hopf algebras of fixed dimension.

AI disclaimer

This text has been produced by ChatGPT at a request by Pavel Etingof. It was repeatedly checked by AI agents and revised in response to their comments. Pavel's substantive prompts are reproduced in chronological order at the end. Feel free to use at your discretion. If you rewrite it or build on it, you may say that it was inspired by this note.

Contents

AI disclaimer 1

1 Introduction 1

2 The Hopf family 2

3 A distinguished block and its radical layers 5

4 The elliptic invariant 7

Acknowledgements 8

References 8

1 Introduction

Etingof and Gelaki ask in [1, p. 2 of the arXiv version] whether there are only finitely many associative algebra isomorphism classes underlying complex Hopf algebras of a given dimension. We construct a continuous family with infinitely many algebra types.

Theorem 1.1. *There exists a Hopf algebra $\mathcal{H}$, free of rank* 1024 *over* $\mathbb{C}[t]$*, with constant underlying coalgebra, whose fibers H_t have the following property. For $t, t' \in T := \mathbb{C} \setminus \{0, 1, -1\}$,*

$$H_t \cong H_{t'} \text{ as associative algebras} \quad \Longrightarrow \quad \mathcal{J}(t) = \mathcal{J}(t'),$$

where

$$\mathcal{J}(t) = 256\frac{(t^4 - t^2 + 1)^3}{t^4(t^2 - 1)^2}. \tag{1}$$

Each associative algebra isomorphism class occurs at most twelve times on T. On some open complex disc $U \subset T$, the fibers H_t are pairwise nonisomorphic as associative algebras.

2 The Hopf family

We use the convention $[x, y] = xy - yx$. The algebra H_t has commuting generators g, r, s satisfying $g^4 = r^2 = s^2 = 1$, and four further generators x_1, y_1, x_2, y_2. Put

$$a = gr, \qquad b = gs, \qquad c = g^2, \qquad d = ab = c\,rs.$$

Thus c is central, $a^2 = b^2 = c$, and $d^2 = 1$. The group action is specified by the table below, where an entry ϵ means $hz = \epsilon zh$:

	x_1	y_1	x_2	y_2
g	-1	1	-1	1
r	1	1	1	-1
s	1	-1	1	1

. (2)

The remaining relations are

$$\begin{aligned} x_1^2 = x_2^2 = \frac{1-c}{2}, &\qquad y_1^2 = y_2^2 = 0, \\ x_1x_2 + x_2x_1 = t(1-d), &\qquad y_1y_2 = y_2y_1, \\ x_1y_2 + y_2x_1 = 0, &\qquad x_2y_1 + y_1x_2 = 0, \\ [x_1, y_1]^2 = 0, &\qquad [x_2, y_2]^2 = 0. \end{aligned} \tag{3}$$

The elements g, r, s are group-like, and

$$\begin{aligned} \Delta(x_1) = x_1 \otimes 1 + a \otimes x_1, &\quad \Delta(y_1) = y_1 \otimes 1 + b \otimes y_1, \\ \Delta(x_2) = x_2 \otimes 1 + b \otimes x_2, &\quad \Delta(y_2) = y_2 \otimes 1 + a \otimes y_2. \end{aligned} \tag{4}$$

The counit is zero on x_i, y_i. For a generator z with $\Delta(z) = z \otimes 1 + h_z \otimes z$, the antipode is $S(z) = -h_z^{-1}z$.

Proposition 2.1. *These formulas define Hopf algebras H_t of dimension* 1024*. Writing $z_i = [x_i, y_i]$, a basis is*

$$x_1^{a_1} z_1^{b_1} y_1^{c_1} x_2^{a_2} z_2^{b_2} y_2^{c_2} g^i r^j s^k, \qquad \begin{array}{c} a_\ell, b_\ell, c_\ell, j, k \in \{0, 1\}, \\ 0 \le i < 4. \end{array} \tag{5}$$

They are the fibers of a free Hopf algebra of rank 1024 *over* $\mathbb{C}[t]$*, with constant coalgebra.*

Proof. Let $\Gamma = C_4(h) \times C_2(v) \times C_2(w)$, and put $u = h^2$. Define an auxiliary Hopf algebra K by adjoining X_1, Y_1, X_2, Y_2 to $\mathbb{C}\Gamma$. The group action and the group degrees q_Z are

$$\begin{array}{c|cccc} & X_1 & Y_1 & X_2 & Y_2 \\ \hline h & \mathrm{i} & \mathrm{i} & \mathrm{i} & \mathrm{i} \\ v & 1 & -1 & -1 & 1 \\ w & -1 & 1 & 1 & -1 \\ \hline q_Z & u & v & u & w \end{array}, \qquad \Delta(Z) = Z \otimes 1 + q_Z \otimes Z. \tag{6}$$

Its defining relations, besides those of the group and its action, are

$$X_i^2 = Y_i^2 = (X_iY_i + Y_iX_i)^2 = 0 \quad (i = 1, 2),$$
$$X_1X_2 + X_2X_1 = X_1Y_2 + Y_2X_1 = X_2Y_1 + Y_1X_2 = 0, \qquad Y_1Y_2 = Y_2Y_1.$$

For either pair X, Y, the algebra $\mathbb{C}\langle X, Y\rangle/(X^2, Y^2, (XY + YX)^2)$ has basis

$$1,\ X,\ Y,\ XY,\ YX,\ XYX,\ YXY,\ XYXY. \tag{7}$$

Indeed, only alternating words survive, $YXYX = -XYXY$, every word of length five vanishes, and there are no further relations in degrees at most four. The cross relations are scalar commutation relations between the two factors, so their product has dimension 64. Adjoining Γ gives $\dim K = 1024$.

The coproduct respects the relations. For the only quartic relation in the first pair, put $Z_1 = X_1Y_1 + Y_1X_1$. Then

$$\Delta(Z_1) = Z_1 \otimes 1 + uv \otimes Z_1 + 2vX_1 \otimes Y_1.$$

The quadratic relations give $[Z_1, X_1] = [Z_1, Y_1] = 0$, while

$$Z_1(uv) = -(uv)Z_1, \qquad Z_1(vX_1) = -(vX_1)Z_1, \qquad (uv)(vX_1) = -(vX_1)(uv).$$

Together with $(uv)^2 = 1$ and $Y_1^2 = 0$, these identities give

$$\Delta(Z_1^2) = Z_1^2 \otimes 1 + 1 \otimes Z_1^2,$$

so the quartic relation is preserved. The second pair is identical with v replaced by w, and the quadratic relations are checked directly from (6). The usual antipode on the skew-primitive generators makes K a Hopf algebra.

The elements u, X_1, X_2 generate the Hopf subalgebra $\bigwedge\langle X_1, X_2\rangle \rtimes \mathbb{C}\langle u\rangle$. Set

$$(b_{ij}) = -\frac{1}{2}\begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}, \qquad T_t = \sum_{i,j=1}^{2} b_{ij} X_i u \otimes X_j, \qquad F_t = e^{T_t} = 1 + T_t + \tfrac{1}{2}T_t^2. \tag{8}$$

Here $T_t^3 = 0$. To check the twist equation, for $L = \sum b_{ij} X_i u \otimes X_j$ consider

$$L_{12} = L \otimes 1, \qquad L_{23} = 1 \otimes L, \qquad L^u_{13} = \sum b_{ij} X_i u \otimes u \otimes X_j.$$

Each summand of one tensor commutes with each summand of the other two: the signs from $uX_i = -X_iu$ and $X_iX_j = -X_jX_i$ cancel. Thus these tensors commute pairwise. Moreover,

$$(\Delta \otimes \mathrm{id})L = L^u_{13} + L_{23}, \qquad (\mathrm{id} \otimes \Delta)L = L_{12} + L^u_{13}.$$

Exponentiation proves the twist equation; the counit normalization is immediate. Thus $K \otimes \mathbb{C}[t]$, with coproduct $F_t^{-1}\Delta(-)F_t$, is a Hopf algebra. Its $\mathbb{C}[t]$-linear dual $\mathcal{H}$ is free of rank 1024, with coalgebra $K^* \otimes \mathbb{C}[t]$. Write $\widetilde{H}_t$ for its fibers.

We identify these fibers with the stated presentation. Take $g, r, s \in \widehat{\Gamma}$ to be the characters

	h	v	w
g	i	1	1
r	1	1	-1
s	1	-1	1

,

extended by zero in positive degree. Define degree-one functionals by

$$x_i(X_j\gamma) = \delta_{ij}, \qquad y_i(Y_j\gamma) = \delta_{ij} \qquad (\gamma \in \Gamma),$$

and zero on the other homogeneous basis elements. Their coproducts and group action are (4) and (2). Together with g, r, s, they generate the graded algebra K^*: on either factor, pairing the words $x, y;\ xy, yx;\ xyx, yxy;\ xyxy$ with the corresponding words in X, Y gives the invertible matrices

$$I_2, \qquad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad 2I_2, \qquad (2),$$

and the assertion follows for K^* by the scalar cross relations and Fourier duality on Γ. The twisted dual multiplication is filtered, with this graded multiplication as its leading term: the positive-degree terms of F_t contribute only lower-degree terms. Hence the same elements generate $\widetilde{H}_t$.

Quadratic products acquire only degree-zero corrections, since $F_t - 1$ starts in total degree two. For $\gamma \in \Gamma$,

$$F_t^{-1}(\gamma \otimes \gamma)F_t = \gamma \otimes \gamma + (\gamma \otimes \gamma)T_t - T_t(\gamma \otimes \gamma) \\ + \text{terms of total degree at least four.}$$

Applying $x_i \otimes x_j$, and writing $\chi_1 = a,\ \chi_2 = b$ for the characters of X_1, X_2, gives

$$(x_ix_j)(\gamma) = b_{ij}(\chi_i(\gamma)\chi_j(\gamma) - 1).$$

Together with the unchanged degree-two terms, this gives the square and linking relations in (3). The other quadratic relations acquire no correction, since the twist involves only X_1, X_2.

For the quartic relations, put $z_1 = [x_1, y_1]$. Centrality of x_1^2 and the relation $y_1^2 = 0$ give $z_1x_1 = -x_1z_1$ and $z_1y_1 = -y_1z_1$. Moreover,

$$\Delta(z_1) = z_1 \otimes 1 + d \otimes z_1 - 2bx_1 \otimes y_1.$$

Using $d^2 = 1$ and (2), its square is $z_1^2 \otimes 1 + 1 \otimes z_1^2$. A finite-dimensional Hopf algebra in characteristic zero has no nonzero primitive elements, as follows by applying the coproduct to a minimal polynomial. Thus $z_1^2 = 0$; the same argument gives $z_2^2 = 0$.

There is therefore a surjection from the presented algebra onto $\widetilde{H}_t$. To obtain a spanning set, move group elements to the right and move the second pair past the first using the cross relations. The linking correction lowers the number of non-group generators. Within each pair, the square relations and

$$y_ix_iy_ix_i = -x_iy_ix_iy_i$$

reduce every word to the span of eight monomials, equivalently $x_i^{a_i}z_i^{b_i}y_i^{c_i}$ with exponents in $\{0, 1\}$. Thus (5) spans. It has 1024 elements, so the surjection is an isomorphism and (5) is a basis. $\square$

3 A distinguished block and its radical layers

Fix $t \in T$. The central involution c gives

$$H_t = H_t^+ \oplus H_t^-, \qquad H_t^\pm = \frac{1 \pm c}{2} H_t, \qquad \dim H_t^\pm = 512.$$

Regard $A_t = H_t^-$ as a unital algebra with identity $(1-c)/2$. In this algebra,

$$g^2 = -1, \qquad x_1^2 = x_2^2 = 1, \qquad x_1 x_2 + x_2 x_1 = t(1 + rs).$$

Give y_1, y_2 degree one and g, r, s, x_1, x_2 degree zero. The basis (5) gives

$$\mathrm{Hilb}(A_t; q) = 32(1+q)^4.$$

3.1 The semisimple quotient

Let $D_t = (A_t)_0$. Its simple modules are indexed by eight triples

$$v = (r_v, s_v, \alpha_v), \qquad r_v, s_v \in \{1, -1\}, \qquad \alpha_v + \alpha_v^{-1} = t(1 + r_v s_v), \tag{9}$$

and are given by

$$g = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad x_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & \alpha_v \\ \alpha_v^{-1} & 0 \end{pmatrix}, \quad r = r_v I_2, \quad s = s_v I_2. \tag{10}$$

For $r_v s_v = -1$, the roots are $\pm\mathrm{i}$; for $r_v s_v = 1$, they are $t \pm \sqrt{t^2 - 1}$. The eight representations are simple and inequivalent. Since $\dim D_t = 32$, it follows that $D_t \cong \mathrm{Mat}_2(\mathbb{C})^{\oplus 8}$. The positive-degree ideal is nilpotent with semisimple quotient, so it is $\operatorname{rad} A_t$.

The elements g, x_1 generate a copy of $\mathrm{Mat}_2(\mathbb{C})$ in degree zero. The corner

$$B_t = pA_tp, \qquad p = \frac{1 - \mathrm{i}g}{2},$$

is basic, and

$$A_t \cong \mathrm{Mat}_2(B_t), \qquad \mathrm{Hilb}(B_t; q) = 8(1+q)^4. \tag{11}$$

Inserting the degree-zero matrix units between factors shows that B_t, like A_t, is generated in degrees zero and one. Consequently, for $J = \operatorname{rad} B_t$,

$$J^k = \bigoplus_{j \geq k} (B_t)_j, \qquad \operatorname{gr}_J B_t \cong B_t. \tag{12}$$

In particular, $(B_t)_0 = \mathbb{C}^8$, $\dim (B_t)_1 = 32$, and $\dim (B_t)_2 = 48$.

3.2 Arrows

Let e_v be the primitive idempotents of $(B_t)_0$, indexed by (9). Under (11), the component of y_i from v to w is a 2×2 matrix with entries in $e_w (B_t)_1 e_v$. The group relations force y_1 to change (r_v, s_v) to $(r_v, -s_v)$, and y_2 to change it to $(-r_v, s_v)$. The relations with g, x_2, respectively g, x_1, force each component to be a single element of $e_w (B_t)_1 e_v$ times, respectively, the matrix

$$Y_{wv} = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha_v/\alpha_w \end{pmatrix}, \qquad Q_{wv} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

Since $(A_t)_1 = D_t y_1 D_t + D_t y_2 D_t$, each of the $8 \cdot 4 = 32$ possible arrow spaces has dimension at most one. As $\dim(B_t)_1 = 32$, every one has dimension one. We take the nonzero upper-left entries as arrow generators. The resulting quiver has underlying graph $K_{4,4}$, with parts distinguished by $r_v s_v$.

The graph is connected, so A_t is a block. It is the unique block of H_t of dimension 512 all of whose simple modules have dimension two. Indeed, the complementary summand H_t^+ also has dimension 512 and contains the one-dimensional counit module. Any other block with the stated dimension would exhaust that summand.

3.3 Products of two arrows

For vertices v, w, put

$$W_{wv} = e_w(J^2/J^3)e_v = e_w(B_t)_2 e_v.$$

There are four length-two paths from v to each w in the same bipartite part, and none to the opposite part.

First suppose $(r_w, s_w) = (r_v, s_v)$. There are two paths of type $y_1 y_1$ and two of type $y_2 y_2$, with intermediate scalars β, β^{-1}. For either type i, write their products as $P_\beta, P_{\beta^{-1}}$. The relations $y_i^2 = z_i^2 = 0$, using (10) and (13), give

$$P_\beta + P_{\beta^{-1}} = 0, \qquad f_i(\beta)P_\beta + f_i(\beta^{-1})P_{\beta^{-1}} = 0,$$

where

$$f_1(\beta) = (1 + \beta/\alpha_w)(1 + \alpha_v/\beta), \qquad f_2(\beta) = \beta + \alpha_v + \alpha_w + \frac{\alpha_v \alpha_w}{\beta}. \tag{14}$$

For the second weight, the coefficient matrix of $z_2 = [x_2, y_2]$ on a y_2-arrow with source scalar γ and target scalar δ is

$$R(\delta, \gamma) = \begin{pmatrix} 0 & -(\delta + \gamma) \\ \delta^{-1} + \gamma^{-1} & 0 \end{pmatrix}.$$

The upper-left entry of the two-step product through β is

$$\big(R(\alpha_w, \beta)R(\beta, \alpha_v)\big)_{11} = -(\alpha_w + \beta)(\beta^{-1} + \alpha_v^{-1}) = -\frac{f_2(\beta)}{\alpha_v},$$

which gives the stated relation.

If $w = v$, the differences of the second coefficients are

$$(\beta - \beta^{-1})(\alpha_v^{-1} - \alpha_v), \qquad (\beta - \beta^{-1})(1 - \alpha_v^2),$$

respectively. Both are nonzero, so all return-path products vanish. If $\alpha_w = \alpha_v^{-1}$, the coefficients agree; each type then spans at most one line. Thus $W_{vv} = 0$, and the other target with the same r, s has dimension at most two.

Now let $(r_w, s_w) = (-r_v, -s_v)$. Write U_β for the product through $(-r_v, s_v, \beta)$, and V_β for the product through $(r_v, -s_v, \beta)$. The two matrix entries of $y_1 y_2 = y_2 y_1$ give

$$\begin{aligned} U_\beta + U_{\beta^{-1}} &= V_\beta + V_{\beta^{-1}}, \\ \beta U_\beta + \beta^{-1} U_{\beta^{-1}} &= \alpha_v \alpha_w (\beta^{-1} V_\beta + \beta V_{\beta^{-1}}). \end{aligned} \tag{15}$$

These relations are independent because $\beta \neq \beta^{-1}$, so each of the two mixed targets has dimension at most two. Since B_t is generated in degrees zero and one, $(B_t)_2$ is spanned by length-two paths.

Thus only the four targets considered above occur for each source, and the exhaustive upper bounds sum to

$$8(0+2+2+2) = 48 = \dim(B_t)_2.$$

Every bound is therefore attained. For each mixed target, the map from the four-dimensional path space onto W_{wv} has a two-dimensional kernel. The two independent relations (15) form a basis of this kernel.

4 The elliptic invariant

Proof of Theorem 1.1. Proposition 2.1 constructs the required algebraic family. We recover $\mathcal{J}(t)$ from the associative algebra.

The radical multiplication maps

$$e_w(J/J^2)e_k \otimes e_k(J/J^2)e_v \longrightarrow W_{wv}$$

attach a product line to each intermediate vertex k, whenever the product is nonzero. Here the e_v may be viewed in B_t/J: each radical layer J^m/J^{m+1} is canonically a B_t/J-bimodule. Indeed, changing a lift by an element of J changes its action only in J^{m+1}. The construction is therefore independent of the chosen idempotent lifts. If W_{wv} is two-dimensional and the four product lines are distinct, their unordered configuration in $\mathbb{P}(W_{wv})$ is intrinsic. Indeed, an algebra isomorphism preserves the radical filtration and permutes the vertices and arrow spaces of its associated graded algebra. Passing to a basic algebra does not affect the construction, since basic algebras in a given Morita class are isomorphic.

We determine all configurations of four distinct lines. The return target has zero product space. For the other target with the same r, s, the relation $y_i^2 = 0$ makes the two paths of each type span the same line, so there are at most two distinct lines. For the mixed target with $\alpha_w = \alpha_v^{-1}$, equation (15) gives

$$V_\beta = U_{\beta^{-1}}, \qquad V_{\beta^{-1}} = U_\beta,$$

again leaving only two lines.

It remains to take $w = (-r_v, -s_v, \alpha_v)$. Put $\alpha = \alpha_v = \alpha_w$. Equation (15) expresses the two V's in terms of the two U's, and conversely; each pair is therefore a basis of the two-dimensional space W_{wv}. Solving these equations gives the cross-ratio of their four spans, up to ordering:

$$\begin{aligned} \lambda &= -\frac{(\alpha^2\beta - \beta^{-1})(\beta - \alpha^2\beta^{-1})}{(\alpha^2-1)^2} \\ &= \frac{(\alpha+\alpha^{-1})^2 - (\beta+\beta^{-1})^2}{(\alpha-\alpha^{-1})^2}. \end{aligned} \tag{16}$$

By (9), the two sums in the numerator are $(0, 2t)$ or $(2t, 0)$. Hence

$$\lambda = t^2 \qquad \text{or} \qquad \lambda = \frac{t^2}{t^2-1}.$$

These values are related by an anharmonic transformation, and neither is $0, 1, \infty$. Thus the four lines are distinct, and every such configuration is projectively equivalent to $\{0, 1, \infty, t^2\}$.

For an unordered quadruple, the expression

$$j(\lambda) = 256\frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda)^2}$$

is independent of ordering: it is unchanged by $\lambda \mapsto 1-\lambda$ and $\lambda \mapsto 1/\lambda$, which generate the six changes of cross-ratio. Every configuration just found has the same value $j(t^2) = \mathcal{J}(t)$. Thus $\mathcal{J}(t)$ is intrinsic to B_t, hence to A_t, and hence to H_t, because A_t is the unique block identified in Section 3. This proves the asserted implication.

For fixed j_0, the equation $\mathcal{J}(t) = j_0$ is a polynomial equation of degree twelve after clearing the denominator. It therefore has at most twelve solutions in T. Since $\mathcal{J}$ is nonconstant, choose $t_0 \in T$ with $\mathcal{J}'(t_0) \neq 0$. The inverse function theorem gives a small complex disc about t_0 on which $\mathcal{J}$ is injective. The corresponding fibers are pairwise nonisomorphic as associative algebras. □

Remark 4.1. The invariant $\mathcal{J}$ is the classical elliptic j-invariant. The smooth projective double cover of the projective line branched over the recovered quadruple has affine equation

$$E_t : \qquad y^2 = x(x-1)(x-t^2).$$

Choosing a ramification point as origin gives an elliptic curve, with isomorphism class independent of this choice. Its invariants are

$$c_4 = 16(1-t^2+t^4), \qquad \Delta = 16t^4(1-t^2)^2, \qquad j(E_t) = c_4^3/\Delta = \mathcal{J}(t).$$

Thus $[H_t] \mapsto [E_t]$ defines a map from the associative algebra isomorphism classes occurring in this family to the moduli of complex elliptic curves.

Acknowledgements

This paper was written in response to the following substantive prompts by Pavel Etingof, reproduced in chronological order:

1. In the paper https://arxiv.org/pdf/math/0110043 (top of p. 2) it is asked if there are finitely many algebra structures for Hopf algebras over $\mathbb{C}$ of a given dimension n. Can you find a counterexample? Work hard!

2. It's a sequence, but is there a continuous family?

3. We don't need the sequence, the continuous family is good. So the moduli of these Hopf algebras are moduli of elliptic curves, and J is the J-invariant? If so, you may make a remark to that extent. Write a high-quality paper with abstract and introduction. State the main result in the introduction. In the body of the paper give a proof. Be concise, absolutely nothing not needed for the proof. Make the presentation clear and easily human-readable!

References

[1] P. Etingof and S. Gelaki, *On families of triangular Hopf algebras*, International Mathematics Research Notices **2002**, no. 14, 757–768. arXiv:math/0110043.