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Spectrally Simple Spectra without Irreducible Nonnegative Realizations

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Abstract

We give a counterexample to a conjecture of Johnson, Marijuán, and Pisonero: we show that the spectrum $\{1, 1/2, \pm i/\sqrt{3}\}$ is realizable and spectrally simple, but has no irreducible nonnegative realization. More generally, we prove their nonexistence conjecture for an order-four boundary family.

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1 Introduction and main results

For an integer $n \geq 1$, let I_n denote the $n \times n$ identity matrix. The **spectrum** of a real $n \times n$ matrix A is the list of its eigenvalues, counted with algebraic multiplicity. A list σ of n complex numbers is **realizable** if it is the spectrum of an entrywise nonnegative real matrix. We write $A \geq 0$ when every entry of A is nonnegative. Such a matrix $A = (a_{ij})$ is **irreducible** if, for every nonempty proper subset S of $\{1, \ldots, n\}$, some $i \in S$ and $j \notin S$ satisfy $a_{ij} > 0$. A spectrum is **irreducibly realizable** if it has an irreducible nonnegative realization.

For a realizable spectrum σ, its **Perron root** is $\rho(\sigma) = \max_{\lambda \in \sigma} |\lambda|$, which, by the Perron–Frobenius theorem, is a nonnegative real member of σ. Its **spectral circle** is $\{z \in \mathbb{C} : |z| = \rho(\sigma)\}$. Following Johnson, Marijuán, and Pisonero [1], we call σ **spectrally simple** if its Perron root occurs once and every other member has strictly smaller modulus. For a matrix A, write $\rho(A)$ for the maximum modulus of its eigenvalues.

Johnson, Marijuán, and Pisonero [1, p. 3] conjectured that every realizable, spectrally simple spectrum is irreducibly realizable. They proved the assertion in orders below four and for real spectra of order four [1, Sections 8–10]. In their analysis of nonreal spectra of order four, they also conjectured that a particular family has no irreducible realizations [1, p. 12]. Our main result proves the latter conjecture and disproves the former.

Theorem 1.1. *Let $a, b, c, d \in \mathbb{R}$ satisfy $a > b$, $a > 0$, and $d > 0$. Put $s = a + 2c$, and suppose*

$$s > 0, \qquad a - c = \sqrt{3}d, \qquad b > \frac{s}{3}. \tag{1.1}$$

Then $\sigma = \{a, b, c + di, c - di\}$ is realizable and spectrally simple, but is not irreducibly realizable.

Corollary 1.2. *The spectrum*

$$\left\{1, \frac{1}{2}, \frac{i}{\sqrt{3}}, -\frac{i}{\sqrt{3}}\right\} \tag{1.2}$$

is realizable and spectrally simple, but has no irreducible nonnegative realization.

The proof uses an eigenvector for the smaller real eigenvalue b. Its positive and negative coordinates force a strict lower bound on the sum of products $a_{ij}a_{ji}$. The coefficient of x^2 in the characteristic polynomial gives an incompatible upper bound. The argument includes eigenvectors with zero coordinates.

2 An eigenvector inequality

For a nonnegative 4×4 matrix $A = (a_{ij})$, put

$$l_i = a_{ii}, \qquad q_{ij} = a_{ij}a_{ji} \quad (i < j), \qquad Q = \sum_{i<j} q_{ij}. \tag{2.1}$$

Lemma 2.1. *Let $A \geq 0$ be an irreducible 4×4 matrix, and let $b < \rho(A)$ be a real eigenvalue. Suppose $0 \leq l_i \leq b$ for every i, with at most one $l_i = b$. Put $r_i = b - l_i$. After simultaneously permuting the rows and columns,*

$$Q > r_1 r_2 + r_3 r_4. \tag{2.2}$$

Proof. Choose a real eigenvector x with $Ax = bx$. The Perron–Frobenius theorem gives a real vector w whose entries are all positive and which satisfies $w^{\mathsf{T}}A = \rho(A)w^{\mathsf{T}}$. Since $(\rho(A) - b)w^{\mathsf{T}}x = 0$ and $b < \rho(A)$, we have $w^{\mathsf{T}}x = 0$. Because w has strictly positive entries, x has both positive and negative coordinates.

Suppose first that no coordinate is zero. A sign cannot occur just once: if x_i were the sole positive coordinate, the equation

$$r_i x_i = \sum_{j \neq i} a_{ij}x_j \leq 0$$

would force $r_i = 0$ and $a_{ij} = 0$ for all $j \neq i$, contrary to irreducibility. Replacing x by $-x$ treats a sole negative coordinate. We may therefore arrange $x_1, x_2 > 0$ and $x_3, x_4 < 0$. Write $u_i = x_i$ for $i = 1, 2$ and $v_i = -x_i$ for $i = 3, 4$. The first two rows give

$$\begin{aligned} a_{12}u_2 &= r_1 u_1 + a_{13}v_3 + a_{14}v_4, \\ a_{21}u_1 &= r_2 u_2 + a_{23}v_3 + a_{24}v_4. \end{aligned} \tag{2.3}$$

Both left sides are positive: if one vanished, its row would have no positive off-diagonal entry. Irreducibility supplies a positive entry from $\{1, 2\}$ to $\{3, 4\}$, so at least one of $a_{13}, a_{14}, a_{23}, a_{24}$ is positive. Since $v_3, v_4 > 0$, the sums

$$a_{13}v_3 + a_{14}v_4, \qquad a_{23}v_3 + a_{24}v_4$$

in (2.3) are nonnegative, and at least one is positive. Consequently,

$$a_{12}u_2 \geq r_1 u_1, \qquad a_{21}u_1 \geq r_2 u_2,$$

with at least one strict inequality. Both left sides are positive, as established above. Multiplying gives

$$q_{12}u_1u_2 = (a_{12}u_2)(a_{21}u_1) > r_1 r_2 u_1 u_2.$$

If $r_1 r_2 = 0$, the strict inequality follows directly from positivity of the left side. Dividing by $u_1u_2 > 0$ yields $q_{12} > r_1 r_2$.

The same argument, using a positive entry from $\{3, 4\}$ to $\{1, 2\}$, gives $q_{34} > r_3 r_4$. All remaining q_{ij} are nonnegative, so (2.2) follows from

$$Q \geq q_{12} + q_{34} > r_1 r_2 + r_3 r_4.$$

If x has two zero coordinates, its nonzero coordinates have opposite signs. After relabelling, suppose $x_1 > 0$, $x_2 < 0$, and $x_3 = x_4 = 0$. The first two row equations become

$$r_1 x_1 = a_{12} x_2, \qquad r_2 x_2 = a_{21} x_1.$$

In the first equation, the left side is nonnegative and the right side is nonpositive; in the second, the signs are reversed. Thus both equations have both sides zero. Since $x_1, x_2 \neq 0$, we obtain $r_1 = r_2 = 0$, hence $l_1 = l_2 = b$, contradicting the assumption that at most one diagonal entry equals b. More than two zero coordinates cannot give both signs.

It remains to consider exactly one zero. After relabelling and, if necessary, replacing x by $-x$, assume $x_1, x_2 > 0$, $x_3 < 0$, and $x_4 = 0$. The third row gives $r_3 = 0$ and $a_{31} = a_{32} = 0$. Irreducibility forces $a_{34} > 0$. Applied to the subset $\{3, 4\}$, it also forces $a_{41} > 0$ or $a_{42} > 0$. The fourth row equation

$$a_{41} x_1 + a_{42} x_2 + a_{43} x_3 = 0$$

therefore implies $a_{43} > 0$. Hence $q_{34} > 0 = r_3 r_4$. The first two row equations, multiplied as above, give $q_{12} \geq r_1 r_2$. This again proves (2.2). □

3 Proof of the main result

For a list $z = (z_1, z_2, z_3, z_4)$, define $e_2(z) = \sum_{i<j} z_i z_j$.

Proof of Theorem 1.1. The definition of s and (1.1) give

$$c = \frac{s - a}{2}, \qquad d = \frac{3a - s}{2\sqrt{3}}, \qquad 0 < s < 3a. \tag{3.1}$$

The nonnegative matrix

$$B = \frac{1}{3} \begin{pmatrix} s & 3a - s & 0 \\ 0 & s & 3a - s \\ 3a - s & 0 & s \end{pmatrix} \tag{3.2}$$

has characteristic polynomial $(x - s/3)^3 - (a - s/3)^3$ and eigenvalues $a, c + di, c - di$. Adjoin a first row and column to B, placing b in the top-left entry and zero in every other new entry. The resulting nonnegative matrix has characteristic polynomial $(x - b)\det(xI_3 - B)$ and realizes σ. Moreover,

$$c^2 + d^2 = a^2 - s\left(a - \frac{s}{3}\right) < a^2,$$

and $0 < b < a$, so σ is spectrally simple.

Suppose now that $A = (a_{ij})$ is an irreducible nonnegative realization. Put

$$l_i = a_{ii}, \qquad q_{ij} = a_{ij} a_{ji} \quad (i < j), \qquad Q = \sum_{i<j} q_{ij},$$

and write $l = (l_1, l_2, l_3, l_4)$. Write the characteristic polynomial as

$$\det(xI_4 - A) = x^4 + k_1 x^3 + k_2 x^2 + k_3 x + k_4.$$

The first two characteristic coefficients are k_1, the coefficient of x^3, and k_2, the coefficient of x^2. Computing them from the matrix entries and from the prescribed eigenvalues gives

$$k_1 = -\sum_i l_i = -(b + s), \qquad k_2 = e_2(l) - Q = e_2(\sigma) = s\left(b + \frac{s}{3}\right). \tag{3.3}$$

Indeed, expanding the determinant gives $\sum_{i<j}(l_i l_j - a_{ij}a_{ji})$ as the coefficient of x^2.

Let $m = \max_i l_i$. The other three diagonal entries have sum $b+s-m$; their pairwise products sum to at most $(b+s-m)^2/3$, with equality only when those entries are equal. Since $Q \geq 0$, (3.3) yields

$$s\left(b+\frac{s}{3}\right) \leq e_2(l) \leq m(b+s-m) + \frac{(b+s-m)^2}{3}. \tag{3.4}$$

The last expression, as a function of m, is strictly decreasing for $m > (b+s)/4$. Since $b > s/3$, we have $b > (b+s)/4$, and its value at $m = b$ is $s(b+s/3)$. Consequently $m \leq b$. If $m = b$, equality holds throughout (3.4), so $Q = 0$ and the other three diagonal entries are $s/3 < b$. Thus $0 \leq l_i \leq b$, with at most one $l_i = b$.

Lemma 2.1 applies, because $b < a = \rho(A)$. Put $r = (r_1, r_2, r_3, r_4)$, where $r_i = b - l_i$, and label the indices as in that lemma. Expanding $e_2(l) = e_2(b-r_1, \ldots, b-r_4)$ in (3.3) gives

$$\sum_i r_i = 3b - s, \qquad Q = e_2(r) - 3b^2 + 2bs - \frac{s^2}{3}. \tag{3.5}$$

Set $X = r_1 + r_2$ and $Z = r_3 + r_4$. Since $X + Z = 3b - s$, we have $XZ \leq (3b-s)^2/4$. It follows that

$$\begin{aligned} 0 < Q - r_1 r_2 - r_3 r_4 &= XZ - 3b^2 + 2bs - \frac{s^2}{3} \\ &\leq \frac{(3b-s)^2}{4} - 3b^2 + 2bs - \frac{s^2}{3} \\ &= -\frac{(3b-s)^2}{12} < 0. \end{aligned} \tag{3.6}$$

This contradiction proves that σ is not irreducibly realizable. □

Proof of Corollary 1.2. Take $(a,b,c,d) = (1, 1/2, 0, 1/\sqrt{3})$. Then $s = 1$, and every hypothesis of Theorem 1.1 holds. The construction in (3.2) gives the explicit realization

$$\begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 2/3 & 0 & 1/3 \end{pmatrix}.$$

Theorem 1.1 excludes every irreducible nonnegative realization. □

Corollary 3.1. *The conjecture of Johnson, Marijuán, and Pisonero [1, p. 3] that every realizable, spectrally simple spectrum is irreducibly realizable is false.*

Proof. Corollary 1.2 gives a realizable, spectrally simple spectrum with no irreducible nonnegative realization. □

Johnson, Marijuán, and Pisonero [1, Section 10, p. 12] also proposed the following family conjecture. Let $a, b, c, d \in \mathbb{R}$ satisfy $a > b$, $a > 0$, and $d > 0$. Suppose $\{a, c \pm di\}$ is realizable and

$$a + 2c > 0, \qquad a - c = \sqrt{3}d, \qquad b > \frac{a+2c}{3}.$$

They conjectured that $\sigma = \{a, b, c \pm di\}$ has no irreducible nonnegative realization.

Corollary 3.2. *The family conjecture of Johnson, Marijuán, and Pisonero stated above is true.*

Proof. Put $s = a + 2c$. The parameters satisfy every hypothesis of Theorem 1.1, which excludes every irreducible nonnegative realization of σ. □

Acknowledgments

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References

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