A LOWER BOUND FOR THE FILLING AREA OF THE CIRCLE

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Abstract. We prove that every compact connected orientable Riemannian surface isometrically filling the circle of length 2π has area at least 6.06571278970403721852. The exact bound is a specified rational multiple of π^2, and no restriction is imposed on the genus. The proof uses alternating forms built from antipodal distance differences. Finite partitions of the distance profiles, a three-level inheritance construction, and positive combinations of two coefficient systems reduce the norm estimate to rational matrix and Laurent-polynomial certificates. We give the geometric reduction and all exact verification criteria. Complete finite data and independent arithmetic checks accompany the paper.

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1. Introduction

Let $S^1_{2\pi} = \mathbb{R}/2\pi\mathbb{Z}$ carry its intrinsic length metric

$$d_{S^1}(s,t) = \min_{m\in\mathbb{Z}} |s - t + 2m\pi|.$$

An *isometric filling* of $S^1_{2\pi}$ is a compact connected Riemannian surface M with one boundary component and an arclength parametrization $\gamma : S^1_{2\pi} \longrightarrow \partial M$ such that

$$d_M(\gamma(s),\gamma(t)) = d_{S^1}(s,t) \qquad (s,t \in S^1_{2\pi}). \tag{1.1}$$

Here d_M is the intrinsic Riemannian distance on M. Equivalently, one may require the left-hand side to be at least the right-hand side: the reverse inequality follows by using a shorter boundary arc.

Gromov's filling area conjecture asserts that every orientable isometric filling of $S^1_{2\pi}$ has area at least 2π; a hemisphere attains this value [4]. The distinction between fixed and unrestricted topology is essential. Classical results establish filling minimality for the disk and for further classes of surfaces [1, 5]. The present paper concerns all orientable fillings, with no bound on their genus.

Theorem 1.1. *Let M be a compact connected orientable Riemannian surface with exactly one boundary component. Suppose that an arclength parametrization of its boundary satisfies* (1.1)*. Then*

$$\mathrm{Area}(M) \geq \rho_* \pi^2, \tag{1.2}$$

where

$$\rho_* = \frac{2003451650178971284257400112098931034076276719384090896484375}{325984363412975006458640005385159668498568647385301907126702 4}. \tag{1.3}$$

In particular,

$$\mathrm{Area}(M) > 6.06571278970403721852.$$

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By scaling, the corresponding bound for a circle of arbitrary length $L > 0$ is

$$\operatorname{Area}(M) \geq \frac{\rho_*}{4} L^2.$$

We do not claim that ρ_* is optimal.

Recent universal estimates use discrete methods [2], Fourier coefficients of boundary distance functions [3], and calibrations of polyhedral fillings in maximum norm spaces [6]. The estimate above is larger than the universal orientable lower bounds established in those papers. Our construction also uses distance coordinates and piecewise constant alternating forms. Its additional ingredients are refinement confined to endpoint bands, inheritance between different numbers of anchors, and a norm estimate for positive combinations of two coefficient systems. Laurent polynomials allow selected matrix norm bounds to exploit the fact that the tangent space has dimension two.

The proof has an analytic part and a finite arithmetic part. We prove the boundary integral formula for the actual step functions by Lipschitz Stokes, and justify the reduction to finitely many distance profiles, including profiles on grid lines. We then give the precise integer and rational tests for the coefficient arrays in the electronic supplement. No numerical optimization or earlier certificate is needed to verify these data. The theorem is a computer-assisted inequality; the supplement forms part of its finite proof.

2. Distance coordinates and boundary integrals

Let M be a compact connected oriented Riemannian surface with one boundary component. We assume that an arclength parametrization $\gamma\colon \mathbb{R}/2\pi\mathbb{Z} \longrightarrow \partial M$, compatible with the boundary orientation, satisfies

$$d_M(\gamma(s), \gamma(t)) = d_{\mathbb{R}/2\pi\mathbb{Z}}(s, t). \tag{2.1}$$

Here the distance on the right is the length of the shorter circular arc. The usual filling condition with $\geq$ in (2.1) is equivalent to this condition: the reverse inequality follows by using a boundary arc as a path in M.

Fix an integer $n \geq 2$, put $h = \pi/n$ and $\alpha_i = ih$, and define

$$F_i(p) = \frac{d_M(p, \gamma(\alpha_i + \pi)) - d_M(p, \gamma(\alpha_i))}{2}, \qquad X_i(p) = \frac{F_i(p)}{h}, \quad 0 \leq i < n. \tag{2.2}$$

We extend the indices by $F_{i+n} = -F_i$ and $X_{i+n} = -X_i$. Each F_i is 1-Lipschitz. Applying the reverse triangle inequality to the two distances in (2.2) gives $|F_{i+1} - F_i| \leq h$, including the last inequality with $F_n = -F_0$. Consequently $X = (X_0, \ldots, X_{n-1})$ takes its values in the polytope

$$\mathcal{E}_n = \{x \in \mathbb{R}^n : |x_{i+1} - x_i| \leq 1 \ (0 \leq i < n),\ x_n = -x_0\}. \tag{2.3}$$

For $0 \leq i < j < n$ and $k = j - i$, summation along the two complementary chains of consecutive indices gives

$$|x_i - x_j| \leq k, \qquad |x_i + x_j| \leq n - k \quad (x \in \mathcal{E}_n). \tag{2.4}$$

Summation over a chain of length n also gives $|x_i| \leq n/2$. Write

$$\mathcal{R}_{n,k} = \{(s, t) \in \mathbb{R}^2 : |s - t| \leq k,\ |s + t| \leq n - k\}, \qquad 1 \leq k < n. \tag{2.5}$$

Thus $(X_i, X_j)(M) \subset \mathcal{R}_{n,j-i}$.

On the boundary, (2.1) yields

$$F_i(\gamma(s)) = \frac{\pi}{2} - d_{\mathbb{R}/2\pi\mathbb{Z}}(s, \alpha_i). \tag{2.6}$$

The pair (X_i, X_j) maps the positively oriented boundary of M onto $\partial\mathcal{R}_{n,k}$ once, with positive orientation. Indeed, the parameters $\alpha_i, \alpha_j, \alpha_i + \pi, \alpha_j + \pi$ give, in that order, the four corners

$$\left(\frac{n}{2}, \frac{n}{2} - k\right), \quad \left(\frac{n}{2} - k, \frac{n}{2}\right), \quad \left(-\frac{n}{2}, -\frac{n}{2} + k\right), \quad \left(-\frac{n}{2} + k, -\frac{n}{2}\right). \tag{2.7}$$

Between consecutive parameters the two coordinates are affine, with successive directions $(-1,1), (-1,-1), (1,-1), (1,1)$. These are the four edges of the rectangle in counterclockwise order.

We shall integrate forms with coefficients that are constant on finitely many rectangles. The following elementary version of Stokes' theorem also addresses the level sets on which such coefficients jump.

Lemma 2.1. *If u, v are Lipschitz functions on M, then*

$$\int_M du \wedge dv = \int_{\partial M} u\, dv. \tag{2.8}$$

For an interval $I = (a, b)$, let

$$P_I(t) = \min\{\max\{t - a, 0\}, b - a\}.$$

Then

$$d(P_I \circ u) = \mathbf{1}_I(u)\, du \quad \textit{almost everywhere on } M. \tag{2.9}$$

Proof. A Lipschitz function has zero differential almost everywhere on any one of its level sets. To see this in a coordinate chart, consider a point where the function is differentiable and the level set has density one. A nonzero differential would be bounded away from zero, relative to the distance from the point, in a cone of positive relative measure. Differentiability would then put that cone outside the level set near the point, a contradiction. The differentiation and density theorems prove the assertion. The ordinary chain rule away from the two levels a, b, together with this assertion on those levels, proves (2.9).

For completeness, extend u, v in boundary charts, mollify, and combine the local approximations with a smooth partition of unity. This gives smooth functions $u_\varepsilon, v_\varepsilon$ converging uniformly and strongly in $W^{1,2}$ to u, v, with bounded Lipschitz constants. Stokes' theorem holds for each pair of smooth approximations. Its left side converges by the Cauchy–Schwarz inequality. On the closed boundary, integration by parts gives

$$\begin{aligned}\left|\int_{\partial M} u_\varepsilon\, dv_\varepsilon - \int_{\partial M} u\, dv\right| &\le \|u_\varepsilon - u\|_\infty \operatorname{Var}_{\partial M}(v_\varepsilon) \\ &\quad + \|v_\varepsilon - v\|_\infty \operatorname{Var}_{\partial M}(u).\end{aligned}$$

The right side tends to zero. This proves (2.8) without requiring convergence of the boundary derivatives. The same argument proves the planar formula on a polygon; alternatively, that case follows by smoothing the two functions and applying Green's theorem. □

Proposition 2.2. *For open bounded intervals $I, J \subset \mathbb{R}$ and $0 \le i < j < n$,*

$$\int_M \mathbf{1}_I(X_i)\mathbf{1}_J(X_j)\, dF_i \wedge dF_j = h^2 \operatorname{Area}\Big(\mathcal{R}_{n,j-i} \cap (I \times J)\Big). \tag{2.10}$$

Proof. Apply Lemma 2.1 to $P_I(X_i), P_J(X_j)$. The left side of (2.10), divided by h^2, is

$$\int_M d(P_I(X_i)) \wedge d(P_J(X_j)) = \int_{\partial M} P_I(X_i)\, d(P_J(X_j)).$$

The boundary calculation (2.7) identifies the last integral with

$$\int_{\partial \mathcal{R}_{n,j-i}} P_I(s)\, d(P_J(t)).$$

The planar formula in Lemma 2.1 makes this $\int_{\mathcal{R}_{n,j-i}} \mathbf{1}_I(s)\mathbf{1}_J(t)\, ds\, dt$, as asserted. □

Let

$$\begin{gathered} G = \{g_0 < g_1 < \cdots < g_N\}, \qquad g_0 = -n/2, \quad g_N = n/2, \\ I_a = (g_a, g_{a+1}) \qquad (0 \le a < N) \end{gathered} \tag{2.11}$$

be a finite grid. For $1 \le k < n$, let f_k have the constant value $c_{k,a,b}$ on $\mathcal{R}_{n,k} \cap (I_a \times I_b)$. Set $f_k(s,t) = 0$ if s or t is a grid value. At an edge or corner of $\mathcal{R}_{n,k}$ the cell value is retained unless one of its coordinates is a grid value. Define the measurable two-form

$$\Omega_f = \sum_{0 \le i < j < n} f_{j-i}(X_i, X_j)\, dF_i \wedge dF_j \tag{2.12}$$

and its boundary action

$$A_n(f) = \frac{1}{n^2} \sum_{k=1}^{n-1} (n-k) \sum_{a,b=0}^{N-1} c_{k,a,b}\, \mathrm{Area}\Big(\mathcal{R}_{n,k} \cap (I_a \times I_b)\Big). \tag{2.13}$$

Proposition 2.2 immediately gives

$$\int_M \Omega_f = \pi^2 A_n(f). \tag{2.14}$$

There are $n-k$ pairs with separation k, which accounts for that factor in (2.13). Both the coefficients and the action may be signed. The formula is unchanged if all the anchors α_i are translated by the same circular parameter.

3. Bounds for alternating forms

For a real skew-symmetric $n \times n$ matrix W, with upper entries w_{ij}, define

$$\|W\|_{\mathrm{disc}} = \max_{a_i^2 + b_i^2 \le 1} \left| a^{\mathsf{T}} W b \right| = \max_{a_i^2 + b_i^2 \le 1} \left| \sum_{i<j} w_{ij}(a_i b_j - a_j b_i) \right|. \tag{3.1}$$

This is the comass on the product of n Euclidean unit disks. It is a norm on skew-symmetric matrices, is invariant under conjugation by a signed permutation matrix, and satisfies the triangle inequality. These properties follow directly by changing the vectors (a_i, b_i) in (3.1).

The matrix of a coefficient system f at $x \in \mathcal{E}_n$ is

$$(W_f(x))_{ij} = f_{j-i}(x_i, x_j) \quad (i<j), \qquad (W_f(x))_{ji} = -(W_f(x))_{ij}. \tag{3.2}$$

At a differentiability point of the distance coordinates, choose a positively oriented orthonormal basis e_1, e_2 and put $a_i = dF_i(e_1)$, $b_i = dF_i(e_2)$. Since F_i is 1-Lipschitz, $a_i^2 + b_i^2 \le 1$, and

$$\Omega_f(e_1, e_2) = a^{\mathsf{T}} W_f(X) b. \tag{3.3}$$

It follows that any uniform upper bound $U > 0$ for $\|W_f(x)\|_{\mathrm{disc}}$, $x \in \mathcal{E}_n$, gives

$$\mathrm{Area}(M) \ge \frac{\pi^2 A_n(f)}{U}. \tag{3.4}$$

Indeed, (3.3) gives $\Omega_f \le U\, dA$ almost everywhere, and (2.14) evaluates its integral.

We use two finite criteria for bounding the norm (3.1).

Proposition 3.1. *Let D be a real diagonal matrix. If*

$$D + \frac{\mathrm{i}}{2} W \succeq 0, \tag{3.5}$$

then $\|W\|_{\mathrm{disc}} \leq \operatorname{tr} D$. Condition (3.5) *is equivalent to positive semidefiniteness of the real symmetric matrix*

$$\begin{pmatrix} 2D & -W \\ W & 2D \end{pmatrix}. \tag{3.6}$$

Proof. The diagonal entries d_i of D are nonnegative by (3.5). With $z = a + \mathrm{i}b$, one has

$$z^* \left(D + \frac{\mathrm{i}}{2} W \right) z = \sum_i d_i (a_i^2 + b_i^2) - a^{\mathsf{T}} W b \geq 0.$$

For vectors in the unit disks this gives $a^{\mathsf{T}} W b \leq \operatorname{tr} D$. Replacing b by $-b$ gives the opposite inequality. The real quadratic form of (3.6) at (a, b) is twice the displayed Hermitian quadratic form, which proves the equivalence. □

Write $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and set

$$C_W(z) = \sum_{i<j} w_{ij} \operatorname{Im}(\overline{z}_i z_j). \tag{3.7}$$

Lemma 3.2. *The maximum of C_W on the product of closed unit disks equals its maximum on $\mathbb{T}^n$, and both equal $\|W\|_{\mathrm{disc}}$. The latter maximum may be taken over the subtorus $z_0 = 1$.*

Proof. For fixed values of the other coordinates, C_W is a real linear function of $z_i \in \mathbb{R}^2$. Its maximum on a closed disk is attained on its boundary, unless the function is constant, in which case any boundary point has the same value. Starting with a maximizer on the compact product of disks, replace the coordinates successively by such boundary maximizers. This proves the first assertion. Complex conjugation changes the sign of C_W, so its maximum also equals the maximum of its absolute value. Finally, multiplication of all coordinates by the same unit complex number leaves C_W unchanged. □

Proposition 3.3. *Let W have rational entries. Put*

$$\widetilde{C}_W(z_1, \ldots, z_{n-1}) = C_W(1, z_1, \ldots, z_{n-1}).$$

Suppose that Gaussian rational Laurent polynomials $p_1, \ldots, p_q$ satisfy, on $\mathbb{T}^{n-1}$, the identity

$$\widetilde{C}_W(z) + \sum_{\ell=1}^{q} |p_\ell(z)|^2 = a_0 + \sum_{\nu \in H} \left(r_\nu \operatorname{Re} z^\nu + s_\nu \operatorname{Im} z^\nu \right), \tag{3.8}$$

where $a_0, r_\nu, s_\nu \in \mathbb{Q}$ and H contains at most one representative of each nonzero pair $\{\nu, -\nu\} \subset \mathbb{Z}^{n-1}$. If $t_\nu \in \mathbb{Q}$ satisfy

$$t_\nu \geq 0, \qquad t_\nu^2 \geq r_\nu^2 + s_\nu^2, \tag{3.9}$$

then

$$\|W\|_{\mathrm{disc}} \leq a_0 + \sum_{\nu \in H} t_\nu. \tag{3.10}$$

Proof. For $z \in \mathbb{T}^{n-1}$, every z^ν has absolute value one. Hence

$$r_\nu \operatorname{Re} z^\nu + s_\nu \operatorname{Im} z^\nu \leq \sqrt{r_\nu^2 + s_\nu^2} \leq t_\nu.$$

Subtracting the nonnegative sum of squares in (3.8) proves the stated upper bound for $\widetilde{C}_W$ on the torus. Lemma 3.2 proves (3.10) on the full product of disks. Only the inequality is extended to the disks; the Laurent identity is used on the torus. □

The identity in (3.8) has a finite exact meaning. If $p(z) = \sum_\alpha c_\alpha z^\alpha$, then on the torus

$$|p(z)|^2 = \sum_{\alpha,\beta} c_\alpha \overline{c}_\beta z^{\alpha-\beta}.$$

Thus the identity can be checked coefficient by coefficient using Gaussian rational arithmetic. In particular a conjugate pair $(u+\mathrm{i}v)z^\nu + (u-\mathrm{i}v)z^{-\nu}$ contributes $2u\,\mathrm{Re}\, z^\nu - 2v\,\mathrm{Im}\, z^\nu$. Both propositions allow coefficients of either sign.

4. Finite partitions and refinement

4.1. **Symmetry and finite coverage.** Assume that the grid (2.11) is centrally symmetric: $g_{N-a} = -g_a$. Then $-I_a = I_{a^*}$, where $a^* = N-1-a$. We use coefficient systems satisfying

$$\begin{aligned} f_k(s,t) &= f_k(t,s) = f_k(-s,-t), \\ f_k(s,t) &= f_{n-k}(t,-s). \end{aligned} \tag{4.1}$$

Each equality is read on the corresponding pair rectangle. The maps $(s,t) \mapsto (t,s)$ and $(s,t) \mapsto (-s,-t)$ preserve $\mathcal{R}_{n,k}$, whereas $(s,t) \mapsto (t,-s)$ maps it onto $\mathcal{R}_{n,n-k}$.

Define the signed permutation matrices Q, R by

$$Qx = (x_1, \dots, x_{n-1}, -x_0), \qquad Rx = (x_{n-1}, \dots, x_0). \tag{4.2}$$

They preserve $\mathcal{E}_n$. The identities (4.1) imply

$$W_f(Qx) = QW_f(x)Q^{\mathsf{T}}, \qquad W_f(Rx) = -RW_f(x)R^{\mathsf{T}}. \tag{4.3}$$

For the first identity, entries not crossing the last coordinate are immediate. The remaining upper entries satisfy

$$f_{n-1-i}(x_{i+1}, -x_0) = f_{i+1}(x_0, x_{i+1}),$$

which is precisely the sign obtained when the order of the two matrix indices is reversed. For the second identity all upper entries reverse order, and the symmetry $f_k(s,t) = f_k(t,s)$ gives the common minus sign. It follows from (3.1) that the norm is constant on the orbits generated by Q, R. Here $Q^{2n} = R^2 = 1$ and $RQR = Q^{-1}$, so there are finitely many transformations. A simultaneous linear combination of coefficient systems obeying (4.1) has the same covariance, using the same transformation for every summand.

Call a label tuple $\mathbf{a} = (a_0, \dots, a_{n-1})$ *admissible* if

$$\mathrm{Area}\Big(\mathcal{R}_{n,j-i} \cap (I_{a_i} \times I_{a_j})\Big) > 0 \quad \text{for every } 0 \le i < j < n. \tag{4.4}$$

The associated matrix has upper entries c_{j-i,a_i,a_j}. In this definition admissibility asserts only the displayed pairwise conditions.

Lemma 4.1. *Every point of $\mathcal{E}_n$ whose coordinates avoid G belongs to a box $I_{a_0} \times \cdots \times I_{a_{n-1}}$ with admissible label tuple. Consequently a norm bound for the matrices of all admissible tuples implies the same bound for all such points. One matrix from each orbit under* (4.2) *suffices.*

Proof. Let x be such a point. For every $i < j$, the point (x_i, x_j) lies in the open rectangle $I_{a_i} \times I_{a_j}$ and the closed rectangle $\mathcal{R}_{n,j-i}$. A sufficiently small neighborhood of this point inside the open rectangle meets the interior of $\mathcal{R}_{n,j-i}$ in positive area. This remains true if the point lies on an edge or corner of $\mathcal{R}_{n,j-i}$, since the latter is a nondegenerate convex rectangle. Thus (4.4) holds. The matrix is constant on the indicated box, restricted to $\mathcal{E}_n$. The transformations Q, R permute the admissible boxes, by the pair-rectangle symmetries above, and (4.3) preserves the norm. □

The finite set in Lemma 4.1 can be enumerated by assigning labels successively and, after every assignment, retaining in each remaining coordinate only the labels compatible with every previous one in (4.4). Induction on the assigned prefix shows that this procedure lists every admissible tuple. Some listed boxes may have empty intersection with $\mathcal{E}_n$; including their matrices imposes additional inequalities and preserves the conclusion of the lemma.

For $0 < \delta < n/2$ define the endpoint band

$$\mathcal{B}_{n,\delta} = \{x \in \mathcal{E}_n : \max_i |x_i| > n/2 - \delta\}. \tag{4.5}$$

Suppose $\pm(n/2-\delta) \in G$. Every open cell is then either inside or outside the corresponding one-dimensional endpoint band. Call a tuple affected if one of its cells is in that band. Any affected tuple can be carried by a power of Q to a tuple whose first cell is in $(-n/2, -n/2+\delta)$. Indeed, move an endpoint coordinate to the first position and use $Q^n = -1$ if its sign is positive. Hence every affected orbit has such a representative; its lexicographically least representative also has its first cell in this initial group of cells. This justifies restricting the first label when enumerating the affected orbits.

4.2. **Dyadic refinement.** Let f be a coefficient system for m anchors. For $n = 2m$, define its refinement by

$$(\mathcal{D}f)_k(s,t) = \begin{cases} \dfrac{1}{2} f_{k/2}(s/2, t/2), & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases} \tag{4.6}$$

If the parent grid is G_m, choose a child grid G_{2m} satisfying

$$2G_m \subset G_{2m}. \tag{4.7}$$

The refinement is constant on the open child cells. In the following identities all coordinate profiles avoid their respective grids.

For $x \in \mathcal{E}_{2m}$ put

$$x^{(0)} = \tfrac{1}{2}(x_0, x_2, \dots, x_{2m-2}), \qquad x^{(1)} = \tfrac{1}{2}(x_1, x_3, \dots, x_{2m-1}). \tag{4.8}$$

Both belong to $\mathcal{E}_m$: each difference is the sum of two consecutive differences divided by two, including the final difference with the antiperiodic convention. By (4.7), if x avoids the child grid, both profiles avoid the parent grid. Ordering the even coordinates before the odd coordinates gives

$$W_{\mathcal{D}f}(x) \simeq \frac{1}{2} \begin{pmatrix} W_f(x^{(0)}) & 0 \\ 0 & W_f(x^{(1)}) \end{pmatrix}, \tag{4.9}$$

where $\simeq$ denotes conjugation by that permutation matrix. In particular

$$\|W_{\mathcal{D}f}(x)\|_{\mathrm{disc}} \le \frac{1}{2}\Big(\|W_f(x^{(0)})\|_{\mathrm{disc}} + \|W_f(x^{(1)})\|_{\mathrm{disc}}\Big). \tag{4.10}$$

The even child distance coordinates are the parent coordinates, and the odd ones are parent coordinates with all anchors translated by $\pi/(2m)$. The normalized child coordinates are twice the corresponding parent coordinates. Formula (4.6) therefore gives the average of these two parent forms. Their actions are equal, by (2.14) and the translation observation following it, and hence

$$A_{2m}(\mathcal{D}f) = A_m(f). \tag{4.11}$$

This equality also follows directly from (2.13), using $\mathcal{R}_{2m,2k} = 2\mathcal{R}_{m,k}$ and the factor $1/2$ in (4.6). The zero values assigned on the extra child grid lines do not change the

action. On M the corresponding differentials vanish almost everywhere on those levels, by Lemma 2.1.

A child system may differ from a positive multiple $r\mathcal{D}f$ inside an endpoint band. More precisely, require its coefficient of a pair with both coordinates in $[-n/2+\delta, n/2-\delta]$ to agree with $r\mathcal{D}f$, whenever the two coordinates avoid the child grid. Then (4.9), multiplied by r, holds for every profile outside $\mathcal{B}_{n,\delta}$ that avoids the child grid. Profiles inside the band are covered by the affected tuples. The qualification about avoiding the grid is necessary when a refinement introduces extra grid values; we remove it after completing the recursive estimates.

If a parent coefficient is p/T_{old}, a child coefficient is q/T_{new}, and $r = K/T_{\mathrm{new}}$, then the required agreement on an unchanged cell of even separation is precisely

$$q(2T_{\mathrm{old}}) = Kp. \tag{4.12}$$

On unchanged cells of odd separation the child coefficient is zero. Thus this part of the refinement condition can be checked by integer multiplication.

4.3. **Positive combinations through several refinements.** Let $n_\ell = 2^\ell n_0$ for $0 \le \ell \le L$. For each source $s \in \{1, \dots, S\}$, suppose that $f_s^{(\ell)}$ is a coefficient system at level ℓ. At a fixed level all sources use the same grid G_ℓ and satisfy (4.1). Assume $2G_{\ell-1} \subset G_\ell$. At levels $\ell \ge 1$, choose endpoint bands $\mathcal{B}_\ell = \mathcal{B}_{n_\ell,\delta_\ell}$ whose thresholds are grid values, and positive constants $r_{s,\ell}$, so that outside $\mathcal{B}_\ell$ the child system agrees with $r_{s,\ell}\mathcal{D}f_s^{(\ell-1)}$ in the sense just specified. Write $W_s^{(\ell)} = W_{f_s^{(\ell)}}$.

For nonnegative numbers $\mu_1, \dots, \mu_S$, put

$$\beta_s^{(\ell)} = \mu_s \prod_{q=\ell+1}^{L} r_{s,q}, \qquad M_\ell(x) = \sum_{s=1}^{S} \beta_s^{(\ell)} W_s^{(\ell)}(x). \tag{4.13}$$

An empty product is one. Thus, for three levels, the weights are μ_s at level 2, $\mu_s r_{s,2}$ at level 1, and $\mu_s r_{s,2} r_{s,1}$ at level 0.

Proposition 4.2. *Suppose that $U \ge 0$ bounds $\|M_0\|_{\mathrm{disc}}$ on every admissible tuple of level 0, and bounds $\|M_\ell\|_{\mathrm{disc}}$ on every affected admissible tuple of each level $1 \le \ell \le L$. Then*

$$\|M_L(x)\|_{\mathrm{disc}} \le U \tag{4.14}$$

for every $x \in \mathcal{E}_{n_L}$ whose coordinates avoid G_L. In each premise it suffices to check one common symmetry representative of the tuple for all sources.

Proof. We prove the assertion at every level by induction, working throughout with profiles off the corresponding grids. At level 0 it is Lemma 4.1. At a higher level, an affected profile has an affected admissible tuple, so the asserted bound is one of the premises. At a profile outside $\mathcal{B}_\ell$, coefficient inheritance and $\beta_s^{(\ell)} r_{s,\ell} = \beta_s^{(\ell-1)}$ give

$$M_\ell(x) \simeq \frac{1}{2} \begin{pmatrix} M_{\ell-1}(x^{(0)}) & 0 \\ 0 & M_{\ell-1}(x^{(1)}) \end{pmatrix}. \tag{4.15}$$

The parity profiles belong to $\mathcal{E}_{n_{\ell-1}}$ and avoid its grid. The induction hypothesis bounds both blocks by U, so (4.10) bounds their average by U. The two parity profiles may correspond to different tuples; the common bound U applies to both. Finally, the simultaneous covariance in (4.3) proves the assertion about representatives. □

The premises have two useful finite forms. If a certified upper bound $u_{s,\ell,\mathbf{a}}$ is available for the matrix of source s at a required tuple $\mathbf{a}$, the triangle inequality gives the bound

$$\sum_{s=1}^{S} \beta_s^{(\ell)} u_{s,\ell,\mathbf{a}} \tag{4.16}$$

for the mixture at that tuple. One may instead apply Proposition 3.3 directly to the matrix $\sum_s \beta_s^{(\ell)} W_s^{(\ell)}(\mathbf{a})$. This second certificate must refer to this exact weighted matrix, with all summands evaluated at the same tuple. Either bound can be used for each tuple, and any common upper bound for the resulting finite list serves as U. The factors $1/2$ occur in the averages (4.15); they are not additional factors in (4.16).

4.4. **Values on the grids.** We now extend the estimate at the finest level to all profiles. The following argument applies to either type of matrix certificate and to their positive combinations.

Lemma 4.3. *Let $W(x)$ be a finite step matrix field on the part of $\mathcal{E}_n$ whose coordinates avoid a grid G. Suppose its (i,j) entry depends only on the cells of x_i, x_j, also when the pair lies on an edge of $\mathcal{R}_{n,j-i}$. Suppose $\|W(x)\|_{\mathrm{disc}} \le U$ there. At an arbitrary $x \in \mathcal{E}_n$, retain these cell values for pairs of coordinates outside G, and set to zero every row and column whose coordinate belongs to G. The extended matrix still satisfies $\|W(x)\|_{\mathrm{disc}} \le U$.*

Proof. The polytope $\mathcal{E}_n$ is full dimensional, since its defining inequalities are strict at the origin. Its interior is dense in it, and removing finitely many coordinate hyperplanes leaves a dense set. Thus every $x \in \mathcal{E}_n$ is a limit of profiles avoiding G. For such a profile y sufficiently close to x, every coordinate of x outside G and the corresponding coordinate of y lie in the same open cell. The submatrix on those indices is therefore the prescribed one at x. Given vectors in the unit disks for all indices, replace those indexed by grid coordinates of x by zero. Evaluating $W(y)$ on the resulting vectors is exactly the same as evaluating the extended $W(x)$ on the original vectors. The former is bounded in absolute value by U. Taking the maximum proves the assertion. □

Apply Lemma 4.3 once to the finest mixture in Proposition 4.2. All ancestor estimates have already been established off their grids, since the grids are nested as in (4.7). The extension is the coefficient convention in (2.12). On the preimage of any grid value of X_i, $dF_i = 0$ almost everywhere; hence this convention also gives the required pulled-back form when such a level set has positive area. There is no need to assume that these preimages have measure zero.

By linearity of the boundary action, the integral of the resulting finest form is

$$\pi^2 \sum_s \mu_s A_{n_L}(f_s^{(L)}).$$

Consequently, if $U > 0$, the complete finite set of premises in Proposition 4.2 yields

$$\mathrm{Area}(M) \ge \frac{\pi^2}{U} \sum_{s=1}^{S} \mu_s A_{n_L}(f_s^{(L)}). \tag{4.17}$$

Every quantity on the right is determined by finitely many cell areas, coefficient values, and matrix certificates. When the grid values, coefficients, and certificate entries are rational, the cell areas in (2.13) are rational, and all remaining assertions reduce to the finite matrix and rational inequalities in the two certificate criteria.

5. The rational certificate

We now specify the finite data used in the proof. The two collections of coefficients will be denoted by $a = 1, 2$. Each collection has three levels, with $n = 14, 28, 56$. These levels are parts of a single construction. All integers and rational arrays in this section are supplied in the electronic supplement. At a fixed level n, abbreviate $\mathcal{R}_{n,k}$ to R_k.

5.1. Grids and coefficient arrays.

Put

$$\begin{aligned} G_{14} &= \{-4, -3, \dots, 4\} \\ &\quad \cup \left\{ \pm\frac{9}{2},\ \pm 5,\ \pm\frac{21}{4},\ \pm\frac{11}{2},\ \pm\frac{23}{4},\ \pm 6,\ \pm\frac{25}{4},\ \pm\frac{13}{2},\ \pm\frac{27}{4},\ \pm 7 \right\}, \\ G_{28} &= 2G_{14} \cup \{-14, -13, \dots, 14\}, \\ G_{56} &= \{-28, -27, \dots, 28\}. \end{aligned} \tag{5.1}$$

The numbers of open intervals are $28, 36, 56$, respectively. In particular, $2G_{14} \subset G_{28}$ and $2G_{28} \subset G_{56}$. The endpoint band has width 2 at each of the last two levels.

For a grid $G_n = \{t_0 < \cdots < t_N\}$, write $I_u = (t_u, t_{u+1})$. The coefficient classes are the equivalence classes of triples (k, u, v), $1 \le k < n$, for which $\mathrm{Area}(R_k \cap (I_u \times I_v)) > 0$, under

$$\begin{aligned} &(k, u, v),\ (k, v, u),\ (k, \bar{u}, \bar{v}),\ (k, \bar{v}, \bar{u}), \\ &(n-k, v, \bar{u}),\ (n-k, \bar{u}, v),\ (n-k, \bar{v}, u),\ (n-k, u, \bar{v}), \end{aligned} \qquad \bar{u} = N - 1 - u. \tag{5.2}$$

Each class is represented by its lexicographically least triple. The list of these representatives in lexicographic order is denoted by $\mathcal{K}_n$. The supplement supplies the integer vector $(p_{a,n,\kappa})_{\kappa \in \mathcal{K}_n}$. On a pair cell with class κ, the coefficient is

$$f_{a,n,k}(s, t) = \frac{p_{a,n,\kappa}}{T_{a,n}}. \tag{5.3}$$

As in the preceding sections, incident coefficients are set to zero on coordinate grid lines. The resulting form is denoted by $\Omega_{a,n}$.

The positive integers used for normalization and inheritance are as follows.

a	n	$K_{a,n}$	$T_{a,n}$
1	14	100000000	100000344
1	28	1250004300000000	1250009954439084
1	56	31250248860977100000000	31251408412709010636800
2	14	100000000	100001194
2	28	5000059700000000	5000141045182364
2	56	12500352612955910000000	12510026086002960705 1264

For $n = 28, 56$, put $r_{a,n} = K_{a,n}/T_{a,n}$. On every pair cell outside the endpoint band, the coefficient vectors satisfy the following integer identities. If k is odd, then $p_{a,n,\kappa} = 0$. If k is even and κ' is the parent cell class obtained by replacing (k, s, t) with $(k/2, s/2, t/2)$, then

$$2T_{a,n/2}\, p_{a,n,\kappa} = K_{a,n}\, p_{a,n/2,\kappa'}. \tag{5.4}$$

The containment of grids in (5.1) makes the parent class unambiguous on every open child cell. Thus, away from the finest grid, $\Omega_{a,n}$ agrees outside its endpoint band with $r_{a,n}$ times the average of the two rotated parent forms.

The sizes of the arrays and the complete coverage counts are

n	$\#\mathcal{K}_n$	Changed classes	Inherited classes	Rooted boxes	Matrix patterns
14	346	346	0	2975284	53538
28	1382	176	1206	69474	29640
56	8092	162	7930	30856	14590

At the first level, all compatible boxes are enumerated. At the last two levels, *rooted boxes* means compatible boxes whose first coordinate lies in the negative endpoint band. The symmetry argument in the preceding section covers every affected box from these rooted boxes. The same ordered pattern arrays are used for $a = 1$ and $a = 2$.

A pattern records the index in $\mathcal{K}_n$ of the class of each pair (i, j), with $i < j$ and pairs in lexicographic order. For each admissible box, take the lexicographically least such vector under the cyclic and reversed orders of its vertices. The covariance identities (4.3) justify these operations. Duplicate vectors are removed, and the distinct vectors are sorted lexicographically. This defines the ordered pattern array used below.

5.2. **Matrix and polynomial witnesses.** For a stored pattern τ, let $P_{a,n,\tau}$ be the real skew-symmetric matrix whose upper entries are the corresponding $p_{a,n,\kappa}$. The data include positive integers $d_{a,n,\tau,0}, \dots, d_{a,n,\tau,n-1}$ satisfying

$$2\,\mathrm{diag}(d_{a,n,\tau}) + \mathrm{i}P_{a,n,\tau} \succ 0, \qquad \sum_i d_{a,n,\tau,i} \le T_{a,n}. \tag{5.5}$$

Consequently, the norm of this pattern of $\Omega_{a,n}$ is bounded by

$$b_{a,n,\tau} = \frac{\sum_i d_{a,n,\tau,i}}{T_{a,n}}.$$

At level 56, selected bounds are reduced by the Laurent-polynomial criterion proved above, for the same coefficients $p_{a,56,\kappa}/T_{a,56}$. There are sixteen such certificates for $a = 1$, and eight for $a = 2$. In this notation $b_{a,56,\tau}$ will mean the smaller of the diagonal bound and the polynomial bound when a polynomial certificate is supplied. The coefficients of the form are unchanged.

Set

$$\mu_1 = \frac{23108519759}{10^{12}}, \qquad \mu_2 = \frac{976891480241}{10^{12}}, \qquad \mu_1 + \mu_2 = 1. \tag{5.6}$$

For $n = 56, 28, 14$, define

$$\begin{aligned} c_{a,56} &= \mu_a, \\ c_{a,28} &= \mu_a r_{a,56}, \\ c_{a,14} &= \mu_a r_{a,56} r_{a,28}. \end{aligned} \tag{5.7}$$

A first bound for a pattern at level n is

$$B_{n,\tau} = \sum_{a=1}^{2} c_{a,n} b_{a,n,\tau}. \tag{5.8}$$

For eighty-four specified patterns, the supplement gives a polynomial certificate directly for the mixed matrix

$$W_{n,\tau} = \sum_{a=1}^{2} \frac{c_{a,n}}{T_{a,n}} P_{a,n,\tau}. \tag{5.9}$$

Of these certificates, sixty-four belong to level 14 and twenty to level 28. For each certificate, the target edge weights are compared, as rational numbers, with (5.9) before its bound is used. Let $\widehat{B}_{n,\tau}$ be the smaller of $B_{n,\tau}$ and this additional bound, when present, and let $\widehat{B}_{n,\tau} = B_{n,\tau}$ otherwise.

Proposition 5.1. *The arrays in the electronic supplement satisfy the following statements.*

(i) *The exact pair-cell tests and the signed symmetries give the complete pattern sets counted above. Corresponding grids, coefficient keys, and ordered patterns of the two collections are identical.*

(ii) *Every unchanged coefficient satisfies* (5.4)*. Every one of the* 195536 *matrices in the two collections satisfies* (5.5)*.*

(iii) *All* 108 *Laurent certificates have their stated rational upper bounds and the exact target coefficients specified above.*

(iv) *The maximum of the* 97768 *final mixed budgets is*

$$U=\frac{29105746733301339862378571909389256115943629230830527422027}{291284869959445374607081859722363834000000000000000000000000}. \tag{5.10}$$

(v) *The boundary action factors of the two level-*56 *forms are*

$$\begin{aligned} A_1&=\frac{76814122456010519630747}{125005633650836042547200},\\ A_2&=\frac{21510614040353910320352 11}{35028073040808289974353 92}. \end{aligned} \tag{5.11}$$

so that $\int_M \Omega_{a,56}=\pi^2 A_a$ *for every filling under consideration.*

Proof. All tests in this proposition are finite integer or rational computations. We describe the tests and their mathematical justification; the complete arrays and executable implementations accompany the paper.

First compute the area of every $R_k\cap(I_u\times I_v)$. Start with the closed rectangle $[t_u,t_{u+1}]\times[t_v,t_{v+1}]$, and successively intersect it with the four half-planes

$$s-t\le k,\quad t-s\le k,\quad s+t\le n-k,\quad -s-t\le n-k.$$

All vertices remain rational. At an edge from x to y meeting $L(z)=c$, the new vertex is

$$x+\frac{c-L(x)}{L(y)-L(x)}(y-x).$$

The shoelace formula gives its area exactly. A pair of labels is compatible if and only if this area is positive. These tests also reconstruct every coefficient key and the boundary weights

$$m_{n,\kappa}=\sum_{k=1}^{n-1}(n-k)\sum_{\substack{u,v\\ [k,u,v]=\kappa}}\operatorname{Area}(R_k\cap(I_u\times I_v)). \tag{5.12}$$

To enumerate boxes, choose the coordinate labels in order. After assigning label u_i, delete from the domain of every later coordinate j the labels incompatible with (u_i,u_j). A branch is discarded only if a domain is empty. At level 14 every initial domain consists of all labels. At levels 28 and 56, only the first domain is restricted to the negative endpoint band. This gives every all-pairs-compatible box in the relevant rooted set by induction on the number of assigned coordinates. Reduce the boxes and their coefficient patterns by the signed symmetries, discard duplicate patterns, and compare with the supplied arrays. This yields the counts in the table. The coverage lemma shows why this computation suffices for all actual distance profiles, without assuming joint realizability of every enumerated box.

For each unchanged coefficient, locate the containing parent cell after dividing its endpoints by two and check (5.4) by integer multiplication. The numbers of identities checked at levels 28 and 56 are 1206 and 7930, respectively, in each collection.

The matrix tests are performed on the actual integer entries. Two exact tests are used. Write $H = 2\,\mathrm{diag}(d) + \mathrm{i}P$. The first test constructs a Gaussian rational matrix L for the normalized matrix $A = H/K$, where $K = K_{a,n} > 0$, and proves

$$(A - LL^*)_{ii} > \sum_{j \neq i} \left(|\mathrm{Re}(A - LL^*)_{ij}| + |\mathrm{Im}(A - LL^*)_{ij}| \right). \tag{5.13}$$

At level 14, the same test instead uses a real rational Gram factor for the real representation of $H/(2K)$. Denominators are cleared before every comparison. Indeed, for a Hermitian matrix R, the estimate $2|z_i||z_j| \leq |z_i|^2 + |z_j|^2$ proves

$$z^* R z \geq \sum_i \left(R_{ii} - \sum_{j \neq i} |R_{ij}| \right) |z_i|^2.$$

Thus (5.13) proves strict positivity of the normalized matrix, and hence of H. The real Gram test has the same implication.

The second test uses the real symmetric representation

$$\mathcal{H} = \begin{pmatrix} 2\,\mathrm{diag}(d) & -P \\ P & 2\,\mathrm{diag}(d) \end{pmatrix}. \tag{5.14}$$

An exact Schur complement may first eliminate positive principal blocks; every zero entry required for a block separation is checked in $\mathcal{H}$ itself. For the remaining rational symmetric matrices, clear a positive common denominator and perform fraction-free elimination. With $p_{-1} = 1$, the elimination step is

$$M_{ij}^{(r+1)} = \frac{p_r M_{ij}^{(r)} - M_{ir}^{(r)} M_{rj}^{(r)}}{p_{r-1}}, \qquad p_r = M_{rr}^{(r)}.$$

Every division is required to be exact and every pivot strictly positive. The determinant identity underlying elimination identifies these pivots with successive leading principal minors. Their positivity is equivalent to a positive diagonal in the LDL^{T} decomposition. The Schur-complement identity then proves positivity of the original matrix. The trace tests in (5.5) are integer sums. Both methods pass for every matrix in the table. The fraction-free elimination is evaluated directly with Python integers at level 14, and with FLINT at levels 28 and 56.

A failed numerical Gram proposal is never accepted. The programs then use exact elimination or an exact characteristic-polynomial test. For a real symmetric matrix S, strict positivity of every coefficient of $\det(tI + S)$ implies $S \succ 0$: the polynomial has no zero for $t \geq 0$, while a nonpositive eigenvalue of S would give such a zero. All matrices in the present data pass the Gram criterion itself.

For completeness, the polynomial test has the following integer form. Write its Gaussian rational polynomials with common denominator $D_{\mathrm{pol}} > 0$, and its rational target edge weights with common denominator $D_{\mathrm{wt}} > 0$. Set $q = \mathrm{lcm}(D_{\mathrm{pol}}^2, D_{\mathrm{wt}})$. Direct convolution of the integer polynomial coefficients gives

$$q \left(\widetilde{C}_W(z) + \sum_\ell |p_\ell(z)|^2 \right) = b_0 + \sum_{\nu \in \mathcal{V}} \Big(b_\nu \,\mathrm{Re}\, z^\nu + c_\nu \,\mathrm{Im}\, z^\nu \Big), \qquad z \in \mathbb{T}^{n-1}, \tag{5.15}$$

where the coordinate z_0 has been fixed to 1, b_0, b_ν, c_ν are integers and $\mathcal{V}$ contains one representative of each nonzero exponent pair $\{\nu, -\nu\}$. The rational upper bound checked by the program is

$$\frac{b_0 + \sum_{\nu \in \mathcal{V}} \left\lceil \sqrt{b_\nu^2 + c_\nu^2} \right\rceil}{q}. \tag{5.16}$$

For an integer $m \geq 0$, the ceiling square root is determined by integer comparisons: it is $\lfloor \sqrt{m} \rfloor$ when m is a square and $\lfloor \sqrt{m} \rfloor + 1$ otherwise. Thus no floating-point inequality enters (5.16). The two supplied polynomial implementations independently expand every square and obtain the same bound. The phase-to-disk lemma makes this a bound for the required two-dimensional norm.

Finally, compare all grids, keys and patterns directly, form the rational weights (5.7), and compute every $\widehat{B}_{n,\tau}$. Their maximum is exactly (5.10). It is attained at pattern 646 of level 28, with patterns indexed from zero as in the supplied arrays. The action is independently computed from (5.12):

$$A_a = \frac{1}{56^2 T_{a,56}} \sum_{\kappa \in \mathcal{K}_{56}} m_{56,\kappa} p_{a,56,\kappa}. \tag{5.17}$$

This gives (5.11), and the boundary integral identity gives the final assertion. □

5.3. **Completion of the proof.**

Proof of Theorem 1.1. Use the level-56 distance functions, with

$$\Omega = \mu_1 \Omega_{1,56} + \mu_2 \Omega_{2,56}.$$

First suppose that the level-56 profile avoids its grid. At a profile in the level-56 endpoint band, its pattern is covered by Proposition 5.1, and $|\Omega(e_1, e_2)| \leq U$. Outside that band, the inheritance identities express this form as the average of two rotated level-28 mixed forms with weights $c_{a,28}$. Each of the two level-28 profiles either lies in its endpoint band, where its mixed budget is at most U, or lies outside the band. In the latter case, inheritance expresses the corresponding mixed form as the average of two rotated level-14 mixed forms with weights $c_{a,14}$. All level-14 profiles are covered. The triangle inequality for each average therefore gives the common bound U on every off-grid profile. The zero-coordinate extension lemma extends it to grid profiles. For the actual distance functions, all omitted rows have zero differential almost everywhere on their corresponding grid level sets. Hence

$$|\Omega(e_1, e_2)| \leq U \quad \text{almost everywhere on } M.$$

The boundary integral identity and (5.11) give

$$U \operatorname{Area}(M) \geq \int_M \Omega = \pi^2 (\mu_1 A_1 + \mu_2 A_2).$$

Exact rational simplification using (5.6), (5.10) and (5.11) yields

$$\frac{\mu_1 A_1 + \mu_2 A_2}{U} = \rho_*.$$

This proves the asserted inequality. □

Appendix A. Exact reproduction and decimal enclosures

The electronic supplement is part of the finite certificate. It contains the six matrix data files, the two lists of source polynomial bounds, the eighty-four mixed polynomial bounds, and all verification programs. The file `proof/certificate_manifest.json` identifies each base array by its source index a and level n, and records the file identities of all finite data. In the base JSON files, `coefficient_numerators` stores the vector p, `diagonal_numerators` stores the rows d, and `scaling_numerator` is T. The auxiliary integer K is the field `denominator` at level 14, and `coefficient_denominator` at levels 28 and 56. The entry point `proof/verify_all.py` regenerates the profile coverage and performs all matrix, polynomial, inheritance and final-budget tests. The original

numerical optimization is not needed for this verification. Python with NumPy and `python-flint`, together with a C++17 compiler, suffices. Every acceptance test is an exact integer or rational comparison; fixed-width products are used only with explicit bounds preventing overflow. Floating-point linear algebra is used only to propose a Gram factor that is subsequently checked exactly.

The following order makes the dependencies explicit:

1. Recompute all clipped pair areas and the complete level-14 coverage; verify both level-14 matrix collections.
2. Recompute the level-28 coverage, exact inherited coefficients, and every level-28 matrix; repeat at level 56.
3. Expand all 108 Gaussian polynomial certificates in two independent implementations and compare every target edge with its actual rational coefficient.
4. Compare the two collections' profile arrays, compute all 97768 mixed budgets, and recompute the boundary action and ρ_*.

Successful verification therefore uses the finite arrays themselves, rather than a record that a previous computation succeeded.

The displayed decimal can also be enclosed using rational arithmetic. For $q > 1$, put

$$S_N(q) = \sum_{j=0}^{N-1} \frac{(-1)^j}{(2j+1)q^{2j+1}}.$$

For even N, the alternating-series estimate gives

$$S_N(q) < \arctan(1/q) < S_N(q) + \frac{1}{(2N+1)q^{2N+1}}.$$

Machin's identity

$$\pi = 16\arctan(1/5) - 4\arctan(1/239)$$

then gives rational lower and upper bounds for π. With $N = 64$, squaring these positive bounds and multiplying by the positive rational number ρ_* proves

$$6.065712789704037218\mathbf{52} < \rho_*\pi^2 < 6.065712789704037218\mathbf{53}.$$

The supplement includes this integer check.

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