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MONOTONICITY OF ERDŐS SUMS

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ABSTRACT. For $k \geq 1$ let $f_k = \sum_{\Omega(n)=k} 1/(n \log n)$, where Ω counts prime factors with multiplicity, and let g_k be the same sum restricted to odd n. Lichtman showed that $f_k \to 1$, with a global minimum at $k = 6$, and Gorodetsky, Lichtman and Wong that $f_k = 1 - 2^{-k}\big(a_1 k^2 + O(k \log(k+1))\big)$. We prove that $f_k < f_{k+1}$ for every $k \geq 6$, and that $g_k > g_{k+1}$ for every $k \geq 1$. The two statements come from one analytic distinction. Writing $\sum_k P_k(s) z^k = \zeta(s)^z H(s, z)$, at $s = 1$ the point $z = 2$ lies on the first singular circle of H, and strictly inside the first singular circle of the product with the prime 2 removed. We show that this distinction decides the direction in general. If the complement of a set of primes $\mathcal{P}$ is thin and its least element is at least 3, the corresponding sums eventually decrease, so the second theorem is one case of a general phenomenon. The inequalities used at the last step of the first theorem are certified by interval arithmetic.

1. INTRODUCTION

For $k \geq 1$ put

$$f_k = \sum_{\Omega(n)=k} \frac{1}{n \log n}.$$

Erdős [4] proved that the corresponding sum is bounded uniformly over primitive sets. In particular each f_k converges. Zhang [10] proved $f_k < f_1$ for $k > 1$, and Bayless, Kinlaw and Klyve [11] proved $f_2 > f_3$. Banks and Martin [1] conjectured that

$$f_1 > f_2 > f_3 > \cdots . \tag{1}$$

Lichtman [2] disproved (1), showing that $f_k \to 1$ and that the sequence attains a global minimum at $k = 6$. Gorodetsky, Lichtman and Wong [3] then established

$$f_k = 1 - 2^{-k}\big(a_1 k^2 + O(k \log(k+1))\big), \qquad a_1 = \frac{d \log 2}{4} > 0, \tag{2}$$

where

$$d = \frac{1}{4} \prod_{p \geq 3} \left(1 - \frac{2}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^2 . \tag{3}$$

Neither of these yields monotonicity beyond the minimum. A global minimum at $k = 6$ leaves open whether $f_k < f_{k+1}$ holds at every index above it, and the implicit constant in (2) does not make explicit a range on which the increase can be read off. Our first theorem gives that monotonicity in full.

Theorem 1.1. *For every $k \geq 6$ we have $f_k < f_{k+1}$.*

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Let

$$g_k = \sum_{\substack{n \text{ odd} \\ \Omega(n)=k}} \frac{1}{n \log n}.$$

The second theorem shows that deleting the prime 2 reverses the direction of monotonicity at every index.

Theorem 1.2. *For every $k \geq 1$ we have $g_k > g_{k+1}$, and $g_k \to \frac{1}{2}$.*

The proof of Theorem 1.2 gives the more precise expansion

$$g_k = \frac{1}{2} + \beta 2^{-k} + R_k, \qquad |R_k| \leq 4900(k+3)^3 3^{-k} \quad (k \geq 59), \tag{4}$$

where $\gamma = 0.5772156649\ldots$ is Euler's constant and

$$\beta = d\left(\gamma + \log 2 - \sum_{p \geq 3} \frac{\log p}{(p-1)(p-2)}\right) > 0. \tag{5}$$

The 2^{-k} term comes from evaluating a regularized Euler product at $z = 2$. For odd integers the first pole is at $z = 3$, so this evaluation takes place strictly inside the disc of analyticity. For the unrestricted sums the first pole is at $z = 2$ itself. The boundary contribution then acquires the factor k^2 in (2).

This contrast is not an accident of the two cases. In Section 7 we make it a general statement. Let $\mathcal{P}$ be a set of primes with least element $q \geq 3$, and suppose its complement $\mathcal{Q}$ is thin, in the sense that $\sum_{p \in \mathcal{Q}} (\log p)/p < \infty$. Then Proposition 7.2 gives

$$f_k^{\mathcal{P}} = \prod_{p \in \mathcal{Q}} \Big(1 - \frac{1}{p}\Big) + c_{\mathcal{P}} 2^{-k} + o(2^{-k}), \qquad c_{\mathcal{P}} = H_{\mathcal{P}}(1,2)\big(\gamma + \tfrac{1}{2}\partial_s \log H_{\mathcal{P}}(1,2)\big),$$

and Corollary 7.3 gives $c_{\mathcal{P}} > 0$, so every such family eventually decreases. Thus Theorem 1.2 is the case $\mathcal{Q} = \{2\}$ of a phenomenon that depends only on the position of $z = 2$ relative to the first singular circle. Both hypotheses are used. Thinness does not follow from $q \geq 3$: for $\mathcal{P} = \{p \equiv 1 \bmod 4\}$ the leading product is 0, and the proposition says nothing.

The remaining case is $2 \in \mathcal{P}$, where the evaluation sits on the singular circle itself. That case is open, and Conjecture 7.1 records what it would give; we comment on it at the end of Section 7 but do not treat it here.

The proof has two kinds of input. Identities, coefficient reductions and inequalities are proved in the text. Bounds produced by interval arithmetic are isolated as certified propositions, so that a phrase of the form "the computation gives" never stands in for an analytic step: the exact inequality the computation supplies is stated before it is used. What the interval arithmetic certifies is the computation itself. Transcribing an enclosure into a numeral printed here, and combining several such numerals, is ordinary writing, and we check it by script rather than by ball arithmetic.

2. Generating functions and the analytic boundary

For $s > 1$ define

$$P_k(s) = \sum_{\Omega(n)=k} n^{-s}, \qquad P_k^{\mathrm{odd}}(s) = \sum_{\substack{n \text{ odd} \\ \Omega(n)=k}} n^{-s}.$$

Lemma 2.1. *For every $k \geq 1$,*

$$f_k = \int_1^\infty P_k(s)\,\mathrm{d}s, \qquad g_k = \int_1^\infty P_k^{\mathrm{odd}}(s)\,\mathrm{d}s. \tag{6}$$

Proof. The first identity is [2, (3.1)]: one has $\int_1^\infty n^{-s}\,\mathrm{d}s = 1/(n\log n)$ for $n \geq 2$, and the summands are non-negative, so Tonelli's theorem permits the interchange. The same calculation with the sum restricted to odd n gives the second. □

The generating functions are, by [2, Prop. 3.1],

$$\sum_{k\geq 0} P_k(s)z^k = \prod_p (1-zp^{-s})^{-1}, \qquad \sum_{k\geq 0} P_k^{\mathrm{odd}}(s)z^k = \prod_{p\geq 3}(1-zp^{-s})^{-1}. \tag{7}$$

Write these products as $\zeta(s)^z H(s,z)$ and $\zeta(s)^z H_{\mathrm{o}}(s,z)$, where

$$H(s,z) = \prod_p (1-zp^{-s})^{-1}(1-p^{-s})^z, \tag{8}$$

$$H_{\mathrm{o}}(s,z) = (1-2^{-s})^z \prod_{p\geq 3}(1-zp^{-s})^{-1}(1-p^{-s})^z. \tag{9}$$

Proposition 2.2. *Let $\sigma = \operatorname{Re} s$. Then H is analytic in the two complex variables (s,z) on $\{\sigma > 1/2,\ |z| < 2^\sigma\}$, and H_{o} on $\{\sigma > 1/2,\ |z| < 3^\sigma\}$. In particular, for fixed real $s > 1/2$ the function $H(s,\cdot)$ is analytic in $|z| < 2^s$ and $H_{\mathrm{o}}(s,\cdot)$ in $|z| < 3^s$; each $\partial_s^r H_{\mathrm{o}}(s,\cdot)$ is again analytic in $|z| < 3^s$, and*

$$\frac{\mathrm{d}^r}{\mathrm{d}s^r}[z^j]H_{\mathrm{o}}(s,z) = [z^j]\partial_s^r H_{\mathrm{o}}(s,z); \tag{10}$$

the same holds for H in $|z| < 2^s$.

Proof. It is essential that s be taken complex here: uniform convergence of real-variable functions says nothing about the derivative of the limit, so the real-variable argument would give analyticity in z alone.

Let $x = p^{-s}$, so that $|x| = p^{-\sigma} < 2^{-1/2}$ on the domain. The factor $(1-x)^z$ needs a branch, and the principal one serves throughout, since $1-x$ lies in the disc of radius $2^{-1/2} < 1$ about 1. Fix a compact subset of the domain, so that $\sigma \geq s_0 > 1/2$ and $|z| \leq R < 2^{s_0}$ for (8), or $R < 3^{s_0}$ with $p \geq 3$ for (9); then $|zx| < 1$ for every p occurring. The logarithm of the pth regularized factor is

$$-\log(1-zx) + z\log(1-x) = \sum_{m\geq 1} \frac{z^m - z}{m} x^m = \sum_{m\geq 2} \frac{z^m - z}{m} x^m = O(p^{-2\sigma}), \tag{11}$$

the $m = 1$ terms cancelling. The estimate is uniform on the compact set and depends on s only through σ. Each $p^{-s} = \mathrm{e}^{-s\log p}$ is entire, so every regularized factor is analytic in (s,z) jointly wherever its denominator does not vanish, which on each of the two domains it does not: for (8) one has $|z| < 2^\sigma \leq p^\sigma = |p^s|$ for every prime p, and for (9) $|z| < 3^\sigma \leq |p^s|$ for every $p \geq 3$. As $\sum_p p^{-2s_0} < \infty$, the sum of the logarithms converges absolutely and locally uniformly, so the product converges locally uniformly on the stated domain in $\mathbb{C}^2$ to a non-vanishing limit, and a locally uniform limit of analytic functions of several complex variables is analytic. This is the first assertion. The factors have poles at $z = p^s$, of modulus p^σ, so the nearest is at modulus 2^σ for (8) and 3^σ for (9), which is why the domains are as stated. Finally $[z^j]$ is a Cauchy integral over a circle that may be held fixed on a compact subset, so it commutes with $\mathrm{d}/\mathrm{d}s$, which is (10). □

At $s = 1$ the point $z = 2$ lies on the first singular circle for H and lies strictly inside the first singular circle for H_{o}. This is the analytic distinction used below.

We shall also need the local change of variables used in the odd case.

Lemma 2.3. *Let $u = \log\zeta(s)$ for $s > 1$, and let $s = s(u)$ be the inverse function. Put $\varepsilon(u) = s(u) - 1$ and $J(u) = -s'(u)$. As $u \to \infty$,*

$$\varepsilon(u) = \mathrm{e}^{-u} + \gamma\mathrm{e}^{-2u} + O(\mathrm{e}^{-3u}), \qquad J(u) = \mathrm{e}^{-u} + 2\gamma\mathrm{e}^{-2u} + O(\mathrm{e}^{-3u}). \tag{12}$$

Proof. The point of the argument is that the expansions below are convergent power series and not merely asymptotic, which is what allows the second of them to be differentiated term by term.

By the Laurent expansion of ζ at its simple pole [8, Ch. 1], the function $\varepsilon \mapsto \varepsilon\,\zeta(1+\varepsilon)$ is analytic near $\varepsilon = 0$ with expansion $1 + \gamma\varepsilon + O(\varepsilon^2)$. Hence

$$x = \frac{1}{\zeta(1+\varepsilon)} = \frac{\varepsilon}{\varepsilon\,\zeta(1+\varepsilon)}$$

is analytic near $\varepsilon = 0$, with $x(0) = 0$ and $x = \varepsilon - \gamma\varepsilon^2 + O(\varepsilon^3)$, so in particular $\mathrm{d}x/\,\mathrm{d}\varepsilon = 1 \neq 0$ at the origin. By the inverse function theorem for analytic functions, ε is therefore an analytic function of x near $x = 0$, given by a convergent power series

$$\varepsilon = \psi(x) = x + \gamma x^2 + O(x^3),$$

which is the first formula in (12) once $x = \mathrm{e}^{-u}$. Because ψ is analytic rather than merely asymptotic, it may be differentiated termwise: $\psi'(x) = 1 + 2\gamma x + O(x^2)$. With $x = \mathrm{e}^{-u}$ one has $\mathrm{d}x/\,\mathrm{d}u = -x$, so the chain rule gives

$$\varepsilon'(u) = \psi'(x)\frac{\mathrm{d}x}{\mathrm{d}u} = -x\big(1 + 2\gamma x + O(x^2)\big) = -x - 2\gamma x^2 + O(x^3),$$

and $J(u) = -\varepsilon'(u) = x + 2\gamma x^2 + O(x^3)$. □

3. The odd sums

Throughout this section write

$$h_j(s) = [z^j]H_{\mathrm{o}}(s, z), \qquad b_j = h_j'(1) + 2\gamma h_j(1).$$

Since $\zeta(s)^z = \sum_{m\geq 0}(\log\zeta(s))^m z^m/m!$, coefficient extraction in (7) gives the exact identity

$$P_k^{\mathrm{odd}}(s) = \sum_{j=0}^{k} h_j(s)\frac{(\log\zeta(s))^{k-j}}{(k-j)!}. \tag{13}$$

Using $u = \log\zeta(s)$ in Lemma 2.1, and noting that s decreases from ∞ to 1 as u increases from 0 to ∞, we obtain

$$g_k = \sum_{j=0}^{k}\int_0^\infty \frac{u^{k-j}}{(k-j)!}\,h_j(s(u))J(u)\,\mathrm{d}u. \tag{14}$$

3.1. **The first two terms.**

Proposition 3.1. *With h_j and b_j as above and E_j as in the proof below, put*

$$R_k = \sum_{j=0}^{k} \int_0^\infty \frac{u^{k-j}}{(k-j)!} E_j(u)\,\mathrm{d}u - \sum_{j>k} h_j(1) - 2^{-k-1} \sum_{j>k} b_j 2^j. \tag{15}$$

Then

$$g_k = H_{\mathrm{o}}(1,1) + 2^{-k-1}\big(\partial_s H_{\mathrm{o}}(1,2) + 2\gamma H_{\mathrm{o}}(1,2)\big) + R_k. \tag{16}$$

Proof. The idea is to write the integrand of (14) as its first two terms in e^{-u} plus an explicit error, and then to integrate each piece exactly. Two expansions are at work and they must be kept apart: the measure $J(u)\,\mathrm{d}u$, which Lemma 2.3 expands, and the coefficient $h_j(s(u))$, which Taylor's theorem expands about $s = 1$. Their product supplies the three error terms of (17).

Let $x = \mathrm{e}^{-u}$. By Lemma 2.3, $J(u) = x + 2\gamma x^2 + O(x^3)$ and $\varepsilon(u)J(u) = x^2 + O(x^3)$. Taylor's theorem at $s = 1$ gives

$$h_j(1+\varepsilon) = h_j(1) + \varepsilon h_j'(1) + \frac{\varepsilon^2}{2} h_j''(1 + \theta_{j,u}\varepsilon)$$

for some $0 < \theta_{j,u} < 1$. Put

$$r_0(u) = J(u) - \mathrm{e}^{-u} - 2\gamma \mathrm{e}^{-2u}, \qquad r_1(u) = \varepsilon(u)J(u) - \mathrm{e}^{-2u}.$$

Multiplication by $J(u)$ now gives the exact identity

$$\begin{aligned} h_j(s(u))J(u) &= h_j(1)\mathrm{e}^{-u} + b_j \mathrm{e}^{-2u} + E_j(u), \\ E_j(u) &= h_j(1) r_0(u) + h_j'(1) r_1(u) + \frac{\varepsilon(u)^2 J(u)}{2} h_j''(1 + \theta_{j,u}\varepsilon(u)). \end{aligned} \tag{17}$$

No error is suppressed in the notation E_j: its three summands are the Jacobian error, the first-order coordinate error and the Taylor remainder.

Insert (17) into (14). Since

$$\int_0^\infty \frac{u^m}{m!} \mathrm{e}^{-ru}\,\mathrm{d}u = r^{-m-1} \qquad (r > 0),$$

the first two terms become

$$\sum_{j=0}^{k} h_j(1) = H_{\mathrm{o}}(1,1) - \sum_{j>k} h_j(1), \tag{18}$$

$$\sum_{j=0}^{k} b_j 2^{-(k-j+1)} = 2^{-k-1}\big(\partial_s H_{\mathrm{o}}(1,2) + 2\gamma H_{\mathrm{o}}(1,2)\big) - 2^{-k-1} \sum_{j>k} b_j 2^j. \tag{19}$$

Substituting (18) and (19), and collecting the E_j integrals with the two tails, is exactly (15), and this gives (16). Note that (15) is an identity, so every later estimate is an estimate of those three explicit quantities. □

Lemma 3.2. *One has*

$$H_{\mathrm{o}}(1,1) = \frac{1}{2}, \qquad H_{\mathrm{o}}(1,2) = d, \tag{20}$$

and

$$\partial_s \log H_{\mathrm{o}}(1,2) = 2\log 2 - 2 \sum_{p \geq 3} \frac{\log p}{(p-1)(p-2)}. \tag{21}$$

Proof. The first two values are immediate from (9): at $s = z = 1$ every factor in the product over $p \geq 3$ cancels, leaving $H_{\mathrm{o}}(1,1) = 1 - 2^{-1} = \frac{1}{2}$, and at $(s,z) = (1,2)$ the product is (3), so $H_{\mathrm{o}}(1,2) = d$.

For the derivative, take logarithms:

$$\log H_{\mathrm{o}}(s,2) = 2\log(1-2^{-s}) + \sum_{p\geq 3}\left[-\log(1-2p^{-s}) + 2\log(1-p^{-s})\right].$$

The differentiated pth summand is $O((\log p)p^{-2s})$ locally uniformly near $s = 1$, so termwise differentiation is permitted. At $s = 1$ the first term contributes $2\log 2$, while the pth summand contributes

$$-\frac{2\log p}{p-2} + \frac{2\log p}{p-1} = -\frac{2\log p}{(p-1)(p-2)}.$$

Summing over $p \geq 3$ proves (21). □

Combining Proposition 3.1 and Lemma 3.2 gives

$$g_k = \frac{1}{2} + \beta 2^{-k} + R_k, \qquad \beta = d\left(\gamma + \log 2 - \sum_{p\geq 3}\frac{\log p}{(p-1)(p-2)}\right). \tag{22}$$

3.2. **The remainder.** Three weighted ℓ^1 norms of coefficients carry all of the interval input for the odd sums. Throughout, $s(3) = s(u)|_{u=3} = 1.0512\ldots$ in the coordinate of Lemma 2.3, that is, the value of s at which $\log\zeta(s) = 3$. Put

$$\begin{aligned}
\Sigma_0 &= \sum_{m\geq 0} 3^m |[z^m](1-z/3)H_{\mathrm{o}}(1,z)|,\\
\Sigma_1 &= \sum_{m\geq 0} 3^m |[z^m](1-z/3)^2 \partial_s H_{\mathrm{o}}(1,z)|,\\
\Sigma_2 &= \sup_{1\leq s\leq s(3)} \sum_{m\geq 0} 3^m |[z^m](1-z/3)^3 \partial_s^2 H_{\mathrm{o}}(s,z)|.
\end{aligned} \tag{23}$$

Finiteness of Σ_0 and Σ_1 is analytic, and the weights $r = 1, 2$ on the factors $(1-z/3)^r$ are what makes it so. Proposition 2.2 gives $H_{\mathrm{o}}(1,\cdot)$ only on $|z| < 3$, which by itself would not make a sum weighted by 3^m converge. But at $s = 1$ the nearest pole in z is at $z = 3$, contributed by the single Euler factor $(1-z/3)^{-1}$, and after $r-1$ derivatives in s that pole has order r; multiplying by $(1-z/3)^r$ cancels it exactly, so both functions continue analytically to $|z| < 5$, the next pole being at $z = 5$. Their coefficients decay like 5^{-m} and the weighted sums converge geometrically.

For Σ_2 the same device applies only at the left endpoint. At $s > 1$ the pole sits at $z = 3^s$, which $(1-z/3)^3$ does not cancel, and the inner sum is finite because the radius of convergence is then $3^s > 3$, a margin that shrinks to nothing as $s \to 1^+$, so this does not by itself bound the supremum. Each inner sum is finite for every s in the range, by cancellation at $s = 1$ and by $3/3^s < 1$ for $s > 1$; the finiteness of the supremum over $[1, s(3)]$ is part of what the interval computation below establishes, not a separate analytic claim. For Σ_0 and Σ_1 only the numerical values are certified.

Next, split the remainder of Proposition 3.1 at $u = 3$: with E_j the explicit expression in (17), put

$$R_{k,\mathrm{far}} = \sum_{j=0}^{k}\int_3^\infty \frac{u^{k-j}}{(k-j)!}E_j(u)\,\mathrm{d}u, \qquad R_{k,\mathrm{near}} = \sum_{j=0}^{k}\int_0^3 \frac{u^{k-j}}{(k-j)!}E_j(u)\,\mathrm{d}u, \tag{24}$$

so that R_k is $R_{k,\mathrm{far}} + R_{k,\mathrm{near}}$ together with the two coefficient tails of (18)–(19).

The far range needs one further constant. Write $\varepsilon = s - 1$ and $J = \zeta(s)/(-\zeta'(s))$; this is the J of Lemma 2.3 expressed in s rather than u, since $\mathrm{d}u/\,\mathrm{d}s = \zeta'(s)/\zeta(s)$ makes $-s'(u)$ equal to $\zeta(s)/(-\zeta'(s))$, and the four quantities below compare it with powers of $\zeta^{-1} = \mathrm{e}^{-u}$. Let

$$C_b = \sup|\varepsilon\zeta|, \qquad C_f = \sup|\zeta^2\big(J - \zeta^{-1}\big)|,$$
$$C_J = \sup\big|\zeta^3\big(J - \zeta^{-1} - 2\gamma\zeta^{-2}\big)\big|, \qquad C_P = \sup\big|\zeta^3\big(\varepsilon J - \zeta^{-2}\big)\big|, \tag{25}$$
$$C_E = C_J + C_P + C_b^2\,(C_f + 1),$$

every supremum being over $1 < s \le s(3)$. Each of the four is a ratio in which the pole of ζ at $s = 1$ cancels. How they are computed matters, and is described in Section 8: evaluating them on boxes of s loses that cancellation and the enclosures diverge however finely the range is partitioned, so a Taylor model is used instead.

Proposition 3.3 (Certified odd input)**.** *With Σ_i as in* (23) *and C_E as in* (25)*, the following are certified by interval arithmetic.*
(i) The coefficient norms satisfy

$$\Sigma_0 \le 44.5987, \qquad \Sigma_1 \le 486.1946, \qquad \Sigma_2 \le 14414, \qquad C_E \le 5.48910. \tag{26}$$

(ii) On the far range,

$$\sum_{j=0}^{k} \int_3^\infty \frac{u^{k-j}}{(k-j)!}|E_j(u)|\,\mathrm{d}u \le \frac{C_E\Sigma_2}{18}(k+3)^3 3^{-k}. \tag{27}$$

(iii) On the near range the two groups of terms in (17) *are bounded separately:*

$$\sum_{j=0}^{k} \int_0^3 \frac{u^{k-j}}{(k-j)!}|h_j(s(u))|J(u)\,\mathrm{d}u \le C_N 3^{-k} \quad (59 \le k \le 200), \quad C_N \le 367.8436, \tag{28}$$

$$\sum_{j=0}^{k} \int_0^3 \frac{u^{k-j}}{(k-j)!}\big(|h_j(1)|\mathrm{e}^{-u} + |b_j|\mathrm{e}^{-2u}\big)\,\mathrm{d}u \le N_U(k+3)^3 3^{-k}, \qquad N_U \le 26.0962. \tag{29}$$

(iv) $\beta > 0$, and

$$4900(62)^3 3^{-59} < \frac{\beta}{3} 2^{-59}. \tag{30}$$

(v) Direct interval evaluation encloses g_k for $1 \le k \le 75$, and gives $g_k > g_{k+1}$ for $1 \le k \le 74$; the last comparison available is between g_{74} and g_{75}, since g_{76} is not computed.

Part (v) is a sweep, not a coefficient bound. The range in (28) is two-sided, and its ends are discharged differently: the lower one is harmless, monotonicity below the analytic threshold coming from (v) and never appealing to (32), while the upper one is removed in the proof of Proposition 3.5, where a Cauchy estimate extends the bound past $k = 200$.

The lower restriction is a feature of the method, not of the inequality. The h_j alternate in sign, so the left-hand side must be majorized rather than evaluated in closed form, and the majorant is loosest where $a(s) = 2|h_1(s)|$ is largest, at the left end of the s-range. The bound is therefore weakest for small k, where the weights $u^{k-j}/(k-j)!$ put least mass away from that end: it exceeds $C_N 3^{-k}$ exactly for $9 \le k \le 25$, peaking at $699.87 \cdot 3^{-k}$ at $k = 14$, and falls to $34.14 \cdot 3^{-k}$ by $k = 59$. We claim nothing about the left-hand side itself outside the stated range — it is the certified bound, not the quantity, that needs the restriction, and $k \ge 26$ would serve — and state $k \ge 59$ because that is the only range in which (32) is applied. The numeral $C_N \le 367.8436$ is carried rather than the 34.14 computed on $k \ge 59$, so that the ledger below is

unchanged. It is not idle: it is the least four-decimal bound on $\sup_{k\ge1} 3^k P_k^{\mathrm{odd}}(s(3))$, the signed quantity whose coefficient identity earlier drafts used; that supremum is $367.84350\ldots$, attained at $k = 11$.

Lemma 3.4. *Let $A(z) = \sum_{j\ge0} a_j z^j$.*
(i) If $\sum_m 3^m|[z^m](1 - z/3)A(z)| \le S$, then $|a_j| \le S3^{-j}$.
(ii) If $\sum_m 3^m|[z^m](1 - z/3)^2A(z)| \le S$, then $|a_j| \le S(j + 1)3^{-j}$.
(iii) If $\sum_m 3^m|[z^m](1 - z/3)^3A(z)| \le S$, then

$$\sum_{j=0}^{k} 3^j|a_j| \le \binom{k+3}{3} S. \tag{31}$$

Sketch. Write $(1 - z/3)^r A(z) = \sum_m w_m z^m$ and expand $(1 - z/3)^{-r}$, so that A is the Cauchy product of $\sum_q \binom{q+r-1}{r-1} 3^{-q} z^q$ with $\sum_m w_m z^m$. Comparing coefficients gives (i) and (ii) at once. For (iii), summing the product bound over $j \le k$ and using $\sum_{q\le k-m} \binom{q+2}{2} = \binom{k-m+3}{3} \le \binom{k+3}{3}$ gives the claim. □

Proposition 3.5. *For every $k \ge 59$,*

$$|R_k| \le 4900(k + 3)^3 3^{-k}. \tag{32}$$

The threshold is not an artefact of the method but of the near range: the coefficients h_j alternate in sign, so the near-range term has to be estimated through the majorant (36), and the resulting bound exceeds (32) for small k. Nothing is lost, since monotonicity for $k \le 75$ is certified directly and does not use (32); see the proof of Theorem 1.2.

Proof. By (15) the remainder has four parts: the error integrals $\int E_j$ over the far range $u \ge 3$ and over the near range $0 < u \le 3$, and the two coefficient tails of (18)–(19). Each has to be brought to the common shape $(k + 3)^3 3^{-k}$, and the four resist to very different degrees, so it is worth saying in advance where the work lies.

The two tails are the easiest: they are geometric series in the h_j and b_j, which Lemma 3.4 bounds directly. The far range is harder only in bookkeeping. The integrand is small there, the certified Taylor model of (27) controls it pointwise, and the only real task is to see how the three summands of (17) depend on k. The near range is where the difficulty is, and it is what forces the threshold $k \ge 59$: there the h_j alternate in sign, so the coefficient identity that would *evaluate* the sum instead bounds it from the wrong side, and one is pushed onto a majorant that is loosest precisely where the weights put most of their mass. We take the four in that order.

Two remarks before starting. The far term and the near-range main term are cubic in k while the other two are constant multiples of 3^{-k}; since $(k + 3)^3 \ge 1$, the cubic factor may be absorbed once, at the end. And every estimate below is an estimate of an explicit quantity: no asymptotic series is summed anywhere.

Far range $u \ge 3$. The exact error here is the sum of the three terms of (17), so the certified Taylor model of (27) applies to the integrand as it stands. What that computation cannot supply is the dependence on k, since it certifies finitely many numbers; that comes instead from Lemma 3.4, and the three terms of (17) call on three different parts of it, since they carry $h_j(1)$, $h_j'(1)$ and $\partial_s^2 h_j$ respectively. The factor 3 in each is not a width, the far range $u \ge 3$ being unbounded, but a rate: the far-range error is $O(\mathrm{e}^{-3u})$, which is what the powers of ζ in (25) encode, and $\int_0^\infty u^m \mathrm{e}^{-3u}\,\mathrm{d}u/m! = 3^{-m-1}$. So each summand needs its own part of the lemma,

and the three parts give exactly the three weights required:

$$\sum_{j\le k} 3^j|h_j(1)| \le (k+1)\Sigma_0, \quad \sum_{j\le k} 3^j|h_j'(1)| \le \tbinom{k+2}{2}\Sigma_1, \quad \sum_{j\le k} 3^j|\partial_s^2 h_j| \le \tbinom{k+3}{3}\Sigma_2,$$

so with C_J, C_P and $C_b^2(C_f+1)$ as the respective prefactors of (25), the last of them halved by the Taylor remainder's $\frac12\varepsilon^2$, the far-range error is at most

$$\frac{3^{-k}}{3}\Big(C_J(k+1)\Sigma_0 + C_P\tbinom{k+2}{2}\Sigma_1 + \tfrac12 C_b^2(C_f+1)\tbinom{k+3}{3}\Sigma_2\Big).$$

Only the third term is cubic in k, and it dominates the other two by three orders of magnitude. The bracket is therefore at most $\big(C_J+C_P+\frac12 C_b^2(C_f+1)\big)\binom{k+3}{3}\Sigma_2$, and this is below $C_E\binom{k+3}{3}\Sigma_2$ since $\frac12 C_b^2(C_f+1) < C_b^2(C_f+1)$. Hence

$$|R_{k,\mathrm{far}}| \le \frac{C_E\Sigma_2}{18}(k+3)^3 3^{-k}, \tag{33}$$

the $18 = 6\cdot 3$ recording $\binom{k+3}{3} \le (k+3)^3/6$ and the rate.

Collapsing the three constants into C_E is wasteful, and deliberately so. The three-term bracket is $7.1\cdot 10^4$ at $k=1$ against the $8.4\cdot 10^5$ of (33), a factor of 12; the factor falls to 4.9 by $k=59$ and tends to $C_E/\big(\frac12 C_b^2(C_f+1)\big) = 4.67$. We keep the single constant because the ledger of Proposition 3.5 closes comfortably with it, and because one constant in the statement is easier to check than three; the slack is recorded here rather than spent.

First tail. By Lemma 3.4(i), $\Big|\sum_{j>k} h_j(1)\Big| \le \Sigma_0 \sum_{j>k} 3^{-j} = \frac{\Sigma_0}{2}3^{-k}$.

Second tail. Put $B(z) = \partial_s H_{\mathrm{o}}(1,z) + 2\gamma H_{\mathrm{o}}(1,z)$. Multiply by $(1-z/3)^2$ and treat the two summands separately. The first contributes Σ_1 by its definition in Proposition 3.3. For the second, write $(1-z/3)^2 H_{\mathrm{o}} = (1-z/3)\cdot\big[(1-z/3)H_{\mathrm{o}}\big]$; the weighted ℓ^1 norm $\sum_m 3^m|[z^m]\cdot|$ is submultiplicative under this extra factor $(1-z/3)$, whose own norm is 2, so the second summand contributes at most $2\gamma\cdot 2\Sigma_0 = 4\gamma\Sigma_0$. This is where the 4 comes from, rather than the 2 of the prefactor. Since $\gamma < 0.578$,

$$\sum_{m\ge 0} 3^m|[z^m](1-z/3)^2 B(z)| \le \Sigma_1 + 4\gamma\Sigma_0 < 590,$$

so $|b_j| \le 590(j+1)3^{-j}$ by Lemma 3.4(ii). Summing the resulting geometric series,

$$2^{-k-1}\sum_{j>k}|b_j|2^j \le 590\cdot 2^{-k-1}\sum_{j>k}(j+1)(2/3)^j = 590(k+4)3^{-k},$$

and $(k+4) \le \frac{5}{64}(k+3)^3$ for $k\ge 1$, with equality at $k=1$.

Near range $0 < u \le 3$. This is the delicate part. Here $|E_j|$ is at most the true integrand plus the two subtracted main terms. The latter contribute at most $N_U(k+3)^3 3^{-k}$ by (29). For the former, (13) and the substitution $s = s(u)$, under which $u = \log\zeta(s)$ and $J(u)\,\mathrm{d}u = \mathrm{d}s$, give

$$\sum_{j=0}^{k}\int_0^3 \frac{u^{k-j}}{(k-j)!}|h_j(s(u))|J(u)\,\mathrm{d}u = \int_{s(3)}^{\infty} A_k(s)\,\mathrm{d}s, \qquad A_k(s) = \sum_{j=0}^{k}\frac{(\log\zeta(s))^{k-j}}{(k-j)!}|h_j(s)|. \tag{34}$$

It is A_k, with the absolute values inside, that has to be estimated, and this is the obstruction. Writing $\Pi(t) = \sum_{p\ge 3} p^{-t}$, the z-linear coefficient of $\log H_{\mathrm{o}}$ in (9) is

$$h_1(s) = \Pi(s) - \log\zeta(s) = \log(1-2^{-s}) - \sum_{m\ge 2}\frac{\Pi(ms)}{m} < 0, \tag{35}$$

both terms being negative. So the h_j are not of one sign, and the identity $\sum_j u^{k-j}h_j(s)/(k-j)! = P_k^{\mathrm{odd}}(s)$, which would evaluate the sum exactly if they were, bounds A_k from below rather than above. This is the whole source of the loss in the near range, and of the threshold in (32).

We majorize instead. What makes this cheap is that, of the coefficients of $\log H_{\mathrm{o}}$, only the first is negative. Indeed by (35) every coefficient of $\log H_{\mathrm{o}}(s,z) = h_1(s)z + \sum_{m\ge 2}\Pi(ms)z^m/m$ beyond the first is positive, so replacing h_1 by $|h_1|$ produces

$$H_{\mathrm{o}}^{+}(s,z) = H_{\mathrm{o}}(s,z)\mathrm{e}^{-2h_1(s)z},$$

a power series with non-negative coefficients dominating $|h_j|$ termwise. Hence, with $a(s) = 2|h_1(s)|$,

$$A_k(s) \le [z^k]\big(\zeta(s)^z H_{\mathrm{o}}^{+}(s,z)\big) = \sum_{i=0}^{k} P_{k-i}^{\mathrm{odd}}(s)\frac{a(s)^i}{i!} =: P_k^{+}(s). \tag{36}$$

It remains to integrate P_k^{+}, and no single device does the whole of $[s(3),\infty)$. We fix a cut point $s_0 > s(3)$ and treat the two pieces differently, each by the feature that makes it tractable: the compact piece by monotonicity, the tail by its decay rate.

On the compact piece $[s(3),s_0]$, every P_j^{odd} is a Dirichlet series with positive terms and, by (35), a is a positive combination of such series; a sum of products of non-negative decreasing functions is decreasing, so P_k^{+} decreases in s. A finite left-endpoint Riemann sum over $[s(3),s_0]$ is therefore an upper bound for $\int_{s(3)}^{s_0} P_k^{+}$, with no quadrature error to estimate, which is what makes this step certifiable without any analysis of the quadrature rule itself.

On the tail $[s_0,\infty)$ the integral converges at a certified rate. Every odd n with $\Omega(n) = k-i$ satisfies $n \ge 3^{k-i}$, so for $s \ge s_0$, writing $t = s - s_0$, we get $P_{k-i}^{\mathrm{odd}}(s) \le 3^{-(k-i)t}P_{k-i}^{\mathrm{odd}}(s_0)$, while every Dirichlet term of a has $n \ge 2$ and hence $a(s) \le 2^{-t}a(s_0)$ by (35). Since $3^{-x} \le 2^{-x}$ for $x \ge 0$, the ith term of P_k^{+} is multiplied by at most $3^{-(k-i)t}2^{-it} \le 2^{-kt}$; all terms being non-negative, Tonelli permits termwise integration, and $\int_0^\infty 2^{-kt}\,\mathrm{d}t = 1/(k\log 2)$ gives

$$\int_{s_0}^{\infty} P_k^{+}(s)\,\mathrm{d}s \le \frac{P_k^{+}(s_0)}{k\log 2}. \tag{37}$$

Adding the Riemann sum over $[s(3),s_0]$ to (37) bounds $\int_{s(3)}^{\infty} P_k^{+}$, hence $\int_{s(3)}^{\infty} A_k$ by (36). For one k this is a finite computation, and (28) is its outcome for each of the 142 indices $59 \le k \le 200$.

That leaves the dependence on k, which no finite computation can settle, and which has to be reduced as the far range's was. The tool this time is Cauchy's estimate, and the reason it is available is again the positivity of the majorant's coefficients. Because $H_{\mathrm{o}}^{+}(s,\cdot)$ has non-negative coefficients, its maximum on a circle $|z| = R$ is attained at $z = R$, so for any $R < 3^{s(3)}$ and any $s \ge s(3)$,

$$A_k(s) \le R^{-k}\zeta(s)^R H_{\mathrm{o}}(s,R)\mathrm{e}^{a(s)R} = R^{-k}\prod_{p\ge 3}\big(1-Rp^{-s}\big)^{-1}\mathrm{e}^{a(s)R}, \tag{38}$$

a bound whose only k-dependence is the factor R^{-k}. One cannot simply integrate it over $[s(3),\infty)$: its right-hand side tends to 1, not to 0, as $s \to \infty$. So we use it on $[s(3),s_1]$ for a second cut point s_1, and fall back beyond s_1 on the decay already established. On $[s(3),s_1]$, (38) gives $\int_{s(3)}^{s_1} A_k \le R^{-k}\Theta_1$ with $\Theta_1 = \int_{s(3)}^{s_1}\prod_{p\ge 3}(1-Rp^{-s})^{-1}\mathrm{e}^{a(s)R}\,\mathrm{d}s$, finite and independent of k. Beyond s_1, (37) applies with $s_0 = s_1$, and bounding $P_k^{+}(s_1)$ by (38) in turn gives $\int_{s_1}^{\infty} A_k \le R^{-k}\Theta_2/(k\log 2)$ with $\Theta_2 = \prod_{p\ge 3}(1-Rp^{-s_1})^{-1}\mathrm{e}^{a(s_1)R}$, again independent of k.

Hence

$$3^k \int_{s(3)}^{\infty} A_k \le \Big(\frac{3}{R}\Big)^k \Big(\Theta_1 + \frac{\Theta_2}{k \log 2}\Big),$$

Take $R = 3.15$, which is below $3^{s(3)} = 3.1738\ldots$, and $s_1 = 8$. The certified values are then $\Theta_1 + \Theta_2/(200 \log 2) \le 175346$, while $(3/3.15)^{201} = 5.5075 \cdot 10^{-5}$, so the right-hand side is at most 9.66 for every $k > 200$. Therefore

$$\int_{s(3)}^{\infty} A_k(s)\,\mathrm{d}s \le C_N 3^{-k} \qquad \text{for all } k \ge 59, \tag{39}$$

which is the range in which (32) is used. Neither branch is anywhere near binding: over $59 \le k \le 200$ the largest value is $34.14 \cdot 3^{-k}$, at $k = 59$, and the Cauchy branch gives at most $9.66 \cdot 3^{-k}$ beyond, against the $C_N = 367.8436$ carried in the ledger.

The companion bound (29), for the two subtracted main terms, reduces the same way and needs no new idea. Here $|h_j(1)| \le \Sigma_0 3^{-j}$ by Lemma 3.4(i), while $b_j = h_j'(1) + 2\gamma h_j(1)$ needs both parts: (ii) gives $|h_j'(1)| \le (j+1)\Sigma_1 3^{-j}$ and (i) again gives the $2\gamma h_j(1)$ term, so $|b_j| \le \big((j+1)\Sigma_1 + 2\gamma\Sigma_0\big)3^{-j}$. These turn each integral into a regularized incomplete gamma value, and $P(m,x) \le x^m/m!$ makes the resulting sums telescope: with $U = 3$ the first is at most $\Sigma_0 \mathrm{e}^9 3^{-k}$ and the second at most $\frac{1}{2}\big((k+1)\Sigma_1 + 2\gamma\Sigma_0\big)\mathrm{e}^9 3^{-k}$. Both are linear in k where the budget $(k+3)^3 3^{-k}$ is cubic, so the ratio of this telescoped majorant to the budget decreases in k; since at $k = 400$ that ratio is already 12.08, below N_U, the maximum over the finite head $1 \le k \le 400$ governs. It is attained at $k = 7$, far outside the range in which (32) is used.

It remains only to add up. Substituting the certified constants, the four contributions are

$$|R_k| \le \Big(\underbrace{\frac{5.48910 \cdot 14414}{18}}_{4395.55} + \underbrace{\frac{44.5987}{2} + \frac{5 \cdot 590}{64} + C_N + N_U}_{462.34}\Big)(k+3)^3 3^{-k} < 4900(k+3)^3 3^{-k},$$

with $C_N = 367.8436$ and $N_U = 26.0962$. □

Proof of Theorem 1.2. The limit follows at once from (22) and Proposition 3.5. For monotonicity,

$$g_k - g_{k+1} = \beta 2^{-k-1} + R_k - R_{k+1}. \tag{40}$$

The main term $\beta 2^{-k-1}$ has to survive two remainders, at k and at $k+1$, so it is enough to give each a third of it: if $|R_m| < \beta 2^{-m}/3$ for $m = k, k+1$, then

$$|R_k| + |R_{k+1}| < \frac{2\beta}{3} 2^{-k-1} + \frac{\beta}{3} 2^{-k-1} = \beta 2^{-k-1}.$$

Define

$$q(k) = \frac{3 \cdot 4900}{\beta}(k+3)^3 (2/3)^k.$$

The certified inequality (30) says $q(59) < 1$. Moreover

$$\frac{q(k+1)}{q(k)} = \frac{2}{3}\left(\frac{k+4}{k+3}\right)^3 < 1 \qquad (k \ge 4).$$

Thus $q(k) < 1$ for every $k \ge 59$, and Proposition 3.5 gives $|R_k| < \beta 2^{-k}/3$ throughout that range. Equation (40) proves monotonicity for $k \ge 59$. The certified direct computation gives the comparison for $1 \le k \le 74$, so the two ranges cover every $k \ge 1$. They overlap on the sixteen indices $59 \le k \le 74$. □

4. An exact criterion for the unrestricted sums

The asymptotic (2) already says that f_k eventually increases, but it does not say from where: its $O(k\log(k+1))$ carries no explicit constant, so no index can be read off it. The remedy is to stop treating the remainder as an error term. Define $E(k)$ by

$$f_k = 1 - 2^{-k}\big(a_1 k^2 + E(k)\big), \qquad a_1 = \frac{d\log 2}{4}, \tag{41}$$

so that $E(k)$ is an exact quantity and not an estimate. Then

$$f_{k+1} - f_k = 2^{-k-1}\left(a_1(k^2 - 2k - 1) + 2E(k) - E(k+1)\right). \tag{42}$$

Thus $f_k < f_{k+1}$ if and only if

$$a_1(k^2 - 2k - 1) + 2E(k) - E(k+1) > 0. \tag{43}$$

We now derive the decomposition of $E(k)$ used to bound the second term.

Set

$$F(s,z) = \prod_p (1 - zp^{-s})^{-1}, \qquad G_2(s,z) = (s-1)^z(1 - z2^{-s})F(s,z),$$

and write $c_i = [z^i]G_2(1,z)$. Here $G_2(s,z) = \big((s-1)\zeta(s)\big)^z H_{\mathrm{o}}(s,z)$ for $s > 1$, and $((s-1)\zeta(s))^z \to 1$ as $s \to 1^+$ because $(s-1)\zeta(s)$ extends analytically to 1 with value 1; the value at $s = 1$ is to be read through that removable singularity, the displayed product being $0 \cdot \infty$ if taken literally. So $G_2(1,z) = H_{\mathrm{o}}(1,z)$, the power series has radius 3 and

$$\sum_{i\geq 0} c_i = \frac{1}{2}, \qquad \sum_{i\geq 0} 2^i c_i = d. \tag{44}$$

The coefficients are those of Section 3:

$$c_i = h_i(1). \tag{45}$$

Define the absolutely convergent signed moments

$$U = \sum_{i\geq 0} i2^i c_i, \qquad V = \sum_{i\geq 0} i^2 2^i c_i. \tag{46}$$

Following [3, §3], put

$$I_k = \int_1^2 [z^k]F(s,z)\,\mathrm{d}s, \tag{47}$$

$$I_k' = \sum_{i+j+\ell=k} c_i \int_1^2 \frac{(-\log(s-1))^\ell}{\ell!} 2^{-js}\,\mathrm{d}s. \tag{48}$$

For $m \geq 0$ define

$$A_m = 2^m \sum_{j+\ell=m} \int_1^2 \frac{(-\log(s-1))^\ell}{\ell!} 2^{-js}\,\mathrm{d}s \tag{49}$$

and

$$\delta_m = A_m - 2^{m+1} + \frac{\log 2}{4} m^2, \qquad \Delta(k) = \sum_{i=0}^{k} 2^i c_i \delta_{k-i}. \tag{50}$$

Proposition 4.1. *Let*

$$E_{S_1}(k) = -2^k(I_k - I'_k), \tag{51}$$

$$L_k = 2\sum_{i>k} c_i, \qquad Q_k = \sum_{i>k} 2^i c_i (k-i)^2, \tag{52}$$

$$T_k = f_k - I_k, \tag{53}$$

$$E_{\exp}(k) = 2^k L_k - \frac{\log 2}{4} Q_k - 2^k T_k. \tag{54}$$

Then the identity

$$E(k) = E_{S_1}(k) + \frac{\log 2}{4}(V - 2kU) - \Delta(k) + E_{\exp}(k) \tag{55}$$

holds for every $k \geq 1$.

Proof. From (48) and (49),

$$I'_k = 2^{-k} \sum_{i=0}^{k} 2^i c_i A_{k-i}.$$

Insert the definition of δ_{k-i}:

$$I'_k = 2\sum_{i=0}^{k} c_i - \frac{\log 2}{4} 2^{-k} \sum_{i=0}^{k} 2^i c_i (k-i)^2 + 2^{-k}\Delta(k). \tag{56}$$

By (44) and (46),

$$2\sum_{i=0}^{k} c_i = 1 - L_k,$$

$$\sum_{i=0}^{k} 2^i c_i (k-i)^2 = \sum_{i\geq 0} 2^i c_i (k-i)^2 - Q_k$$
$$= dk^2 - 2kU + V - Q_k.$$

Since $a_1 = \frac{\log 2}{4} d$, substitution into (56) gives

$$I'_k = 1 - 2^{-k}\left(a_1 k^2 + \frac{\log 2}{4}(V - 2kU) - \Delta(k) + 2^k L_k - \frac{\log 2}{4} Q_k \right).$$

Finally $f_k = I'_k + (I_k - I'_k) + T_k$. Comparing with (41) yields (55). □

5. Bounds for the four source terms

By (55) the criterion (43) needs a lower bound for $2E(k) - E(k+1)$, and E is a sum of four quantities of quite different character. Two of them, S_1 and the signed moments, are linear in k and are bounded by suprema of coefficients. The third, Δ, grows like $k \log k$, and getting the logarithm rather than a power of k is the only real work in this section. The fourth is exponentially small and is the only one needing a certificate uniform in k. We treat them in that order.

One point of method recurs. Each bound is taken on the *combination* $2E_\bullet(k) - E_\bullet(k+1)$ that the criterion produces, rather than on $E_\bullet(k)$ and $E_\bullet(k+1)$ separately, since the combination enjoys cancellation that absolute values would destroy. The treatment of the signed moments below is the clearest instance.

5.1. **The S_1 term.** Set

$$\alpha_i = \sup_{1\le t\le 2} |[z^i]\partial_s G_2(t,z)|, \qquad \alpha_+ = \frac{1}{2}\sum_{i\ge 0}\alpha_i 2^i.$$

The series converges, and a pole location at one value of t is not enough to see it: what is needed is a bound on the coefficients uniform in t.

First, G_2 is analytic in s on a neighbourhood of $[1,2]$, the apparent branch point of $(s-1)^z$ at $s=1$ notwithstanding. Indeed $F(s,z)=\zeta(s)^z H(s,z)$ by (8), so

$$G_2(s,z) = \big((s-1)\zeta(s)\big)^z(1-z2^{-s})H(s,z),$$

and $(s-1)\zeta(s)$ is analytic near $s=1$ with value 1 there, the pole of ζ being simple with residue 1; hence $\big((s-1)\zeta(s)\big)^z = \exp\big(z\log((s-1)\zeta(s))\big)$ is analytic there too, and H is analytic by Proposition 2.2. (At $s=1$ this also gives $G_2(1,z)=H_{\mathrm{o}}(1,z)$, as used above.)

The first z-pole of $\partial_s G_2(t,\cdot)$ is at $z=3^t$, so for $t\in[1,2]$ all of these functions are analytic in $|z|<3$. Fix r with $2<r<3$. Then $M=\sup_{1\le t\le 2}\sup_{|z|=r}|\partial_s G_2(t,z)|$ is finite, the set $[1,2]\times\{|z|=r\}$ being compact and $\partial_s G_2$ continuous on it by the previous paragraph, and Cauchy's estimate on the fixed circle $|z|=r$ gives $\alpha_i\le Mr^{-i}$ uniformly in t. Hence $\sum_{i\ge0}\alpha_i 2^i \le M(1-2/r)^{-1}<\infty$.

Lemma 5.1. *For every $k\ge 1$,*

$$|E_{S_1}(k)| \le \alpha_+(k+1), \tag{57}$$

and hence

$$|2E_{S_1}(k) - E_{S_1}(k+1)| \le \alpha_+(3k+4). \tag{58}$$

Proof. Since $[z^i]G_2(\cdot,z)$ is complex-valued, the mean value theorem is not available; we integrate instead. By (10) and the analyticity of G_2 in s established above,

$$[z^i]G_2(s,z) - c_i = \int_1^s [z^i]\partial_t G_2(t,z)\,\mathrm{d}t, \qquad \text{so} \qquad |[z^i]G_2(s,z)-c_i| \le (s-1)\alpha_i$$

for $1\le s\le 2$, by the definition of α_i as a supremum over that interval. Subtracting (48) from the coefficient expansion for I_k and then setting $s=1+\mathrm{e}^{-t}$ gives

$$\begin{aligned}
|I_k - I_k'| &\le \sum_{i+j+\ell=k}\alpha_i\int_0^\infty \frac{t^\ell}{\ell!}2^{-j(1+\mathrm{e}^{-t})}\mathrm{e}^{-2t}\,\mathrm{d}t \\
&\le \sum_{i+j+\ell=k}\alpha_i 2^{-j}\int_0^\infty \frac{t^\ell}{\ell!}\mathrm{e}^{-2t}\,\mathrm{d}t \\
&= \sum_{i+j+\ell=k}\alpha_i 2^{-j-\ell-1}.
\end{aligned}$$

Multiplying by 2^k and using $k=i+j+\ell$,

$$|E_{S_1}(k)| \le \frac{1}{2}\sum_{i=0}^{k}\alpha_i 2^i(k-i+1) \le \alpha_+(k+1).$$

The triangle inequality now gives

$$|2E_{S_1}(k) - E_{S_1}(k+1)| \le 2\alpha_+(k+1) + \alpha_+(k+2) = \alpha_+(3k+4).$$

□

5.2. **The signed moment term.** The combination of signed moments that has to be bounded is not chosen; it is what the difference of two consecutive source terms produces:

$$2(V-2kU)-\big(V-2(k+1)U\big)=V+(2-2k)U. \tag{59}$$

Since this is an identity, bounding the right-hand side is an exact optimization over the rectangle of admissible (U,V), not a termwise absolute-value estimate. That is the reason for evaluating the moments before absolute values are taken: replacing them by $\sum_i 2^i|c_i|$ would discard the cancellation in (59) and cost a factor in k.

Lemma 5.2. *If $|U|\le U_{\max}$ and $|V|\le V_{\max}$, then for $k\ge 2$,*

$$\frac{\log 2}{4}\big(V+(2-2k)U\big)\ge -\frac{\log 2}{4}\big(V_{\max}+(2k-2)U_{\max}\big). \tag{60}$$

Proof. The coefficient of V is 1, while the coefficient $2-2k$ of U is negative. The minimum over the rectangle $[-U_{\max},U_{\max}]\times[-V_{\max},V_{\max}]$ is therefore attained at $U=U_{\max}$ and $V=-V_{\max}$, giving (60). □

5.3. **The δ term.** The next lemma supplies the self-contained bound used for $\Delta(k)$.

Lemma 5.3. *For $j\ge 0$ and $u\ge 0$ put*

$$B_j(u)=2^{-j\mathrm{e}^{-u}}-1+j\log 2\,\mathrm{e}^{-u}$$

and

$$\eta_i(j)=\frac{1}{i!}\int_0^\infty u^i\mathrm{e}^{-u}B_j(u)\,\mathrm{d}u.$$

Then $B_j(u)\ge 0$, $\eta_i(j)\ge 0$, and

$$\delta_m=-1-\frac{\log 2}{4}m+\sum_{i=0}^{m}2^i\eta_i(m-i). \tag{61}$$

Proof. With $s=1+\mathrm{e}^{-u}$ in (49),

$$A_m=\sum_{i=0}^{m}\frac{2^i}{i!}\int_0^\infty u^i\mathrm{e}^{-u}2^{-(m-i)\mathrm{e}^{-u}}\,\mathrm{d}u.$$

Use $2^{-j\mathrm{e}^{-u}}=1-j\log 2\,\mathrm{e}^{-u}+B_j(u)$. The constant term contributes

$$\sum_{i=0}^{m}2^i=2^{m+1}-1.$$

The linear term contributes

$$\begin{aligned}-\log 2\sum_{i=0}^{m}\frac{2^i(m-i)}{i!}\int_0^\infty u^i\mathrm{e}^{-2u}\,\mathrm{d}u&=-\frac{\log 2}{2}\sum_{i=0}^{m}(m-i)\\&=-\frac{\log 2}{4}m(m+1).\end{aligned}$$

The remaining term is $\sum_{i=0}^{m}2^i\eta_i(m-i)$. Subtracting 2^{m+1} and adding $(\log 2)m^2/4$ gives (61). Finally, for $x\ge 0$, $\mathrm{e}^{-x}\ge 1-x$; applying this with $x=j\log 2\,\mathrm{e}^{-u}$ proves $B_j(u)\ge 0$. □

Lemma 5.4. *For every $m\ge 0$,*

$$|\delta_m|\le D(m),\qquad D(m)=(\log 2)m\log(m+1)+1. \tag{62}$$

Proof. For the lower bound, (61) gives $\delta_m \geq -1 - \frac{\log 2}{4} m \geq -D(m)$ for $m \geq 1$, because $\log(m+1) \geq \log 2 > 1/4$; and $\delta_0 = -1 = -D(0)$.

The upper bound is where the $\log(m+1)$ comes from, and it comes from a split. Put $\Lambda_m = \sum_{i=0}^{m} 2^i \eta_i (m-i)$. The elementary inequality $\mathrm{e}^{-x} - 1 + x \leq \min(x, x^2/2)$ offers two bounds. Here $x = (m-i)\log 2\,\mathrm{e}^{-u}$ is large for small u and small for large u, so the linear bound is the better one near $u = 0$ and the quadratic one is better out in the tail; using each where it is better is what produces a logarithm rather than a power of m. Taking $x = (m-i)\log 2\,\mathrm{e}^{-u}$ gives

$$\Lambda_m \leq \int_0^\infty \mathrm{e}^{-u} \sum_{i=0}^{m} \frac{(2u)^i}{i!} \min\left((m-i)\log 2\,\mathrm{e}^{-u}, \frac{(m-i)^2 (\log 2)^2 \mathrm{e}^{-2u}}{2} \right) \mathrm{d}u. \tag{63}$$

The crossover is at $v_m = \max\big(0, \log \frac{m+1}{3}\big)$. On $0 \leq u \leq v_m$ use the linear bound together with $\sum_{i=0}^{m} (m-i)(2u)^i/i! \leq m\mathrm{e}^{2u}$, and on $u \geq v_m$ the quadratic bound together with $\sum_{i=0}^{m} (m-i)^2 (2u)^i/i! \leq m^2 \mathrm{e}^{2u}$. Then

$$\Lambda_m \leq m \log 2\, v_m + \frac{m^2 (\log 2)^2}{2} \mathrm{e}^{-v_m}. \tag{64}$$

For $m \geq 2$ we have $v_m = \log \frac{m+1}{3}$ and $\mathrm{e}^{-v_m} = \frac{3}{m+1}$, so (64) and $\frac{3m^2}{2(m+1)} \leq \frac{3}{2} m$ give

$$\begin{aligned} \Lambda_m &\leq m \log 2 \log \frac{m+1}{3} + \frac{3m^2 (\log 2)^2}{2(m+1)} \\ &\leq m \log 2 \big(\log(m+1) - \log 3 + \tfrac{3}{2} \log 2 \big) \leq m \log 2 \log(m+1), \end{aligned}$$

the last step because $3 > 2^{3/2}$. For $m \leq 1$ we have $v_m = 0$, and (64) reads $\Lambda_m \leq m^2 (\log 2)^2/2$, which is at most $m \log 2 \log(m+1)$ since $\frac{m \log 2}{2} \leq \log(m+1)$ there. In both cases

$$\delta_m = -1 - \frac{\log 2}{4} m + \Lambda_m \leq \Lambda_m \leq (\log 2) m \log(m+1) < D(m),$$

which with the lower bound proves (62). □

Lemma 5.5. *If $S^* = \sum_{i \geq 0} 2^i |c_i|$, then, with D as in Lemma 5.4,*

$$|2\Delta(k) - \Delta(k+1)| \leq S^* \big(2D(k) + D(k+1) \big). \tag{65}$$

Sketch. Since D is increasing, $|\Delta(k)| \leq \sum_{i \leq k} 2^i |c_i| D(k-i) \leq S^* D(k)$. Now apply the triangle inequality to $2\Delta(k) - \Delta(k+1)$. □

5.4. **The exponentially small term.** The three terms in (54) are respectively a coefficient tail at $z = 1$, a weighted coefficient tail at $z = 2$, and the integral over $s \geq 2$. The following is the exact numerical input used for them.

Proposition 5.6 (Certified unrestricted input)**.** *The interval computations give*

$$\begin{gathered} d \geq 0.3786949, \qquad S^* \leq 6.9764, \qquad \alpha_+ \leq 15.425661, \\ |U| \leq U_{\max} = 0.08436, \qquad |V| \leq V_{\max} = 1.907462. \end{gathered} \tag{66}$$

For every $k \geq 900$,

$$2^k |T_k| \leq \frac{1.45}{k}, \tag{67}$$

and a contour bound for the pole-free part *of the generating function of the c_i: writing ρ for the residue of $G_2(1,\cdot)$ at its nearest pole $z=3$, so that $|\rho|\le 0.1749324$,*

$$\sup_{|z|=4.2}\Big|G_2(1,z)-\frac{\rho}{3-z}\Big|\le 101.245. \tag{68}$$

Here the subtraction matters. For the regularized product $\widetilde{G}_2(z)=(1-z/3)\,G_2(1,z)$ one has $228.94\le\sup_{|z|=4.2}|\widetilde{G}_2|\le 228.95$, so 101.245 is not a bound for $\widetilde{G}_2$. It is a bound for (68)*, with about 5.7% to spare. Both suprema are certified in* `numerics/contour_certified.py` *(log* `contour.log`*), which also recovers ρ from the same series and matches the value the sweep scripts use. Together these give*

$$|E_{\exp}(k)|\le\frac{1.45}{k}+10^{-3}\qquad(k\ge 900), \tag{69}$$

as derived in §5.4. Of the three terms of $E_{\exp}$, two have their dependence on k reduced analytically there, to geometric bounds whose constant is (68)*; only T_k needs a certificate that is itself uniform in k, namely* (67)*. Both are recorded here rather than inside that derivation so that every certified input for the unrestricted case appears in one place. Finally, $f_k<f_{k+1}$ is certified on*

$$6\le k\le 199,\qquad 78\le k\le 899,\qquad 900\le k\le 2439, \tag{70}$$

whose union is the single range $6\le k\le 2439$. These three ranges are not established the same way, and the distinction matters to anyone checking the certificates. On the first, f_k itself is computed as an enclosure for each k and the two enclosures are compared; the quadrature behind those values, and the six error sources entering each, are described in Appendix A. (That computation in fact encloses f_k for $2\le k\le 500$, but the pairwise comparison closes only up to $k=297$: past there the gap $f_{k+1}-f_k$ falls below the common radius, and the certificate's own `monotonicity` *entries record* `verified: false` *from $k=298$ on. Only $6\le k\le 199$ is needed here, and that range is fully verified.) On the other two it is the left-hand side of* (43) *that is evaluated in ball arithmetic at each k and found positive. This is the same quantity whose closed-form lower bound φ drives Section 6, evaluated per k rather than bounded uniformly. That is why it closes well below the threshold 2440 which the uniform bound requires. Within the middle range two assemblies of that quantity are involved: the refined one for $78\le k\le 734$, and for the rest of the range the same coarser assembly that covers $900\le k\le 2439$, whose bound on E_1 is the one built termwise from the β_i rather than from their envelope α_+. That coarser sweep is in fact positive from $k=728$, so the two overlap rather than abut; only $735\le k\le 899$ is needed here, and the overlap is recorded because it is what makes the union of* (70) *robust to a change in either sweep.*

The coefficient tails in (69) are obtained only after the pole at $z=3$ has been removed, and the removal is by subtraction of the principal part rather than by multiplication. Put

$$H(z)=G_2(1,z)-\frac{\rho}{3-z}=\sum_{m\ge 0}w_m z^m,$$

which is holomorphic on $|z|<5$, the next pole of $G_2(1,\cdot)$ being at $z=5$. Cauchy's estimate on $|z|=4.2$ together with (68) gives $|w_m|\le 101.245\,4.2^{-m}$, while $[z^i]\rho/(3-z)=\rho\,3^{-i-1}$, so

$$|c_i|\le\frac{|\rho|}{3}3^{-i}+101.245\,(3/4.2)^i 3^{-i}\le 101.31\,3^{-i}<354.36\,3^{-i}. \tag{71}$$

The weaker constant 354.36 is the one carried below; the sharper 101.31 is recorded only so that the chain can be checked. By (45) the c_i are the coefficients $h_i(1)$, so Lemma 3.4(i) sharpens

(71) to $|c_i| \le \Sigma_0 3^{-i}$, dispensing with the contour bound; the weaker form suffices below. With $C = 354.36$,

$$2^k|L_k| \le C(2/3)^k,$$
$$\frac{\log 2}{4}|Q_k| \le \frac{\log 2}{4} C(2/3)^k \sum_{r\ge 1} r^2 (2/3)^r = \frac{30C\log 2}{4}(2/3)^k.$$

Both terms are less than 10^{-3} in total for $k \ge 900$, and both bounds are geometric in k, so no computation is needed to make them uniform.

The third term admits an elementary bound, which we give because (67) is otherwise the one input of Proposition 5.6 not visibly tied to a computation. Since $[z^k]F(s,z) = \sum_{\Omega(n)=k} n^{-s}$, (53) and (47) give

$$T_k = \int_2^\infty \sum_{\Omega(n)=k} n^{-s}\,\mathrm{d}s = \sum_{\Omega(n)=k} \frac{n^{-2}}{\log n},$$

a sum of positive terms, so $T_k > 0$ and nothing is lost in absolute value. Every such n is at least 2^k, whence $\log n \ge k\log 2$ and

$$2^k|T_k| \le \frac{2^k P_k(2)}{k\log 2}, \qquad P_k(2) = [z^k]F(2,z).$$

As $F(2,\cdot)$ is holomorphic on $|z| < 4$, Cauchy's estimate on $|z| = 3$ gives $P_k(2) \le M3^{-k}$ with $M = \prod_p (1-3p^{-2})^{-1} \le 7.96896$, so

$$2^k|T_k| \le \frac{M}{k\log 2}\Big(\frac{2}{3}\Big)^k, \tag{72}$$

which is below $1.45/k$ for every $k \ge 6$ and at $k = 900$ is smaller by a factor of about 10^{158}. So (67) is true with room of that order; it is stated in that form because $1.45/k$ is what the criterion consumes, not because the constant is near optimal, and the threshold 900 is inherited from (70) rather than forced here. (The first two terms of (54) are below 10^{-3} from $k \ge 37$ on, so no term of $E_{\exp}$ requires $k \ge 900$.) The computation behind M is `numerics/tk_certified.py`, log `tk.log`.

Substituting (67) into (54) proves (69), and no asymptotic O-term enters the final criterion.

6. Assembly of the unrestricted theorem

The four bounds of Section 5 now combine into a single explicit function φ whose positivity is equivalent to the inequality wanted, and the proof of Theorem 1.1 becomes two statements about φ: that it is positive at one index, and that it increases thereafter. Neither is an asymptotic statement, and the remaining finitely many indices are settled by computation.

For $k \ge 900$ define

$$R(k) = \frac{\log 2}{4}\big((2k-2)U_{\max} + 2V_{\max} + 4U_{\max}\big) + S^*\big(2D(k) + D(k+1)\big) + \alpha_+(3k+4) + 3\left(\frac{1.45}{k} + 10^{-3}\right), \tag{73}$$

and

$$\varphi(k) = a_1(k^2 - 2k - 1) - R(k). \tag{74}$$

Proposition 6.1. *For $k \geq 900$,*

$$2E(k) - E(k+1) \geq -R(k). \tag{75}$$

Consequently $\varphi(k) > 0$ implies $f_k < f_{k+1}$.

Proof. Apply (55) at k and $k+1$. The S_1 contribution is bounded by (58). The signed moment contribution is

$$\frac{\log 2}{4}\big(V + (2-2k)U\big),$$

which is bounded below by (60). The first line of (73) is slightly larger than the resulting quantity, since

$$V_{\max} + (2k-2)U_{\max} \leq (2k-2)U_{\max} + 2V_{\max} + 4U_{\max}.$$

The Δ contribution is bounded by (65). Finally, (69) gives

$$|2E_{\exp}(k) - E_{\exp}(k+1)| \leq 2\left(\frac{1.45}{k} + 10^{-3}\right) + \left(\frac{1.45}{k+1} + 10^{-3}\right) \leq 3\left(\frac{1.45}{k} + 10^{-3}\right).$$

Adding the four lower bounds proves (75). Equation (42) then gives

$$f_{k+1} - f_k \geq 2^{-k-1}\varphi(k),$$

which proves the last assertion. □

Lemma 6.2. *With the certified constants in Proposition 5.6, $\varphi(2440) > 1155$.*

Proof. Substitution in (74) gives the following certified one-sided bounds:

$$\begin{aligned} a_1(2440^2 - 2\cdot 2440 - 1) &> 390371.6, \\ \frac{\log 2}{4}\big(4878U_{\max} + 2V_{\max} + 4U_{\max}\big) &< 72.1, \\ S^*\big(2D(2440) + D(2441)\big) &< 276166.8, \\ \alpha_+(3\cdot 2440 + 4) &< 112977.6, \\ 3\left(\frac{1.45}{2440} + 10^{-3}\right) &< 0.005. \end{aligned}$$

Therefore

$$\varphi(2440) > 390371.6 - 72.1 - 276166.8 - 112977.6 - 0.005 > 1155.$$

The same substitution at $k = 2433$ gives $\varphi(2433) > 135$, so the criterion is already in force there. Direct evaluation puts the sign change just below that, at $\varphi(2432) < 0 < \varphi(2433)$; we do not rely on it, since certifying $\varphi(2432) < 0$ would need a *lower* bound on $R(2432)$, whereas every constant in (66) is one-sided the other way. Nothing below requires the first index: we carry the threshold at 2440, which is one past the end of (70), so that the sweep and the criterion meet without a gap. □

Lemma 6.3. *The function $k \mapsto \varphi(k)$ is strictly increasing for integers $k \geq 2440$.*

Proof. First,

$$a_1\big((k+1)^2 - 2(k+1) - 1 - (k^2 - 2k - 1)\big) = a_1(2k-1).$$

Moreover

$$\begin{aligned} D(m+1) - D(m) &= \log 2\left(\log(m+2) + m\log\left(1 + \frac{1}{m+1}\right)\right) \\ &\leq \log 2\big(\log(m+2) + 1\big). \end{aligned}$$

The last term in (73) decreases with k. Hence

$$R(k+1) - R(k) \le \frac{\log 2}{2} U_{\max} + 3\alpha_+ + S^* \log 2\big(2\log(k+2) + \log(k+3) + 3\big). \tag{76}$$

Define, for real $x \ge 2440$,

$$M(x) = a_1(2x-1) - \frac{\log 2}{2} U_{\max} - 3\alpha_+ - S^* \log 2\big(2\log(x+2) + \log(x+3) + 3\big).$$

Then

$$M'(x) = 2a_1 - S^* \log 2 \left(\frac{2}{x+2} + \frac{1}{x+3} \right) > 0,$$

because the certified lower bound for d gives $2a_1 > 0.1312$, whereas the subtracted term is < 0.006 for $x \ge 2440$. Direct substitution gives $M(2440) > 146$. Thus $M(k) > 0$ for every $k \ge 2440$. Together with (76), this yields $\varphi(k+1) - \varphi(k) > 0$. □

Proof of Theorem 1.1. By Lemmas 6.2 and 6.3, $\varphi(k) > 0$ for every $k \ge 2440$. Proposition 6.1 therefore gives $f_k < f_{k+1}$ on that range. The certified ranges in (70) cover every integer from 6 through 2439. Together the analytic and certified ranges cover all $k \ge 6$. □

7. A general principle

Conjecture 7.1. Let $\mathcal{P}$ be a set of primes with least element q and thin complement,

$$\sum_{p \notin \mathcal{P}} \frac{\log p}{p} < \infty, \tag{77}$$

and let $f_k^{\mathcal{P}}$ be the sum of $1/(n \log n)$ over integers n with $\Omega(n) = k$ and all prime factors in $\mathcal{P}$. The sequence $f_k^{\mathcal{P}}$ is eventually increasing if and only if $q = 2$.

Some hypothesis is needed. For $\mathcal{P} = \{2\}$ one has $f_k^{\mathcal{P}} = 1/(2^k k \log 2)$, and for $\mathcal{P} = \{2, 3\}$ the sequence again decreases to 0; in both cases $q = 2$ and the conclusion fails. What goes wrong is that a sparse $\mathcal{P}$ leaves $\sum_{p \in \mathcal{P}} p^{-s}$ bounded as $s \to 1^+$, so the generating function has no $\zeta(s)^z$ factor and $f_k^{\mathcal{P}}$ decays geometrically whatever q is.

Those examples, however, are already excluded by the weaker condition $\sum_{p \notin \mathcal{P}} 1/p < \infty$, and that condition is what governs the size of the limit. The candidate limit is $\prod_{p \notin \mathcal{P}} (1 - 1/p)$, which is the value $H_{\mathcal{P}}(1, 1)$ of (79) below, and it is a convergent product with positive value precisely when $\sum_{p \notin \mathcal{P}} 1/p < \infty$, and 0 otherwise. That it is indeed the limit is proved below for $q \ge 3$, in Proposition 7.2(iv); for $\mathcal{P}$ all primes it is Lichtman's $f_k \to 1$ [2]; and for the two sparse examples the product is 0, as are their limits. So the unweighted condition separates a sequence with a positive limit, whose monotonicity is then decided by a correction term, from one that decays to 0. The sparse examples are the degenerate case of a single product, and not a separate phenomenon.

Why then does (77) carry the weight $\log p$? Of the two products in (79) it is the second, over the complement, that needs a hypothesis at all. The first converges locally uniformly on $\sigma > 1/2$ for any set of primes whatever, since $\log L_p = O(p^{-2s})$, while the logarithm of the second is $\asymp -zp^{-s}$ and so requires $\sum_{p \in \mathcal{Q}} p^{-\sigma} < \infty$. That is the unweighted condition of the paragraph above. The weight is needed instead for the s-derivative at $s = 1$, where the constant multiplying 2^{-k} lives: that constant involves $\sum_{p \notin \mathcal{P}} \frac{\log p}{p-1}$, finite precisely under (77), and the weaker condition does not give it. Take the complement to be blocks of primes in

$[X_j, X_j^{1+\eta_j}]$ with $\eta_j \to 0$ slowly and $\log X_j$ growing geometrically; then $\sum_{p\notin\mathcal{P}} 1/p$ converges while $\sum_{p\notin\mathcal{P}} (\log p)/p$ diverges. We have not verified that such a $\mathcal{P}$ violates the conclusion. We state the hypothesis in the safe direction, where strengthening it makes the conjecture claim less.

The singularity structure is the one the two theorems above exploit. On $|z| < q^\sigma$ the product $\prod_{p\in\mathcal{P}}(1 - zp^{-s})^{-1}$ converges directly, by (11), so its first z-singularity is at $z = q^s$; the factorization

$$\prod_{p\in\mathcal{P}}(1 - zp^{-s})^{-1} = \prod_{p}(1 - zp^{-s})^{-1} \prod_{p\notin\mathcal{P}}(1 - zp^{-s})$$

then relates it to $\zeta(s)^z H_{\mathcal{P}}(s, z)$ on the smaller disc $|z| < 2^\sigma$ where the middle product also converges. At $s = 1$ the point $z = 2$ lies on the singular circle exactly when $q = 2$. Note that $z = 2$ is distinguished here not because 2 is prime, but because it is where the γe^{-2u} term of J in Lemma 2.3 evaluates the regularized product. If $q \geq 3$ the extraction at $z = 2$ is interior, and gives a constant multiple of 2^{-k} as in Section 3. If $q = 2$ it is at the boundary, and the mechanism producing a factor k^2 is the one identified by Gorodetsky, Lichtman and Wong [3] for $\mathcal{P} = \{\text{all primes}\}$.

In the interior case $q \geq 3$ the regularized product does supply the control the extraction needs, and the constant can be identified. Write

$$L_p(s, z) = (1 - p^{-s})^z(1 - zp^{-s})^{-1}, \qquad \log L_p(s, z) = \sum_{m\geq 2} \frac{z^m - z}{m} p^{-ms}, \tag{78}$$

the second identity being (11). Define $H_{\mathcal{P}}$ by the regularized product, as (8) and (9) define H and H_{o}:

$$H_{\mathcal{P}}(s, z) = \prod_{p\notin\mathcal{P}}(1 - p^{-s})^z \prod_{p\in\mathcal{P}}(1 - zp^{-s})^{-1}(1 - p^{-s})^z = \prod_{p\notin\mathcal{P}}(1 - p^{-s})^z \prod_{p\in\mathcal{P}} L_p(s, z), \tag{79}$$

which is (8) when $\mathcal{P}$ is all primes and (9) when $\mathcal{P} = \{p : p \geq 3\}$. For $s > 1$, where $\zeta(s)^z$ is finite and non-zero, (7) extends to $\mathcal{P}$ as

$$\sum_{k\geq 0} P_k^{\mathcal{P}}(s) z^k = \prod_{p\in\mathcal{P}}(1 - zp^{-s})^{-1} = \zeta(s)^z H_{\mathcal{P}}(s, z), \qquad P_k^{\mathcal{P}}(s) = \sum_{\substack{\Omega(n)=k \\ p|n\Rightarrow p\in\mathcal{P}}} n^{-s}, \tag{80}$$

by the same computation, since $\zeta(s)^{-z} = \prod_p (1 - p^{-s})^z$ there. It is (79) and not the factor $\zeta(s)^{-z}\prod_{\mathcal{P}}(1-zp^{-s})^{-1}$ of (80) that is used below: the latter is 0 at $s = 1$, where every constant in this section is evaluated, while the former is finite and non-zero there. Put $h_j^{\mathcal{P}}(s) = [z^j]H_{\mathcal{P}}(s, z)$.

Proposition 7.2. *Let $\mathcal{P}$ be a set of primes with least element $q \geq 3$ and complement $\mathcal{Q}$ satisfying* (77)*, and fix $R \in (2, 3)$, which is possible with $R < q$ since $q \geq 3$. Then:*

(i) *for every real $s \geq 1$ both products in* (79) *converge absolutely and locally uniformly on $|z| < q^s$, where $H_{\mathcal{P}}(s, \cdot)$ is therefore analytic and zero-free;*

(ii) *$M := \sup_{s\geq 1} \sup_{|z|=R} |H_{\mathcal{P}}(s, z)| < \infty$, and hence $|h_j^{\mathcal{P}}(s)| \leq MR^{-j}$ for all $j \geq 0$ and all $s \geq 1$;*

(iii) *each $h_j^{\mathcal{P}}$ is C^1 on $[1, \infty)$ with $\sup_{s\geq 1} |(h_j^{\mathcal{P}})'(s)| \leq M'R^{-j}$ for some $M' < \infty$, and*

$$\tfrac{1}{2}\partial_s \log H_{\mathcal{P}}(1, 2) = -\sum_{p\in\mathcal{P}} \frac{\log p}{(p - 1)(p - 2)} + \sum_{p\in\mathcal{Q}} \frac{\log p}{p - 1}, \tag{81}$$

both sums finite, the second precisely by (77)*;*

(iv) *consequently*

$$f_k^{\mathcal{P}} = \prod_{p\in\mathcal{Q}}\Big(1-\frac{1}{p}\Big) + c_{\mathcal{P}}2^{-k} + o(2^{-k}), \qquad c_{\mathcal{P}} = H_{\mathcal{P}}(1,2)\big(\gamma + \tfrac{1}{2}\partial_s \log H_{\mathcal{P}}(1,2)\big), \tag{82}$$

$$\text{with } H_{\mathcal{P}}(1,2) = \textstyle\prod_{p\in\mathcal{P}}(1-p^{-1})^2(1-2p^{-1})^{-1}\prod_{p\in\mathcal{Q}}(1-p^{-1})^2 \in (0,\infty).$$

Proof. Parts (i)–(iii) are the analytic groundwork: that $H_{\mathcal{P}}$ exists, that it and its s-derivative are bounded on a circle $|z| = R$ with $R \in (2,3)$, and that the coefficients $h_j^{\mathcal{P}}$ inherit geometric decay R^{-j} from that. Part (iv) is the substance, and it uses nothing of $\mathcal{P}$ beyond those three facts. The reason the radius must lie strictly between 2 and 3 emerges there: $R > 2$ is what makes the coefficient tails $o(2^{-k})$, and $R < 3$ is what makes a geometric sum in the error terms dominated by its last term.

(i) Fix $s \geq 1$ and a compact $K \subset \{|z| < q^s\}$, say $|z| \leq R' < q^s$. For the product over $\mathcal{P}$, (78) gives $\log L_p(s,z) = O_{R'}(p^{-2s})$ uniformly on K by (11), whose hypothesis $|zp^{-s}| < 1$ holds because $|z| \leq R' < q^s \leq p^s$ for every $p \in \mathcal{P}$; and $\sum_p p^{-2s} < \infty$ since $s \geq 1$. For the product over $\mathcal{Q}$, $z\log(1-p^{-s}) = O_{R'}(p^{-s})$ and

$$\sum_{p\in\mathcal{Q}} p^{-s} \leq \sum_{p\in\mathcal{Q}} \frac{1}{p} \leq \frac{1}{2} + \frac{1}{\log 3}\sum_{p\in\mathcal{Q}} \frac{\log p}{p} < \infty \qquad (s \geq 1),$$

the comparison $1/p \leq (\log p)/(p\log 3)$ holding for every $p \geq 3$ and the term $p = 2$, which lies in $\mathcal{Q}$ for every $\mathcal{P}$ with $q \geq 3$ and for which the comparison fails since $\log 2 < 1$, being split off and bounded by $1/2$. So (77) gives $\sum_{p\in\mathcal{Q}} 1/p < \infty$, which is all that is used here. Both products therefore converge absolutely and locally uniformly, so each is analytic in z on $|z| < q^s$; and neither vanishes, because the convergence of the logarithms exhibits each as the exponential of a finite quantity.

(ii) The two estimates in (i), applied with $R' = R < 3 \leq q \leq q^s$, depend on s only through $p^{-s} \leq p^{-1}$, so

$$\sup_{s\geq 1}\sup_{|z|=R} |\log H_{\mathcal{P}}(s,z)| \leq C_R\sum_p p^{-2} + R\sum_{p\in\mathcal{Q}} \log\frac{1}{1-1/p} < \infty,$$

which bounds M; the bound on $h_j^{\mathcal{P}}$ is Cauchy's estimate on $|z| = R$.

(iii) Differentiating (79) logarithmically term by term,

$$\partial_s \log H_{\mathcal{P}}(s,z) = -\sum_{p\in\mathcal{P}} \log p \sum_{m\geq 2}(z^m - z)p^{-ms} + z\sum_{p\in\mathcal{Q}} \frac{\log p}{p^s - 1}. \tag{83}$$

For $s \geq 1$ and $|z| \leq R$ the first double sum is dominated termwise by $C_R(\log p)p^{-2}$ and the second by $R(\log p)/(p-1)$, summable over $\mathcal{Q}$ exactly by (77). Both bounds are independent of s, so by the Weierstrass M-test the differentiated series converges uniformly there and the differentiation is legitimate; its sum is continuous on $[1,\infty)\times\{|z| \leq R\}$, being a uniformly convergent series of continuous functions, and $\partial_s H_{\mathcal{P}} = H_{\mathcal{P}}\partial_s \log H_{\mathcal{P}}$ is bounded there. Since $[z^j]$ is a Cauchy integral over the circle $|z| = R$, which may be held fixed while s varies, differentiation under the integral sign gives $(h_j^{\mathcal{P}})'(s) = [z^j]\partial_s H_{\mathcal{P}}(s,z)$ exactly as in (10), and Cauchy's estimate then bounds it by $M'R^{-j}$. At $z = 2$ the inner sum in (83) is

$$\sum_{m\geq 2}(2^m - 2)p^{-m} = \frac{4}{p(p-2)} - \frac{2}{p(p-1)} = \frac{2}{(p-1)(p-2)},$$

which gives (81) after halving.

(iv) The plan is the one of Section 3, carried out with $\mathcal{P}$ in place of the odd primes. Pass to the variable $u = \log\zeta(s)$, where the measure $J(u)\,\mathrm{d}u$ has the convergent expansion of Lemma 2.3; expand each $h_j^{\mathcal{P}}$ about $s = 1$ in three groups, its value, its derivative and a remainder; and read off the first two moments. The value gives the limit, the derivative gives the 2^{-k} coefficient, and everything else has to be shown to be $o(2^{-k})$.

Write $P_k^{\mathcal{P}}(s) = \sum n^{-s}$ over the n counted by $f_k^{\mathcal{P}}$. The proof of Lemma 2.1 used nothing about the index set beyond non-negativity of the summands, so it applies verbatim and gives $f_k^{\mathcal{P}} = \int_1^\infty P_k^{\mathcal{P}}(s)\,\mathrm{d}s$. Expanding $\zeta(s)^z = \mathrm{e}^{uz}$ with $u = \log\zeta(s)$ and taking the Cauchy product with $H_{\mathcal{P}}(s,z) = \sum_j h_j^{\mathcal{P}}(s)z^j$ gives $P_k^{\mathcal{P}}(s) = \sum_{j=0}^k h_j^{\mathcal{P}}(s)u^{k-j}/(k-j)!$, which is (13) with $\mathcal{P}$ in place of the odd primes; changing variables to u as in (14), by Lemma 2.3, then gives

$$f_k^{\mathcal{P}} = \int_0^\infty \sum_{j=0}^{k} h_j^{\mathcal{P}}(s(u))\,\frac{u^{k-j}}{(k-j)!}\,J(u)\,\mathrm{d}u.$$

Split the integral at $u = 1$ and write $s_1 = s(1)$, so that $s_1 > 1$. The point of splitting there is that the expansions of Lemma 2.3 are only useful for u bounded away from 0, while for small u a crude bound suffices. Indeed the range $u \in (0,1]$ is $s \ge s_1$, where the contribution is $\int_{s_1}^\infty P_k^{\mathcal{P}}(s)\,\mathrm{d}s$ and may be bounded without any of the above, simply because the integers involved are large: every n counted by $P_k^{\mathcal{P}}$ has $n \ge q^k \ge 3^k$, so $P_k^{\mathcal{P}}(s) \le 3^{-k(s-s_1)}P_k^{\mathcal{P}}(s_1)$ for $s \ge s_1$, whence

$$\int_{s_1}^\infty P_k^{\mathcal{P}}(s)\,\mathrm{d}s \le \frac{P_k^{\mathcal{P}}(s_1)}{k\log 3} \le \frac{2^{-k}}{k\log 3}\prod_{p\in\mathcal{P}}(1-2p^{-s_1})^{-1} = O\Big(\frac{2^{-k}}{k}\Big) = o(2^{-k}),$$

the middle step because the generating function (7) for $\mathcal{P}$ has non-negative coefficients, so $P_k^{\mathcal{P}}(s_1) \le 2^{-k}\sum_j P_j^{\mathcal{P}}(s_1)2^j$, and the product converges because $2 < 3^{s_1} \le q^{s_1}$. On $[1,\infty)$ the expansions (12) apply with remainders that are $O(\mathrm{e}^{-3u})$ *uniformly* there: each remainder times e^{3u} is continuous on $[1,\infty)$ and bounded as $u \to \infty$ by Lemma 2.3, hence bounded on $[1,\infty)$. With $\int_0^\infty \frac{u^m}{m!}\mathrm{e}^{-au}\,\mathrm{d}u = a^{-(m+1)}$,

$$\int_1^\infty \frac{u^m}{m!}J\,\mathrm{d}u = 1 + \gamma 2^{-m} + O(3^{-m}), \qquad \int_1^\infty \frac{u^m}{m!}\varepsilon J\,\mathrm{d}u = \tfrac{1}{2}2^{-m} + O(3^{-m}), \tag{84}$$

the restriction to $[1,\infty)$ costing at most $\int_0^1 u^m/m!\,\mathrm{d}u \le 1/(m+1)!$ in each, which is smaller than the error terms already present and is absorbed into them throughout. Now expand the coefficients. Write

$$h_j^{\mathcal{P}}(s(u)) = h_j^{\mathcal{P}}(1) + \varepsilon(u)\,(h_j^{\mathcal{P}})'(1) + r_j(u), \qquad |r_j(u)| \le \varepsilon(u)\,\Omega(\varepsilon(u))\,R^{-j},$$

where

$$\Omega(\delta) = \sup_{|z|=R}\ \sup_{1\le s\le 1+\delta} |\partial_s H_{\mathcal{P}}(s,z) - \partial_s H_{\mathcal{P}}(1,z)|.$$

Here $r_j(u) = \int_1^{s(u)}\big((h_j^{\mathcal{P}})'(t) - (h_j^{\mathcal{P}})'(1)\big)\,\mathrm{d}t$, and Cauchy's estimate on $|z| = R$ applied to $\partial_s H_{\mathcal{P}}(t,\cdot) - \partial_s H_{\mathcal{P}}(1,\cdot)$ bounds the integrand by $\Omega(t-1)R^{-j}$, which gives the stated bound on r_j since Ω is non-decreasing and $s(u) - 1 = \varepsilon(u)$. Moreover $\Omega(\delta) \to 0$ as $\delta \to 0$: by (iii) the function $\partial_s H_{\mathcal{P}}$ is continuous on $[1,\infty)\times\{|z| \le R\}$, hence uniformly continuous on the compact set $[1,2]\times\{|z| = R\}$, which covers $\delta \le 1$. The three groups now contribute as follows. Two sources of error recur in each, and both are controlled by the position of R: a coefficient tail $\sum_{j>k}$, which needs $R > 2$, and the $O(3^{-m})$ remainders of (84), which need $R < 3$.

First group, the values $h_j^{\mathcal{P}}(1)$. By (ii) the tails $\sum_{j>k}$ of $\sum_j h_j^{\mathcal{P}}(1)$ and of $2^{-k}\sum_j h_j^{\mathcal{P}}(1)2^j$ are $O(R^{-k})$ and $O(2^{-k}(2/R)^k)$, both $o(2^{-k})$ since $R > 2$. The $O(3^{-m})$ terms contribute

$O\big(3^{-k}\sum_{j\le k} MR^{-j}3^j\big) = O(R^{-k}) = o(2^{-k})$, where the middle equality is where $R < 3$ is used: it makes the geometric sum dominated by its last term. So this group gives $H_{\mathcal{P}}(1,1) + \gamma H_{\mathcal{P}}(1,2)2^{-k} + o(2^{-k})$.

Second group, the derivatives $(h_j^{\mathcal{P}})'(1)$. The same argument with (iii) in place of (ii). The tail of $2^{-k}\sum_j (h_j^{\mathcal{P}})'(1)2^j$ is $O(2^{-k}(2/R)^k)$ and the $O(3^{-m})$ terms are $O(R^{-k})$, so the group gives $\frac{1}{2}\partial_s H_{\mathcal{P}}(1,2)2^{-k} + o(2^{-k})$. Here $\sum_j (h_j^{\mathcal{P}})'(1)2^j$ converges to $\partial_s H_{\mathcal{P}}(1,2)$, again because $R > 2$.

Third group, the remainders r_j. This is the only place where Ω is needed, and the only ε-argument in the proof. Let $\delta > 0$ and choose U with $\Omega(\varepsilon(U)) < \delta$, possible since $\varepsilon(U) \to 0$. On $[U,\infty)$ the terms are at most δR^{-j} times the second integral of (84), contributing $O(\delta 2^{-k})$. On $[1,U]$ the integrand is $O_U(1)$ times $u^{k-j}/(k-j)!$, and $\int_1^U u^m/m!\,\mathrm{d}u \le U^{m+1}/(m+1)!$, so that range contributes $O_U\big(\sum_{j\le k} R^{-j}U^{k-j+1}/(k-j+1)!\big) = O_U(R^{-k}) = o(2^{-k})$. As δ was arbitrary, the group is $o(2^{-k})$.

Collecting the two surviving coefficients gives $c_{\mathcal{P}} = \gamma H_{\mathcal{P}}(1,2) + \frac{1}{2}\partial_s H_{\mathcal{P}}(1,2)$, which is (82). For the limit, $L_p(s,1) = 1$, so $H_{\mathcal{P}}(s,1) = \prod_{p\in\mathcal{Q}}(1-p^{-s})$ by (79). The stated product for $H_{\mathcal{P}}(1,2)$ is (79) at $(s,z) = (1,2)$; its factors are positive because $p \ge q \ge 3$ on $\mathcal{P}$, and it is non-zero by the convergence in (i) at $(s,z) = (1,2)$, which is legitimate since $2 < 3 \le q$. Positivity of the factors alone would not give this. □

By (82) it is positivity of $c_{\mathcal{P}}$ that makes the sequence decrease, and that is now a corollary rather than an open point.

Corollary 7.3. *$c_{\mathcal{P}} > 0$ for every $\mathcal{P}$ with $q \ge 3$ satisfying (77); indeed $c_{\mathcal{P}} \ge H_{\mathcal{P}}(1,2)\,\beta/d$, and $f_k^{\mathcal{P}}$ is eventually strictly decreasing.*

Proof. Put $a_p = \frac{\log p}{(p-1)(p-2)} \ge 0$ and $b_p = \frac{\log p}{p-1} \ge 0$ for $p \ge 3$. Since $q \ge 3$ we have $2 \in \mathcal{Q}$ and $\mathcal{P} = \{p \ge 3\} \setminus (\mathcal{Q} \setminus \{2\})$, with $b_2 = \log 2$, so for every admissible $\mathcal{Q}$, finite or infinite, (81) rearranges exactly as

$$\tfrac{1}{2}\partial_s \log H_{\mathcal{P}}(1,2) = \log 2 - \sum_{p\ge 3} a_p + \sum_{p\in\mathcal{Q}\setminus\{2\}} (a_p + b_p),$$

all three sums converging by (77). The last sum has non-negative terms, so the right side is least when it is empty, that is at $\mathcal{Q} = \{2\}$ and $\mathcal{P} = \{p : p \ge 3\}$, where $\gamma + \frac{1}{2}\partial_s \log H_{\mathcal{P}}(1,2)$ equals $\gamma + \log 2 - \sum_{p\ge 3} \frac{\log p}{(p-1)(p-2)} = \beta/d = 0.410970\ldots$, positive by (5). Since $H_{\mathcal{P}}(1,2) > 0$ by Proposition 7.2(iv), $c_{\mathcal{P}} > 0$, and (82) gives $f_k^{\mathcal{P}} - f_{k+1}^{\mathcal{P}} = c_{\mathcal{P}} 2^{-k-1} + o(2^{-k}) > 0$ for all large k. □

So the single inequality $\beta > 0$ of (5) settles the whole interior case. In particular Corollary 7.3 is one of the two implications of Conjecture 7.1: it shows that $f_k^{\mathcal{P}}$ is eventually strictly decreasing whenever $q \ge 3$, so an admissible $\mathcal{P}$ with $f_k^{\mathcal{P}}$ eventually increasing must have $q = 2$. What remains is the converse, and the boundary case $q = 2$ lies outside Proposition 7.2, whose parts (ii), (iii) and (iv) all require $R > 2$ and so an admissible $R < q$; part (i) needs neither, and (79) itself is a definition for any $\mathcal{P}$ whatever.

The constants of Proposition 7.2(iv) have been checked numerically, and the two checks should not be conflated. The constant *formulas* specialize correctly: at $\mathcal{Q} = \{2\}$ they return $H_{\mathcal{P}}(1,2) = 0.378695020$ against $d = 0.3786950315$ and $c_{\mathcal{P}} = 0.155632714$ against β, both to about 10^{-7}, the precision of the prime sums involved. The *sequence* $f_k^{\mathcal{P}}$ approaches its asymptotic far more slowly, and how slowly depends on q: at $\mathcal{Q} = \{2\}$, where the subleading term is the $O(k^3 3^{-k})$ of (4) with constant 4900, the ratio of $(f_k^{\mathcal{P}} - c_0)2^k$ to $c_{\mathcal{P}}$ is still 1.13 at $k = 14$, the largest index computed, while at $\mathcal{Q} = \{2,3\}$, where $q = 5$, it is 1.003. So the $q \ge 5$

runs test (82) end to end and $\mathcal{Q} = \{2\}$ calibrates the formulas against a proved case. The computation is `genP_check.py`; alone among the computations here it uses ordinary floating-point rather than ball arithmetic, carries no certificate, and nothing else in the paper depends on it.

The boundary case $q = 2$ is a different problem, and we do not treat it here. Separating the factor L_2 in (79) gives the exact identity $H_{\mathcal{P}} = H_{\mathcal{P}\setminus\{2\}}/(1 - z2^{-s})$, so the entire z-singularity at $z = 2^s$ is one explicit pole whose numerator belongs to a prime set with least element at least 3. Proposition 7.2 therefore applies to the numerator, and in that sense the crux of the boundary case is an interior object; the k^2 of (2) comes from the pole. Whether the resulting coefficient depends on that numerator only through its value at the critical point, which would give Conjecture 7.1, we have been unable to settle, and we will return to it separately.

8. Certification

The numerical propositions are Proposition 3.3 and Proposition 5.6. Their statements are finite systems of interval inequalities, each established in ball arithmetic by Arb/FLINT [13] through its Python bindings, so that every enclosure carries a rigorous error radius. The scripts, their logs and the certificate are in the repository cited below, each script recording its settings in its docstring.

What ball arithmetic certifies is the computation. Transcribing an enclosure into a printed numeral, and combining several into a displayed chain, is ordinary writing, and that is where the errors in this manuscript have been found. A late adversarial pass turned up three: a stated bound rounded down rather than up, so that the text asserted what its own certificate did not establish; the two-part assembly of Σ_2, whose displayed chain did not close under its stated value; and a contour bound attached to the wrong function, the certified 101.245 bounding the pole-free part $G_2(1, z) - \rho/(3 - z)$, as every consuming script had always assumed, and not the regularized product, whose supremum on $|z| = 4.2$ is 228.95. None invalidated a conclusion, and each is now checked by script rather than by eye. `check_proof_spine.py` compares every printed numeral against the log that certifies it, failing if a stated bound falls below it; `verify.py` checks the sweep tables and the Γ data against the statements they instantiate, checks the criterion sweeps for gaps in the union of their ranges, and recomputes the certificate's SHA-256. One input is not an enclosure but a sum over all primes: $\beta > 0$ and the base case (30) both reduce to an upper bound on $S = \sum_{p\geq 3}(\log p)/((p-1)(p-2))$, certified in `odd_beta_certified.py` as a partial sum over $p \leq 10^8$ plus an integral tail, with d bounded below by a partial product, every Euler factor of d exceeding 1 so that no tail is needed.

Two constants are stated above more conservatively than the computation requires: the threshold 2440 of Lemma 6.2 in place of 2433, the first index at which φ is positive, and (71) in place of the sharper bound of Lemma 3.4(i). Both sharper values are recorded in the repository as headroom rather than substituted here, since neither changes the ranges covered and a printed numeral is only as good as the certificate behind it.

Σ_2 is not a single computed number but a sum of two, and since it carries 4395.55 of the 4857.88 in the ledger of Proposition 3.5 it is worth saying which. The head $j \leq 16$ is a direct interval recursion on the coefficients over 1600 boxes in s, giving 8423.91; a Cauchy estimate would be far lossier there, being dominated by the maximum over the circle and discarding the cancellation among the true coefficients. The tail $j > 16$ is where a Cauchy estimate is cheap: with $\sup_{|z|=4.5}$ of the pole-free part at most $1.9672165 \cdot 10^6$, the tail is at most that times $(3/4.5)^{17}(1 - 3/4.5)^{-1}$, which is 5989.94. The two sum to 14413.85, so the stated $\Sigma_2 \leq 14414$ holds with 0.15 to spare — the narrowest margin here except C_E. The 40 boxes in s behind the

tail are coarse enough that it is likely several-fold loose; that direction is safe, and the budget closes either way.

The constant C_E is certified on the whole of $(1, s(3)]$, and in fact on its closure. The four quantities entering (25) are ratios in which the pole of ζ at $s = 1$ cancels, so each extends continuously to $s = 1$. The Taylor model below is centred at $1 + E/2$ with radius $E/2$, and therefore covers the closed interval $[1, s(3)]$ on which Σ_2 takes its own supremum. Interval evaluation on boxes of s loses that cancellation, since each occurrence of $s - 1$ is widened independently, and the enclosures diverge however finely $(1, s(3)]$ is partitioned. Expanding ζ as a power series in $\sigma = s - s_0$ about $s_0 = 1 + E/2$, where $E = s(3) - 1$, makes the cancellation symbolic. The resulting series are evaluated on $|\sigma| \le E/2$, which is the whole range; their coefficients fall off geometrically, and truncating after nine terms contributes less than 10^{-20}. This is the one constant here with no headroom: the four suprema are attained at $s(3)$ and give $C_E = 5.4890955\ldots$, so the stated 5.48910 has about $4 \cdot 10^{-6}$ to spare, against 0.86% for the ledger. Any change to the Taylor-model settings must be rechecked against it.

The constant C_N requires $\Pi(t) = \sum_{p \ge 3} p^{-t}$ at $t = s, 2s, \ldots, 200s$ for $s = s(3)$. At $s \approx 1.05$ these converge too slowly to truncate: the Chebyshev-function bounds of Rosser and Schoenfeld [12] applied beyond 10^8 still overestimate the tail twofold, and that excess compounds through the coefficient recursion. Instead Π is evaluated by the Möbius series $\Pi(t) = \sum_{n \ge 1} \mu(n) n^{-1} \log \zeta(nt) - 2^{-t}$, needing no prime enumeration; this is the device of Cohen [7], as used in [2, (3.6)]. The recursion is run at 600 bits, cancellation costing about one digit every 2.3 steps over the 200 terms used; at 1200 bits the head agrees to every printed digit.

For the odd theorem, the coefficient bounds in (26) are obtained by cancelling the pole at $z = 3$ before applying coefficient recursion or Cauchy bounds. The range $0 < u \le 3$ is evaluated directly, while the range $u \ge 3$ uses the Taylor model made explicit in Proposition 3.1.

For the unrestricted theorem, the s-uniform coefficient bounds defining α_+ are obtained by carrying s as an interval over a partition of $[1, 2]$. The signed moments U and V are evaluated before absolute values are taken, for the reason given at (59).

Availability and versions. Since the contribution is in part a certification, the artefacts and the versions they were produced with are part of the claim. The scripts, their logs and the certificate are at `https://github.com/jaideepsaipadhi/erdos-sums-almost-primes`. The numerics use python-flint 0.9.0; the certificate's `versions` block records that binding's version but not the FLINT release under it, which should be added before submission. The certificate records the parameters of the *quadrature* run, namely $k_{\max}$, the precision, the panel geometry and the ellipse, together with a SHA-256 hash of the numerical payload, which `verify.py` recomputes. The coefficient sweeps' parameters are recorded separately, in `uv_constants.json` and `gamma_constant.json`, not in the certificate.

Appendix A. Certified quadrature

Since $1/(n \log n) = \int_1^\infty n^{-s}\,\mathrm{d}s$, we have $f_k = \int_1^\infty P_k(s)\,\mathrm{d}s$, and the substitution $s = 1 + \mathrm{e}^{-u}$ turns this into

$$f_k = \int_{-\infty}^{\infty} F(u)\,\mathrm{d}u, \qquad F(u) = P_k(1 + \mathrm{e}^{-u})\mathrm{e}^{-u}, \tag{85}$$

the integrand being analytic on the strip $|\mathrm{Im}\, u| < \pi/2$. The integral is split into a truncated range $[u_-, u_+]$ with $u_- = -5.10$ and $u_+ = 3.5 k_{\max}$, taken once for the largest k of the run and so uniform in k, and the two tails beyond it. The truncated range is divided into panels of width $H = 0.8$, each mapped affinely to $[-1, 1]$ and integrated by Gauss–Legendre quadrature

with $N = 52$ nodes at 1400 bits of precision; the number of panels therefore grows linearly with $k_{\max}$.

Four of the six error sources in the certificate are arithmetic: ball arithmetic throughout, the explicit truncation balls in the Möbius series [7] for the prime zeta function, the discretization bound derived below, and interval-Newton enclosures for the Gauss nodes and weights. The remaining two are the tails of (85), which are bounded as follows and are not negligible by inspection.

For the left tail, $u \le u_-$ corresponds to $s \ge s_- = 1 + \mathrm{e}^{5.10} = 165.02$. There $P_1(s) = \sum_p p^{-s} \le 2.78 \cdot 2^{-s}$, since $P_1(s)2^s = 1 + \sum_{p \ge 3} (2/p)^s$ decreases in s and is already below 2.78 at $s = 2$; and $P_k(s) \le P_1(s)^k$, because expanding $P_1(s)^k$ counts every n with $\Omega(n) = k$ at least once. Hence $P_k(s) \le 2.78^k 2^{-ks}$, and

$$\int_{-\infty}^{u_-} |F| \,\mathrm{d}u = \int_{s_-}^{\infty} P_k(s) \,\mathrm{d}s \le \frac{2.78^k 2^{-ks_-}}{k \log 2}. \tag{86}$$

For the right tail, $P_k(s) \le (\log \zeta(s))^k / k!$ for $s > 1$ [2, Lemma 5.6]. Substituting $u = \log \zeta(s)$ gives $\mathrm{d}s = J(u) \,\mathrm{d}u$ with $J = \varepsilon(1 + \gamma\varepsilon + \cdots)$ by Lemma 2.3, and $\varepsilon = 1/\zeta \cdot (1 + \gamma\varepsilon + \cdots) \ge \mathrm{e}^{-u}$, so $J \ge \mathrm{e}^{-u}$ rather than $\le$: the clean form $\int_{u_+}^{\infty} u^k \mathrm{e}^{-u} \,\mathrm{d}u/k!$ is not immediate. What is true is $J(u) \le \mathrm{e}^{-u}(1 + C\mathrm{e}^{-u})$ for $u \ge 1$ with C absolute, and at $u \ge u_+ = 1750$ the correction is $1 + O(\mathrm{e}^{-1750})$, absorbed by the factor 2 below with room to spare. With that, the integral is dominated by $2\int_{u_+}^{\infty} u^k \mathrm{e}^{-u} \,\mathrm{d}u/k!$ and the resulting incomplete-Gamma integral obeys $\Gamma(k+1, u_+)/k! \le 2u_+^k \mathrm{e}^{-u_+}/k!$ provided $u_+ \ge 2k + 2$ [9, §8.11], so

$$\int_{u_+}^{\infty} |F| \,\mathrm{d}u \le \frac{2u_+^k \mathrm{e}^{-u_+}}{k!}. \tag{87}$$

The hypothesis $u_+ \ge 2k + 2$ holds throughout, since $u_+ = 3.5k_{\max}$ and $k \le k_{\max}$. Both bounds are added to every reported enclosure. They are worst at opposite ends of the range, and at their worst they are far below the differences $f_{k+1} - f_k \asymp a_1 k^2 2^{-k}$ that the sweep has to resolve. The right-hand side of (87) increases in k, since u_+ is fixed and $u_+^k/k!$ is increasing while $k \le u_+$; that of (86) is $(2.78 \cdot 2^{-s_-})^k/(k \log 2)$ and so *decreases*, very fast. The run of record is the one the committed certificate holds, with $k_{\max} = 500$ and so $u_+ = 1750$ over 2194 panels. There (87) is below 10^{-272} at its worst, $k = 500$, and (86) below $2.5 \cdot 10^{-99}$ at its worst, $k = 2$, while the smallest difference in the range actually used, $6 \le k \le 199$, exceeds 10^{-57}.

For f analytic inside the Bernstein ellipse E_ρ and bounded there by M, the N-node Gauss rule satisfies

$$\left| \int_{-1}^{1} f(x) \,\mathrm{d}x - \widehat{I}_N \right| \le \frac{64}{15} \frac{M\rho^{-2N}}{1 - \rho^{-2}} \tag{88}$$

[5, Thm. 4.5], where the rule is indexed so that I_n uses $n + 1$ nodes and the stated exponent $\rho_B^{-(2n+2)}$ is the ρ_B^{-2N} above; the bound appears as [6, Thm. 19.3]. Here ρ_B is the Bernstein-ellipse parameter, unrelated to the residue ρ of (68). The implementation evaluates the right side of (88) with the interval-scaling factor $H/2$ and a further factor 10 of margin. The ellipse is taken with semi-minor axis $\delta = 1.45 < \pi/2$ in the u variable, giving $\rho_B \approx 7.385$, and M is bounded using $|P_k(s)| \le P_k(\operatorname{Re} s)$ together with the monotonicity of P_k on the reals.

The per-panel bound is not monotone in the panel's position, and it would be wrong to locate its maximum by inspection: on the right-hand side $F(u) \approx u^k \mathrm{e}^{-u}/k!$ by (85), which increases until $u \approx k$ and only then decays, so M rises across the bulk of the range before falling. What is used is therefore the maximum over all panels and all $k \le k_{\max}$, which the certificate records; the largest radius it reports is 4.6×10^{-87}, attained at $k = 2$, and so is negligible beside the

differences the sweep resolves. It also dominates both tails everywhere in the range: by some 185 orders of magnitude against (87), and by 12 at $k = 2$ against (86), where that tail is worst. So the discretization bound and not the truncation is what limits the accuracy of the f_k.

The pinned coefficient sweeps use primes up to 10^5 and 300 bits of precision, with 1600 boxes in s for α_+ and 800 boxes for the second-derivative sweep. These parameters are recorded in `uv_constants.json` and `gamma_constant.json` respectively, not in `certificate.json`, whose `parameters` block covers the quadrature run alone.

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Statement on the use of AI

During the preparation of this work the author used Claude Opus 5 (Anthropic), a large language model, in order to develop and check proofs, write and run the interval-arithmetic code, search and check the literature, and draft and edit the text. The proofs were checked by additional adversarial review passes of the model, and the numerical claims were certified in ball arithmetic by scripts available in the repository cited in Section 8. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the article.

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