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THE NECKLACE OF A RATIONAL RIGHT TRIANGLE: EXACT COUNTING AND COMPLETE PERIODICITY

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ABSTRACT. Let T be a right triangle with acute angle α, $2\alpha/\pi = P/Q$ in lowest terms. The Katok–Zemlyakov unfolding of T, quotiented by its holonomy involution, is a **necklace of Q kites**: an explicit genus-zero half-translation surface in the stratum $Q(P-2, Q-P-2, -1^Q)$, whose kites glue in steps of P and whose interfaces have explicit transverse measures. We use it to count the cells into which the vertices of T cut the beam perpendicular to each side. For a completely periodic direction class the separatrix diagram has exactly Q edges, and sorting them by the involution gives the identity $Q = h + f + 2p$ (h sides of T in the class, f extra cells of the perpendicular beams, p involution-swapped edge pairs). When $p = 0$ it is a **covering law**: the perpendicular beams of a class cut $Q-1$ or $Q+1$ cells in total at odd Q, according to the class, and Q at even Q; with no hypothesis the shortfall is even, non-negative, and zero only on a completely periodic class. Complete periodicity of a class — that almost every trajectory in it meets a side at a right angle — is equivalent to $p = 0$ and to the covering law (on two-beam classes off the Veech locus, one direction assumes an orphan branch, which companion papers prove at broad classes of centres); it fails at $8/15$. For the beam perpendicular to a leg the count $n(P/Q)$ is at most $Q-1$ at every centre and is known exactly on the Veech locus $\min(P, Q-P) \le 2$; off it we prove $n(3/Q) \le 2\cdot\mathrm{round}(Q/3)-1$ for $Q \ge 5$, with equality exactly under complete periodicity, and $n(5/(5m+1)) = 5$ for every $m \ge 3$. Finally, every periodic billiard path of a rational right triangle in a mirror direction meets a side at a right angle.

1. INTRODUCTION

1.1. **The question.** A billiard trajectory in a triangle that meets a side perpendicularly retraces its path. The set of such trajectories therefore organises the dynamics: fixing a side σ of a triangle T and shooting perpendicularly from each point of σ, one obtains a partition of σ into finitely or infinitely many *cells* — the maximal subintervals of σ on which the orbit's combinatorics (the sequence of sides it meets) is constant, equivalently the components of σ minus the launch points whose orbit meets a vertex. Two cells may carry the same orbit, launched from mirror points; they are counted separately. When T is rational the partition is finite, and one may ask for the number of cells,

$$n_\sigma := \text{the number of cells the perpendicular beam cuts on the side } \sigma,$$

as a function of the triangle. For a general rational triangle this is an unsolved counting problem. This paper solves it — as an *exact identity*, at every triangle in a family, and in a strong enough form to pin the individual counts — for **right** triangles, and it does so by producing an explicit model of the unfolded surface rather than by estimating anything.

The family is the following. Let T be a right triangle with acute angle α, legs L_1, L_2 and hypotenuse H, and write

$$2\alpha/\pi = P/Q, \quad \gcd(P,Q) = 1, \quad Q \geq 3.$$

Directions are measured in units of $\pi/2Q$, in which the three sides have directions $s(L_2) = 0$, $s(L_1) = Q$, $s(H) = P$; a *class* is a pair $(P/Q, \mathfrak{e})$ with $\mathfrak{e} \in \{0,1\}$ the direction parity.

There is no standing parity hypothesis: every statement below carries its own scope, and the scopes are collected in §1.5. The model of §2, the counting identity of §3, the interval model of §4, the rigidity theorems of §6.1, Theorem 6.7 on the pole prongs, the overlap theorem and cylinder count of §§5.2 and 7.2, the three sufficient conditions of §7.1, the bounds of §8 and the sweep law of §12 hold at **every** Q, both parities. What is developed at odd Q only is the **PAIRED/LONE dichotomy** — at even Q a side supplies both arcs of Fix(iota) or neither, so the two signatures there are $(h,v) = (1,1)$ and $(2,2)$ and the dichotomy does not exist — together with the statement that reads it, the Veech-locus case of $CP \implies p = 0$ (§6.3).

1.2. **The model: a necklace of kites.** Everything below is read off one picture: an explicit presentation of a surface Apisa identified [Api21, §1.2, §7], and making it explicit is the paper's first contribution.

Unfolding T in the sense of Katok–Zemlyakov [ZK75] gives a translation surface S_α; quotienting by the holonomy involution τ gives a half-translation surface $B = S_\alpha/\tau$, a meromorphic quadratic differential on $\mathbb{CP}^1$ in the stratum $Q(P-2, Q-P-2, -1^Q)$ (at $P \in \{2, Q-2\}$ the entry 0 is a marked point). Concretely (Proposition 2.1):

> B **is a cyclic chain of** Q **kites.** A kite is two copies of T glued along *both* legs — a bigon with corners 2α and $\pi - 2\alpha$ and one interior simple pole. The Q kites are glued along hypotenuse edges, all O-corners meeting at a single cone point of angle $P\pi$ and all A-corners at one of angle $(Q-P)\pi$.

The kite is where the right angle enters, and it enters as an equivalence rather than a convenience (§1.3). The chain carries an explicit transverse coordinate: the interface between the kites at positions t and $t+1$ has transverse measure $\mathrm{ell}_t = \sin(\pi c_t/2Q)$, with c_t running through one parity class modulo $2Q$ in steps of $2P$ (Proposition 2.2). Two features of the model are easy to get wrong. **The kites glue in steps of** P**, not of** 1**:** kite K_j has its hypotenuse edges at directions $\{P+2j, -P+2j\}$, so matching them forces $j' = j \pm P \pmod{Q}$, and the naive $+1$ reading is right only at $P = 1$

and $P = Q - 1$. And on the class $\mathfrak{e} = 1$ the interface directions carry an offset, without which every interface of that class is misstated.

The class involution iota reverses the necklace, so it acts as a dihedral reflection of the Q-cycle, and its axis meets the cycle in two places, each a kite or an interface. Writing $n_{\mathrm{fix}} \in \{0, 1, 2\}$ for the number of iota-**fixed kites** and m for the number of kites the quotient carries in its interior,

> $n_{\mathrm{fix}} + 2m = Q$, with $(n_{\mathrm{fix}}, m) = (1, (Q-1)/2)$ at odd Q; $(2, Q/2 - 1)$ at even Q with $\mathfrak{e} \not\equiv P \pmod 2$; and $(0, Q/2)$ at even Q with $\mathfrak{e} \equiv P \pmod 2$.

In every case the quotient $D = B/\mathrm{iota}$ is a *chain* of m whole kites, capped at each end either by a half-kite (at an iota-fixed kite, contributing a boundary simple pole) or by an iota-fixed interface. At odd Q — one cap of each kind — the phrase *one triangle with both legs on the boundary, followed by m kites in a row* is literally right, and it is the picture to hold. The first-return map of the class direction on this chain is a finite composition of interval maps, which we write down explicitly and show to close up (Theorem 4.1); the rest of the paper is the exploitation of that closed form.

1.3. **Why a right triangle: the hypothesis is not a simplification.** It is worth saying at the outset what the right angle buys, because the honest answer makes the hypothesis a feature of the construction rather than a restriction on it.

A kite is the triangle doubled along the two sides meeting at some angle γ, so that vertex becomes an **interior** point of cone angle 2γ. A half-translation surface admits only cone angles that are integer multiples of π — a quadratic-differential singularity of order $k \geq -1$ has cone angle $(k+2)\pi$ — so $2\gamma = m\pi$ with $0 < \gamma < \pi$ forces $\gamma = \pi/2$, and then $m = 1$: the vertex is a **simple pole**. *No other triangle makes this construction a half-translation surface at all.* The orders then sum to $(P-2)+(Q-P-2)-Q = -4$, which forces genus zero, by Gauss–Bonnet for quadratic differentials ([Orph] Proposition 7.1). So the right angle is what buys the sphere, and the sphere is what buys the explicit interval model. In the parametrisation $(r\pi/n, s\pi/n, t\pi/n)$ with $r+s+t = n$ used by Kenyon and Smillie [KS00], this family is $n = 2Q$, $(r, s, t) = (P, Q-P, Q)$, and the right angle is exactly the condition $t = n/2$.

What this does *not* say is that other rational triangles are intractable. The Katok–Zemlyakov unfolding always exists; what fails off the right-angled family is the *kite*, hence the necklace, hence the interval sequence. Indeed several of the results below use none of it: the counting identity of Theorem 3.4 is an involution count, the total-length identity of Theorem 3.9 is a boundary double count valid on any completely periodic surface, and the reversibility theorems of §6.1 read only that the return map is reversible and that the pole is the fold's fixed point. Each generalises once a separatrix diagram is supplied — and producing one off this family is the work.

1.4. **Relation to previous work.** The stratum this paper works in is a well-studied one. Three programmes — Masur–Veech volumes [AEZ16], configurations of saddle connections [Boi09], and the statistics of cylinder decompositions [DGZZ20a] — have been carried out over genus-zero strata of quadratic differentials, and all three are *asymptotic or statistical*: they compute averages over a stratum, growth rates, or almost-everywhere statements. This paper is exact and individual: it fixes one triangle and one direction class, and computes.

That contrast is not by itself what is new. Exact per-direction results are available for *lattice* (Veech) surfaces, where a Teichmüller curve and its cusps supply the rigidity; McMullen's analysis of billiards in regular polygons [McM25] is the sharpest such statement. What distinguishes the present results is the **object**: B is a lattice surface only on the locus $\min(P, Q-P) \leq 2$ [KS00, Thm 2], and the theorems below hold at every coprime (P, Q), with no lattice, no cusp and no congruence invariant available off that locus. The specialisation to right triangles (§1.3) is what makes this possible.

Saddle connections and separatrix diagrams. Boissy's classification of configurations of $\hat{h}$omologous saddle connections on $\mathbb{CP}^1$ [Boi09], the genus-zero case of Masur and Zorich's classification for quadratic differentials [MZ08] (itself the quadratic analogue of Eskin, Masur and Zorich's [EMZ03]), admits a connection joining two poles only in one configuration, accompanied by a longer closed connection that cuts off a half-pillowcase — the shape of the reduction step (ZZ) of §6.2. It is a resemblance and not an import: the classification assumes that every saddle connection parallel to a member of the collection is $\hat{h}$omologous to it, and in a completely periodic direction of our surfaces almost none are. The separatrix diagram of §3 is Kontsevich and Zorich's [KZ03], in the form of [DGZZ19, App. A]; a direction all of whose leaves are closed or saddle connections is a Jenkins–Strebel direction [Str84], and that is our hypothesis of complete periodicity. What [DGZZ19, DGZZ20a, DGZZ20b] do with these diagrams is count square-tiled surfaces — in genus zero in [DGZZ20a], in general strata in the other two — and ours are not square-tiled.

Counting cylinders. Naveh's and Lindsey's bounds on the number of invariant components [Nav08, Lin15] apply to S_α as a translation surface, unmarked — at $P = 1$, odd Q, Lindsey's bound gives $(Q-1)/2$ cylinders — but they do not see the marked right-angle images or the branch partition. Closer, for the regular n-gon with n odd, McMullen [McM25] shows that in every periodic direction the geodesics through the Weierstrass points are one zero connection and $(n-1)/2$ closed geodesics, with lengths proportional to $|\sin(2\pi\ell/n)|$; the decomposition into g cylinders of equal modulus goes back to Veech [Vee89]. On the locus $\min(P, Q-P) = 2$ this is $n = Q$, and $(n-1)/2$ is the value $\lfloor Q/2 \rfloor$ that §7.2 records for a lone class. **We regard this agreement as a check, and do not claim the shape of the count as new.** What is new is where it holds: off the lattice locus, where no cusp is available, uniformly in (P, Q); the lone/paired dichotomy,

and with it the branch $\lceil Q/2 \rceil$, has no counterpart in the lattice picture. Nor does McMullen's count supply our values on the locus: it counts trajectory types rather than branches of one beam (Proposition 5.1). The values in §10.2 are ours.

The chain of rhombi. The cell of §2 and its cyclic gluing are not new. Troubetzkoy reduces billiards in the triangle to billiards in the rhombus obtained by reflecting it in its legs [Tro05, §2]; Hooper describes the invariant surface of a right triangle as a chain of rhombi, each glued to its two neighbours, with removable singularities at their centres and an involution acting by a half-turn about each centre [Hoo07, §3]; and Cipra, Hanson and Kolan [CHK95] replace the right triangle by "a rhombus consisting of four copies of the triangle", for the reason §2 gives — the right-angle vertex is "a removable singularity" — and follow a trajectory through a chain of such rhombi. **Their width $|\sin((2n+1)\alpha)|$ for the interface between consecutive rhombi is Proposition 2.2's ell$_t$ at $\mathfrak{e} = 0$ — the same expression and the same object.** The rhombus is the lift of our kite to S_α: the kite is the rhombus folded by Hooper's half-turn. The quotient B itself is Apisa's — the pull-back of the pillowcase double of T under $z \mapsto z^Q$ [Api21, §1.2, §7]. What we add, beyond these, is the explicit kite-chain presentation of B — the gluing in steps of P, the transverse measures, and the class $\mathfrak{e} = 1$ with the offset of Proposition 2.2, which has no counterpart in the rhombus chain — and what it makes available: §3's counting identity and §6.1's rigidity. The earlier work on the perpendicular problem itself — numerical, irrational and elementary — is surveyed in [Orph] §1.5.

1.5. **What is conditional.** Two hypotheses appear, each carried on the face of every statement that uses it. **Complete periodicity (CP)** of the direction class is a genuine hypothesis: it fails at 8/15 (§11). **Hypothesis (OB)** (orphan branch) is the existence of an orphan branch: the $\perp L_1$ beam of $T(P'/Q)$ has one, where $P' \coloneqq P$ if P is odd and $P' \coloneqq Q - P$ if P is even (§5.2). It is consumed by Corollary 5.5, and through it in the proof of Corollary 7.11 — to separate the two zeros on the classes with two perpendicular beams off the Veech locus — and in Remark 7.12, Remark 8.3 and the last clause of Proposition 9.3. (OB) holds at every prime-power Q ([Orph] Theorem 4.11), when Q is twice or four times a prime and at the edges of every row ([Orph] Theorems 4.18, 4.24, Corollary 4.20), at $P' = 3$ (Corollary 10.9, and [Orph] Theorem 4.11 at $Q = 7$), at $P' = Q - 2$ (Theorem 10.3), at $P' = 5$, $Q \equiv 1 \pmod 5$, $Q \geq 16$ (Theorem 10.11), and at the centres of [Meth] Theorems 2–3: $\min(P, Q-P) \geq 2$ and either $\gcd(P \pm 2, Q) = 1$ or Q with at most one odd prime factor. It is conjectured everywhere ([Orph] Conjecture 4.12).

hypothesis	statements
none	§2; §4; Lemma 3.3; Proposition 3.6 (one direction through Theorem 7.6); §5.1; Theorem 5.4; Lemma 5.6; §6 except Theorem 6.5; Proposition 7.1, Lemma 7.2, Corollaries 7.3–7.5, Theorem 7.6, Corollaries 7.7, 7.8, Lemma 7.9; §8 except Remark 8.3; §9 except the last clause of Proposition 9.3; Theorems 10.2, 10.3, 10.5, 10.7, 10.11, Corollaries 10.8, 10.9; §12
(OB) alone	Corollary 5.5; the last clause of Proposition 9.3; Remark 8.3
CP	Theorems 3.1, 3.4, 3.9, Remark 3.11, Lemma 3.7, Proposition 3.12 (its $\kappa = 2$ clause under the hypotheses of Corollary 7.11); Theorem 7.10; Corollary 7.11 on the classes with one perpendicular beam and on the Veech locus; Remark 7.13; Theorem 7.15, Corollary 7.16
CP and (OB)	Corollary 7.11 on the classes with two perpendicular beams off the Veech locus, and what reads $p = 0$ there: Remark 7.12, §7.2 except Theorem 7.15 and Corollary 7.16
CP and (PP)	Theorem 6.5
$p = 0$ alone	the covering law (Corollary 3.5), since $p = 0$ implies CP (Theorem 7.6)
certificate	Proposition 11.1 (the count $n(8/15) = 6$, Appendix A.2; [DR22] corroborates); the counts of Remark 10.12 (Appendix A); the row $Q = 7$ of Theorem 10.7 and Corollary 10.9 (Appendix A.2); the inequalities of Theorems 10.7 and 10.11 (Appendix A.3)

At the centres of [Meth] Theorems 2–3 complete periodicity, $p = 0$ and (OB) all hold, so every row of the table holds outright there ([Meth] Corollaries 4.1 and 5.13; there $p = 0$ is Corollary 7.11 applied to [Meth]'s complete periodicity); [Meth] Corollary 7.15 adds further certified classes.

The computation enters only as a **certificate** at named centres — complete periodicity decided by the exact flow decomposition of Appendix A — or as data (§13).

The parity of Q. There is no standing parity hypothesis. The PAIRED/LONE labels exist only at odd Q: at even Q a side of T supplies both arcs of Fix(iota) or neither, so $h = v$ and the two signatures are $(1, 1)$ and $(2, 2)$. §6.3 is the one odd-Q-only section (apart from statements on rows where coprimality forces Q odd). Three statements have parity in their *content*, and quoting the odd-Q half alone is a scope error: the shape of D and the triple (n_{fix}, m) of §1.2; the right-hand side of the covering law, $Q \mp 1$ at odd Q and Q at even; and the cylinder count of Corollary 7.14, whose two branches collapse to $C = Q/2$ at even Q.

What this paper imports. From [Orph]: Theorem 4.6, the turnaround type of an orphan branch, and Theorem 4.7, that there is at most one; Theorems 4.11, 4.18, 4.24 and Corollary 4.20, existence of the orphan, as cases of (OB); Lemma 2.1, Theorem 2.2 and Lemma 4.2 (the retrace criterion and the orphan); Lemmas 3.1 and 4.9 and Proposition 3.4 (the development and finiteness); Corollary 4.14; Corollary 5.1 and Theorem 5.3 (parity, and the two ends of a double normal); the leg's avoidance of minimal components ([Orph] §2.1); and the cone data, Propositions 7.5–7.6, Lemma 7.8 and the conventions of [Orph] §§6–7 — each cited where used. [Orph] uses nothing from here, so the dependence runs one way. Nothing is imported from [MirA], which is deposited after this paper: the two results of it that Corollary 12.5 needs, [MirA] Corollary 4.1 and Proposition 4.3(1)–(4), are reproved as Lemma 12.4 from Hooper's theorem [Hoo07]. [MirA] is cited only for context, and [Meth] only for the status of hypotheses (§§1.5, 5.2, 8, 11, 13, Remarks 10.10, 12.6).

1.6. **Organisation.** §2 constructs the necklace and its coordinates. §3 proves the edge count, the counting identity and the covering law, identifies the edges of the separatrix diagram with the billiard prongs, and shows that each pole–pole edge pins a cylinder. §4 gives the interval model and proves that it closes. §5 compares the branch partition with the cylinder decomposition: the crossing number and the overlap theorem. §6 proves the rigidity theorems at a pole, reduces $p = 0$ to two statements, proves the first (no prong of an interior pole ends at a zero), and proves the second, (ZZ), on the Veech locus at odd Q. §7 proves three sufficient conditions for complete periodicity and, by one global edge count, $CP \iff p = 0$ at every centre, and draws the consequences of $p = 0$ for the cylinders. §8 bounds the count. §9 describes the involution on the cylinders. §10 computes: the whole lattice locus, the numerator three, and the family $n(5/(5m+1)) = 5$, $m \geq 3$. §11 treats the centres where complete periodicity fails. §12 draws the consequences for billiard paths in mirror directions, among them the sweep law. §13 lists what is open. Appendix A describes the computation.

2. The necklace

Throughout, $T = T(\alpha)$ is the right triangle of §1.1, with legs L_1, L_2 and hypotenuse H, and n_{fix}, m are as in §1.2, so that $n_{\text{fix}} + 2m = Q$ and $m = (Q-1)/2$ at odd Q.

Proposition 2.1 (the model of B)**.** *B is a cyclic chain of Q **kites**. A kite is two copies of T glued along both legs — a bigon with corners 2α and $\pi - 2\alpha$ and one interior simple pole R. The Q kites are glued cyclically along hypotenuse edges, all O-corners meeting at a single cone point Z_O of angle $Q \cdot 2\alpha = P\pi$ and all A-corners at a single cone point Z_A of angle $(Q-P)\pi$. The 1-skeleton is*

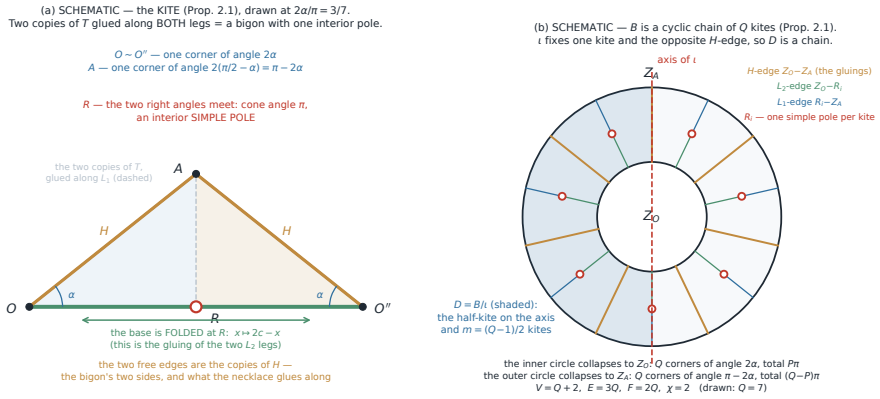


FIGURE 1. (a) The kite, built. Two copies of T glued along both legs develop as the isosceles triangle $OO''A$, whose base is then folded at R by $x \mapsto 2c - x$ — that fold is the gluing of the two L_2 legs. It makes R an interior point of cone angle π, identifies O with O'' into a single corner of angle 2α, and leaves the two copies of H free as the bigon's two edges. Drawn at $2\alpha/\pi = 3/7$. (b) B as the cyclic chain of Q kites, glued along those H-edges: the inner circle collapses to Z_O and the outer to Z_A. iota is the reflection in the dashed axis; D is one side of it. At the odd Q drawn here the axis fixes one H-edge and one kite, which is the case $n_{\mathrm{fix}} = 1$ of §1.2 — at even Q it fixes two of one kind and none of the other.

$V = \{Z_O, Z_A, R_1, \ldots, R_Q\}$, $E = Q$ *H-edges* Z_O–$Z_A \oplus Q$ L_1*-edges* Z_A–$R_i \oplus Q$ L_2*-edges* Z_O–R_i, $F = 2Q$.

Proof. The Katok–Zemlyakov surface S_α is the union of $4Q$ copies of T, indexed by the dihedral group generated by the reflections in the three sides, glued along the reflections. The holonomy involution τ is rotation by π, which acts freely on the index set and identifies each copy with its antipode; the quotient therefore has Q pairs of copies. Two copies related by reflection in a *leg* are identified in the quotient into a single bigon — the kite — whose leg edges have become interior; the L_1-fold and the L_2-fold of the pair meet at a point interior to the bigon, of total angle $2 \cdot (\pi/2) = \pi$, which is a simple pole. Reflection in the hypotenuse is what remains, and it glues the kites into a cycle — a single one, since by Proposition 2.2 below the neighbours of K_j are $K_{j\pm P}$ and $\gcd(P, Q) = 1$. Summing the angles at the O-corners of the Q kites gives $Q \cdot 2\alpha = P\pi$ and at the A-corners $Q \cdot (\pi - 2\alpha) = (Q - P)\pi$. Finally $V - E + F = (Q + 2) - 3Q + 2Q = 2$, confirming $g = 0$. □

The identification of the resulting quadratic differential with the stratum $Q(P-2, Q-P-2, -1^Q)$ is Apisa's [Api21].

The Z/Q deck rotation ρ rotates the necklace. The **class involution** iota of a class with direction ω is the involution of B induced by the reflection r_ω of S_α whose axis is perpendicular to ω (Lemma 5.6): r_ω commutes with τ, the rotation by π, so it descends to B, and it preserves the foliation of direction ω. Its fixed set Fix(iota) is the union of the side copies fixed by r_ω (perpendicular to ω) or by τr_ω (parallel to ω). iota is a dihedral reflection of the Z/Q-action. Its axis meets the cycle twice, at a kite or at an interface each time, so with n_{fix} and m as in §1.2:

> $D \coloneqq B/$**iota is a chain** of m whole kites, capped at each of its two ends by a half-kite (one triangle with both legs on ∂D) or by an iota-fixed interface, with n_{fix} caps of the first kind — and $n_{\text{fix}} + 2m = Q$.

D has m interior simple poles and n_{fix} further simple poles on ∂D, and its boundary corners have angles $P\pi/2$, $(Q-P)\pi/2$ and, at each half-kite cap, $\pi/2$. At odd Q, $n_{\text{fix}} = 1$: one cap of each kind, three boundary corners and $m = (Q-1)/2$. The two even-Q shapes — $n_{\text{fix}} = 2$, $m = Q/2 - 1$ when $\mathfrak{e} \not\equiv P \pmod 2$, and $n_{\text{fix}} = 0$, $m = Q/2$ when $\mathfrak{e} \equiv P$ — are the same chain with the caps exchanged, and no statement of §§2–4 or §6.1 reads which it has.

Proposition 2.2 (the gluing step, and the interface coordinate)**.** *Kite $K_j \coloneqq \rho^j K_0$ has its hypotenuse edges at directions $\{P+2j, -P+2j\}$ in the units of §1.1. Kites glue only along a shared direction, and $-P+2j' = P+2j$ forces*

$$j' = j \pm P \pmod Q\text{.}$$

Consequently, indexing kites by geometric position t along the chain — so that position t holds the kite K_{tP} — the interface between positions t and $t+1$ is the hypotenuse copy at direction $a + P(2t+1) \pmod{2Q}$, and its transverse measure is

$$\boldsymbol{ell}_t = |\sin((a + P(2t+1))\pi/2Q)| = \sin(\pi c_t/2Q)\text{,}\ c_t = (c_0 + 2Pt) \bmod 2Q\text{,}$$

with $a = 0$, $c_0 = P$ for $\mathfrak{e} = 0$; for $\mathfrak{e} = 1$, $a = Q$, $c_0 = P + Q$ at odd Q and $a = -P$, $c_0 = 0$ at even Q. The rule behind the choice: the interface directions have the parity of P and the class direction that of $\mathfrak{e}$, so c_t must have the parity of $P + \mathfrak{e}$. Any c_0 of that parity describes the same chain — relabelling t shifts c_0 by $2P$, which reaches every residue of the parity — but $P + Q$ has it only when Q is odd; at even Q it has the parity of P and would re-model the class $\mathfrak{e} = 0$. In each case $c_t \equiv u_t - \omega \pmod{2Q}$, where $u_t \coloneqq P(2t+1)$ is the direction of the interface's hypotenuse copy in the units of §1.1 and $\omega \coloneqq -a$ is a class direction ($\omega = 0$, Q, P in the three cases): ell_t is the sine of the angle between the interface and the class direction. The involution iota maps kites to kites and reverses the necklace,

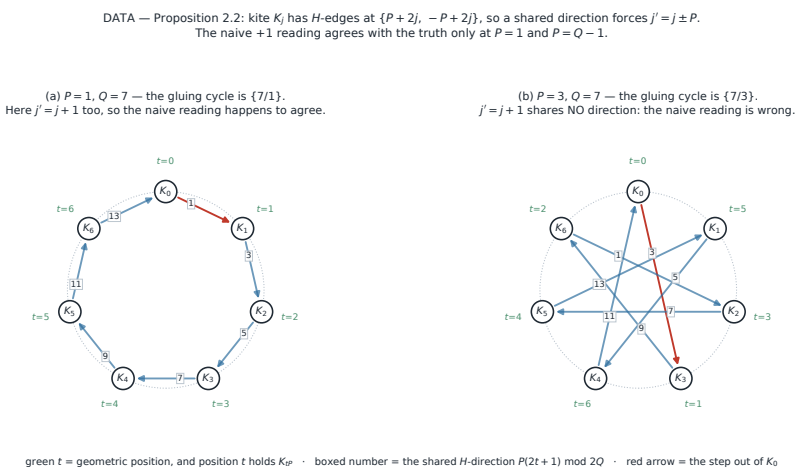


FIGURE 2. The gluing cycle on the kite indices, at $Q = 7$. Since K_j carries H-edges at $\{P + 2j, -P + 2j\}$, a shared direction forces $j' = j \pm P$, so the cycle is the star polygon $\{Q/P\}$ and geometric position t holds K_{tP}; the boxed number on each arrow is the shared direction $P(2t + 1) \bmod 2Q$. (a) $P = 1$: the cycle is the heptagon, and the naive $+1$ reading happens to agree. (b) $P = 3$: consecutive indices share no direction at all, and the naive reading is wrong at every step.

so it acts on interface positions by a reflection $t \mapsto r - t$; and r_ω maps a line of direction u to one of direction $2\omega - u$, so it carries the interface with c_t to the one with $\omega - u_t = -c_t$. Interface directions are distinct modulo $2Q$, so $c_{r-t} \equiv -c_t \pmod{2Q}$, i.e. $r \equiv -c_0 P^{-1} \pmod{Q}$: $t \mapsto -1 - t$ when $c_0 \equiv P \pmod{Q}$, and $t \mapsto -t$ when $c_0 \equiv 0$. Solving for its fixed points recovers the three shapes of §1.2: at odd Q the fixed kite is at position $t = 0$ and the fixed interface at $t = m$, and D consists of half of kite 0 together with kites $1, \dots, m$; at even Q with $\mathfrak{e} \not\equiv P \pmod 2$ the two fixed kites are at $t = 0$ and $t = Q/2$; and at even Q with $\mathfrak{e} \equiv P$ the two fixed interfaces are at $t = 0$ and $t = Q/2$. In all three, D carries m whole kites and the quotient interface sequence is a path of $m + 1$ intervals.

Proposition 2.3 (the per-kite maps)**.** *Fix a class direction. On each kite the fold lives on the **larger** of the kite's two interfaces, and the smaller interface is entirely a through band; which sub-interval of the larger the fold occupies is decided by which corner's angular sector contains the direction (O-sector versus A-sector): in the O-sector the two interfaces are aligned at their Z_A-ends and the fold occupies the Z_O-end of the larger, in the A-sector they are aligned at their Z_O-ends and the fold occupies the Z_A-end, and the kite at position k is in the O-sector iff $(-c_{k-1}) \bmod 2Q \in (0, 2P)$. The*

pole's single prong is the fold's ***fixed point****, and the corner's prong is the fold image of that corner. Two degeneracies are forced rather than imposed, and both are the arms of* $\mathrm{Fix}(\mathit{iota})$*:*

- *the* ***tie*** *— a kite whose two interfaces are equal, so that there is no fold — occurs exactly at the* n_{fix} *iota-fixed kites, hence at kite* 0 *alone when* Q *is odd; there the pole's prong runs along a diagonal of the kite into a corner;*
- *the empty through band occurs at the two kites flanking an iota-fixed interface whose measure degenerates — at odd* Q *that is the interface* m *on the LONE class, which is why* $h = 2$ *there.*

The cone angles $P\pi$ *and* $(Q-P)\pi$ *of Proposition 2.1 give* Z_O *exactly* P *prongs and* Z_A *exactly* $Q-P$ *in any fixed direction, and that is where those two counts come from.* ***They are not read off the sector split****, which is what one would expect from the previous paragraph and is a count of a different thing: the number of* O*-sector kites is* P *on the class* $\mathfrak{e} \neq P \bmod 2$ *but* $P-1$ *on the other, where the two degenerate kites above carry no fold to take a corner image of. The discrepancy is in the bookkeeping of the last prong, not in either count: on that class the degenerate interface is an* H*-edge parallel to the direction, a* Z_O*–*Z_A *saddle connection, and it carries the missing prong of each corner.*

Proof. Develop the kite's two copies of T as the isosceles triangle $\Delta = AOO'$ with R the midpoint of its base. The gluing of the second pair of legs, RO to RO', is the half-turn h about R: in S_α the copy T is glued along OR to $r_{OR}(T)$, and the holonomy involution τ identifies that copy with $r_{AR}(T)$ because $r_{AR}r_{OR} = h$, the legs being perpendicular. So the kite is the rhombus $Rh = \Delta \cup h(\Delta) = AOA^*O'$ modulo h. Its interfaces are the two pairs of opposite sides, $e_1 = [AO] = [A^*O']$ and $e_2 = [AO'] = [A^*O]$; its corners are $\{A, A^*\}$ (Z_A) and $\{O, O'\}$ (Z_O); and R, a simple pole, is its only interior singular point.

Let y be the coordinate normal to the direction, with $y(R) = 0$, so that h acts by $y \mapsto -y$, and put $a = y(A)$, $o = y(O)$. A chord of Rh at a generic level y joins the two sides whose projections contain y. If $|o| < |a|$ — after renaming, $0 \le o < a$ — the chords at $y \in (o, a)$ join AO to AO', crossing from e_1 to e_2 while preserving the transverse distance from the common vertex A (their h-images at $(-a, -o)$ are the same leaves), and the chords at $y \in (-o, o)$ join AO' to $A^*O = h(AO')$: a leaf entering through e_2 there leaves through e_2, at the point of AO' at level $-y$. So e_1, of measure $a-o$, is all through band; e_2, of measure $a+o$, is through on its Z_A-end and folds on the tail $(-o, o)$ at its Z_O-end, by reflection about the tail's midpoint $y(R)$. If $|a| < |o|$ — after renaming, $0 \le a < o$ — the chords at (a, o) join AO to A^*O, preserving distance from the common vertex O, and those at $(-a, a)$ join A^*O to $AO' = h(A^*O)$; now the tail $(-a, a)$ sits at the Z_A-end of e_2. (In Δ alone the two interfaces project to $[a, o]$ and $[-o, a]$, side by side, and every leaf bounces off the base; reading the bounce through h restores

the nesting.) Orienting every interface with its Z_O-end down, the first case aligns the two at their Z_A-ends and the second at their Z_O-ends.

The vertex O is not extreme in y iff the direction lies in the angle 2α of Rh at O, i.e. in the arc of directions of size 2α between the two interface directions; in units of $\pi/2Q$ these are c_{k-1} and $c_k \equiv c_{k-1} + 2P$ (Proposition 2.2), and since $2Q - 2P \neq 2P$ the arc of size $2P$ is $(c_{k-1}, c_{k-1} + 2P)$, which contains the direction 0 iff $(-c_{k-1}) \bmod 2Q \in (0, 2P)$. The level 0 runs through R and, unless the tail is empty, through the tail, at its midpoint; the level of the non-extreme corner is the fold image of the tail's corner end. The tail is empty iff $o = 0$ or $a = 0$ — the direction is a diagonal of Rh, and the pole's prong runs along it into a corner — and this is $|\sin(\pi c_{k-1}/2Q)| = |\sin(\pi c_k/2Q)|$, i.e. $c_k \equiv -c_{k-1} \pmod{2Q}$; since iota acts on positions by $t \mapsto r - t$ with $c_{r-t} \equiv -c_t$ and the c_t are distinct mod $2Q$, this holds iff iota swaps the kite's two interfaces. The through band is empty iff an interface has $c_t \equiv 0$, which forces $r - t \equiv t$. Finally, every corner prong enters exactly one kite in whose open sector the direction lies, or runs along an interface with $c_t \equiv 0$; the c_t fill the parity class of $c_0 \equiv P + \mathfrak{e}$, so counting residues in $(0, 2P)$ and $(2P, 2Q)$ gives the sector counts above and the last prong. □

3. The counting identity and the covering law

Write G for the separatrix diagram of the class on B: the union of the saddle connections in the class direction, viewed as a graph with the $Q + 2$ points of Proposition 2.1 as vertices — all of them marked, including Z_O at $P = 2$ and Z_A at $P = Q - 2$, where the cone angle is 2π. Put

- $h := \#\{\sigma : s(\sigma) \equiv \mathfrak{e} \pmod 2\}$ — the **horizontal** side-copies in Fix(iota), i.e. the sides of T whose own direction lies in the class;
- $v := \#\{\sigma : s(\sigma) \equiv \mathfrak{e} + Q \pmod 2\}$ — the **vertical** ones, i.e. the sides whose perpendicular beam lies in the class;
- $p :=$ the number of iota-**swapped** edge pairs of G.

At odd Q exactly one of the two $\mathfrak{e}$-classes has $v = 2$; that one is the **PAIRED** class ($h = 1$), and the other is **LONE** ($h = 2$, $v = 1$). The lone class is characterised by $\mathfrak{e} \equiv P \pmod 2$; geometrically, it is the class containing the hypotenuse's own direction.

> **The dichotomy is an odd-Q phenomenon.** At even Q a side supplies both arcs of Fix(iota) or neither, so $h = v$ and the two signatures are $(h, v) = (1, 1)$ and $(2, 2)$; the mixed signatures $(1, 2)$ and $(2, 1)$ do not occur, and the lone $\iff$ $\mathfrak{e} \equiv P$ label is h-inverted there. Statements below that name a class as PAIRED or LONE are therefore odd-Q statements, and apart from rows where coprimality alone forces Q odd (P even, or $P = Q - 2$) they are the only ones in the paper that are: everything phrased in h, v and p is parity-uniform.

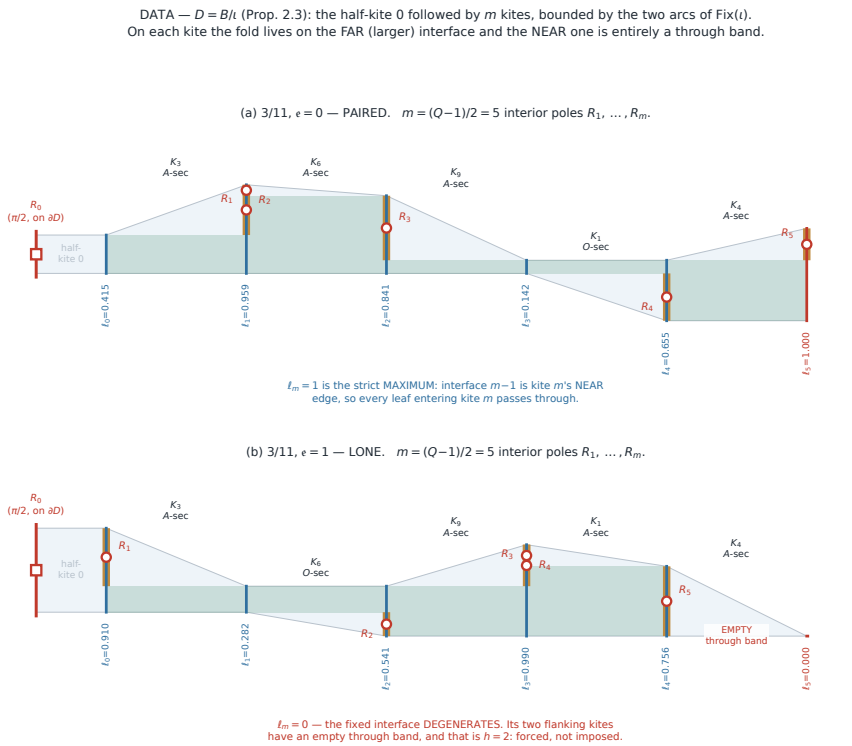


FIGURE 3. D at 3/11 — an odd Q, so $n_{\text{fix}} = 1$ — drawn in the transverse coordinate of Theorem 4.1 below: the half of the tie kite that survives iota, then kites $1, \ldots, m$ in geometric position, bounded on the right by the fixed interface m. Green is the through band — the intersection of the kite's two interfaces — and gold the fold, on the far (larger) one, with the pole at its fixed point. Since every kite touches both cone points, the strip is a drawing convention: the whole lower boundary is Z_O and the whole upper boundary is Z_A. (a) the class with $\text{ell}_m = 1$, its strict maximum, so no leaf can fold in kite m. (b) the class with $\text{ell}_m = 0$: the fixed interface degenerates, the two kites flanking it have an empty through band, and that is the second horizontal arm — $h = 2$.

Theorem 3.1 (the edge count)**.** *Let the class be completely periodic. Then G has exactly $E = Q$ edges, at every centre and in both classes; and the number of cylinders of the decomposition is*

$$C = (Q - b)/2 + \kappa - 1,$$

where b is the number of pole–zero edges of G and $\kappa \in \{1, 2\}$ is the number of components of G containing a zero. Here, and in Corollary 7.14, a ***zero*** *means one of the two named cone points Z_O, Z_A whatever its order, and a* ***pole*** *one of the R_i: at $P \in \{1, Q-1\}$ one named cone point has angle π, and the count below reads only the names; at $P \in \{2, Q-2\}$ the named cone point is a marked regular point, and C counts the cylinders of B with it marked (on the PAIRED class the unmarked count is $C-1$, Remark 3.11).*

Proof. The vertex set of G is that of Proposition 2.1: Q simple poles, each carrying one separatrix germ — which, the class being completely periodic, is a saddle connection — and hence of degree 1, together with Z_O of degree P and Z_A of degree $Q-P$. Summing degrees, $2E = P + (Q-P) + Q = 2Q$, so $E = Q$ and $V - E = (Q+2) - Q = 2$.

For the cylinder count, complete periodicity makes $B \setminus G$ a disjoint union of C open cylinders. Let G have k connected components. Each component G_i has a regular neighbourhood N_i, a compact planar surface with $\chi(N_i) = V_i - E_i$ and therefore $\partial_i = 2 - V_i + E_i$ boundary circles. Every boundary circle of an N_i is a boundary circle of exactly one cylinder, and each cylinder has two, so $2C = \sum_i (2 - V_i + E_i) = 2k - V + E$, i.e. $C = k - 1$.

It remains to count components. A pole is a leaf of G. A component containing no zero therefore has all degrees equal to 1, so it is a single edge joining two poles; write a for the number of such pole–pole edges, so that $k = a + \kappa$. Counting the degree of the poles, $2a + b = Q$, whence $a = (Q-b)/2$ and $C = (Q-b)/2 + \kappa - 1$. □

Remark 3.2. Complete periodicity is load-bearing here, and it enters everything downstream through this theorem. (A direction of a half-translation surface decomposes into cylinders and minimal components, bounded by saddle connections [Str84, §11.4], [MT02].) A minimal component makes the complement of the singular set non-annular, and neither the Euler count nor the component count survives.

The classification of edges that the counting identity rests on is local — it looks at one saddle connection at a time — and it uses no complete periodicity. Because §7.1 needs it in a setting where complete periodicity is the *conclusion* rather than the hypothesis, we isolate it first.

Lemma 3.3 (the trichotomy for a saddle connection, with no hypothesis)**.** Fix(*iota*) *consists of the h horizontal side-copies together with the v vertical ones. Let e be a saddle connection of the class direction. Then exactly one of the following holds.*

1. *e lies inside* Fix(*iota*)*, and is then one of the h* ***horizontal*** *arcs.*
2. *e is iota-invariant and does not lie inside* Fix(*iota*)*; then e meets* Fix(*iota*) *in exactly one interior point of e, its midpoint, and that point lies in the interior of a* ***vertical*** *arc.*
3. *e is not iota-invariant, and iota(e) is a second saddle connection distinct from it.*

Conversely, every point of the critical graph lying in the interior of a vertical arc lies on an edge of type (2).

Proof. A reflection of S_α fixes no triangle copy, so Fix(iota) is the union of the side-copies it contains, which are the h horizontal and v vertical ones: the copies of a side σ in K_j have directions $\pm s(\sigma) + 2j$, so σ has exactly one horizontal copy if $s(\sigma) \equiv \mathfrak{e}$ and exactly one vertical copy if $s(\sigma) \equiv \mathfrak{e} + Q \pmod 2$. The three cases are exhaustive and disjoint by definition. In case (1), e is a leaf of the class direction lying in Fix(iota), so it is one of the horizontal arcs — a vertical arc is transverse to the class direction. In case (2), iota maps e to itself without fixing it pointwise, and an involution of an arc that is not the identity reverses it and fixes exactly its midpoint among its interior points (a loop's endpoint is also fixed); that fixed point lies in Fix(iota), and it cannot lie on a horizontal arc, since that arc is itself a leaf and two leaves of a foliation do not cross. So it lies on a vertical arc, and, not being a vertex, in its interior. For the converse, a point of the critical graph interior to a vertical arc is not a vertex, so it lies on some edge e; then iota(e) is an edge through the same point in the same direction, and two leaves cannot cross, so iota(e) $= e$, which is case (2). □

Nothing in this lemma is conditional. It is a statement about one saddle connection, and it is what §7.1 consumes; what complete periodicity supplies to Theorem 3.4 below is only that the critical graph has Q edges in the first place, which is Theorem 3.1.

Theorem 3.4 (the counting identity)**.** *For a completely periodic class,*

$$Q = h + f + 2p\text{, }\textbf{\textit{where}}\ f \coloneqq \#(G \cap\ \textbf{\textit{open vertical arcs}}) = \sum_\sigma (n_\sigma - 1)\text{,}$$

and equivalently

$$\sum_{\sigma : s(\sigma) \equiv \mathfrak{e} + Q} n_\sigma = Q - h + v - 2p\text{.}$$

Proof. By Theorem 3.1 the critical graph has Q edges — this is the one place complete periodicity enters — and Lemma 3.3 classifies each of them: h of type (1), the type-(2) edges being in bijection with the points of G on the vertical arcs, of which there are f, and the type-(3) edges falling into iota-swapped pairs, of which there are p. Counting edges by type gives $Q = h + f + 2p$. Finally, the points of G on the vertical copy of a side σ are exactly the interior cell boundaries of the $\perp \sigma$ beam, so $f = \sum_\sigma (n_\sigma - 1)$ with the sum over the v vertical sides. □

Corollary 3.5 (the covering law — conditional on $p = 0$)**.** *(As stated this is a corollary of Theorem 3.4 and so appears to need CP as well; it does not. By Theorem 7.6, $p = 0 \implies CP$, so the single hypothesis $p = 0$ delivers both.) If $p = 0$, then*

$$\sum_{\sigma : s(\sigma) \equiv \mathfrak{e} + Q} n_\sigma = Q - h + v\text{,}$$

which is $Q-1$ on the LONE class and $Q+1$ on the PAIRED class at odd Q (and Q for the even-Q sibling). In particular, on the PAIRED class the two perpendicular beams satisfy $n_\sigma + n'_\sigma = Q+1$.

Proposition 3.6 (three faces of $p = 0$)**.** *For any class the following are equivalent:*

1. *$p = 0$;*
2. *every saddle connection of the class direction is iota-invariant;*
3. ***(M1)** every prong of the class — a billiard trajectory launched from a vertex of T into the interior along a class direction (Lemma 3.7) — meets a side of T perpendicularly at a point that is not a vertex, and therefore retraces itself and ends at the vertex where it started.*

Proof. (1) $\iff$ (2): by Lemma 3.3 every saddle connection is of type (1), (2) or (3); the first two are iota-invariant, and the type-(3) ones fall into the p swapped pairs. (3) $\implies$ (2): a saddle connection of the class direction projects to a billiard path from a vertex of T to a vertex with no vertex in between, so it is either a side-leaf, of type (1), or the lift of a prong. Under (3) that prong meets a side σ perpendicularly at a point that is not a vertex, so its lift passes through an interior point of the vertical copy of σ, and by the converse in Lemma 3.3 it is of type (2). (2) $\implies$ (3): by (1) and Theorem 7.6 the class is completely periodic. (Theorem 7.6 is proved in §7.1 from Lemma 3.3 and Corollaries 7.3–7.4 and does not use this proposition.) By Lemma 3.7 every prong then runs to a vertex and lifts to an edge of G that is not a side-leaf. By (2) that edge is iota-invariant, so it is of type (2) and meets a vertical arc at its midpoint, which is not a vertex of G. There the prong meets the side perpendicularly, so it reverses exactly and returns along itself to its launch vertex. □

Lemma 3.7 (what the edges are)**.** *For a completely periodic class, every edge of G is either the lift of a class **prong** — a billiard trajectory launched from a vertex of T into the interior along a class direction and run to its first return to a vertex — or a **side-leaf**, a side of T whose own direction lies in the class. The correspondence is a bijection orbit by orbit under time reversal (a self-reverse prong carries one edge, a reversal pair two), and the lift is a local isometry, so an edge's length is the prong's traced arc length ℓ_p, respectively the side's length $|\sigma|$.*

Proof. A prong **slot** is a pair (V, u_0) with u_0 an inward grid direction at the vertex V of the class parity. The slot supplies two of the ray-ends of G at V — the two lifts of its launch, exchanged by iota. A side-leaf supplies one ray-end at each of its two endpoints. So the two families account for $2N_V + b_V$ ray-ends at V, where N_V is the number of slots at V and b_V the number of sides incident to V that lie in the class, and we claim

$$2N_O + b_O = P, \quad 2N_A + b_A = Q - P, \quad 2N_R + b_R = Q,$$

which are exactly the ray degrees at Z_O, Z_A and at the poles computed in Theorem 3.1. Each is elementary. The inward window at O is the open

interval $(0, P)$, so $N_O = \#\{u \in (0,P) : u \equiv \mathfrak{e} \pmod 2\}$; the two sides at O have $s(L_2) = 0$ and $s(H) = P$, so the four cases of $(P \bmod 2, \mathfrak{e})$ give $b_O = 2, 0, 1, 1$ respectively, and the identity holds in each. The same computation at A (window $(2Q+P, 3Q)$, sides H and L_1) and at a pole R (window $(Q, 2Q)$, sides L_2 and L_1) gives the other two. Summing over the vertices, the two families exhaust all $2Q$ ray-ends. None of this uses complete periodicity. Under it, each launch runs to a vertex, and time reversal $s \mapsto s'$ (the launch at the far end of s's path; class directions are closed under reversal) is an involution on slots. If $s' \neq s$, the four ray-ends of s and s' pair into two edges, exchanged by iota; if $s' = s$, the path is its own reversal and its two ray-ends form one iota-invariant edge. Either way a segment carries as many edges as slots, each of the segment's length, so the two families account for all Q edges, bijectively. □

Remark 3.8. The resulting count $\#\text{prongs} + \#\text{side} - \text{leaves} = Q$ is Theorem 3.4 read in this classification: an edge inside Fix(iota) is exactly a side-leaf, so $\#\text{side} - \text{leaves} = h$ and $\#\text{prongs} = f + 2p$.

Theorem 3.9 (the edge-length identity)**.** *For a completely periodic class,*

$$\textstyle\sum_c \ \boldsymbol{circ}_c = \sum_{e \in G} \ell_e = \sum_{prongs} \ell_p + \sum_{\sigma : s(\sigma) \equiv \mathfrak{e}} |\sigma|,$$

the side-leaf sum running over exactly the h sides of §3 — at odd Q, the complement of the v sides whose perpendicular beam lies in the class.

Proof. Under complete periodicity the surface is the union of the closed cylinders, glued along the singular leaf set, which is the union of the edges of G. Every edge has exactly two sides; each side lies on exactly one boundary component of exactly one cylinder; and every boundary component of a cylinder c is a closed curve of length circ_c concatenated out of edge-sides. Counting the total boundary length in two ways,

$$\sum_c (|\partial^+ c| + |\partial^- c|) = 2 \sum_c \ \mathrm{circ}_c \quad \text{and} \quad \sum_c (|\partial^+ c| + |\partial^- c|) = \sum_e 2\ell_e.$$

The second equality of the statement is Lemma 3.7. □

Remark 3.10. A fold at a simple pole puts **both** sides of one edge on the same boundary component; the count is untouched. This is not a corner case — every L_1- and L_2-side-leaf folds, at its pole endpoint. Note also that the theorem needs neither $p = 0$ nor (M1): the boundary-length count does not read which vertex an edge lands on.

Remark 3.11 (the unmarked surface). Theorem 3.1 counts the cylinders of B with every vertex of G marked. On the unmarked surface a boundary component all of whose vertices are regular is a regular closed leaf, and the two cylinders it separated merge. A vertex is regular on B iff its order is 0, and the orders are $\mathrm{ord}(Z_O) = P-2$, $\mathrm{ord}(Z_A) = Q-P-2$ and $\mathrm{ord}(R_i) = -1$; so this can happen only at $P = 2$ or $P = Q-2$, and the width-2 inward window there carries a prong only in the PAIRED class. That prong returns to the named point: these rows lie on the Veech locus, where Corollary 7.11

gives $p = 0$, hence (M1) of Proposition 3.6. So a B-regular leaf exists on exactly the rows with $P \in \{2, Q-2\}$ in the PAIRED class, and there the two cylinders it separates have equal circumference — the length of the leaf — and merge into one. At $P \in \{1, Q-1\}$ the named point is a simple pole, and there is no merge.

Theorem 3.9 matches the two sides in aggregate. On the pole–pole edges a much finer statement holds: each one individually *is* a circumference.

Proposition 3.12 (the pinning — conditional on CP)**.** *Let the class be completely periodic and let e be one of the a pole–pole edges of G. Then e bounds a single cylinder $c(e)$, of circumference*

$$\boldsymbol{circ}_{c(e)} = 2\ell_e\boldsymbol{,}$$

*and the assignment $e \mapsto c(e)$ is injective. Consequently $C - a = \kappa - 1$: on a class with $\kappa = 1$ (the LONE class at odd Q, and $\mathfrak{e} = 1$ at even Q) the multiset of cylinder circumferences is exactly $\{2\ell_e : e \text{ pole–pole}\}$, and on a class with $\kappa = 2$ (the classes with two perpendicular beams, under the hypotheses of Corollary 7.11; Remark 7.12) it is that multiset together with **exactly one** further cylinder.*

Proof. (i) A simple pole has degree 1, so a pole–pole edge is a whole component of G — this is the component count already made in the proof of Theorem 3.1. As a plane graph it has $V = 2$ and $E = 1$, so its regular neighbourhood has $2 - V + E = 1$ boundary circle, which therefore traverses e on both sides and has length $2\ell_e$. (This is Remark 3.10's fold, seen from the other side.)

(ii) Every boundary circle of the regular neighbourhood of G is a boundary component of exactly one cylinder, and every boundary component of a cylinder c is a closed curve of length circ_c — both steps of the count in the proof of Theorem 3.9. Hence $\mathrm{circ}_{c(e)} = 2\ell_e$.

(iii) Injectivity is *forced*, not observed. Each circle of (i) is one boundary circle of $c(e)$, so $e \mapsto c(e)$ fails to be injective only if some cylinder c has both of its boundary circles produced by pole–pole edges $e \neq e'$. Then the closure $c \cup e \cup e'$ is open in B: at an interior point of e both sides lie in c, since the circle traverses e on both sides, and at a pole the punctured disc cut along its one germ is a single sector lying in c (Proposition 7.1's fold). Being also closed, it is all of the connected surface B; but it contains no zero, while Z_O and Z_A are vertices of G. Hence the assignment is injective.

(iv) With $C = k - 1$ and $k = a + \kappa$ from the proof of Theorem 3.1, $C - a = \kappa - 1$, which is 0 when $\kappa = 1$ — on the LONE class the side-leaf H joins the two zeros, and at even Q, $\mathfrak{e} = 1$, the horizontal copy of H does (one of the two H-copies of the bigon Fix(iota), proof of Corollary 7.4, is horizontal) — and 1 when $\kappa = 2$. □

$\kappa = 2$ on the classes with two perpendicular beams is proved in §7.1 (Corollary 7.11, Remark 7.12).

The hypothesis is CP alone: the proposition does not use (ZZ), and it does not use $p = 0$. It is strictly stronger than Theorem 3.9 on this set of edges, and it is the form in which the identity is used later — in Theorem 7.15, where what is needed is not the *length* match but the fact that e **bounds** $c(e)$, so that the leaves of $c(e)$ are the trajectories parallel to e and inherit its word.

4. The interval model, and that it closes

In the coordinate measured from the Z_O end, Proposition 2.3's maps shift the transverse coordinate by $M - m'$ on a through crossing in an O-sector and by 0 in an A-sector, where $M \geq m'$ are the two interface measures of the kite. That says exactly that a kite's two interfaces are **aligned** — they share their Z_A end in an O-sector and their Z_O end in an A-sector. Laying every interface on a single line as $I_t = [u_t, u_t + \ \mathrm{ell}_t]$ by that rule makes consecutive interfaces **nested**, the through band their intersection, and the fold the reflection of the larger interval's protruding tail about its own midpoint. Hence a leaf's coordinate is **constant between folds**: the leaf traverses the maximal arc of $S(y) = \{t : y \in I_t\}$ through its start, then reflects and reverses.

This construction needs the offsets u_t to close up around the cycle. They do.

Theorem 4.1 (the global coordinate closes)**.** *With $c_t = (c_0 + 2Pt) \bmod 2Q$, $ell_t = \sin(\pi c_t/2Q)$ and $osec[k] = [(-c_{k-1}) \bmod 2Q \in (0, 2P)]$,*

$$S := \textstyle\sum_{k \in \mathbb{Z}/Q} \ \boldsymbol{osec}[k] \cdot (\boldsymbol{ell}_{k-1} - \ \boldsymbol{ell}_k) = 0,$$

*at every $Q \geq 3$ — **both parities** — every coprime P and either $\mathfrak{e}$.*

Proof. Kite k imposes $u_{\mathrm{near}} - u_{\mathrm{far}} = \ \mathrm{osec}[k] \cdot \mathrm{delta}[k]$ with $\mathrm{delta}[k] = \ \mathrm{ell}_{\mathrm{far}} - \ \mathrm{ell}_{\mathrm{near}}$, which reads $u_h i + \ \mathrm{ell}_h i = u_l o + \ \mathrm{ell}_l o$ in the O-sector and $u_h i = u_l o$ in the A-sector. Both are symmetric in far/near, so the only place at which the *order* of the two lengths enters drops out, and the cycle telescopes to the single identity above. Now:

1. reindexing $t = k - 1$ gives $S = \sum_t w_t(\mathrm{ell}_t - \ \mathrm{ell}_{t+1})$;
2. the window is $w_t = 1 \iff 2Q - 2P < c_t < 2Q$;
3. $c_{t+1} = (c_t + 2P) \bmod 2Q$ for every t, the wrap included, since $2PQ \equiv 0 \pmod{2Q}$;
4. the reduction mod $2Q$ therefore bites exactly on the support of w and flips a sign: $w_t = 1$ gives $c_t + 2P \in (2Q, 2Q + 2P)$, so $\mathrm{ell}_{t+1} = -\sin(\pi(c_t + 2P)/2Q)$ and the summand becomes a *sum*, $\mathrm{ell}_t - \ \mathrm{ell}_{t+1} = 2\cos(\pi P/2Q) \cdot \sin(\pi(c_t + P)/2Q)$;
5. $\gcd(2P, 2Q) = 2$ makes $\{c_t\}$ the full parity class mod $2Q$, so the support is $\{2Q - r : 0 < r < 2P, r \equiv c_0 \pmod 2\}$ and $\sin(\pi(c_t + P)/2Q) = -\sin(\pi(P - r)/2Q)$;

6. with $s = P - r$ the index set is $\{s : |s| \le P-1, s \equiv P - c_0 \pmod 2\}$, symmetric under $s \mapsto -s$, and $\sin(\pi s/2Q)$ is odd, so the sum cancels in pairs.

□

The coincidence behind the proof is that **the osec window half-width and the gluing step are the same number $2P$.** That is exactly why the support of w is precisely the set of indices that wrap at step 4.

The proof uses neither the parity of Q nor which $\mathfrak{e}$ produced c_0 — only $\gcd(P, Q) = 1$ and $0 < P < Q$ — so the closure is an arithmetic fact about the gluing rule rather than a fact about the billiard, and it is that width which delivers the even-Q half of the statement. That extra width is not vacuity: each perturbation of the rule breaks the identity in general, including replacing the sector test by a shift or by a parity, changing the gluing step from P to $P+2$, and widening or narrowing the window to $P \pm 1$.

Corollary 4.2 (the quotient path)**.** *With $\rho(t) = r - t$ the reflection of $\mathbb{Z}/Q$ induced by iota (Proposition 2.2: $\rho(t) = -1 - t$ when $c_0 \equiv P \pmod Q$, $\rho(t) = -t$ when $c_0 = 0$), $ell_{\rho(t)} = ell_t$ is algebra, and $u_{\rho(t)} = u_t$ follows: iota fixes Z_O, so it maps O-sector kites to O-sector kites, and each kite's alignment rule (Theorem 4.1) is unchanged when the order of its two interfaces is reversed; so $t \mapsto u_{\rho(t)}$ solves the same system as u, and the two differ by a constant, which vanishes at a ρ-fixed interface, or at a ρ-fixed kite (equal lengths, so both rules give $u_t = u_{t-1}$). Hence $I_{\rho(t)} = I_t$ and the cyclic interval sequence is a **palindrome**. At odd Q, $\rho(t) = -1 - t$ and its two axis points are the interface m and the kite-0 gap — exactly the two caps of §2; at even Q they are the two fixed kites when $\mathfrak{e} \not\equiv P \pmod 2$ and the two fixed interfaces when $\mathfrak{e} \equiv P$, as in §1.2. In every case quotienting gives a **path** $J_0, \dots, J_m$ of $m+1$ intervals, at odd Q with $J_i := I_{m-i} = I_{m+i}$, carrying one interior pole on each of its m **edges** $\kappa_0, \dots, \kappa_{m-1}$ — κ_i joining J_i to J_{i+1}, and being the kite at position $m - i$ — and*

$$|J_i| = |\cos(iP\pi/Q)| = \sin(\pi\|i\beta - 1/2\|) \text{ (PAIRED) } / \; |\sin(iP\pi/Q)| = \sin(\pi\|i\beta\|) \text{ (LONE), } \beta = P/Q,$$

with edge κ_i top-aligned iff $\mathrm{frac}((i+1)\beta + 1/2) < \beta$ (PAIRED), respectively $\mathrm{frac}((i+1)\beta) < \beta$ (LONE). Consequently

$$p = 0 \implies \textbf{every interior pole of the path escapes to an end of the path} —$$

a one-dimensional statement about $m+1$ intervals in place of a two-dimensional one about Q kites. (The implication is proved later: $p = 0$ gives complete periodicity by Theorem 7.6, and then Theorems 6.1, 6.3 and 6.7 leave a closing interior-pole prong no end but one of the path's. The converse is open in general. The escape of the m interior poles says nothing directly about the corner prongs, and that gap is (ZZ) (§6.2). Under complete periodicity the converse holds at every Q and in both classes, by

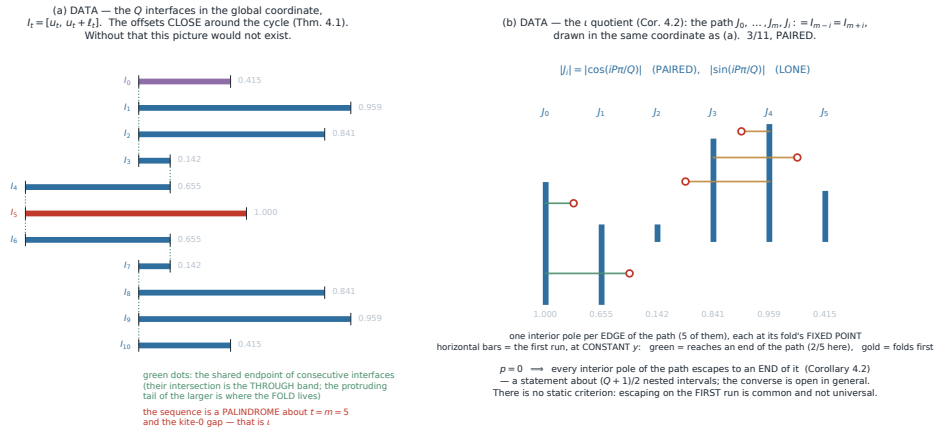


FIGURE 4. The interval model at 3/11. (a) The Q interfaces laid on one line as $I_t = [u_t, u_t + \ell_t]$ by the alignment rule of Proposition 2.3. Consecutive intervals share an endpoint and nest: their intersection is the through band, and the protruding tail of the larger is where the fold lives. The sequence is a palindrome about $t = m$ and the kite-0 gap, which is iota. That the offsets u_t close around the cycle is Theorem 4.1 — without it the picture would not be well posed. (b) The quotient path $J_0, \dots, J_m$ in the same coordinate, with its m interior poles, each at its fold's fixed point. The horizontal bar at each pole is the first run of its prong, at constant y: the leaf crosses the maximal arc of $S(y)$ through its start and then reflects. $p = 0$ requires every one of these poles eventually to reach an end of the path (Corollary 4.2); some reach an end on the first run and some do not.

Corollary 7.11, assuming (OB) of §5.2 on the two classes named there.) (The $|J_i|$ formula above is written in the odd-Q PAIRED/LONE labels; at even Q the interval lengths are Proposition 2.2's ℓ_t, read along the path from a fixed interface or kite.)

The LONE class is the harder half on both counts.

5. Branches and cylinders

This section compares the branch partition of the leg with the cylinder decomposition of the perpendicular direction. §5.1 shows that the branch count is not the number of closed geodesics the beam meets — the difference is exactly [Orph] Theorem 2.2 — and computes the crossing number; §5.2 proves the overlap theorem for the two perpendicular transversals of a class; §5.3 records the reflection that fixes the perpendicular side copies of a class, which §12 uses.

5.1. **Branches and cylinders are different partitions.** The perpendicular direction on S_α decomposes the periodic part of the surface into maximal cylinders — families of parallel closed geodesics bounded by saddle connections through genuine singularities. It is tempting to read a branch as a cylinder, since both are maximal objects partitioning the same transversal. They are not the same, and the count differs.

Two objects first, because §9 uses both and they are easy to run together. Downstairs is the billiard leg L_1, which a closed orbit may strike obliquely many times and meets perpendicularly twice. Upstairs is the segment Λ of [Orph] Proposition 7.5, which consists of exactly two leg-copies λ_1, λ_2 exchanged by τ ([Orph] Proposition 7.6), so that $L_1 = \Lambda/\tau$; Λ is orthogonal to the perpendicular direction, and a *leaf* of that direction on S_α crosses it orthogonally wherever it meets it. A closed orbit downstairs is a τ-pair of leaves upstairs, and its two perpendicular meetings with L_1 are one meeting of λ_1 by each leaf of the pair. **Statements about branches and their words are downstairs; statements about cylinders and leaf spaces are upstairs.**

One word next, because it is used throughout §5.1 and again, of a different object, in §9. A **crossing** of a closed orbit γ by the leg is a connected component of $L_1 \cap \gamma$ — a single point of the leg at which γ meets it perpendicularly, together with the branch of the perpendicular flow through that point. An orbit may be crossed by the leg more than once, and Proposition 5.1 is the statement that off the orphan it is crossed exactly twice. This is not §9's *crossing interval*, which is a maximal interval of the leg free of a certain kind of branch boundary; that term is defined where it is introduced and the two should not be run together.

Roadmap. §5.1 proves two propositions and four lemmas, and each has a named consumer. Proposition 5.1 (every branch lies in one cylinder, and off the orphan two branches share it) and Proposition 5.3 (a closed geodesic is crossed once or twice, and an interior boundary never splits a crossing) convert a branch count into a count of the cylinders met. Lemma ($\star$), *graze precedence*, is the input to Proposition 5.3(ii); Lemma 5.2, *word parity*, says on which copy of the leg a leaf returns; Lemma (†), *reachability*, removes the acute vertices from Proposition 5.3(ii) and from Proposition 9.1 at small P; Lemma ($\diamond$), *one interval per cylinder*, makes the correspondence between branches and met cylinders exact; and Lemma (‡) identifies the billiard's cylinders with those of the unmarked surface at $P = 2$. Together they give the count $n(2/Q) = Q - 1$ (Theorem 10.5) and, through §9, the odd orbits of an isosceles triangle (Corollary 12.5).

Proposition 5.1 (a branch is not a cylinder)**.** *Every branch lies in a single cylinder of the perpendicular direction, but the map from branches to cylinders is* ***not injective****. A closed geodesic met by L_1 at a non-orphan height s is met again at $J(s) \neq s$, and the two crossings carry reversed words and*

equal widths; they are two branches of one cylinder C, and $\tau(C) = C$. A branch is an orphan exactly when J fixes it, and such a branch is met once.

Proof. That each branch lies in one cylinder is immediate: the orbits in a branch are parallel and combinatorially identical, hence contained in one maximal parallel family. For non-injectivity, [Orph] Theorem 2.2 gives $J(s) \neq s$ off the orphan, and s and $J(s)$ are the two ends of one closed orbit, hence two crossings of one geodesic; [Orph] Theorem 2.2 also gives them equal widths, and reversing time reverses the word. They lie in one cylinder, and τ fixes it. By [Orph] Corollary 4.14 the two branches are adjacent on the leg, and their common endpoint c is a height whose orbit reaches R before any other vertex; that orbit retraces at R and closes at c, so its leaf is a regular closed leaf, interior to one cylinder C, which contains the leaves on either side of c and hence both branches. τ fixes the copy of R on that leaf ([Orph] Proposition 7.2), so $\tau(C) = C$. The orphan branches are by definition those J fixes ([Orph] Theorem 2.2). □

Note that the proposition says *a* fixed point, not *the*. How many there are is [Orph] Theorem 4.6 (none at even P) and [Orph] Theorem 4.7 (at most one at odd P; exactly one under (OB), [Orph] §7), and the count below uses that.

One case settles the matter with no estimate at all, and it is worth giving because it is decisive rather than suggestive. The centre 2/3 is the 30-60-90 triangle. By [Orph] Theorem 6.1 its unfolding has $Q = 3$, $g = 1$ and every cone order equal to 0: it is a **flat torus with no singularities**, so every *periodic* direction on it is a single cylinder — there being no saddle connection to bound a second — and the perpendicular direction here is periodic, $n(2/3)$ being finite. Yet $n(2/3) = 2$. So two branches sit on one cylinder, and no count of cylinders can equal n.

The natural correction is to count each non-orphan cylinder twice, giving $(n + \nu)/2$ for the number of cylinders the leg meets, where $\nu \in \{0, 1\}$ is the number of orphan branches — at most one by [Orph] Theorem 4.7, so $\nu = [P \text{ odd}]$ under (OB), and $(n+\nu)/2 = \lceil n/2 \rceil$ since n is odd exactly when there is an orphan ([Orph] Corollary 5.1) — and this is the formula used in the proof of Theorem 10.5 and in Remark 10.4 (where the orphan exists: Theorem 10.2 at $P = 1$, [Orph] Theorems 4.11 and 4.18 at $P = Q - 1$). At even P, which is the case of Theorem 10.5, there is no orphan and $\nu = 0$. It requires one further input, and that input in turn needs a small statement about where a boundary's graze sits along its own passage. We isolate it first, because it is what makes the retrace argument below run and because it splits into two halves of quite different character — one of which is a consequence of how the word was defined in the first place.

Throughout, the **defining graze** of a boundary is the first vertex its trajectory meets. Recall from [Orph] §2.1 that the word $W(s)$ records the sides struck *before* the first perpendicular return, so a vertex met after that return is invisible to the word.

Lemma ($\star$) (graze precedence)**.** *Let y be an interior branch boundary. Then its defining graze occurs strictly before the trajectory's first perpendicular return to L_1 — strictly inside the passage, rather than at its end.*

Proof. Write k for the index of the defining graze. **Not before.** Suppose the first perpendicular return were at some index $j < k$. Every strike of γ_y before index k is at an interior point of a side, so for all s near y the first j strikes lie on the same sides, at interior points; the side sequence being the same and each reflection a fixed linear map, the directions agree as well. So for all such s the strike at j is a head-on L_1 hit and is the first one, and $W(s)$ is constant near y — making y interior to a branch rather than a boundary. Nor can a nearby passage end sooner by retracing instead: by [Orph] Lemma 2.1 an interior head-on on L_2 or H is a condition on the word alone, and the first j letters do not change, so such a head-on occurs for all these s or for none, and either way $W(s)$ is again constant near y.

Not at. Suppose the strike at index k were both the graze and a head-on L_1 hit. Head-on to L_1 means the velocity is horizontal, L_1 being the leg $x = \cot\alpha$; and a strike that is also a graze is at a vertex lying on L_1, hence at $R = (\cot\alpha, 0)$ or at $A = (\cot\alpha, 1)$. If at A, the segment arriving there lies on the horizontal line through A, which meets the closed triangle in that single point, and a segment cannot. If at R, that segment lies on the horizontal line carrying L_2, which meets the triangle in L_2 exactly, so the segment lies inside the wall L_2; its other endpoint is then a strike at an interior point of L_2 — the vertex O being excluded, as no vertex is met before index k — and reflection in L_2 reverses only the vertical component, so the segment before it is horizontal and lies on that same line. Iterating reaches the launch segment, which is horizontal at height $s \in (0, 1)$. Contradiction. □

The proof uses only that the angle at R is a right angle, so that *perpendicular to* L_1 and *along* L_2 are the same condition there; that A is the apex of the leg; and that the launch height is positive. It says nothing about which vertex the graze is at, though the use below is at R; and the first half is not geometry at all but a restatement of [Orph] §2.1's definition, which is worth saying because the two halves together read like a single assertion about grazes and are not one.

Lemma 5.2 (word parity)**.** *Let W be a branch's word — the sides struck between the launch and the first perpendicular return — let g be the product of the side reflections it names (*[Orph] *Lemma 3.1), and let ε be the slope of J on the branch, the sign with which* $\mathrm{Lin}(g)$ *acts on the vertical direction (the proof of* [Orph] *Theorem 2.2). Then $\varepsilon = (-1)^{|W|+1}$: the length $|W|$ is* ***odd*** *on the orphan branch and* ***even*** *on every mirror-paired branch.*

Proof. The developed ray travels in the direction $(-1, 0)$ for its whole length, while the folded orbit arrives at its perpendicular return moving in the direction $(+1, 0)$ — it must, to meet L_1. By [Orph] Lemma 3.1(1) the folded velocity inside the last copy is $\mathrm{Lin}(g)^{-1}$ applied to the developed one, so

$\mathrm{Lin}(g)^{-1}(-1,0) = (+1,0)$: $\mathrm{Lin}(g)$ reverses the horizontal direction. It sends the vertical direction to $(0,\varepsilon)$ and it is orthogonal, so $\mathrm{Lin}(g) = \mathrm{diag}(-1,\varepsilon)$ and $\det \mathrm{Lin}(g) = -\varepsilon$. But g is the product of $|W|$ side reflections, each of determinant -1, so $\det \mathrm{Lin}(g) = (-1)^{|W|}$. Hence $\varepsilon = (-1)^{|W|+1}$. On the orphan $\varepsilon = +1$, and on a mirror pair $\varepsilon = -1$ by [Orph] Lemma 4.9. □

Proposition 5.3 (crossing number)**.** *(i) Every closed geodesic of the perpendicular direction that meets L_1 is met either once or twice — once exactly for the orphan; equivalently, no branch word contains an interior perpendicular reflection, except the orphan's, which contains exactly one. (ii)* ***Every*** *interior branch boundary separates two* ***different*** *crossings, never two pieces of one.*

Part (ii) is proved according to the vertex the defining graze is at. The case that needs a genuine argument is the right angle, which we give first; the acute cases — a graze at an image of positive order, and the four rows at which the acute image is regular — follow it.

Proof of (i); ($\star$) is not used. The perpendicular direction is by definition orthogonal to L_1, so every meeting of an orbit with L_1 is a head-on incidence and the orbit retraces there. [Orph] Lemma 2.1 splits the passage from s two ways and the split is exhaustive. If the orbit meets some side head-on before completing the passage, it reverses and retraces to s; the closed orbit is the out-and-back path through that side, and it meets L_1 once. By [Orph] Lemma 4.2 this is exactly the orphan case. Otherwise the orbit completes the passage, arriving at a re-met copy of L_1 perpendicular at $J(s) \neq s$, and retraces from there; the closed orbit is the double normal with feet s and $J(s)$. The case hypothesis is that it meets no side perpendicularly in between, and a copy of L_1 is such a side, so the two feet are all of its meetings with L_1. The slope is constant on a branch, so the alternative is a property of the branch rather than of the point. That is the ≤ 2 half; Proposition 5.1 is the ≥ 2 half, and it also identifies the once-met geodesic as the orphan's. □

Proof of (ii), from ($\star$). This is the clause the correspondence actually needs, because the mechanism that could break it is a grazing of a *regular* vertex image: the right angle unfolds to cone angle exactly 2π, so a ray grazing one of the Q points over R changes the reflection word while the flat geometry passes straight through — cutting a branch without bounding a cylinder. Such grazes occur, and are common. So let y be an interior boundary whose defining graze, at index k, is at a copy of R. At R the reflections in the two legs commute and compose to $-\mathrm{id}$, the corner retroreflector, and R is the only vertex of T with that property — O and A give rotations by 2α and $\pi - 2\alpha$, neither of which is π. In the development, therefore, the half-turn r about the copy of R met at the graze is an element of the reflection group: it carries the development to itself, and each copy of a side onto a copy of the same side. For $\delta > 0$ small, the rays launched at $y + \delta$ and at $y - \delta$ make

their first $k-1$ strikes on the same sides as γ_y, at interior points, and meet no vertex. They develop to the two lines parallel to γ_y at offsets $\pm\delta$, which pass on either side of the copy of R. The half-turn r exchanges those two lines and reverses their direction. So the part of the $(y+\delta)$-trajectory after it passes the copy of R is the image under r of the part of the $(y-\delta)$-trajectory before it, traversed backwards — and that part begins perpendicular to the launch copy of L_1, at height $y-\delta$. Hence the $(y+\delta)$-trajectory arrives perpendicular to the copy $r(L_1)$ of L_1, at height $y-\delta$.

No head-on event comes earlier. Before the copy of R, $(\star)$'s *not before* half, whose proof also excludes a head-on on L_2 or H, applies to every s near y. After it, a head-on on the $(y+\delta)$-trajectory would be the r-image of a head-on on the $(y-\delta)$-trajectory before the copy of R, because perpendicularity survives an isometry and a reversal of direction; and there is none. So $J(y+\delta) = y-\delta$, and the same argument with the signs exchanged gives $J(y-\delta) = y+\delta$. By [Orph] Theorem 2.2 J is affine of slope ± 1 on each branch. So on both flanking cells $J(s) = 2y - s$, neither of them is the orphan (on which $J = \mathrm{id}$), and near y the map J carries each flanking cell onto the other. The two cells flanking y are therefore the two members of one mirror pair — the branches at s and at $J(s)$ — and by Proposition 5.1 these are two *different* crossings of one geodesic. □

The argument uses no hypothesis on the direction; in particular it does not use that there is at most one orphan.

The right angle is the case that needs an argument of that shape; it is also the common case, and the one §5.1 identified as the mechanism that could break the count. For the other two vertices the question is usually settled without any of the above: a graze at an image of *positive order* runs into a genuine singularity, so the leaf through y is singular and the leaves immediately on either side of it are therefore different leaves — which is what clause (ii) asserts, and it separates distinct crossings for that reason alone. We put it that way rather than saying the saddle connection *bounds cylinders on both sides*, which would presuppose that the direction is completely periodic near y; clause (ii) is claimed at every centre, including the non-periodic ones, and $8/15$ has both its acute orders positive and so exercises this branch. [Orph] Theorem 6.1 decides which images those are, row by row, and the orders it lists are not always positive: $P-1$ vanishes at $P=1$, $P/2-1$ at $P=2$, $Q-P-1$ at $P=Q-1$, and $(Q-P)/2-1$ at $P=Q-2$. At such a centre the acute image in question is regular, the saddle-connection argument is unavailable, and the retroreflector argument does not replace it, since O and A rotate by 2α and $\pi-2\alpha$ and neither reverses a velocity.

At three of those four rows the graze cannot occur, for a reason that is arithmetic on the directions of the beam and answers the same question at every vertex and every centre; the fourth is settled by the explicit partition of Theorem 10.3.

Lemma (†) (reachability)**.** *A trajectory of the perpendicular beam can meet a copy of a vertex V only along a direction whose index on the circle $\mathbb{Z}/2Q$ of directions modulo 2π, one unit being π/Q, lies strictly inside the reverse of V's interior cone, namely $(Q, Q+P/2)$ at O, $(P/2, Q/2)$ at A and $(-Q/2, 0)$ at R — intervals of index-width the angle at V. So the number of directions along which V can be met is $\lceil P/2\rceil - 1$ at O, $\lceil Q/2\rceil - \lfloor P/2\rfloor - 1$ at A and $\lceil Q/2\rceil - 1$ at R. In particular a copy of O is unreachable when $P \le 2$, and a copy of A exactly when $P = Q - 1$.*

Proof. Reflection in a line of orientation θ sends a direction ψ to $2\theta - \psi$, and the sides of T have orientations $\pi/2$, 0 and $\alpha = P\pi/2Q$; so in units of π/Q the three reflections are $\psi \mapsto Q - \psi$, $\psi \mapsto -\psi$ and $\psi \mapsto P - \psi$ modulo $2Q$. The launch direction is the head-on one, π, which is the integer index Q; so every direction of the beam is an integer index. A segment ending at a copy of V has the points just before its end inside the triangle near V, so its direction lies in the reverse of the interior cone at V, an interval whose two endpoints are the directions along the two sides meeting at V. Those endpoints are excluded: a segment along a side σ lies *inside* σ, the line carrying it meeting the closed triangle in σ exactly, so its other endpoint is a strike at an interior point of σ — no vertex is met before the first one — and reflection in σ preserves the tangential component, so the segment before it lies along σ as well; induction reaches the launch segment, which is horizontal at height $s \in (0, 1)$ and lies in no side. The counts are the integers strictly inside the three intervals, and the two vanishing conditions are $P/2 \le 1$ and $\lfloor P/2\rfloor \ge \lceil Q/2\rceil - 1$. □

Applied to the four rows: at $P = 1$ and $P = 2$ no O-graze exists, and at $P = Q - 1$ no A-graze does, so at those three clause (ii) has nothing to cover and the proposition holds without exemption. At $P = 2$ in particular — the family of Theorem 10.5 — A has order $Q - 3$, positive for $Q \ge 5$, and the regular O is simply unreachable; so every boundary there is a graze at an image of R or of A, a consequence of (†) rather than an assumption.

One row survives, $P = Q - 2$ with $Q \ge 5$ (at $Q = 3$ it is the row $P = 1$), where the two points over A are regular and (†) leaves exactly one admissible direction, the bisector of the corner at A. There the proof of Theorem 10.3 computes the partition outright: its only A-graze, at the centre of the second polygon (there is one exactly when $Q \equiv 1 \pmod 4$), separates the orphan, met once, from a member of a mirror pair, met twice, so the two flanking cells are crossings of different geodesics. (Theorem 10.3 uses nothing from this section.)

One input remains, and it is not either proposition. To pass from branches to cylinders we need that the leg meets a cylinder in **one** interval: Proposition 5.1 gives only non-injectivity, and Proposition 5.3(i) constrains a single orbit, so nothing said so far prevents two mirror pairs from lying in one cylinder over disjoint ranges of its **leaf space** — the interval of closed leaves making up the cylinder, parametrised transversally, one point per

leaf — with every leaf still crossed twice. We isolate it, because §9 proves it only for $P \geq 3$, $Q - P \geq 3$ (Lemma 9.2), and Remark 10.4 applies the correspondence at $P = 1$ and $P = Q - 1$, where that restriction fails.

Lemma ($\diamond$) (one interval per cylinder)**.** *Let C be a cylinder of the perpendicular direction met by the leg. Then $\lambda_1 \cap C$ is a single interval. No periodicity hypothesis and no restriction on P; the only exclusions are $Q \leq 3$.*

Proof. Write h for the height of C. In cylinder coordinates the flow is horizontal and Λ is perpendicular to it, so the leaf-space coordinate advances along λ_1 at unit rate: each component of $\lambda_1 \cap C$ maps to the leaf space of C by a strictly monotone isometry onto its image.

Each leaf of C meets λ_1 at most once. By Proposition 5.3(i) a closed orbit is crossed twice, or once for the orphan; by the branch-word parity of Lemma 5.2 a non-orphan leaf launched in λ_1 returns to λ_2, so of the pair's two crossings exactly one lies in λ_1, and the orphan's single crossing lies in λ_1. Hence the images of distinct components of $\lambda_1 \cap C$ are pairwise **disjoint**.

A component with both endpoints on ∂C has image the whole leaf space. Its endpoints have leaf-space coordinate 0 or h, and a strictly monotone injection cannot send both to the same one; so they go to opposite ends and the image is all of $(0, h)$.

Combining the two: if any component is of that kind, disjointness leaves room for no other, and $\lambda_1 \cap C$ is a single interval. Otherwise every component has an endpoint which is an endpoint of λ_1 rather than a point of ∂C. There are two such endpoints, so there are at most two such components, and having two forces **both** endpoints of λ_1 to be interior to cylinders. The endpoint at A lies on ∂C unless the image of A is regular; the endpoint at the R-junction lies on ∂C unless the image of O is regular, the leaf through it containing the side L_2 and terminating at an image of O. By [Orph] Theorem 6.1 the image of O is regular only for $P \leq 2$, and the image of A only for $Q - P \leq 2$; with $\gcd(P, Q) = 1$ those hold together only at $(P, Q) \in \{(1, 2), (1, 3), (2, 3)\}$. So for $Q \geq 4$ a second component is impossible. □

With ($\diamond$) the map from branches to cylinders has fibres the crossing intervals, one per **met cylinder** — a cylinder of the perpendicular direction that L_1 meets at all, which is the only kind the count below ever sees, the direction having cylinders the leg misses (Remark 10.4) — and Proposition 5.3(ii) makes each such interval exactly one mirror pair — or, at the orphan, one branch. Propositions 5.1 and 5.3 together with ($\diamond$) therefore give the number of perpendicular cylinders met by L_1 as $(n+\nu)/2$, and at even P, where there is no orphan, every cylinder met contributes exactly two branches — which is the correspondence the proof of Theorem 10.5 (§10.3) uses.

What the correspondence rests on. ($\diamond$) is unconditional, and so is Proposition 5.3(ii): its right-angle case is the half-turn argument, which

reads no hypothesis on the direction, and its acute cases are the saddle-connection remark, (†) and Theorem 10.3. So the correspondence is **unconditional at both parities of** P.

That correspondence converts a branch count into a count of cylinders *met*. Theorem 10.5 counts the cylinders of the surface S_α, whereas the billiard's partition sees that surface with its vertex images *marked*, and a marked point interior to a cylinder cuts it in two. At $P = 2$ the two lists agree, and it is worth saying why, because on the other leg of the same triangle they do not.

[Orph] Theorem 6.1 says which points are at stake. At $P = 2$ the marked points are the Q images of R, the two images of O and the single image of A; the last has order $Q - 3$, positive for $Q \geq 5$, and is the only genuine singularity, so the question is about the first two — and they are settled by different arguments. The images of R *do* lie inside cylinders, all but one of them, and what makes them harmless is Proposition 5.3(ii): the two branches flanking an R-graze are the two crossings of one closed geodesic, so such a graze cuts a branch without cutting a cylinder. (The same fact in the language of [Orph] §7: by [Orph] Proposition 7.2 the images of R are Weierstrass points, so τ fixes each of them and hence fixes any cylinder one lies in, and a half-turn of a cylinder has exactly two fixed points, both on its core.) The images of O are the ones that could cut, and at $P = 2$ they do not.

Lemma (‡) (the regular acute images lie on the boundary)**.** *Let $P = 2$ and $Q \geq 5$, so that Q is odd. (At $Q = 3$ the image of A has order $Q - 3 = 0$ and the leaf below is not a saddle connection; nothing is lost, since (†) there makes both acute vertices unreachable.) The leaf of the perpendicular direction through a copy of O is a* ***saddle connection****: it runs from a copy of A along the hypotenuse to O, along L_2 to a copy of R, and symmetrically back out to a copy of A. It carries* ***both*** *images of O and one image of R, and its length is $2(|OA| + |OR|) = 2\cot(\pi/2Q)$. In particular neither image of O is interior to a cylinder.*

Proof. The angle of T at O is $\alpha = \pi/Q$, so $2Q$ corners meet at each image of O and fill the cone angle 2π that [Orph] Theorem 6.1 records there. Each corner is bounded by one image of L_2 and one of H, so the $2Q$ side-images around the point **alternate** between the two types. The perpendicular direction is orthogonal to L_1, and at the right angle that is the direction of L_2; the side L_2 is the segment from O to R, so the leaf through O contains it and arrives at O along an L_2-image. Its two rays at a point of cone angle 2π are half the cone apart, that is Q steps of α in a cycle of $2Q$, and Q is odd — so they are of *different* types and the leaf leaves O along an H-image. The hypotenuse runs from O to A, so that ray ends at an image of A, a cone point. Following the leaf the other way, along L_2, it reaches an image of R; there the cone angle is 2π as well, the four rays are two L_1-images and two L_2-images alternately, and the opposite ray in a cycle of four has the *same*

type, so the leaf continues along a second L_2-image into a second copy of O, and thence, by the step just made, out to an image of A. The four segments have lengths $|OA|, |OR|, |OR|, |OA|$; with $L_1 = 1$ the leg L_2 has length $\cot\alpha$ and the hypotenuse $\csc\alpha$, so the total is $2(\cot\alpha + \csc\alpha) = 2\cot(\alpha/2)$. Finally the two copies of O on it are the two distinct images: τ fixes the image of R ([Orph] Proposition 7.2) and reverses the flow, so it fixes the connection and exchanges the two copies of O about that midpoint, and by [Orph] Theorem 6.1 and Proposition 7.2 it exchanges the two images of O, of which there are only two. □

So at $P = 2$ no marked point cuts a cylinder that the singularity does not already bound, and the billiard's list of perpendicular cylinders *is* the decomposition of the unmarked surface — which is what the last step of the proof of Theorem 10.5 (§10.3) needs.

The proof also says why the other leg differs, and the asymmetry is not an accident of bookkeeping. At $P = Q - 2$ the launch leg is L_2, so the perpendicular direction is orthogonal to L_2: at a copy of O it makes angle $\pi/2$ with the L_2-images instead of lying along them, and $\pi/2$ is not a multiple of π/Q for odd Q. The leaf therefore leaves through the interior of a copy of T and has no reason to reach a singularity — and it does not. The two regular acute images there lie inside one cylinder of that direction, on two distinct closed leaves exchanged by τ, and marking them cuts that cylinder in two. That is the extra cylinder on that leg (Theorem 10.3 and §10.3's first closing comment), and the reason the identification above is a statement about a leg rather than about the surface.

5.2. **The overlap theorem.** The covering law's companion half concerns the two perpendicular transversals of a class. In Theorem 5.4 and Corollary 5.5 the class is one with $v = 2$ — at odd Q the PAIRED class, with verticals $\{L_1, H\}$ at P odd and $\{L_2, H\}$ at P even; the even-Q sibling, with verticals $\{L_1, L_2\}$, is the third case and is included in the statements. Write s for the number of cylinders met by both transversals. The LONE class, which has a single transversal, re-enters at Corollary 7.14.

Theorem 5.4. *At most one cylinder is met by both perpendicular transversals: $s \le 1$. The direction need not be completely periodic; s counts cylinders only.*

Proof (four steps, all on the disk D). **(0)** The arcs of Fix(iota) are side-copies and its corners vertex-copies, so the only two zeros lie *on* Fix(iota) and **every singularity of int D is a simple pole**. Gauss–Bonnet then fixes the interior count per class, and $2n_{\text{int}} + \#(\text{poles on Fix}) = Q$ as pure arithmetic.

(1) An iota-invariant cylinder can be fixed neither freely nor horizontally — not freely, since Fix(iota) separates B into two discs exchanged by iota (Corollary 7.4), so a connected invariant set meets it; not along a closed leaf, since Fix(iota) is a finite graph whose arcs end at vertices of G and

contains no closed regular leaf — so it meets Fix(iota) in exactly two vertical segments, and its intersection with D is a rectangle.

(2) Hence *met by* $\perp\,\sigma$ means *has a segment on* σ*'s vertical arc*, and the overlap is the set of **straddlers** — rectangles joining the two vertical arcs.

(3) Straddlers are disjoint rectangles each joining the two vertical arcs, hence **nested**.

(4) Two consecutive straddlers are impossible. If the region U between them is empty, their common horizontal boundary bounds two distinct maximal cylinders and so carries a singularity; but an interior singularity is a simple pole, whose total cone angle π cannot supply the π that *each* of the two cylinders needs. If U is non-empty, it is a disk whose four corners have interior angle $\pi/2$, so Gauss–Bonnet's whole 2π is spent on $4 \times \pi/2$, every remaining term is ≥ 0, and by (0) no zero can appear — so U has no singularity, whence one of its boundary components is a closed leaf carrying no singularity, which no maximal cylinder has. □

For the reverse inequality we need an orphan, an input from the companion paper.

> **Theorem A, turnaround clause (the orphan theorem;** [Orph]**, Theorem 4.6). Every** orphan orbit of the $\perp\,L_1$ beam turns around at a single head-on collision, on a side of type L_2 when Q is even and of type H when Q is odd.

That statement is unconditional and elementary, a congruence count on the velocity set of the unfolded beam. It does not say that an orphan exists. [Orph] proves existence at prime-power Q, when Q is twice or four times a prime, and at the edges of every row (its Theorems 4.11, 4.18, 4.24, Corollary 4.20), [Meth] at the centres of [Meth] Theorems 2–3 (§1.5), and the general case is [Orph] Conjecture 4.12. We therefore carry existence as a named hypothesis. Write $P' \coloneqq P$ if P is odd and $P' \coloneqq Q - P$ if P is even, so that P' is odd.

> **Hypothesis (OB) at** P/Q**.** *The* $\perp\,L_1$ *beam of* $T(P'/Q)$ *has an orphan branch.*

(OB) holds when Q is a prime power ([Orph], Theorem 4.11); when $P' = 3$ (Corollary 10.9 below, which uses only Theorem 10.7 and the turnaround clause, for $Q \neq 7$; $Q = 7$ is prime); and when $P' = 5$ and $Q \equiv 1 \pmod 5$, $Q \geq 16$ (Theorem 10.11: an odd branch count forces a branch fixed by the involution J, which is an orphan by [Orph], Theorem 2.2); when $P' = Q - 2$ (Theorem 10.3, whose proof exhibits the orphan at every odd Q); when Q is twice or four times a prime and at the edges of every row ([Orph] Theorems 4.18, 4.24, Corollary 4.20); and at the centres of [Meth] Theorems 2–3 (§1.5). At even Q it holds at P/Q exactly when it holds at $(Q - P)/Q$, the two orphans being the two ends of one double normal ([Orph], Theorem 5.3). It is conjectured everywhere, and it holds at every centre whose partition has been computed. [Orph] uses nothing from this paper, so there is no cycle:

§§2–4, then [Orph], then this section, then §7, whose Corollary 7.11 uses Corollary 5.5.

Corollary 5.5 (conditional on (OB))**.** *$s \geq 1$ in all three classes; with Theorem 5.4, $s = 1$:* ***exactly one cylinder is met by both perpendicular transversals, the one carrying the orphan*** *— uniformly in P, with no parity caveat.*

Proof. For P **odd** this is the assertion that an orphan lies in the overlap. (OB) gives the $\perp L_1$ beam an orphan, [Orph] Theorem A's turnaround clause puts its turnaround perpendicular hit on L_2 (Q even) or H (Q odd), and time-reversal makes the cylinder carrying that double normal meet both transversals.

For P **even** — necessarily Q odd, the $\{L_2, H\}$ class — the leg swap supplies the statement, with no new machinery. $T(P/Q)$ and $T((Q-P)/Q)$ are the *same* triangle with the two legs interchanged and H fixed, so the $\perp L_2$ beam of the first **is** the $\perp L_1$ beam of the second, and the two $\perp H$ beams coincide: a relabelling, not a transport. Now $Q - P = P'$ is odd, and by (OB) that beam has an orphan, whose far turnaround is on H because Q is odd; the cylinder carrying it is met by $\perp L_1$ and $\perp H$ there, hence by $\perp L_2$ and $\perp H$ here. The classes match, and the pairing is forced rather than chosen: the paired class of P/Q is $\mathfrak{e} = 1$, with verticals $\{L_2, H\}$, and that of $(Q-P)/Q$ is $\mathfrak{e} = 0$, with verticals $\{L_1, H\}$, while the partner's *same*-$\mathfrak{e}$ class is the lone one and has no $\{L_1, H\}$ reading at all. With Theorem 5.4 this gives $s = 1$. □

Neither half needs complete periodicity, in either parity: §5.2 uses (OB) and nothing else.

5.3. **The reflection perpendicular to a class direction.** Theorem 5.4 and Corollary 5.5 count the cylinders met by *both* perpendicular transversals of a class. Whether every cylinder is met by *some* perpendicular transversal is answered in §12 (Theorem 12.2), with no hypothesis on the direction, and the answer is a statement on S_α rather than on B. This subsection records the reflection it turns on. The case $Q = 2$, which §1.1's standing assumption excludes, is allowed here.

Write $\mathcal{G}$ for the group generated by the linear reflections ρ_σ in the three sides σ of T. As in the proof of Proposition 2.1, S_α is glued from copies (T, g), $g \in \mathcal{G}$. The copy (T, g) carries the flat structure of $g(T)$, and its side σ is glued to the side σ of $(T, g\rho_\sigma)$, so that $(x, g) = (x, g\rho_\sigma)$ for $x \in \sigma$. An element $k \in \mathcal{G}$ acts on S_α by $(x, g) \mapsto (x, kg)$. This is an isometry with linear part k, and it carries each vertex of each copy to the same vertex of another copy. In this subsection we **mark** every point of S_α lying over a vertex of T. A *cylinder* of a direction is then a maximal cylinder of S_α minus its marked points, and $\mathrm{ncyl}(S_\alpha)$ is the number of them. By [Orph] Theorem 6.1, the marked points that are not cone points of S_α are the lifts of R, those of O when $P \leq 2$, and those of A when $Q - P \leq 2$. Marking them is

what makes the count of Theorem 12.2 a count of cells: a cell boundary of a perpendicular beam is a launch point whose orbit meets a vertex, whether that vertex is a cone point or not. Marking points creates no cylinder out of a minimal component, so *completely periodic* means the same with or without the marks.

Lemma 5.6 (the reflection perpendicular to a class direction)**.** *Fix a class $(P/Q, \mathfrak{e})$ and a direction ω of it. Exactly one reflection $r = r_\omega \in \mathcal{G}$ has its axis perpendicular to ω. Its fixed set on S_α is the union V of the side copies perpendicular to ω, together with finitely many marked points. V consists of exactly two copies of each side σ with $s(\sigma) \equiv \mathfrak{e} + Q \pmod 2$ — the v vertical sides of §3 — and of no copy of any other side, and τ exchanges the two copies of each such side. Finally, r preserves the foliation of direction ω and reverses its orientation, and it permutes the marked points, hence the cylinders of direction ω.*

Proof. In the units of §1.1, take oriented directions modulo $4Q$. The reflection in a line of index u acts on them by $w \mapsto 2u - w$. The products of two side reflections are the rotations by $\pm 2Q$, $\pm 2P$ and $\pm 2(Q - P)$. Since $\gcd(P, Q) = 1$ these generate the rotations by every even number of units. So the reflections of $\mathcal{G}$ are the maps $w \mapsto 2(s(\sigma) + k) - w$, $k \in \mathbb{Z}$, and their axes run through all $2Q$ line directions, each exactly once, $\mathcal{G}$ having order $4Q$. The line direction perpendicular to ω is $\omega + Q$, and r is the reflection in it.

No $k \neq 1$ fixes an interior point of a copy. Let x be interior to the side σ. The point (x, g) is the point $(x, g\rho_\sigma)$, so it is fixed by k exactly when $kg \in \{g, g\rho_\sigma\}$, that is when $k = g\rho_\sigma g^{-1}$, the reflection in the line of the copy $g(\sigma)$. Hence, off the marked points, $\mathrm{Fix}(r)$ is the union of the side copies whose line is the axis of r. These are exactly the copies perpendicular to ω.

The copies of σ have line directions $\pm s(\sigma) + 2k$, and these are the Q line directions of the parity of $s(\sigma)$. The copies of σ with a given line direction are the edges $\{g, g\rho_\sigma\}$ for which $g\rho_\sigma g^{-1}$ is a given reflection. When such g exist they form a coset of the centraliser $\{1, \rho_\sigma, \tau, \tau\rho_\sigma\}$ of ρ_σ, which gives two edges. The element τ is central and is not a reflection, so it exchanges these two edges. Hence V holds two copies of σ when $s(\sigma) \equiv \omega + Q \pmod 2$, that is when $s(\sigma) \equiv \mathfrak{e} + Q$, and none otherwise.

Finally, r has as linear part the reflection whose axis is perpendicular to ω, so it maps ω to $-\omega$. It carries each vertex of a copy to the same vertex of another copy, so it permutes the marked points. It therefore permutes the cylinders of direction ω. □

6. Rigidity at a pole, and the local route to $p = 0$

6.1. **Rigidity at a pole. Scope.** Every statement in this section concerns the chain $D = B/\mathrm{iota}$, not B. An *interior pole* is one of D's m interior simple poles — never one of the n_{fix} iota-fixed poles, which lie on ∂D. On a completely periodic class, B does carry pole-to-pole saddle connections:

m of them, one for each iota-swapped pair of poles, and nothing below excludes them. Off complete periodicity there can be fewer: at $8/15$, class $\mathfrak{e} = 0$, there are 3, against $m = 7$. Each is iota-invariant and projects to a half-edge from its interior pole of D to ∂D, which is exactly what these theorems permit. The remaining n_{fix} pole ray-ends belong to the fixed poles, each a pole–zero side-leaf, and $n_{\text{fix}} + 2m = Q$ accounts for all Q. The values $(n_{\text{fix}}, m) = (1, (Q-1)/2)$, with the one fixed pole R_0, are the odd-Q case (§1.2); at even Q they are $(2, Q/2-1)$ when $\mathfrak{e} \not\equiv P \pmod 2$ and $(0, Q/2)$ when $\mathfrak{e} \equiv P$, and every statement below holds in all three cases. In particular nothing here says that B has no pole-to-pole connection; under complete periodicity step (1) of the proof of Theorem 7.10 proves the count m, every interior pole's prong closing on its partner.

Work in the path of Corollary 4.2. The fold at edge j — the edge joining nodes J_j and J_{j+1} — is the reflection $y \mapsto s_j - y$, whose fixed point $s_j/2$ *is* the interior pole there. Call a state of the walk **Z-free** if it has not yet arrived at a corner (a zero, Z_O or Z_A).

Theorem 6.1 (transverse holonomy: $j' \neq j$)**.** *No interior pole prong of D reaches a* ***different*** *interior pole of D —* ***at every $Q \geq 3$, both parities, both classes, prime or composite.***

Proof. **The lattice step**, which reads no parity. After n folds at edges $a_1, \ldots, a_n$, the coordinate is $y_n = (-1)^n y_0 + \sum_i (-1)^{n-i} s_{a_i}$. Put $\Lambda := \sum_{a,b} \mathbb{Z}(s_a - s_b)$. If the pole at edge j reaches the pole at edge j', then n even gives coefficient sum 0 and hence $(s_{j'} - s_j)/2 \in \Lambda$, while n odd gives coefficient sum 1 and hence $(s_{j'} + s_j)/2 - s_{a_n} \in \Lambda$ — the same condition in either parity, namely $s_{j'} - s_j \in 2\Lambda$. So it suffices that the classes $v_j := [s_j - s_0] \in \Lambda/2\Lambda$ be pairwise distinct, and since $2\Lambda \subseteq 2\mathbb{Z}[\zeta_{4Q}]$ it suffices that $s_a - s_b \notin 2\mathbb{Z}[\zeta_{4Q}]$.

The closed form. Normalise lengths by the factor $2i$, so that $L_t = \zeta^{c_t} - \zeta^{-c_t} \in \mathbb{Z}[\zeta_{4Q}]$, $\zeta = e^{i\pi/2Q}$. Every interface endpoint is then in $\mathbb{Z}[\zeta_{4Q}]$: the coordinate starts at 0 and steps by $\pm(L_{\text{far}} - L_{\text{near}})$ (proof of Theorem 4.1). Adjacent intervals nest and share exactly one endpoint (Proposition 6.10). The fold reflects the protruding tail of the larger interval about its midpoint (§4), so s_e is the sum of the two unshared endpoints: $s_e = 2a + L_{\text{near}} + L_{\text{far}}$ if the shared endpoint is the low one, a, and $s_e = 2b - L_{\text{near}} - L_{\text{far}}$ if it is the high one, b. Both are $\equiv L_j + L_{j+1}$ modulo $2\mathbb{Z}[\zeta_{4Q}]$. The test below is applied in this normalisation, in which $\Lambda \subseteq \mathbb{Z}[\zeta_{4Q}]$ and the s_e are not all even; in Theorem 6.7's doubled units $\text{ell}_t = 2L_t$ every s_e is even and the test would be vacuous.

The criterion. The closed form $s_j \equiv L_j + L_{j+1}$ modulo $2\mathbb{Z}[\zeta_{4Q}]$ reduces that to the non-vanishing, in $R_2 := \mathbb{Z}[\zeta_{4Q}]/2 = \mathbb{F}_2[x]/\Phi_{4Q}(x)$, of

$$g_{a,b} := \sum_{i \in \{a, a+1, b, b+1\}} (x^{c_i} + x^{-c_i}), \qquad c_i = \gamma + 2Pi \pmod{2Q},$$

the c_i being the chain's node offsets and γ the first node's.

Since $x^{2Q} = 1$ in R_2 at **both** parities, writing $y = x^{2P}$ and carrying the offset rather than normalising it away groups the four terms exactly:

$$g_{a,b} = x^{-\gamma-2P(b+1)} \cdot (1+y) \cdot (1+y^e) \cdot (1+x^{2Pw}),\ e \coloneqq b-a,$$
$$w \coloneqq (a+b+1) + \gamma P^{-1} \pmod{Q} \text{ — so } w = a+b+1,$$
except at even Q with $\mathfrak{e} \not\equiv P \pmod 2$, where $\gamma = P+Q$ and $w = a+b+2$.

The valuation step, which replaces the root of unity. Write $Q = 2^v Q_0$ with Q_0 odd. Then $\Phi_{4Q} \equiv \Phi_{Q_0}^N \pmod 2$ with $N \coloneqq \varphi(2^{v+2}) = 2^{v+1}$, so R_2 is a product of **local** rings of nilpotency degree N, one for each irreducible factor f of $\Phi_{Q_0} \pmod 2$. Every root of Φ_{Q_0} has order exactly Q_0, so the valuation

$$T(k) \coloneqq \operatorname{ord}_f(1+x^{2Pk}) = [Q_0|k] \cdot 2^{1+\operatorname{ord}_2(k)} \textbf{ for } 1 \le k \le Q-1$$

is the same for every f, and therefore $g_{a,b} = 0$ in R_2 **iff** $T(1) + T(e) + T(w) \ge N$. The admissible ranges are $e \in [1, m-1]$ and $w \in [2, Q-3]$ (odd Q), $[2, Q-2]$ (even Q, $\mathfrak{e} \equiv P$), $[3, Q-3]$ (even Q, $\mathfrak{e} \not\equiv P$). Three cases.

- $Q_0 > 1$ — every odd Q, and every even Q that is not a power of two. $\gcd(P, Q_0) = 1$ gives $T(1) = 0$; $T(e) \neq 0$ forces $e = Q_0 s$ with $s < 2^{v-1}$, so $T(e) \le 2^{v-1}$, and likewise $T(w) \le 2^v$. The total is $\le 3 \cdot 2^{v-1} < 2^{v+1} = N$. At odd Q one has $v = 0$ and all three terms vanish, which is the classical evaluation at a primitive Q-th root of unity read as a valuation.
- $Q_0 = 1$**,** $Q = 2^v$**,** $v \ge 3$**.** Here $T(1) = 2$. At $\mathfrak{e} \equiv P$, $e + w = 2b + 1$ is odd, so one of $T(e), T(w)$ is 2 and the other at most 2^v: total $\le 4 + 2^v < N$. At $\mathfrak{e} \not\equiv P$, $w - e = 2a + 2$ is even; both odd gives 6, and both even gives total $\le 2 + 2^{v-1} + 2^v = N - (2^{v-1} - 2) < N$.
- $Q = 4$**.** At $\mathfrak{e} \not\equiv P$ there is no admissible pair, m being 1. At $\mathfrak{e} \equiv P$ the bound is exactly tight and the criterion genuinely **fails**: $(a, b) = (0, 1)$ gives $e = 1$, $w = 2$ and $2 + 2 + 4 = 8 = N$. The conclusion holds there with no criterion at all, since $m = 2$ makes $\Lambda = \mathbb{Z}(s_1 - s_0)$ with $s_1 \neq s_0$, so $\Lambda/2\Lambda = \mathbb{Z}/2$ and $v_0 \neq v_1$.

□

Remark 6.2. Three things are worth separating. **(i)** The zero set of $g_{a,b}$ is non-empty off the admissible range, so the theorem is precisely the observation that the range excludes it. **(ii)** The $Q_0 = 1$ bound is **attained** — the margin is exactly $2^{v-1} - 2$ — which is why $Q = 4$ and only $Q = 4$ fails, and why that failure is predicted rather than excused. **(iii)** The valuation step is what removes any parity hypothesis, as the factorisation removes any primality one: no primitive root of unity, no separability of Φ_Q modulo 2 — which is in fact false at $Q \equiv 0 \pmod 4$ and is never needed — and no count of exponents.

Theorem 6.3 (reversibility: $j' = j$)**.** *No interior pole prong of D returns to* ***its own*** *pole,* ***at every*** *$Q \ge 3$****, both parities, both classes.***

Proof. Let T be the step map on states (i, d, y) — node, direction, coordinate — and let $R(i, d, y) \coloneqq (i, -d, y)$.

Step 1 (reversibility). $T^{-1} = RTR$ on every state of a Z-free orbit. For a crossing this is immediate. For a fold, adjacent intervals share exactly one endpoint, so being blocked at edge e means $y \in J_i \setminus J_{i2}$, and the reflection $y \mapsto s_e - y$ maps that set to itself; the reflected point is blocked by the *same* edge, so the backward step of a fold is a fold.

Step 2. $s_e/2$ is interior to exactly one of the two intervals at edge e, namely the larger. Hence the start $u_0 = (i_0, d_0, s_j/2)$ and the state u_M at which a self-hit would be detected have the same node and opposite direction, so $u_M = R(u_0)$.

Step 3 (the palindrome). Put $w_k \coloneqq R(u_{M-k})$. Then $w_0 = u_0$ and $T(w_k) = RT^{-1}(u_{M-k}) = w_{k+1}$, so $u_k = R(u_{M-k})$ for all k.

Step 4. If M is even then $u_{M/2} = R(u_{M/2})$, i.e. $d = -d$, which is impossible. If M is odd then $u_{(M+1)/2} = R(u_{(M-1)/2})$ — same node, same y, flipped direction — which only a fold can produce, and a fold preserves y only when $y = s_e/2$. So the state at index $(M-1)/2$ sits on a pole, the prong **ends** there, and it never reaches u_M. □

Remark 6.4. Nothing in this proof reads Q, P, a length, the lattice Λ or a cyclotomic fact — only that T is reversible and that the pole is the fold's fixed point. It therefore holds for arbitrary, non-arithmetic interval data. It also settles rows that no computation could: a prong whose fold count exceeds any feasible cap is settled for free, because the argument never looks at the orbit's length. What it settles is that such a prong does not end *at a pole*, not that it ends.

Theorem 6.5 (assembly — conditional on CP and on (PP))**.** *By Theorems 6.1 and 6.3 there is no pole-to-pole saddle connection interior to D at all,* ***at either parity of*** *Q (interior to D: on B there are m of them under complete periodicity; see the scope paragraph opening this section). If in addition $p \neq 0$ implies the existence of a pole-to-pole saddle connection interior to D — this is the hypothesis* ***(PP)*** *(pole to pole), which §6.2 reduces to the single statement* ***(ZZ)****, that no iota-swapped edge joins two zeros (a consequence of Corollary 7.11 under CP, not true without it in general: it fails at* $8/15$*, Remark 6.8) — then the contrapositive gives*

$$CP \implies p = 0, \text{ in both classes and at every } Q \geq 3, \text{ both parities,}$$

and with it Corollary 3.5, statement (M1) of Proposition 3.6, and the count law of Corollary 7.14. The conclusion in fact needs no (PP) anywhere: Corollary 7.11 proves $CP \implies p = 0$ from CP at every Q, both classes — with the hypothesis (OB) of §5.2 on two of them — by a global count (Theorem 7.10).

This route proves $p = 0$ from (ZZ), which is itself obtained under CP only through §7.1 (Corollary 7.11). The implication of Theorems 6.1 and 6.3 also ranges over interior *pole* prongs only — m of the $Q - h$ — and §6.2 shows that the rest is not a technicality but the other half of the statement.

6.2. **The reduction step: the pole prongs and (ZZ).** Throughout this section $w \coloneqq \zeta_{2Q}$, $\zeta \coloneqq \zeta_{4Q}$, $K \coloneqq \mathbb{Q}(\zeta_{4Q})^+$ is the maximal real subfield, and O_K its ring of integers. Recall $L \coloneqq \cot\alpha$.

Proposition 6.6 (the reduction)**.** *Write W for the region an iota-swapped pair of edges cuts off inside D. The step invoked in Theorem 6.5 is a Gauss–Bonnet count on W, requiring four inputs:* ***(PP)*** *interiority — that $p \neq 0$ yields a configuration whose region is interior to D at all;* ***(G1)*** *$\chi(W) = 1$;* ***(G2)*** *no zero in int(W);* ***(G3)*** *the boundary contributes 0. Of these:*

- ***(G2) follows from (PP).*** *The boundary corners of D have angles $P\pi/2$, $(Q-P)\pi/2$ and $\pi/2$, so Z_O and Z_A sit on ∂D, carrying half of their B-angles, and are not interior.*
- ***(G1) is free, and the apparatus is redundant.*** *Given (PP), both ends of the swapped edge are interior poles, so the edge* ***is*** *the pole-to-pole connection the step is invoked for; (G1)–(G3) exist only to evaluate $\#\text{poles}(W) = 2$, and nothing downstream consumes that number. In any case $\chi(W) = 1$ holds: W is a regular neighbourhood of the arc and retracts to it, so $\chi = 1$ unless the arc is a loop, and a simple pole carries exactly one separatrix germ, so a leaf leaving a pole can only return along its own outgoing germ; such a saddle connection would be its own reverse, and a leaf cannot reverse at a regular point, so there is no such loop.*

So the hypothesis of Theorem 6.5 is exactly (PP), and (PP) splits. A swapped edge cannot meet Fix(*iota*) *at a regular point (two leaves would cross), nor lie along a horizontal* Fix *arc (that arc is itself a leaf), so it meets* Fix(*iota*) *only possibly at a singular endpoint. Hence*

> ***(PP)*** $\iff$ *no prong of an interior pole ends at a zero (the pole–zero side-leaves of the fixed poles lie in* Fix(*iota*) *and are not swapped) — Theorem 6.7 below —* $\oplus$ ***(ZZ)*** *no swapped zero–zero edge — under CP it holds at every Q, both classes, with (OB) on two of them (Corollary 7.11); without CP it is false (Remark 6.8).* □

(The direction $\impliedby$ is elementary; $\implies$ also uses Theorems 6.1 and 6.3, which exclude a swapped pole–pole edge.)

Theorem 6.7 (the pole prongs, at every Q and both parities)**.** *No prong launched at an interior pole of the chain ends at a zero.*

Proof. Work in $\mathbb{Z}[\zeta_{4Q}]$ with $G(c) \coloneqq \zeta^c - \zeta^{-c}$, so that $L_t = G(c_t)$ in Theorem 6.1's normalisation, and double the units: $\text{ell}_t \coloneqq 2L_t = 2G(c_t)$. Write

s_e for the fold sums and Λ_S for their $\mathbb{Z}$-span in these doubled units. The pole at edge j is the fixed point of its fold, $s_j/2$, which is the fold sum of Theorem 6.1's undoubled units; call it $\hat{s}_j$. An ending of the excluded kind requires end $\mp y_0 \in \Lambda_S$, where end is an interface endpoint and $y_0 = \hat{s}_j$ is the prong's start at the pole j; so it is enough to exhibit an invariant separating the two. The invariant is a **parity**, and the module to test it against is not Λ_S but the coarser $2\mathbb{Z}[\zeta_{4Q}] \supseteq \Lambda_S$ — the same coarsening that makes Theorem 6.1's pole-to-pole criterion need no membership test at all.

The endpoints are even. By construction $\mathrm{ell}_t = 2G(c_t)$, so each increment $\mathrm{delta}[k] = \mathrm{ell}_{\mathrm{far}} - \mathrm{ell}_{\mathrm{near}}$ is even; the global coordinate starts at $u[0] = 0$ and steps by $\pm\mathrm{delta}$. Hence every $lo[t] = u[t]$ and every $hi[t] = u[t] + \mathrm{ell}_t$ is an even vector. Consequently each s_k, being $hi[a] + hi[b]$ or $lo[a] + lo[b]$, is even, which gives $\Lambda_S \subseteq 2\mathbb{Z}[\zeta_{4Q}]$ and puts every interface endpoint in the **zero coset**.

The interior-pole prong is odd. Reducing modulo 2 and writing x for the image of ζ, $\hat{s}_j \equiv (x^P + x^{-P})(x^{b_j} + x^{-b_j})$ with b_j the kite bisector of edge j, by two elementary identities: the fold sum is the bisector sine, $G(c) + G(c+2P) = (\zeta^P + \zeta^{-P})G(c+P)$, and modulo 2 the interface positions drop out of the undoubled fold sum (Theorem 6.1's closed form $s_j \equiv L_j + L_{j+1}$). In $\mathbb{F}_2[x]/\Phi_{Q_0}^N$ — Theorem 6.1's own ring, $Q = 2^v Q_0$, $N = 2^{v+1}$ — the valuation criterion gives $T(x^b + x^{-b}) \geq N \iff Q|b$. The first factor has $T(x^P + x^{-P}) = 0$ when $Q_0 > 1$ and $= 2$ when $Q = 2^v$, and there $T(x^b + x^{-b}) \leq 2^v$ unless $Q|b$, while $2 + 2^v < 2^{v+1}$ for $v \geq 2$; so in every case $\hat{s}_j$ vanishes exactly when $b_j \in \{0, Q\}$. Putting $\beta(p) \coloneqq c_0 + P \cdot p$ on positions $p \in \mathbb{Z}/2Q$, one has $\beta(2k-1) = b_k$ and $Q|\beta(p) \iff p \equiv -c_0 P^{-1} \pmod{Q}$, which is precisely iota-fixedness (Proposition 2.2, with c_0 the class offset there); and the iota-fixed positions are the chain's two **ends**. So at an interior edge $b_j \notin \{0, Q\}$ and $\hat{s}_j$ is odd. (At even Q on $\mathfrak{e} \equiv P$ the two fixed positions are both interfaces, so every kite is interior and every $\hat{s}_j$ is odd. The step needs c_0 of the right parity: with the odd-Q value $c_0 = P + Q$ carried to that class, $Q|b_k$ would land on two kites that are not fixed and the argument would fail as written.)

Hence end $\mp y_0$ is even minus odd, so it lies outside $2\mathbb{Z}[\zeta_{4Q}]$ and a fortiori outside Λ_S, at every endpoint, every interior pole and either sign. □

The reading to keep is the coset one: **the prong of an interior pole lives permanently in $\hat{s}_j + 2\mathbb{Z}[\zeta_{4Q}]$** — every fold adds an even vector and negation preserves the class — **while the interface endpoints are exactly the zero coset, and an ending of the excluded kind would be a point in both.** No lattice, no rank and no cutoff enters. The criterion is one-sided, as it must be: membership is *necessary* for such an ending, so exclusion proves impossibility while a hit would prove nothing.

Why the same argument cannot reach (ZZ) — and this is a theorem, not a gap in coverage. A corner prong launches *from* an interface endpoint, so its y_0 is even by the first step above, where an interior

pole's $\hat{s}_j$ is odd; and its target is another endpoint. Both sit in the zero coset, and the parity functional separates nothing. The asymmetry is the whole of why Theorem 6.7 closes here and (ZZ) does not.

(ZZ) is not a side-condition; it is the other half of $p = 0$. Theorems 6.1 and 6.3 range over interior *pole* prongs. (ZZ) is precisely the O- and A-launched half of the statement *every prong launched from a vertex returns to that vertex*, about half of all prongs. Discharging it is not a step towards $p = 0$; it is the remainder of it. (Under CP, and (OB) where needed, Corollary 7.11 reaches $p = 0$ at every Q by a global count and obtains (ZZ) as a consequence. That does not contradict this paragraph: the count sees both halves at once, and it discharges neither separately.)

Remark 6.8 ((ZZ) is false without CP). At 8/15 the class $\mathfrak{e} = 0$ is not completely periodic (Proposition 11.1), so by Theorem 7.6 it carries an iota-swapped saddle connection. It meets Fix(iota) at no regular point (after Proposition 6.6), so a pole end would make its image a prong of D from an interior pole, and Theorems 6.1, 6.3 and 6.7 exclude every end such a prong can have but a fixed pole; and a fixed pole's one germ is a horizontal arc of Fix(iota) (proof of Corollary 7.7), which is not swapped. So both its ends are zeros: (ZZ) fails there. Explicitly: the O-prong of the slot $u_0 = 2$ (Lemma 3.7, units of §1.1) meets $L_1, H, L_2, H, L_1, H, L_1, L_2, H, L_1, L_2, H, L_2$ and arrives at A with no perpendicular incidence, so its lift is a zero–zero saddle connection that neither lies in Fix(iota) nor crosses a vertical arc — of type (3) in Lemma 3.3. So no argument that ignores which row it is on can prove (ZZ), and a candidate criterion must admit 8/15; conversely, that row lies outside complete periodicity, under which (ZZ) is asserted, and a test that fails only there has refuted nothing.

6.3. $CP \implies p = 0$ **on the Veech locus.** This section is unconditional. It proves (ZZ) — hence, with §§6.1–6.2, $CP \implies p = 0$ outright — on a named region, at every odd Q. **It is the paper's one odd-Q-only section** (§1.5): the locus it resolves is cut out by the turn law in the PAIRED/LONE labels, which by §3 are an odd-Q phenomenon, and the even-Q analogue is not developed here.

Proposition 6.9 (the profile is a folded rotation)**.** *Along the quotient path of Corollary 4.2,*

$$|J_i| = \sin(w(x_i)\pi/2Q),\ w(x) = \min(x, 2Q - x),\ x_i = (c_0 + 2P(m - i)) \bmod 2Q.$$

The start $x_0 = c_m$ is always 0 or Q — a fold point of w — and the value set is an arithmetic progression of step 2 with $m + 1$ distinct terms.

Proposition 6.10 (nesting is definitional)**.** *Adjacent path interfaces* ***nest and share exactly one endpoint****, at every kite, every P, every $\mathfrak{e}$ and every Q, both parities — the proof below reads the chain construction only.*

Proof. The chain construction — Proposition 2.3's alignment rule — sets $\mathrm{delta}[k] = \mathrm{ell}_{\mathrm{far}} - \mathrm{ell}_{\mathrm{near}}$ with far the larger edge and propagates $u_{\mathrm{near}} =$

$u_{\text{far}} +$ delta$[k]$ if osec$[k]$ and $u_{\text{near}} = u_{\text{far}}$ otherwise. Substituting $lo_t = u_t$ and $hi_t = u_t +$ ell$_t$ puts J_{near} inside J_{far}, sharing the low endpoint when osec$[k]$ is false and the high endpoint when it is true. □

At odd Q the unique tie kite is kite 0 — delta$[k] = 0$ forces $2Pk \equiv 0 \pmod{Q}$, hence $k = 0$ — and the path's adjacencies are kites $1, \ldots, m$, so **the path never meets it**. At even Q the tie kites are the n_{fix} fixed kites (Proposition 2.3), which are the path's ends, so the same holds.

Theorem 6.11 (no interior local maximum $\implies$ escape)**.** *On a class row whose profile has* ***no interior local maximum****, every corner prong of an edge with a non-empty through band (every path edge on the PAIRED class, edges $1, \ldots, m-1$ on the LONE class, where edge 0's corner germ is the H side-leaf) escapes with* ***zero folds****: in exactly $j+1$ steps when the larger node of edge j is j, and in $m-j$ steps when it is $j+1$.*

Proof. The launch puts the prong on the larger node heading away from the edge, so the first step is uphill unless that node is a local maximum. By Proposition 6.10 an uphill entry is a **forced cross** — nesting makes $J_i \subset J_{i2}$, so every $y \in J_i$ lies in J_{i2} — and it leaves y unchanged. With no interior local maximum, every later step is uphill as well. □

This covers the **monotone** rows $P \in \{1, Q-1\}$ and also the **valleys**. The step-count formula is keyed on which node of the edge is the larger one; a row-level version of it is false.

Theorem 6.12 (the turn law)**.** *The number of interior extrema of the profile is*

$$\textbf{\textit{turns}} \;= \min(P, Q-P) - 1\textbf{.}$$

Proof. w is strictly monotone on each of $(0, Q)$ and $(Q, 2Q)$, so an interior extremum occurs exactly when a step crosses a fold point. The step is $2p$ with $p \coloneqq \min(P, Q-P) \le m < Q$, so it crosses at most one fold point. Total travel is $2pm = p(Q-1)$, starting **on** a fold point, and the fold points are Q apart, giving $\lfloor (pQ-p)/Q \rfloor = p-1$ crossings. Each lands on an interior index: the first cannot land on $i = 0$, which sits a full Q from the fold point while x_1 sits $Q - 2p$ from it, and the last lands on $i = m$ only if $2p \ge Q$, which is false. □

So turns $= 0 \iff P \in \{1, Q-1\}$, the monotone rows, and turns $= 1 \iff P \in \{2, Q-2\}$, which have one interior extremum — either a **valley** or a **tent**. Theorem 6.11 covers the valleys; the tents need their own argument.

Theorem 6.13 (the tent)**.** *On a tent row, every corner prong of an edge with a non-empty through band (as in Theorem 6.11; so $\nu_j \ge 2$) has fold levels exactly $\nu_j - 2, \nu_j - 4, \ldots, 2, 0$, folds exactly $\nu_j/2$ times, and* ***escapes****.*

Proof. Write $\ell(\nu) \coloneqq \sin(\nu\pi/2Q)$ for the profile value at level ν, the levels being the arithmetic progression of Proposition 6.9. Four inputs and one estimate.

- **(T1)** The tents are exactly $(P, \mathfrak{e}) = (2, 0)$ and $(Q-2, 1)$, two per odd Q. Indeed a turns $= 1$ row is a tent iff $x_0 = c_m = 0$; since $c_m \equiv c_0 + P(Q-1) \pmod{2Q}$, $\mathfrak{e} = 0$ gives $x_0 = PQ \equiv 0$ iff P is even and $\mathfrak{e} = 1$ gives $(P+1)Q \equiv 0$ iff P is odd; at odd Q, 2 is even and $Q-2$ odd.
- **(T2) Concentricity**, which is exact arithmetic on osec rather than a measurement. osec$[k]$ holds iff $c_{k-1} > 2Q - 2P$; evaluating that on the two families gives osec **false** on every path kite for $(2, 0)$, and **true** on path kites $1, \ldots, m-1$ with kite m false for $(Q-2, 1)$. So all path edges but edge 0 share the *same* endpoint, and the nodes are **nested with a common endpoint** c.
- **(T3)** Hence $r \coloneqq |y - c|$ is a single scalar, with $y \in \text{ int } J_i \iff r < \ell(a_i)$, and every fold obeys $r \mapsto \ell(\text{near}) + \ell(\text{far}) - r$.
- **(T4) The two ramps interleave mod 4.** x steps by 4 and w is affine on each arc, so the up-ramp values are the residues $\equiv 0 \pmod 4$ and the down-ramp values those $\equiv 2 \pmod 4$. With the progression of step 2, the two ramps **alternate** in sorted order, so $\nu + 2$ always lies on the other ramp and ν's same-ramp neighbour toward the peak is $\nu + 4$ — the sole exception being $\nu = Q - 3$, whose neighbour is the peak $\nu + 2$.
- **(T5) Strict concavity.** Every value lies in $[0, Q]$, so every angle lies in $[0, \pi/2]$, giving $\ell(\nu-2) + \ell(\nu+2) < 2\ell(\nu)$**.**

The invariant is $I(\nu)$: *after the fold at level* ν, $\ell(\nu) < r < \ell(\nu+2)$. The launch puts the prong at $r_0 = \ell(\nu_j)$, ν_j being the level of the smaller node of edge j; the step below uses only $\ell(\nu) \le r < \ell(\nu+2)$, so it applies at the launch, and from the first fold on $I(\nu)$ holds strictly. Given $I(\nu)$, the prong crosses the $\nu{+}2$ node, since $r < \ell(\nu{+}2)$, and folds at the $\nu{-}2$ node, since $r > \ell(\nu) > \ell(\nu{-}2)$, giving $r' = \ell(\nu{-}2) + \ell(\nu{+}2) - r$. Then $r' > \ell(\nu{-}2)$ restates the upper half of $I(\nu)$; and $r' < \ell(\nu) \iff r > \ell(\nu-2) + \ell(\nu+2) - \ell(\nu)$, whose right-hand side is $< \ell(\nu) < r$ by (T5) alone. So $I(\nu) \implies I(\nu-2)$, the levels descend by exactly 2, and the ladder bottoms out at 0. The level-2 node is the path's end J_m ($x_m = c_0 \in \{2, 2Q-2\}$), so after the level-0 fold $r < \ell(2)$ and the prong crosses J_m and leaves the path. □

> The missing ingredient here is analytic, not order-theoretic. Re-run the margin in (T5) with the linear surrogate $\tilde{\ell}(a) = a$ and it is **exactly** 0: the induction has zero slack without strict concavity, so the order facts (T1)–(T4) prove nothing on their own. The genuine margin is $2\ell(\nu)(1 - \cos(\pi/Q))$, at least of order π^3/Q^3 (the worst case, $\nu = 2$).

Corollary 6.14. *By Theorem 6.11 (monotone and valley) and Theorem 6.13 (tent), every corner prong on the turns* ≤ 1 *locus escapes; an*

escaping prong leaves the path across a vertical arc of Fix(*iota*) *(on the PAIRED class the fixed interface or the vertical leg of the half-kite, on the LONE class the vertical leg of the half-kite), so its lift is iota-invariant (Lemma 3.3) and lies on no swapped edge, and the remaining zero germs are horizontal side-leaves. Hence (ZZ) holds on the whole turns* ≤ 1 *locus, at every odd* Q*. By Theorem 6.12 that locus is* $P \in \{1, 2, Q-2, Q-1\}$*, and by* [KS00, Thm 2] *— a right triangle is a lattice triangle iff its smallest angle is* π/n*, which here reads* $\min(P, Q-P) \in \{1, 2\}$ *— those are exactly the* ***Veech (lattice) right triangles*** *in this family:* $\{1, Q-1\}$ *the regular* $2Q$*-gon and* $\{2, Q-2\}$ *the double regular* Q*-gon, each pair a leg swap.*

> With Theorem 6.7 and Theorem 6.5, $CP \implies p = 0$ **is therefore a theorem on the Veech locus at every odd** Q**.** (At even Q the Veech rows $P \in \{1, Q-1\}$ are in Corollary 7.11.)

7. Complete periodicity and $p = 0$

Complete periodicity is a genuine hypothesis of Theorem 6.5 and of the covering law, not a caveat. It enters through Theorem 3.1, as the **finiteness** input that Theorems 6.1 and 6.3 do not supply: those two forbid a prong *ending at a pole*, not its *failing to end*.

§7.1 gives three *sufficient* conditions for the hypothesis, each unconditional; the first of them, $p = 0 \implies CP$, lets Corollary 3.5 drop one of its two hypotheses. It then proves the converse, $CP \implies p = 0$, by one global edge count (Theorem 7.10, Corollary 7.11). §7.2 draws the consequences of $p = 0$. §11 exhibits the two class rows at which complete periodicity is false.

7.1. Three routes to the hypothesis.

The reversing symmetry iota is what makes these available. A minimal component is constrained not by what iota does to a leaf's two sides but by what it does to a leaf's own return dynamics, and nothing that crosses Fix(iota) can bound one. We begin with a local fact about the poles, which is independent of everything else here.

Proposition 7.1 (the pole fold)**.** *An edge of* G *with a simple-pole endpoint has both of its sides in the same component of the complement of the singular leaves; in particular it cannot separate a minimal component from a cylinder, nor two distinct components of any kind.*

Proof. A simple pole has cone angle π and carries exactly one horizontal separatrix germ (Proposition 2.1). A punctured disc about it, cut along that germ, is therefore a **single** sector of angle π whose two boundary rays are the two sides of the edge. They are joined around the pole, so both lie in one component of the complement; and being a cylinder or a minimal component is a property of the component. □

This removes the pole–pole edges and the pole–zero side-leaves at a stroke. It is unconditional and reads only the stratum.

Lemma 7.2 (reversibility on Fix(iota))**.** *Let V_0 be a vertical arc of* Fix(*iota*)*. Then every recurrent leaf meeting V_0 is closed; hence almost every point of V_0 lies on a closed leaf, and V_0 is covered, up to a set of measure zero, by cylinders.*

Proof. iota is an isometry of the half-translation structure preserving the class foliation and fixing V_0 pointwise, so at $x \in V_0$ its differential fixes the tangent to V_0 and, being an involution other than the identity, negates the horizontal direction. Let $\tilde{V}_0 = V_0 \times \{\pm\}$ be the transversal doubled by a choice of leaf direction, R the first-return map on it, and $S(x, d) = (x, \bar{d})$. Time reversal gives $RSR = S$ on any transversal: if the leaf from (x, d) first meets V_0 at y leaving in direction e, then the leaf from $(y, \bar{e})$ retraces it and arrives at x leaving in direction $\bar{d}$. And iota-equivariance gives $RS = SR$, which is special to Fix(iota): iota carries the leaf segment from (x, d) to the one from $(x, \bar{d})$ and fixes y, so $R(x, \bar{d}) = (y, \bar{e})$. Together $R^2 = \mathrm{id}$ — the leaf from x runs to y and returns to x in its original direction, that is, it closes. Poincaré recurrence for the transverse measure makes almost every point of V_0 return, and a positive-measure set of closed leaves fills cylinders. □

Attribution. Lemma 7.2 is not new and is not this paper's. Its mechanism — time reversal at a perpendicular hit, so that a symmetric trajectory is periodic or singular — is Stepin's [GSV92, §5], and in the affine-reflection form used here it is [Vor97, Prop. 2.1]. It is the statement that the first perpendicular-return map of a beam is an involution, together with the branch dichotomy — every branch of a perpendicular beam is a mirror pair or the orphan, and both are parabolic cylinders — which is also how [Orph] reaches it. The proof above is self-contained and is not a citation: nothing in Lemma 7.2 through Corollary 7.8 depends on the companion paper. A vertical arc of Fix(iota) **is** a perpendicular beam's base: the vertical sides are by definition those whose perpendicular beam lies in the class, and Theorem 3.4's proof identifies the points of G on such an arc with that beam's interior cell boundaries. What is new below is only the transport. The *separatrix* face of this reversibility is (M1) of Proposition 3.6, which is $p = 0$ restated and is the hypothesis direction; Lemma 7.2 and its consequences are the conclusion direction and are unconditional.

Corollary 7.3. *A saddle connection meeting a vertical arc of* Fix(*iota*) *at an interior point of both has a cylinder on each side. In particular it lies on the boundary of no minimal component.*

Proof. Let x be such a crossing, interior to both arcs. There are finitely many saddle connections in the class, so a small disc about x meets only this one, and the points of V_0 just to either side of x lie in the two components adjacent to the edge. By Lemma 7.2 almost all of them lie on closed leaves, so each of those components contains a closed leaf and is a cylinder. □

Corollary 7.4. *No minimal component meets* Fix(*iota*)*. Hence a minimal component M lies in one of the two open discs of $B \setminus \mathrm{Fix}(iota)$, and M, iota(M) are two exchanged minimal components.*

Proof. The h horizontal arcs are leaves with singular endpoints, hence saddle connections, hence in the critical graph and in no component. If M met a vertical arc V_0, then $M \cap V_0$ would be a non-empty open subset of V_0, of positive measure, and by Lemma 7.2 almost all of it would lie on closed leaves — but a minimal component contains none. So M misses Fix(iota) and, being connected, lies in one complementary disc; iota preserves the decomposition, so iota(M) is the other.

That the complement has exactly two discs is a count on the sphere, uniform in the parity of Q. Fix(iota) is a graph whose arcs are two per iota-fixed **kite** — an R–Z_O copy of L_2 and an R–Z_A copy of L_1 — and one per iota-fixed **interface**, a copy of H, itself a Z_O–Z_A connection. So every fixed position contributes exactly one strand from Z_O to Z_A, and there are always exactly two fixed positions. With $V = 2 + \#\text{fixkites}$, $E = 2 \cdot \#\text{fixkites} + \#\text{fixints}$, Euler gives $F = \#\text{fixkites} + \#\text{fixints} = 2$: a triangle at odd Q, a square at even Q with $\mathfrak{e} \not\equiv P$, a bigon at even Q with $\mathfrak{e} \equiv P$, and $E = h + v$ throughout. □

Corollary 7.5 (the head-on branch)**.** *A head-on loop has a cylinder on each side, so it bounds no minimal component; likewise a side-leaf with a pole endpoint.*

Proof. Head-on means a perpendicular bounce, $u_{\text{out}} = u_i n + 2Q$, which is a crossing of a vertical arc of Fix(iota); apply Corollary 7.3. More generally every iota-invariant edge outside Fix(iota) crosses one, by Lemma 3.3(2), so the same applies to it. For a side-leaf L_1 or L_2: Proposition 7.1 puts its two sides in one component C, while the leaf lies in Fix(iota), so its two sides lie in the two *different* discs of Corollary 7.4; hence C meets both, and by Corollary 7.4 C is not minimal. □

Theorem 7.6 ($p = 0 \implies CP$)**.** *A class with no iota-swapped edge is completely periodic. In particular, since (M1) implies $p = 0$ (Proposition 3.6, (3) $\implies$ (1)),* ***(M1)*** $\implies$ ***CP****.*

Proof. Suppose $p = 0$ and a minimal component M exists. By Corollary 7.4, M lies in one open disc Δ of $B \setminus \mathrm{Fix}$(iota). At $p = 0$ no saddle connection is of Lemma 3.3's type (3), so each is of type (1) — lying in Fix(iota), hence one of the h horizontal arcs of $\partial\Delta$ — or of type (2), crossing a vertical arc, and Corollary 7.3 keeps the latter off ∂M. **Lemma 3.3 carries no hypothesis, so no step here assumes what is being proved.** So no arc of ∂M meets int Δ: the frontier of M inside int Δ is at most a set S of singular points of the foliation, which is finite. int $\Delta \setminus S$ is connected (a finite set does not disconnect an open disc), and M is open **and** relatively closed in it, whence $M =$ int $\Delta \setminus S$. But $\partial\Delta$ carries a vertical arc V_0, since $v \geq 1$ at every centre, and by Lemma 7.2 almost every point of V_0 lies on

a closed leaf, which crosses V_0 transversally and so has regular points in $\operatorname{int}\Delta \setminus S = M$. A minimal component contains no closed leaf. □

Three consequences, and one warning.

The contrapositive is $\neg CP \implies p \geq 1$: a minimal component *forces* a swapped edge. Second, every step transfers to even Q: Corollaries 7.3 and 7.4 and Lemma 7.2 read no parity, Lemma 3.3 is parity-free, and the one further input, $v \geq 1$, holds at every centre ($v = 1$ at even Q with $\mathfrak{e} \equiv P$, $v = 2$ with $\mathfrak{e} \not\equiv P$). Theorem 7.6 is the converse of $CP \implies p = 0$ and does not prove it; that direction is Corollary 7.11 below.

The second route is a condition at the two cone points, and it is a remark of Troubetzkoy's [Tro05] to which we supply a proof.

Corollary 7.7 (the vertex condition; Troubetzkoy's remark)**.** *Let* ***(V)*** *be the condition that every corner prong is* ***head-on****: launched at O or A along a class direction, it meets a side perpendicularly before any vertex, and so retraces its path to its vertex. (This is the reading of Troubetzkoy's condition that "the perpendicular orbits starting at the (non-right angle) vertices are simple periodic loops"* [Tro05, end of §3] *used here; a closed corner loop that does not retrace is a swapped pair and is not admitted.) Then* ***(V)*** $\implies$ ***CP****, unconditionally, at every Q, both parities and in both classes.*

Proof. Lemma 3.7's ray-end identity $2N_Z + b_Z = \deg(Z)$ is elementary counting on the inward windows and uses no complete periodicity. Under (V) each of the N_Z slots at a zero Z is a head-on loop at Z itself, which consumes both of its ray-ends and is iota-invariant; the remaining b_Z ray-ends are the horizontal side-leaves, which lie in Fix(iota). So no edge with a zero endpoint is iota-swapped. A swapped edge meets Fix(iota) only possibly at a singular endpoint, and no iota-fixed pole can be that endpoint: a simple pole carries exactly one class separatrix germ, and at an iota-fixed pole that germ is the horizontal one of the two Fix(iota) diagonals issuing from its kite, hence an iota-invariant edge. The argument is per pole, so it is indifferent to how many fixed poles there are — two at even Q with $\mathfrak{e} \not\equiv P$, none when $\mathfrak{e} \equiv P$, one at odd Q. So both endpoints of a swapped pole–pole edge are non-fixed poles, and the edge projects to a prong of D running from an interior pole of D to an interior pole of D, which Theorems 6.1 and 6.3 forbid unconditionally. Hence $p = 0$, and Theorem 7.6 gives CP. □

(V) is (ZZ) at the vertex index, so Corollary 7.7 and Theorem 7.6 are the same implication read through two dictionaries; what Corollary 7.7 adds is the identification. It consumes neither Theorem 6.7 nor $CP \implies p = 0$.

The third route is a numerical certificate, and it reverses the arrow of Corollary 3.5.

Corollary 7.8 (the covering identity is a CP certificate)**.** *Write*

$$D \coloneqq (Q - h + v) - \textstyle\sum_{\sigma vertical} n_\sigma = Q - h - f.$$

Then $D = 2p + (z - X)$ where z counts the vertices lying off G, $X \coloneqq \sum_M \chi_c(M)$ over the minimal components M, and $z - X = \sum_M (n_M - \chi_c(M)) \geq 0$, with n_M the number of vertices interior to M. In particular D is even and non-negative, and $D = 0 \implies CP$, unconditionally.

Proof. The edge-type count $E = h + f + 2p$ is Lemma 3.3's trichotomy and reads no complete periodicity; what CP supplies to Theorem 3.4 is $z = 0$ and $\sum_{\min} \chi_c = 0$. Dropping those, additivity of χ_c over $B = G \sqcup$ faces gives $E = V - 2 + \sum_{\min} \chi_c$ and $V = Q + 2 - z$, hence $Q = h + f + 2p + z - \sum_{\min} \chi_c$. The identification $f = \sum_\sigma (n_\sigma - 1)$ uses no complete periodicity either: iota fixes a vertical arc pointwise and reverses the leaf through each of its points, so a leaf from the arc that meets a singularity in one direction meets one in the other, and is a type-(2) edge meeting Fix(iota) only there (Lemma 3.3(2)); hence the points of G on the open arc are exactly the interior cell boundaries of the beam. Every vertex off G is a non-fixed pole: Z_O, Z_A and the iota-fixed poles are endpoints of arcs of Fix(iota); a horizontal arc is an edge, and the end cell of a vertical arc lies in one cylinder (Lemma 7.2), whose boundary passes through the arc's endpoint. So the interior vertices of a minimal component M are simple poles, and Gauss–Bonnet on $\bar{M}$, with $a_i\pi$ its angles at its boundary corners, gives $2\chi = n_M - \Sigma(a_i - 1)$, so $n_M - \chi = (n_M + \Sigma(a_i - 1))/2 \geq 0$; the blocks come in iota-pairs of equal value by Corollary 7.4, so the sum is even. $D = 0$ forces $p = 0$, and Theorem 7.6 gives CP. □

So the covering law of Corollary 3.5 is not only a consequence of complete periodicity: its identity is *sufficient* for it. And what D counts when it does not vanish is exact too: the non-closing ray count is $s = 2(z - X)$, so $2p = D - s/2$, a swapped-edge count free of any cutoff. At $8/15, \mathfrak{e} = 0$ the covering law is short by $D = 8$ against the certified $n_{L_1} = 6$, and $D = 2p + s/2 = 2 + 6$.

The same count proves the converse of Theorem 7.6 under complete periodicity, by one argument for every class. It needs one piece of bookkeeping about the shape of Fix(iota).

Lemma 7.9 (the fixed-kite count)**.** *In every class, $n_{\text{fix}} = h + v - 2$.*

Proof. At odd Q, $n_{\text{fix}} = 1$ in both classes (§1.2), and (h, v) is $(2, 1)$ on the LONE class and $(1, 2)$ on the PAIRED class (§3). At even Q, P is odd; the class $\mathfrak{e} = 0$ has $\mathfrak{e} \not\equiv P$, so $n_{\text{fix}} = 2$, and $h = v = 2$; the class $\mathfrak{e} = 1$ has $n_{\text{fix}} = 0$ and $h = v = 1$ (§1.2, §3). □

Theorem 7.10 ($p = v - \kappa$ — conditional on CP)**.** *Let the class be completely periodic, at odd Q with $\min(P, Q - P) \geq 3$ or at any even $Q \geq 4$, and let $\kappa \in \{1, 2\}$ be the number of components of G containing a zero (Theorem 3.1). Then*

$$p = v - \kappa.$$

Proof. Write V for the union of the vertical arcs of Fix(iota).

(1) Every interior pole's prong closes on its partner; $b = n_{\mathrm{fix}}$ and $a = m$. Let R be an interior pole of D. Under CP its prong is a saddle connection; follow it in D to its first arrival at Fix(iota) or at a singularity. It does not first reach an interior pole of D (Theorems 6.1 and 6.3), nor a zero (Theorem 6.7; an interior pole is not the pole of an end kite, so the exception there does not arise). Nor does it reach a fixed pole, or a named point Z_O, Z_A that is a simple pole (at even Q, $P \in \{1, Q-1\}$): each of these carries a single germ, already used by a horizontal arc of Fix(iota) — at a fixed pole the horizontal diagonal of its kite (proof of Corollary 7.7), at the named simple pole the horizontal arc of Fix(iota) at it (e_0 of step (3) on the class $\mathfrak{e} = 0$, the horizontal copy of H on $\mathfrak{e} = 1$). It does not run inside a horizontal arc, which is itself a leaf. So it meets a vertical arc at an interior point x. On B, iota fixes x and reverses the leaf through it, so past x the leaf is the iota-image of its incoming half, traversed backwards, and it ends at $\mathrm{iota}(R) \neq R$; by Lemma 3.3 the edge is iota-invariant. This gives one pole–pole edge for each of the m swapped pairs of interior poles, so $a = m$. Each fixed pole's one germ ends at a named zero, so $b = n_{\mathrm{fix}}$, and $2a + b = Q$ is the degree count of Theorem 3.1.

(2) Every cylinder is iota-invariant. A pole–pole edge e is iota-invariant by (1), so $\mathrm{iota}(c(e))$ is a cylinder bounded by e, hence $c(e)$ itself (Proposition 3.12: e bounds a single cylinder). By Proposition 3.12 (iv) at most $\kappa - 1 \leq 1$ cylinders are not of the form $c(e)$; the set of the $c(e)$ is iota-stable, so the remaining one, if any, is mapped to itself.

(3) Each cylinder meets $V \setminus G$ in exactly two segments. An iota-invariant cylinder meets Fix(iota) in exactly two segments transverse to the leaves (step (1) of the proof of Theorem 5.4), which therefore lie in V. For $c(e)$, iota preserves the boundary circle traced by e, and its only fixed point there is x, the midpoint of e; so the two segments run from the far circle to x, and with x they form one arc meeting G only at x. For the leftover cylinder c_0, when $\kappa = 2$, the two segments cross c_0 from one boundary circle to the other. (At even Q with $P \in \{1, Q-1\}$ the leftover is the cylinder $c(e_0)$ of the horizontal arc e_0 joining the fixed pole R_1 to the named point that is a simple pole: iota fixes e_0 pointwise, and its fixed set in $c(e_0)$ is two segments issuing from the two ends of e_0, one into each vertical arc.)

(4) $f + v = 2C$. Under CP, $V \setminus G$ lies in the open cylinders, so by (3) it has exactly $2C$ components. Each of the v vertical arcs has both endpoints at vertices of G and $n_\sigma - 1$ interior points on G, so it contributes n_σ components (at odd Q on the PAIRED class the two vertical arcs share one endpoint — Z_A if P is odd, Z_O if P is even — and are counted separately). Summing, $f + v = 2C$.

(5) The identity. Theorem 3.1 with (1) gives $C = (Q - n_{\mathrm{fix}})/2 + \kappa - 1$, so by (4) $f = Q - n_{\mathrm{fix}} + 2\kappa - 2 - v$. Theorem 3.4 gives $2p = Q - h - f = n_{\mathrm{fix}} + v - h + 2 - 2\kappa$, and by Lemma 7.9 $n_{\mathrm{fix}} + v - h = 2v - 2$. So $p = v - \kappa$. □

On a class with one perpendicular beam, $p = 1 - \kappa \leq 0$, so $p = 0$ follows from CP alone. On a class with two, $p = 2 - \kappa$, and $p = 1$ is exactly a swapped pair of edges joining the two zeros — the case (ZZ) is about; so the zeros must be separated, and that is where an orphan enters. The route is global: the local route of §6.1 needs (ZZ), and here (ZZ) is never examined — the edge budget $E = Q$ of Theorem 3.1 is used up by the side-leaves, the pole–pole edges and the edges crossing V.

Corollary 7.11 ($CP \iff p = 0$ at every Q)**.** *At every $Q \geq 3$, in both classes and at every P — assuming (OB) on the classes with two perpendicular beams off the Veech locus, namely the PAIRED class at odd Q with $\min(P, Q - P) \geq 3$ and the class $\mathfrak{e} = 0$ at even Q with $3 \leq P \leq Q - 3$ —*

$$\boldsymbol{CP} \iff p = 0 \iff D = 0\boldsymbol{,}$$

where D is the defect of Corollary 7.8. By class, $D = 0$ reads:

class	*vertical sides*	$D = 0$
LONE, odd Q	*L_1 (P even) / L_2 (P odd)*	$n_{L_1} = Q - 1$ / $n_{L_2} = Q - 1$
PAIRED, odd Q	*σ and H, $\sigma = L_2$ (P even) / L_1 (P odd)*	$n_\sigma + n_H = Q + 1$
$\mathfrak{e} = 0$, *even Q*	*L_1, L_2*	$n_{L_1} + n_{L_2} = Q$, *that is* $n(P/Q) + n((Q - P)/Q) = Q$
$\mathfrak{e} = 1$, *even Q*	*H*	$n_H = Q$

Hence (ZZ) holds under complete periodicity, and so do the covering law (Corollary 3.5), (M1) of Proposition 3.6 and the count law (Corollary 7.14), with (OB) on those two classes. The backward implications, and the forward one on every other class, need no (OB); and (OB) holds at every prime-power Q and at the centres named in §1.5, so there the corollary is unconditional.

Proof. The backward implications are Theorem 7.6 and Corollary 7.8, at every centre. Forward, on the odd-Q Veech locus $\min(P, Q - P) \leq 2$ it is Corollary 6.14 with Theorem 6.5, whose turn law (Theorem 6.12) does not read the class. Everywhere else Theorem 7.10 applies. If $v = 1$, then $p = 1 - \kappa \leq 0$, so $p = 0$. If $v = 2$, then $p = 2 - \kappa$, and $p = 0$ if and only if $\kappa = 2$. At even Q with $P \in \{1, Q - 1\}$ one named point is a simple pole whose one germ is e_0, so e_0 is a whole component of G and $\kappa = 2$. Otherwise use (OB): by Corollary 5.5, which assumes (OB) and not CP, some cylinder is met by both perpendicular beams, so its intersection with Fix(iota) meets both vertical arcs (step (2) of the proof of Theorem 5.4). If $\kappa = 1$, every cylinder is a $c(e)$ (Proposition 3.12 (iv)), whose intersection with Fix(iota) is one arc meeting G only at x, by step (3) of the proof of Theorem 7.10; the corners of Fix(iota) are vertices of G, so that arc lies in a single vertical arc — a contradiction. So $\kappa = 2$ and $p = 0$. Finally, under CP $D = Q - h - f = 2p$ by Theorem 3.4, so $p = 0 \iff D = 0$; and

$D = (Q - h + v) - \Sigma n_\sigma$ reads as in the table with the values of (h, v) of §3, the leg swap $n_{L_2}(P/Q) = n((Q-P)/Q)$ (§8) giving the last form in the row $\mathfrak{e} = 0$. □

At even Q with $P \in \{1, Q-1\}$, the Veech locus there [KS00, Thm 2], the class direction contains a saddle connection (a horizontal arc of Fix(iota)), so Veech's dichotomy [Vee89, Proposition (2.10)] makes both classes completely periodic, and Corollary 7.11 holds there outright. At 8/15 and its leg-swap partner 7/15, which are not completely periodic (§11), it is consistent with what §11 finds: the leg count is $6 < 14$.

Remark 7.12 (the leftover is the overlap). On a class with two perpendicular beams, under CP and (OB), step (3) of the proof of Theorem 7.10, as used in that of Corollary 7.11, identifies the cylinder met by both beams: it is not a $c(e)$, so it is the unique leftover c_0 of Proposition 3.12, and for P odd it carries every orphan double normal ([Orph] Theorem 4.6). The identification, and with it $\kappa = 2$, also holds at odd Q with $\min(P, Q-P) = 2$. There a fixed pole's one germ is the horizontal arc of Fix(iota) (proof of Corollary 7.7), which ends at Z_O or Z_A, neither of them a pole; so a pole–pole edge e joins two non-fixed poles. Were it iota-swapped it would meet Fix(iota) at no point (§6.2, after Proposition 6.6) and project to a prong of D between interior poles, which Theorems 6.1 and 6.3 exclude; it is not a horizontal arc, every horizontal arc having a fixed-pole endpoint. So e is of type (2) in Lemma 3.3 and crosses a vertical arc at its midpoint x; iota$(c(e))$ is a cylinder bounded by e, hence $c(e)$ (Proposition 3.12), and step (3) of the proof of Theorem 7.10 applies to it — that step reads neither $\min(P, Q-P)$ nor the parity of Q. Corollary 5.5 needs (OB) at P/Q, that is an orphan of the $\perp L_1$ beam of $T((Q-2)/Q)$, and Theorem 10.3, which uses nothing from §§5–7, exhibits one at every odd Q; so on these rows the identification needs CP only.

Remark 7.13 ($p = v - \kappa$ on the odd-Q Veech locus). At odd Q with $\min(P, Q-P) \le 2$, outside Theorem 7.10's range, the formula still holds under CP. There $p = 0$ (§6.3), so it says $\kappa = v$. On the LONE class the horizontal side H joins Z_O to Z_A, so $\kappa = 1$. On the PAIRED class at $P = 1$ (resp. $Q-1$) the named simple pole Z_O (resp. Z_A) has one germ, the horizontal copy of L_2 (resp. L_1) at it, which ends at a copy of R; that edge is a whole component of G, as at even Q in the proof of Corollary 7.11, so $\kappa = 2$. At $\min = 2$, $\kappa = 2$ is Remark 7.12. So $p = v - \kappa$ holds under CP at every $Q \ge 3$.

Throughout what follows, *under CP, by Corollary 7.11* carries that corollary's proviso: (OB) on the two classes named there.

7.2. **Consequences of $p = 0$: the cylinder count and the strip set.** Under $p = 0$ the cylinder count follows at once from §3.

Corollary 7.14 (the cylinder count law — conditional on CP, with (OB) on the two classes of Corollary 7.11)**.** *For a completely periodic class,*

$$C(\textbf{\textit{lone}}) = \lfloor Q/2 \rfloor \ \textbf{\textit{and}}\ C(\textbf{\textit{paired}}) = \lceil Q/2 \rceil\textbf{,}$$

uniformly in the parity of Q, the even-Q case collapsing both branches to $C = Q/2$.

Proof. Theorem 3.1 gives $C = (Q - b)/2 + \kappa - 1$ under CP alone, with b the number of pole–zero edges and κ the number of components of G holding a zero. By Corollary 7.11, $p = 0$, hence (M1) of Proposition 3.6: every prong ends where it started. So there is no interior pole–zero edge and no interior edge joining the two zeros, whence b reduces to the boundary count $b_b d$, and $\kappa = 1$ exactly when the two zeros are joined, which only the H side-leaf can do. Both remaining quantities are then parity arithmetic on the three side directions $s(L_2) = 0$, $s(L_1) = Q$, $s(H) = P$: one has $b_b d = [\mathfrak{e} \equiv 0] + [\mathfrak{e} \equiv Q]$, and H is a leaf iff $\mathfrak{e} \equiv P$, which by §3 is exactly the LONE condition. Substituting, $\delta := b - 2(\kappa - 1)$ equals $+1$ on the LONE class and -1 on the PAIRED class at odd Q, and 0 at even Q, so $C = (Q - \delta)/2$ and the parity case split evaporates. □

Theorem 5.4 and Corollary 5.5 count the *overlap* of the two perpendicular transversals. The pinning of Proposition 3.12 answers the finer question — for a single transversal, which cylinders it meets — and the answer is a parity condition on one word.

Throughout, for a pole–pole edge e write $\sigma_{\mathrm{mid}}(e)$ for the side carrying the central letter of its word, which is an odd palindrome (step (1) of the proof of Theorem 7.15).

Theorem 7.15 (the strip set of a pinned cylinder — conditional on CP)**.** *Let e be a pole–pole edge and $c(e)$ the cylinder it pins. Then $c(e)$ is met by exactly one perpendicular transversal, namely the one perpendicular to $\sigma_{\mathrm{mid}}(e)$.*

Proof. (1) *Exactly one head-on, at the midpoint.* A prong that meets a side head-on retraces from that instant, so it returns to its launch vertex and stops; it cannot reach a second head-on. That it reaches one at all needs CP and no more. In the range of Theorem 7.10 it is step (1) of that proof, which uses CP alone: e crosses a vertical arc of Fix(iota) at an interior point, its midpoint, and a vertical arc is a copy of a side perpendicular to the leaves. At odd Q with $\min(P, Q - P) \le 2$ it is §6.3: there CP gives $p = 0$, hence (M1) of Proposition 3.6. So there is exactly one, at the midpoint of the word, on a side $\sigma_{\mathrm{mid}}(e)$, and the word is an odd palindrome returning to its own launch vertex.

(2) *The parity.* In an odd palindrome every letter other than the central one is matched, so for each side X one has $\#X \equiv [X = \sigma_{\mathrm{mid}}(e)] \pmod 2$.

(3) *The prong bounds the cylinder.* By Proposition 3.12, e is a boundary saddle connection of $c(e)$; the leaves of $c(e)$ are therefore the trajectories

parallel to e, and they carry its word. This is the step that needs the pinning in its *bounding* form and not merely as the length match circ $= 2\ell$.

(4) *The far end contributes no strip.* Since e retraces, it arrives back at R with direction $u_0 + 2Q$. The inward window at R is the open interval $(Q, 2Q)$, so $u_0 \in (Q, 2Q)$ strictly and $u_0 + 2Q \in (3Q, 4Q)$ strictly. A head-on on L_1 requires direction $\equiv 0 \pmod{2Q}$ and one on L_2 requires $\equiv Q \pmod{2Q}$, and the open interval $(3Q, 4Q)$ contains neither. So the leaves of $c(e)$ reverse at R by the right-angle retroreflector — obliquely on both legs — and contribute no perpendicular strip there.

(5) By (3) and (4) the only head-on of $c(e)$ is the $\sigma_{\mathrm{mid}}(e)$ one inherited from e, and it sits at a regular interior point of that side. Hence $c(e)$ meets the transversal perpendicular to $\sigma_{\mathrm{mid}}(e)$ and no other. □

Corollary 7.16 (a parity criterion for being missed — conditional on CP)**.** *A cylinder pinned by a pole–pole edge is met by the L_1-perpendicular beam if and only if $\#L_1$ of its prong word is **odd**.*

Proof. Immediate from Theorem 7.15 and step (2): $\#L_1 \equiv [\sigma_{\mathrm{mid}} = L_1] \pmod 2$. □

Theorem 7.15 needs CP and nothing more. Step (1) is the only place it enters, and there CP supplies the head-on of a pole–pole edge directly, without $p = 0$ on the whole class; so (OB) is not needed, even on the two classes of Corollary 7.11 where $p = 0$ needs it.

8. Bounds

This section bounds the branch count $n(P/Q)$ from both sides, at every coprime centre and with no hypothesis on the direction. Throughout, $n(P/Q) = n_{L_1}(P/Q)$: the cells of §1.1 and the branches of [Orph] §2.1 are cut at the same points, namely the launch points whose trajectory meets a vertex — a graze at a copy of R included, since the launches on its two sides strike the two different legs and so carry different words. Since $T(P/Q)$ and $T((Q-P)/Q)$ are one triangle with its legs exchanged, $n_{L_2}(P/Q) = n((Q-P)/Q)$. To avoid a clash with the chain $D = B/\mathrm{iota}$, we write **the defect** for the quantity D of Corollary 7.8.

The frame. We use the coordinates of [Orph] §1.1 and of the proof of Theorem 10.2: $O = (0,0)$, $R = (\cot\alpha, 0)$, $A = (\cot\alpha, 1)$, so that $L_1 = RA$, $L_2 = OR$, $H = OA$ and $|L_1| = 1$. Reflecting successively in the lines through O that carry H and L_2 produces copies T_k, $k \in \mathbb{Z}$, $T_0 = T$, with T_k filling the sector $k\alpha \le \arg z \le (k+1)\alpha$. The ray $\arg z = k\alpha$ carries an image of L_2 when k is even and of H when k is odd, and on it lies the vertex image V_k: an image of R at distance $\cot\alpha$ from O when k is even, an image of A at distance $\csc\alpha$ when k is odd. As in the proof of Theorem 10.2, the images of L_1 in T_{k-1} and T_k are collinear when k is even (both are perpendicular to the ray at V_k). Hence, for $k < l$ with $(l-k)\alpha < \pi$, the union $U_{k,l} := T_k \cup \cdots \cup T_{l-1}$ is a **convex** polygon: its sides are OV_k, OV_l

and images of L_1, and its angles are $(l-k)\alpha$ at O, $\pi - 2\alpha$ at an interior image of A, π at an interior image of R, and $\pi/2$ or $\pi/2-\alpha$ at V_k, V_l. Every vertex image in $U_{k,l}$ lies on its boundary. We use repeatedly the following consequence of convexity: *if X lies in the relative interior of one side of $U_{k,l}$ and V is a vertex of $U_{k,l}$ not on the line of that side, the open segment XV lies in the interior of $U_{k,l}$, so a trajectory developed along it meets no vertex image and no image of L_1 before V.*

8.1. **The branch count.** Write $M \coloneqq \lceil Q/P \rceil - 1$, the number of $m \geq 1$ with $mP < Q$, equivalently with $m\alpha < \pi/2$.

Lemma 8.1 (wall grazes — no hypothesis)**.** *Let* $\gcd(P,Q) = 1$, $Q \geq 3$.

1. *(The leg beam.) Let $P \geq 2$, let $m \coloneqq \lceil 2Q/P \rceil - 1$ be the largest integer with $m\alpha < \pi$, and put $y \coloneqq |OV_m| \sin(m\alpha)$. Then $0 < y < 1$, and the trajectory launched perpendicular to L_1 at height y reaches the vertex R (m even) or A (m odd) before it meets L_1 again. In particular y is an interior cell boundary of the $\perp L_1$ beam, and $n(P/Q) \geq 2$.*
2. *(The hypotenuse beam.) For $0 \leq m \leq M$ put $d_m \coloneqq \cot\alpha \cdot \cos(m\alpha)$ if m is odd and $d_m \coloneqq \csc\alpha \cdot \cos(m\alpha)$ if m is even. Then $|H| = d_0 > d_1 > \cdots > d_M > 0$, and for $1 \leq m \leq M$ the trajectory launched perpendicular to H from the point at distance d_m from O reaches R (m odd) or A (m even) before it meets L_1. Hence $n_H \geq M+1$.*
3. *(The other leg.) For $1 \leq m \leq M$ put $e_m \coloneqq \cot\alpha \cdot \cos(m\alpha)$ if m is even and $e_m \coloneqq \csc\alpha \cdot \cos(m\alpha)$ if m is odd. Then $|L_2| = e_1 > e_2 > \cdots > e_M > 0$, and for $2 \leq m \leq M$ the trajectory launched perpendicular to L_2 from $(e_m, 0)$ reaches R (m even) or A (m odd) before it meets L_1. Hence $n_{L_2} \geq M$.*

Proof. (1) Since $P \nmid 2Q$ unless $P \leq 2$, $\pi - \alpha \leq m\alpha < \pi$, and $m \geq 2$ because $Q > P$. Hence $0 < \sin(m\alpha) \leq \sin\alpha$, with equality only if $(m+1)P = 2Q$, that is $P = 2$, in which case Q is odd and $m = Q-1$ is even. If m is even, $y = \cot\alpha \cdot \sin(m\alpha) \leq \cos\alpha < 1$; if m is odd, $P \neq 2$ and $y = \sin(m\alpha)/\sin\alpha < 1$. The point $X = (\cot\alpha, y)$ lies in the relative interior of the side RA of the convex polygon $U_{0,m}$, and V_m, whose abscissa $|OV_m|\cos(m\alpha)$ is negative ($m\alpha > \pi/2$), is not on the line of that side; and V_m has height y. The trajectory launched at X with velocity $(-1,0)$ develops along the horizontal segment XV_m, so by convexity it meets no vertex image and no image of L_1 before V_m. So the launch at y meets a vertex, and y is a cell boundary interior to L_1.

(2) Work with the copies $T_{1-m}, \ldots, T_0$, that is in $U_{1-m,1}$, whose angle at O is $m\alpha < \pi/2$; H is its side on the ray $\arg z = \alpha$, and its vertex V_{1-m} is an image of R at distance $\cot\alpha$ when m is odd and of A at distance $\csc\alpha$ when m is even. With $X \coloneqq d_m e^{i\alpha}$, $\Re((V_{1-m}-X)e^{-i\alpha}) = |OV_{1-m}|\cos(m\alpha) - d_m = 0$ and $\Im((V_{1-m}-X)e^{-i\alpha}) < 0$, so V_{1-m} lies on the normal to H at X, on the side of T. The gaps are $d_m - d_{m+1} = \sin((m+1)\alpha)$ for m even and $\sin(m\alpha)$ for m odd (expand $\cos(m\alpha) = \cos((m+1)\alpha)\cos\alpha + \sin((m+1)\alpha)\sin\alpha$,

respectively $\cos((m+1)\alpha) = \cos(m\alpha)\cos\alpha - \sin(m\alpha)\sin\alpha$), positive because $(m+1)\alpha \le M\alpha + \alpha < \pi$; and $d_M > 0$ because $M\alpha < \pi/2$. So X lies in the relative interior of H, and V_{1-m} is not on the line of H. By convexity the trajectory launched perpendicular to H at X meets a vertex — at V_{1-m} — before any image of L_1, so d_m is an interior cell boundary of the $\perp H$ beam; the M points $d_1, \dots, d_M$ are distinct, and $n_H \ge M + 1$.

(3) The same in $U_{0,m}$, with $X := (e_m, 0)$ on its side OR and the beam direction $(0, 1)$: the abscissa of V_m is e_m. The gaps are $e_m - e_{m+1} = \sin((m+1)\alpha)$ for m odd and $\sin(m\alpha)$ for m even, by the same two expansions, and $e_1 = \csc\alpha\cos\alpha = \cot\alpha = |L_2|$ is the endpoint R; so the interior boundaries are $e_2, \dots, e_M$, and $n_{L_2} \ge M$. □

The case $m = 1$ of (2) is the foot of the altitude from R, whose perpendicular runs straight to R.

Theorem 8.2 (an upper bound on n — no hypothesis)**.** *Let* $\gcd(P, Q) = 1$, $Q \ge 3$.

1. $n(P/Q) \le Q - 1$**.** *At even P, equality holds if and only if the LONE class is completely periodic. At odd P and even Q, equality holds if and only if $P = Q - 1$.*
2. *If P is odd,*
   $$n(P/Q) \le Q + 1 - \lceil Q/P \rceil\text{,}$$
   with equality if and only if the defect of the class $\mathfrak{e} = 0$ vanishes and $n_H = M + 1$ (Q odd), respectively $n_{L_2} = M$ (Q even). Equality therefore forces the class $\mathfrak{e} = 0$ — the class of the $\perp L_1$ beam — to be completely periodic.
3. $n(P/Q) \ge 2$ *for* $P \ge 2$*. In particular* $n(P/Q) = 1$ *if and only if* $P = 1$.

Proof. The class of the $\perp L_1$ beam is $\mathfrak{e} = 0$, since $s(L_1) = Q$ and the beam has direction 0. By §3's signature count on $s(L_2) = 0$, $s(L_1) = Q$, $s(H) = P$, this class has $h = 2$, $v = 1$ with vertical side L_1 when P is even (it is then the LONE class at odd Q); $h = 1$, $v = 2$ with vertical sides L_1, H when P and Q are odd (the PAIRED class); and $h = v = 2$ with vertical sides L_1, L_2 when Q is even. This reads only the definitions of §3 and holds on the Veech locus as well. Corollary 7.8, which carries no hypothesis, gives in the three cases

$$n = Q - 1 - D, \quad n + n_H = Q + 1 - D, \quad n + n_{L_2} = Q - D,$$

with the defect D even and non-negative, and $D = 0 \implies CP$.

(1) at even P is the first identity. For its equality case, $D = 0 \implies CP$ is Corollary 7.8, and conversely CP gives $D = 0$ on the LONE class at every odd Q and every P, with no (OB) (Corollary 7.11).

(2) At odd Q, Lemma 8.1(2) gives $n_H \ge M + 1$, so $n = Q + 1 - D - n_H \le Q - M$; at even Q, Lemma 8.1(3) gives $n_{L_2} \ge M$, so $n = Q - D - n_{L_2} \le Q - M$. And $Q - M = Q + 1 - \lceil Q/P \rceil$. Equality holds exactly when $D = 0$

and the lemma's inequality is an equality, and $D = 0$ forces CP by Corollary 7.8.

(1) at odd P follows from (2), since $\lceil Q/P \rceil \geq 2$. Its equality case at even Q: $n = Q - 1$ forces $n_{L_2} = n((Q-P)/Q) = 1$, which by (3), applied to the numerator $Q - P$, forces $Q - P = 1$; and $n((Q-1)/Q) = Q - 1$ by Theorem 10.2.

(3) is Lemma 8.1(1), and $n(1/Q) = 1$ is Theorem 10.2 — or (2) at $P = 1$, where $\lceil Q/P \rceil = Q$. □

At odd P and odd Q, (1) is attained only if $D = 0$ and $n_H = 2$, so only if $n = Q - 1$ is even, which is excluded wherever the orphan exists (Remark 8.3). At even P, equality in (1) is complete periodicity of the LONE class: it fails at 8/15 (§11), and the companion [Meth] proves it at the centres of [Meth] Theorems 2–3 (§1.5). Corollary 10.8 is the instance $P = 3$ of the same bookkeeping, with n_H and n_{L_2} computed exactly by Theorem 10.7 rather than bounded. The bound (2) is far from sharp at large P: it reads roughly $Q(1 - 1/P)$, while $n(5/31) = 5$ (Theorem 10.11) against 25.

Remark 8.3 (the parity rounding, under (OB)). At odd P the count $n(P/Q)$ is odd if and only if the orphan exists ([Orph] Corollary 5.1), and the defect D is even (Corollary 7.8). So under (OB) at P/Q the partner count — $n_H = Q + 1 - D - n$ at odd Q, $n_{L_2} = Q - D - n$ at even Q — is odd, and Lemma 8.1 and Theorem 8.2(3) round it up: $n(P/Q)$ is at most the largest odd integer $\leq Q + 1 - \lceil Q/P \rceil$, with equality if and only if the class $\mathfrak{e} = 0$ is completely periodic and $n_H = M + 1 + [M \text{ odd}]$ (Q odd), respectively $n_{L_2} = M + [M \text{ even}]$ (Q even) — $D = 0 \implies CP$ by Corollary 7.8, and conversely by Corollary 7.11, whose (OB) is the hypothesis here. This bound is at least 3 for $P \geq 3$ and at most $Q - 2$ at odd Q; at even Q with $P \neq Q - 1$, $n(P/Q)$ itself is at most $Q - 3$, since it is odd and Theorem 8.2(1) gives $n \leq Q - 2$ there. (OB) holds at the centres listed in §1.5, and there these bounds are unconditional. At $P = 3$ and $Q \equiv 2 \pmod 3$ the first is attained exactly under complete periodicity (Corollary 10.8).

9. Adjacency, and the involution on cylinders

Two statements about the perpendicular cylinders: that the two members of a mirror pair always sit next to one another on the leg — this is [Orph] Corollary 4.14, at every centre and with no hypothesis — and that [Orph] Theorem 2.2's splitting of the *branches* is matched by a splitting of the *cylinders* into τ-orbits. This section proves the second and reads the first on the cylinders, **with no hypothesis on the direction**. The perpendicular direction may carry minimal components, and Proposition 11.1 exhibits a centre where it does. What makes this possible is that the two inputs one might take from complete periodicity are available without it. The first is [Orph] §2.1's observation that the leg misses every minimal component. The second is the count of [Orph] Lemma 7.8, which admits minimal components.

Throughout, $P \geq 3$ and $Q - P \geq 3$; we refer to that standing restriction as **(R)**. By Theorem 10.1, (R) is exactly the complement of the lattice locus, and it puts outside this section the families the paper pins down separately, the four rows of the lattice locus (Theorems 10.2, 10.3 and 10.5). At even P the same inputs give the identity $n(P/Q) = 2 \cdot C_{\mathrm{met}}$ (Theorem 9.6).

Write Λ for the segment of [Orph] Proposition 7.5 and λ_1, λ_2 for its two leg-copies, which τ exchanges ([Orph] Proposition 7.6); we read the leg as λ_1. Call a **crossing interval** a maximal subinterval of the leg containing no branch boundary whose leaf meets a singularity. (This is a different object from a *crossing* in the sense of §5.1, which is a single point of the leg on one geodesic; the two terms are unrelated and we never use one for the other.) We take this longer definition rather than the shorter reading *no branch boundary whose leaf is a saddle connection*, because a separatrix running into a minimal component is neither a saddle connection nor a regular closed leaf. With the longer definition everything below is hypothesis-free; under complete periodicity the two readings agree.

Proposition 9.1. *Suppose $Q - P \geq 3$. Then a crossing interval contains at most **two** branches; and if two, they are the two members of one mirror pair, adjacent, with a boundary grazing a copy of R between them.*

Proof. A branch boundary interior to a crossing interval has, by the definition just given, a leaf meeting no singularity; and a branch boundary's leaf grazes a vertex image, so that image is regular. By [Orph] Theorem 6.1 the regular images are the copies of R, the images of O when $P \leq 2$, and the images of A when $Q - P \leq 2$. The hypothesis removes the last, and the first of the other two is removed not by it but by (†): a copy of O is unreachable when $P \leq 2$, so where the O-images are regular there is no O-graze to consider. So every interior boundary of the interval is an R-graze, and Proposition 5.3(ii) makes its two flanking branches one mirror pair. Were there three consecutive branches B_1, B_2, B_3, the two boundaries between them would give $J(B_2) = B_1$ and $J(B_2) = B_3$, so $B_1 = B_3$, and they are distinct. □

One further fact is used twice below. The perpendicular direction is a single direction and $d\tau = -\mathrm{id}$, so a leaf crosses each of λ_1, λ_2 in one sense only, and a leaf launched from a given copy returns to *that* copy exactly when its passage word has odd length; that much is elementary. The parity itself is **Lemma 5.2**: $|W|$ is odd on the orphan and even on every mirror-paired branch. The fact is an input to the last clause of Lemma 9.2, and it carries no hypothesis, so neither complete periodicity nor (R) is involved.

Lemma 9.2 (crossing intervals are cylinders)**.** *A crossing interval lies in a single cylinder C, is a whole component of $\lambda_1 \cap C$, and maps monotonically onto the entire leaf space of C; in particular its length is the height of C. Moreover the cylinders met by the leg correspond bijectively to the crossing intervals of the leg.*

Proof. The perpendicular direction decomposes S_α into its saddle connections, finitely many cylinders and finitely many minimal components [MT02]. A point of λ_1 whose leaf meets a singularity is a branch boundary of the kind the definition excludes. The leaf through a point of λ_1 is the forward passage from that height together with its σ_v-image, since σ_v fixes λ_1 pointwise and reverses the beam ([Orph] §7.6), and σ_v permutes the cone points. So the leaf meets a singularity only if the passage meets a vertex image, and then the height is not interior to a branch ([Orph] §2.1). By [Orph] §2.1 the leg meets no minimal component, and it meets the closure of one only at finitely many points on boundary saddle connections, which are such boundaries. So the interior of a crossing interval lies in the union of the open cylinders. It is connected, so it lies in one cylinder C, and it is a whole component of $\lambda_1 \cap C$, because a point of λ_1 on ∂C has a leaf running into a singularity. In cylinder coordinates the flow is horizontal and Λ is perpendicular to it, so the leaf-space coordinate advances along the interval at unit rate: the map to the leaf space of C is strictly monotone and preserves length. Both endpoints of the interval lie on ∂C — an interior one because its leaf meets a genuine singularity, the far endpoint because it is the image of A, singular since $Q - P \geq 3$, and the endpoint at the R-junction because the leaf through it contains the side L_2 and terminates at an image of O, singular since $P \geq 3$. A strictly monotone map of a closed interval whose endpoints go to the two ends of the leaf space is onto it.

For the last clause, a closed leaf of C meets λ_1 once for each component of $\lambda_1 \cap C$. By Proposition 5.3(i) such a leaf meets the leg twice, or once for the orphan; and by the parity remark a non-orphan leaf launched from λ_1 returns to λ_2, so of its two crossings exactly one lies in λ_1, while the orphan's single crossing lies in λ_1. Either way $\lambda_1 \cap C$ has exactly one component. □

We now compute the τ-action, and the computation is downstairs.

Proposition 9.3 (the involution on the cylinders)**.** *Let a_R be the number of saddle connections of the base (S, q) of* [Orph] *Proposition 7.1 in the perpendicular direction that join two images of R. Then the perpendicular direction on S_α has exactly a_R τ-invariant cylinders; its other cylinders form τ-exchanged pairs:* ***none*** *if P is even, and if P is odd* ***at most one****, with exactly one when the $\perp L_1$ beam has an orphan branch (so under (OB)). No hypothesis on the direction is made.*

Here *cylinder* is meant in [Orph] §7.6's convention, which cuts along the leaves through the images of O and A, including those of order 0, and does not mark or cut at the images of R. In (R) no image of O or A has order 0 ([Orph] Theorem 6.1, Corollary 6.2), so the convention cuts only leaves through genuine singularities, which meet no cylinder's interior, and its cylinders are the maximal cylinders of §5.1. That is the sense used in the rest of this section.

Proof. We use [Orph] Lemma 7.8 in the following form, which collects its statement and what its proof establishes.

> [Orph] **Lemma 7.8 and its proof.** *Take a direction on S_α containing a side copy, with cylinders cut as above. On the base (S, q) — a sphere with simple poles at the Q images of R and the points Z_O, Z_A of orders $P-2$, $Q-P-2$ — let G be the graph of the saddle connections in that direction, C_B the number of base cylinders, a the number of components of G that are a single pole–pole edge, a_R the number of those joining two images of R, and κ'' the number of components with an edge and a vertex emitting at least two prongs (necessarily Z_O or Z_A). Then, minimal components allowed, $C_B - a_R \le (a - a_R) + \kappa'' - 1 \le 1$. A base cylinder bounded by an edge between two images of R lifts to one τ-invariant cylinder of S_α, and there is exactly one such base cylinder per such edge; every other base cylinder lifts to two cylinders exchanged by τ.*

The inequality comes from an Euler-characteristic count on the base in which minimal components are allowed ([Orph] Lemma 7.8's proof bounds their contribution by 0), so nothing here assumes complete periodicity. The perpendicular direction contains the side L_2, so the block applies: the τ-invariant cylinders are the a_R lifts of the base cylinders bounded by an edge between two images of R, and the others form $C_B - a_R \le 1$ exchanged pairs. If P is odd and the orphan exists, its cylinder is exchanged (the paragraph before [Orph] Lemma 7.8), so there is exactly one pair.

Let P be even. We show that some copy of the hypotenuse is parallel to the perpendicular direction. The base is tiled by $2Q$ copies of T, the $4Q$ copies of [Orph] Theorem 6.1 modulo $\tau = \rho^Q$ ([Orph] §7.6), with all O-corners meeting at Z_O and all A-corners at Z_A, since [Orph] Proposition 7.1 has one image of each; so every copy of the hypotenuse is a saddle connection joining Z_O to Z_A, with no vertex in its interior. Index the directions of unoriented lines in units of $\pi/2Q$, so that a direction is an element of $\mathbb{Z}/2Q$ — this is *not* the circle of §5.1's (†), which indexes oriented directions in units of π/Q. Placing L_2 at index 0 puts L_1 at Q and the hypotenuse at P. Reflection in a line of index u sends v to $2u - v$, so the reflections in the three sides generate the maps $v \mapsto \pm v + 2kP$. Since $\gcd(P, Q) = 1$ the residues $2kP$ exhaust the even classes, so the copies of a side realise exactly the directions congruent to that side's own index modulo 2. The perpendicular direction is orthogonal to L_1, hence of index $Q + Q = 0$, which is even; so when P is even some hypotenuse copy is parallel to it, and that copy is an edge of G joining Z_O to Z_A.

In the notation of the block above, Z_O and Z_A therefore lie in one component of G. At even P the vertex Z_O emits $P \ge 2$ prongs. Z_A is a simple pole only if $Q - P = 1$, and then its one prong is the hypotenuse edge, so

it ends no pole–pole edge; $Q - P = 2$ cannot occur, since Q is odd. So $(a - a_R) + \kappa'' = 1$ and $C_B - a_R \leq 0$: there is no exchanged pair. □

The cover of a base cylinder is not a cylinder-by-cylinder double cover. Over a base cylinder whose boundary carries an R–R edge, the two sheets are the two halves of the single τ-invariant cylinder upstairs, on either side of its core, and the core is folded two-to-one onto the edge. Over any other base cylinder the two sheets are two distinct cylinders exchanged by τ. In both cases the preimage of a base cylinder is a single τ-orbit.

Lemma 9.4 (a τ-invariant cylinder holds a mirror pair)**.** *Assume (R). If a crossing interval I lies in a τ-invariant cylinder C, then I holds exactly two branches, the two members of one mirror pair, separated by a boundary grazing a copy of R on the core of C.*

Proof. Since $d\tau = -\mathrm{id}$, τ acts on C as a half-turn, which has exactly two fixed points, both on the core. They are Weierstrass points interior to a cylinder, hence regular, so by [Orph] Proposition 7.2 and the lift rule of [Orph] §7.1 they lie over simple poles of the base. In (R) the only simple poles are the images of R, so the fixed points are copies of R. By Lemma 9.2, I meets every leaf of C, the core included. The point x where it meets the core has a regular closed leaf containing a copy of R. That leaf is the passage from x together with its σ_v-image (proof of Lemma 9.2), and σ_v permutes the images of R, so the passage itself meets a copy of R. It does not meet it at an end: the only copy of R on Λ is the junction of λ_1 and λ_2, which lies on ∂C (proof of Lemma 9.2), while x and the core are interior. So x is a branch boundary interior to I, and Proposition 9.1 gives exactly two branches, one mirror pair. □

Adjacency itself needs no cylinders: the two members of every mirror pair are adjacent on the leg, their common endpoint a height whose orbit reaches R before any other vertex, at every centre and with no hypothesis ([Orph] Corollary 4.14). Read on the surface, each mirror pair occupies one τ-invariant cylinder, whose core is the regular closed leaf through that endpoint (Proposition 5.1) — while at odd P the orphan's cylinder is τ-swapped (Proposition 9.3): the orphan retraces at a wall, which is a reflection, rather than at a corner, which is the half-turn. So the leg meets $(n + \nu)/2$ τ-orbits of cylinders, ν the number of orphan branches, in agreement with §5.1.

Remark 9.5 (the certificate)*.* Adjacency can be decided at a single centre, with no hypothesis, by a count. The proof of Proposition 5.3(ii) shows that the two cells flanking an interior boundary whose defining graze is at a copy of R are the two members of one mirror pair. Every such boundary is therefore pair-internal, and distinct ones give distinct pairs, since two intervals of a line share at most one endpoint. Hence

$$\#\{\text{interior boundaries grazing a copy of } R\} \leq \#\{\text{mirror pairs}\},$$

always, with equality exactly when every pair is adjacent, which by [Orph] Corollary 4.14 is always the case.

The count at even P. At even P the same two inputs give the branch count as a cylinder count.

Theorem 9.6 (the even-P count)**.** *Assume (R) and P even, and let C_{met} be the number of perpendicular cylinders the leg meets. Then*

$$n(P/Q) = 2 \cdot C_{\mathrm{met}} \leq 2a_R \leq Q - 1.$$

No hypothesis on the direction is made.

Proof. Every branch lies in exactly one crossing interval, and by Lemma 9.2 the crossing intervals correspond bijectively to the cylinders the leg meets. By Proposition 9.3 every cylinder is τ-invariant, so by Lemma 9.4 each crossing interval holds exactly two branches. Hence $n = 2 \cdot C_{\mathrm{met}}$, and $C_{\mathrm{met}} \leq a_R$ by Proposition 9.3. A simple pole lies on at most one edge, so the a_R edges between images of R have $2a_R$ distinct endpoints among the Q images; Q is odd because P is even and $\gcd(P, Q) = 1$, so $a_R \leq (Q-1)/2$. □

The bound is Theorem 8.2(1) again, whose equality case is complete periodicity of the LONE class; what is new here is the identity $n = 2 \cdot C_{\mathrm{met}}$. At $8/15$, $n = 6$, so the leg meets exactly three cylinders, all τ-invariant. (These are cylinders with the images of R unmarked, as in [Orph]; §11 marks them, which cuts cylinders, and quotes 6.)

10. Explicit counts

10.1. **The lattice locus.**

Theorem 10.1 (the lattice locus, and the dichotomy on it; imported)**.** *S_α is a lattice (Veech) surface if and only if $\min(P, Q-P) \in \{1, 2\}$. For $\min(P, Q-P) > 2$ the $GL_2(R)$-orbit closure is the full locus of holonomy double covers of $Q(P-2, Q-P-2, -1^Q)$, of rank at least two. On a lattice surface, Veech's dichotomy holds: every direction is either completely periodic or uniquely ergodic.*

Given $\gcd(P, Q) = 1$ the criterion $\min(P, Q-P) \in \{1, 2\}$ is elementarily equivalent to the smallest angle of T being of the form π/n, so the first assertion is Kenyon and Smillie's classification of lattice right triangles [KS00, Thm 2] (sufficiency: Veech [Vee89] and Vorobets [Vor96, §4]). Apisa's classification of the orbit closures of rational right triangles recovers it and gives the second [Api21] (Theorem 7.6 with Lemma 7.4). The two lattice families are Veech's regular- and double-regular-polygon loci, and [Orph] Corollary 6.2 shows that at the centres of the locus the cone data of S_α degenerates. [Orph] Corollary 6.2 covers all four families of the locus — $P = 1$, $P = 2$, $P = Q - 2$ and $P = Q - 1$, the last being the $P = 1$ triangle read from its other leg (Remark 10.4) — but it is one implication only and should not be read as a characterisation of the locus; the characterisation is Theorem 10.1's, imported. The third assertion is Veech's [Vee89, Propositions

(2.10)–(2.11)] and is quoted here because the proof of Theorem 10.2 uses it on the row $P = Q - 1$; at $P = 2$ Theorem 10.5 exhibits the decomposition by hand, without the dichotomy. We keep the two notions apart — "lattice surface" is a property of S_α, the dichotomy is a statement about its directions, and only the first is what $\min(P, Q - P) \in \{1, 2\}$ decides.

The dichotomy is the structural reason to expect no closed form for $n(P/Q)$ in general. On a lattice surface the Veech group is a lattice in $SL_2(R)$, the perpendicular direction is one cusp among finitely many orbits, and a single-direction count is finite arithmetic data. Off the locus the orbit closure is positive-dimensional, there is no renormalisation of the direction available, and the count is being asked of one direction on a surface with no finite symmetry group organising its directions. We state this as the reason the deficit resists a closed form, not as a proof that none exists.

10.2. **The counts on the lattice locus.** The lattice locus of Theorem 10.1 is $P \in \{1, 2, Q-2, Q-1\}$. The numerator $P = 2$ is Theorem 10.5. The two rows with $\min(P, Q - P) = 1$ are one triangle read from its two legs, and both counts are theorems.

Theorem 10.2 (the rows $\min(P, Q - P) = 1$)**.** *For every $Q \geq 2$,*

$$n(1/Q) = 1 \text{ and } n((Q-1)/Q) = Q - 1.$$

At $P = 1$ the whole leg is the orphan branch and J is the identity.

Proof. The row $P = 1$. Place T as in [Orph] §1.1 — $O = (0, 0)$, $R = (\cot\alpha, 0)$, $A = (\cot\alpha, 1)$, the beam leaving $(\cot\alpha, s)$, $s \in (0, 1)$, with velocity $(-1, 0)$ — so that here $\alpha = \pi/2Q$. Reflecting successively in the lines through O that carry H and L_2 produces copies $T_0 = T, T_1, \ldots, T_{2Q-1}$, T_k filling the sector $k\alpha \leq \arg z \leq (k+1)\alpha$; since $2Q\alpha = \pi$ they fill the upper half-plane near O. The ray $\arg z = k\alpha$ carries an image of L_2 for k even and of H for k odd. The image of L_1 in T_k is perpendicular to the L_2-ray of T_k at its image of R, at distance $\cot\alpha$ from O; for $1 \leq j \leq Q - 1$ the copies T_{2j-1} and T_{2j} share the L_2-ray $\arg z = 2j\alpha$ and the image R_{2j} of R on it, so their two images of L_1 are collinear and together form one segment, tangent at R_{2j} to the circle $|z| = \cot\alpha$. Hence $U = T_0 \cup \ldots \cup T_{2Q-1}$ is the polygon with vertices $R_0 = (\cot\alpha, 0)$, the images $A_1, A_3, \ldots, A_{2Q-1}$ of A (at arguments $\alpha, 3\alpha, \ldots, (2Q-1)\alpha$), and $R_{2Q} = (-\cot\alpha, 0)$. It is convex — the intersection of the upper half-plane with the half-planes, containing O, bounded by the lines carrying the images of L_1 — and its boundary consists of the segment $R_{2Q}R_0$ through O together with the images of L_1, the first and last of which, R_0A_1 and $A_{2Q-1}R_{2Q}$, are vertical. Every image of R and of A lies on ∂U, and O lies on its lower side.

Unfolded, γ_s runs along the line $y = s$. Since $0 < s < 1$ and $A_1 = (\cot\alpha, 1)$, $A_{2Q-1} = (-\cot\alpha, 1)$, the segment from $(\cot\alpha, s)$ to $(-\cot\alpha, s)$ has its endpoints on the two vertical sides of U, so by convexity its interior lies in the interior of U. It therefore meets no image of a vertex — those lie on ∂U — and no image of L_1 before its far end: γ_s makes $2Q - 1$ reflections,

alternately in H and L_2, and first returns to L_1 at the point of T_{2Q-1} over $(-\cot\alpha, s)$, travelling perpendicular to it. The reflection $z \mapsto -\bar{z}$ maps the sector of T_k to that of T_{2Q-1-k} and preserves the type of each ray, so it carries $T_0 = T$ onto T_{2Q-1} vertex for vertex; the return point is therefore $(\cot\alpha, s)$ itself. So $J(s) = s$ for every $s \in (0,1)$, no point of L_1 is a branch boundary, and the partition has the single cell L_1, the orphan branch. (In the billiard, the segment meets the side copy on the ray $\arg z = \pi/2$ head-on, at the Q-th strike — on H for odd Q, on L_2 for even Q — and retraces from there; its far end is where the retraced path returns.)

The row $P = Q-1$. The perpendicular direction of $T((Q-1)/Q)$ carries closed orbits: a trajectory launched off the finitely many branch boundaries ([Orph] Proposition 3.4) meets L_1 perpendicularly at $J(s)$, reverses there, and retraces its path back to its launch point, where it reverses again. So the direction is not uniquely ergodic, and since S_α is a lattice surface (Theorem 10.1) Veech's dichotomy makes it completely periodic — and with it the class on B (proof of Proposition 11.1). At $Q = 2$ the row is the row $P = 1$. At odd Q the numerator $Q-1$ is even, the $\perp L_1$ beam lies on the LONE class with L_1 its only vertical side (§3), and Corollary 7.11, with no (OB), gives $n_{L_1} = Q-1$. At even $Q \geq 4$ the $\perp L_1$ beam lies on the class $\mathfrak{e} = 0$, whose vertical sides are L_1 and L_2, and Corollary 7.11, with no (OB) at $P \in \{1, Q-1\}$, gives $n_{L_1} + n_{L_2} = Q$; the L_2-beam of $T((Q-1)/Q)$ is the L_1-beam of $T(1/Q)$, the same triangle with its legs exchanged, so $n_{L_2} = n(1/Q) = 1$ and $n((Q-1)/Q) = Q-1$. □

The third row is $P = Q-2$, which forces Q odd; its model is the double regular Q-gon, in $H(2g-2)$ with Veech group the Hecke group $\Delta(2, Q, \infty)$ [Vee89, Proposition (7.8)] (the triangle group is misprinted (n, n, ∞) on p. 571 there, corrected in [Vee91]).

Theorem 10.3 (the row $P = Q-2$)**.** *For every odd $Q \geq 3$,*

$$n((Q-2)/Q) = 2\lfloor (Q-1)/4 \rfloor + 1.$$

Proof. Here $\alpha = \pi/2 - \pi/Q$, so the angle of T at A is π/Q. The $2Q$ copies of T obtained by reflecting about A in the lines carrying L_1 and H fill a regular Q-gon P_1 centred at the image of A: its sides are images of L_2, each the union of two of them meeting at an image of R (two right angles make a straight angle), its vertices are images of O, and its inradius is $|AR| = |L_1| = 1$. The remaining $2Q$ copies fill a second regular Q-gon P_2, and S_α is the double regular Q-gon: in coordinates centred at the respective images of A, $P_2 = -P_1$, and each side of P_1 is glued by translation to the parallel side of P_2. Place the side of P_1 through the launch image of R horizontally at height -1. The perpendicular direction is then horizontal; the two copies of the leg orthogonal to it are $\lambda_1 = \{0\} \times (-1, 0)$ in P_1 and $\lambda_2 = \{0\} \times (0, 1)$ in P_2, exchanged by $\tau : z \mapsto -z$ ([Orph] Propositions 7.5 and 7.6), and the point of L_1 at distance s from R lifts to $(0, y) \in \lambda_1$ with $y = s - 1$.

The circumradius is $1/\cos(\pi/Q)$, and the vertices of P_1 lie at the levels $\ell_i = -\cos((2i+1)\pi/Q)/\cos(\pi/Q)$, $0 \le i \le (Q-1)/2$, from $\ell_0 = -1$ to $\ell_{(Q-1)/2} = 1/\cos(\pi/Q)$. In the band $\ell_{i-1} < y < \ell_i$ the left and the right boundary of P_1 each lie on a single side, and both sides have their midpoint — an image of R — at the mid-level $\mu_i = (\ell_{i-1} + \ell_i)/2 = -\cos(2\pi i/Q)$. Let K be the band containing the centre, $\ell_{K-1} < 0 < \ell_K$, that is $2K-1 < Q/2 < 2K+1$.

Follow the leaf from $(0, y) \in \lambda_1$, y in the band i, leftwards. It crosses P_1 to its left side at height y. The gluing translation carries that side onto the parallel side of P_2, and in centred coordinates it is $z \mapsto z - 2m$ with m the side's midpoint, so the leaf enters P_2 on its right side at height $h = y - 2\mu_i$ and runs to P_2's vertical axis. If $h > 0$ it meets λ_2 there, a return to the leg, and τ carries the point to $(0, -h)$: $J(y) = 2\mu_i - y$. If $h < 0$ it crosses the image of H below P_2's centre, leaves P_2 on its left side at height h, and re-enters P_1 on its right side at height $h + 2\mu_i = y$ — the side of P_1 at height $-h = 2\mu_i - y$ is the right side of the same band, $2\mu_i - y$ being the reflection of y in the band's mid-level — and so returns to λ_1 at its starting point: $J(y) = y$. On the way the leaf meets the image of a vertex exactly when y is a level (an image of O), when $y = \mu_i$ (the two images of R on the sides it crosses), or when $h = 0$ (the centre of P_2, an image of A), and nowhere else; these are the branch boundaries.

Since $2\mu_i = \ell_{i-1} + \ell_i$, a band below the centre ($i < K$, so $\ell_i < 0$) has $h = y - 2\mu_i > -\ell_i > 0$ throughout: it lies in $(-1, 0)$ and contributes the two branches (ℓ_{i-1}, μ_i) and (μ_i, ℓ_i), exchanged by J. In the band K the leg occupies $(\ell_{K-1}, 0)$, and $h = 0$ at $y = 2\mu_K$, which exceeds ℓ_{K-1} because $\ell_K > 0$. If $\mu_K > 0$, that is $4K > Q$, then $h < 0$ on the whole of $(\ell_{K-1}, 0)$, which is one branch, the orphan. If $\mu_K < 0$, that is $4K < Q$, the boundaries $2\mu_K$ and μ_K cut it into the orphan $(\ell_{K-1}, 2\mu_K)$ and the pair $(2\mu_K, \mu_K)$, $(\mu_K, 0)$, exchanged by J. ($\mu_K = 0$ would need $4K = Q$.) Hence $n = 2(K-1) + 1 + 2 \cdot [4K < Q]$. For $Q = 4t+1$ one has $K = t$ and $4K < Q$; for $Q = 4t+3$, $K = t+1$ and $4K > Q$; in both cases $n = 2t+1$. □

So every count on the lattice locus is a theorem: $n = 1$, $Q-1$, $Q-1$ and $2\lfloor (Q-1)/4 \rfloor + 1$ on the rows $P = 1$, 2, $Q-1$ and $Q-2$. The certificate of Appendix A confirms all four at every $Q \le 60$.

The rows $P = 2$ and $P = Q-2$ describe the same triangles seen from the two legs — $T(P/Q)$ and $T((Q-P)/Q)$ are one triangle with its legs interchanged — and the counts genuinely differ, $Q-1$ against $2\lfloor (Q-1)/4 \rfloor + 1$. The branch count is a property of the (triangle, leg) pair, not of the triangle, in contrast with the orphan, which belongs to the triangle ([Orph] Theorem 5.3).

Remark 10.4 (the extreme leg swap). $P = 1$ and $P = Q-1$ are one triangle seen from its two legs, with model the regular $2Q$-gon. By Theorem 10.2 and the correspondence of §5.1 — Propositions 5.1 and 5.3 with Lemma ($\diamond$) (for $Q \ge 4$; at $Q \le 3$ the counts are immediate), unconditional on both legs

— the $P = Q - 1$ leg meets $\lfloor Q/2 \rfloor$ cylinders of its perpendicular direction and the $P = 1$ leg exactly one. The direction perpendicular to the $P = 1$ leg carries more than one for $Q \geq 5$: by [McM25] Corollaries 4.2–4.4 (the billiard form is its Theorems 2.5 and 2.7, by its Theorem 5.1) its closed geodesics through Weierstrass points fall into at least $(Q-1)/2$ classes at odd Q and $Q/2-1$ at even Q; distinct such geodesics lie in distinct cylinders, τ fixing each and acting on its cylinder as a half-turn with both fixed points on the core; and marking the vertex images, as the billiard does, can only cut cylinders. So meeting every cylinder of the perpendicular direction, which the leg does at $P = 2$ (Theorem 10.5), is a property of the leg: it is not implied by the surface being a lattice surface, nor by the lattice locus.

The lattice heuristic of §10.1 does not reverse: $n(3/Q)$ has a closed form under complete periodicity (Corollary 10.8), proved outright whenever Q has at most two odd prime factors (Remark 10.10), while $P = 3$ is non-lattice for every $Q \geq 6$, and at even Q the odd-numerator partner count $n((Q-3)/Q)$ is closed-form outright (Theorem 10.7). So a closed form is not evidence that the surface is Veech, nor is non-lattice-ness evidence that none exists.

10.3. **The surjectivity at $P = 2$.** The count $n(2/Q) = Q - 1$ comes down to one geometric statement — that the launch leg meets *every* cylinder of the perpendicular direction — and the statement is a property of the leg rather than of the surface (Remark 10.4). We prove it here. What makes the difference is that at $P = 2$ the leg is exactly *half* of the shortest saddle connection of its own perpendicular direction, and the two halves are exchanged by a reflection.

Theorem 10.5 (the leg meets every perpendicular cylinder at $P = 2$)**.** *Let $Q \geq 5$ be odd and $\alpha = \pi/Q$. Then the direction perpendicular to L_1 on S_α decomposes into exactly $(Q-1)/2$ cylinders, and L_1 meets each of them in exactly one interval. Consequently $C_{\mathrm{met}} = C_{\mathrm{total}} = (Q-1)/2$ and $n(2/Q) = Q - 1$, where C_{total} is the number of cylinders the perpendicular direction carries and C_{met} the number of them that L_1 meets.*

Proof. The model is the double regular Q-gon (as in the proof of Theorem 10.3, or [Vee89, §7]): two copies Π and Π' of the regular Q-gon, related by $z \mapsto -z$, with side r of Π glued to side r of Π' by a translation. Normalise so that side 0 is horizontal and the circumradius is $1/\sin(\pi/Q)$, so that a half-side has length 1. Write S_0 for side 0; its midpoint is an image of R, its two halves are the two copies of L_1 carried by a horizontal saddle connection, and the direction perpendicular to L_1 is vertical. Write $a = \pi/2Q$, σ for the reflection $x \mapsto -x$ and Γ for the vertical segment $x = 0$. The proof is in four steps; only the second is not immediate.

Step 1: the horizontal direction is elementary. The vertices of Π have y-coordinates $-\cos(k\pi/Q)/\sin(\pi/Q)$ for odd $k \leq Q$, which is $(Q+1)/2$ distinct levels, each carrying two vertices except the top one; the horizontal

chord at a level carrying two vertices joins them and is therefore a saddle connection. No cone point lies strictly between two consecutive levels, so each of the $(Q-1)/2$ open strips of Π between them is a union of horizontal leaves; a strip is a trapezoid — the topmost a triangle, the top level being a single vertex — and the gluing carries its two slanted sides to those of the point-reflected strip of Π', so the two together form a parallelogram — a single cylinder, crossed once in each polygon. Its height is the level gap $[\cos((2i-1)\pi/Q) - \cos((2i+1)\pi/Q)]/\sin(\pi/Q)$ and its circumference is the sum of the two parallel sides, so

$h_i = 2\sin(2\pi i/Q)$, $w_i = 4\cot(\pi/Q)\sin(2\pi i/Q)$, $i = 1, \dots, (Q-1)/2$.

Every cylinder therefore has the *same* modulus $\mu = h_i/w_i = \tan(\pi/Q)/2$. And the heights are pairwise distinct: $\sin(2\pi i/Q) = \sin(2\pi i'/Q)$ with $1 \le i, i' \le (Q-1)/2$ forces $2(i+i') = Q$, which is impossible for odd Q.

Step 2: an affine automorphism carries the horizontal direction to the vertical one. Rotation by $2\pi/Q$ preserves each polygon and the gluing, and $z \mapsto -z$ exchanges them and preserves the gluing; since Q is odd, $\pi/Q = ((1-Q)/2)(2\pi/Q) + \pi$, so *rotation by π/Q is itself an automorphism.* Call it ρ. By Step 1 the horizontal direction is completely periodic with a single modulus, so the simultaneous single Dehn twist in all its cylinders is an affine automorphism $\mathcal{T}$ with derivative $D\mathcal{T} = [[1, 2\cot(\pi/Q)], [0, 1]]$ (the Thurston–Veech construction [Thu88]; [Vee89, (5.2) and §7] — an import, like Veech's dichotomy, Theorem 10.1). Put $m = (Q-1)/2$, so that $D\rho^m$ is the rotation by $m\pi/Q = \pi/2 - a$. A two-line computation with $t = \tan a$ gives the exact identity

$Rot(\pi/2-a) \cdot D\mathcal{T} \cdot Rot(\pi/2-a) = [[0, -\tan a], [\cot a, 0]] =: M$, $\det M = 1$, $M^2 = -I$.

So $\Phi \coloneqq \rho^m \circ \mathcal{T} \circ \rho^m$ is an affine automorphism of S_α whose derivative interchanges the horizontal and vertical directions. Consequently Φ carries the horizontal cylinder decomposition onto the vertical one. A cylinder of core holonomy $(w, 0)$ and height h is carried to one of core holonomy $(0, w\cot a)$, hence of circumference $w\cot a$ and — the area being preserved — of height $h\tan a$. So the vertical direction has exactly $(Q-1)/2$ cylinders, of heights

$h_i^\perp = 2\tan(a)\sin(2\pi i/Q)$,

pairwise distinct, with $\sum_i h_i^\perp = 2\tan(a)\cdot\cot(a)/2 = 1$, the length of L_1.

Step 3: S_0 crosses every vertical cylinder exactly twice. By Step 1 the horizontal saddle connections are exactly the vertex-level chords, of lengths $2\sin(k\pi/Q)/\sin(\pi/Q)$ for odd $k \le Q-2$; the shortest length is 2, attained at $k = 1$, since $\sin(k\pi/Q) > \sin(\pi/Q)$ for $3 \le k \le Q-2$. The chord at a given level occurs once in Π and once in Π', so each length with $3 \le k \le Q-2$ is realised by *two* distinct saddle connections; the shortest is the exception, and that is what makes it unique. The $k = 1$ level is the bottom one, its chord is side 0 of each polygon, and side 0 of Π is glued to side 0 of Π' — so the two $k = 1$ chords are one and the same saddle connection,

namely S_0. Hence S_0 is the unique shortest horizontal saddle connection. Φ carries vertical saddle connections to horizontal ones scaling lengths by $\tan a$, so the shortest vertical saddle connection is unique and has length $2\cot a$. By §5.1 (‡) the segment Γ *is* a vertical saddle connection of length $2(\csc\alpha+\cot\alpha) = 2\cot a$; hence Γ is that shortest one and $\Phi(\Gamma) = S_0$. Now $\Gamma\cap\Pi$ is the full vertical chord of Π at $x = 0$, running from the bottom vertex level to the top one, so it crosses each horizontal strip of Π in exactly one segment, and likewise in Π': Γ crosses each horizontal cylinder exactly twice, once per polygon. Applying Φ, S_0 crosses each vertical cylinder exactly twice.

Step 4: the reflection. The reflection $\sigma : x \mapsto -x$ carries Π to Π and Π' to Π' — the regular Q-gon with a horizontal side has a vertical axis — and carries side r to side $Q - r$ in each, so it preserves the gluing and is an isometry of S_α. Its derivative fixes the vertical direction, so it permutes the vertical cylinders preserving their heights; by Step 2 those heights are pairwise distinct, so σ **fixes every vertical cylinder**. Let C be one. Its intersection with S_0 is two open intervals by Step 3, and neither contains $x = 0$, since Γ is a boundary of the decomposition. σ maps $C\cap S_0$ to itself and an interval it fixed would have to contain 0; therefore σ interchanges the two intervals, and exactly one of them lies in the half $x > 0$, which is L_1. So L_1 meets C, in exactly one interval.

That gives $C_{\mathrm{met}} = C_{\mathrm{total}} = (Q-1)/2$. By Proposition 5.1 the mirror partner $J(s)$ of a point $s \in L_1\cap C$ lies in C again (here also because the vertical cylinders have pairwise distinct heights, Step 2, so τ fixes each of them). By Propositions 5.1 and 5.3, unconditional at $P = 2$, each met cylinder therefore contributes exactly two branches, and by §5.1 (‡) the cylinders of the billiard partition are those of the unmarked surface; so $n(2/Q) = 2C_{\mathrm{met}} = Q - 1$. □

Two comments. First, the argument says exactly why the other leg behaves differently, and the reason is visible in Step 4. At $P = Q - 2$ the launch leg is L_2, whose perpendicular direction is horizontal; the leg is the sub-segment of Γ from a polygon centre to the image of R, and the *other* half of $\Gamma\cap\Pi$ is an image of the hypotenuse, not a second copy of the leg. There is no reflection exchanging the two, and the leg reaches only those strips that dip below the centre — the levels $-\cos((2i-1)\pi/Q)/\sin(\pi/Q) < 0$, that is $2i - 1 < Q/2$, an initial segment of the index. That is the prefix shape of the count, and with the cylinder through the two centres cut in two it returns $2\lfloor(Q-1)/4\rfloor + 1$ — Theorem 10.3, whose proof follows exactly this picture. The $P = 2$ leg is special because it is one of *two* copies of L_1 on a single saddle connection, and the $P = Q - 2$ leg is not.

Second, Step 4 is where the distinctness of the heights is spent, and nothing weaker will do. Any quantity attached to a single cylinder — its height, its circumference, its modulus, the number of sub-strips in it — takes the same value on C and on $\sigma(C)$, because σ is an isometry. So no such quantity

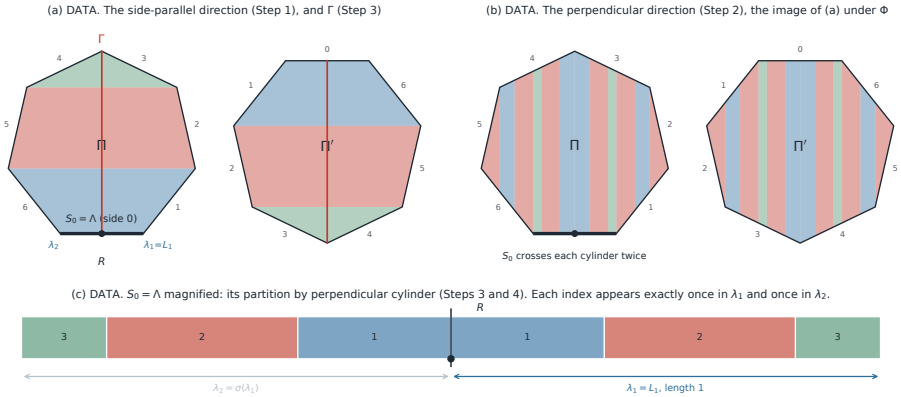


FIGURE 5. The double regular Q-gon at $Q = 7$, and the four steps of Theorem 10.5. Side r of Π is glued to side r of Π', and $\Pi' = -\Pi$. **(a)** The side-parallel (horizontal) direction of Step 1: each of the $(Q-1)/2$ strips of Π between consecutive vertex levels pairs with the point-reflected strip of Π' — same shade — to make one cylinder, and Γ, the vertical chord $x = 0$, crosses each of them exactly twice, once per polygon (Step 3). **(b)** The perpendicular (vertical) direction, which is the image of (a) under the affine automorphism Φ of Step 2; shades match under Φ, and a cylinder here is a union of several strips. **(c)** The side S_0 — which is the segment Λ of [Orph] Proposition 7.5, its marked midpoint the regular point over R and its two halves the leg-copies λ_1, λ_2 of [Orph] Proposition 7.6 — magnified, and partitioned by which perpendicular cylinder each point lies in. Each shade occurs exactly once in $\lambda_1 = L_1$ and once in λ_2, the two exchanged by σ: that is Step 4, and with it $C_{\mathrm{met}} = C_{\mathrm{total}}$ and $n(2/Q) = Q-1$. The interval lengths are the cylinder heights $h_i^{\perp} = 2\tan(\pi/2Q)\sin(2\pi i/Q)$, which sum to 1, the length of the leg.

can decide whether $\sigma(C) = C$; only a quantity that *separates* the cylinders from one another can, which is what Step 2 supplies.

10.4. **The numerator three.** At $P = 3$ the interval model of §4 is explicit enough to be followed by hand. Doing so counts the beam of the second vertical arc exactly, with no hypothesis, and with the covering identity of §7.1 this gives the count $n(3/Q)$ under complete periodicity, and an upper bound for it without.

Setting. $P = 3$, $Q \geq 5$, $3 \nmid Q$, and the class $\mathfrak{e} = 0$: at odd Q the PAIRED class, whose vertical arcs are the copy of L_1 in the fixed kite 0 and the fixed interface H_m; at even Q the class of both leg beams, whose vertical arcs

are the copy of L_1 in kite 0 and the copy of L_2 in kite $Q/2$. Call the arc in kite 0 the **near** arc and the other one the **far** arc Φ, so that its beam count n_Φ is $n_H(3/Q)$ at odd Q and $n_{L_2}(3/Q) = n((Q-3)/Q)$ at even Q (the leg swap, §8). The path of D runs over the interfaces $0, \dots, M$, with $M = m$ at odd Q and $M = Q/2 - 1$ at even Q; by Proposition 2.2 ($\mathfrak{e} = 0$, $c_0 = P$) $c_t = 3 + 6t \pmod{2Q}$ and $\mathrm{ell}_t = \sin(\pi c_t/2Q)$. Write $x = \pi/2Q$, $s_k = \sin(kx)$, $t_* \coloneqq \lfloor Q/3 \rfloor$ and $r_3 \coloneqq Q \bmod 3$.

Lemma 10.6 (the profile)**.** *(i) $c_{t_*} = 2Q - 1$ if $r_3 = 2$ and $c_{t_*} = 1$ if $r_3 = 1$, so $\mathrm{ell}_{t_*} = s_1$; and $\mathrm{ell}_{t_*-1} = s_7$, resp. s_5. (ii) Put $T \coloneqq t_*$ if $r_3 = 2$ and $T \coloneqq t_* - 1$ if $r_3 = 1$. For $0 \le t \le T$, $c_t = 3 + 6t \in (0, 2Q)$, so $\mathrm{ell}_0, \dots, \mathrm{ell}_T$ is unimodal — the **hump** — with $\mathrm{ell}_0 = s_3$ and $\mathrm{ell}_1 = s_9$. Kites $1, \dots, T$ are A-sector, so $I_0, \dots, I_T$ share their low endpoint; kite t_* is O-sector exactly when $r_3 = 1$. (iii) For $t_* < t \le M$, $0 < c_t \le Q$ and c_t increases, so ell_t increases strictly from ell_{t_*} to ell_M and $I_{t_*} \subset I_{t_*+1} \subset \cdots \subset I_M$.*

Proof. Arithmetic on $c_t = 3 + 6t$: $6t_* + 3$ is $2Q - 1$ at $r_3 = 2$ and $2Q + 1$ at $r_3 = 1$. The sector test is Theorem 4.1's $\mathrm{osec}[k] = [(-c_{k-1}) \bmod 2Q \in (0, 6)]$, which is false for $c_{k-1} \le 2Q - 7$ and true for $c_{k-1} = 2Q - 5$. The shared endpoint is the alignment rule of §4 ($u_h i = u_l o$ in the A-sector), and nesting is Proposition 6.10. At the far end $c_M = Q$ (odd Q) or $Q - 3$ (even Q). □

On the hump we use the coordinate $p \coloneqq y - u_0$, in which $I_t = [0,\ \mathrm{ell}_t]$ for $0 \le t \le T$; at $r_3 = 1$, $I_{t_*} = [s_5 - s_1, s_5]$ shares the high endpoint of $I_{t_*-1} = [0, s_5]$. Recall the three moves of §4: a crossing preserves the coordinate; entering a kite through its smaller interface is a forced crossing; entering through the larger one crosses if the coordinate lies in the smaller interface and otherwise folds, $y \mapsto \mu_k - y$, on the protruding tail, whose midpoint $\mu_k/2$ is the pole (μ_k is the fold sum, written s_k in §6.1; renamed here to keep s_k for $\sin(kx)$).

Theorem 10.7 (the far beam, unconditional)**.** *The beam of the far arc has*

$$n_\Phi = 2(M - t_*) + 1 + 2 \cdot [r_3 = 1],$$

that is, $n_H(3/Q) = (Q+4)/3$ or $(Q+8)/3$ at odd Q, and $n((Q-3)/Q) = n_{L_2}(3/Q) = (Q+1)/3$ or $(Q+5)/3$ at even Q, for $r_3 = 2$, resp. $r_3 = 1$. No complete periodicity is assumed.

Proof. The cells of the Φ beam are the components of Φ minus the points whose leaf meets a singularity — the identification in the last sentence of Theorem 3.4's proof, which (unlike the theorem, stated for a completely periodic class) reads no complete periodicity: iota fixes Φ pointwise and reverses the leaf through each of its points, so a leaf from Φ with one singular end has two, and is a saddle connection of type (2) meeting Φ once (Lemma 3.3). Φ is carried to I_M by the leaves (at even Q through the half of the tie kite $Q/2$), so it is enough to follow the leaf from each point y of I_M into D.

(a) The tails. For $t_* < t \le M$ let $F_t \coloneqq I_t \setminus I_{t-1}$, one interval by nesting. From $y \in \ \mathrm{int}\ F_t$ the leaf crosses kites $M, \dots, t+1$ (since $y \in I_t \subset I_{t'-1}$

for $t' > t$), folds in kite t to $\mu_t - y \in F_t$, and climbs back to Φ by forced crossings. It is regular except at the midpoint of F_t, where it meets kite t's pole; the ends of the F_t are interval endpoints, hence singular. So $I_M \setminus I_{t_*}$ carries exactly $2(M - t_*)$ cells.

(b) The window, $r_3 = 2$. Here $I_{t_*} = [0, s_1]$ in the hump coordinate. From $p \in (0, s_1)$ the leaf crosses to interface t_* and into the hump, where every interval contains $[0, s_1]$, since every hump value is at least $\min(s_3, s_1) = s_1$ and $s_1 < s_3$; so it crosses to interface 0 with no fold, transits kite 0, and meets the near arc at a regular point. One cell.

(c) The window, $r_3 = 1$, $Q \geq 10$. Follow $p \in (s_5 - s_1, s_5)$. On the hump, a step towards the peak is forced, and a step away from it crosses if and only if the coordinate is below the next interface; the values beyond the peak are at least s_9 on the left (down to interface 1) and at least s_{11}, then s_5, on the right (down to interfaces $t_* - 2$, $t_* - 1$).

1. *Left.* $p < s_5 < s_9$ (C2) crosses to interface 1; $p > s_3$ (C1) folds in kite 1: $p' = s_3 + s_9 - p$.
2. *Right.* $p' < s_{11}$ (C3) crosses to interface $t_* - 2$; $p' > s_5$ (C4) folds in kite $t_* - 1$: $p'' = s_5 + s_{11} - p'$ (in the hump coordinate the fold sum of kite k is the sum of its two interface values).
3. *Left.* $p'' < s_9$ (C6) crosses to interface 1; $p'' > s_3$ (C7) folds in kite 1: $p''' = s_3 + s_9 - p'' = 2s_3 + 2s_9 - s_5 - s_{11} - p$.
4. *Right.* $0 < p''' < s_5$ (C11, C8) crosses to interface $t_* - 1$ and enters kite t_* through its larger interface. Put $p^* \coloneqq s_1 + 2s_3 - 2s_5 + 2s_9 - s_{11}$, which lies strictly inside the window (C9a, C9b). For $p < p^*$, $p''' > s_5 - s_1$, so $p''' \in I_{t_*}$ and the leaf crosses into $(t_*, M]$ and climbs to Φ — three folds, from Φ back to Φ. For $p > p^*$, p''' lies in kite t_*'s tail $[0, s_5 - s_1]$ and folds: $p'''' = s_5 - s_1 - p'''$.
5. *Left.* $p'''' < s_3$ (C10) crosses to interface 0 and meets the near arc.

The singular points of the window are p^*, whose leaf meets the endpoint $s_5 - s_1$ of I_{t_*}, and the point at which step 2's fold is at kite $t_* - 1$'s pole, $p' = (s_5 + s_{11})/2$, which is the midpoint of $(s_5 - s_1, p^*)$; no other fold is at its fixed point (C12a–c), and no crossing test is met with equality. So the window carries three cells.

Summing, $n_\Phi = 2(M - t_*) + 1 + 2[r_3 = 1]$, and the explicit forms are arithmetic on M and t_*. The conditions C1–C12c, listed in Appendix A.3, are each an odd polynomial in $S = \sin x$, and each has no root in $(0, \sin(\pi/20)]$ and is positive there, by exact real-root isolation; so all hold for $Q \geq 10$. (C6 and C9a vanish to first order in x and are positive at third order, $16x^3$ and $32x^3$; C6 becomes an equality at $Q = 7$, whose factor $8S^3 - 4S^2 - 4S + 1$ vanishes at $S = \sin(\pi/14)$. There is no condition C5.) The one remaining row, $Q = 7$, is decided by exact computation: $n_H(3/7) = 5$. (At $Q = 7$ C3, C4 and C12b fail and C2 and C6 are equalities, C2 since $9x = \pi - 5x$.) □

Corollary 10.8 (the count at $P = 3$)**.** *For $Q \geq 5$ with $3 \nmid Q$,*

$n(3/Q) \le 2 \cdot \textbf{\textit{round}}(Q/3) - 1$***, with equality if and only if the class is completely periodic.***

At even Q the leg-swap partner's count $n((Q-3)/Q) = Q + 1 - 2 \cdot round(Q/3)$ holds unconditionally, by Theorem 10.7 — an off-locus family at an odd numerator. (At $Q = 4$ the formula fails: $n(3/4) = 3$.)

Proof. By Corollary 7.8, $D = (Q - h + v) - \Sigma n_\sigma \ge 0$, and $D = 0$ if and only if the class is completely periodic (Corollary 7.8 and Corollary 7.11; (OB) holds here by Corollary 10.9, and at $Q = 7$ by [Orph] Theorem 4.11). At odd Q, $h = 1$ and $v = 2$, so $n(3/Q) = Q + 1 - n_H - D$; at even Q, $h = v = 2$, so $n(3/Q) = Q - n_{L_2} - D$. Substituting Theorem 10.7 gives $(2Q-1)/3$ at $r_3 = 2$ and $(2Q-5)/3$ at $r_3 = 1$ in both parities, which is $2 \cdot \mathrm{round}(Q/3) - 1$. The partner count is Theorem 10.7 at even Q. □

Corollary 10.9 (the orphan width at $P = 3$)**.** *In the normalisation $|L_1| = 1$, the orphan branch of the $\perp L_1$ beam of $T(3/Q)$ has width*

$$s_1/s_3 \text{ if } r_3 = 2, \text{ and } (-s_1 - 2s_3 + 3s_5 - 2s_9 + s_{11})/s_3 \text{ if } r_3 = 1,$$
$$Q \ge 10.$$

(At $Q = 7$, outside step (c), the same value holds by exact computation; the proof below does not cover it.)

Proof. The cell of Theorem 10.7 whose leaves reach the near arc — the window at $r_3 = 2$, the interval (p^*, s_5) at $r_3 = 1$ — is a band of regular leaves perpendicular to L_1 at one end and to the far side at the other. As billiard orbits they leave L_1 perpendicularly and meet the far side head-on, so they retrace: they form an orphan branch, by [Orph] Lemma 4.2. It is the only one, with no appeal to the uniqueness half of [Orph] Theorem A: by the turnaround clause as quoted in §5.2, every orphan orbit turns around head-on on H (odd Q) or L_2 (even Q), and Φ is the only copy of that side perpendicular to the class (at odd Q the only H-interface with $c_t = Q$ is $t = m$; at even Q Fix(iota) is a square with one vertical copy of L_2, by the proof of Corollary 7.4 and the quarter turn $\rho^{Q/2}$). So every orphan branch is a cell of the Φ beam reaching the near arc, and Theorem 10.7 finds exactly one. Its boundary leaves are singular, so the band is the whole branch, of transverse width s_1 resp. $s_5 - p^*$ in units of $|H|$; and $|H| = 1/\sin\alpha = 1/s_3$ when $|L_1| = 1$. □

Remark 10.10 (the equality, where it is known). The companion [Meth] proves equality in Corollary 10.8 — hence complete periodicity — whenever $5 \nmid Q$ or Q has at most one odd prime factor (its Theorems 2, 3), and, from the far beam of Theorem 10.7 and its code criterion, whenever Q has at most two distinct odd prime factors (its Corollary 7.8). The first Q outside all three is $Q = 385$, where the class is completely periodic by exact computation; and every $P = 3$ centre with $Q \le 60$ is certified completely periodic by Appendix A.

The same bookkeeping sometimes closes at $P = 5$, and when it closes on the **near** arc — the $\perp L_1$ beam itself — it gives the count outright, with no hypothesis at all.

Theorem 10.11 ($n(5/Q) = 5$ at $Q \equiv 1 \pmod 5$, unconditional)**.** *For $Q = 5m + 1$ with $m \geq 3$,*

$$n(5/Q) = 5.$$

(At $m = 2$, $n(5/11) = 7$; at $m = 1$, $n(5/6) = 5$.)

Proof. The class is $\mathfrak{e} = 0$ and $c_t = 5 + 10t \pmod{2Q}$. Arithmetic as in Lemma 10.6: the profile has interior minima at $t = m$ ($c_m = 3$) and $t = 2m$ ($c_{2m} = 1$), with $c_{m-1} = 2Q - 7$, $c_{m+1} = 13$, $c_{2m-1} = 2Q - 9$; the kites m and $2m$ are the only O-sector kites of the path; c increases on each of the two humps $0..m - 1$, $m..2m - 1$ and on the tail $2m..M$ (where $c \leq Q$; M is the last interface, as in the Setting). Here $m = (Q - 1)/5$, not §2's m. So the hump intervals $I_0, \ldots, I_{m-1}$ share their low endpoint 0; $I_m = [a, s_7]$ with $a \coloneqq s_7 - s_3$; the second hump shares the low endpoint a, so $I_{m+1} = [a, a + s_{13}]$ and $I_{2m-1} = [a, a + s_9]$; and $I_{2m} = [a + s_9 - s_1, a + s_9]$. The near arc — the L_1 copy in kite 0, of transverse length $\mathrm{ell}_0 = s_5 = \sin\alpha = |L_1|$ (with $|H| = 1$), which pins the leg — is carried to $I_0 = [0, s_5]$; follow $p \in (0, s_5)$.

1. Right across the first hump ($p < s_5 < s_7$, D2) to interface $m - 1$; kite m is entered through its larger interface, and p crosses into I_m iff $p > a$.
2. *Band 1*, $p \in (0, a)$: it folds in kite m to $a - p < s_5$ (D1) and returns to the near arc. Two cells, cut at $a/2$, kite m's pole.
3. $p \in (a, s_5)$ crosses the second hump ($p < a + s_9$, D3) and folds in kite $2m$ ($p < a + s_9 - s_1$, D4): $p' = 2a + s_9 - s_1 - p$. Left across the second hump ($p' < a + s_{13}$, D6), it folds in kite $m + 1$ ($p' > a + s_3$, D5): $p'' = s_1 + s_3 + s_{13} - s_9 + p$. Right again ($p'' < a + s_9$, D7): with $p^* \coloneqq a + 2s_9 - 2s_1 - s_3 - s_{13} \in (a, s_5)$ (D8, D9),
   - *band 3*, $p \in (p^*, s_5)$: $p'' \in I_{2m}$, so it crosses into the tail and climbs to the far arc — one cell (its fold at kite $m + 1$ misses that kite's pole, which lies in band 2's image);
   - *band 2*, $p \in (a, p^*)$: it folds in kite $2m$ again to $p''' \in (a, a + s_3)$ (D10) and returns across the second hump ($p''' < a + s_3 < a + s_{13}$, D11), kite $m + 1$, kite m (entered through its smaller interface) and the first hump ($p''' < p^* < s_5$) to the near arc. Its middle fold is at kite $m + 1$, whose pole is the midpoint of the band's image there (an identity), so two cells.

No other fold meets its pole (D12, D13) and no test is met with equality, so the near arc carries $2 + 2 + 1 = 5$ cells. The conditions D1–D13, each the strict inequality of one step above (Appendix A.3), are odd polynomials in $\sin x$, root-free on $(0, \sin(\pi/28)]$, so they hold for $Q \geq 14$; the tightest, D9 ($p^* < s_5$), fails at $Q = 11$, which is why $Q = 11$ is the exception (its value $n(5/11) = 7$ is by exact computation). □

This is a family of the kind §13 item 1 asks for: an odd numerator ≥ 5, off the Veech locus ($\min(5, Q-5) = 5$), with the count proved rather than computed.

The method's reach is limited, and we say so: in our computations (an observation, not a theorem) across $P \leq 13$ and every residue class, the only families whose walks stay bounded as m grows are this one and two more, at $(5, 5m+4)$ and $(7, 7m+6)$, which we do not pursue; for every other residue at least one leaf wanders a number of folds that grows with Q, and the bookkeeping does not close.

Two remarks. First, the argument uses no complete periodicity until Corollary 10.8, and there only through the equality case, by Corollaries 7.8 and 7.11: Theorem 10.7, Corollary 10.9 and Theorem 10.11 are unconditional, and the upper bound in Corollary 10.8 is unconditional too. Second, it is the interval model and not the cylinder picture that makes $P = 3$ tractable. The hump is short in the sense that matters: a leaf entering it through the window either never folds ($r_3 = 2$) or folds a bounded number of times ($r_3 = 1$), independently of Q, because the window is s_1 wide against hump values s_3 and s_9 at the two ends where a fold can occur. At $P \geq 5$ the profile has more humps and the same bookkeeping has no reason to close in general — the three families above are the cases where it does.

10.5. **The count is not maximal in general, and not regular.**

Remark 10.12. For general even P the count is not maximal: $n(8/15) = 6$, against $Q - 1 = 14$. For odd $P \geq 5$ the count is arithmetically irregular: $n(5/22) = 5$ while $n(5/24) = 17$. Figure 6, in §13, plots both statements against every certified centre in a window.

All four counts are certified by the exact computation of Appendix A, so the irregularity is a fact about the triangle and not an artefact of a truncated computation; §10.1 gives the structural reason to expect no closed form off the lattice locus, and §13 records it as open. The two failures are not alike. $8/15$ is the only even-P centre at which we have found the count to be non-maximal: among the 307 even-P centres off the lattice locus with $Q \leq 60$ it is the sole exception. At even P, by Theorem 8.2(1), the count is maximal exactly when the LONE class is completely periodic, so $8/15$'s deficit is the signature of its minimal component (§11).

11. Where complete periodicity fails

The CP set on the band $2 \leq Q \leq 60$ (the centre $1/2$ included) is completely determined: all 2202 class rows — both classes at each of the 1101 coprime centres — are decided, with none left unresolved.

- 2200 **are certified completely periodic** by the exact flow decomposition of Appendix A.2, which returns every component of the direction as a cylinder.

- 2 **are proved *not* completely periodic** — the leg-swap pair 8/15 with $\mathfrak{e} = 0$ and 7/15 with $\mathfrak{e} = 1$ — unconditionally, from one certified branch count (Proposition 11.1), and independently from a certified minimal component.

Each class row is reached through one perpendicular direction. The class $\mathfrak{e} = 0$ contains the $\perp L_1$ direction, and the decomposition in that direction certifies complete periodicity at every centre except 8/15, where it returns 6 cylinders and 2 minimal components. At odd Q the class $\mathfrak{e} = 1$ of P/Q is the class $\mathfrak{e} = 0$ of $(Q - P)/Q$, the same triangle with its legs exchanged, so the same computation decides it. At even Q the class $\mathfrak{e} = 1$ has H as its only vertical side (§3), and the decomposition in the $\perp H$ direction certifies it completely periodic at all 373 centres with even $2 \le Q \le 60$. On each of those rows $n_H = Q$, as Corollary 7.11 requires. At odd $Q \le 31$ the $\perp H$ decomposition was run as well, and it agrees with the $\perp L_1$ verdict on all 212 rows. An independent exact closure certificate in $\mathbb{Z}[\zeta_{4Q}]$ gives the same verdicts on the 424 class rows with odd $Q \le 31$.

The pair is one direction, on the LONE class, at $Q = 15$.

Proposition 11.1. *The class rows* $8/15, \mathfrak{e} = 0$ *and* $7/15, \mathfrak{e} = 1$ *are not completely periodic.*

Proof. At 8/15 the class $\mathfrak{e} = 0$ is the LONE class (P even, Q odd), with vertical side L_1. By Theorem 8.2(1) it is completely periodic if and only if $n(8/15) = Q - 1 = 14$; but $n(8/15) = 6$ (Remark 10.12, certified by Appendix A.2). The class $\mathfrak{e} = 1$ of 7/15 is the class $\mathfrak{e} = 0$ of 8/15, the same triangle with its legs exchanged. □

Two independent witnesses agree, and neither uses $p = 0$. First, a direction perpendicular to a side of the triangle with angles $7\pi/30$, $8\pi/30$ carries a minimal component: Appendix A.2's decomposition of the $\mathfrak{e} = 0$ direction finds on S_α 6 cylinders (with the images of R marked) and 2 minimal components; Delecroix and Rüth certify it by an exact minimality test [DR22], and [Meth] Proposition 6.4 proves it by hand. Complete periodicity is a property of B and transfers both ways across the degree-2 cover $\pi : S_\alpha \to B$: a cylinder contains no singularity, so π restricted to its preimage is an unbranched cover of an annulus; and conversely the holonomy involution permutes the cylinders of a decomposition. Second, that direction is the row $8/15, \mathfrak{e} = 0$, because the other class is completely periodic: $n_{L_2}(8/15) + n_H(8/15) = 11 + 5 = 16 = Q + 1$, so $D = 0$ and Corollary 7.8 applies.

The argument is a one-row result — it rests on one certified count — and decides exactly the two rows named. At 8/15 the leg nevertheless has a finite branch partition whose widths sum to 1, as [Orph] §2.1 requires: by [Orph] Proposition 3.4 the leg misses the minimal component, meeting its closure in at most finitely many points. That is why [Orph] §2.1 rests on finiteness and covering and not on complete periodicity: the stronger

statement is false here, and the weaker one suffices for every count in this paper.

For a right triangle, Boshernitzan's Conjecture 2.2(e) [Bos92] asks that every direction in $S^1(2d')$ be purely periodic; the perpendicular direction of $8/15$ lies in $S^1(60)$ and carries a minimal component, so $8/15$ is a counterexample, due to Delecroix and Rüth [DR22]. The companion [Meth] treats the conjecture.

12. Mirror-direction corollaries

A **mirror direction** of T is the axis of a reflection in the group $\mathcal{G}$ of §5.3, generated by the reflections in the lines of the sides of T. (Galperin and Zvonkine's *mirror trajectories* [GZ03, §3] are a different notion.) By the first paragraph of the proof of Lemma 5.6, the mirror directions are exactly the integer directions in the units of §1.1. So they are the directions of the two classes $(P/Q, 0)$ and $(P/Q, 1)$, and the perpendicular to a mirror direction is again one. Every statement of this paper about *both classes* is therefore a statement about every mirror direction. This section draws two consequences for billiard paths. As in §5.3, we allow $Q = 2$ here. Every rational right triangle has $2\alpha/\pi = P/Q$ with $\gcd(P, Q) = 1$, $0 < P < Q$ and $Q \geq 2$: if $\alpha/\pi = a/b$ in lowest terms, take $(P, Q) = (a, b/2)$ for b even and $(2a, b)$ for b odd.

Corollary 12.1 (periodic paths in mirror directions are perpendicular)**.** *Let T be a right triangle whose acute angles are rational multiples of π. Every periodic billiard path of T whose direction is a mirror direction meets some side of T at a right angle, at a point that is not a vertex. No hypothesis on the direction is made: the flow in it need not be completely periodic.*

Proof. Let γ be such a path and $(P/Q, \mathfrak{e})$ the class of its direction. First take $Q \geq 3$. Let ω be the direction of the lift of γ to S_α. That lift is a closed leaf avoiding the vertex lifts, and we let $C \subset S_\alpha$ be the maximal cylinder of direction ω containing it. In this proof cylinders are those of S_α itself, with no marks. Put $U \coloneqq \pi_B(C) \subset B$, where $\pi_B : S_\alpha \to B$ is the quotient by τ, and let r and V be as in Lemma 5.6.

Step 1: if C meets V, every closed leaf of C that avoids the vertex lifts meets a side copy perpendicular to ω at an interior point. r fixes a point of C (Lemma 5.6), and $r(C)$ is a maximal cylinder of direction ω meeting C, so $r(C) = C$. Since r preserves C, reverses the flow direction and fixes the transverse one (Lemma 5.6), it acts on $C = (\mathbb{R}/a\mathbb{Z}) \times (0, h)$ as $(t, y) \mapsto (t_0 - t, y)$. Its fixed set there is two full-height transversals contained in V together with the marked points. Every closed leaf of C crosses both transversals, and a leaf that avoids the vertex lifts crosses them at interior points of side copies perpendicular to ω.

Step 2: the two shapes of U. The linear part of τ is -1, so τ maps leaves to leaves. Hence $\tau(C)$ is a maximal cylinder of the same direction, and either $\tau(C) = C$ or $\tau(C) \cap C = \emptyset$.

(a) Suppose $\tau(C) = C$. Then τ acts on $C = (\mathbb{R}/a\mathbb{Z}) \times (0, h)$ as the orientation-preserving isometry $(t, y) \mapsto (t_0 - t, h - y)$, which exchanges the two boundary components. Its fixed points are exactly $(t_0/2, h/2)$ and $(t_0/2 + a/2, h/2)$, both on the mid-height leaf. They are interior to C, hence regular points of S_α, and τ fixes them. So their images are two distinct simple poles $p \neq p'$ of B. The image of either half of the mid-height leaf is a saddle connection $[p, p']$ of B with no singular point in its interior. Every other leaf of U is closed and free of singular points.

(b) Suppose $\tau(C) \cap C = \emptyset$. Then π_B is injective on C, and U is a flat open annulus of closed leaves free of singular points. It is a maximal cylinder of B. Indeed, let $U \subseteq U'$ with U' a cylinder of B. The branch points of π_B lie over the points of B of odd order: the simple poles, and Z_O or Z_A when P or $Q - P$ is odd ([Orph] Proposition 7.2). All of these are singular points of B, and U' contains none, so $\pi_B^{-1}(U') \to U'$ is an unbranched double cover. Its component containing C is an annulus of closed leaves containing C, hence equal to C by maximality, and so $U' = \pi_B(C) = U$.

Step 3: if C misses V, then U misses Fix(*iota*). By Lemma 3.3, Fix(iota) is the union of whole side copies of B, horizontal or vertical.

- *Vertical arcs.* Suppose a vertical arc meets the open set U. Then it meets U at a point x interior to the arc. Write $x = \pi_B(y)$ with $y \in C$. Then y is an interior point of a side copy of S_α over that vertical side copy. That copy is perpendicular to ω, so $y \in V \cap C$, against the hypothesis.
- *Horizontal arcs.* Suppose a horizontal arc meets U at a point x whose leaf in U is closed and free of singular points. The arc lies along that leaf, since both are geodesic segments of direction ω through x. But every arc of Fix(iota) is a whole side copy with a singular endpoint. A copy of L_1 or L_2 ends at a pole R_i. A copy of H joins Z_O to Z_A, and these are not both regular, because $P = 2 = Q - P$ is excluded by coprimality. A singular endpoint cannot lie on a leaf free of singular points: contradiction.
- *The segment $[p, p']$.* It remains to treat, in case (a), a horizontal arc through a point of $[p, p']$. Such an arc runs along $[p, p']$ to both ends, so it is a side copy joining p and p'. No side copy joins two kite poles (Proposition 2.1). So one of the two, say p, is Z_O with $P = 1$ or Z_A with $Q - P = 1$, the named cone point being then a simple pole. In that case $p \in U \cap$ Fix(iota). The boundary corner of D at p has angle $P\pi/2 = \pi/2$, respectively $(Q - P)\pi/2 = \pi/2$ (§2), and one of its two arms is horizontal. So the other arm of Fix(iota) at p is vertical. It enters the open set U, and the vertical case applies.

Step 4: if U misses Fix(*iota*)*, then an interior pole prong of D reaches a different interior pole.* By the proof of Corollary 7.4, $B \setminus$ Fix(iota) is two open discs. The curve Fix(iota) is a Jordan curve, and iota reverses orientation and fixes it pointwise, so near each point of it iota exchanges the

two sides. Hence iota exchanges the two discs, and each maps isometrically onto int D. U is connected, so it lies in one of them, say D_1. The singular points in D_1 are interior poles of D (step (0) of the proof of Theorem 5.4). In case (a), $[p, p'] \subset U \subset D_1$ is the prong of the interior pole p, and it reaches the different interior pole p'. In case (b), let γ_U be a closed leaf of U. It is a Jordan curve in the open disc D_1, so it bounds a closed disc $\Delta \subset D_1$, and the boundary component $\partial_1 U$ of U on Δ's side lies in Δ. By Step 2(b), U is maximal, so $\partial_1 U$ contains a singular point of B, and the singular points in Δ are interior poles of D. A simple pole carries a single prong, so the boundary path arrives at it and leaves it along that prong. Hence $\partial_1 U$ is a single saddle connection traversed twice, from a pole p to a singular point $p' \in \Delta$, which is again a pole. If $p' = p$, a prong of p returns to p, which Theorem 6.3 excludes. So $p' \neq p$, and again an interior pole prong of D reaches a different interior pole.

Assembly. Theorem 6.1 excludes the conclusion of Step 4. So by Step 3, C meets V. Then, by Step 1, the lift of γ — a closed leaf of C avoiding the vertex lifts — meets a side copy perpendicular to ω at an interior point. So γ meets a side of T at a right angle, at a point that is not a vertex.

The case $Q = 2$. Here T is the isosceles right triangle, and the mirror directions are the multiples of $\pi/4$. Place the legs on the coordinate axes and H on $x + y = 1$.

- *Axis-parallel directions.* Every segment of the path is horizontal or vertical. A periodic path strikes a leg at some point, since it cannot strike H twice in succession. A horizontal or vertical segment that ends on a leg is not parallel to it, so it meets that leg at a right angle.
- *Diagonal directions.* Every segment has direction $\pm(1, 1)$ or $\pm(1, -1)$. A segment of direction $(1, 1)$ increases $x + y$, so it ends on H, and it meets H at a right angle. A segment of direction $\pm(1, -1)$ is parallel to H, so it ends on a leg and leaves in direction $(1, 1)$. A segment of direction $(-1, -1)$ ends on a leg and leaves in direction $\pm(1, -1)$. So within two reflections the path travels in direction $(1, 1)$ and meets H at a right angle.

In both cases the meeting point is not a vertex, because a periodic billiard path meets no vertex. □

The corollary is the input to a complete answer to the companion question of Theorem 5.4: every cylinder of a class direction is met by *some* perpendicular transversal, and the perpendicular beams together count the cylinders.

For a side σ and a cylinder c, write $k_c(E)$ for the number of components of $E \cap c$, where E is a copy of σ in V. Write $Sat(V)$ for the union of the leaves of direction ω that meet V.

Theorem 12.2 (the sweep law)**.** *Let $0 < P < Q$ be coprime, $Q \geq 2$. Fix a class $(P/Q, \mathfrak{e})$ and a direction ω of it, and let V be as in Lemma 5.6. Cylinders are those of S_α with every vertex lift marked (§5.3). Then:*

1. *every cylinder of direction ω meets V, and each of its closed leaves crosses V* ***exactly twice*** *per circuit;*
2. *up to the finitely many saddle connections of direction ω, $Sat(V)$ is the union of the cylinders, and its complement is the union of the minimal components. In particular $Sat(V)$ has full area if and only if the class is completely periodic. Equivalently: almost every billiard trajectory of T with direction in the class meets a side of T at a right angle if and only if the class is completely periodic;*
3. $\sum_{\sigma : s(\sigma) \equiv \mathfrak{e}+Q \pmod 2} n_\sigma = \boldsymbol{ncyl}(S_\alpha)$**.**

No hypothesis on the direction is made. Complete periodicity, $p = 0$, (PP), (ZZ), (OB) and Veech's dichotomy are not used.

Proof. (i) Let c be a cylinder and ℓ a closed leaf of it. ℓ avoids the marked points, so it projects to a periodic billiard path of T in a direction of the class, and every such direction is a mirror direction. By Corollary 12.1 that path meets a side of T at a right angle, at a point that is not a vertex. So ℓ crosses a side copy perpendicular to ω at an interior point, which is a point of V. By Lemma 5.6, r fixes that point, and $r(c)$ is a cylinder of direction ω meeting c; so $r(c) = c$. Write $c = (\mathbb{R}/a\mathbb{Z}) \times (0, h)$ with the flow along the first coordinate. The isometry r preserves c, reverses the flow direction and fixes the transverse one, so it acts as $(t, y) \mapsto (t_0 - t, y)$. Its fixed set in c is the two full-height transversals $t = t_0/2$ and $t = t_0/2 + a/2$. That fixed set is $\mathrm{Fix}(r) \cap c = V \cap c$, since c contains no marked point. Every closed leaf of c crosses each transversal exactly once.

(ii) $\subseteq$. r is an isometry of S_α that preserves the foliation of direction ω, fixes V pointwise, and reverses the flow direction (Lemma 5.6). These are the only properties of iota that the proof of Lemma 7.2 reads. That proof therefore applies word for word with $(B,\ \text{iota}, V_0)$ replaced by (S_α, r, V), the first-return map being taken to V. So almost every point of V lies on a closed leaf. No minimal component meets V, by the argument of Corollary 7.4: if a minimal component M met V, then $M \cap V$ would be a non-empty relatively open subset of V, of positive length, and almost all of it would lie on closed leaves, which a minimal component does not contain. A direction of a compact translation surface with finitely many marked points decomposes into finitely many cylinders and finitely many minimal components, together with finitely many saddle connections [MT02]. Hence every point of V lies in a cylinder or on a saddle connection, and $Sat(V)$ is contained in the union of the cylinders and the saddle connections. (In measure the same follows because the saturation of a null subset of a transversal is null: the transverse measure is invariant, and flow boxes cover the saturation.) $\supseteq$. By (i) every leaf of every cylinder meets V. So up to the saddle connections $Sat(V)$ is the union of the cylinders, and by the same decomposition its complement is the union of the minimal components. Minimal components are open and non-empty, so they have positive area. Hence $Sat(V)$ has full area if and only if there is no minimal component, that is, if and only if

the direction is completely periodic. Complete periodicity of the direction on S_α is complete periodicity of the class on B (proof of Proposition 11.1). The billiard reading follows: unfolding sends a trajectory with direction in the class to a leaf of direction ω (the class is one $\mathcal{G}$-orbit of directions), and the trajectory meets a side at a right angle exactly when the leaf meets V.

(iii) Let σ be a vertical side and E either of its two copies in V. The map $(x, g) \mapsto x$ is an isometry of E onto σ. The leaf through (x, g) projects to the billiard orbit launched from x perpendicular to σ, and it meets a marked point exactly when that orbit meets a vertex before its first perpendicular return: a leaf through a point of V that meets V again is carried onto itself, reversed, by r, which fixes both points, so it closes up there and meets no marked point. Write S_E for the set of these points of E. Then the cells of the $\perp \sigma$ beam are carried onto the components of $E \setminus S_E$, and n_σ is the number of those components, for either copy (the two copies are also exchanged by τ). By the proof of (ii), a point of E lies in a cylinder or on a saddle connection. A saddle connection runs between marked points, and every cone point of S_α lies over a vertex and is marked. So $E \setminus S_E = E \cap (\cup_c c)$. The cylinders are disjoint open sets, so the components of $E \setminus S_E$ are the components of the sets $E \cap c$, and

$$n_\sigma = \sum_c k_c(E).$$

By (i), $V \cap c$ is two full-height transversals. Each is connected and contains no marked point, and distinct copies in V meet only at marked points, so each transversal lies in the interior of a single copy. Hence $\sum_{E \subset V} k_c(E) = 2$ for every cylinder c. Summing over c, and using that each vertical side has two copies in V,

$$2 \sum_\sigma n_\sigma = \sum_{E \subset V} \sum_c k_c(E) = \sum_c 2 = 2 \cdot \mathrm{ncyl}(S_\alpha).$$

□

Clause (i) is the *set* form of the covering half of §3, at every centre and with no hypothesis; the *count* form, Corollary 3.5, needs $p = 0$. Clause (iii) says exactly what the count form loses off complete periodicity: Corollary 7.8's defect is $D = (Q - h + v) - \mathrm{ncyl}(S_\alpha)$, the shortfall of the marked cylinder count from $Q - h + v$; at $8/15, \mathfrak{e} = 0$ the direction carries 6 cylinders and 2 minimal components (§11), and $D = 14 - 6 = 8$. Clause (ii) detects whether the class is completely periodic but does not locate where it fails, and it needs the whole class: for one transversal of a class with $v = 2$ a swept fraction below 1 does not imply a minimal component. Nothing here bears on the value of n_σ.

Remark 12.3. (1) The proof of Corollary 12.1 uses Theorems 6.1 and 6.3, which concern only the prongs launched at interior poles. It needs neither complete periodicity nor $p = 0$. The only part of the boundary that Step 4 reads is the segment $[p, p']$ in case (a) and the boundary component inside

the disc Δ in case (b), and no zero lies on either. So the corner-launched half, (ZZ), is never consulted. This is why the conclusion holds on directions with minimal components, such as the one of §11. (2) For the cylinders of a fixed direction the corollary says more: by Step 1, every closed leaf of every cylinder crosses the perpendicular side copies exactly twice per circuit, which is clause (i) of Theorem 12.2. (3) The corollary is the rational counterpart of [MirA] Theorem 3.1, which reaches the same conclusion for right triangles with $\cos 2\alpha$ transcendental by a different argument. At irrational α/π with $\cos 2\alpha$ algebraic it can fail ([MirA] §5). The cylinders that witness the failure there have the shape of case (a) of Step 2, a τ-invariant cylinder with two right-angle-vertex lifts on its core. What has no counterpart there is Theorem 6.1's arithmetic.

The second consequence concerns isosceles triangles. Let $I = I(\alpha)$ be the isosceles triangle with base angle α, base midpoint M, apex N and axis MN, and let ϱ be the reflection in MN. ([MirA] §4.2 writes O, B, Δ, σ for M, N, MN, ϱ, and [MirA] §2.1 writes O, A, B for the vertices R, O, A of T and OB, OA for the legs L_1, L_2.) Folding I along MN gives T with the labelling of [Orph] §2.1: $M \mapsto R$, each base endpoint $\mapsto O$, $N \mapsto A$, the axis $MN \mapsto L_1$, each half of the base $\mapsto L_2$, and each leg of $I \mapsto H$. A periodic billiard path of I has **odd period** if it makes an odd number of bounces per period. For a point y of L_1, γ_y is the trajectory launched from y perpendicular to L_1, as in [Orph] §2.1. The input is the following lemma. It is [MirA] Corollary 4.1 and Proposition 4.3(1)–(4), (3) without its last clause; the proofs are short, and we reproduce them so that this paper is self-contained. Part (0) answers a question of Holt, who excluded the periods 3, 5 and 7 [Hol93, §4.3]. Both of its ingredients are known — odd orbits are stable ([GSV92, §8A], [HH00, Thm 1]), and Hooper's theorem — and their combination is [MirA]'s. Periodic orbits under the fold of I onto T are studied by Becker [Bec13].

Lemma 12.4 (odd orbits of I)**.** *(0) No right triangle, rational or irrational, has a periodic billiard path of odd period. Let $\alpha \in (0, \pi/2)$ and let γ be a periodic billiard path of $I(\alpha)$ of odd period. Then:*

1. *γ bounces at M;*
2. *$\varrho\gamma$ is γ reversed;*
3. *if γ is primitive, then in each period it bounces exactly once at M and crosses MN at a right angle exactly once;*
4. *conversely, a path leaving M that meets MN at a right angle after b bounces, without meeting a vertex or returning to M first, closes up with its ϱ-image into a primitive periodic path of period $2b + 1$.*

Proof. (0) Let β be a periodic path of odd period p in a triangle T', and unfold one period along its orbit type. The composition F of the p reflections is orientation-reversing, fixes the direction v of β, and moves the starting point z to $z + Lv$, $L > 0$. So F is a glide reflection: the reflection in a line ℓ

parallel to v followed by a translation cv along it. Its displacement $F(z)-z$ is cv plus a vector normal to v of length twice the distance from z to ℓ; as $F(z)-z=Lv$, that distance is 0, so the unfolded path is the axis ℓ and $c=L$. For a triangle near T' the same orbit type gives a glide reflection near F, with axis near ℓ and positive glide; since ℓ crosses each edge at positive distance from the vertices, the new axis crosses the same edges in the same order and folds to a periodic path of the same orbit type. So β is *stable*. By Hooper's theorem [Hoo07, Thm 1] no periodic path of a right triangle is stable (Hooper reduces the rational case to the irrational one, [Hoo07, p. 6]). (1) Otherwise fold γ along MN. The image is a billiard path of T that meets no vertex: O and A are images of vertices of I, which γ avoids, and R is the image of M alone. It closes after $p+c$ bounces, where p is the period of γ and c its number of crossings of MN per period; c is even because MN separates I. So its minimal period divides the odd number $p+c$ and is odd, contradicting (0). (2) At a bounce at M the outgoing direction is the reflection of the incoming one in the base, and ϱ composed with the reflection in the base is -1. So $\varrho\gamma$ reversed passes through M, which γ does by (1), with the same outgoing direction as γ; a billiard path is determined by a point and an outgoing direction. (3) Parametrise γ by arclength t, on a circle of the length of one period. By (2), $\varrho\gamma(s-t)=\gamma(t)$ for some s; the involution $t\mapsto s-t$ of the circle has exactly two fixed points, and the bounces it does not fix are exchanged in pairs. A bounce at M, or a right-angle crossing of MN, at time t_1 gives $\varrho\gamma(2t_1-t)=\gamma(t)$ (the computation in (2), and ϱ reverses a direction normal to MN); then γ is invariant under the shift by $2t_1-s$, so as γ is primitive $2t_1\equiv s$ and t_1 is a fixed point. Conversely a fixed point lies on MN with its direction reversed by ϱ: it is a right-angle crossing, or a bounce at $MN\cap\partial I=\{M,N\}$, and N is a vertex. As the period is odd, the number of fixed points that are bounces is odd: exactly one fixed point is the bounce at M, and the other is the right-angle crossing. (4) Reflect the path in MN and reverse it. The two halves join at the right-angle crossing, where ϱ reverses the direction, and at M, by the computation in (2). The result is a closed path of $2b+1$ bounces that meets no vertex. Neither half returns to M, so it bounces at M once per period, and a shift preserving it preserves that bounce: it is primitive. □

Corollary 12.5 (odd orbits of isosceles triangles)**.**

1. *For every $\alpha\in(0,\pi/2)$, sending a primitive periodic path of $I(\alpha)$ of odd period to the point where it crosses MN at a right angle induces a bijection between two sets: the primitive odd-period orbits of $I(\alpha)$ up to reversal, and the points y interior to L_1 whose trajectory γ_y in T meets the vertex R before any other vertex.*
2. *If $\alpha/\pi\in\mathbb{Q}$, those points are exactly the interior branch boundaries of the $\perp L_1$ partition of T whose defining graze (§5.1) is at R.*

3. *Let Q be odd and P even with $4 \le P \le Q-3$, and let $2\alpha/\pi = P/Q$, so that $\alpha = m\pi/Q$ with $P = 2m$. Then $I(\alpha)$ has exactly $n(P/Q)/2$ primitive odd-period orbits up to reversal. This is at most $(Q-1)/2$, with equality if and only if the LONE class of P/Q, the class of the $\perp L_1$ beam, is completely periodic. Equality fails at $\alpha = 4\pi/15$.*
4. *If $\alpha = \pi/Q$ with $Q \ge 5$ odd, then $I(\alpha)$ has exactly $(Q-1)/2$ primitive odd-period orbits up to reversal.*

Proof. (i) *Orbit to point.* Let γ be a primitive periodic path of odd period. By Lemma 12.4(1)–(3), in each period it bounces exactly once at M and crosses MN at a right angle exactly once, at a point x. The two fixed points of ϱ composed with time reversal are this crossing and the bounce at M, because the apex N is a vertex. So x is interior to MN, and the two arcs of γ from x to M are exchanged by that involution. Fold one of them along MN. It becomes the trajectory γ_x of T, leaving $x \in L_1$ perpendicular to L_1 into T, the only inward perpendicular there. That trajectory ends at R, the fold of M. Before its end it meets no other vertex: the vertices of I, whose images are O and A, are avoided by a periodic path, and the open arc contains no bounce at M.

Point to orbit. Let γ_y meet R before any other vertex. It then has no head-on hit on any side before R: such a hit makes it retrace to y and close up without meeting any vertex. Lift γ_y to I, unfolding across MN at each oblique strike of L_1, and reverse it. The result is a path leaving M that meets MN at a right angle at y and meets no vertex and no second copy of M before. By Lemma 12.4(4) it closes up into a primitive periodic path of odd period, whose right-angle crossing of MN is y.

The two maps are inverse. By Lemma 12.4(3) an orbit has exactly one right-angle crossing of MN, and its reversal has the same one. The two lifts of γ_y, one on either side of MN, are exchanged by ϱ, and by Lemma 12.4(2) they give an orbit and its reversal. So, up to reversal, the two maps are inverse.

(ii) Let $\alpha/\pi \in \mathbb{Q}$, and let γ_y meet R before any other vertex. For heights s near y, the trajectories γ_s make the same strikes as γ_y, at interior points of sides, until they pass R on one side or the other. Then one strikes L_1 and then L_2, the other L_2 and then L_1, and none has returned perpendicularly to L_1 before. So their words ([Orph] §2.1) differ, and y is interior to no branch: it is an interior branch boundary. Its defining graze, the first vertex its trajectory meets, is R. Conversely, the trajectory of an interior boundary whose defining graze is at R meets R before any other vertex, by definition. By $(\star)$ of §5.1 it does so before its first perpendicular return.

(iii) The hypotheses give $P \ge 3$ and $Q - P \ge 3$. By [Orph] Corollary 4.14, which needs no hypothesis, the two members of each mirror pair are adjacent on the leg and their common endpoint is a height whose orbit reaches R before any other vertex, and conversely every such height in $(0, 1)$ is the common endpoint of a mirror pair. By (ii) these heights are the

interior boundaries whose defining graze is at R, so those boundaries are as many as the mirror pairs. At even P there is no orphan ([Orph] Theorem 4.6), so by [Orph] Theorem 2.2 the $n(P/Q)$ branches form $n(P/Q)/2$ mirror pairs, and by (i) and (ii) $I(\alpha)$ has $n(P/Q)/2$ primitive odd-period orbits up to reversal. By Theorem 8.2(1), $n(P/Q) \le Q-1$, with equality if and only if the LONE class is completely periodic. At $\alpha = 4\pi/15$, $P/Q = 8/15$ is in range, and its LONE class, the row $8/15, \mathfrak{e} = 0$, is not completely periodic (Proposition 11.1). So equality fails there.

(iv) Here $P/Q = 2/Q$. By Theorem 10.5, $n(2/Q) = Q-1$, and L_1 meets each of the $(Q-1)/2$ perpendicular cylinders in exactly one interval, a component of $L_1 \cap C$.

These components are the crossing intervals of §9. The interior of a component lies in an open cylinder, so it contains no boundary whose leaf meets a singularity. Each end of a component is one of three things: a point of ∂C, whose leaf meets a genuine singularity; the image of A, which is singular since $Q-3 > 0$; or the R-junction, whose leaf is the saddle connection of §5.1 (‡).

Proposition 9.1 is stated under $Q-P \ge 3$ alone, and its proof treats $P \le 2$ through (†). So it applies, and each crossing interval holds at most two branches, and if two, a mirror pair separated by a boundary grazing a copy of R. There are $Q-1$ branches in $(Q-1)/2$ intervals. So each interval holds exactly two, a mirror pair split by such a boundary, and there are at least $(Q-1)/2$ interior boundaries with defining graze at R. By Remark 9.5 there are at most as many as mirror pairs, which is $(Q-1)/2$, as P is even and there is no orphan ([Orph] Theorem 4.6). By (i) and (ii), $I(\pi/Q)$ has exactly $(Q-1)/2$ primitive odd-period orbits up to reversal. □

Remark 12.6. (1) [MirA] Proposition 4.3(5) bounds the number of primitive odd-period orbits of $I(m\pi/Q)$, Q odd, by $(Q-1)/2$, and [MirA] §8 leaves open when the bound is attained. Corollary 12.5 answers this at every rational angle with $2m \ne Q-1$: at $m=1$ for every odd $Q \ge 5$, and for $4 \le 2m \le Q-3$ exactly when the LONE class of $2m/Q$ is completely periodic — whenever Q is a prime power or $2m \pm 2$ are prime to Q, by [Meth] Theorems 2–3 — but not at $4\pi/15$, where there are 3 orbits against the bound 7. (2) Part (i) holds at every α. At irrational α/π it turns the question whether $I(\alpha)$ has infinitely many odd-period orbits into the question whether infinitely many points of L_1 have perpendicular trajectories reaching R first.

13. Open problems, and the data

1. The count at odd $P \ge 5$. This is the main open problem. What is known is the parity ([Orph] Corollary 5.1); the bound $n \le Q-1$ and its sharpening at odd P (§8); the closed forms on the whole lattice locus (§10.2) and at $P=3$ (Corollary 10.8, its equality outright wherever Remark 10.10 applies); the family $n(5/(5m+1)) = 5$ for $m \ge 3$ (Theorem 10.11); and, at

even Q under complete periodicity, with (OB) off the Veech locus (so outright at the centres of §1.5), the relation $n(P/Q)+n((Q-P)/Q) = Q$ (Corollary 7.11). That relation ties two numerators at one denominator, while the irregularity runs across denominators — $n(5/22) = 5$ against $n(5/24) = 17$, both consistent with it ($n(17/22) = 17$, $n(19/24) = 7$). §10.1 gives a reason, not an obstruction, to expect no closed form in general.

2. Complete periodicity off the lattice locus. By §7 complete periodicity of a class is equivalent to $p = 0$ and to the covering identity, with (OB) on two classes. The data say it almost always holds — on the class of the $\perp L_1$ beam at every coprime centre with $Q \leq 60$ except 8/15 (§11) — and 8/15 shows that an arithmetic condition has to enter any general proof. It would give, at even P, the maximal count $n = Q - 1$. The companion [Meth] proves it on both classes at the centres of [Meth] Theorems 2–3 (§1.5), and conjectures that 8/15 is the only failure (its Conjecture 6.5).

3. Hypothesis (OB). The existence of an orphan branch at odd P is proved at every prime-power Q, at Q twice or four times a prime and the edges of every row ([Orph] Theorems 4.11, 4.18, 4.24, Corollary 4.20), at $P' = 3$ (Corollary 10.9; [Orph] Theorem 4.11 at $Q = 7$), at $P' = Q - 2$ (Theorem 10.3), at $P' = 5$, $Q \equiv 1 \pmod 5$, $Q \geq 16$ (Theorem 10.11), and at the centres of [Meth] Theorems 2–3 (§1.5); it is conjectured everywhere ([Orph] Conjecture 4.12). Every statement here that assumes (OB) becomes unconditional with it.

4. The base without the orphan. At odd P in (R), under (OB), no perpendicular saddle connection joins Z_O to Z_A: Proposition 9.3 gives $C_B - a_R = 1$, the two acute images are zeros so $a = a_R$, and the identity of [Orph] Lemma 7.8's proof gives $\kappa'' = 2 + (B - M) \geq 2$; as each κ''-component contains Z_O or Z_A, in fact $\kappa'' = 2$ and $B = M$, so the two zeros lie in different components of G. The argument routes through the orphan; an argument on the base that does not is missing.

The data. Figure 6 shows item 1 as data.

Appendix A. The computation

A.1. **The fixed-angle backend, and the cylinder data.** The branch partition of [Orph] Appendix A is computed exactly at each centre, in $\mathbb{Z}[\zeta_{4Q}]$ ([Orph] §A.1). The cylinder data of §5.1 — the decomposition of the perpendicular direction and, for each closed geodesic met by L_1, the number of times it is met — is computed by a fixed-angle backend that works directly in $\mathbb{Z}[\zeta_{2Q}]$, at a cost per reflection depending on $\varphi(2Q)$ but not on depth. The crossing number of Proposition 5.3 is read off this computation as the ratio of a geodesic's period to its first return to L_1, and it is exact. Remark 9.5 reads the branch partition, and §11 the flow decomposition of A.2.

A.2. **Deciding complete periodicity.** Complete periodicity is a property of a direction, not of the partition, so the completeness certificate of [Orph] §A.3 says nothing about it; it is decided by a separate instrument, an **exact**

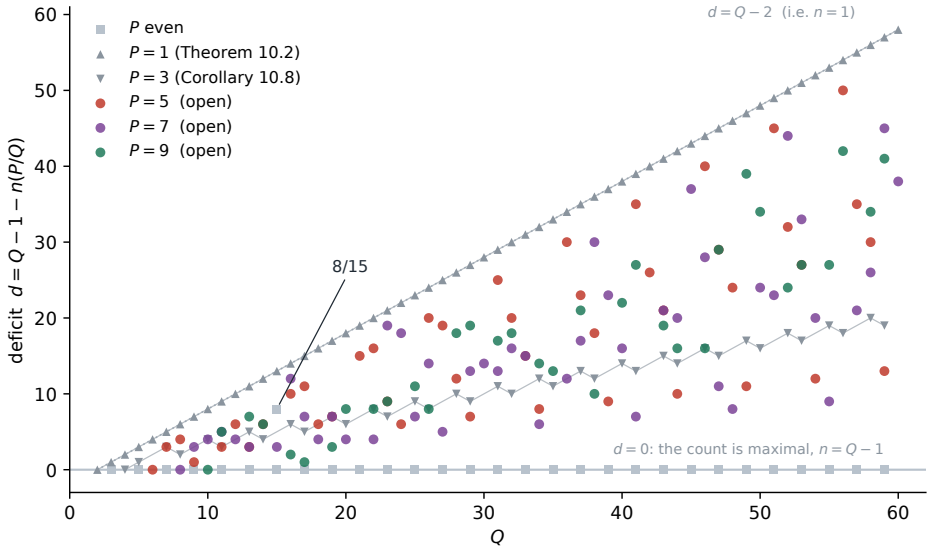


FIGURE 6. The deficit $d = Q - 1 - n(P/Q)$ at every coprime centre with $P \le 9$, $2 \le Q \le 60$ — all 322, each count certified by Appendix A. We plot the deficit rather than the count so that the families with a known count form the picture's edges and the open range its interior. The axis $d = 0$ is the maximal count $n = Q-1$. Every even P lies on it except $8/15$, the one square off the axis: $P = 2$ by Theorem 10.5, $P = Q - 1$ by Theorem 10.2, and the rest because the class is completely periodic there, by Corollary 7.11 and Theorem 8.2(1), which make the maximal count at even P equivalent to complete periodicity. The upper edge $d = Q - 2$ is $n = 1$, reached by $P = 1$ at every Q and only there (Theorem 8.2(3)), and the sawtooth between the edges is the count at $P = 3$ (Corollary 10.8). The odd numerators $P = 5, 7, 9$ fill the interior with no visible law: that is item 1. The one constraint they obey is arithmetic — $d \equiv Q \pmod 2$ at odd P, equivalently [Orph] Corollary 5.1 — and it is checked on all 322 rows, as is every other quantity named here, before the figure is drawn.

flow decomposition. The first return of the perpendicular flow on S_α to a transversal made of all the side-images, laid end to end, is an interval exchange whose lengths and translations lie in $\mathbb{Z}[\zeta_{4Q}]$. It is decomposed by Zorich-accelerated Rauzy induction, with saddle connections detected and merged and reducible permutations split, every comparison made exactly in that ring — the decision procedure of Delecroix and Rüth's intervalxt [DRixt], the method of [DR22]. The procedure ends with every component

either a **cylinder**, reported with its exact period and transverse measure, or a component **without periodic trajectory**, certified by Boshernitzan's criterion [Bos88, Lemma 6.7]: an exact rational functional y with $y(t_j) > 0$ for every translation t_j. A periodic orbit has total displacement $\Sigma n_j t_j = 0$ with $n_j \geq 0$ not all zero, so no such y exists when there is one. Both outcomes are certificates; nothing is sampled, and no tolerance enters. (A component on which neither outcome is reached by the step cap is reported undetermined; within the band below none was.) The images of the right angle are kept as marked points, so a cylinder count is convention-dependent; a count of minimal components is not.

On the class of the $\perp L_1$ beam at every coprime centre with $Q \leq 60$ the decomposition certifies complete periodicity except at $8/15$, where it finds two minimal components, agreeing with [DR22] (§11). The same run returns the branch partition of L_1 — the count $n(P/Q)$ and the exact multiset of branch widths — which is how the counts quoted as certificates in §10 were obtained.

A.3. **The certificates of Theorems 10.7 and 10.11.** Two proofs in §10 end in finitely many explicit inequalities checked by machine, and nothing else in them is computational, except the row $Q = 7$ of Theorem 10.7 and Corollary 10.9, which is certified by the exact computation of A.2 ($n_H(3/7) = 5$).

Theorem 10.7. The conditions of step (c) are

> C1 $s_5 > s_1 + s_3$; C2 $s_9 > s_5$; C3 $s_{11} > s_1 + s_3 + s_9 - s_5$; C4 $s_3 + s_9 > 2s_5$; C6 $2s_9 + s_3 > 2s_5 + s_{11}$; C7 $2s_5 + s_{11} > s_1 + 2s_3 + s_9$; C8 $3s_5 + s_{11} > s_1 + 2s_3 + 2s_9$; C9a $2s_1 + 2s_3 + 2s_9 > 3s_5 + s_{11}$; C9b $3s_5 + s_{11} > s_1 + 2s_3 + 2s_9$; C10 $3s_3 + s_1 + 2s_9 > 3s_5 + s_{11}$; C11 $2s_3 + 2s_9 > 2s_5 + s_{11}$; C12a $s_3 + s_9 > 2s_5$ (= C4; C9b likewise repeats C8); C12b $4s_5 + 2s_{11} > 2s_1 + 3s_3 + 3s_9$; C12c $4s_3 + 4s_9 + s_1 > 5s_5 + 2s_{11}$,

each an odd polynomial in $S = \sin x$, root-free and positive on $(0, \sin(\pi/20)]$ by exact real-root isolation.

Theorem 10.11. With $s_k = \sin(kx)$, $a = s_7 - s_3$ and $p^* = a + 2s_9 - 2s_1 - s_3 - s_{13}$, the conditions of the walk are

> D1 $s_5 > a$; D2 $s_7 > s_5$; D3 $a + s_9 > s_5$; D4 $a + s_9 > s_1 + s_5$; D5 $s_7 + s_9 > s_1 + 2s_3 + s_5$; D6 $s_1 + s_{13} > s_9$; D7 $s_7 + 2s_9 > s_1 + 2s_3 + s_5 + s_{13}$; D8 $p^* > a$, that is $2s_9 > 2s_1 + s_3 + s_{13}$; D9 $s_5 > p^*$; D10 $2s_1 + 2s_3 + s_{13} > 2s_9$; D11 $s_{13} > s_3$; D12 $a + (s_9 - s_1)/2 > s_5$; D13 $s_1 + s_3 + s_{13} > s_9 + (s_9 - s_1)/2$,

each an odd polynomial in $S = \sin x$, root-free and positive on $(0, \sin(\pi/28)]$ by exact real-root isolation of the polynomial with integer coefficients. As a control, a generator that derives such conditions mechanically from the walk reproduces the hand-derived lists of Theorem 10.7 ($P = 3$, $r_3 = 1$, $Q \geq 10$) and Theorem 10.11 ($Q \geq 14$), all but

D11, a link of the proof's chain $p''' < a + s_3 < a + s_{13}$ that the generator tests in one step.

Code availability. The computations of this paper — the exact flow decomposition, the branch partitions and the certificates of A.3 — are deposited at `https://github.com/josh-boorstein/billiards`, whose manifest maps each one to the section it supports, and which also holds the code of [Orph]; this paper's is in the directories `lib/` and `neck/`. The version used here is release v1.0, archived at Zenodo as doi:10.5281/zenodo.24037744.

Use of AI

This paper — its text, its proofs and the code behind Appendix A — was drafted by an AI system (Claude, Anthropic) working under the author's direction, and checked by independent AI review passes and by the exact computations described in Appendix A. The author has read parts of it in detail and takes responsibility for its content.

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