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CONSECUTIVE DIVISORS OF SUMS OF FIBONACCI POLYNOMIALS: A LUCA–MARQUES ANALOGUE IN FINITE-FIELD POLYNOMIAL RINGS

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To the memory of Florian Luca

ABSTRACT. For every prime power q with $\gcd(q,6)=1$, we construct infinitely many pairs of consecutive monic polynomials in $\mathbb{F}_q[T]$ whose product divides the Fibonacci-polynomial sum with upper index one less than their common norm. This gives a function-field counterpart of the divisibility condition in the Luca–Marques theorem on consecutive integral arithmetic means of Fibonacci numbers. A short extension of the construction gives infinitely many triples of consecutive monic polynomials with the same property.

1. INTRODUCTION

Answering a question of Fatehizadeh and Yaqubi [FY22], Luca and Marques [LM25] proved that there are infinitely many positive integers n for which both

$$\frac{1}{n}\sum_{j=1}^{n} F_j \qquad \text{and} \qquad \frac{1}{n+1}\sum_{j=1}^{n+1} F_j$$

are integers, where F_n denotes the Fibonacci sequence, with $F_0=0$ and $F_1=1$.[1] Since $\sum_{j=1}^{n} F_j = F_{n+2}-1$, the Luca–Marques result equivalently asserts that

$$n \mid F_{n+2}-1, \qquad n+1 \mid F_{n+3}-1$$

for infinitely many n.

This note proves a norm-indexed polynomial analogue over finite fields. Consecutive nonconstant monic polynomials have the same degree, so the analogue uses a common

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I used LLMs to assist with computations and analysis in the preparation of this article, particularly GPT-5.5 Pro and Claude 4.7 Opus (accessed in part via Poe with the support of Quora, where I am an advisor). The problem, methods, and final written form are my own, and of course any errors remain my responsibility. I thank the referee for a careful reading and for suggesting the question that led to Corollary 8.1. This work was conducted while I was visiting the Technological Innovation, Entrepreneurship, and Strategic Management (TIES) Group at the MIT Sloan School of Management; I greatly appreciate their hospitality.

[1] Flynn-Connolly [FC25] proves related results.

summation index. Fix a prime power q with $\gcd(q,6)=1$. In $\mathbb{F}_q[T]$, define the Fibonacci-polynomial sequence

$$G_0(T)=0, \qquad G_1(T)=1, \qquad G_n(T)=TG_{n-1}(T)+G_{n-2}(T) \qquad (n\geq 2),$$

and its Lucas companion

$$L_0(T)=2, \qquad L_1(T)=T, \qquad L_n(T)=TL_{n-1}(T)+L_{n-2}(T) \qquad (n\geq 2).$$

We suppress the argument T and write simply G_n and L_n when doing so will not introduce confusion.

For a nonzero polynomial $A\in\mathbb{F}_q[T]$, put $|A|=q^{\deg A}$. For $N\geq 0$, set

$$\mathcal{E}_N(T):=G_{N+1}(T)+G_N(T)-1.$$

The recurrence defining G_j gives

$$TG_j=G_{j+1}-G_{j-1} \qquad (j\geq 1),$$

and hence, with the empty-sum convention,

$$\mathcal{E}_N(T)=T\sum_{j=1}^{N}G_j(T) \qquad (N\geq 0).$$

We say that a nonzero polynomial $A\in\mathbb{F}_q[T]$ is *norm-Fibonacci-summable* if $\gcd(A,T)=1$ and $A\mid \mathcal{E}_{|A|-1}(T)$. Since $\mathcal{E}_N=T\sum_{j=1}^{N}G_j(T)$, the coprimality condition $\gcd(A,T)=1$ makes the second condition equivalent to

$$A\mid \sum_{j=1}^{|A|-1}G_j(T).$$

Our main result is the following explicit construction.

Theorem 1.1. *Let q be a prime power with $\gcd(q,6)=1$. For every integer $a\geq 1$, set $k=2^a$ and $B=L_{2k}$. Then B and $B+1$ are monic of degree $2k$ and norm-Fibonacci-summable over $\mathbb{F}_q[T]$. More precisely, $B+1=G_{3k}/G_k$, and, with $N=q^{2k}-1$,*

$$B(B+1)\mid \sum_{j=1}^{N}G_j(T).$$

In particular, there are infinitely many such monic polynomials B.

Corollary 8.1 extends the construction to three consecutive monic polynomials, answering a question suggested by the referee.

The proof of Theorem 1.1 separates into two elementary parts. The polynomial part is a period-doubling lemma: if a polynomial divides G_r with r even, then it divides $\mathcal{E}_N$ whenever $2r\mid N$. The integer part is the simple observation that, for $k=2^a$, the hypothesis $\gcd(q,6)=1$ implies

$$24k\mid q^{2k}-1.$$

Together the preceding facts force the required divisibilities for

$$B=L_{2k} \qquad \text{and} \qquad B+1=\frac{G_{3k}}{G_k}.$$

2. Preliminaries

All polynomial divisibilities and congruences in what follows are taken in $\mathbb{F}_q[T]$. Integer divisibilities, such as $24k \mid q^{2k}-1$, are taken in $\mathbb{Z}$. Congruences between matrices are understood entrywise.

A direct induction shows that G_n is monic of degree $n-1$ for $n \geq 1$, and that L_n is monic of degree n for $n \geq 1$.

Set

$$M = \begin{pmatrix} T & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q[T]).$$

Then

$$M^n = \begin{pmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{pmatrix} \qquad (n \geq 1).$$

Taking determinants gives Cassini's identity

$$G_{n+1}G_{n-1} - G_n^2 = (-1)^n.$$

In a splitting field of $X^2 - TX - 1$ over $\mathbb{F}_q(T)$, let α and β be the two roots. Then

$$\alpha + \beta = T, \qquad \alpha\beta = -1, \qquad \beta = -\alpha^{-1}.$$

The discriminant T^2+4 is nonzero, so the roots are distinct. The standard Binet formulas give

$$G_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n.$$

Thus

$$G_{2m} = G_m L_m.$$

Also, for $k \geq 1$, the identity $x^3 - y^3 = (x-y)(x^2+xy+y^2)$ gives

$$G_{3k} = G_k \left(\alpha^{2k} + (\alpha\beta)^k + \beta^{2k}\right).$$

Hence $G_k \mid G_{3k}$, and

$$\frac{G_{3k}}{G_k} = \alpha^{2k} + (\alpha\beta)^k + \beta^{2k} = L_{2k} + (-1)^k.$$

For even k, this becomes

$$\frac{G_{3k}}{G_k} = L_{2k} + 1.$$

Henceforth, for even k, set

$$B_k := L_{2k}, \qquad C_k := B_k + 1 = \frac{G_{3k}}{G_k}.$$

Both B_k and C_k are monic of degree $2k$. Also, evaluating the Lucas recurrence at $T = 0$ gives

$$B_k(0) = L_{2k}(0) = 2, \qquad C_k(0) = 3.$$

Because $\gcd(q, 6) = 1$, both constants are nonzero in $\mathbb{F}_q$. Therefore

$$\gcd(B_k, T) = \gcd(C_k, T) = 1.$$

Moreover,

$$|B_k| - 1 = |C_k| - 1 = q^{2k} - 1.$$

3. A period-doubling lemma

Lemma 3.1. *Let $H \in \mathbb{F}_q[T]$ be nonzero, and let r be a positive even integer. If $H \mid G_r$, then*

$$M^{2r} \equiv I \pmod{H}.$$

Consequently, for every positive integer N with $2r \mid N$, we have

$$H \mid \mathcal{E}_N.$$

Proof. Since $H \mid G_r$ and $G_{r+1} - G_{r-1} = TG_r$, the matrix formula gives

$$M^r \equiv \begin{pmatrix} G_{r+1} & 0 \\ 0 & G_{r-1} \end{pmatrix} \equiv G_{r+1} I \pmod{H}.$$

Taking determinants and using that r is even, we obtain

$$G_{r+1}^2 \equiv \det(M^r) = (-1)^r = 1 \pmod{H}.$$

Squaring the matrix congruence therefore gives $M^{2r} \equiv I \pmod{H}$. If $2r \mid N$, then $M^N \equiv I \pmod{H}$. Its entries yield $G_N \equiv 0 \pmod{H}$ and $G_{N+1} \equiv 1 \pmod{H}$, so

$$\mathcal{E}_N = G_{N+1} + G_N - 1 \equiv 0 \pmod{H}. \qquad \square$$

Remark. The factor of 2 in Lemma 3.1 cannot in general be omitted. For example, let $H = T^2 + 2$ and $r = 4$. Then $G_4 = TH$, while $G_5 = T^4 + 3T^2 + 1 \equiv -1 \pmod{H}$. Thus $M^4 \equiv -I \pmod{H}$ and $\mathcal{E}_4 \equiv -2 \not\equiv 0 \pmod{H}$, since the characteristic is odd.

4. An elementary integer input

Lemma 4.1. *Let q be a prime power with $\gcd(q, 6) = 1$. If $a \geq 1$ and $k = 2^a$, then*

$$24k \mid q^{2k} - 1.$$

Proof. Since q is odd, $q^2 \equiv 1 \pmod{8}$. If $x \equiv 1 \pmod{2^m}$ with $m \geq 3$, writing $x = 1 + 2^m u$ gives

$$x^2 - 1 = 2^{m+1}u + 2^{2m}u^2 \equiv 0 \pmod{2^{m+1}}.$$

Starting from $q^2 \equiv 1 \pmod{8}$ and iterating the preceding observation a times, we obtain

$$q^{2k} = (q^2)^{2^a} \equiv 1 \pmod{2^{a+3}}.$$

Also, $3 \nmid q$ gives $q^2 \equiv 1 \pmod{3}$, and hence $q^{2k} \equiv 1 \pmod{3}$. Since 3 and 2^{a+3} are coprime,

$$24k = 3 \cdot 2^{a+3} \mid q^{2k} - 1. \qquad \square$$

5. A sufficient criterion

Proposition 5.1. *Let q be a prime power with $\gcd(q, 6) = 1$, and let k be an even positive integer such that $24k \mid q^{2k} - 1$. Set*

$$B_k = L_{2k}, \qquad C_k = B_k + 1 = \frac{G_{3k}}{G_k}.$$

Then B_k and C_k are norm-Fibonacci-summable. More precisely, with $N = q^{2k} - 1$, we have

$$B_k C_k \mid \sum_{j=1}^{N} G_j(T).$$

Proof. As noted in Section 2, both B_k and C_k are monic of degree $2k$ and coprime to T, so their common norm minus 1 is $N = q^{2k} - 1$. The identities

$$C_k = \frac{G_{3k}}{G_k}, \qquad G_{4k} = G_{2k} B_k$$

show that $C_k \mid G_{3k}$ and $B_k \mid G_{4k}$. Since k is even, Lemma 3.1 applies with $r = 3k$ and $r = 4k$, respectively. Both $6k$ and $8k$ divide $24k$, which by hypothesis divides N. Consequently, $C_k \mid \mathcal{E}_N$ and $B_k \mid \mathcal{E}_N$.

Because $C_k = B_k + 1$, the polynomials B_k and C_k are coprime. Their product is also coprime to T. Cancelling T from $\mathcal{E}_N = T \sum_{j=1}^{N} G_j(T)$ therefore gives the required divisibility. □

6. Proof of Theorem 1.1

Proof. Let $a \geq 1$ and $k = 2^a$. Then k is even, and Lemma 4.1 gives $24k \mid q^{2k} - 1$. Proposition 5.1 therefore applies to $B = L_{2k}$ and $B + 1 = G_{3k}/G_k$, proving the asserted divisibility and norm-Fibonacci-summability. Finally, $\deg B = 2k = 2^{a+1}$ tends to infinity with a, so these polynomials are distinct. □

7. An example: $q = 5$, $k = 2$

Let $q = 5$ and $a = 1$, so $k = 2$. Then $N = 5^4 - 1 = 624$, and $24k = 48 \mid 624$. The construction gives

$$B_2 = L_4 = T^4 + 4T^2 + 2, \qquad B_2 + 1 = T^4 + 4T^2 + 3 = \frac{G_6}{G_2}$$

in $\mathbb{F}_5[T]$. Since $B_2 \mid G_8$ and $B_2 + 1 \mid G_6$, and since $16 \mid 624$ and $12 \mid 624$, Lemma 3.1 shows that both factors divide $\mathcal{E}_{624}$. They are coprime to each other and to T, so their product satisfies

$$B_2(B_2 + 1) = T^8 + 3T^6 + T^4 + 1 \mid \sum_{j=1}^{624} G_j(T).$$

8. Remarks

The identity $G_{3k}/G_k = L_{2k} + 1$ produces the consecutive pair before any congruence argument is made. The period-doubling lemma then reduces the proof to the integer-divisibility condition

$$\operatorname{lcm}(6k, 8k) = 24k \mid q^{2k} - 1.$$

No primitive-divisor theorem is needed. Proposition 5.1 also applies to any other even positive integer k satisfying the divisibility condition; the choice $k = 2^a$ gives a uniform elementary infinite family for every q with $\gcd(q, 6) = 1$.

The hypothesis $\gcd(q, 6) = 1$ supplies both the congruences $q^2 \equiv 1 \pmod 8$ and $q^2 \equiv 1 \pmod 3$ used in Lemma 4.1. It also ensures that the constant terms $B_k(0) = 2$

and $(B_k + 1)(0) = 3$ are nonzero, allowing cancellation of T. In characteristics 2 or 3, both the coprimality condition and the period-forcing argument would require modification.

The choice of the index $|A| - 1$, rather than $|A|$, is not merely an artifact of the construction. Appendix A shows that, for q odd and $d \geq 1$, the polynomial $\sum_{j=1}^{q^d} G_j$ has no divisor of degree d. In particular, no polynomial $A \in \mathbb{F}_q[T]$ of degree d and coprime to T can satisfy

$$A \mid \mathcal{E}_{|A|}.$$

The referee suggested considering three consecutive monic polynomials, in analogy with the question Luca and Marques [LM25, p. 510] pose regarding the possibility of achieving three consecutive integral arithmetic means. While the Luca–Marques question remains open for the Fibonacci numbers, our argument yields a positive answer to the analogous question for Fibonacci polynomials. In fact, shifting our construction by -1 gives an affirmative answer at the same norm-indexed upper limit.

Corollary 8.1. *Let q be a prime power with $\gcd(q, 6) = 1$. For every integer $a \geq 1$, set $k = 2^a$ and $B = L_{2k} - 1$. Then B, $B + 1$, and $B + 2$ are monic of degree $2k$ and norm-Fibonacci-summable, and*

$$B(B + 1)(B + 2) \mid \sum_{j=1}^{q^{2k}-1} G_j(T);$$

in particular, we have infinitely many such triples.

Proof. Put $N = q^{2k} - 1$. Theorem 1.1 shows that $(B + 1)(B + 2) \mid \sum_{j=1}^{N} G_j(T)$. For even k, the Binet formulas give

$$L_{3k} = L_k(L_{2k} - (-1)^k) = L_k(L_{2k} - 1) = L_k B.$$

Hence $B \mid L_{3k} \mid G_{6k}$. Since $12k \mid N$ by Lemma 4.1, Lemma 3.1 with $r = 6k$ gives $B \mid \mathcal{E}_N$. Moreover, $B(0) = 1$, so B is coprime to T and therefore divides $\sum_{j=1}^{N} G_j(T)$. The three polynomials are pairwise coprime, because their differences are 1 or 2 and the characteristic is odd. Their product thus divides the sum. Finally, all three have degree $2k$, which tends to infinity with a. □

Appendix A. A shifted-index obstruction

For $N \geq 1$, write

$$S_N(T) = \sum_{j=1}^{N} G_j(T).$$

Proposition A.1. *Assume that q is odd. Let $d \geq 1$, and put $Q = q^d$. Then S_Q has no divisor of degree d in $\mathbb{F}_q[T]$. Consequently, no polynomial $A \in \mathbb{F}_q[T]$ with $\deg A = d$ and $\gcd(A, T) = 1$ satisfies*

$$A \mid \mathcal{E}_{|A|}.$$

Proof. The key point is that every root of S_Q lies in $\mathbb{F}_{q^{2d}}$, but no root lies in $\mathbb{F}_{q^d}$. We first prove this root-field statement, and then translate it into a degree obstruction.

Put

$$D = T^2 + 4, \qquad h = \frac{Q-1}{2}.$$

Let α and β be the roots of $X^2 - TX - 1$ in a splitting field over $\mathbb{F}_q(T)$. Since Q is a power of $\mathrm{char}(\mathbb{F}_q)$, the Q-power map is Frobenius. Hence the Binet formulas give

$$G_Q = \frac{\alpha^Q - \beta^Q}{\alpha - \beta} = \frac{(\alpha-\beta)^Q}{\alpha-\beta} = (\alpha-\beta)^{Q-1} = D^h,$$

and

$$L_Q = \alpha^Q + \beta^Q = (\alpha+\beta)^Q = T^Q.$$

The identity

$$2G_{n+1} = L_n + TG_n$$

therefore gives

$$G_{Q+1} = \frac{T^Q + TD^h}{2}.$$

Also, summing $TG_j = G_{j+1} - G_{j-1}$ from $j = 1$ to Q gives

$$TS_Q = G_{Q+1} + G_Q - 1.$$

Combining these identities, we obtain

$$TS_Q = \frac{T^Q + (T+2)D^h - 2}{2}. \tag{A.1}$$

At $T = 0$, the recurrence gives $G_{2m}(0) = 0$ and $G_{2m+1}(0) = 1$. Since Q is odd, direct summation gives

$$S_Q(0) = \sum_{m=0}^{(Q-1)/2} 1 = \frac{Q+1}{2} = \frac{1}{2}$$

in $\mathbb{F}_q$, because the characteristic divides Q. Thus $T \nmid S_Q$.

Now let P be an irreducible factor of S_Q, put $r = \deg P$, and let $\theta \in \overline{\mathbb{F}_q}$ be a root of P. Since $T \nmid S_Q$, we have $\theta \neq 0$. Set

$$\delta = \theta^2 + 4, \qquad u = \delta^h.$$

Evaluating (A.1) at $T = \theta$ gives

$$\theta^Q + (\theta+2)u - 2 = 0. \tag{A.2}$$

We first rule out $\delta = 0$. If $\delta = 0$, then $u = 0$, and (A.2) gives $\theta^Q = 2$. Squaring and using $\theta^2 = -4$, we obtain

$$4 = \theta^{2Q} = (-4)^Q = -4,$$

which is impossible because q is odd. Therefore $\delta \neq 0$.

Since $\delta \neq 0$, we have

$$u^2 = \delta^{Q-1} = \frac{\delta^Q}{\delta} = \frac{\theta^{2Q}+4}{\theta^2+4}.$$

Using $\theta^Q = 2 - (\theta+2)u$ from (A.2), this becomes

$$(\theta^2+4)u^2 = (2-(\theta+2)u)^2 + 4.$$

After simplification,

$$0 = 4\big(\theta u^2 - (\theta+2)u + 2\big) = 4(u-1)(\theta u - 2).$$

Since q is odd, either $u = 1$ or $\theta u = 2$. The root equation (A.2) therefore yields

$$\theta^Q = \begin{cases} -\theta, & \text{if } u = 1, \\ -4/\theta, & \text{if } \theta u = 2. \end{cases}$$

Applying the Q-power map once more gives $\theta^{Q^2} = \theta$. Indeed, the two alternatives give, respectively,

$$\theta^{Q^2} = (-\theta)^Q = -\theta^Q = \theta, \qquad \theta^{Q^2} = \left(-\frac{4}{\theta}\right)^Q = -\frac{4}{\theta^Q} = \theta.$$

Thus $\theta \in \mathbb{F}_{q^{2d}}$. On the other hand, if $\theta^Q = \theta$, the first alternative would imply $\theta = 0$, and the second would imply $\theta^2 = -4$. Both have been excluded. Consequently,

$$\theta \in \mathbb{F}_{q^{2d}} \setminus \mathbb{F}_{q^d}.$$

Since θ has degree r over $\mathbb{F}_q$, the standard finite-field criterion gives

$$r \mid 2d \qquad \text{but} \qquad r \nmid d.$$

Write $d = 2^\nu d_0$, where d_0 is odd. If $r \mid 2d$ but $r \nmid d$, then $2^{\nu+1} \mid r$. Hence every irreducible factor of S_Q has degree divisible by $2^{\nu+1}$. It follows that the degree of every divisor of S_Q, including those with repeated irreducible factors, is divisible by $2^{\nu+1}$. But $d = 2^\nu d_0$ is not divisible by $2^{\nu+1}$. Therefore S_Q has no divisor of degree d.

For the final assertion, let $A \in \mathbb{F}_q[T]$ have degree d and satisfy $\gcd(A, T) = 1$. Then $|A| = q^d = Q$ and $\mathcal{E}_{|A|} = \mathcal{E}_Q = TS_Q$. If $A \mid \mathcal{E}_Q$, then $A \mid TS_Q$. Since $\gcd(A, T) = 1$, this would imply $A \mid S_Q$, contradicting the first part of the proposition. □

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