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LUSZTIG'S SHARP BOUND FOR HECKE ALGEBRAS

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Abstract. We prove Lusztig's sharp boundedness conjecture for Hecke algebras of finite-rank Coxeter systems with positive integer-valued weight functions.

1. Introduction

1.1. **The boundedness conjecture.** Kazhdan and Lusztig introduced their basis of a Hecke algebra and the associated cell relations in [7]. Lusztig later introduced the a-function in his study of cells in affine Weyl groups [9]. Its value at a group element is the supremum of the degrees of the corresponding Kazhdan–Lusztig structure constants. A uniform bound on these degrees makes the a-function finite-valued. The boundedness problem can also be expressed in the standard basis, where the multiplication rule gives a direct path interpretation of degree. We determine the optimal uniform bound in this form.

Let (W, S) be a Coxeter system with S finite. Write e for its identity, ℓ for its length function, $\leq$ for Bruhat order, and m_{st} for the order of st for $s, t \in S$. A weight function is a map $L : W \longrightarrow \mathbb{Z}$ such that $L(xy) = L(x) + L(y)$ whenever $\ell(xy) = \ell(x)+\ell(y)$. We assume $L(s) > 0$ for every $s \in S$. Let v be an indeterminate and put $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. The Hecke algebra $\mathcal{H}$ has standard basis $(T_w)_{w\in W}$, with $T_e = 1$ and relations

$$T_xT_y = T_{xy} \quad \text{if } \ell(xy) = \ell(x) + \ell(y), \qquad T_s^2 = 1 + (v^{L(s)} - v^{-L(s)})T_s \quad (s \in S).$$

Define its standard structure constants by

$$T_xT_y = \sum_{z\in W} f_{x,y,z}T_z.$$

These sums have finite support. For $X \in \mathcal{H}$, write $[T_z]X$ for its T_z-coefficient. The degree of a nonzero Laurent polynomial is its largest exponent of v, and $\deg 0 = -\infty$.

A subset $I \subseteq S$ is *spherical* if the standard parabolic subgroup $W_I = \langle I\rangle$ is finite. For such I, let w_I be the longest element of W_I, and set

$$N_L(W) = \max_{I\subseteq S,\, W_I \text{ finite}} L(w_I). \tag{1.1}$$

The empty subset is included, with $w_\varnothing = e$. The maximum is finite because S is finite. Lusztig conjectured that

$$\deg f_{x,y,z} \leq N_L(W) \qquad (x, y, z \in W)$$

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in [10, §13.4]. In equal parameters, normalized by $L = \ell$, the prediction is related to Xi's earlier Question 1.13(iv) in [15], as Xi explains in [16, Introduction]. Lusztig's comparison of the two bases shows that the same uniform bound then holds for the Kazhdan–Lusztig structure constants [10, Lemma 13.5].

1.2. **Previous results.** For a finite Coxeter group, the bound is $L(w_S)$ [10, Lemma 13.3]. Lusztig proved the equal-parameter affine case in his work on affine cells [9, §7]; in [10, §13.4], he observes that the proof also applies to positive weight functions. These cases suggested that finite parabolic subgroups should control the bound even when W is infinite.

Several families of infinite Coxeter groups were subsequently treated. In equal parameters, Belolipetsky proved the bound for right-angled groups, where $m_{st} \in \{2, \infty\}$ for distinct s, t [1, §4.2]. Xi proved it for complete Coxeter graphs, where $m_{st} \neq 2$ for distinct s, t [16, Theorem 2.1], and Zhou established the rank-three case [17].

For positive weights, Shi and Yang computed the a-function of universal Coxeter groups, where $m_{st} = \infty$ for all distinct s, t, and obtained the expected bound [13, Theorem 3.7]. They then proved the bound for complete Coxeter graphs [14, Theorem 3.2]. Li and Shi treated Coxeter systems with $m_{st} \neq 3$ for all distinct s, t [8, Theorem 6.2]. Their hypothesis allows commuting generators and groups of arbitrary finite rank. Gao proved the bound for every weighted Coxeter group of rank three [6, Theorem 2.1].

In his 2020 list of open problems, Lusztig asked whether the standard structure constants admit some uniform finite degree bound [11, §2(b)]. Chen and Hu proved this for every finite-rank Coxeter system with a positive integer-valued weight function [4, Theorem 1.3]. The bound they obtain can exceed $N_L(W)$, even in equal parameters; see [4, Example 5.3]. Their theorem supplies the general finiteness result used here. We identify the resulting maximum with the value in (1.1).

1.3. **Main result.**

Theorem 1.1. *Let (W, S) be a Coxeter system of finite rank, and let L be a positive integer-valued weight function. Then*

$$\max_{x,y,z \in W} \deg f_{x,y,z} = N_L(W).$$

The bound is attained inside a finite standard parabolic subgroup. Indeed, for every spherical $I \subseteq S$,

$$\deg f_{w_I, w_I, w_I} = L(w_I);$$

we prove this in Lemma 6.1. Taking I that attains (1.1) proves the lower bound in the theorem. The main argument establishes the upper bound.

1.4. **Outline of the proof.** By the theorem of Chen and Hu, the degrees of the standard structure constants have an attained global maximum D. Trace symmetry produces a coefficient $f_{a,y,y}$ of degree D. We choose y of minimum length among all such inputs and follow a path of fold weight D in the standard multiplication expansion of $T_a T_y$.

The first step is geometric. Every state of this path has the same continuation cone, consisting of the right factors that extend it with additive length. We use the boundary description of these cones from Parkinson and Yau [12]. Every boundary

root must occur at a fold: a missing boundary root would give a common right reflection that preserves the path and shortens y. Dyer's criterion [5] then identifies the boundary as the canonical simple system of a finite reflection subgroup. An accepted continuation places it in a finite standard parabolic component. Two common-suffix lemmas show that the cone is the continuation cone of w_J for a spherical J, and give

$$y = uw_J, \qquad \ell(uw_Ju^{-1}) = 2\ell(u) + \ell(w_J),$$

where u is the minimal representative of its right coset uW_J.

The second step takes place in the Hecke algebra. Expanding $T_yT_{y^{-1}}$ bounds the degree of every coefficient of $T_uT_hT_{u^{-1}}$ by $D - L(h)$ for $h \in W_J$. We compress left multiplication by T_a to the standard coordinates indexed by uW_J. Minimality of y forces a coefficient indexed by an element whose support is all of J to attain its degree bound. A finite power of that element reaches w_J under the Demazure rule, which crosses at an ascent and stays at a descent. Positivity of the fold expansions compares powers of the compressed operator with the original powers. For a positive integer K, this gives

$$(K-1)D \;\geq\; \deg\Big([T_{uw_Ju^{-1}}]\,T_a^K\Big) \;\geq\; KD - L(w_J).$$

Hence $D \leq L(w_J) \leq N_L(W)$, which proves the upper bound.

2. Standard multiplication and a maximum-degree closed path

For $s \in S$, write $\xi_s = v^{L(s)} - v^{-L(s)}$. The standard multiplication rules are

$$T_sT_w = \begin{cases} T_{sw}, & \ell(sw) = \ell(w) + 1, \\ T_{sw} + \xi_sT_w, & \ell(sw) = \ell(w) - 1, \end{cases} \tag{2.1}$$

and the analogous formulas hold for right multiplication. To compute T_xT_y using a fixed reduced expression $x = s_m \cdots s_1$, start at y and apply $s_1, \ldots, s_m$ on the left. At each step a *crossing* replaces the current state w by s_iw and contributes 1. If $s_iw < w$, a *fold* may instead leave the state unchanged and contributes ξ_{s_i}. We refer to the sequence of these choices as a path. Its fold weight is the sum of $L(s_i)$ over its folds. A path is closed if its initial and final states agree.

Each coefficient $f_{x,y,z}$ is the sum of the fold monomials of the paths from y to z. The same statement holds for a finite product of standard basis elements, by concatenating their reduced words. A *positive fold polynomial* is an element of $\mathcal{A}$ admitting a polynomial expression in the ξ_s with nonnegative integer coefficients; zero is allowed.

Lemma 2.1. *A nonzero polynomial with nonnegative integer coefficients in the elements ξ_s has positive leading Laurent coefficient. Its degree is the largest weighted degree of its displayed monomials. Consequently, in a finite sum of such nonzero polynomials the degree is the largest degree of a summand.*

Proof. The monomial $\prod_s \xi_s^{n_s}$ is monic of degree $\sum_s n_sL(s)$. All its other exponents are strictly smaller because each $L(s)$ is positive. The terms of largest displayed degree therefore have positive leading coefficients and cannot cancel. □

Define the $\mathcal{A}$-linear map $\tau : \mathcal{H} \longrightarrow \mathcal{A}$ by $\tau(T_w) = \delta_{w,e}$, where δ is the Kronecker delta. Define the $\mathcal{A}$-bilinear form $\langle T_x, T_y \rangle = \delta_{x,y}$ on $\mathcal{H}$. For $sx > x$, left multiplication by T_s on the ordered pair (T_x, T_{sx}) has the symmetric matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & \xi_s \end{pmatrix}.$$

Thus $\langle T_s X, Y \rangle = \langle X, T_s Y \rangle$ for all $X, Y \in \mathcal{H}$. Moving the simple factors of a reduced expression of x from the first argument to the second gives

$$\delta_{x,y} = \langle T_x, T_y \rangle = \langle T_e, T_{x^{-1}} T_y \rangle = \tau(T_{x^{-1}} T_y).$$

Consequently $\tau(T_x T_y) = \delta_{x^{-1},y} = \tau(T_y T_x)$. By bilinearity, we obtain the trace identities

$$\tau(XY) = \tau(YX), \qquad \tau(T_{x^{-1}} T_y) = \delta_{x,y}, \qquad [T_z]\, X = \tau(T_{z^{-1}} X). \tag{2.2}$$

These are the identities of [10, Lemma 10.4(a), (b)]. Reversing the defining relations gives the $\mathcal{A}$-linear anti-involution $T_w \mapsto T_{w^{-1}}$, which preserves τ. It follows that

$$f_{x,y,z} = f_{x^{-1},z,y}. \tag{2.3}$$

Indeed, the two sides are respectively $\tau(T_{z^{-1}} T_x T_y)$ and $\tau(T_{y^{-1}} T_{x^{-1}} T_z)$, which agree by the anti-involution and cyclicity of τ.

By [4, Theorem 1.3], there is a finite bound for all $\deg f_{x,y,z}$. The degrees of the nonzero structure constants are integers, and $f_{e,e,e} = 1$. Thus the maximum

$$D = \max_{x,y,z \in W} \deg f_{x,y,z} \tag{2.4}$$

exists and is attained, with $D \geq 0$.

Lemma 2.2. *If* $\deg f_{x,y,z} = D$ *for some* x, y, z, *then* $\deg f_{a,y,y} = D$ *for some* $a \in W$.

Proof. Extracting the T_y coefficient from $T_{x^{-1}}(T_x T_y) = (T_{x^{-1}} T_x) T_y$ and using (2.3) gives

$$\sum_z f_{x,y,z}^2 = \sum_a f_{x^{-1},x,a} f_{a,y,y}. \tag{2.5}$$

The sums are finite. The left side has degree $2D$ and positive leading coefficient. On the right, both factors in each summand have degree at most D. Lemma 2.1 shows that at least one summand has degree $2D$, so both its factors have degree D. □

Fix an element y of minimum ordinary Coxeter length among all inputs for which $\deg f_{a,y,y} = D$ for some $a \in W$. Fix such an a, a reduced expression $a = s_m \cdots s_1$, and a closed path from y to y of fold weight D for that expression. Such a path exists by Lemma 2.1. If $S = \varnothing$, then $W = \{e\}$ and $\mathcal{H} = \mathcal{A}$, so both sides of Theorem 1.1 are zero.

3. Coxeter conventions and cone boundaries

We use the real geometric representation of (W, S) on the vector space with basis $\Pi = \{\alpha_s : s \in S\}$ and symmetric bilinear form

$$(\alpha_s, \alpha_t) = \begin{cases} -\cos(\pi/m_{st}), & m_{st} < \infty, \\ -1, & m_{st} = \infty. \end{cases}$$

For a vector λ, the action of $s \in S$ is $s\lambda = \lambda - 2(\lambda, \alpha_s)\alpha_s$. This representation is faithful and preserves the form. The root system $\Phi = W\Pi$ is the disjoint union $\Phi^+ \sqcup (-\Phi^+)$, where

$$\Phi^+ = \Phi \cap \sum_{s \in S} \mathbb{R}_{\geq 0}\alpha_s.$$

We write $\beta > 0$ for $\beta \in \Phi^+$ and $\beta < 0$ for $\beta \in -\Phi^+$. Every root has squared norm one. For a root $\beta = w\alpha_s$, its reflection is $r_\beta = wsw^{-1} \in W$, acting by $r_\beta(\lambda) = \lambda - 2(\lambda, \beta)\beta$. For multiplication by a simple reflection, strict Bruhat comparisons are exactly the corresponding length comparisons. We use the two inversion conventions

$$N(w) = \{\beta \in \Phi^+ : w\beta \in -\Phi^+\}, \qquad \mathrm{Inv}(w) = N(w^{-1}).$$

Both inversion sets are finite and have cardinality $\ell(w)$. For $\beta \in \Phi^+$, reflection descent is characterized by $\ell(wr_\beta) < \ell(w)$ if and only if $\beta \in N(w)$. If $sw < w$, then

$$N(sw) = N(w) \setminus \{-w^{-1}\alpha_s\}. \tag{3.1}$$

For these root-sign and inversion facts, see [2, Section 4.4]. An inversion set and its complement in Φ^+ are closed under positive linear combinations which are roots. For example, if two roots are sent to positive roots by w, then any positive linear combination of them which is a root is also sent to a positive root. This proves the assertion for the complement; the other assertion is identical.

For $J \subseteq S$, put $\Pi_J = \{\alpha_s : s \in J\}$ and $\Phi_J = W_J\Pi_J$, with $\Phi_J^+ = \Phi_J \cap \Phi^+$. The set

$$W^J = \{q \in W : \ell(qs) > \ell(q) \text{ for all } s \in J\}$$

consists of the minimal representatives of the cosets wW_J. Every w has a unique factorization $w = qp$, with $q \in W^J$, $p \in W_J$, and $\ell(w) = \ell(q) + \ell(p)$. Moreover, $\ell(qk) = \ell(q) + \ell(k)$ for every $k \in W_J$. For $q \in W^J$, the roots $q\alpha_s$, $s \in J$, are positive; their nonnegative linear combinations give $q\Phi_J^+ \subseteq \Phi^+$. We write $D_L(w) = \{s \in S : sw < w\}$ and $D_R(w) = \{s \in S : ws < w\}$. These sets are spherical. If J is spherical, w_J permutes Π_J after a change of sign and $\ell(w_Jp) = \ell(pw_J) = \ell(w_J) - \ell(p)$ for $p \in W_J$. We use the Bruhat subword property and the existence of joins of bounded subsets in weak order. These standard facts hold for arbitrary Coxeter systems; see [2, Chapters 2–3]. The support $\mathrm{supp}(w)$ is the set of simple reflections occurring in any reduced expression for w; it is independent of the expression.

We record the simple-coset rule in a form needed later. If $q \in W^J$ and $s \in S$, either $sq \in W^J$, or $sq = qt$ for a simple reflection $t \in J$. To see the second assertion, if sq is not J-minimal, choose $t \in J$ with $sq\alpha_t < 0$. Since $q\alpha_t > 0$ and a simple reflection changes the sign of only its own positive simple root, $q\alpha_t = \alpha_s$, whence $sq = qt$. It follows that a simple left ascent on qp either changes the minimal representative q by an ascent and leaves p fixed, or leaves q fixed and makes a simple left ascent on p. The alternative $sq \in W^J$ with $sq < q$ is a descent on qp for every $p \in W_J$.

Two reflection-subgroup facts will also be used. Every reflection subgroup has its canonical simple system and the positive roots induced from Φ^+. Dyer's criterion says that a subset $B \subseteq \Phi^+$ is such a canonical simple system if the inner products

of distinct members belong to

$$\{-\cos(\pi/m) : m \in \mathbb{Z},\ m \geq 2\} \ \cup\ (-\infty, -1]. \tag{3.2}$$

Equivalently, it is enough that each pair be the canonical simple pair of its two-generated reflection subgroup [5, Theorem 3.9 and Corollary 3.10]. All positive roots of the generated subgroup are then nonnegative linear combinations of B. Finally, every finite subgroup of W is contained in a conjugate of a finite standard parabolic subgroup [3, Proposition 29].

Define the continuation cone of w by

$$\mathcal{C}(w) = \{c \in W : \ell(wc) = \ell(w) + \ell(c)\}.$$

The basic inversion criterion is

$$c \in \mathcal{C}(w) \quad \Longleftrightarrow \quad N(w) \cap \mathrm{Inv}(c) = \varnothing. \tag{3.3}$$

For a cone C, define its boundary ∂C to be the set of positive roots γ for which there exist $c \notin C$ and $s \in S$ such that $cs \in C$ and $r_\gamma c = cs$. This definition depends only on C. The following boundary facts are due to Parkinson and Yau [12, Theorems 2.6 and 2.8, Corollary 2.7]; we include the elementary proofs in our inversion conventions.

Lemma 3.1. *Let C be a continuation cone and put $B = \partial C$.*

(i) For every $\gamma \in B$, there exists a single $c_\gamma \in W$ such that

$$N(w) \cap \mathrm{Inv}(c_\gamma) = \{\gamma\} \quad \textit{for every } w \textit{ with } \mathcal{C}(w) = C.$$

In particular, $B \subseteq N(w)$ for every such w, and B is finite.

(ii) For every $\gamma \in B$ and every representative w of C, $\ell(wr_\gamma) = \ell(w) - 1$.

(iii) The boundary gives the exact half-space description

$$C = \{c \in W : c^{-1}\gamma > 0 \textit{ for every } \gamma \in B\}. \tag{3.4}$$

No boundary root can be omitted from this description.

Proof. Choose c, s in the definition of a boundary root. Necessarily $cs < c$. Indeed, if $cs > c$, for every representative w of C the membership $cs \in C$ and nonmembership $c \notin C$ would give

$$\ell(w) + \ell(c) + 1 = \ell(wcs) \leq \ell(wc) + 1 < \ell(w) + \ell(c) + 1.$$

Thus $\gamma = -c\alpha_s > 0$ and $\mathrm{Inv}(cs) = \mathrm{Inv}(c) \setminus \{\gamma\}$. Applying (3.3) to c and cs proves $N(w) \cap \mathrm{Inv}(c) = \{\gamma\}$ simultaneously for every representative. This proves (i).

For (ii), use the same pair c, s. Since $cs = r_\gamma c \in C$,

$$\ell(w) + \ell(c) - 1 = \ell(wr_\gamma c) \leq \ell(wr_\gamma) + \ell(c).$$

On the other hand, $\gamma \in N(w)$, so reflection descent gives $\ell(wr_\gamma) < \ell(w)$. These two inequalities give (ii).

If $c \in C$, its inversion set avoids $N(w)$ and hence avoids B, which proves one inclusion in (iii). Conversely, take $c \notin C$ and a reduced word for c. Let b be the first prefix which does not belong to C, and write $b = b's$, where $b' \in C$ is the preceding prefix. Then $\gamma = -b\alpha_s$ lies in B, and $\gamma \in \mathrm{Inv}(b) \subseteq \mathrm{Inv}(c)$. Hence $c^{-1}\gamma < 0$. This proves (3.4), including the case $b' = e$. Finally, the witness from (i) has inversion set meeting B only in γ. It violates precisely that boundary inequality, proving irredundancy. □

For later use, if $c \in \mathcal{C}(w)$, then the cone $\mathcal{C}(wc)$ is determined by $\mathcal{C}(w)$ and c. Indeed, $d \in \mathcal{C}(wc)$ is equivalent to $\ell(cd) = \ell(c) + \ell(d)$ and $cd \in \mathcal{C}(w)$. Thus the same accepted continuation applied to elements with equal cones again gives elements with equal cones.

4. Maximal paths and finite reflection subgroups

We call a coefficient *maximal* if its degree is the global maximum D in (2.4). In this section, a path for a fixed reduced expression of a single left input is called maximal if its fold weight is D. Throughout this section, contributions to products are compared using Lemma 2.1.

Lemma 4.1. *If* $\deg f_{x,b,c} = D$, *then* $\mathcal{C}(b) = \mathcal{C}(c)$.

Proof. Suppose that $bs > b$ and $cs < c$ for a simple reflection s. In

$$T_x T_{bs} = (T_x T_b) T_s,$$

the coefficient of T_c has the positive fold contribution $\xi_s f_{x,b,c}$. Its degree is at least $D + L(s) > D$, a contradiction. Applying the same argument to $f_{x^{-1},c,b} = f_{x,b,c}$, by (2.3), excludes the opposite discrepancy. Thus b and c have the same simple right ascents.

If s is such a common ascent, the crossing from c to cs in the same identity shows that $f_{x,bs,cs}$ contains the contribution $f_{x,b,c}$. Therefore $\deg f_{x,bs,cs} = D$. This argument can be iterated along any common accepted prefix. If the two continuation cones differed, take a reduced expression for an element belonging to one cone but not the other. At its first unaccepted letter, the states reached after the preceding common prefix would have different simple right ascents, contrary to what was just proved. Hence the cones agree. □

Lemma 4.2. *Suppose that* $x = hk$ *is reduced and that*

$$\deg f_{k,b,d} + \deg f_{h,d,c} = D. \tag{4.1}$$

Then

$$\mathcal{C}(b) = \mathcal{C}(d) = \mathcal{C}(c).$$

Consequently, every state of a path of fold weight D *for a reduced expression of* x *has the same continuation cone as its endpoints.*

Proof. Associativity and positivity imply $\deg f_{x,b,c} = D$, so Lemma 4.1 gives $\mathcal{C}(b) = \mathcal{C}(c)$. We first compare the simple right ascents of b and d. If $bs > b$ and $ds < d$, expand

$$T_x T_{bs} = T_h\Big((T_k T_b) T_s\Big).$$

The contribution through T_d, followed by the fold at s, and then through T_c under left multiplication by T_h, is $\xi_s f_{k,b,d} f_{h,d,c}$. By (4.1), this gives degree at least $D + L(s)$ in one ordinary two-factor coefficient, which is impossible.

For the other discrepancy, suppose that $bs < b$ and $ds > d$. The endpoint equality gives $cs < c$. The right multiplication rule implies the exact identities

$$f_{k,b,d} = f_{k,bs,ds}, \qquad f_{h,ds,c} = f_{h,d,cs} + \xi_s f_{h,d,c}. \tag{4.2}$$

Indeed, the first identity follows by taking the T_d coefficient in $T_k T_b = (T_k T_{bs}) T_s$; a fold at d is unavailable because $ds > d$. For the second, take the T_c coefficient in $T_h T_{ds} = (T_h T_d) T_s$, using $cs < c$. The contribution through ds in $T_h(T_k T_{bs}) = T_x T_{bs}$

now has degree at least $D + L(s)$. This is again a contradiction. Hence b and d have the same simple right ascents.

Let s be a common right ascent of b, d, c. Crossing at the output in each of the two cut coefficients gives contributions

$$f_{k,b,d} \quad \text{in } f_{k,bs,ds}, \qquad f_{h,d,c} \quad \text{in } f_{h,ds,cs}.$$

The new degrees have sum at least D. Their product is a positive summand of $f_{x,bs,cs}$, so the global bound makes the sum exactly D. Thus (4.1) remains valid after any common accepted prefix. The preceding comparison of simple ascents can be repeated at each prefix. The first-discrepancy argument from Lemma 4.1 proves equality of the full cones.

Finally, cut a maximal path at any one of its states d. Its two subpaths have fold weights with sum D, so the two corresponding coefficients have degrees with sum at least D. Since $x = hk$ is reduced, their product is a positive summand of $f_{x,b,c}$, and the sum cannot exceed D. The cut therefore satisfies (4.1), proving the last assertion. □

An accepted continuation preserves a maximal path itself. More precisely, let c belong to the common cone of all its states and multiply each state w on the right by c. A crossing remains a crossing. If a step is a fold at $sw < w$, then

$$\ell(swc) \le \ell(sw) + \ell(c) = \ell(wc) - 1,$$

so it is still a fold at a descent. All fold labels and weights are unchanged. The resulting path is maximal, and its states again have equal cones, either by Lemma 4.2 or by the accepted continuation observation following Lemma 3.1.

Return to the fixed closed maximal path for the reduced expression $a = s_m \cdots s_1$, with initial state y chosen above. Set

$$C = \mathcal{C}(y), \qquad B = \partial C.$$

At a fold with letter s and state w, call

$$\eta = -w^{-1}\alpha_s \in \Phi^+$$

its *fold root*. Thus $\eta \in N(w)$ and $N(sw) = N(w) \setminus \{\eta\}$.

Lemma 4.3. *Let $\gamma \in B$, and let η be any fold root of the fixed path. If $\eta \neq \gamma$, then $r_\gamma \eta \in \Phi^+$.*

Proof. Let w be the fold state, with letter s, and use the common witness c_γ from Lemma 3.1. Both inversion sets

$$N(sw), \qquad \mathrm{Inv}(c_\gamma)$$

contain γ and omit η. For the first set this follows from the single-root deletion formula and $\gamma \neq \eta$; for the second it follows from $N(w) \cap \mathrm{Inv}(c_\gamma) = \{\gamma\}$.

Suppose instead that $r_\gamma \eta < 0$, and put $\delta = -r_\gamma \eta > 0$. The reflection formula gives

$$\eta + \delta = \lambda\gamma \quad \text{for some } \lambda > 0.$$

The coefficient is positive because both sides lie in the pointed positive cone spanned by Π. If δ were omitted by either of the displayed inversion sets, closure of its complement under positive root combinations would imply that it also omits $\gamma = \lambda^{-1}(\eta + \delta)$. Thus δ belongs to both sets. Moreover $\delta \neq \gamma$: equality would

imply $r_\gamma \eta = -\gamma$ and hence $\eta = \gamma$. Since $N(sw) \subseteq N(w)$, this contradicts the singleton witness intersection. Therefore $r_\gamma \eta > 0$. □

Lemma 4.4. *Every root in the complete boundary B occurs as a fold root of the fixed closed maximal path.*

Proof. If $\gamma \in B$ never occurs, multiply every path state on the right by r_γ. Crossings remain crossings. At a fold with root $\eta \neq \gamma$, its new descent test is

$$(wr_\gamma)^{-1}\alpha_s = -r_\gamma \eta < 0$$

by Lemma 4.3. Thus all folds survive, with their original weights. The transformed path is closed, has the same reduced left expression a, and has fold weight D. It follows that $\deg f_{a,yr_\gamma,yr_\gamma} = D$. But Lemma 3.1 gives $\ell(yr_\gamma) = \ell(y) - 1$, contrary to the minimality of y. □

For a reflection subgroup R of W, put

$$\Psi_R = \{\beta \in \Phi : r_\beta \in R\}, \qquad \Psi_R^+ = \Psi_R \cap \Phi^+.$$

These are its induced root system and positive roots. For $h \in R$, the inversion set in its canonical Coxeter system is

$$N_R(h) = \{\beta \in \Psi_R^+ : h\beta < 0\}.$$

Lemma 4.5. *For every $w \in W$, the intersection $N(w) \cap \Psi_R^+$ is an inversion set of R.*

Proof. Choose q of minimum ambient Coxeter length in the coset wR, and write $w = qh$ with $h \in R$. For $\beta \in \Psi_R^+$, the reflection r_β belongs to R. Minimality gives $\ell(qr_\beta) \geq \ell(q)$, and equality is impossible because a reflection changes length parity. Thus $\ell(qr_\beta) > \ell(q)$, so $q\beta \in \Phi^+$ by the reflection descent criterion. It follows also that q sends every negative R-root to an ambient negative root. Since h permutes Ψ_R, for every $\beta \in \Psi_R^+$ we have

$$w\beta < 0 \quad \Longleftrightarrow \quad h\beta < 0.$$

Consequently $N(w) \cap \Psi_R^+ = N_R(h)$. □

Proposition 4.6. *The set B is the canonical simple system of a finite reflection subgroup*

$$H = \langle r_\gamma : \gamma \in B \rangle.$$

Proof. First take distinct $\gamma, \eta \in B$, and let $R = \langle r_\gamma, r_\eta \rangle$. Its canonical Coxeter system is dihedral [5, Corollary 3.16 and the following discussion]. By Lemma 4.4, there is a path state at which γ is a fold root. Restrict the inversion sets before and after its single-root deletion to Ψ_R^+, using Lemma 4.5. We obtain an inversion set A of R containing both γ and η, such that $A \setminus \{\gamma\}$ is also an inversion set. Here both roots belong to the original inversion set because every path state has cone C. Similarly, another fold gives an inversion set A' which contains both roots and from which η alone can be removed.

Recall explicitly the inversion sets of a dihedral system. In infinite dihedral type, the two alternating reduced words give two disjoint strings of positive roots, each starting at one canonical simple root. Every nonempty inversion set is a finite initial segment of one string; the root which can be removed while leaving an inversion set is its terminal root. These descriptions follow directly by adding

the one new inversion root at each letter of an alternating reduced word. Thus A would place η strictly before γ in one string, whereas A' would place γ strictly before η in that same string. This is impossible. Hence R is finite.

In finite dihedral type of order $2m$, write its positive roots in angular order as $\rho_1, \dots, \rho_m$, with the two canonical simple roots at the ends. The same alternating-word calculation says that its inversion sets are the initial and final segments of this ordered list, including the empty and full sets. Suppose that $\gamma = \rho_i$ and $\eta = \rho_j$, with $i < j$. Because A contains both and permits removal of ρ_i, it must be the final segment $\{\rho_i, \dots, \rho_m\}$. Likewise, $A' = \{\rho_1, \dots, \rho_j\}$. In particular, $A \cup A' = \Psi_R^+$.

The common witness for γ meets the ambient inversion set at each of these two states exactly in γ. Thus the restriction of $\mathrm{Inv}(c_\gamma) = N(c_\gamma^{-1})$ meets A and A' exactly in γ. Since their union is all of Ψ_R^+, that restriction is the singleton $\{\gamma\}$. By Lemma 4.5 it is an inversion set of R. A singleton inversion set is the root of a canonical simple generator. The same reasoning applies to η. Therefore γ, η are precisely the canonical simple pair of R.

Dyer's criterion (3.2) now shows that the entire set B is the canonical simple system of H; the assertion is immediate when B has at most one member. Every positive H-root is a nonnegative linear combination of B. Since $B \subseteq N(y)$, its image under y is a nonzero nonpositive combination of ambient simple roots. It is therefore a negative root. Hence

$$\Psi_H^+ \subseteq N(y).$$

The set on the right is finite. The canonical length of any $h \in H$ is the cardinality of its H-inversion set and is consequently bounded by $|N(y)|$. Since B is finite, there are only finitely many words of that bounded length in the canonical generators. Thus H is finite. □

5. Parabolic boundaries and common suffixes

Proposition 5.1. *There exist a spherical subset $J \subseteq S$ and an element $d \in W$ such that $B = d\Pi_J$.*

Proof. By finite-subgroup containment [3, Proposition 29], choose a spherical $I \subseteq S$ and $g \in W$ with $H \subseteq gW_I g^{-1}$. Then $g^{-1}B \subseteq \Phi_I$. Indeed, the reflection in $g^{-1}\beta$ belongs to W_I for each $\beta \in B$. The positive choice of this root is inverted by its own reflection, and the inversions of an element of W_I all lie in Φ_I, as follows from a reduced expression in I.

Choose a linear functional δ strictly positive on every ambient simple root. The functional $f = \delta \circ g$ on $\mathrm{span}(\Phi_I)$ satisfies $f(g^{-1}\beta) > 0$ for all $\beta \in B$. Perturb f to a regular functional while preserving these finitely many strict inequalities. The chambers of the finite reflection arrangement of W_I are its translates of the standard chamber. Thus there is $k \in W_I$ such that the roots on which this functional is positive are exactly $k\Phi_I^+$. Put $c = gk$. We have

$$c^{-1}B \subseteq \Phi_I^+, \qquad c \in C, \tag{5.1}$$

where the second assertion follows from (3.4). If $B = \varnothing$, one may take $I = \varnothing$ and $c = e$ instead.

Extend every state of the fixed path by c. All folds survive and all extended states have the same cone. Their right W_I-components are therefore equal. To

verify this last assertion, write an extended state as $w = qb$, with $q \in W^I$ and $b \in W_I$. For $r \in W_I$, parabolic length additivity gives

$$r \in \mathcal{C}(w) \quad \Longleftrightarrow \quad \ell(br) = \ell(b) + \ell(r).$$

Within W_I, this accepted set has the unique element of greatest length $b^{-1}w_I$. Its length is $\ell(w_I) - \ell(b)$, and any accepted r of that length satisfies $br = w_I$. Thus $\mathcal{C}(w) \cap W_I$ determines b. Equality of the extended cones proves that the same b occurs at every extended state.

Suppose that an extended fold has letter $t \in S$ and root $\beta' = -b^{-1}q^{-1}\alpha_t$ lying in Φ_I^+. Then $q^{-1}\alpha_t \in \Phi_I$. This root is positive: if it were negative, the fact that $q \in W^I$ sends Φ_I^+ to positive roots would imply $\alpha_t < 0$. Let $r = r_{q^{-1}\alpha_t} \in W_I$. We have $tq = qr$ and $\ell(tq) = \ell(q) + 1$. On the other hand, parabolic additivity gives $\ell(qr) = \ell(q) + \ell(r)$. Hence $\ell(r) = 1$, so $r = s$ for some $s \in I$. The fold condition $tqb < qb$ becomes $sb < b$, and its root is

$$\beta' = -b^{-1}\alpha_s, \qquad s \in D_L(b).$$

Each original boundary root occurs at a fold, and its extended fold root is its image under c^{-1}. By (5.1), the preceding argument therefore gives a subset $K \subseteq D_L(b) \subseteq I$ such that

$$c^{-1}B = \{-b^{-1}\alpha_s : s \in K\}.$$

The subset K is spherical. Its longest element satisfies $w_K\Pi_K = -\Pi_K$, so

$$B = (cb^{-1}w_K)\Pi_K,$$

which proves the assertion. □

For the remaining argument use left weak order in its right-suffix form: write $d \preccurlyeq x$ if $x = hd$ and $\ell(x) = \ell(h) + \ell(d)$. We call d an *additive right suffix* of x. In particular, $N(d) \subseteq N(x)$. We will use bounded joins for this order, obtained by inversion from the usual right weak order [2, Theorem 3.2.1 and the following discussion].

Lemma 5.2. *Let J be spherical and $q \in W^J$. If d is an additive right suffix of qw_J and of every qsw_J, $s \in J$, then it is an additive right suffix of every element of qW_J.*

Proof. For each $t \in J$, opposition gives $s = w_Jtw_J \in J$ and $qsw_J = qw_Jt$. Since $w_J\alpha_t \in -\Phi_J^+$ and $q\Phi_J^+ \subseteq \Phi^+$, we have

$$(qw_Jt)\alpha_t = -qw_J\alpha_t \in \Phi^+.$$

Thus qw_Jt does not invert α_t. Since d is a right suffix of this element, $\alpha_t \notin N(d)$. It follows that $d \in W^J$.

Choose a reduced left-ascent gallery from d to qw_J: such a gallery exists because $d \preccurlyeq qw_J$. Write each state in its right parabolic decomposition rp, with $r \in W^J$ and $p \in W_J$. The initial decomposition is $d \cdot e$, and the final one is qw_J. The simple-coset rule in Section 3 shows that a step with letter s has exactly one of the following forms:

(i) $sr \in W^J$, $\ell(sr) = \ell(r) + 1$, and $rp \longmapsto (sr)p$;
(ii) $sr = rt$ for $t \in J$, and $rp \longmapsto r(tp)$ with $\ell(tp) = \ell(p) + 1$.

The possibility $sr \in W^J$ with $sr < r$ is excluded because it would be a descent on rp for every $p \in W_J$.

Record the labels of the steps of the second type in chronological order as $t_1, \dots, t_n$. They give the reduced expression

$$w_J = t_n \cdots t_1.$$

For an arbitrary $k \in W_J$, choose a reduced subword representing k in this expression, using the Bruhat subword property. Retain every step of the first type and precisely those steps of the second type selected by this subword.

This selection still gives a left-ascent gallery. Indeed, discarding a step of the second type leaves the minimal representative r unchanged. The sequence of minimal representatives at every retained position is consequently the original sequence. The simple-coset alternative at a position, and the label $t = r^{-1}sr$ in the second alternative, depend only on s and r, so these labels remain unchanged. As the retained finite steps are executed chronologically, their successive products are suffixes of the selected reduced expression for k. Each is therefore a left ascent. Every step of the first type is an ascent independently of the current finite component, by parabolic length additivity. The new gallery starts at d, ends at qk, and is reduced. Hence $d \preccurlyeq qk$ for every $k \in W_J$. □

Lemma 5.3. *Let J be spherical and $q \in W^J$. There exist a spherical subset K, an element $p \in W^K$, and $z \in W$ such that*

$$z\Pi_J = \Pi_K, \qquad qk = p(zkz^{-1})z \quad (k \in W_J), \tag{5.2}$$

where the outer factorizations are length additive:

$$\ell(qk) = \ell(p) + \ell(zkz^{-1}) + \ell(z) = \ell(p) + \ell(k) + \ell(z).$$

The element z is the greatest common additive right suffix of qW_J. In particular, if d is a common right suffix and $z = z_0 d$ is reduced, then

$$qw_J d^{-1} = pw_K z_0 \quad \text{is reduced}, \qquad d\Pi_J = z_0^{-1}\Pi_K. \tag{5.3}$$

Proof. We describe one shortening step. Suppose that every element of qW_J has a common simple right descent t. Since $q \in qW_J$ and $q \in W^J$, we have $t \notin J$. The right descent set of qw_J contains $I = J \cup \{t\}$, so I is spherical. Its right parabolic decomposition has the form

$$qw_J = rw_I, \qquad r \in W^I.$$

Put $u = w_I w_J$ and $J' = w_I J w_I \subseteq I$. Finite opposition gives $u\Pi_J = \Pi_{J'}$. In particular, $u \in W^J$, and conjugation by u identifies the simple generators of W_J and $W_{J'}$, preserving their lengths. Since $\ell(u) = \ell(w_I) - \ell(w_J)$, cancellation in the preceding reduced equality gives $q = ru$ reduced.

For every $k \in W_J$, we have

$$qk = r(uku^{-1})u. \tag{5.4}$$

This expression is reduced. Indeed, $u \in W^J$ gives $\ell(uk) = \ell(u) + \ell(k)$, while $\ell(uku^{-1}) = \ell(k)$. Thus the last two factors in (5.4) are length additive. Their product belongs to W_I, and $r \in W^I$ makes the whole product additive. The new minimal representative r lies in $W^{J'}$ and is strictly shorter than q, because $J \subsetneq I$ implies $\ell(u) > 0$.

Repeat this step for the new coset whenever it has a common simple right descent. The length of the minimal representative strictly decreases, so the procedure terminates.

To verify that the factors compose with additive lengths, consider two successive steps. The first gives (5.4); write the second as

$$rj = p(u'j(u')^{-1})u' \quad (j \in W_{J'}), \qquad u'\Pi_{J'} = \Pi_{J''},$$

with the displayed product reduced and $p \in W^{J''}$. Substitute $j = uku^{-1} \in W_{J'}$, whose length is $\ell(k)$. Then

$$\begin{aligned} qk &= p\Big((u'u)k(u'u)^{-1}\Big)u'u, \\ \ell(qk) &= \ell(rj) + \ell(u) \\ &= \ell(p) + \ell(k) + \ell(u') + \ell(u). \end{aligned}$$

For $k = e$, this gives $q = pu'u$ with $\ell(q) = \ell(p)+\ell(u')+\ell(u)$. The length inequalities

$$\ell(q) \le \ell(p) + \ell(u'u) \le \ell(p) + \ell(u') + \ell(u)$$

are therefore equalities. Thus $u'u$ is reduced, and $(u'u)\Pi_J = \Pi_{J''}$. Induction gives K, p, z satisfying (5.2), with all concatenations reduced. At the terminal stage pW_K has no common simple right descent. It therefore has no nonidentity common additive right suffix: the final simple letter of such a suffix would be a common right descent.

The element z in (5.2) is a common suffix. Let d be any other common suffix. The pair d, z is bounded above in left weak order, for example by q, so its join $b = d \vee z$ exists. Every element of qW_J is an upper bound for this pair; therefore b is also a common suffix. As $z \preccurlyeq b$, write $b = cz$ reduced. For every $k \in W_J$, the comparisons $b \preccurlyeq qk$ and (5.2) give

$$p(zkz^{-1})z = h_k cz$$

with both expressions reduced. Cancelling z preserves length additivity and yields

$$p(zkz^{-1}) = h_k c, \qquad \ell\Big(p(zkz^{-1})\Big) = \ell(h_k) + \ell(c).$$

Conjugation by z maps W_J onto W_K, so c is a common right suffix of pW_K. The terminal property forces $c = e$. Hence $b = z$ and $d \preccurlyeq z$, proving that z is greatest.

Finally, conjugation by z takes w_J to w_K. Substituting $z = z_0 d$ in the reduced equality for qw_J and cancelling d gives the first assertion of (5.3). The equality $z\Pi_J = \Pi_K$ gives the second. □

Theorem 5.4. *Let C be a continuation cone. Suppose that its complete boundary has the form $\partial C = d\Pi_J$ for a spherical subset J and some $d \in W$. Then there is a spherical subset $K \subseteq S$ such that*

$$\partial C = \Pi_K, \qquad C = \mathcal{C}(w_K).$$

Every representative g of C has a reduced factorization $g = pw_K$ with $p \in W^K$.

Proof. Put $B = \partial C = d\Pi_J$. Positivity of B gives $d \in W^J$, and (3.4) gives $d \in C$, since $d^{-1}B = \Pi_J$. Choose a representative g of C. The product gd is reduced and sends every simple J-root to a negative root, because $B \subseteq N(g)$. Its right parabolic component is thus w_J, and we may write

$$gd = qw_J, \qquad q \in W^J.$$

For $s \in J$, put $\gamma = d\alpha_s$. Boundary shortness gives $\ell(gr_\gamma) = \ell(g) - 1$, whereas

$$gr_\gamma d = gds = qw_J s, \qquad \ell(qw_J s) = \ell(g) + \ell(d) - 1.$$

It follows that $gr_\gamma d$ is reduced. Thus d is a common right suffix of qw_J and every $qw_J s$. Opposition permutes J, so these are also the elements qsw_J used in Lemma 5.2. That lemma makes d a common right suffix of the whole coset qW_J.

Apply Lemma 5.3, and write its greatest common suffix as $z = z_0 d$ reduced. Formula (5.3) becomes

$$g = pw_K z_0 \quad \text{reduced}, \qquad B = z_0^{-1}\Pi_K. \tag{5.5}$$

The full boundary formula now puts z_0^{-1} in $\mathcal{C}(g)$, since $z_0 B = \Pi_K \subseteq \Phi^+$. Hence

$$\ell(gz_0^{-1}) = \ell(g) + \ell(z_0).$$

But the reduced factorization in (5.5) gives $\ell(gz_0^{-1}) = \ell(g) - \ell(z_0)$. Therefore $z_0 = e$. This proves $B = \Pi_K$ and the asserted factorization of g.

Finally, (3.4) says that C consists exactly of the elements c for which $c^{-1}\Pi_K$ is positive. For each such c, the inverse c^{-1} sends every positive root of Φ_K to a positive root, since those roots are nonnegative combinations of Π_K. As $N(w_K) = \Phi_K^+$, the inversion criterion (3.3) identifies this set with $\mathcal{C}(w_K)$. The proof also includes $J = \varnothing$; equivalently, an empty boundary gives $C = W$, whose only representative is e. □

Proposition 5.5. *Let y have minimum length among all inputs supporting a closed coefficient of the global maximum degree D, as chosen after Lemma 2.2. Then there exist a spherical subset $J \subseteq S$ and $u \in W^J$ such that*

$$y = uw_J, \qquad \mathcal{C}(y) = \mathcal{C}(w_J), \qquad \ell(uw_J u^{-1}) = 2\ell(u) + \ell(w_J).$$

Proof. Propositions 4.6 and 5.1, followed by Theorem 5.4, give a spherical J with $\mathcal{C}(y) = \mathcal{C}(w_J)$ and $y = uw_J$ reduced, $u \in W^J$. Inverting this factorization shows that $w_J u^{-1}$ is reduced. Thus $u^{-1} \in \mathcal{C}(w_J) = \mathcal{C}(y)$, and consequently

$$\ell(uw_J u^{-1}) = \ell(y) + \ell(u) = 2\ell(u) + \ell(w_J).$$

□

6. Parabolic compression and the sharp bound

Apply Proposition 5.5 to the minimum-length input y supporting a closed path of degree D, chosen above. Thus there are a spherical subset $J \subseteq S$ and an element $u \in W^J$ such that

$$y = uw_J, \qquad \ell(uw_J u^{-1}) = 2\ell(u) + \ell(w_J).$$

Retain the fixed a with $\deg f_{a,y,y} = D$, and put

$$w = w_J, \qquad g = uwu^{-1}.$$

We shall prove $D \le L(w)$.

Lemma 6.1 (Finite longest-element coefficients)**.** *Let $I \subseteq S$ be spherical, and let $b \in W_I$. The coefficients f_{b,w_I,w_I} and $f_{w_I,w_I,b}$ are monic Laurent polynomials of degree $L(b)$.*

Proof. Choose a reduced expression for b and expand $T_b T_{w_I}$ by successive left multiplication. Every simple reflection in I is a left descent of w_I. Consequently, the path which folds at every letter is legal, remains at w_I, and contributes a product of fold factors of total weight $L(b)$. Its leading coefficient is one. Every other path uses at least one crossing, and hence omits at least one strictly positive letter weight from its fold weight. Thus the all-fold path is the unique path of weight $L(b)$, proving the assertion for f_{b,w_I,w_I}.

Since $w_I^{-1} = w_I$, the trace pairing and cyclicity give

$$f_{w_I,w_I,b} = \tau(T_{b^{-1}} T_{w_I} T_{w_I}) = \tau(T_{w_I} T_{b^{-1}} T_{w_I}) = f_{b^{-1},w_I,w_I}.$$

The first assertion, applied to b^{-1}, completes the proof because $L(b^{-1}) = L(b)$. □

Lemma 6.2 (A degree estimate from a square)**.** *For $h \in W_J$ and $z \in W$, one has*

$$\deg[T_z]\, T_u T_h T_{u^{-1}} \le D - L(h). \tag{6.1}$$

Proof. The expressions uw and wu^{-1} are reduced. Expanding their product gives

$$T_y T_{y^{-1}} = T_u T_w^2 T_{u^{-1}} = \sum_{h \in W_J} f_{w,w,h}\, T_u T_h T_{u^{-1}}. \tag{6.2}$$

For a fixed z, each summand in its standard coordinate has a positive fold expansion. By Lemma 6.1, the factor $f_{w,w,h}$ has degree $L(h)$. If the coordinate of the remaining triple product is nonzero, their product therefore has degree

$$L(h) + \deg[T_z]\, T_u T_h T_{u^{-1}}.$$

Lemma 2.1 shows that this leading degree cannot be cancelled by the other summands in (6.2). The coordinate on the left is $f_{y,y^{-1},z}$ and has degree at most D. This proves (6.1); the zero coordinate satisfies it under the convention $\deg 0 = -\infty$. □

Let $\mathcal{H}_J = \bigoplus_{h \in W_J} \mathcal{A} T_h$ be the parabolic Hecke subalgebra, and let $E_J : \mathcal{H} \longrightarrow \mathcal{H}_J$ be the standard-coordinate projection:

$$E_J(T_x) = \begin{cases} T_x, & x \in W_J, \\ 0, & x \notin W_J. \end{cases}$$

Define $\varphi : \mathcal{H} \longrightarrow \mathcal{H}_J$ by

$$\varphi(X) = E_J(T_{u^{-1}} X T_u). \tag{6.3}$$

Let $\mathcal{A}^{(W)}$ and $\mathcal{A}^{(W_J)}$ be the free $\mathcal{A}$-modules with coordinate bases $(\delta_x)_{x \in W}$ and $(\delta_h)_{h \in W_J}$, respectively. We write λ_W and λ_J for the left regular representations in the standard bases, identified with these coordinate bases. Thus

$$[\lambda_W(X)]_{x,z} = [T_x]\, X T_z.$$

These matrices are column-finite. Define $\mathcal{A}$-linear maps $V : \mathcal{A}^{(W_J)} \longrightarrow \mathcal{A}^{(W)}$ and $V^* : \mathcal{A}^{(W)} \longrightarrow \mathcal{A}^{(W_J)}$ by

$$V\delta_h = \delta_{uh} \quad (h \in W_J), \qquad V^*\delta_x = \begin{cases} \delta_h, & x = uh \text{ for } h \in W_J, \\ 0, & x \notin uW_J. \end{cases}$$

In particular, $V^*V = 1$.

Proposition 6.3 (Compression to the parabolic algebra)**.** *For every $X \in \mathcal{H}$,*

$$\lambda_J(\varphi(X)) = V^* \lambda_W(X) V. \tag{6.4}$$

If

$$P = \varphi(T_a) = \sum_{h \in W_J} p_h T_h,$$

then every p_h has a positive fold expansion, and

$$p_h = f_{a,u,uh} = [T_{a^{-1}}]\, T_u T_{h^{-1}} T_{u^{-1}}, \qquad \deg p_h \le D - L(h). \tag{6.5}$$

Proof. First, E_J is an $\mathcal{H}_J$-bimodule map. Indeed, if $x \notin W_J$ and $s \in J$, then sx and xs also lie outside W_J. The multiplication rules therefore show that left or right multiplication by T_s takes the span of the outside standard basis elements into itself. On $\mathcal{H}_J$, the projection is the identity. The asserted bimodule property follows by linearity and multiplication of the generators.

For $h, k \in W_J$, the trace pairing now yields

$$\begin{aligned}
[\lambda_J(\varphi(X))]_{h,k} &= \tau\Big(T_{h^{-1}} E_J(T_{u^{-1}} X T_u) T_k\Big) \\
&= \tau\Big(T_{h^{-1}} T_{u^{-1}} X T_u T_k\Big) \\
&= \tau\Big(T_{(uh)^{-1}} X T_{uk}\Big) \\
&= [T_{uh}]\, X T_{uk}.
\end{aligned}$$

For the second equality, use the bimodule property and $\tau \circ E_J = \tau$. For the third, use $u \in W^J$, which gives $T_u T_k = T_{uk}$ and $T_{h^{-1}} T_{u^{-1}} = T_{(uh)^{-1}}$. This proves (6.4) with the stated row and column indices.

Taking the (h, e) entry with $X = T_a$ gives $p_h = f_{a,u,uh}$, so p_h has a positive fold expansion. Alternatively, cyclicity of the trace gives

$$\begin{aligned}
p_h &= \tau(T_{h^{-1}} T_{u^{-1}} T_a T_u) \\
&= \tau(T_u T_{h^{-1}} T_{u^{-1}} T_a) \\
&= [T_{a^{-1}}]\, T_u T_{h^{-1}} T_{u^{-1}}.
\end{aligned}$$

Applying Lemma 6.2 to h^{-1} proves the degree estimate, since $L(h^{-1}) = L(h)$. □

Lemma 6.4 (A coefficient of full support)**.** *There exists $h \in W_J$ such that*

$$\deg p_h = D - L(h). \tag{6.6}$$

Every such h satisfies $\operatorname{supp}(h) = J$.

Proof. The diagonal entry of (6.4) indexed by $k \in W_J$ reads

$$[T_k]\, P T_k = f_{a,uk,uk}. \tag{6.7}$$

At $k = w$, this becomes

$$f_{a,y,y} = \sum_{h \in W_J} p_h f_{h,w,w}.$$

Every nonzero summand has a positive fold expansion, and Lemma 6.1 gives $\deg f_{h,w,w} = L(h)$. By (6.5), the degree of each summand is at most D. Since the left side has degree D, at least one summand attains that degree. This is precisely (6.6).

Now let h satisfy (6.6), and put $I = \operatorname{supp}(h) \subseteq J$. Suppose $I \subsetneq J$. Then W_I is finite and $h \in W_I$. Lemma 6.1 shows that $p_h f_{h,w_I,w_I}$ has degree D. It is a

positive fold contribution to $[T_{w_I}]\,PT_{w_I}$, so that coordinate has degree at least D. By (6.7), it is f_{a,uw_I,uw_I}, whose degree is at most D by the definition of D. Thus uw_I supports a closed path of degree D. Since $u \in W^J$ and $I \subsetneq J$,

$$\ell(uw_I) = \ell(u) + \ell(w_I) < \ell(u) + \ell(w_J) = \ell(y).$$

This contradicts the global minimality of y and proves the claim. □

Lemma 6.5 (A finite power reaches the longest element)**.** *Let $h \in W_J$ have support J. There is an integer $K \geq 1$ such that*

$$[T_w]\,T_h^K \text{ is monic of degree } KL(h) - L(w). \tag{6.8}$$

One may take $K \leq \ell(w) - \ell(h) + 1$.

Proof. Fix a reduced expression for h. Starting at e, read successive copies of this expression by the right Demazure rule: cross at an ascent, and remain at the current element at a descent. The first copy ends at h. Length never decreases. If a complete subsequent copy causes no increase, the state stays fixed throughout that copy. Every letter in the expression is then a right descent of this state. The support assumption says that all of J are right descents. This state lies in W_J and sends Π_J, hence Φ_J^+, to negative roots. Its length is therefore $|\Phi_J^+| = \ell(w)$, so it equals w. Hence, until w is reached, each complete copy increases length by at least one. It follows that some $K \leq \ell(w) - \ell(h) + 1$ copies end at w.

Expand T_h^K using the concatenation of these K reduced words, starting at T_e and multiplying on the right. For any path ending at w, let U_+, U_-, and F denote the total letter weights of its ascending crossings, descending crossings, and folds, respectively. Each letter belongs to exactly one of these three classes. Moreover, ascending and descending crossings change the weight of the state by plus and minus the corresponding letter weight. Therefore

$$KL(h) = U_+ + U_- + F, \qquad L(w) = U_+ - U_-, \qquad F = KL(h) - L(w) - 2U_-.$$

There is exactly one path with no descending crossing: it crosses at every ascent and folds at every descent. This is the Demazure path just constructed, and it ends at w. It has fold weight $KL(h) - L(w)$. Every other path ending at w has $U_- > 0$, because the simple weights are strictly positive, and consequently has smaller fold weight. The unique path of greatest fold weight contributes a fold monomial with leading coefficient one. This proves (6.8).

If $J = \varnothing$, then $h = w = e$ and the same conclusion holds with $K = 1$. □

Choose h as in Lemma 6.4, and choose K as in Lemma 6.5. Selecting $p_h T_h$ in every factor of P^K gives the positive fold contribution $p_h^K T_h^K$. Its T_w coordinate has degree

$$K\Big(D - L(h)\Big) + KL(h) - L(w) = KD - L(w).$$

All other contributions also have positive fold expansions. Thus

$$\deg[T_w]\,P^K \geq KD - L(w). \tag{6.9}$$

Proposition 6.6 (Comparison of powers)**.** *For every integer $m \geq 1$, each matrix entry of*

$$\lambda_J\Big(\varphi(T_a^m) - P^m\Big)$$

is a positive fold polynomial, possibly zero. In particular, for the chosen K,

$$\deg[T_w]\,\varphi(T_a^K) \geq KD - L(w). \tag{6.10}$$

Proof. Put $M = \lambda_W(T_a)$ and $Q = VV^*$. The latter is the coordinate projection onto uW_J. Equation (6.4) and multiplicativity of the regular representations give

$$\lambda_J(\varphi(T_a^m)) = V^*M^mV, \qquad \lambda_J(P^m) = V^*MQM\cdots QMV,$$

where the second product contains m copies of M and $m-1$ copies of Q. For $h,k \in W_J$, set $x_0 = uk$ and $x_m = uh$. The difference of the two (h,k) entries is exactly

$$\sum_{\substack{x_1,\dots,x_{m-1}\in W \\ x_i\notin uW_J \text{ for some } i}} \prod_{i=1}^{m} f_{a,x_{i-1},x_i}. \tag{6.11}$$

Indeed, V^*M^mV allows all intermediate states, whereas each inserted Q requires the corresponding state to belong to uW_J. Every factor in (6.11) has a positive fold expansion. The sum is finite: every standard multiplication has finite support, and only finitely many successive multiplications are involved. For $m = 1$ the difference is zero.

Take the (w,e) entry and apply Lemma 2.1. The coefficient $[T_w]\,\varphi(T_a^K)$ is the sum of $[T_w]\,P^K$ and a positive fold polynomial. Its degree is therefore at least that in (6.9), proving (6.10). □

Lemma 6.7 (An involution coefficient)**.** *For every $X \in \mathcal{H}$,*

$$[T_w]\,\varphi(X) = [T_g]\,X. \tag{6.12}$$

Proof. The expression $g = uwu^{-1}$ is reduced by Proposition 5.5, and $g^{-1} = g$ because $w^{-1} = w$. Using the bimodule property of E_J, cyclicity, and the trace pairing, we obtain

$$\begin{aligned}
[T_w]\,\varphi(X) &= \tau(T_wT_{u^{-1}}XT_u)\\
&= \tau(T_uT_wT_{u^{-1}}X)\\
&= \tau(T_gX)\\
&= [T_g]\,X.
\end{aligned}$$

The reducedness assertion is what gives $T_uT_wT_{u^{-1}} = T_g$ in the third line. □

Proof of Theorem 1.1. For each integer $m \geq 1$, a product of m standard basis elements has every coefficient of degree at most $(m-1)D$. To see this, the assertion for $m = 1$ follows from its single coefficient 1. If $T_{x_1}\cdots T_{x_m} = \sum_z c_zT_z$ satisfies the asserted bound, then the coefficient of T_t after multiplication by $T_{x_{m+1}}$ is

$$\sum_z c_z f_{z,x_{m+1},t}.$$

Each summand has degree at most $(m-1)D + D = mD$, proving the induction.

Apply this upper bound to T_a^K, and combine (6.10) with (6.12). It follows that

$$(K-1)D \geq \deg[T_g]\,T_a^K = \deg[T_w]\,\varphi(T_a^K) \geq KD - L(w).$$

Subtracting $(K-1)D$ gives

$$D \leq L(w_J) \leq N_L(W).$$

Conversely, for every spherical $I \subseteq S$, Lemma 6.1 gives

$$\deg f_{w_I,w_I,w_I} = L(w_I).$$

The definition of D therefore implies $D \geq N_L(W)$. Hence $D = N_L(W)$, proving the theorem. □

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