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PRETORSION THEORIES AND IMAGE-CLOSED SUBCATEGORIES

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Abstract. A pretorsion theory generalizes a torsion pair by allowing its two components to have a nonzero intersection, with morphisms factoring through the intersection playing the role of zero morphisms. In the module category of a finite-dimensional algebra, the first and second components of pretorsion theories are known to be the quotient-closed and submodule-closed subcategories, respectively. We show that the intersections of the two components are precisely the image-closed subcategories, namely those containing the image of every morphism between their objects, and that the pretorsion theories form a complete lattice. For a linearly oriented quiver of type A, we prove that the numbers of pretorsion theories and of image-closed subcategories are the Genocchi and median Genocchi numbers, respectively. Counting pretorsion theories by the number of indecomposable modules in their intersection, and image-closed subcategories by their number of indecomposable modules, yields the Gandhi polynomial refinements of Han and Zeng.

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1. Introduction

Let Λ be a finite-dimensional algebra over a field $\Bbbk$, and consider full, additive subcategories of $\mathsf{mod}\,\Lambda$ closed under isomorphisms and direct summands. We ask which subcategories can occur as the intersection $\mathcal{T} \cap \mathcal{F}$ of the two components of a pretorsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathsf{mod}\,\Lambda$. Recall that a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathsf{mod}\,\Lambda$ satisfies $\mathrm{Hom}_\Lambda(\mathcal{T}, \mathcal{F}) = 0$, and every module $X \in \mathsf{mod}\,\Lambda$ fits into a short exact sequence $0 \to T \to X \to F \to 0$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$, where $T \to X$ is a kernel of $X \to F$ and $X \to F$ is its cokernel; in particular, $\mathcal{T} \cap \mathcal{F} = 0$. Pretorsion theories relax this orthogonality condition by allowing the intersection $\mathcal{Z} = \mathcal{T} \cap \mathcal{F}$ to be nonzero: morphisms factoring through $\mathcal{Z}$ replace zero morphisms both in the orthogonality condition between $\mathcal{T}$ and

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$\mathcal{F}$ and in the kernel and cokernel universal properties (Definition 2.2). Facchini and Finocchiaro [10] introduced this notion for categories without a zero object—starting from the category of nonempty preordered sets, where equivalence relations and partial orders form the two components and sets equipped with equality form their intersection—and Facchini, Finocchiaro, and Gran [11, Section 2] developed the theory for general categories.

Characterizing the intersections $\mathcal{T} \cap \mathcal{F}$ in $\mathsf{mod}\,\Lambda$ brings together the study of quotient-closed and submodule-closed subcategories in the author's earlier work [8] and that of image-closed subcategories in joint work with Sakai [9]. Indeed, Campanini, Fedele, and Yıldırım [4, Lemma 2.6, Proposition 2.8, and Section 3.2] proved that the subcategories occurring as first and second components of pretorsion theories in $\mathsf{mod}\,\Lambda$ are precisely the quotient-closed and submodule-closed subcategories, respectively. Earlier, Campanini and Fedele [3, Theorem 4.2] constructed pretorsion theories whose intersections are prescribed Serre subcategories. Here we determine all possible intersections, allowing subcategories that are not Serre.

A subcategory $\mathcal{Z} \subseteq \mathsf{mod}\,\Lambda$ is *image-closed* if $\operatorname{Im} f \in \mathcal{Z}$ for every morphism $f\colon X \to Y$ with $X, Y \in \mathcal{Z}$. Write $\mathsf{Gen}\,\mathcal{Z}$ and $\mathsf{Cogen}\,\mathcal{Z}$ for the subcategories consisting of the quotients and the submodules, respectively, of finite direct sums of objects of $\mathcal{Z}$. Auslander and Smalø [1, Proposition 4.10(i)] proved that a subcategory $\mathcal{Z}$ is image-closed if and only if $\mathcal{Z} = \mathsf{Gen}\,\mathcal{Z} \cap \mathsf{Cogen}\,\mathcal{Z}$. Our first main result shows that although $(\mathsf{Gen}\,\mathcal{Z}, \mathsf{Cogen}\,\mathcal{Z})$ need not itself be a pretorsion theory (Example 4.2), every image-closed subcategory $\mathcal{Z}$ is the intersection $\mathcal{T} \cap \mathcal{F}$ of some pretorsion theory $(\mathcal{T}, \mathcal{F})$.

Theorem A (= Theorem 4.4)**.** *Let Λ be a finite-dimensional algebra, and let $\mathcal{Z}$ be a subcategory of $\mathsf{mod}\,\Lambda$. The following are equivalent.*

(1) *$\mathcal{Z}$ is image-closed.*
(2) *$\mathcal{Z} = \mathsf{Gen}\,\mathcal{Z} \cap \mathsf{Cogen}\,\mathcal{Z}$.*
(3) *$\mathcal{Z} = \mathcal{T} \cap \mathcal{F}$ for a pretorsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathsf{mod}\,\Lambda$.*

Theorem A may be compared with a result of the author and Sakai [9, Proposition 2.3], who proved that the subcategories of $\mathsf{mod}\,\Lambda$ closed under both images and extensions are precisely the intersections of a torsion class and a torsion-free class (which need not form a torsion pair).

To construct a pretorsion theory with a prescribed image-closed intersection, we use Theorem 2.12, which characterizes when a quotient-closed subcategory $\mathcal{T}$ and a submodule-closed subcategory $\mathcal{F}$ form a pretorsion theory by comparing, for each module, its largest submodule in $\mathcal{T}$ (its trace) with its largest quotient in $\mathcal{F}$ (the quotient by its reject).

We next study the poset $\mathsf{PT}(\Lambda)$ of pretorsion theories in $\mathsf{mod}\,\Lambda$, ordered by inclusion of first components and reverse inclusion of second components. Complementing the complete lattices of individual components studied in [4, Sections 3.1–3.2], we prove that $\mathsf{PT}(\Lambda)$ is a complete lattice and describe its meet and join (Theorem 3.6). A pretorsion theory with intersection $\mathcal{Z}$ is called a *realization* of $\mathcal{Z}$. For each image-closed subcategory $\mathcal{Z}$, its realizations form a complete lattice with greatest and least elements (Corollaries 4.5 and 4.7), whose nonempty meets and joins agree with those in $\mathsf{PT}(\Lambda)$ (Proposition 3.8). In particular, for $\mathcal{Z} = 0$, the realizations are precisely the torsion pairs, which form a complete sublattice containing the least and greatest elements of $\mathsf{PT}(\Lambda)$ and isomorphic to the lattice of torsion classes (Corollary 3.9).

To study the poset $\mathcal{L}_{\mathrm{im}}(\Lambda)$ of image-closed subcategories of $\mathsf{mod}\,\Lambda$, ordered by inclusion, we use convex geometry [7] to describe its generating sets and covering relations. Write $\mathsf{ind}\,\Lambda$ for a set of representatives of the isomorphism classes of indecomposable Λ-modules, and $\mathsf{ind}\,\mathcal{C}$ for those belonging to $\mathcal{C}$. The *size* of $\mathcal{C}$ is $|\mathsf{ind}\,\mathcal{C}|$, the number of isomorphism classes of its indecomposable objects. Sending a subset of $\mathsf{ind}\,\Lambda$ to the indecomposable objects of the smallest image-closed subcategory containing it defines a closure operator icl on $\mathsf{ind}\,\Lambda$. Extending the corresponding result for quotient-closed and submodule-closed subcategories in [8], we show that $(\mathsf{ind}\,\Lambda, \mathsf{icl})$ is a finitary *convex geometry* (Theorem 4.11): we have $\mathsf{icl}(\varnothing) = \varnothing$, the closure of any subset is the union of the closures of its finite subsets, and icl satisfies the anti-exchange property that if $\mathcal{C}$ is image-closed and $X, Y \in \mathsf{ind}\,\Lambda \setminus \mathsf{ind}\,\mathcal{C}$ are distinct, then $X \in \mathsf{icl}(\mathsf{ind}\,\mathcal{C} \cup \{Y\})$ implies $Y \notin \mathsf{icl}(\mathsf{ind}\,\mathcal{C} \cup \{X\})$.

As in the geometry of convex hulls and their extreme points, this convex-geometric structure has direct consequences for $\mathcal{L}_{\mathrm{im}}(\Lambda)$ (Corollary 4.14). Every image-closed subcategory $\mathcal{Z}$ of finite size has a unique smallest generating set of indecomposables, consisting of its *extreme points*—the indecomposables in $\mathcal{Z}$ that do not belong to the image closure of the remaining ones. Proposition 4.13 identifies this set as the union of the splitting projective and splitting injective objects of Auslander and Smalø [1, Section 2]: an indecomposable $M \in \mathcal{Z}$ is splitting projective if every epimorphism to M from an object of $\mathcal{Z}$ splits, and splitting injective if every monomorphism from M to an object of $\mathcal{Z}$ splits. The existence of this smallest generating set also follows from [1, Proposition 2.1 and Corollary 2.4(b),(d)], together with the description of image closure as $\mathsf{Gen} \cap \mathsf{Cogen}$. Moreover, every covering relation in $\mathcal{L}_{\mathrm{im}}(\Lambda)$ adds a single indecomposable module, so $\mathcal{L}_{\mathrm{im}}(\Lambda)$ is graded by size whenever Λ is representation-finite.

The linearly oriented quiver of type A. Campanini, Fedele, and Yıldırım [4, Appendix A.2] listed the 17 pretorsion theories for the linearly oriented quiver of type A_2. In the second half of this paper, we consider the path algebra

$$\Lambda_n = \Bbbk(1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n)$$

of a linearly oriented quiver of type A_n (with $\Lambda_0 = 0$), and determine the numbers of pretorsion theories and of image-closed subcategories in $\mathsf{mod}\,\Lambda_n$ for every $n \geq 0$, together with their natural q-refinements. These enumeration formulas are established only for the linearly oriented quiver: unlike the number of torsion pairs, the numbers of pretorsion theories and of image-closed subcategories for a quiver of type A_n depend on the orientation (Remark 8.8). For small values of n, the cardinalities of $\mathsf{PT}(\Lambda_n)$ and $\mathcal{L}_{\mathrm{im}}(\Lambda_n)$ are given in the following table.

n	0	1	2	3	4	5
$\lvert\mathsf{PT}(\Lambda_n)\rvert$	1	3	17	155	2073	38227
$\lvert\mathcal{L}_{\mathrm{im}}(\Lambda_n)\rvert$	1	2	8	56	608	9440

These are the sequences A110501 and A005439 in the OEIS [17, 18], given by the positive Genocchi numbers g_{2n+4} and the median Genocchi numbers H_{2n+3}. Following Han and Zeng [15, p. 63], the positive Genocchi numbers g_{2i} and the median Genocchi numbers H_{2m+1} are defined by

$$\begin{aligned} \frac{2t}{e^t+1} &= \sum_{i\geq 0} G_i \frac{t^i}{i!}, \qquad g_{2i} = |G_{2i}|, \\ H_{2m+1} &= \sum_{0\leq i\leq (m-1)/2} (-1)^i g_{2m-2i} \binom{m}{2i+1} \quad (m \geq 1). \end{aligned} \tag{1.1}$$

To refine these numbers to polynomials in q, we recall the q-Gandhi polynomials introduced by Han and Zeng [14, 15].

Definition 1.1 ([14, p. 120, Equations (3)–(4)])**.** For a polynomial $f \in \mathbb{Z}[q,x]$, put

$$\Delta_q f(x) = \frac{f(1+qx) - f(x)}{1+(q-1)x}. \tag{1.2}$$

The *q-Gandhi polynomials of the first and of the second kind* are defined by $B_1(x,q) = C_1(x,q) = 1$ and, for $m \geq 2$,

$$B_m(x,q) = \Delta_q\big(x^2 B_{m-1}(x,q)\big), \tag{1.3}$$

$$C_m(x,q) = (1+qx)\Delta_q\big(xC_{m-1}(x,q)\big). \tag{1.4}$$

Since $1+(q-1)x = (1+qx) - x$, the operator Δ_q preserves $\mathbb{Z}[q,x]$ and specializes at $q=1$ to the forward difference operator $f(x) \mapsto f(x+1) - f(x)$. The classical specializations are $B_m(1,1) = g_{2m+2}$ [14, p. 120, Equation (1)] and $C_m(1,1) = H_{2m+1}$ [15, pp. 63–64, Equation (2)].

The quotient–submodule equidistribution conjecture of [8] compares the numbers of quotient-closed and submodule-closed subcategories of each size for a representation-finite algebra, predicting that they agree. Counting subcategories by size in this way motivates our q-refinement, which

enumerates pretorsion theories by the size of their intersections and image-closed subcategories by their size.

Theorem B (= Theorems 7.5 and 8.5)**.** *For every $n \geq 0$,*

$$\sum_{(\mathcal{T},\mathcal{F})\in \mathsf{PT}(\Lambda_n)} q^{|\mathsf{ind}(\mathcal{T}\cap\mathcal{F})|} = B_{n+1}(1,q),$$

$$\sum_{\mathcal{C}\in \mathcal{L}_{\mathrm{im}}(\Lambda_n)} q^{|\mathsf{ind}\,\mathcal{C}|} = C_{n+1}(1,q).$$

Since $\mathcal{L}_{\mathrm{im}}(\Lambda_n)$ is graded by size, $C_{n+1}(1,q)$ is its rank-generating polynomial. At $q = 1$, Theorem B gives $|\mathsf{PT}(\Lambda_n)| = g_{2n+4}$ and $|\mathcal{L}_{\mathrm{im}}(\Lambda_n)| = H_{2n+3}$.

To prove Theorem B, we describe both families of subcategories combinatorially. Each indecomposable Λ_n-module is determined by its support interval in $\{1,\dots,n\}$, and we show in Section 5 that the image-closed subcategories of $\mathsf{mod}\,\Lambda_n$ correspond bijectively to the families of intervals in $\{1,\dots,n\}$ closed under nonempty intersections (Theorem 5.5). Pilaud [19, p. 3] also obtained the count H_{2n+3} for these intersection-closed interval families. In Section 6, we parametrize quotient-closed and submodule-closed subcategories of $\mathsf{mod}\,\Lambda_n$ by endpoints recording how far the support intervals extend when one endpoint is fixed, and characterize which pairs form a pretorsion theory (Theorem 6.5). Adjoining one vertex at a time in Sections 7 and 8, we then establish two-variable refinements of Theorem B (Theorems 7.5 and 8.5) in which the auxiliary variable x records the number of indecomposable modules whose support ends at vertex n, in $\mathcal{T}$ for $B_{n+1}(x,q)$ and in $\mathcal{C}$ for $C_{n+1}(x,q)$.

For comparison, a *Genocchi permutation* of $\{1,\dots,2m\}$ is a permutation σ satisfying $\sigma(2i-1) > 2i-1$ and $\sigma(2i) \leq 2i$ for $1 \leq i \leq m$. Writing σ_{odd} and σ_{even} for its subwords in odd and even positions and inv for the number of inversions, Han and Zeng [14, Proposition 19 and Corollary 21] show that $B_m(1,q)$ is the sum, over all Genocchi permutations σ of $\{1,\dots,2m\}$, of the weights

$$q^{m^2-\sum_{i=1}^{m}\sigma(2i)-\mathrm{inv}(\sigma_{\mathrm{odd}})-\mathrm{inv}(\sigma_{\mathrm{even}})},$$

and that restricting the sum to fixed-point-free Genocchi permutations gives $C_m(1,q)$. Our identifications in Theorem B are proved by recurrences, and no statistic-preserving bijection between our subcategories and these permutation models is known to us.

Organization. Section 2 recalls pretorsion theories and proves the trace–reject criterion (Theorem 2.12), and Section 3 shows that $\mathsf{PT}(\Lambda)$ is a complete lattice (Theorem 3.6). Section 4 proves Theorem A and studies the convex geometry of image closure. Section 5 describes morphisms between indecomposable Λ_n-modules and characterizes the image-closed subcategories of $\mathsf{mod}\,\Lambda_n$ by their support intervals. Section 6 classifies the pretorsion theories in $\mathsf{mod}\,\Lambda_n$. Sections 7 and 8 prove the two formulas of Theorem B.

Conventions and notation. Throughout this paper, $\Bbbk$ is an arbitrary field and Λ is a finite-dimensional $\Bbbk$-algebra. All modules are finite-dimensional right modules, and $\mathsf{mod}\,\Lambda$ denotes the category of finite-dimensional right Λ-modules. Paths in a quiver are composed from left to right, so Λ_n-modules are representations of the quiver $1 \to 2 \to \cdots \to n$.

All subcategories of $\mathsf{mod}\,\Lambda$ are assumed to be full, additive, and closed under isomorphisms and direct summands; 0 denotes the zero subcategory. For a collection of modules $\mathcal{M}$, let $\mathsf{add}\,\mathcal{M}$ denote the subcategory of direct summands of finite direct sums of modules in $\mathcal{M}$. We write $\mathsf{ind}\,\Lambda$ for a fixed set of representatives of the isomorphism classes of indecomposable Λ-modules, and $\mathsf{ind}\,\mathcal{C}$ for the subset of representatives belonging to a subcategory $\mathcal{C} \subseteq \mathsf{mod}\,\Lambda$.

For integers a and d, the interval $[a,d]$ denotes the set $\{a, a+1,\dots,d\}$, which is empty when $a > d$. Inequalities between integer vectors are understood coordinatewise.

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the text. The author directed the research and the writing and is responsible for the mathematical content and the final manuscript.

2. Pretorsion theories, trace, and reject

In this section, we recall the definition of a pretorsion theory in $\mathsf{mod}\,\Lambda$ and characterize it in terms of the trace and the reject of its two components (Theorem 2.12).

Definition 2.1. A subcategory $\mathcal{C} \subseteq \mathsf{mod}\,\Lambda$ is *quotient-closed* if every quotient of an object of $\mathcal{C}$ belongs to $\mathcal{C}$, and it is *submodule-closed* if every submodule of an object of $\mathcal{C}$ belongs to $\mathcal{C}$. It is *image-closed* if $\operatorname{Im} f \in \mathcal{C}$ for every morphism $f\colon X \to Y$ with $X, Y \in \mathcal{C}$. Write $\mathcal{L}_{\mathrm{quot}}(\Lambda)$, $\mathcal{L}_{\mathrm{sub}}(\Lambda)$, and $\mathcal{L}_{\mathrm{im}}(\Lambda)$ for the sets of quotient-closed, submodule-closed, and image-closed subcategories of $\mathsf{mod}\,\Lambda$, each ordered by inclusion.

Every quotient-closed subcategory and every submodule-closed subcategory is image-closed, because the image of a morphism $f\colon X \to Y$ is both a quotient of X and a submodule of Y. More generally, if $\mathcal{T} \in \mathcal{L}_{\mathrm{quot}}(\Lambda)$ and $\mathcal{F} \in \mathcal{L}_{\mathrm{sub}}(\Lambda)$, every morphism f from an object of $\mathcal{T}$ to an object of $\mathcal{F}$ satisfies

$$\operatorname{Im} f \in \mathcal{T} \cap \mathcal{F}. \tag{2.1}$$

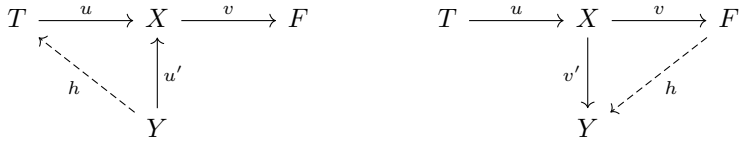
In particular, $\mathcal{T} \cap \mathcal{F} \in \mathcal{L}_{\mathrm{im}}(\Lambda)$.

We now recall the relative kernel and cokernel conditions that define a pretorsion theory.

Definition 2.2 ([11, Definition 2.6], specialized to $\mathsf{mod}\,\Lambda$)**.** Fix a subcategory $\mathcal{Z} \subseteq \mathsf{mod}\,\Lambda$. A morphism is *$\mathcal{Z}$-trivial* if it factors through an object of $\mathcal{Z}$. For composable morphisms

$$T \xrightarrow{u} X \xrightarrow{v} F,$$

the morphism u is a *$\mathcal{Z}$-prekernel* of v if vu is $\mathcal{Z}$-trivial and every morphism $u'\colon Y \to X$ for which vu' is $\mathcal{Z}$-trivial factors as $u' = uh$ for a unique morphism $h\colon Y \to T$. Dually, v is a *$\mathcal{Z}$-precokernel* of u if vu is $\mathcal{Z}$-trivial and every morphism $v'\colon X \to Y$ for which $v'u$ is $\mathcal{Z}$-trivial factors as $v' = hv$ for a unique morphism $h\colon F \to Y$. In diagrams,



where vu is $\mathcal{Z}$-trivial, as are vu' on the left and $v'u$ on the right, and h is the unique morphism making the respective triangle commute. The sequence $T \xrightarrow{u} X \xrightarrow{v} F$ is *short $\mathcal{Z}$-preexact* if u is a $\mathcal{Z}$-prekernel of v and v is a $\mathcal{Z}$-precokernel of u.

A *pretorsion theory* in $\mathsf{mod}\,\Lambda$ is a pair $(\mathcal{T}, \mathcal{F})$ of subcategories of $\mathsf{mod}\,\Lambda$ such that, for $\mathcal{Z} = \mathcal{T} \cap \mathcal{F}$, the following two conditions hold.

(1) Every morphism from an object of $\mathcal{T}$ to an object of $\mathcal{F}$ is $\mathcal{Z}$-trivial.
(2) Every module $X \in \mathsf{mod}\,\Lambda$ occurs in a short $\mathcal{Z}$-preexact sequence $T \to X \to F$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

The definition of a pretorsion theory in [11, Definition 2.6] and [4, Section 2] requires only full subcategories closed under isomorphisms, without assuming closure under finite direct sums or direct summands. The following result of Campanini, Fedele, and Yıldırım [4] shows that these closure properties hold automatically in $\mathsf{mod}\,\Lambda$.

Proposition 2.3 ([4, Lemma 2.6 and Proposition 2.8; dual statement in Section 3.2])**.** *Let $(\mathcal{T}, \mathcal{F})$ be a pretorsion theory of full subcategories closed under isomorphisms in $\mathsf{mod}\,\Lambda$. Then $\mathcal{T}$ is closed under quotients and finite direct sums, and $\mathcal{F}$ is closed under submodules and finite direct sums.*

Since any direct summand of a module is both a quotient and a submodule of it, the components $\mathcal{T}$ and $\mathcal{F}$ are also closed under direct summands, and hence satisfy our standing subcategory conventions. Thus the two conventions give the same pretorsion theories in $\mathsf{mod}\,\Lambda$.

We write $\mathsf{PT}(\Lambda)$ for the set of pretorsion theories in $\operatorname{\mathsf{mod}}\Lambda$, ordered by

$$(\mathcal{T},\mathcal{F}) \le (\mathcal{T}',\mathcal{F}') \quad \Longleftrightarrow \quad \mathcal{T} \subseteq \mathcal{T}' \text{ and } \mathcal{F} \supseteq \mathcal{F}'. \tag{2.2}$$

When $\mathcal{T} \cap \mathcal{F} = 0$, Definition 2.2 reduces to the classical notion of a torsion pair.

Definition 2.4. A *torsion pair* in $\operatorname{\mathsf{mod}}\Lambda$ is a pair $(\mathcal{T},\mathcal{F})$ of subcategories such that $\operatorname{Hom}_\Lambda(\mathcal{T},\mathcal{F}) = 0$ and every module X occurs in a short exact sequence

$$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

The following observation identifies the torsion pairs among pretorsion theories.

Proposition 2.5. *A pair $(\mathcal{T},\mathcal{F})$ of subcategories of $\operatorname{\mathsf{mod}}\Lambda$ is a torsion pair if and only if it is a pretorsion theory with $\mathcal{T} \cap \mathcal{F} = 0$.*

Proof. ($\Rightarrow$): If $(\mathcal{T},\mathcal{F})$ is a torsion pair, then $\operatorname{Hom}_\Lambda(\mathcal{T},\mathcal{F}) = 0$ implies $\mathcal{T} \cap \mathcal{F} = 0$, and the defining short exact sequences are short 0-preexact sequences.

($\Leftarrow$): When $\mathcal{Z} = 0$, a morphism is $\mathcal{Z}$-trivial if and only if it is zero, so $\mathcal{Z}$-prekernels and $\mathcal{Z}$-precokernels coincide with ordinary kernels and cokernels. Thus a pretorsion theory with $\mathcal{T} \cap \mathcal{F} = 0$ is a torsion pair. □

Over $\Lambda_1 = \Bbbk$, one can list the pretorsion theories directly.

Example 2.6. The category $\operatorname{\mathsf{mod}}\Lambda_1$ has a single indecomposable module up to isomorphism, namely the simple module S, so each of $\mathcal{T}$ and $\mathcal{F}$ is either 0 or $\operatorname{\mathsf{mod}}\Lambda_1$. The pair $(0,0)$ has intersection 0, so by Proposition 2.5 it would have to be a torsion pair; however, S does not fit into a short exact sequence $0 \to 0 \to S \to 0 \to 0$. For $(\operatorname{\mathsf{mod}}\Lambda_1, \operatorname{\mathsf{mod}}\Lambda_1)$, we have $\mathcal{Z} = \operatorname{\mathsf{mod}}\Lambda_1$, so every morphism is $\mathcal{Z}$-trivial and the identity maps give a short $\mathcal{Z}$-preexact sequence

$$X \xrightarrow{1_X} X \xrightarrow{1_X} X$$

for every $X \in \operatorname{\mathsf{mod}}\Lambda_1$. The remaining two pairs $(0, \operatorname{\mathsf{mod}}\Lambda_1)$ and $(\operatorname{\mathsf{mod}}\Lambda_1, 0)$ are the ordinary torsion pairs, which are pretorsion theories by Proposition 2.5. Thus $\mathsf{PT}(\Lambda_1)$ consists of the three pairs

$$(0, \operatorname{\mathsf{mod}}\Lambda_1), \qquad (\operatorname{\mathsf{mod}}\Lambda_1, \operatorname{\mathsf{mod}}\Lambda_1), \qquad (\operatorname{\mathsf{mod}}\Lambda_1, 0),$$

whose intersections are 0, $\operatorname{\mathsf{mod}}\Lambda_1$, and 0, respectively.

We next recall the trace and reject constructions of Auslander and Smalø [1, Proposition 4.8 and Corollary 4.9].

Definition 2.7. Let $\mathcal{C} \subseteq \operatorname{\mathsf{mod}}\Lambda$ be a subcategory and let $X \in \operatorname{\mathsf{mod}}\Lambda$. The *trace of $\mathcal{C}$ in X* and the *reject of $\mathcal{C}$ in X* are the submodules

$$t_{\mathcal{C}}(X) = \sum_{Y \in \mathcal{C},\ f\colon Y \to X} \operatorname{Im} f, \qquad r_{\mathcal{C}}(X) = \bigcap_{Y \in \mathcal{C},\ g\colon X \to Y} \operatorname{Ker} g$$

of X, where the sum and the intersection range over all objects $Y \in \mathcal{C}$ and all morphisms $f\colon Y \to X$ and $g\colon X \to Y$.

Remark 2.8. The standard duality $D = \operatorname{Hom}_\Bbbk(-,\Bbbk)$ sends $\mathcal{Z}$-prekernels to $D\mathcal{Z}$-precokernels and conversely, and therefore induces an order-reversing bijection

$$\mathsf{PT}(\Lambda) \longrightarrow \mathsf{PT}(\Lambda^{\mathrm{op}}), \qquad (\mathcal{T},\mathcal{F}) \longmapsto (D\mathcal{F}, D\mathcal{T}),$$

with respect to (2.2). We will use this bijection to deduce dual assertions in the proofs below. The functor D also exchanges trace submodules with reject quotients:

$$D(t_{\mathcal{C}}(X)) \cong DX/r_{D\mathcal{C}}(DX), \qquad D(X/r_{\mathcal{C}}(X)) \cong t_{D\mathcal{C}}(DX).$$

These identifications follow by dualizing the sums of images and intersections of kernels in Definition 2.7.

Although Definition 2.7 takes the sum and the intersection over all morphisms, the following observation for modules of finite length shows that finitely many images or kernels suffice.

Lemma 2.9. *Every sum or intersection of submodules of a module of finite length is the sum or intersection of finitely many of them.*

Proof. This follows from the ascending and descending chain conditions on submodules. □

We collect the basic properties of the trace and the reject when $\mathcal{T}$ is quotient-closed and $\mathcal{F}$ is submodule-closed.

Proposition 2.10 ([1, Proposition 4.8 and Corollary 4.9])**.** *Let $\mathcal{T} \in \mathcal{L}_{\mathrm{quot}}(\Lambda)$, let $\mathcal{F} \in \mathcal{L}_{\mathrm{sub}}(\Lambda)$, and let $X \in \mathsf{mod}\,\Lambda$.*

(1) *$t_{\mathcal{T}}(X)$ is the largest submodule of X belonging to $\mathcal{T}$.*
(2) *$X/r_{\mathcal{F}}(X)$ is the largest quotient of X belonging to $\mathcal{F}$.*
(3) *$X \in \mathcal{T}$ if and only if $t_{\mathcal{T}}(X) = X$, and $X \in \mathcal{F}$ if and only if $r_{\mathcal{F}}(X) = 0$.*
(4) *Both operations commute with finite direct sums:*

$$t_{\mathcal{T}}(X_1 \oplus X_2) = t_{\mathcal{T}}(X_1) \oplus t_{\mathcal{T}}(X_2), \qquad r_{\mathcal{F}}(X_1 \oplus X_2) = r_{\mathcal{F}}(X_1) \oplus r_{\mathcal{F}}(X_2). \tag{2.3}$$

(5) *$t_{\mathcal{T}}(W) \subseteq t_{\mathcal{T}}(X)$ and $r_{\mathcal{F}}(W) \subseteq r_{\mathcal{F}}(X)$ whenever $W \subseteq X$.*

Proof. (1) This is [1, Proposition 4.8(a)]: by Lemma 2.9, the trace is a sum of finitely many images of morphisms from objects of $\mathcal{T}$, hence a quotient of a finite direct sum of objects of $\mathcal{T}$, so it belongs to $\mathcal{T}$ and contains every $\mathcal{T}$-submodule of X.
(2) This is dual to (1), or [1, Proposition 4.8(b)].
(3) This follows immediately from (1) and (2).
(4) By [1, Corollary 4.9], $t_{\mathcal{T}}$ is right adjoint to the inclusion $\mathcal{T} \hookrightarrow \mathsf{mod}\,\Lambda$, and $X \mapsto X/r_{\mathcal{F}}(X)$ is left adjoint to $\mathcal{F} \hookrightarrow \mathsf{mod}\,\Lambda$. These additive functors preserve finite biproducts; taking kernels of the quotient maps gives the assertion for $r_{\mathcal{F}}$.
(5) Composing any morphism $Y \to W$ ($Y \in \mathcal{T}$) with the inclusion $W \hookrightarrow X$ shows that $t_{\mathcal{T}}(W) \subseteq t_{\mathcal{T}}(X)$. For the reject, restricting any morphism $g\colon X \to Y$ ($Y \in \mathcal{F}$) to W gives a morphism $g|_W\colon W \to Y$, so $r_{\mathcal{F}}(W) \subseteq \operatorname{Ker}(g|_W) \subseteq \operatorname{Ker} g$; intersecting over all such g yields $r_{\mathcal{F}}(W) \subseteq r_{\mathcal{F}}(X)$. □

To describe short preexact sequences in terms of submodules and quotients, we use the following property of their maps.

Lemma 2.11 ([11, Propositions 2.1–2.2])**.** *Let $\mathcal{Z} \subseteq \mathsf{mod}\,\Lambda$ be a subcategory. Every $\mathcal{Z}$-prekernel in $\mathsf{mod}\,\Lambda$ is a monomorphism, and every $\mathcal{Z}$-precokernel is an epimorphism.*

Not every pair $(\mathcal{T}, \mathcal{F}) \in \mathcal{L}_{\mathrm{quot}}(\Lambda) \times \mathcal{L}_{\mathrm{sub}}(\Lambda)$ is a pretorsion theory. For any such pair, condition (1) of Definition 2.2 holds automatically: by (2.1), each map factors through its image in $\mathcal{T} \cap \mathcal{F}$. Thus only the existence of short $(\mathcal{T} \cap \mathcal{F})$-preexact sequences in condition (2) is at issue. The following theorem characterizes the pairs that form pretorsion theories by two equalities relating the trace and the reject.

Theorem 2.12. *Let $\mathcal{T} \in \mathcal{L}_{\mathrm{quot}}(\Lambda)$ and $\mathcal{F} \in \mathcal{L}_{\mathrm{sub}}(\Lambda)$, and write $t = t_{\mathcal{T}}$ and $r = r_{\mathcal{F}}$. Then $(\mathcal{T}, \mathcal{F})$ is a pretorsion theory if and only if*

$$r(X) = r(t(X)), \qquad t(X/r(X)) = t(X)/r(X) \tag{2.4}$$

for every $X \in \mathsf{mod}\,\Lambda$. Here the first equality implies $r(X) \subseteq t(X)$, so that $t(X)/r(X)$ is defined and the second equality is an equality of submodules of $X/r(X)$. When (2.4) holds, the subquotient $t(X)/r(X)$ belongs to $\mathcal{T} \cap \mathcal{F}$, and the canonical sequence

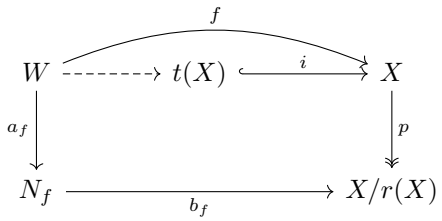
$$t(X) \longrightarrow X \longrightarrow X/r(X)$$

is short $(\mathcal{T} \cap \mathcal{F})$-preexact.

Proof. Put $\mathcal{Z} = \mathcal{T} \cap \mathcal{F}$. Under the identifications in Remark 2.8, dualizing either equality in (2.4) gives the other equality for $(D\mathcal{F}, D\mathcal{T})$. We use this to pass between the two universal properties.

($\Leftarrow$): Suppose that (2.4) holds for $X \in \operatorname{mod}\Lambda$. The first equality gives $r(X) = r(t(X)) \subseteq t(X)$. Moreover, $t(X)/r(X)$ belongs to $\mathcal{T}$ as a quotient of $t(X)$, and to $\mathcal{F}$ as a submodule of $X/r(X)$. Hence $t(X)/r(X) \in \mathcal{Z}$.

Let $i\colon t(X) \hookrightarrow X$ be the inclusion and $p\colon X \twoheadrightarrow X/r(X)$ the quotient map. The composite pi has image $t(X)/r(X) \in \mathcal{Z}$ and is therefore $\mathcal{Z}$-trivial. If $f\colon W \to X$ is a morphism such that pf is $\mathcal{Z}$-trivial, choose a factorization $pf = b_f a_f$ through an object $N_f \in \mathcal{Z}$:



Since $N_f \in \mathcal{Z} \subseteq \mathcal{T}$ and $\mathcal{T}$ is quotient-closed, we have $\operatorname{Im} b_f \in \mathcal{T}$. By Proposition 2.10(1) and the second equality in (2.4), it follows that

$$\operatorname{Im}(pf) \subseteq \operatorname{Im} b_f \subseteq t(X/r(X)) = t(X)/r(X).$$

Consequently, $\operatorname{Im} f \subseteq p^{-1}(t(X)/r(X)) = t(X)$, so f factors through i, uniquely because i is a monomorphism. Thus i is a $\mathcal{Z}$-prekernel of p.

Dually, p is a $\mathcal{Z}$-precokernel of i, by Remark 2.8 and the first equality in (2.4). Thus the canonical sequence is short $\mathcal{Z}$-preexact.

($\Rightarrow$): Conversely, assume that $(\mathcal{T}, \mathcal{F})$ is a pretorsion theory, and choose a short $\mathcal{Z}$-preexact sequence

$$T_X \xrightarrow{u} X \xrightarrow{v} F_X$$

with $T_X \in \mathcal{T}$ and $F_X \in \mathcal{F}$. By Lemma 2.11, u is a monomorphism and v is an epimorphism, so we may identify T_X with $\operatorname{Im} u \subseteq X$ and F_X with $X/\operatorname{Ker} v$.

Since $\operatorname{Im} u$ is a quotient of $T_X \in \mathcal{T}$, we have $\operatorname{Im} u \in \mathcal{T}$, and hence $\operatorname{Im} u \subseteq t(X)$. On the other hand, the restriction of v to $t(X)$ is a morphism from $t(X) \in \mathcal{T}$ to $F_X \in \mathcal{F}$, so it is $\mathcal{Z}$-trivial by condition (1) of Definition 2.2. By the $\mathcal{Z}$-prekernel property of u, the inclusion $t(X) \hookrightarrow X$ factors through u, giving $t(X) \subseteq \operatorname{Im} u$. Thus $\operatorname{Im} u = t(X)$, and u is isomorphic over X to the inclusion $i\colon t(X) \hookrightarrow X$.

Dually, $\operatorname{Ker} v = r(X)$, so we may take $u = i\colon t(X) \hookrightarrow X$ and $v = p\colon X \twoheadrightarrow X/r(X)$.

Put $U = t(X/r(X))$ and $P = p^{-1}(U)$. Since $U \in \mathcal{T}$ and $U \subseteq X/r(X) \in \mathcal{F}$, we have $U \in \mathcal{Z}$. The composite $P \hookrightarrow X \xrightarrow{p} X/r(X)$ has image $U \in \mathcal{Z}$ and is therefore $\mathcal{Z}$-trivial; since $i\colon t(X) \hookrightarrow X$ is a $\mathcal{Z}$-prekernel of p, the inclusion $P \hookrightarrow X$ factors through i, giving $P \subseteq t(X)$. Conversely, $p(t(X))$ is a quotient of $t(X) \in \mathcal{T}$, so it belongs to $\mathcal{T}$ and is therefore contained in U; hence $t(X) \subseteq p^{-1}(U) = P$. Thus $P = t(X)$, so $r(X) \subseteq t(X)$ and $t(X/r(X)) = U = t(X)/r(X)$. Dually, because p is a $\mathcal{Z}$-precokernel of i, we obtain $r(X) = r(t(X))$, completing (2.4). □

For pretorsion theories in a general category, Facchini, Finocchiaro, and Gran [11, Proposition 2.7] showed that any two of the three classes $\mathcal{T}, \mathcal{F}, \mathcal{Z}$ determine the third. Specializing [11, Proposition 3.9] and its dual to $\operatorname{mod}\Lambda$ yields the following reconstruction formulas in terms of the trace and the reject. We will use them to construct a pair with a prescribed intersection in Section 4.

Corollary 2.13 ([11, Proposition 3.9 and its dual])**.** *If $(\mathcal{T}, \mathcal{F})$ is a pretorsion theory and $\mathcal{Z} = \mathcal{T} \cap \mathcal{F}$, then*

$$\mathcal{F} = \{Y \in \operatorname{mod}\Lambda : t_{\mathcal{T}}(Y) \in \mathcal{Z}\}, \qquad \mathcal{T} = \{X \in \operatorname{mod}\Lambda : X/r_{\mathcal{F}}(X) \in \mathcal{Z}\}. \tag{2.5}$$

Proof. By Proposition 2.3, $\mathcal{T} \in \mathcal{L}_{\mathrm{quot}}(\Lambda)$ and $\mathcal{F} \in \mathcal{L}_{\mathrm{sub}}(\Lambda)$, so Proposition 2.10 and Theorem 2.12 apply to $(\mathcal{T}, \mathcal{F})$. For $Y \in \mathcal{F}$, we have $r_{\mathcal{F}}(Y) = 0$, so $r_{\mathcal{F}}(t_{\mathcal{T}}(Y)) = r_{\mathcal{F}}(Y) = 0$ by Theorem 2.12,

and hence $t_{\mathcal{T}}(Y) \in \mathcal{T} \cap \mathcal{F} = \mathcal{Z}$. Conversely, if $t_{\mathcal{T}}(Y) \in \mathcal{Z} \subseteq \mathcal{F}$, then $r_{\mathcal{F}}(Y) = r_{\mathcal{F}}(t_{\mathcal{T}}(Y)) = 0$, so $Y \in \mathcal{F}$. This proves the first equality in (2.5).

The second equality follows dually. □

3. The lattice of pretorsion theories

In this section, we use Theorem 2.12 to prove that the poset $\mathsf{PT}(\Lambda)$, ordered by (2.2), is a complete lattice whose nonempty meets and joins preserve a fixed intersection (Theorem 3.6 and Proposition 3.8).

Campanini, Fedele, and Yıldırım [4, Proposition 3.4 and Section 3.2] showed that the individual first and second components of pretorsion theories, namely $\mathcal{L}_{\mathrm{quot}}(\Lambda)$ and $\mathcal{L}_{\mathrm{sub}}(\Lambda)$, form complete lattices under inclusion. Here we study the pairs $(\mathcal{T}, \mathcal{F})$ forming pretorsion theories.

By Theorem 2.12, $(\mathcal{T}, \mathsf{mod}\,\Lambda)$ is a pretorsion theory for every $\mathcal{T} \in \mathcal{L}_{\mathrm{quot}}(\Lambda)$: the reject $r_{\mathsf{mod}\,\Lambda}(X)$ vanishes for every module X, so both equalities in (2.4) hold trivially. Dually, $(\mathsf{mod}\,\Lambda, \mathcal{F})$ is a pretorsion theory for every $\mathcal{F} \in \mathcal{L}_{\mathrm{sub}}(\Lambda)$. Together with Proposition 2.3, these examples show that the possible first and second components of pretorsion theories are precisely the members of $\mathcal{L}_{\mathrm{quot}}(\Lambda)$ and $\mathcal{L}_{\mathrm{sub}}(\Lambda)$, respectively. In particular, $(0, \mathsf{mod}\,\Lambda)$ and $(\mathsf{mod}\,\Lambda, 0)$ are the least and greatest elements of $\mathsf{PT}(\Lambda)$, respectively.

To construct the meet of a nonempty family $(\mathcal{T}_i, \mathcal{F}_i)_{i \in I}$ in $\mathsf{PT}(\Lambda)$, we pair the intersection $\bigcap_{i \in I} \mathcal{T}_i$ of its first components with the smallest subcategory $\mathcal{F}$ containing every $\mathcal{F}_i$ such that $(\bigcap_i \mathcal{T}_i, \mathcal{F})$ is a pretorsion theory. The existence of such a smallest $\mathcal{F}$ follows once we show that, for a fixed $\mathcal{T} \in \mathcal{L}_{\mathrm{quot}}(\Lambda)$, the subcategories $\mathcal{F}$ with $(\mathcal{T}, \mathcal{F}) \in \mathsf{PT}(\Lambda)$ are closed under arbitrary intersections (Proposition 3.2), since $\mathsf{mod}\,\Lambda$ always belongs to this collection. To prove that the resulting pair is the greatest lower bound, we then show in Proposition 3.5 that if $(\mathcal{T}'', \mathcal{F}'')$ is any lower bound of the family, then $(\bigcap_{i \in I} \mathcal{T}_i, \mathcal{F}'')$ is also a pretorsion theory.

For two second components $\mathcal{F}_1, \mathcal{F}_2$ with the same first component, the sum $r_{\mathcal{F}_1}(X) + r_{\mathcal{F}_2}(X)$ is contained in $r_{\mathcal{F}_1 \cap \mathcal{F}_2}(X)$. We will repeatedly divide by these sums until the quotient belongs to both second components. The next lemma shows that, for $R(X) = r_{\mathcal{F}_1}(X) + r_{\mathcal{F}_2}(X)$, the trace on $X/R(X)$ is $t_{\mathcal{T}}(X)/R(X)$. This allows us to recover the second equality in (2.4) at X from the same equality at $X/R(X)$.

Lemma 3.1. *Let $(\mathcal{T}, \mathcal{F}_1)$ and $(\mathcal{T}, \mathcal{F}_2)$ be pretorsion theories, write $t = t_{\mathcal{T}}$ and $r_i = r_{\mathcal{F}_i}$, and put $R(X) = r_1(X) + r_2(X)$ for a module X. Then $R(X) \subseteq t(X)$ and*

$$t\big(X/R(X)\big) = t(X)/R(X).$$

Proof. By Theorem 2.12, we have $r_i(X) \subseteq t(X)$ for $i = 1, 2$, so $R(X) \subseteq t(X)$. Since $t(X)/R(X)$ is a quotient of $t(X) \in \mathcal{T}$, it belongs to $\mathcal{T}$ and is therefore contained in $t(X/R(X))$ by Proposition 2.10(1).

For the reverse inclusion, we first divide by $r_1(X)$ and compare $X/R(X)$ with a further quotient whose trace is known from Theorem 2.12. Put $Y = X/r_1(X)$, let $p\colon X \twoheadrightarrow Y$ be the quotient map, and put $K = p(R(X)) = p(r_2(X))$, so that $X/R(X) \cong Y/K$. Every morphism $g\colon Y \to G$ with $G \in \mathcal{F}_2$ satisfies $gp(r_2(X)) = 0$, so $K \subseteq r_2(Y)$. Applying Theorem 2.12 to $(\mathcal{T}, \mathcal{F}_1)$ at X and to $(\mathcal{T}, \mathcal{F}_2)$ at Y gives

$$t(Y) = t(X)/r_1(X), \qquad r_2(Y) \subseteq t(Y), \qquad t\big(Y/r_2(Y)\big) = t(Y)/r_2(Y).$$

Let $q\colon Y/K \twoheadrightarrow Y/r_2(Y)$ be the canonical quotient map. The image $q(t(Y/K))$ belongs to $\mathcal{T}$ and hence is contained in $t(Y/r_2(Y)) = t(Y)/r_2(Y)$. Thus the restriction of q fits into the commutative diagram

$$\begin{array}{ccc} t(Y/K) & \longrightarrow & t(Y)/r_2(Y) \\ \downarrow & & \downarrow \\ Y/K & \xrightarrow{\;q\;} & Y/r_2(Y). \end{array}$$

Taking the inverse image under q and using $r_2(Y) \subseteq t(Y)$ yields

$$t(Y/K) \subseteq q^{-1}\big(t(Y)/r_2(Y)\big) = \big(t(Y) + r_2(Y)\big)/K = t(Y)/K.$$

Under the canonical isomorphism $Y/K \cong X/R(X)$, the submodule

$$t(Y)/K = \big(t(X)/r_1(X)\big)/\big(R(X)/r_1(X)\big)$$

corresponds to $t(X)/R(X)$. Consequently, $t(X/R(X)) \subseteq t(X)/R(X)$, which gives the reverse inclusion. □

We now show that the second components forming a pretorsion theory with a fixed $\mathcal{T}$ are closed under arbitrary intersections.

Proposition 3.2. *Let $\mathcal{T}$ be quotient-closed and let $(\mathcal{T}, \mathcal{F}_i)$, $i \in I$, be a family of pretorsion theories. Then $(\mathcal{T}, \bigcap_{i\in I} \mathcal{F}_i)$ is a pretorsion theory.*

Proof. For the empty family, $(\mathcal{T}, \operatorname{mod}\Lambda)$ is a pretorsion theory by Theorem 2.12, since $r_{\operatorname{mod}\Lambda} = 0$. Assume $I \neq \varnothing$. We first prove closure under finite intersections by repeatedly dividing out two rejects, and then pass to arbitrary intersections using finite length.

Step 1: Finite intersections. Let $I = \{1, 2\}$, and put $\mathcal{F}' = \mathcal{F}_1 \cap \mathcal{F}_2$, $r' = r_{\mathcal{F}'}$, $t = t_{\mathcal{T}}$, $r_i = r_{\mathcal{F}_i}$, and $R(M) = r_1(M) + r_2(M)$ for $M \in \operatorname{mod}\Lambda$. Every morphism from M to an object of $\mathcal{F}'$ vanishes on both $r_1(M)$ and $r_2(M)$, and hence on $R(M)$. Composing with the quotient map $\pi_M \colon M \twoheadrightarrow M/R(M)$ therefore identifies the morphisms from $M/R(M)$ to objects of $\mathcal{F}'$ with those from M, so that

$$r'(M) = \pi_M^{-1}\big(r'(M/R(M))\big), \qquad M/r'(M) \cong \big(M/R(M)\big)/r'\big(M/R(M)\big). \tag{3.1}$$

We verify (2.4) for $(\mathcal{T}, \mathcal{F}')$ at every module X by induction on $\dim_{\Bbbk} X$, using (3.1) to reduce to $X/R(X)$ whenever $R(X) \neq 0$. If $R(X) = 0$, then $r_1(X) = r_2(X) = 0$, so $X \in \mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}'$ by Proposition 2.10(3); since $\mathcal{F}'$ is submodule-closed, we also have $t(X) \in \mathcal{F}'$, so $r'(X) = r'(t(X)) = 0$ and both equalities in (2.4) hold.

Suppose now that $R(X) \neq 0$, put $V = X/R(X)$, and let $\pi \colon X \twoheadrightarrow V$ be the quotient map. Since $\dim_{\Bbbk} V < \dim_{\Bbbk} X$, the equalities (2.4) hold for $(\mathcal{T}, \mathcal{F}')$ at V by the induction hypothesis, and $t(V) = t(X)/R(X)$ by Lemma 3.1. Moreover, $r_i(t(X)) = r_i(X)$ for $i = 1, 2$ by Theorem 2.12, so $R(t(X)) = R(X)$. The quotient map $t(X) \twoheadrightarrow t(X)/R(t(X))$ therefore coincides with the restriction $\pi_t \colon t(X) \twoheadrightarrow t(V)$ of π, fitting into the commutative square

$$\begin{array}{ccc} t(X) & \xrightarrow{\pi_t} & t(V) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\pi} & V \end{array}$$

with $\pi^{-1}(t(V)) = t(X)$. Consequently, any submodule of $t(V)$ has the same inverse image in X under π_t and under π. Applying (3.1) to $M = X$ and to $M = t(X)$ gives

$$r'(X) = \pi^{-1}(r'(V)), \qquad r'(t(X)) = \pi_t^{-1}(r'(t(V))).$$

Since $r'(V) = r'(t(V))$ by the induction hypothesis at V, these two inverse images coincide, so $r'(X) = r'(t(X))$.

For the second equality, the isomorphism $X/r'(X) \cong V/r'(V)$ from (3.1) carries $t(X)/r'(X)$ onto $t(V)/r'(V)$, and $t(V/r'(V)) = t(V)/r'(V)$ by the induction hypothesis at V. Hence $t(X/r'(X)) = t(X)/r'(X)$, completing the induction on $\dim_{\Bbbk} X$. By Theorem 2.12, $(\mathcal{T}, \mathcal{F}_1 \cap \mathcal{F}_2)$ is a pretorsion theory, and induction on $|I|$ extends this to any finite index set I.

Step 2: Arbitrary intersections. Put $\mathcal{F} = \bigcap_{i\in I} \mathcal{F}_i$, and for each finite subset $J \subseteq I$ put $\mathcal{F}_J = \bigcap_{j\in J} \mathcal{F}_j$ and $r_J = r_{\mathcal{F}_J}$. By the finite case, $(\mathcal{T}, \mathcal{F}_J)$ is a pretorsion theory for every finite $J \subseteq I$. Fix a module X. Since $r_J(X) \subseteq r_{J'}(X)$ whenever $J \subseteq J'$ and X is finite-dimensional, we may choose a finite subset $J_0 \subseteq I$ for which $\dim_{\Bbbk} r_{J_0}(X)$ is maximal, and put $N = r_{J_0}(X)$. For every $i \in I$, maximality gives $r_{J_0\cup\{i\}}(X) = N$, so $X/N \in \mathcal{F}_{J_0\cup\{i\}} \subseteq \mathcal{F}_i$ by Proposition 2.10(2). Thus $X/N \in \mathcal{F}$, which gives $r_{\mathcal{F}}(X) \subseteq N$; since $\mathcal{F} \subseteq \mathcal{F}_{J_0}$, we also have $N = r_{J_0}(X) \subseteq r_{\mathcal{F}}(X)$, and hence $r_{\mathcal{F}}(X) = r_J(X)$ for every finite $J \supseteq J_0$. Choosing such a finite subset for X and another for $t(X)$, and taking their union J, we obtain $r_{\mathcal{F}}(X) = r_J(X)$ and $r_{\mathcal{F}}(t(X)) = r_J(t(X))$. The equalities (2.4) for $(\mathcal{T}, \mathcal{F}_J)$ at X therefore imply those for $(\mathcal{T}, \mathcal{F})$ at X, and Theorem 2.12 shows that $(\mathcal{T}, \mathcal{F})$ is a pretorsion theory. □

To construct meets in $\mathsf{PT}(\Lambda)$, we must also compare pretorsion theories with different first components. The next lemma shows that the first equality in (2.4), combined with the second reconstruction formula in (2.5), suffices for a pair $(\mathcal{T},\mathcal{F})$ to be a pretorsion theory.

Lemma 3.3. *Let $\mathcal{T}\in\mathcal{L}_{\mathrm{quot}}(\Lambda)$ and $\mathcal{F}\in\mathcal{L}_{\mathrm{sub}}(\Lambda)$, put $\mathcal{Z}=\mathcal{T}\cap\mathcal{F}$, and write $t=t_{\mathcal{T}}$ and $r=r_{\mathcal{F}}$. Suppose that $r(X)=r(t(X))$ for every module X and that $\mathcal{T}=\{X\in\operatorname{mod}\Lambda : X/r(X)\in\mathcal{Z}\}$. Then $(\mathcal{T},\mathcal{F})$ is a pretorsion theory.*

Proof. Fix $X\in\operatorname{mod}\Lambda$, put $U=t(X/r(X))$, and let $P\subseteq X$ be the inverse image of U under the quotient map $X\twoheadrightarrow X/r(X)$. Since $r(X)=r(t(X))\subseteq t(X)$, the submodule $t(X)/r(X)$ of $X/r(X)$ belongs to $\mathcal{T}$ and is therefore contained in U, so $t(X)\subseteq P$. Moreover, U belongs to $\mathcal{T}$ and is a submodule of $X/r(X)\in\mathcal{F}$, so $U\in\mathcal{Z}$. By definition of P, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & r(X) & \longrightarrow & P & \longrightarrow & U & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & r(X) & \longrightarrow & X & \longrightarrow & X/r(X) & \longrightarrow & 0. \end{array}$$

Since $t(X)\subseteq P\subseteq X$, Proposition 2.10(5) gives $r(t(X))\subseteq r(P)\subseteq r(X)$, and hence $r(P)=r(X)$ because $r(t(X))=r(X)$. Consequently, $P/r(P)=P/r(X)=U\in\mathcal{Z}$, so the hypothesis on $\mathcal{T}$ gives $P\in\mathcal{T}$ and therefore $P\subseteq t(X)$. Thus $P=t(X)$ and $t(X/r(X))=U=t(X)/r(X)$, so $(\mathcal{T},\mathcal{F})$ is a pretorsion theory by Theorem 2.12. □

To apply Lemma 3.3 to an intersection of first components, we must recover membership in each component from a quotient by the reject of a larger second component. The following lemma gives this implication.

Lemma 3.4. *Let $(\mathcal{T},\mathcal{F})$ be a pretorsion theory, let $\mathcal{G}\in\mathcal{L}_{\mathrm{sub}}(\Lambda)$ contain $\mathcal{F}$, and let X be a module with $X/r_{\mathcal{G}}(X)\in\mathcal{T}$. Then $X\in\mathcal{T}$.*

Proof. Since $\mathcal{F}\subseteq\mathcal{G}$, we have $r_{\mathcal{G}}(X)\subseteq r_{\mathcal{F}}(X)$, so $X/r_{\mathcal{F}}(X)$ is a quotient of $X/r_{\mathcal{G}}(X)\in\mathcal{T}$ and therefore belongs to $\mathcal{T}$. By Theorem 2.12, $X/r_{\mathcal{F}}(X)=t_{\mathcal{T}}(X/r_{\mathcal{F}}(X))=t_{\mathcal{T}}(X)/r_{\mathcal{F}}(X)$, which gives $t_{\mathcal{T}}(X)=X$ and hence $X\in\mathcal{T}$ by Proposition 2.10(3). □

Combining Lemmas 3.3 and 3.4, we show that a lower bound $(\mathcal{T}'',\mathcal{F}'')$ of a nonempty family $(\mathcal{T}_i,\mathcal{F}_i)_{i\in I}$ yields a pretorsion theory $(\bigcap_{i\in I}\mathcal{T}_i,\mathcal{F}'')$.

Proposition 3.5. *Let $(\mathcal{T}_i,\mathcal{F}_i)$, $i\in I$, be a nonempty family in $\mathsf{PT}(\Lambda)$, and let $(\mathcal{T}'',\mathcal{F}'')\in\mathsf{PT}(\Lambda)$ be a lower bound of the family. Then $(\bigcap_{i\in I}\mathcal{T}_i,\mathcal{F}'')$ is a pretorsion theory.*

Proof. Put $\widehat{\mathcal{T}}=\bigcap_{i\in I}\mathcal{T}_i\in\mathcal{L}_{\mathrm{quot}}(\Lambda)$, $\widehat{\mathcal{Z}}=\widehat{\mathcal{T}}\cap\mathcal{F}''$, and $r=r_{\mathcal{F}''}$. We verify the two hypotheses of Lemma 3.3 for $(\widehat{\mathcal{T}},\mathcal{F}'')$. First, since $\mathcal{T}''\subseteq\widehat{\mathcal{T}}$, every morphism from an object of $\mathcal{T}''$ to a module X has image contained in $t_{\widehat{\mathcal{T}}}(X)$, so $t_{\mathcal{T}''}(t_{\widehat{\mathcal{T}}}(X))=t_{\mathcal{T}''}(X)$. Applying Theorem 2.12 for $(\mathcal{T}'',\mathcal{F}'')$ to X and to $t_{\widehat{\mathcal{T}}}(X)$ yields

$$r(X)=r\big(t_{\mathcal{T}''}(X)\big)=r\big(t_{\mathcal{T}''}(t_{\widehat{\mathcal{T}}}(X))\big)=r\big(t_{\widehat{\mathcal{T}}}(X)\big).$$

Second, if $X\in\widehat{\mathcal{T}}$, then $X/r(X)$ belongs to $\widehat{\mathcal{T}}$ by quotient closure and to $\mathcal{F}''$ by Proposition 2.10(2), so $X/r(X)\in\widehat{\mathcal{Z}}$. Conversely, if $X/r(X)\in\widehat{\mathcal{Z}}$, then $X/r(X)\in\mathcal{T}_i$ and $\mathcal{F}_i\subseteq\mathcal{F}''$ for every $i\in I$, so Lemma 3.4 gives $X\in\mathcal{T}_i$ for every $i\in I$, that is, $X\in\widehat{\mathcal{T}}$. Thus $\widehat{\mathcal{T}}=\{X\in\operatorname{mod}\Lambda : X/r(X)\in\widehat{\mathcal{Z}}\}$, and Lemma 3.3 shows that $(\widehat{\mathcal{T}},\mathcal{F}'')$ is a pretorsion theory. □

Propositions 3.2 and 3.5 together yield the complete lattice structure on $\mathsf{PT}(\Lambda)$.

Theorem 3.6. *For every finite-dimensional algebra Λ, the poset $\mathsf{PT}(\Lambda)$ is a complete lattice. For a nonempty family $(\mathcal{T}_i,\mathcal{F}_i)$, $i\in I$, its meet is*

$$\Big(\bigcap_{i\in I}\mathcal{T}_i,\ \bigvee\{\mathcal{F} : (\textstyle\bigcap_i\mathcal{T}_i,\mathcal{F})\in\mathsf{PT}(\Lambda),\ \mathcal{F}\supseteq\mathcal{F}_i \text{ for every } i\}\Big). \tag{3.2}$$

Its join is

$$\Big(\bigcap\{\mathcal{T} : (\mathcal{T}, \textstyle\bigcap_i \mathcal{F}_i) \in \mathsf{PT}(\Lambda),\ \mathcal{T} \supseteq \mathcal{T}_i \text{ for every } i\},\ \textstyle\bigcap_{i\in I} \mathcal{F}_i\Big). \tag{3.3}$$

The empty meet is $(\mathsf{mod}\,\Lambda, 0)$*, and the empty join is* $(0, \mathsf{mod}\,\Lambda)$.

Proof. Put $\widehat{\mathcal{T}} = \bigcap_{i\in I} \mathcal{T}_i$. The set of second components in (3.2) is nonempty because it contains $\mathsf{mod}\,\Lambda$, so Proposition 3.2 shows that $(\widehat{\mathcal{T}}, \widehat{\mathcal{F}}) \in \mathsf{PT}(\Lambda)$ for their intersection $\widehat{\mathcal{F}}$. By construction, $(\widehat{\mathcal{T}}, \widehat{\mathcal{F}})$ is a lower bound of the family $(\mathcal{T}_i, \mathcal{F}_i)_{i\in I}$. If $(\mathcal{T}'', \mathcal{F}'') \in \mathsf{PT}(\Lambda)$ is any lower bound of the family, then $\mathcal{T}'' \subseteq \widehat{\mathcal{T}}$, and Proposition 3.5 shows that $(\widehat{\mathcal{T}}, \mathcal{F}'') \in \mathsf{PT}(\Lambda)$. Since $\mathcal{F}'' \supseteq \mathcal{F}_i$ for every $i \in I$, the subcategory $\mathcal{F}''$ belongs to the family intersected in (3.2), so $\mathcal{F}'' \supseteq \widehat{\mathcal{F}}$ and therefore $(\mathcal{T}'', \mathcal{F}'') \le (\widehat{\mathcal{T}}, \widehat{\mathcal{F}})$. Thus every nonempty family in $\mathsf{PT}(\Lambda)$ has a meet given by (3.2). Applying formula (3.2) to Λ^{op} and using Remark 2.8 gives (3.3). The least and greatest elements give the empty join and meet, respectively. □

We will use the following criterion both to construct meets with a fixed intersection and to realize image-closed subcategories in Theorem 4.4.

Lemma 3.7. *Let* $\mathcal{T} \in \mathcal{L}_{\mathrm{quot}}(\Lambda)$*,* $\mathcal{F} \in \mathcal{L}_{\mathrm{sub}}(\Lambda)$*,* $\mathcal{Z} = \mathcal{T} \cap \mathcal{F}$*, and* $t = t_{\mathcal{T}}$*. Suppose that*

$$\mathcal{F} = \{Y \in \mathsf{mod}\,\Lambda : t(Y) \in \mathcal{Z}\}, \tag{3.4}$$

$$\mathcal{T} = \{E \in \mathsf{mod}\,\Lambda : \text{every morphism } E \to F,\ F \in \mathcal{F},\ \text{is } \mathcal{Z}\text{-trivial}\}. \tag{3.5}$$

Then $(\mathcal{T}, \mathcal{F})$ *is a pretorsion theory.*

Proof. For $X \in \mathsf{mod}\,\Lambda$, put $K = r_{\mathcal{Z}}(t(X))$. By Lemma 2.9, the quotient $t(X)/K$ embeds in a finite direct sum of objects of $\mathcal{Z}$, so $t(X)/K \in \mathcal{F}$ since $\mathcal{Z} \subseteq \mathcal{F}$ and $\mathcal{F}$ is submodule-closed; since $t(X)/K$ is also a quotient of $t(X) \in \mathcal{T}$, we have $t(X)/K \in \mathcal{T} \cap \mathcal{F} = \mathcal{Z}$. Put $U = t(X/K)$ and let E be the inverse image of U in X. Since $t(X)/K \in \mathcal{T}$, we have $t(X)/K \subseteq U$ and hence $t(X) \subseteq E$. We obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & U & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & X/K & \longrightarrow & 0. \end{array}$$

By (3.4), every morphism from $t(X)$ to an object $F \in \mathcal{F}$ factors through $t(F) \in \mathcal{Z}$ and therefore vanishes on K. In particular, every morphism $E \to F$ with $F \in \mathcal{F}$ vanishes on $K \subseteq t(X)$ and thus descends to a morphism $U \to F$, which is $\mathcal{Z}$-trivial by (3.5) because $U \in \mathcal{T}$. Hence every morphism $E \to F$ is $\mathcal{Z}$-trivial, so (3.5) gives $E \in \mathcal{T}$. Consequently, $E = t(X)$ and

$$t(X/K) = t(X)/K \in \mathcal{Z}, \tag{3.6}$$

which implies $X/K \in \mathcal{F}$ by (3.4).

Since every morphism from $t(X)$ or X to an object $F \in \mathcal{F}$ vanishes on K, we have $K \subseteq r_{\mathcal{F}}(t(X))$ and $K \subseteq r_{\mathcal{F}}(X)$. Conversely, since $t(X)/K \in \mathcal{Z} \subseteq \mathcal{F}$ and $X/K \in \mathcal{F}$, we obtain the reverse inclusions. Therefore $r_{\mathcal{F}}(t(X)) = K = r_{\mathcal{F}}(X)$, and (3.6) then gives both equalities of Theorem 2.12. □

The lattice operations also preserve a prescribed intersection. For a subcategory $\mathcal{Z}$, write

$$\mathsf{PT}_{\mathcal{Z}}(\Lambda) = \{(\mathcal{T}, \mathcal{F}) \in \mathsf{PT}(\Lambda) : \mathcal{T} \cap \mathcal{F} = \mathcal{Z}\}.$$

Theorem 4.4 shows that this set is nonempty precisely when $\mathcal{Z}$ is image-closed.

Proposition 3.8. *For every subcategory* $\mathcal{Z}$*, the set* $\mathsf{PT}_{\mathcal{Z}}(\Lambda)$ *is closed under all nonempty meets and joins in* $\mathsf{PT}(\Lambda)$.

Proof. Let $(\mathcal{T}_i, \mathcal{F}_i)_{i\in I}$ be a nonempty family in $\mathsf{PT}_{\mathcal{Z}}(\Lambda)$. Put $\mathcal{T} = \bigcap_{i\in I} \mathcal{T}_i$, $t = t_{\mathcal{T}}$, and

$$\mathcal{F} = \{Y \in \mathsf{mod}\,\Lambda : t(Y) \in \mathcal{Z}\}.$$

We first show that $(\mathcal{T}, \mathcal{F})$ is a pretorsion theory with intersection $\mathcal{Z}$, and then compare it with the meet in $\mathsf{PT}(\Lambda)$. We repeatedly use the following observation: if a module $M \in \mathcal{T}$ embeds

in an object of $\mathcal{Z}$, then for any $i \in I$ we have $M \in \mathcal{T}_i$ and $M \in \mathcal{F}_i$ (because $\mathcal{Z} \subseteq \mathcal{F}_i$ and $\mathcal{F}_i$ is submodule-closed), so $M \in \mathcal{T}_i \cap \mathcal{F}_i = \mathcal{Z}$.

Step 1: Constructing a pretorsion theory with intersection $\mathcal{Z}$. If $Y' \subseteq Y \in \mathcal{F}$, then $t(Y') \subseteq t(Y) \in \mathcal{Z}$, so the observation gives $t(Y') \in \mathcal{Z}$. Thus $\mathcal{F}$ is submodule-closed; additivity follows from that of t. We have $\mathcal{Z} \subseteq \mathcal{T} \cap \mathcal{F}$, and $X \in \mathcal{T} \cap \mathcal{F}$ implies $X = t(X) \in \mathcal{Z}$, so $\mathcal{T} \cap \mathcal{F} = \mathcal{Z}$. If $Y \in \mathcal{F}_i$, then $t(Y) \subseteq t_{\mathcal{T}_i}(Y) \in \mathcal{Z}$ by (2.5); since $t(Y) \in \mathcal{T}$, the observation gives $t(Y) \in \mathcal{Z}$ and hence $Y \in \mathcal{F}$. Thus $\mathcal{F}_i \subseteq \mathcal{F}$ for every $i \in I$.

To apply Lemma 3.7, it remains to verify (3.5). Every morphism $T \to Y$ with $T \in \mathcal{T}$ and $Y \in \mathcal{F}$ factors through $t(Y) \in \mathcal{Z}$. Conversely, suppose that every morphism from a module E to an object of $\mathcal{F}$ factors through $\mathcal{Z}$. For each $i \in I$, we have $E/r_{\mathcal{F}_i}(E) \in \mathcal{F}_i \subseteq \mathcal{F}$, so the canonical projection $p_i \colon E \twoheadrightarrow E/r_{\mathcal{F}_i}(E)$ is $\mathcal{Z}$-trivial. By Theorem 2.12, the inclusion $t_{\mathcal{T}_i}(E) \hookrightarrow E$ is the $\mathcal{Z}$-prekernel of p_i; the identity 1_E therefore factors through $t_{\mathcal{T}_i}(E)$, giving $E \in \mathcal{T}_i$. Since $i \in I$ was arbitrary, $E \in \mathcal{T}$.

This proves (3.5), so Lemma 3.7 gives $(\mathcal{T}, \mathcal{F}) \in \mathsf{PT}(\Lambda)$.

Step 2: The ambient meet and join. By (3.2), the meet in $\mathsf{PT}(\Lambda)$ has the form $(\mathcal{T}, \mathcal{G})$, where $\mathcal{F}_i \subseteq \mathcal{G} \subseteq \mathcal{F}$ for every $i \in I$. Fixing $i \in I$, we obtain

$$\mathcal{Z} = \mathcal{T} \cap \mathcal{F}_i \subseteq \mathcal{T} \cap \mathcal{G} \subseteq \mathcal{T} \cap \mathcal{F} = \mathcal{Z},$$

so $\mathcal{T} \cap \mathcal{G} = \mathcal{Z}$ and the meet belongs to $\mathsf{PT}_{\mathcal{Z}}(\Lambda)$. The assertion for nonempty joins follows by duality (Remark 2.8). □

When $\mathcal{Z} = 0$, Proposition 3.8 shows that torsion pairs form a complete sublattice of $\mathsf{PT}(\Lambda)$.

Corollary 3.9. *The torsion pairs form a complete sublattice containing the least and greatest elements of* $\mathsf{PT}(\Lambda)$. *Sending a torsion pair to its first component identifies this sublattice with the lattice of torsion classes ordered by inclusion.*

Proof. By Proposition 2.5, the torsion pairs in $\mathsf{mod}\,\Lambda$ are the elements of $\mathsf{PT}_0(\Lambda)$. Proposition 3.8 gives closure under nonempty meets and joins, and the greatest and least elements $(\mathsf{mod}\,\Lambda, 0)$ and $(0, \mathsf{mod}\,\Lambda)$ of $\mathsf{PT}(\Lambda)$ also belong to $\mathsf{PT}_0(\Lambda)$. Finally, the reconstruction formula (2.5) at $\mathcal{Z} = 0$ gives $\mathcal{F} = \{Y \in \mathsf{mod}\,\Lambda : \mathrm{Hom}_\Lambda(\mathcal{T}, Y) = 0\}$. Thus $\mathcal{T}$ determines $\mathcal{F}$, and $\mathcal{T}_1 \subseteq \mathcal{T}_2$ implies $\mathcal{F}_1 \supseteq \mathcal{F}_2$, so the projection $(\mathcal{T}, \mathcal{F}) \mapsto \mathcal{T}$ is an order isomorphism. □

4. Image-closed subcategories as intersections of the two components

In this section, we prove Theorem A by characterizing the subcategories of $\mathsf{mod}\,\Lambda$ that occur as intersections of pretorsion theories, and we construct the greatest and least pretorsion theories with a prescribed intersection. We call a pretorsion theory $(\mathcal{T}, \mathcal{F})$ with $\mathcal{T} \cap \mathcal{F} = \mathcal{Z}$ a *realization* of $\mathcal{Z}$.

Definition 4.1. For a subcategory $\mathcal{Z} \subseteq \mathsf{mod}\,\Lambda$, let $\mathsf{Gen}\,\mathcal{Z}$ consist of the quotients of finite direct sums of objects of $\mathcal{Z}$, and let $\mathsf{Cogen}\,\mathcal{Z}$ consist of the submodules of such direct sums.

For an image-closed subcategory $\mathcal{Z}$, Theorem 4.4 below shows that the quotient-closed subcategory $\mathsf{Gen}\,\mathcal{Z}$ and the submodule-closed subcategory $\mathsf{Cogen}\,\mathcal{Z}$ have intersection $\mathcal{Z}$, but the pair $(\mathsf{Gen}\,\mathcal{Z}, \mathsf{Cogen}\,\mathcal{Z})$ need not itself be a pretorsion theory.

Example 4.2. Let $\Lambda_2 = \Bbbk(1 \to 2)$ and let $\mathcal{Z}$ be the subcategory of semisimple Λ_2-modules. Then $\mathsf{Gen}\,\mathcal{Z} = \mathsf{Cogen}\,\mathcal{Z} = \mathcal{Z}$. Writing $t = t_{\mathcal{Z}}$ and $r = r_{\mathcal{Z}}$, we see that for the indecomposable representation $X = (\Bbbk \xrightarrow{1} \Bbbk)$, both $t(X)$ and $r(X)$ are equal to the simple submodule $S_2 = (0 \to \Bbbk)$, whereas $r(S_2) = 0$ because $S_2 \in \mathcal{Z}$. Thus $r(X) \neq r(t(X)) = 0$, so $(\mathcal{Z}, \mathcal{Z})$ is not a pretorsion theory by Theorem 2.12.

To obtain a pretorsion theory with intersection $\mathcal{Z}$, we enlarge the first component. If $(\mathcal{T}, \mathcal{F})$ is a pretorsion theory with $\mathcal{T} \cap \mathcal{F} = \mathcal{Z}$, then for any morphism $f \colon X \to N$ with $X \in \mathcal{T}$ and $N \in \mathcal{Z} \subseteq \mathcal{F}$, the image $\mathrm{Im}\, f$ lies in $\mathcal{T}$ by quotient closure and in $\mathcal{F}$ by submodule closure, and hence belongs to $\mathcal{Z}$. This necessary condition on $\mathcal{T}$, together with the reconstruction formula (2.5) for $\mathcal{F}$, motivates the following construction.

Construction 4.3. For an image-closed subcategory $\mathcal{Z} \subseteq \mathsf{mod}\,\Lambda$ put

$$\mathcal{T}^+(\mathcal{Z}) = \{X \in \mathsf{mod}\,\Lambda : \operatorname{Im} f \in \mathcal{Z} \text{ for every } f\colon X \to N \text{ with } N \in \mathcal{Z}\},$$
$$\mathcal{F}^-(\mathcal{Z}) = \{Y \in \mathsf{mod}\,\Lambda : t_{\mathcal{T}^+(\mathcal{Z})}(Y) \in \mathcal{Z}\}.$$

The superscripts anticipate the extremality proved in Corollary 4.5: among realizations of $\mathcal{Z}$, $\mathcal{T}^+(\mathcal{Z})$ is the largest possible first component and $\mathcal{F}^-(\mathcal{Z})$ the smallest possible second component.

The equivalence of (1) and (2) in the following theorem is due to Auslander and Smalø [1, Proposition 4.10(i)]. We include its short proof before showing that $(\mathcal{T}^+(\mathcal{Z}), \mathcal{F}^-(\mathcal{Z}))$ is a pretorsion theory with intersection $\mathcal{Z}$.

Theorem 4.4. *For a subcategory $\mathcal{Z} \subseteq \mathsf{mod}\,\Lambda$, the following are equivalent.*

(1) *$\mathcal{Z}$ is image-closed.*
(2) *$\mathcal{Z} = \mathsf{Gen}\,\mathcal{Z} \cap \mathsf{Cogen}\,\mathcal{Z}$.*
(3) *$\mathcal{Z} = \mathcal{T} \cap \mathcal{F}$ for a pretorsion theory $(\mathcal{T}, \mathcal{F})$.*

Proof. (1) $\Rightarrow$ (2): Let $X \in \mathsf{Gen}\,\mathcal{Z} \cap \mathsf{Cogen}\,\mathcal{Z}$. By Definition 4.1, there exist finite direct sums N_1 and N_2 of objects of $\mathcal{Z}$, an epimorphism $N_1 \twoheadrightarrow X$, and a monomorphism $X \hookrightarrow N_2$:

$$N_1 \twoheadrightarrow X \hookrightarrow N_2.$$

Since $\mathcal{Z}$ is additive, both N_1 and N_2 belong to $\mathcal{Z}$. The composite morphism $N_1 \to N_2$ has image X, so $X \in \mathcal{Z}$ because $\mathcal{Z}$ is image-closed. The reverse inclusion $\mathcal{Z} \subseteq \mathsf{Gen}\,\mathcal{Z} \cap \mathsf{Cogen}\,\mathcal{Z}$ is immediate.

(2) $\Rightarrow$ (1): For any morphism $f\colon X \to Y$ with $X, Y \in \mathcal{Z}$, the image $\operatorname{Im} f$ is a quotient of X and a submodule of Y, so $\operatorname{Im} f \in \mathsf{Gen}\,\mathcal{Z} \cap \mathsf{Cogen}\,\mathcal{Z} = \mathcal{Z}$.

(3) $\Rightarrow$ (1): If $(\mathcal{T}, \mathcal{F})$ is a pretorsion theory with $\mathcal{T} \cap \mathcal{F} = \mathcal{Z}$, then $\mathcal{T} \in \mathcal{L}_{\mathrm{quot}}(\Lambda)$ and $\mathcal{F} \in \mathcal{L}_{\mathrm{sub}}(\Lambda)$ by Proposition 2.3, so $\mathcal{Z} = \mathcal{T} \cap \mathcal{F}$ is image-closed by (2.1).

(1) $\Rightarrow$ (3): Assume that $\mathcal{Z}$ is image-closed, and consider the pair $(\mathcal{T}^+(\mathcal{Z}), \mathcal{F}^-(\mathcal{Z}))$ of Construction 4.3. We first verify the closure properties of the two components and show that their intersection is $\mathcal{Z}$, and then apply Lemma 3.7.

Step 1: Closure properties and the intersection. Write $t = t_{\mathcal{T}^+(\mathcal{Z})}$. We first verify that $\mathcal{T}^+(\mathcal{Z}) \in \mathcal{L}_{\mathrm{quot}}(\Lambda)$, $\mathcal{F}^-(\mathcal{Z}) \in \mathcal{L}_{\mathrm{sub}}(\Lambda)$, and both subcategories contain $\mathcal{Z}$.

Let $X_1, X_2 \in \mathcal{T}^+(\mathcal{Z})$ and let $f\colon X_1 \oplus X_2 \to N$ be a morphism with $N \in \mathcal{Z}$. The images of the two components $f_i\colon X_i \to N$ $(i = 1, 2)$ belong to $\mathcal{Z}$, and factoring each f_i through $\operatorname{Im} f_i$ yields a commutative triangle

$$\begin{array}{ccc} X_1 \oplus X_2 & \twoheadrightarrow & \operatorname{Im} f_1 \oplus \operatorname{Im} f_2 \\ & \searrow^{f} & \downarrow{\scriptstyle (\iota_1, \iota_2)} \\ & & N, \end{array}$$

where $\iota_i\colon \operatorname{Im} f_i \hookrightarrow N$ is the inclusion. The vertical morphism has source $\operatorname{Im} f_1 \oplus \operatorname{Im} f_2 \in \mathcal{Z}$, target $N \in \mathcal{Z}$, and image $\operatorname{Im} f$, so $\operatorname{Im} f \in \mathcal{Z}$ because $\mathcal{Z}$ is image-closed. Thus $X_1 \oplus X_2 \in \mathcal{T}^+(\mathcal{Z})$. Clearly $0 \in \mathcal{T}^+(\mathcal{Z})$, and $\mathcal{T}^+(\mathcal{Z})$ is closed under isomorphisms. If $X \in \mathcal{T}^+(\mathcal{Z})$ and $\pi\colon X \twoheadrightarrow X/K$ is a quotient map, then any morphism $f\colon X/K \to N$ with $N \in \mathcal{Z}$ satisfies $\operatorname{Im} f = \operatorname{Im}(f\pi) \in \mathcal{Z}$. Thus $\mathcal{T}^+(\mathcal{Z})$ is quotient-closed and in particular closed under direct summands, so $\mathcal{T}^+(\mathcal{Z}) \in \mathcal{L}_{\mathrm{quot}}(\Lambda)$.

Next, we record the identity

$$\mathcal{T}^+(\mathcal{Z}) \cap \mathsf{Cogen}\,\mathcal{Z} = \mathcal{Z}. \tag{4.1}$$

Indeed, if $X \in \mathcal{T}^+(\mathcal{Z}) \cap \mathsf{Cogen}\,\mathcal{Z}$, there is a monomorphism $X \hookrightarrow N$ into a finite direct sum N of objects of $\mathcal{Z}$; since $N \in \mathcal{Z}$ and $X \in \mathcal{T}^+(\mathcal{Z})$, the image X of this monomorphism belongs to $\mathcal{Z}$. Conversely, image closure of $\mathcal{Z}$ implies $\mathcal{Z} \subseteq \mathcal{T}^+(\mathcal{Z})$, and clearly $\mathcal{Z} \subseteq \mathsf{Cogen}\,\mathcal{Z}$.

We now check that $\mathcal{F}^-(\mathcal{Z}) \in \mathcal{L}_{\mathrm{sub}}(\Lambda)$. Since t commutes with finite direct sums by Proposition 2.10(4) and $\mathcal{Z}$ is additive and closed under direct summands, $\mathcal{F}^-(\mathcal{Z})$ is additive and closed under direct summands. If $W \subseteq Y$ with $Y \in \mathcal{F}^-(\mathcal{Z})$, then $t(W) \subseteq t(Y) \in \mathcal{Z}$ by Proposition 2.10(5), so $t(W) \in \mathsf{Cogen}\,\mathcal{Z}$; since $t(W) \in \mathcal{T}^+(\mathcal{Z})$ by Proposition 2.10(1), equation (4.1) gives $t(W) \in \mathcal{Z}$, and thus $W \in \mathcal{F}^-(\mathcal{Z})$. Hence $\mathcal{F}^-(\mathcal{Z}) \in \mathcal{L}_{\mathrm{sub}}(\Lambda)$.

Finally, since $\mathcal{Z} \subseteq \mathcal{T}^+(\mathcal{Z})$, every $N \in \mathcal{Z}$ satisfies $t(N) = N \in \mathcal{Z}$ by Proposition 2.10(3), so $\mathcal{Z} \subseteq \mathcal{F}^-(\mathcal{Z})$ as well. Conversely, if $X \in \mathcal{T}^+(\mathcal{Z}) \cap \mathcal{F}^-(\mathcal{Z})$, then $X = t(X) \in \mathcal{Z}$, so $\mathcal{T}^+(\mathcal{Z}) \cap \mathcal{F}^-(\mathcal{Z}) = \mathcal{Z}$.

Step 2: The pretorsion property. The definition of $\mathcal{F}^-(\mathcal{Z})$ gives (3.4). For (3.5), every morphism from an object of $\mathcal{T}^+(\mathcal{Z})$ to $Y \in \mathcal{F}^-(\mathcal{Z})$ factors through $t(Y) \in \mathcal{Z}$. Conversely, suppose that every morphism from E to an object of $\mathcal{F}^-(\mathcal{Z})$ is $\mathcal{Z}$-trivial. If $Y \in \mathsf{Cogen}\,\mathcal{Z}$, then $t(Y) \in \mathcal{T}^+(\mathcal{Z}) \cap \mathsf{Cogen}\,\mathcal{Z} = \mathcal{Z}$ by (4.1), so $\mathsf{Cogen}\,\mathcal{Z} \subseteq \mathcal{F}^-(\mathcal{Z})$. For a morphism $f\colon E \to N$ with $N \in \mathcal{Z}$, put $I = \operatorname{Im} f$. Since $I \in \mathsf{Cogen}\,\mathcal{Z} \subseteq \mathcal{F}^-(\mathcal{Z})$, the epimorphism $E \twoheadrightarrow I$ factors through an object of $\mathcal{Z}$, giving $I \in \mathsf{Gen}\,\mathcal{Z}$. Thus $I \in \mathsf{Gen}\,\mathcal{Z} \cap \mathsf{Cogen}\,\mathcal{Z} = \mathcal{Z}$ by (2), and $E \in \mathcal{T}^+(\mathcal{Z})$. Lemma 3.7 now shows that $(\mathcal{T}^+(\mathcal{Z}), \mathcal{F}^-(\mathcal{Z}))$ is a pretorsion theory. □

The realizing pair $(\mathcal{T}^+(\mathcal{Z}), \mathcal{F}^-(\mathcal{Z}))$ has the following extremal property in $\mathsf{PT}(\Lambda)$.

Corollary 4.5. *Let $\mathcal{Z}$ be image-closed. Then $(\mathcal{T}^+(\mathcal{Z}), \mathcal{F}^-(\mathcal{Z}))$ is the greatest pretorsion theory with intersection $\mathcal{Z}$, in the order (2.2).*

Proof. Let $(\mathcal{T}, \mathcal{F}) \in \mathsf{PT}(\Lambda)$ satisfy $\mathcal{T} \cap \mathcal{F} = \mathcal{Z}$. For $X \in \mathcal{T}$ and $N \in \mathcal{Z}$, the image of every map $X \to N$ belongs to $\mathcal{T} \cap \mathcal{F} = \mathcal{Z}$. Hence $\mathcal{T} \subseteq \mathcal{T}^+(\mathcal{Z})$ by Construction 4.3. If $Y \in \mathcal{F}^-(\mathcal{Z})$, then $t_{\mathcal{T}}(Y) \subseteq t_{\mathcal{T}^+(\mathcal{Z})}(Y) \in \mathcal{Z} \subseteq \mathcal{F}$. Since $\mathcal{F}$ is submodule-closed, we obtain $t_{\mathcal{T}}(Y) \in \mathcal{T} \cap \mathcal{F} = \mathcal{Z}$, and Corollary 2.13 gives $Y \in \mathcal{F}$. Thus $\mathcal{F}^-(\mathcal{Z}) \subseteq \mathcal{F}$, so $(\mathcal{T}, \mathcal{F}) \leq (\mathcal{T}^+(\mathcal{Z}), \mathcal{F}^-(\mathcal{Z}))$. □

For example, when $\mathcal{Z}$ is the subcategory of semisimple Λ_2-modules in Example 4.2, every morphism to a semisimple module has semisimple image, so $\mathcal{T}^+(\mathcal{Z}) = \mathsf{mod}\,\Lambda_2$ and $\mathcal{F}^-(\mathcal{Z}) = \mathcal{Z}$; thus $(\mathsf{mod}\,\Lambda_2, \mathcal{Z})$ is the greatest realization of $\mathcal{Z}$.

The dual construction yields the least realization.

Construction 4.6. For an image-closed subcategory $\mathcal{Z} \subseteq \mathsf{mod}\,\Lambda$, put

$$\mathcal{F}^+(\mathcal{Z}) = \{Y \in \mathsf{mod}\,\Lambda : \operatorname{Im} g \in \mathcal{Z} \text{ for every } g\colon N \to Y \text{ with } N \in \mathcal{Z}\},$$
$$\mathcal{T}^-(\mathcal{Z}) = \{X \in \mathsf{mod}\,\Lambda : X/r_{\mathcal{F}^+(\mathcal{Z})}(X) \in \mathcal{Z}\}.$$

Corollary 4.7. *Let $\mathcal{Z}$ be image-closed. Then $(\mathcal{T}^-(\mathcal{Z}), \mathcal{F}^+(\mathcal{Z}))$ is the least pretorsion theory with intersection $\mathcal{Z}$. Every pretorsion theory $(\mathcal{T}, \mathcal{F})$ with $\mathcal{T} \cap \mathcal{F} = \mathcal{Z}$ satisfies*

$$(\mathcal{T}^-(\mathcal{Z}), \mathcal{F}^+(\mathcal{Z})) \leq (\mathcal{T}, \mathcal{F}) \leq (\mathcal{T}^+(\mathcal{Z}), \mathcal{F}^-(\mathcal{Z})).$$

Moreover, $\mathsf{PT}_{\mathcal{Z}}(\Lambda)$ is a complete lattice: its nonempty meets and joins agree with those in $\mathsf{PT}(\Lambda)$, and its empty meet and join are the greatest and least realizations above.

Proof. Applying Corollary 4.5 to the image-closed subcategory $D\mathcal{Z} \subseteq \mathsf{mod}\,\Lambda^{\mathrm{op}}$ and using Remark 2.8 shows that $(\mathcal{T}^-(\mathcal{Z}), \mathcal{F}^+(\mathcal{Z}))$ is the least realization of $\mathcal{Z}$. Proposition 3.8 gives closure under nonempty meets and joins in $\mathsf{PT}(\Lambda)$, while the greatest and least realizations provide the empty meet and join. □

4.1. **The convex geometry of image closure.** We now study $\mathcal{L}_{\mathrm{im}}(\Lambda)$ as a complete lattice and use convex geometry to describe its generating sets and covering relations. While meets in $\mathcal{L}_{\mathrm{im}}(\Lambda)$ are intersections, forming joins requires the smallest image-closed subcategory containing a given family, described by the image closure of Auslander and Smalø [1, Proposition 4.3].

Definition 4.8. For a subcategory $\mathcal{C} \subseteq \mathsf{mod}\,\Lambda$, its *image closure* is

$$\mathsf{icl}\,\mathcal{C} = \mathsf{Gen}\,\mathcal{C} \cap \mathsf{Cogen}\,\mathcal{C}.$$

Corollary 4.9 ([1, Proposition 4.3])**.** *Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\,\Lambda$. Then $\mathsf{icl}\,\mathcal{C}$ is the smallest image-closed subcategory of $\mathsf{mod}\,\Lambda$ containing $\mathcal{C}$. In particular a subcategory is image-closed if and only if it equals its own image closure.*

Proof. Since $\mathsf{Gen}\,\mathcal{C}$ is quotient-closed and $\mathsf{Cogen}\,\mathcal{C}$ is submodule-closed, their intersection $\mathsf{icl}\,\mathcal{C}$ is image-closed by (2.1), and clearly $\mathcal{C} \subseteq \mathsf{icl}\,\mathcal{C}$. If $\mathcal{D} \in \mathcal{L}_{\mathrm{im}}(\Lambda)$ contains $\mathcal{C}$, then $\mathsf{Gen}\,\mathcal{C} \subseteq \mathsf{Gen}\,\mathcal{D}$ and $\mathsf{Cogen}\,\mathcal{C} \subseteq \mathsf{Cogen}\,\mathcal{D}$, so the equivalence of (1) and (2) in Theorem 4.4 gives

$$\mathsf{icl}\,\mathcal{C} \subseteq \mathsf{Gen}\,\mathcal{D} \cap \mathsf{Cogen}\,\mathcal{D} = \mathcal{D}.$$

Thus $\mathsf{icl}\,\mathcal{C}$ is the smallest image-closed subcategory containing $\mathcal{C}$, and $\mathcal{C} \in \mathcal{L}_{\mathrm{im}}(\Lambda)$ if and only if $\mathcal{C} = \mathsf{icl}\,\mathcal{C}$. □

The theory of convex geometries on finite sets goes back to Edelman and Jamison [7]; we use the following finitary formulation.

Definition 4.10 ([8, Definitions 2.1 and 2.2])**.** A *closure operator* on a set E is a map $c\colon 2^E \to 2^E$ such that $S \subseteq c(S)$, $c(S) \subseteq c(S')$ whenever $S \subseteq S'$, and $c(c(S)) = c(S)$ for all $S, S' \subseteq E$. A subset $C \subseteq E$ is *closed* if $c(C) = C$. The operator c is *finitary* if $c(S)$ is the union of $c(S_0)$ over all finite subsets $S_0 \subseteq S$, and it satisfies *anti-exchange* if

$$x \in c(C \cup \{y\}) \quad \Longrightarrow \quad y \notin c(C \cup \{x\})$$

for every closed subset $C \subseteq E$ and all distinct $x, y \in E \setminus C$. A *convex geometry* is a pair (E, c) where c is a closure operator on E satisfying $c(\varnothing) = \varnothing$ and anti-exchange; it is a *finitary convex geometry* if c is also finitary. For a closed subset $C \subseteq E$, the set of *extreme points* of C is

$$\mathsf{ex}(C) = \{x \in C : x \notin c(C \setminus \{x\})\}.$$

For $S \subseteq \mathsf{ind}\,\Lambda$, put

$$\mathsf{qcl}(S) = \mathsf{ind}\,\mathsf{Gen}(\mathsf{add}\,S), \qquad \mathsf{scl}(S) = \mathsf{ind}\,\mathsf{Cogen}(\mathsf{add}\,S), \tag{4.2}$$

$$\mathsf{icl}(S) = \mathsf{ind}\big(\mathsf{Gen}(\mathsf{add}\,S) \cap \mathsf{Cogen}(\mathsf{add}\,S)\big) = \mathsf{qcl}(S) \cap \mathsf{scl}(S). \tag{4.3}$$

Thus $\mathsf{icl}(S) = \mathsf{ind}(\mathsf{icl}(\mathsf{add}\,S))$; we use the same symbol icl for the operator on subcategories and for the induced operator on subsets of $\mathsf{ind}\,\Lambda$. Under the bijection $\mathcal{C} \mapsto \mathsf{ind}\,\mathcal{C}$ between subcategories of $\mathsf{mod}\,\Lambda$ and subsets of $\mathsf{ind}\,\Lambda$, the closed sets of qcl correspond to $\mathcal{L}_{\mathrm{quot}}(\Lambda)$, since $\mathsf{Gen}\,\mathcal{C}$ is the smallest quotient-closed subcategory containing $\mathcal{C}$. Dually, the closed sets of scl correspond to $\mathcal{L}_{\mathrm{sub}}(\Lambda)$, and the closed sets of icl correspond to $\mathcal{L}_{\mathrm{im}}(\Lambda)$ by Corollary 4.9.

On a finite ground set, the pointwise intersection of two closure operators of convex geometries is again the closure operator of a convex geometry, by Edelman and Jamison [7, Theorem 5.1]. We verify the finitary version directly from (4.3).

Theorem 4.11. *Let Λ be a finite-dimensional algebra over a field. Then $(\mathsf{ind}\,\Lambda, \mathsf{icl})$ is a finitary convex geometry whose closed sets correspond via $\mathcal{C} \mapsto \mathsf{ind}\,\mathcal{C}$ to the image-closed subcategories of $\mathsf{mod}\,\Lambda$.*

Proof. By Corollary 4.9, icl is a closure operator on $\mathsf{ind}\,\Lambda$ whose closed sets correspond to $\mathcal{L}_{\mathrm{im}}(\Lambda)$, and $\mathsf{icl}(\varnothing) = \varnothing$ because the zero subcategory is image-closed.

To see that icl is finitary, let $X \in \mathsf{icl}(S)$. Then X is a quotient of a finite direct sum of modules in S and a submodule of another such finite direct sum; if $S_0 \subseteq S$ is the finite set of modules appearing in these two direct sums, then $X \in \mathsf{icl}(S_0)$.

For anti-exchange, recall from [8, Proposition 2.13 and Corollary 2.14] that both qcl and scl satisfy anti-exchange. Let $C \subseteq \mathsf{ind}\,\Lambda$ be icl-closed, and suppose for contradiction that there exist distinct $x, y \in \mathsf{ind}\,\Lambda \setminus C$ with

$$x \in \mathsf{icl}(C \cup \{y\}), \qquad y \in \mathsf{icl}(C \cup \{x\}).$$

Since $x \notin C = \mathsf{qcl}(C) \cap \mathsf{scl}(C)$, choose $c \in \{\mathsf{qcl}, \mathsf{scl}\}$ with $x \notin c(C)$. There are two cases.

Case 1: $y \in c(C)$**.** Then $c(C \cup \{y\}) = c(C)$, contradicting $x \in \mathsf{icl}(C \cup \{y\}) \subseteq c(C \cup \{y\})$.

Case 2: $y \notin c(C)$**.** The two assumed inclusions imply $x \in c(c(C) \cup \{y\})$ and $y \in c(c(C) \cup \{x\})$, contradicting anti-exchange for c at the closed set $c(C)$. □

To describe the extreme points of icl, we recall the splitting projective and splitting injective objects introduced by Auslander and Smalø [1, Section 2].

Definition 4.12 ([1, Section 2])**.** Let $\mathcal{C} \subseteq \mathsf{mod}\,\Lambda$ be a subcategory and let $X \in \mathsf{ind}\,\mathcal{C}$. Then X is *splitting projective* in $\mathcal{C}$ if every epimorphism $N \to X$ with $N \in \mathcal{C}$ splits, and *splitting injective* in $\mathcal{C}$ if every monomorphism $X \to N$ with $N \in \mathcal{C}$ splits. Write $\mathcal{P}_0(\mathcal{C})$ and $\mathcal{I}_0(\mathcal{C})$ for the sets of splitting projective and splitting injective objects in $\mathsf{ind}\,\mathcal{C}$.

Parts (1) and (2) of the next proposition follow from Auslander and Smalø [1, Proposition 2.2] and apply to arbitrary subcategories; combining them yields a characterization of the extreme points of an image-closed subcategory.

Proposition 4.13. *Let $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ be a subcategory and let $X \in \operatorname{ind}\mathcal{C}$.*

(1) *$X \notin \operatorname{qcl}(\operatorname{ind}\mathcal{C} \setminus \{X\})$ if and only if $X \in \mathcal{P}_0(\mathcal{C})$.*
(2) *$X \notin \operatorname{scl}(\operatorname{ind}\mathcal{C} \setminus \{X\})$ if and only if $X \in \mathcal{I}_0(\mathcal{C})$.*

If $\mathcal{C}$ is image-closed, so that $\operatorname{ind}\mathcal{C}$ is icl-closed, the following are equivalent.

(3) *$X \in \operatorname{ex}(\operatorname{ind}\mathcal{C})$ with respect to icl.*
(4) *$\operatorname{add}(\operatorname{ind}\mathcal{C} \setminus \{X\})$ is image-closed.*
(5) *$X \in \mathcal{P}_0(\mathcal{C}) \cup \mathcal{I}_0(\mathcal{C})$.*

Proof. (1) We prove both directions by contraposition. If $X \notin \mathcal{P}_0(\mathcal{C})$, choose a nonsplit epimorphism $M \twoheadrightarrow X$ with $M \in \mathcal{C}$, and write $M \cong X^n \oplus M'$ where X is not a direct summand of M', so that $M' \in \operatorname{add}(\operatorname{ind}\mathcal{C} \setminus \{X\})$. By Auslander and Smalø [1, Proposition 2.2(a)], the existence of a nonsplit epimorphism $M' \oplus X^n \twoheadrightarrow X$ with X indecomposable implies the existence of an epimorphism $(M')^m \twoheadrightarrow X$ for some $m \geq 1$, whence $X \in \operatorname{qcl}(\operatorname{ind}\mathcal{C} \setminus \{X\})$. Conversely, if $X \in \operatorname{qcl}(\operatorname{ind}\mathcal{C} \setminus \{X\})$, there is an epimorphism $f\colon M \twoheadrightarrow X$ with $M \in \operatorname{add}(\operatorname{ind}\mathcal{C} \setminus \{X\}) \subseteq \mathcal{C}$. If f split, then X would be isomorphic to a direct summand of M, contradicting the Krull–Schmidt theorem; thus f does not split and $X \notin \mathcal{P}_0(\mathcal{C})$.

(2) This is dual to (1).

(3) $\Leftrightarrow$ (4): Since $\mathcal{C}$ is image-closed, we have $\operatorname{icl}(\operatorname{ind}\mathcal{C} \setminus \{X\}) \subseteq \operatorname{ind}\mathcal{C}$, so $\operatorname{icl}(\operatorname{ind}\mathcal{C} \setminus \{X\})$ equals either $\operatorname{ind}\mathcal{C} \setminus \{X\}$ or $\operatorname{ind}\mathcal{C}$. Thus $X \notin \operatorname{icl}(\operatorname{ind}\mathcal{C} \setminus \{X\})$ if and only if $\operatorname{icl}(\operatorname{ind}\mathcal{C} \setminus \{X\}) = \operatorname{ind}\mathcal{C} \setminus \{X\}$, which by Corollary 4.9 means that $\operatorname{add}(\operatorname{ind}\mathcal{C} \setminus \{X\})$ is image-closed.

(3) $\Leftrightarrow$ (5): By (4.3), we have

$$\operatorname{icl}(\operatorname{ind}\mathcal{C} \setminus \{X\}) = \operatorname{qcl}(\operatorname{ind}\mathcal{C} \setminus \{X\}) \cap \operatorname{scl}(\operatorname{ind}\mathcal{C} \setminus \{X\}).$$

Hence $X \notin \operatorname{icl}(\operatorname{ind}\mathcal{C} \setminus \{X\})$ if and only if $X \notin \operatorname{qcl}(\operatorname{ind}\mathcal{C} \setminus \{X\})$ or $X \notin \operatorname{scl}(\operatorname{ind}\mathcal{C} \setminus \{X\})$, which is equivalent to $X \in \mathcal{P}_0(\mathcal{C}) \cup \mathcal{I}_0(\mathcal{C})$ by (1) and (2). □

The closed sets of any closure operator form a complete lattice, giving part (1) below directly from Corollary 4.9. The convex geometry of Theorem 4.11 gives parts (2) and (3).

Corollary 4.14. *Let Λ be a finite-dimensional algebra over a field.*

(1) *$\mathcal{L}_{\mathrm{im}}(\Lambda)$ is a complete lattice, in which the meet of a family $(\mathcal{Z}_i)_{i\in I}$ is its intersection and the join is $\operatorname{icl}\big(\operatorname{add}\bigcup_i \mathcal{Z}_i\big)$.*
(2) *If $\mathcal{Z}$ is an image-closed subcategory with finitely many indecomposable objects, then there is a unique smallest subset $S \subseteq \operatorname{ind}\mathcal{Z}$ with $\operatorname{icl}(S) = \operatorname{ind}\mathcal{Z}$, namely $S = \mathcal{P}_0(\mathcal{Z}) \cup \mathcal{I}_0(\mathcal{Z})$.*
(3) *If $\mathcal{Z}' \subsetneq \mathcal{Z}$ is a covering relation in $\mathcal{L}_{\mathrm{im}}(\Lambda)$, then $\operatorname{ind}\mathcal{Z} \setminus \operatorname{ind}\mathcal{Z}'$ has exactly one element. If Λ is representation-finite, then $\mathcal{L}_{\mathrm{im}}(\Lambda)$ is graded by the size $\mathcal{Z} \mapsto |\operatorname{ind}\mathcal{Z}|$.*

Proof. (1) Any intersection of image-closed subcategories is image-closed and is therefore the meet, while $\operatorname{icl}(\operatorname{add}\bigcup_i \mathcal{Z}_i)$ is the smallest image-closed subcategory containing every $\mathcal{Z}_i$ by Corollary 4.9 and is therefore the join.

(2) Since $C = \operatorname{ind}\mathcal{Z}$ is finite and icl-closed, restricting icl to subsets of C defines a finite convex geometry. By Edelman and Jamison [7, Theorem 2.1(d)–(e)], we have $C = \operatorname{icl}(\operatorname{ex}(C))$, and every subset $S \subseteq C$ with $\operatorname{icl}(S) = C$ must contain $\operatorname{ex}(C)$ because no $x \in \operatorname{ex}(C)$ belongs to $\operatorname{icl}(C \setminus \{x\})$. Thus $\operatorname{ex}(C)$ is the unique smallest generating set of C, and $\operatorname{ex}(C) = \mathcal{P}_0(\mathcal{Z}) \cup \mathcal{I}_0(\mathcal{Z})$ by Proposition 4.13.

(3) Put $C = \operatorname{ind}\mathcal{Z}'$ and $D = \operatorname{ind}\mathcal{Z}$. For every $x \in D \setminus C$, we have $C \subsetneq \operatorname{icl}(C \cup \{x\}) \subseteq D$; since $\mathcal{Z}' \subsetneq \mathcal{Z}$ is a covering relation in $\mathcal{L}_{\mathrm{im}}(\Lambda)$, it follows that $\operatorname{icl}(C \cup \{x\}) = D$. If $D \setminus C$ contained two distinct elements x and y, we would have $y \in \operatorname{icl}(C \cup \{x\})$ and $x \in \operatorname{icl}(C \cup \{y\})$, contradicting the anti-exchange property at C. Thus $|D \setminus C| = 1$, and when $\operatorname{ind}\Lambda$ is finite, $\mathcal{Z} \mapsto |\operatorname{ind}\mathcal{Z}|$ is a rank function on $\mathcal{L}_{\mathrm{im}}(\Lambda)$. □

5. Images and closure conditions for interval modules

In this section, we characterize image-closed subcategories for Λ_n, with arrows $1 \to 2 \to \cdots \to n$, in terms of support intervals (Theorem 5.5), reducing the second count in Theorem B to an enumeration of interval families.

Definition 5.1. For $1 \le a \le d \le n$, the *interval module* $M[a,d]$ is one-dimensional at the vertices $a, a+1, \dots, d$, zero at all other vertices, and has identity maps along the arrows between consecutive vertices in $[a,d]$. By convention, we also set $M[a,a-1] = 0$ for $1 \le a \le n+1$. We call the modules with fixed left endpoint a *row* a, and those with fixed right endpoint d *column* d.

The nonzero interval modules $M[a,d]$ $(1 \le a \le d \le n)$ form a complete set of representatives of the isomorphism classes of indecomposable Λ_n-modules, and we often identify $M[a,d]$ with its support interval $[a,d]$. The modules in row a and in column d are

$$\begin{aligned} \text{row } a: &\quad M[a,a], M[a,a+1], \dots, M[a,n], \\ \text{column } d: &\quad M[1,d], M[2,d], \dots, M[d,d]. \end{aligned}$$

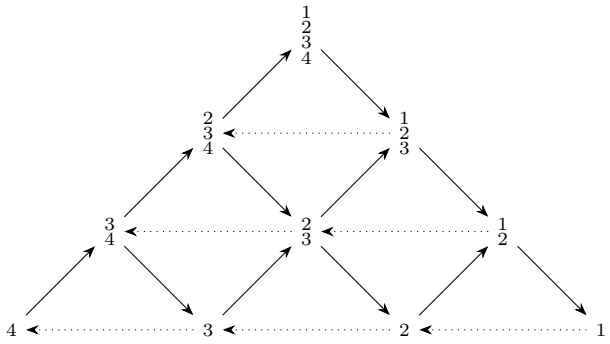


FIGURE 1. The Auslander–Reiten quiver of $\mathsf{mod}\,\Lambda_4$. A stack $\begin{smallmatrix} a \\ a+1 \\ \vdots \\ d \end{smallmatrix}$ denotes $M[a,d]$, with top S_a and socle S_d. Solid arrows are irreducible maps; dotted arrows point from a nonprojective module X to τX.

Rows and columns are the two families of sectional paths in Figure 1: row arrows are the quotients $M[a,d] \to M[a,d-1]$, and column arrows are the inclusions $M[a+1,d] \to M[a,d]$. Diagrams of a single subcategory show the full ambient Auslander–Reiten quiver, with black vertices for indecomposable modules in the subcategory and open vertices for the remaining indecomposable modules. The quivers for Λ_2 and Λ_3 use the same arrangement, restricted to composition factors $1,2$ and $1,2,3$, respectively.

5.1. **Morphisms and images of interval modules.** We compute morphisms and their images from the endpoints of the source $M[c,d]$ and target $M[a,b]$.

Proposition 5.2 ([21, Proposition 6.1])**.** *For two indecomposable modules $M[c,d]$ and $M[a,b]$,*

$$\dim_{\Bbbk} \mathrm{Hom}_{\Lambda_n}(M[c,d], M[a,b]) = \begin{cases} 1, & a \le c \le b \le d, \\ 0, & \text{otherwise.} \end{cases}$$

When $\mathrm{Hom}_{\Lambda_n}(M[c,d], M[a,b]) \neq 0$, every nonzero morphism $M[c,d] \to M[a,b]$ has image $M[c,b]$, the interval module supported on $[c,d] \cap [a,b]$.

Proof. At each vertex $i \in [c,d] \cap [a,b]$, a morphism $M[c,d] \to M[a,b]$ acts by a scalar, and by commutativity along the arrows $i \to i+1$ on the overlap, all these scalars are equal. Commutativity at the arrows entering $\max(a,c)$ and leaving $\min(b,d)$ requires $a \le c$ and $b \le d$ for this scalar to be nonzero, in which case the image is supported on $[c,b] = [c,d] \cap [a,b]$. □

5.2. **Closure conditions on support intervals.** We translate the three closure conditions of Definition 2.1 into conditions on support intervals.

A submodule of $M[a,d]$ is spanned by a subset of its nonzero vertex spaces that is closed under the arrows $i \to i+1$, and is therefore supported on a suffix $[\ell,d]$ of $[a,d]$; the corresponding

quotient is supported on the complementary prefix $[a, \ell-1]$. Consequently, the nonzero quotients and submodules of $M[a,d]$ are indecomposable and form chains:

$$\begin{aligned} &\text{indecomposable quotients of } M[a,d]: && M[a,\ell] \quad (a \le \ell \le d), \\ &\text{indecomposable submodules of } M[a,d]: && M[\ell,d] \quad (a \le \ell \le d). \end{aligned} \tag{5.1}$$

Thus, if a quotient-closed subcategory contains $M[a,d]$, it contains $M[a,\ell]$ for every $a \le \ell \le d$; dually, if a submodule-closed subcategory contains $M[a,d]$, it contains $M[\ell,d]$ for every $a \le \ell \le d$. In Figure 1, the indecomposable modules in a quotient-closed subcategory therefore form a prefix $M[a,a], \dots, M[a,b]$ in each row a, and those in a submodule-closed subcategory form a suffix $M[c,d], \dots, M[d,d]$ in each column d.

To apply these interval chains to arbitrary direct sums, we use the following reduction to a single summand, noted by Campanini, Fedele, and Yıldırım [4, Section 4.4] and in the proof of [8, Theorem 4.2].

Lemma 5.3. *Let Y be an indecomposable Λ_n-module and let $X = \bigoplus_i X_i$ be a finite direct sum of Λ_n-modules.*

(1) *If Y is a quotient of X, then Y is a quotient of some X_i.*
(2) *If Y is a submodule of X, then Y is a submodule of some X_i.*

Proof. By (5.1), the submodules of Y form a finite chain under inclusion and therefore have a largest and a smallest member in any nonempty collection.

(1) Let $g\colon X \twoheadrightarrow Y$ be an epimorphism. The images $g(X_i)$ are submodules of Y whose sum is Y, so the largest of them equals Y and the restriction of g to the corresponding summand X_i is an epimorphism.

(2) Let $h\colon Y \hookrightarrow X$ be a monomorphism with components $h_i\colon Y \to X_i$. The kernels $\operatorname{Ker} h_i$ are submodules of Y whose intersection is $\operatorname{Ker} h = 0$, so the smallest of them is zero and the corresponding h_i is a monomorphism. □

We now characterize image-closed subcategories of $\mathsf{mod}\,\Lambda_n$ by their families of support intervals.

Definition 5.4. For a subcategory $\mathcal{C} \subseteq \mathsf{mod}\,\Lambda_n$, write

$$\mathcal{I}(\mathcal{C}) = \{[a,d] : M[a,d] \in \mathsf{ind}\,\mathcal{C}\}$$

for its family of support intervals. A family $\mathcal{D}$ of intervals is *intersection-closed* if

$$I, J \in \mathcal{D}, \quad I \cap J \neq \varnothing \quad \Longrightarrow \quad I \cap J \in \mathcal{D}.$$

The empty family is intersection-closed.

Theorem 5.5. *Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\,\Lambda_n$. Then $\mathcal{C}$ is image-closed if and only if $\mathcal{I}(\mathcal{C})$ is intersection-closed.*

Proof. ($\Rightarrow$): Suppose that $\mathcal{C}$ is image-closed, and let $I, J \in \mathcal{I}(\mathcal{C})$ satisfy $I \cap J \neq \varnothing$. If one of I and J contains the other, then $I \cap J$ is already one of them and belongs to $\mathcal{I}(\mathcal{C})$. Otherwise, after interchanging I and J if necessary, we may write

$$I = [a,b], \qquad J = [c,d], \qquad a < c \le b < d.$$

Since $a \le c \le b \le d$, Proposition 5.2 gives a nonzero morphism $M[c,d] \to M[a,b]$ with image $M[c,b]$. Because $M[c,d], M[a,b] \in \mathcal{C}$ and $\mathcal{C}$ is image-closed, we obtain $M[c,b] \in \mathcal{C}$, so $I \cap J = [c,b] \in \mathcal{I}(\mathcal{C})$.

($\Leftarrow$): Assume that $\mathcal{I}(\mathcal{C})$ is intersection-closed, let $f\colon X \to Y$ be a morphism with $X, Y \in \mathcal{C}$, and let $M[a,b]$ be an indecomposable direct summand of $\operatorname{Im} f$. Composing the canonical epimorphism $X \twoheadrightarrow \operatorname{Im} f$ with the projection onto $M[a,b]$ exhibits $M[a,b]$ as a quotient of X. Writing $X = \bigoplus_i X_i$ with each $X_i \in \mathsf{ind}\,\mathcal{C}$, we see from Lemma 5.3(1) and (5.1) that $M[a,b]$ is a quotient of some $X_i = M[a,d]$ with $d \ge b$, so $[a,d] \in \mathcal{I}(\mathcal{C})$.

Dually, composing the inclusion of $M[a,b]$ into $\operatorname{Im} f$ with $\operatorname{Im} f \hookrightarrow Y$ exhibits $M[a,b]$ as a submodule of Y. Writing $Y = \bigoplus_j Y_j$ with each $Y_j \in \mathsf{ind}\,\mathcal{C}$, we see from Lemma 5.3(2) and (5.1) that $M[a,b]$ is a submodule of some $Y_j = M[c,b]$ with $c \le a$, so $[c,b] \in \mathcal{I}(\mathcal{C})$.

Since $c \le a \le b \le d$, the intersection

$$[a,d] \cap [c,b] = [a,b]$$

is nonempty and therefore belongs to $\mathcal{I}(\mathcal{C})$, which means $M[a,b] \in \mathcal{C}$. Since every indecomposable direct summand of $\operatorname{Im} f$ belongs to $\mathcal{C}$, we conclude that $\operatorname{Im} f \in \mathcal{C}$. □

For $n = 2$, Theorem 5.5 imposes no restriction on a subset of $\mathsf{ind}\,\Lambda_2$.

Example 5.6. The category $\mathsf{mod}\,\Lambda_2$ has three indecomposable modules up to isomorphism: the simple modules $M[1,1]$ and $M[2,2]$, and the projective-injective module $M[1,2]$. Every subset of $\{[1,1],[1,2],[2,2]\}$ is intersection-closed because the only nonempty intersections of distinct intervals in $\{1,2\}$ are inclusions. Hence $\mathcal{L}_{\mathrm{im}}(\Lambda_2)$ is the Boolean lattice of subsets of a three-element set and has $2^3 = 8$ elements.

For $n = 3$, Theorem 5.5 restricts which families of intervals can occur, as the next example shows.

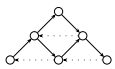
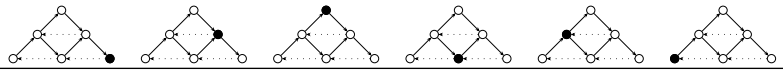
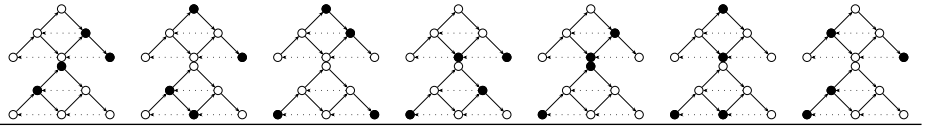
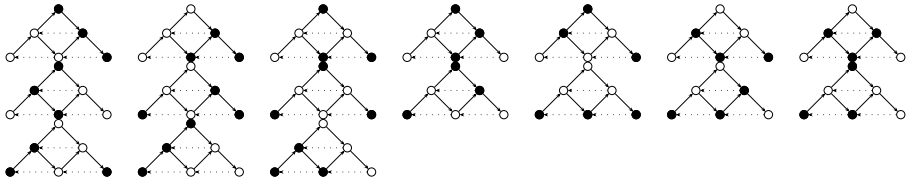
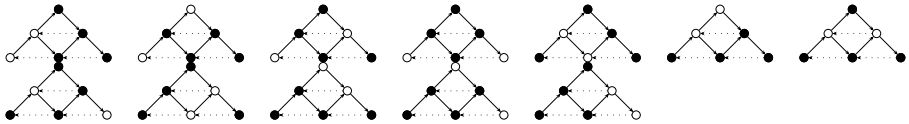
Example 5.7. In $\mathsf{mod}\,\Lambda_3$, the subcategory $\mathcal{C} = \mathsf{add}\{M[1,2], M[2,3]\}$ has support family $\mathcal{I}(\mathcal{C}) = \{[1,2],[2,3]\}$. Since $[1,2] \cap [2,3] = [2,2] \notin \mathcal{I}(\mathcal{C})$, Theorem 5.5 shows that $\mathcal{C}$ is not image-closed; indeed, the nonzero morphism $M[2,3] \to M[1,2]$ has image $M[2,2] \notin \mathcal{C}$. Adjoining $M[2,2]$ produces the image closure $\mathsf{icl}\,\mathcal{C} = \mathsf{add}\{M[1,2], M[2,2], M[2,3]\}$.

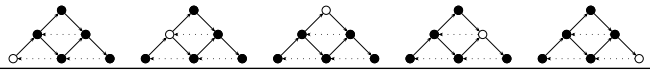
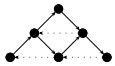
Table 1 lists all image-closed subcategories of $\mathsf{mod}\,\Lambda_3$, grouped by size. Following the convention of Figure 1, each diagram has $M[1,3]$ at the top, $M[2,3]$ and $M[1,2]$ in the middle from left to right, and $M[3,3], M[2,2], M[1,1]$ along the bottom. Among the six indecomposable modules, the only restriction is that including both $M[1,2]$ and $M[2,3]$ requires $M[2,2]$, while the remaining three modules can be chosen independently; thus the table has $(8-1)\cdot 8 = 56$ entries. The numbers of image-closed subcategories of sizes $0, 1, \ldots, 6$ are $1, 6, 14, 17, 12, 5, 1$, so their generating polynomial by size is

$$1 + 6q + 14q^2 + 17q^3 + 12q^4 + 5q^5 + q^6.$$

By Theorem B, this equals $C_4(1,q)$, whose coefficients are also listed in Table 3.

Table 1: All 56 image-closed subcategories of $\mathsf{mod}\,\Lambda_3$, grouped by size.

Size	Subcategories
0	
1	
2	
3	
4	

Size	Subcategories
5	
6	

6. Classification of pretorsion theories by cutoff coordinates

Having characterized the image-closed subcategories of $\mathsf{mod}\,\Lambda_n$ by their support intervals, we now classify the pretorsion theories that realize them. We parametrize the quotient-closed and submodule-closed subcategories of $\mathsf{mod}\,\Lambda_n$ by cutoff coordinates on the rows and columns of Figure 1—where row a fixes the left endpoint of $M[a,d]$ and column d fixes its right endpoint—and characterize which pairs form pretorsion theories (Theorem 6.5).

6.1. **Row and column cutoffs for quotient and submodule closure.** The parametrization in [8, Theorem 4.2(1)–(2)] specializes here to choosing an endpoint on each row or column of Figure 1, as follows.

Proposition 6.1. *There are bijections*

$$\prod_{a=1}^{n}\{a-1,a,\dots,n\} \longrightarrow \mathcal{L}_{\mathrm{quot}}(\Lambda_n), \qquad \beta \longmapsto \mathcal{T}_\beta,$$

$$\prod_{d=1}^{n}\{1,2,\dots,d+1\} \longrightarrow \mathcal{L}_{\mathrm{sub}}(\Lambda_n), \qquad \alpha \longmapsto \mathcal{F}_\alpha,$$

where

$$M[a,d]\in\mathcal{T}_\beta \iff d\le\beta_a, \tag{6.1}$$

$$M[a,d]\in\mathcal{F}_\alpha \iff \alpha_d\le a. \tag{6.2}$$

The values $\beta_a=a-1$ and $\alpha_d=d+1$ specify an empty row and an empty column, respectively. Moreover,

$$\mathcal{T}_\beta\subseteq\mathcal{T}_{\beta'}\iff\beta\le\beta', \qquad \mathcal{F}_\alpha\supseteq\mathcal{F}_{\alpha'}\iff\alpha\le\alpha'.$$

Proof. By (5.1), a set of indecomposable modules is closed under nonzero quotients if and only if its intersection with each row is an initial segment $[a,a],\dots,[a,\beta_a]$, and closed under nonzero submodules if and only if its intersection with each column is a final segment $[\alpha_d,d],\dots,[d,d]$. Lemma 5.3 extends both characterizations to finite direct sums, giving the two bijections, and the inclusion relations follow directly from these segments. □

Definition 6.2. For $\mathcal{T}_\beta$ in Proposition 6.1, the coordinate β_a is the *row cutoff* of row a: it is the right endpoint of the last interval of row a belonging to $\mathcal{T}_\beta$, with $\beta_a=a-1$ when row a contains no object of $\mathcal{T}_\beta$. For $\mathcal{F}_\alpha$, the coordinate α_d is the *column cutoff* of column d: the intervals in column d belonging to $\mathcal{F}_\alpha$ have left endpoints $\alpha_d,\alpha_d+1,\dots,d$, with $\alpha_d=d+1$ when column d contains no object of $\mathcal{F}_\alpha$. We call (β,α) the pair of *cutoff coordinates* of $(\mathcal{T}_\beta,\mathcal{F}_\alpha)$. When $\alpha_d>1$, we call $e=\alpha_d-1$ the *pivot row* in column d: it is the last row excluded from $\mathcal{F}_\alpha$ in that column, and Theorem 6.5 compares it with the row cutoffs of $\mathcal{T}_\beta$.

By Proposition 6.1, the order (2.2) on $\mathsf{PT}(\Lambda_n)$ corresponds to the coordinatewise order on cutoff coordinates (β,α). However, the coordinatewise minimum and maximum of two cutoff pairs in $\mathsf{PT}(\Lambda_n)$ need not define pretorsion theories, so the meet and join of Theorem 3.6 are not given by coordinatewise minimum and maximum in general (see Example 6.9).

Next, we express the trace and the reject of an interval module in terms of cutoff coordinates. For a vector β and $1\le d\le n$, put

$$E_d(\beta)=\{a\le d:\beta_a\ge d\}. \tag{6.3}$$

By (6.1), $E_d(\beta)$ is the set of row indices $a\le d$ such that $M[a,d]\in\mathcal{T}_\beta$.

Lemma 6.3. *Let $X = M[c,d]$. For $\mathcal{T} = \mathcal{T}_\beta$,*

$$t_{\mathcal{T}}(X) = \begin{cases} M[u,d], & u = \min(E_d(\beta) \cap [c,d]), \\ 0, & E_d(\beta) \cap [c,d] = \varnothing. \end{cases}$$

For $\mathcal{F} = \mathcal{F}_\alpha$,

$$r_{\mathcal{F}}(X) = M[b+1,d], \qquad b = \max\big(\{j \in [c,d] : \alpha_j \le c\} \cup \{c-1\}\big).$$

Here $b = d$ gives $M[d+1,d] = 0$ by Definition 5.1, while $b = c-1$ gives $r_{\mathcal{F}}(X) = X$ when no nonzero quotient of X belongs to $\mathcal{F}$.

Proof. By Proposition 2.10(1), $t_{\mathcal{T}}(X)$ is the largest submodule of X belonging to $\mathcal{T}$. By (5.1), the nonzero submodules of $X = M[c,d]$ are the modules $M[a,d]$ with $c \le a \le d$, and $M[a,d] \in \mathcal{T}$ if and only if $a \in E_d(\beta)$. Among these submodules, the largest has the smallest left endpoint u, giving the formula for $t_{\mathcal{T}}(X)$.

Dually, by Proposition 2.10(2), $X/r_{\mathcal{F}}(X)$ is the largest quotient of X belonging to $\mathcal{F}$. The nonzero quotients of X are the modules $M[c,j]$ with $c \le j \le d$, and $M[c,j] \in \mathcal{F}$ if and only if $\alpha_j \le c$. The largest such quotient has the greatest right endpoint b and kernel $r_{\mathcal{F}}(X) = M[b+1,d]$; when no such quotient exists, $X/r_{\mathcal{F}}(X) = 0$ and $r_{\mathcal{F}}(X) = X$, which corresponds to $b = c-1$. □

6.2. **Cutoff pairs defining pretorsion theories.** We characterize the cutoff pairs of pretorsion theories one column at a time, using the earliest row whose cutoff is $d-1$.

For $1 \le d \le n$, define

$$s_d(\beta) = \min\big(\{a : \beta_a = d-1\} \cup \{d+1\}\big). \tag{6.4}$$

Since $\beta_a \ge a-1 \ge d$ for all $a > d$, any row a with $\beta_a = d-1$ satisfies $a \le d$. For $a < d$, the equality $\beta_a = d-1$ means that $M[a,d-1]$ is the last module of row a in $\mathcal{T}_\beta$; for $a = d$, it means that row d is empty in $\mathcal{T}_\beta$. When no row has cutoff $d-1$, we have $s_d(\beta) = d+1$. If $\beta_a \ge d$, then $M[a,d]$ belongs to $\mathcal{T}_\beta$, whereas if $\beta_a < d-1$, row a already ends before column $d-1$. The new obstruction in column d therefore comes from rows with cutoff $\beta_a = d-1$: for such a row, $M[a,d] \notin \mathcal{T}_\beta$ while its quotient $M[a,d-1]$ belongs to $\mathcal{T}_\beta$, which can cause the equality $t(X/r(X)) = t(X)/r(X)$ to fail for $X = M[a,d]$.

The following example exhibits this failure for a row preceding the pivot.

Example 6.4. In $\mathsf{mod}\,\Lambda_2$, take $\beta = (1,2)$ and $\alpha = (1,3)$, so that $\mathcal{T} = \mathcal{T}_\beta = \mathsf{add}\{M[1,1], M[2,2]\}$ and $\mathcal{F} = \mathcal{F}_\alpha = \mathsf{add}\, M[1,1]$. In column 2, the pivot row is $e = \alpha_2 - 1 = 2 \in E_2(\beta)$, but the earlier row $a = 1$ has cutoff $\beta_1 = 1 = 2-1$. Writing $t = t_{\mathcal{T}}$ and $r = r_{\mathcal{F}}$, we find from Lemma 6.3 that $X = M[1,2]$ satisfies

$$t(X) = r(X) = r(t(X)) = M[2,2], \qquad t(X/r(X)) = t(M[1,1]) = M[1,1] \neq 0 = t(X)/r(X),$$

so the second equality in (2.4) fails.

For a pair $(\mathcal{T}_\beta, \mathcal{F}_\alpha)$, if $\alpha_d = e+1 > 1$, then every $M[a,d]$ with $a > e$ belongs to $\mathcal{F}_\alpha$, so both equalities in (2.4) hold automatically on these modules. The criterion in column d therefore concerns only the pivot row e and the rows preceding it.

Theorem 6.5. *The pair $(\mathcal{T}_\beta, \mathcal{F}_\alpha)$ is a pretorsion theory if and only if, for every $1 \le d \le n$, either $\alpha_d = 1$, or the row $e = \alpha_d - 1$ has cutoff at least d while no earlier row has cutoff exactly $d-1$:*

$$\beta_e \ge d, \qquad \beta_a \neq d-1 \quad (1 \le a < e). \tag{6.5}$$

Equivalently,

$$\alpha_d \in \{1\} \cup \{e+1 : e \in E_d(\beta),\ e < s_d(\beta)\}. \tag{6.6}$$

Remark 6.6. For a fixed β, each allowed set in (6.6) depends only on β and d, so the second components $\mathcal{F}_\alpha$ for which $(\mathcal{T}_\beta, \mathcal{F}_\alpha)$ is a pretorsion theory, ordered by reverse inclusion, form the product of these chains of allowed values; see Example 6.7.

Before proving Theorem 6.5, we illustrate the condition (6.5) for $n = 2$ and $n = 3$.

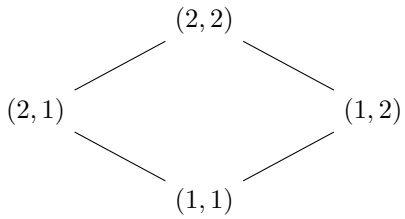
Example 6.7. In $\mathsf{mod}\,\Lambda_2$, fix

$$\beta = (2,1), \qquad \mathcal{T}_\beta = \mathsf{add}\{M[1,1], M[1,2]\},$$

so that row 1 of $\mathcal{T}_\beta$ is full ($\beta_1 = 2$) and row 2 is empty ($\beta_2 = 1$). In column 1, we have $E_1(\beta) = \{1\}$ and $s_1(\beta) = 2$ since no row has $\beta_a = 0$; in column 2, we have $E_2(\beta) = \{1\}$ and $s_2(\beta) = 2$ since $\beta_2 = 1$. Because $1 < s_d(\beta)$ for $d = 1, 2$, condition (6.6) permits $\alpha_d \in \{1, 2\}$ in both columns, yielding four pretorsion theories with first component $\mathcal{T}_\beta$:

α	$\mathcal{F}_\alpha$	$\mathcal{T}_\beta \cap \mathcal{F}_\alpha$
$(1,1)$	$\mathsf{mod}\,\Lambda_2$	$\mathsf{add}\{M[1,1], M[1,2]\}$
$(2,1)$	$\mathsf{add}\{M[1,2], M[2,2]\}$	$\mathsf{add}\, M[1,2]$
$(1,2)$	$\mathsf{add}\{M[1,1], M[2,2]\}$	$\mathsf{add}\, M[1,1]$
$(2,2)$	$\mathsf{add}\, M[2,2]$	0

The product of the two chains $\{1, 2\}$ in Remark 6.6 gives a Boolean square in $\mathsf{PT}(\Lambda_2)$:



Campanini and Fedele [3, Remark 6.3] use the pair with $\alpha = (1, 2)$ to exhibit a pretorsion theory whose second component, $\mathsf{add}\{M[1,1], M[2,2]\}$, is not closed under extensions.

Example 6.8. Figure 2 displays the pretorsion theory in $\mathsf{mod}\,\Lambda_3$ with cutoff coordinates $\beta = (3,2,2)$ and $\alpha = (2,1,2)$. Column 2 has $\alpha_2 = 1$, while columns 1 and 3 have pivot row $e = \alpha_d - 1 = 1$ with $\beta_1 = 3 \geq d$ and no earlier rows, so (6.5) holds in every column.

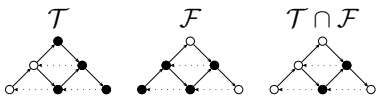


FIGURE 2. The pretorsion theory with $\beta = (3,2,2)$ and $\alpha = (2,1,2)$; black vertices mark membership in each subcategory, and open vertices mark the remaining indecomposables.

Proof of Theorem 6.5. Put $\mathcal{T} = \mathcal{T}_\beta$, $\mathcal{F} = \mathcal{F}_\alpha$, $t = t_\mathcal{T}$, and $r = r_\mathcal{F}$. By Theorem 2.12, $(\mathcal{T}, \mathcal{F})$ is a pretorsion theory if and only if $r(X) = r(t(X))$ and $t(X/r(X)) = t(X)/r(X)$ for every module X.

($\Rightarrow$): Suppose that $(\mathcal{T}, \mathcal{F})$ is a pretorsion theory, and fix a column d with $e = \alpha_d - 1 > 0$. For $X = M[e, d]$, we have $X \notin \mathcal{F}$ by (6.2), whereas every nonzero proper submodule $M[c, d]$ ($e + 1 \leq c \leq d$) belongs to $\mathcal{F}$. If $t(X) \subsetneq X$, then $t(X) \in \mathcal{F}$, so $r(t(X)) = 0 \neq r(X)$ by Proposition 2.10(3), contrary to (2.4). Hence $t(X) = X$, which gives $\beta_e \geq d$.

Suppose next that $1 \leq a < e$ and $\beta_a = d - 1$. For $X = M[a, d]$, we have $X \notin \mathcal{T}$ since $\beta_a = d - 1 < d$, and $X \notin \mathcal{F}$ since $a < e < \alpha_d$. By Proposition 2.10(3) and the equality $r(X) = r(t(X))$ in (2.4), it follows that

$$0 \neq r(X) \subseteq t(X) \subsetneq X.$$

Thus $X/r(X) = M[a, b]$ for some $a \leq b \leq d - 1 = \beta_a$, so $X/r(X) \in \mathcal{T}$. Consequently,

$$t(X/r(X)) = X/r(X) \neq t(X)/r(X),$$

again contradicting (2.4). This proves condition (6.5).

($\Leftarrow$): Suppose that condition (6.5) holds in every column d with $\alpha_d > 1$. Since t and r commute with finite direct sums by Proposition 2.10(4), it suffices to verify (2.4) for each indecomposable

module $X = M[c,d]$. If $X \in \mathcal{F}$, then $t(X) \in \mathcal{F}$ by submodule closure, so $r(X) = r(t(X)) = 0$ and both equalities in (2.4) hold.

Assume now that $X = M[c,d] \notin \mathcal{F}$, so that $\alpha_d > c$. The pivot row $e = \alpha_d - 1$ in column d satisfies $c \le e \le d$ and $\beta_e \ge d$, so $e \in E_d(\beta) \cap [c,d]$. We may therefore put

$$u = \min(E_d(\beta) \cap [c,d]), \qquad b = \max\big(\{\ell \in [c,d] : \alpha_\ell \le c\} \cup \{c-1\}\big),$$

and Lemma 6.3 gives

$$t(X) = M[u,d], \qquad r(X) = M[b+1,d],$$

with $b < d$ because $\alpha_d > c$. We first show that $\beta_i < b$ for $c \le i < u$, ruling out a trace contribution starting before u in the quotient $X/r(X) = M[c,b]$. We then use this bound to verify the reject equality and the trace equality.

Step 1: Excluding earlier rows. If a row before u reached column b, it would end in some column $j-1 < d$, and condition (6.5) in column j would produce an even earlier row with a larger cutoff. We use this observation to show that

$$\beta_i < b \qquad (c \le i < u). \tag{6.7}$$

Suppose for contradiction that (6.7) fails, choose $i \in [c, u-1]$ with β_i maximal, so that $\beta_i \ge b$, and put $j = \beta_i + 1$. By the minimality of u, we have $i \notin E_d(\beta)$ and hence $\beta_i < d$, so $b < j \le d$; since $b \ge c-1$, we also have $j \ge c$. The definition of b then gives $\alpha_j > c$, so the pivot row $e' = \alpha_j - 1$ in column j satisfies $e' \ge c$ and $\beta_{e'} \ge j$. Since $\beta_i = j-1$ and $\beta_{e'} \ge j > \beta_i$, condition (6.5) in column j implies $e' < i$. Thus $e' \in [c, u-1]$ and $\beta_{e'} \ge j > \beta_i$, contradicting the maximality of β_i and proving (6.7).

Step 2: The equality $r(t(X)) = r(X)$. From (6.7), we obtain $u \le b+1$: this is immediate if $u = c$, while if $u > c$, the bound $\beta_a \ge a-1$ at $a = u-1$ gives $u-2 \le \beta_{u-1} < b$. Hence $r(X) = M[b+1,d] \subseteq M[u,d] = t(X)$. To compute $r(t(X))$, note that for $b < j \le d$ we have $\alpha_j > c$; if we also had $\alpha_j \le u$, the pivot row $e' = \alpha_j - 1$ in column j would satisfy $c \le e' < u$ and $\beta_{e'} \ge j > b$, contradicting (6.7). Thus $\alpha_j > u$ for all $b < j \le d$, so no quotient $M[u,j]$ of $t(X) = M[u,d]$ with $j > b$ belongs to $\mathcal{F}$. If $b \ge u$, then $\alpha_b \le c \le u$, so $M[u,b] \in \mathcal{F}$; if $b = u-1$, no nonzero quotient of $M[u,d]$ belongs to $\mathcal{F}$. In both cases, Lemma 6.3 gives

$$r(t(X)) = M[b+1,d] = r(X).$$

Step 3: The equality $t(X/r(X)) = t(X)/r(X)$. We have $X/r(X) = M[c,b]$ (which is zero when $b = c-1$). If $b \ge u$, then (6.7) shows that $i \notin E_b(\beta)$ for $c \le i < u$, whereas $\beta_u \ge d \ge b$ gives $u \in E_b(\beta)$, so $t(M[c,b]) = M[u,b]$ by Lemma 6.3. If $b = u-1$, then (6.7) gives $\beta_i < b$ for every $c \le i \le b$, so $E_b(\beta) \cap [c,b] = \varnothing$ and $t(M[c,b]) = 0$. With the convention $M[u,u-1] = 0$ of Definition 5.1, both cases give

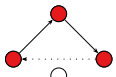
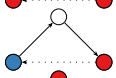
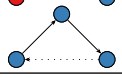
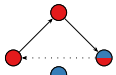
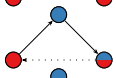
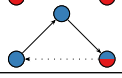
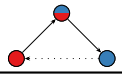
$$t(X/r(X)) = t(M[c,b]) = M[u,b] = t(X)/r(X).$$

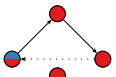
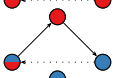
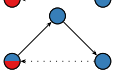
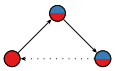
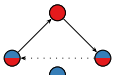
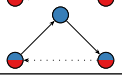
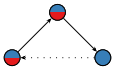
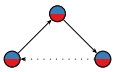
This establishes (2.4), so $(\mathcal{T},\mathcal{F})$ is a pretorsion theory.

Finally, for $\alpha_d = e+1 > 1$, the inequality $\beta_e \ge d$ means that $e \in E_d(\beta)$. Since $s_d(\beta) \notin E_d(\beta)$, the absence of an earlier row $a < e$ with $\beta_a = d-1$ is equivalent to $e < s_d(\beta)$. This proves (6.6). □

6.3. **All pretorsion theories for two vertices.** The complete list for Λ_2 displays both the realizations of each image-closed subcategory and the order on pretorsion theories. Table 2 groups all 17 pretorsion theories by their intersection $\mathcal{Z}$, marking the least and greatest realization from Corollaries 4.5 and 4.7. The eight groups correspond to the eight subcategories in Example 5.6, and Figure 3 gives the Hasse diagram. In each diagram the vertices, from left to right, are $M[2,2], M[1,2], M[1,1]$, with $M[1,2]$ at the apex, as in Figure 1; the membership colours are explained in Table 2.

TABLE 2. All pretorsion theories of Λ_2, grouped by their intersection. Blue: $\mathcal{T}$ only; red: $\mathcal{F}$ only; blue top half and red bottom half: $\mathcal{T} \cap \mathcal{F}$; open: neither. L and G mark the least and greatest realization of each intersection.

No.	$(\mathcal{T}, \mathcal{F})$	β	α	
1		$(0,1)$	$(1,1)$	L
2		$(0,2)$	$(1,3)$	
3		$(1,1)$	$(2,1)$	
4		$(2,1)$	$(2,2)$	
5		$(2,2)$	$(2,3)$	G
6		$(1,1)$	$(1,1)$	L
7		$(2,1)$	$(1,2)$	
8		$(2,2)$	$(1,3)$	G
9		$(2,1)$	$(2,1)$	L, G

No.	$(\mathcal{T}, \mathcal{F})$	β	α	
10		$(0,2)$	$(1,1)$	L
11		$(1,2)$	$(2,1)$	
12		$(2,2)$	$(2,2)$	G
13		$(2,1)$	$(1,1)$	L, G
14		$(1,2)$	$(1,1)$	L
15		$(2,2)$	$(1,2)$	G
16		$(2,2)$	$(2,1)$	L, G
17		$(2,2)$	$(1,1)$	L, G

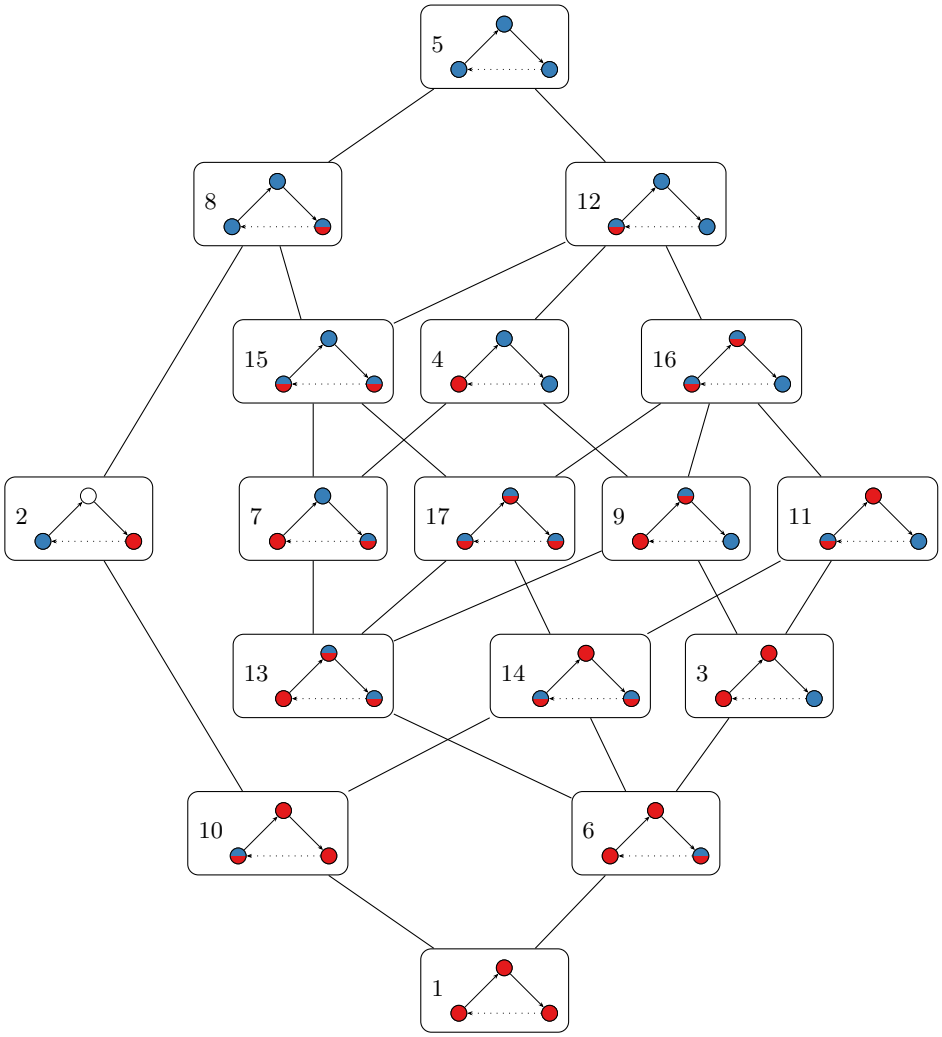


FIGURE 3. The Hasse diagram of $\mathsf{PT}(\Lambda_2)$, increasing upwards; numbers and membership colours are as in Table 2.

The following example computes a meet and a join in Figure 3.

Example 6.9. Let P_i denote entry No. i of Table 2. The coordinatewise minimum of

$$P_2 : \ ((0,2),(1,3)), \qquad P_4 : \ ((2,1),(2,2))$$

is $((0,1),(1,2))$, which violates (6.5) in column 2, since its pivot $e = 1$ has $\beta_1 = 0 < 2$. Their meet is instead P_1, as seen in Figure 3. Similarly, the coordinatewise maximum of P_2 and P_3 : $((1,1),(2,1))$ is $((1,2),(2,3))$, which violates (6.5) in column 2 because its pivot row $e = 2$ is preceded by row 1 with $\beta_1 = 1 = 2 - 1$. Their join is P_5.

7. Enumeration of pretorsion theories by adjoining vertices

In this section, we use the cutoff classification of Theorem 6.5 to prove the first identity of Theorem B (Theorem 7.5) by adjoining one vertex at a time.

Definition 7.1. For $m \geq 1$ and any subcategory $\mathcal{C} \subseteq \mathsf{mod}\,\Lambda_m$, let

$$\rho_m(\mathcal{C}) = |\{a \in \{1,\dots,m\} : M[a,m] \in \mathcal{C}\}| \tag{7.1}$$

be the number of indecomposable modules in $\mathcal{C}$ whose support ends at vertex m, and set $\rho_0(0) = 0$. To each pretorsion theory $(\mathcal{T},\mathcal{F}) \in \mathsf{PT}(\Lambda_m)$, we assign the weight

$$x^{\rho_m(\mathcal{T})} q^{|\,\mathsf{ind}(\mathcal{T}\cap\mathcal{F})|}.$$

For $\mathcal{T} = \mathcal{T}_\beta \subseteq \mathsf{mod}\,\Lambda_m$, the bound $\beta_a \leq m$ gives $\rho_m(\mathcal{T}_\beta) = |E_m(\beta)| = |\{a \in \{1,\dots,m\} : \beta_a = m\}|$. In a fixed column d, the membership rule (6.2) and the pivot row of Definition 6.2 determine how many indecomposable modules in column d belong to $\mathcal{T}_\beta \cap \mathcal{F}_\alpha$. If $\alpha_d = 1$, then every $M[a,d]$ with $a \in E_d(\beta)$ belongs to $\mathcal{T}_\beta \cap \mathcal{F}_\alpha$; if $\alpha_d = e + 1 > 1$, then precisely the modules $M[a,d]$ with $a \in E_d(\beta)$ and $a > e$ belong to $\mathcal{T}_\beta \cap \mathcal{F}_\alpha$. Thus the number of indecomposable modules in column d of $\mathcal{T}_\beta \cap \mathcal{F}_\alpha$ is

$$\begin{cases} |E_d(\beta)|, & \alpha_d = 1, \\ |\{a \in E_d(\beta) : a > e\}|, & \alpha_d = e+1. \end{cases} \tag{7.2}$$

For $n = 1$, the three pretorsion theories of Example 2.6 have cutoff coordinates

$$(\beta,\alpha) = (0,1), \qquad (1,1), \qquad (1,2).$$

In the first pair, $E_1(\beta) = \varnothing$, so $\rho_1(\mathcal{T}) = 0$ and $\mathcal{T} \cap \mathcal{F} = 0$; in the second, $\rho_1(\mathcal{T}) = 1$ and $|\mathsf{ind}(\mathcal{T} \cap \mathcal{F})| = 1$; in the third, $\rho_1(\mathcal{T}) = 1$ and $\mathcal{T} \cap \mathcal{F} = 0$. Their weights are 1, xq, and x, respectively, and, by (1.3), their sum is

$$1 + (1+q)x = B_2(x,q). \tag{7.3}$$

To pass inductively from Λ_{d-1} to Λ_d for $d \geq 1$, identify $\mathsf{mod}\,\Lambda_{d-1}$ with the full subcategory of $\mathsf{mod}\,\Lambda_d$ consisting of modules supported on $\{1,\dots,d-1\}$. The *restriction* of $(\mathcal{T}',\mathcal{F}') \in \mathsf{PT}(\Lambda_d)$ is

$$(\mathcal{T},\mathcal{F}) = (\mathcal{T}' \cap \mathsf{mod}\,\Lambda_{d-1},\ \mathcal{F}' \cap \mathsf{mod}\,\Lambda_{d-1}).$$

It is a pretorsion theory over Λ_{d-1}, and we call $(\mathcal{T}',\mathcal{F}')$ an *extension* of $(\mathcal{T},\mathcal{F})$. To describe all extensions of a fixed $(\mathcal{T},\mathcal{F}) \in \mathsf{PT}(\Lambda_{d-1})$ to Λ_d, let

$$p_1 < \cdots < p_k$$

be the rows $a \in \{1,\dots,d-1\}$ for which $M[p_i,d-1] \in \mathcal{T}$, so that $k = \rho_{d-1}(\mathcal{T})$. In cutoff coordinates, restricting a pretorsion theory $(\mathcal{T}_\beta,\mathcal{F}_\alpha) \in \mathsf{PT}(\Lambda_d)$ to $\mathsf{mod}\,\Lambda_{d-1}$ deletes β_d and α_d and replaces each β_a $(1 \leq a \leq d-1)$ by $\min(\beta_a, d-1)$, by (6.1) and (6.2). Because each row of a quotient-closed subcategory is a prefix, any extension $(\mathcal{T}_\beta,\mathcal{F}_\alpha)$ of $(\mathcal{T},\mathcal{F})$ satisfies $E_d(\beta) \subseteq \{p_1,\dots,p_k,d\}$. We say that row p_i is *continued to column* d when $\beta_{p_i} = d$, that is, when $M[p_i,d] \in \mathcal{T}_\beta$.

Lemma 7.2. *Restriction defines a surjection $\mathsf{PT}(\Lambda_d) \to \mathsf{PT}(\Lambda_{d-1})$. The extensions of a pretorsion theory $(\mathcal{T},\mathcal{F}) \in \mathsf{PT}(\Lambda_{d-1})$ to Λ_d correspond bijectively to the choices of which rows among $p_1,\dots,p_k$ are continued to column d, whether $M[d,d]$ is included in the first component, and a value of α_d permitted by (6.6). Moreover, the indecomposable modules in the intersection of an extension are those in $\mathsf{ind}(\mathcal{T}\cap\mathcal{F})$ together with those in column d belonging to both new components.*

Proof. For any module X supported on $\{1,\dots,d-1\}$, its largest $\mathcal{T}'$-submodule and its largest $\mathcal{F}'$-quotient are also supported on $\{1,\dots,d-1\}$, so its trace and reject with respect to $(\mathcal{T}',\mathcal{F}')$ agree with those with respect to the restricted components $(\mathcal{T},\mathcal{F})$, which remain quotient-closed and submodule-closed. Restriction therefore preserves (2.4) and maps $\mathsf{PT}(\Lambda_d)$ into $\mathsf{PT}(\Lambda_{d-1})$. For any column $j<d$, continuing a row p_i from cutoff $d-1$ to d changes neither the condition $\beta_a \geq j$ nor the condition $\beta_a = j-1$, and therefore leaves both $E_j(\beta)$ and $s_j(\beta)$ unchanged. The new row d also changes neither $E_j(\beta)$ nor $s_j(\beta)$, since $j<d$. By Theorem 6.5, a pair $(\mathcal{T}_\beta,\mathcal{F}_\alpha)\in \mathcal{L}_{\mathrm{quot}}(\Lambda_d)\times\mathcal{L}_{\mathrm{sub}}(\Lambda_d)$ restricting to $(\mathcal{T},\mathcal{F})$ belongs to $\mathsf{PT}(\Lambda_d)$ if and only if α_d satisfies (6.6). In particular, continuing no row p_i, omitting $M[d,d]$, and setting $\alpha_d=1$ yields an extension of $(\mathcal{T},\mathcal{F})$, so restriction is surjective. Distinct choices of continued rows, of $M[d,d]$, and of α_d produce distinct cutoff coordinates (β,α) and hence distinct pretorsion theories. Finally, for $j<d$, we have $\beta_a\geq j$ if and only if $\min(\beta_a,d-1)\geq j$, while α_j is unchanged, so a module $M[a,j]$ with $j<d$ belongs to $\mathcal{T}_\beta\cap\mathcal{F}_\alpha$ if and only if it belongs to $\mathcal{T}\cap\mathcal{F}$. □

We enumerate extensions for arbitrary $k=\rho_{d-1}(\mathcal{T})\geq 0$, using the q-integers

$$[r]_q = 1+q+\cdots+q^{r-1} \qquad (r\geq 1). \tag{7.4}$$

Fix $(\mathcal{T},\mathcal{F})\in\mathsf{PT}(\Lambda_{d-1})$ with $\rho_{d-1}(\mathcal{T})=k$, and sum over all extensions $(\mathcal{T}',\mathcal{F}')\in\mathsf{PT}(\Lambda_d)$ of $(\mathcal{T},\mathcal{F})$:

$$\Phi_{\mathcal{T},\mathcal{F}}(x,q)=\sum_{(\mathcal{T}',\mathcal{F}')} x^{\rho_d(\mathcal{T}')}q^{|\,\mathsf{ind}(\mathcal{T}'\cap\mathcal{F}')|-|\,\mathsf{ind}(\mathcal{T}\cap\mathcal{F})|}. \tag{7.5}$$

By Lemma 7.2, the exponent $|\mathsf{ind}(\mathcal{T}'\cap\mathcal{F}')|-|\mathsf{ind}(\mathcal{T}\cap\mathcal{F})|$ is the number of modules in column d belonging to $\mathcal{T}'\cap\mathcal{F}'$, given by (7.2).

We record two elementary identities used in the recurrences of this section and Section 8. For an indeterminate w, put $U=1+qw$ and let Δ_q act on w. By the binomial theorem,

$$\sum_{v=0}^{r}\binom{r}{v}(qw)^v=\sum_{v=0}^{r}\binom{r}{v}(qw)^{r-v}=U^r \qquad (r\geq 0), \tag{7.6}$$

and since $U-w=1+(q-1)w$ is the denominator of Δ_q in (1.2), the geometric sum formula gives

$$\sum_{i=0}^{\ell} w^iU^{\ell-i}=\frac{U^{\ell+1}-w^{\ell+1}}{U-w}=\Delta_q\big(w^{\ell+1}\big) \qquad (\ell\geq 0). \tag{7.7}$$

The next lemma evaluates $\Phi_{\mathcal{T},\mathcal{F}}(x,q)$ in terms of Δ_q. In [14, p. 132, Lemma 13], Han and Zeng obtain a recurrence by summing weights over the fibres of a restriction map; we use the same method for the restriction of pretorsion theories from Λ_d to Λ_{d-1}.

Lemma 7.3. *For $(\mathcal{T},\mathcal{F})\in\mathsf{PT}(\Lambda_{d-1})$, the sum $\Phi_{\mathcal{T},\mathcal{F}}(x,q)$ in (7.5) depends only on $k=\rho_{d-1}(\mathcal{T})$. Writing $\Phi_k(x,q)$ for this common value, we have*

$$\Phi_k(x,q)=\Delta_q(x^{k+2}). \tag{7.8}$$

Consequently, if

$$f(x)=\sum_{(\mathcal{T},\mathcal{F})\in\mathsf{PT}(\Lambda_{d-1})} x^{\rho_{d-1}(\mathcal{T})}q^{|\,\mathsf{ind}(\mathcal{T}\cap\mathcal{F})|},$$

then

$$\sum_{(\mathcal{T},\mathcal{F})\in\mathsf{PT}(\Lambda_d)} x^{\rho_d(\mathcal{T})}q^{|\,\mathsf{ind}(\mathcal{T}\cap\mathcal{F})|}=\Delta_q\big(x^2f(x)\big). \tag{7.9}$$

Proof. Fix $(\mathcal{T},\mathcal{F})$ and put $k=\rho_{d-1}(\mathcal{T})$. We first compute $\Phi_{\mathcal{T},\mathcal{F}}(x,q)$ by summing over all extensions in column d, and then simplify the resulting expression.

Step 1: Counting the extensions. We compute $\Phi_{\mathcal{T},\mathcal{F}}(x,q)$ by partitioning the extensions according to the first row among $p_1,\dots,p_k$ that is not continued to column d. First, suppose that all k rows $p_1,\dots,p_k$ are continued to column d. If $M[d,d]$ is added to $\mathcal{T}'$, then no row has cutoff $d-1$, so $s_d(\beta)=d+1$; if $M[d,d]$ is not added, then row d is the first row with cutoff $d-1$, so $s_d(\beta)=d$. In either case, every row $e\in E_d(\beta)$ satisfies $e<s_d(\beta)$, so $\alpha_d=e+1$ is allowed

by (6.6). When $M[d,d]$ is added, $|E_d(\beta)| = k+1$ and the $k+2$ choices of α_d yield $k+1, k, \dots, 0$ modules in column d of $\mathcal{T}' \cap \mathcal{F}'$, contributing $x^{k+1}[k+2]_q$; when $M[d,d]$ is not added, $|E_d(\beta)| = k$ and the $k+1$ choices of α_d contribute $x^k[k+1]_q$.

Second, suppose that p_j $(1 \le j \le k)$ is the first row among $p_1, \dots, p_k$ that is not continued to column d, and that v of the $k-j$ later rows $p_{j+1}, \dots, p_k$ are also not continued to column d; there are $\binom{k-j}{v}$ ways to choose these v rows. Here $s_d(\beta) = p_j$, so the permitted values of α_d in (6.6) are $\alpha_d = 1$ and $\alpha_d = e+1$ for the continued rows $e = p_1, \dots, p_{j-1}$ preceding p_j. If $M[d,d]$ is not added, then $|E_d(\beta)| = k-1-v$. The $k-j-v$ continued rows $a > p_j$ satisfy $a > e$ for every allowed choice $\alpha_d = e+1$ and therefore belong to $\mathcal{T}' \cap \mathcal{F}'$ for every choice of α_d, contributing a factor of q^{k-j-v}, while the j choices of α_d leave $j-1, j-2, \dots, 0$ of the rows $p_1, \dots, p_{j-1}$ in $\mathcal{T}' \cap \mathcal{F}'$, contributing $[j]_q$. Thus the contribution when $M[d,d]$ is not added is

$$\binom{k-j}{v} q^{k-j-v}[j]_q x^{k-1-v}.$$

When $M[d,d]$ is added, $d > p_i$ for every $i < j$, so $M[d,d]$ also belongs to both components for every choice of α_d. This increases both $\rho_d(\mathcal{T}')$ and the number of modules in column d of $\mathcal{T}' \cap \mathcal{F}'$ by 1, contributing

$$\binom{k-j}{v} q^{k-j-v+1}[j]_q x^{k-v}.$$

The resulting sum depends only on k, proving the first assertion. Denoting it by $\Phi_k(x,q)$, we obtain

$$\begin{aligned}\Phi_k(x,q) &= x^{k+1}[k+2]_q + x^k[k+1]_q \\ &\quad + \sum_{j=1}^{k}\sum_{v=0}^{k-j}\binom{k-j}{v} q^{k-j-v}[j]_q\bigl(x^{k-1-v} + qx^{k-v}\bigr). \end{aligned} \tag{7.10}$$

Step 2: Evaluating the sum. Put $U = 1+qx$. In the double sum in (7.10), we factor $x^{k-1-v} + qx^{k-v} = Ux^{j-1}x^{k-j-v}$ and apply (7.6) with $w = x$ and $r = k-j$ to obtain

$$\begin{aligned}&\sum_{j=1}^{k}[j]_q(1+qx)\sum_{v=0}^{k-j}\binom{k-j}{v} q^{k-j-v}x^{k-1-v} \\ &\qquad = U\sum_{j=1}^{k}[j]_q x^{j-1}U^{k-j}.\end{aligned}$$

Consequently,

$$\Phi_k = x^{k+1}[k+2]_q + x^k[k+1]_q + U\sum_{j=1}^{k}[j]_q x^{j-1}U^{k-j}. \tag{7.11}$$

For $k = 0$, equation (7.11) gives $\Phi_0 = 1 + (1+q)x$. For $k \ge 1$, subtracting $U\Phi_{k-1}$ from (7.11) cancels the terms with $1 \le j < k$ in the sum as well as the $j = k$ term $Ux^{k-1}[k]_q$, leaving

$$\begin{aligned}\Phi_k - U\Phi_{k-1} &= x^{k+1}[k+2]_q + x^k[k+1]_q - Ux^k[k+1]_q \\ &= x^{k+1}\bigl([k+2]_q - q[k+1]_q\bigr) = x^{k+1}.\end{aligned}$$

Thus Φ_k satisfies the recurrence

$$\Phi_k = U\Phi_{k-1} + x^{k+1} \qquad (k \ge 1). \tag{7.12}$$

On the other hand, by (7.7), $\Delta_q(x^{k+2}) = (U^{k+2} - x^{k+2})/(U-x)$ equals $1 + (1+q)x$ for $k = 0$ and satisfies the same recurrence:

$$U \cdot \frac{U^{k+1} - x^{k+1}}{U-x} + x^{k+1} = \frac{U^{k+2} - Ux^{k+1} + x^{k+1}(U-x)}{U-x} = \frac{U^{k+2} - x^{k+2}}{U-x}.$$

Hence $\Phi_k(x,q) = \Delta_q(x^{k+2})$ for every $k \ge 0$.

By Lemma 7.2, every pretorsion theory $(\mathcal{T}', \mathcal{F}') \in \mathsf{PT}(\Lambda_d)$ is an extension of a unique $(\mathcal{T}, \mathcal{F}) \in \mathsf{PT}(\Lambda_{d-1})$. Summing $q^{|\operatorname{ind}(\mathcal{T}\cap\mathcal{F})|}\Phi_{\rho_{d-1}(\mathcal{T})}(x,q)$ over all $(\mathcal{T}, \mathcal{F}) \in \mathsf{PT}(\Lambda_{d-1})$ and using (7.8) together with the $\mathbb{Z}[q]$-linearity of Δ_q yields (7.9). □

Example 7.4. By Lemma 7.3 and (7.3), the weighted count for $n = 2$ is $\Delta_q(x^2 B_2(x,q)) = B_3(x,q)$. Applying (7.7) with $U = 1 + qx$ gives

$$\Delta_q(x^2) = U + x = 1 + (1+q)x, \qquad \Delta_q(x^3) = U^2 + Ux + x^2 = 1 + (2q+1)x + (q^2+q+1)x^2,$$

and hence

$$B_3(x,q) = \Delta_q\big(x^2 + (1+q)x^3\big) = (2+q) + 2(1+q)^2 x + (1+q)(q^2+q+1)x^2.$$

Setting $x = 1$ yields

$$B_3(1,q) = 5 + 7q + 4q^2 + q^3,$$

so among the 17 pretorsion theories in $\operatorname{mod}\Lambda_2$, there are 5 torsion pairs (intersection size 0), 7 pretorsion theories with intersection of size 1, 4 with intersection of size 2, and 1 with intersection of size 3, as can also be read from Table 2.

Iterating Lemma 7.3 identifies the weighted generating polynomial of $\mathsf{PT}(\Lambda_n)$ with the q-Gandhi polynomial of the first kind.

Theorem 7.5. *Let $B_m(x,q)$ be the polynomials of (1.3). For every $n \geq 0$,*

$$\sum_{(\mathcal{T},\mathcal{F})\in\mathsf{PT}(\Lambda_n)} x^{\rho_n(\mathcal{T})} q^{|\operatorname{ind}(\mathcal{T}\cap\mathcal{F})|} = B_{n+1}(x,q). \tag{7.13}$$

In particular,

$$\sum_{(\mathcal{T},\mathcal{F})\in\mathsf{PT}(\Lambda_n)} q^{|\operatorname{ind}(\mathcal{T}\cap\mathcal{F})|} = B_{n+1}(1,q). \tag{7.14}$$

Proof. We proceed by induction on $n \geq 0$. For $n = 0$, we have $\operatorname{mod}\Lambda_0 = 0$, so $\mathsf{PT}(\Lambda_0)$ consists of the single pair $(0,0)$, which has weight $1 = B_1(x,q)$ by Definition 7.1. For $n \geq 1$, Lemma 7.3 with $d = n$ and the induction hypothesis show that the left-hand side of (7.13) equals $\Delta_q(x^2 B_n(x,q)) = B_{n+1}(x,q)$ by (1.3). Evaluating (7.13) at $x = 1$ gives (7.14). □

Remark 7.6. By Proposition 2.5, the constant term of $B_{n+1}(1,q)$ counts the torsion pairs in $\operatorname{mod}\Lambda_n$. By Corollary 3.9, these pairs are in bijection with torsion classes. Thomas [21, section "Torsion classes in rep A_n"] identifies their poset with the Tamari lattice on bracketings of $n+2$ letters, so this constant term is the Catalan number $\mathrm{Cat}_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$. At the opposite extreme, the coefficient of $q^{n(n+1)/2}$ is 1, contributed only by $(\operatorname{mod}\Lambda_n, \operatorname{mod}\Lambda_n)$. Unlike size on $\mathcal{L}_{\mathrm{im}}(\Lambda_n)$ (Corollary 4.14(3)), however, the intersection size $|\operatorname{ind}(\mathcal{T}\cap\mathcal{F})|$ is not a rank function on the lattice $\mathsf{PT}(\Lambda_n)$: already for $n = 1$, the three pretorsion theories of Example 2.6 form a chain $(0, \operatorname{mod}\Lambda_1) < (\operatorname{mod}\Lambda_1, \operatorname{mod}\Lambda_1) < (\operatorname{mod}\Lambda_1, 0)$ in the order (2.2), with intersection sizes $0, 1, 0$. Table 3 lists the coefficients of $B_{n+1}(1,q)$ and of the rank-generating polynomial $C_{n+1}(1,q)$ from Theorem 8.5 for $n \leq 4$.

TABLE 3. Coefficients in ascending powers of q.

n	$B_{n+1}(1,q)$	$C_{n+1}(1,q)$
0	1	1
1	2, 1	1, 1
2	5, 7, 4, 1	1, 3, 3, 1
3	14, 36, 45, 35, 18, 6, 1	1, 6, 14, 17, 12, 5, 1
4	42, 165, 330, 440, 433, 330, 198, 93, 33, 8, 1	1, 10, 40, 90, 132, 137, 105, 60, 25, 7, 1

The identity $B_m(1,1) = g_{2m+2}$, conjectured by Gandhi [13], was proved by Carlitz [5] and by Riordan and Stein [20]; see also Barsky [2, Section 1, Lemma 1] and Han and Zeng [14, p. 120, Equation (1)] for the indexing used here. Thus (7.14) at $q = 1$ gives

$$|\mathsf{PT}(\Lambda_n)| = g_{2n+4}. \tag{7.15}$$

8. Enumeration of image-closed subcategories by adjoining vertices

In this section, we prove the second identity of Theorem B (Theorem 8.5) by adjoining one vertex at a time, parallel to Section 7.

Identifying $\mathsf{mod}\,\Lambda_{d-1}$ with the subcategory of Λ_d-modules supported on $\{1,\dots,d-1\}$, we define the *restriction* of a subcategory $\mathcal{C}' \subseteq \mathsf{mod}\,\Lambda_d$ to be $\mathcal{C}' \cap \mathsf{mod}\,\Lambda_{d-1}$. Since the image of any morphism between modules supported on $\{1,\dots,d-1\}$ is again supported on $\{1,\dots,d-1\}$, restriction maps $\mathcal{L}_{\mathrm{im}}(\Lambda_d)$ into $\mathcal{L}_{\mathrm{im}}(\Lambda_{d-1})$. An *extension* of $\mathcal{C} \in \mathcal{L}_{\mathrm{im}}(\Lambda_{d-1})$ to Λ_d is an image-closed subcategory $\mathcal{C}' \in \mathcal{L}_{\mathrm{im}}(\Lambda_d)$ whose restriction to $\mathsf{mod}\,\Lambda_{d-1}$ is $\mathcal{C}$; it is obtained by adjoining a subset of the indecomposable modules $M[a,d]$ $(1 \le a \le d)$ in column d.

By Theorem 5.5, we may work with the intersection-closed families of intervals in Definition 5.4. Whether a new interval $[a,d]$ can be adjoined to an intersection-closed family $\mathcal{D}$ in $\{1,\dots,d-1\}$ without generating new intervals in $\{1,\dots,d-1\}$ is governed by the following condition on the left endpoint a.

Definition 8.1. Let $\mathcal{D}$ be an intersection-closed family of intervals in $\{1,\dots,d-1\}$. Define its set of *admissible starts for vertex d* by

$$\mathsf{Adm}_d(\mathcal{D}) = \{a \in \{1,\dots,d\} : [b,\ell] \in \mathcal{D},\ b \le a \le \ell \Longrightarrow [a,\ell] \in \mathcal{D}\}. \tag{8.1}$$

Thus $a \in \mathsf{Adm}_d(\mathcal{D})$ if and only if every interval in $\mathcal{D}$ containing a has its suffix beginning at a in $\mathcal{D}$. In particular, $d \in \mathsf{Adm}_d(\mathcal{D})$ always holds vacuously because no interval in $\mathcal{D}$ contains d.

The following example illustrates how an inadmissible start arises.

Example 8.2. Let $d = 3$. For the intersection-closed family

$$\mathcal{D} = \{[1,2]\}$$

in $\{1,2\}$, we have $\mathsf{Adm}_3(\mathcal{D}) = \{1,3\}$: the start $a = 2$ is not admissible because $2 \in [1,2] \in \mathcal{D}$ whereas $[2,2] \notin \mathcal{D}$. Thus one may adjoin $[1,3]$, $[3,3]$, or both while preserving intersection closure and keeping the restriction to $\{1,2\}$ equal to $\mathcal{D}$, whereas adjoining $[2,3]$ would require the intersection

$$[1,2] \cap [2,3] = [2,2]$$

to belong to the family, altering its restriction to $\{1,2\}$. By contrast, for $\mathcal{D} = \{[1,2],[2,2]\}$, all three starts $1,2,3$ are admissible.

8.1. Extensions of an intersection-closed interval family.

We first parametrize all subcategories of $\mathsf{mod}\,\Lambda_d$ with a prescribed restriction to $\mathsf{mod}\,\Lambda_{d-1}$, and then characterize which of them are image-closed and compute the admissible starts of the resulting family.

Construction 8.3. Let $\mathcal{C} \in \mathcal{L}_{\mathrm{im}}(\Lambda_{d-1})$, put $\mathcal{D} = \mathcal{I}(\mathcal{C})$, and choose a subset $J \subseteq \{1,\dots,d\}$. Define a subcategory $\mathcal{C}_J \subseteq \mathsf{mod}\,\Lambda_d$ by

$$\mathcal{I}(\mathcal{C}_J) = \mathcal{D}_J = \mathcal{D} \cup \{[a,d] : a \in J\}. \tag{8.2}$$

Every subcategory of $\mathsf{mod}\,\Lambda_d$ whose restriction to $\mathsf{mod}\,\Lambda_{d-1}$ is $\mathcal{C}$ has the form $\mathcal{C}_J$ for a unique subset $J \subseteq \{1,\dots,d\}$.

The next lemma characterizes these extensions and computes the admissible starts for the next vertex. This update will let us iterate the count using only the number of admissible starts.

Lemma 8.4. *Let $\mathcal{D}$ be an intersection-closed family in $\{1,\dots,d-1\}$ and write $\mathsf{Adm}_d(\mathcal{D}) = \{c_1 < \cdots < c_k\}$.*

(1) *$\mathcal{D}_J$ is intersection-closed if and only if $J \subseteq \mathsf{Adm}_d(\mathcal{D})$.*

(2) *Suppose* $J \subseteq \mathsf{Adm}_d(\mathcal{D})$. *If* $J = \varnothing$, *then*

$$\mathsf{Adm}_{d+1}(\mathcal{D}_J) = \mathsf{Adm}_d(\mathcal{D}) \cup \{d+1\}. \tag{8.3}$$

If $J \neq \varnothing$ *and* $\min J = c_j$, *then*

$$\mathsf{Adm}_{d+1}(\mathcal{D}_J) = \{c_1, \dots, c_j\} \cup \big(J \cap \{c_{j+1}, \dots, c_k\}\big) \cup \{d+1\}. \tag{8.4}$$

Proof. (1) The nonempty intersection of two intervals in $\mathcal{D}$ already belongs to $\mathcal{D}$, and the intersection of two intervals $[a, d]$ and $[a', d]$ ending at d is $[\max(a, a'), d]$, which belongs to $\mathcal{D}_J$ whenever $a, a' \in J$. For $[b, \ell] \in \mathcal{D}$ and $[a, d]$ with $a \in J$, their intersection is nonempty if and only if $a \leq \ell$, and when $a \leq b$ it equals $[b, \ell] \in \mathcal{D}$; when $b \leq a \leq \ell$, it equals

$$[b, \ell] \cap [a, d] = [a, \ell],$$

which belongs to $\mathcal{D}_J$ if and only if $[a, \ell] \in \mathcal{D}$. Thus $\mathcal{D}_J$ is intersection-closed if and only if every $a \in J$ satisfies the condition in (8.1), that is, $J \subseteq \mathsf{Adm}_d(\mathcal{D})$.

(2) Recall that $c_k = d \in \mathsf{Adm}_d(\mathcal{D})$. If $a \in \{1, \dots, d\} \setminus \mathsf{Adm}_d(\mathcal{D})$, there exists $[b, \ell] \in \mathcal{D} \subseteq \mathcal{D}_J$ with $b \leq a \leq \ell$ and $[a, \ell] \notin \mathcal{D}$, and since $\ell \leq d-1$ we still have $[a, \ell] \notin \mathcal{D}_J$; hence $a \notin \mathsf{Adm}_{d+1}(\mathcal{D}_J)$, so $\mathsf{Adm}_{d+1}(\mathcal{D}_J) \subseteq \mathsf{Adm}_d(\mathcal{D}) \cup \{d+1\}$. If $J = \varnothing$, then $\mathcal{D}_J = \mathcal{D}$ and every $c_i \in \mathsf{Adm}_d(\mathcal{D})$ remains admissible for vertex $d+1$, proving (8.3).

Suppose now that $J \neq \varnothing$ and $\min J = c_j$. Every old interval containing c_i already has its suffix beginning at c_i in $\mathcal{D} \subseteq \mathcal{D}_J$, since $c_i \in \mathsf{Adm}_d(\mathcal{D})$. It remains to check the new intervals. For $1 \leq i \leq j$, any new interval $[a, d]$ $(a \in J)$ containing c_i satisfies $a \leq c_i \leq c_j = \min J \leq a$, so $a = c_i$ and the suffix $[c_i, d]$ belongs to $\mathcal{D}_J$; thus $c_1, \dots, c_j \in \mathsf{Adm}_{d+1}(\mathcal{D}_J)$. For $j < i \leq k$, the new interval $[c_j, d] \in \mathcal{D}_J$ contains c_i, so c_i belongs to $\mathsf{Adm}_{d+1}(\mathcal{D}_J)$ if and only if its suffix $[c_i, d]$ belongs to $\mathcal{D}_J$, that is, $c_i \in J$. Finally, $d+1$ belongs to $\mathsf{Adm}_{d+1}(\mathcal{D}_J)$ vacuously. This proves (8.4). □

Figure 4 illustrates the update rule (8.4): any admissible start c_i after $c_j = \min J$ that is not included in J is no longer admissible at the next step.

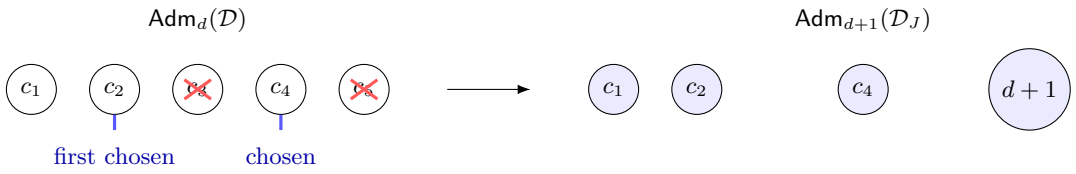


FIGURE 4. The update of admissible starts when $J = \{c_2, c_4\}$ and $c_5 = d$. Red crosses mark starts that cease to be admissible.

8.2. **A recurrence counting intersection-closed families.** We enumerate interval families by their size and number of admissible starts, then recover the statistic ρ_n by substitution.

For $m \geq 0$, define

$$Q_m(z, q) = \sum_{\mathcal{D}} q^{|\mathcal{D}|} z^{|\mathsf{Adm}_{m+1}(\mathcal{D})|}, \tag{8.5}$$

where the sum ranges over all intersection-closed families $\mathcal{D}$ of intervals in $\{1, \dots, m\}$. For $m = 0$, the only family is $\mathcal{D} = \varnothing$, which has $\mathsf{Adm}_1(\varnothing) = \{1\}$, so $Q_0(z, q) = z$. Adjoining vertex n to families on $\{1, \dots, n-1\}$ will express the generating polynomial for $\mathcal{L}_{\mathrm{im}}(\Lambda_n)$ as $Q_{n-1}(1 + qx, q) = C_{n+1}(x, q)$.

Fix $d \geq 1$ and an intersection-closed family $\mathcal{D}$ in $\{1, \dots, d-1\}$ with $|\mathsf{Adm}_d(\mathcal{D})| = k$. By Lemma 8.4, the extensions $\mathcal{D}_J$ of $\mathcal{D}$ to $\{1, \dots, d\}$ correspond to subsets $J \subseteq \mathsf{Adm}_d(\mathcal{D})$. The choice $J = \varnothing$ has $|\mathcal{D}_\varnothing| = |\mathcal{D}|$ and $|\mathsf{Adm}_{d+1}(\mathcal{D}_\varnothing)| = k + 1$, contributing $q^{|\mathcal{D}|} z^{k+1}$ to $Q_d(z, q)$. For $J \neq \varnothing$, let $\min J = c_j$ $(1 \leq j \leq k)$ and choose v of the $k - j$ later elements $c_{j+1}, \dots, c_k$ to belong to J;

there are $\binom{k-j}{v}$ such subsets J, each having $|J| = 1+v$ and $|\mathsf{Adm}_{d+1}(\mathcal{D}_J)| = j+v+1$ by (8.4). Summing over all $J \subseteq \mathsf{Adm}_d(\mathcal{D})$ shows that the extensions of $\mathcal{D}$ contribute $q^{|\mathcal{D}|}$ times

$$\begin{aligned} z^{k+1} + \sum_{j=1}^{k}\sum_{v=0}^{k-j}\binom{k-j}{v} q^{1+v} z^{j+v+1} &= z^{k+1} + qz^2 \sum_{j=1}^{k} z^{j-1}(1+qz)^{k-j} \\ &= z^{k+1} + qz^2 \Delta_q(z^k), \end{aligned} \tag{8.6}$$

where the first equality uses (7.6) with $w = z$ and $r = k-j$, and the second uses (7.7) with $w = z$ and $\ell = k-1$ (with Δ_q acting on z). Summing (8.6) over all $\mathcal{D}$ in $\{1, \dots, d-1\}$ gives the recurrence

$$Q_d(z,q) = zQ_{d-1}(z,q) + qz^2 \Delta_q Q_{d-1}(z,q). \tag{8.7}$$

Recall from Definition 7.1 that $\rho_n(\mathcal{C})$ counts the indecomposable modules in $\mathcal{C}$ whose support ends at n. By Construction 8.3, Lemma 8.4(1), and Theorem 5.5, every $\mathcal{C}' \in \mathcal{L}_{\mathrm{im}}(\Lambda_n)$ with $n \geq 1$ is of the form $\mathcal{C}_J$ for a unique $\mathcal{C} \in \mathcal{L}_{\mathrm{im}}(\Lambda_{n-1})$ and a unique subset $J \subseteq \mathsf{Adm}_n(\mathcal{I}(\mathcal{C}))$, with $\rho_n(\mathcal{C}_J) = |J|$ and $|\mathsf{ind}\,\mathcal{C}_J| = |\mathsf{ind}\,\mathcal{C}| + |J|$. Summing over $\mathcal{C} \in \mathcal{L}_{\mathrm{im}}(\Lambda_{n-1})$ and $J \subseteq \mathsf{Adm}_n(\mathcal{I}(\mathcal{C}))$ therefore yields

$$\sum_{\mathcal{C}' \in \mathcal{L}_{\mathrm{im}}(\Lambda_n)} x^{\rho_n(\mathcal{C}')} q^{|\,\mathsf{ind}\,\mathcal{C}'|} = \sum_{\mathcal{C} \in \mathcal{L}_{\mathrm{im}}(\Lambda_{n-1})} q^{|\,\mathsf{ind}\,\mathcal{C}|} \sum_{J \subseteq \mathsf{Adm}_n(\mathcal{I}(\mathcal{C}))} (qx)^{|J|}.$$

Since the inner sum equals $(1+qx)^{|\mathsf{Adm}_n(\mathcal{I}(\mathcal{C}))|}$, comparing with (8.5) gives

$$\sum_{\mathcal{C} \in \mathcal{L}_{\mathrm{im}}(\Lambda_n)} x^{\rho_n(\mathcal{C})} q^{|\,\mathsf{ind}\,\mathcal{C}|} = Q_{n-1}(1+qx, q). \tag{8.8}$$

In the proof of the following theorem, we verify that substituting $z = 1+qx$ in (8.7) gives the defining recurrence (1.4) for the q-Gandhi polynomials of the second kind.

Theorem 8.5. *Let $C_m(x,q)$ be defined by (1.4). For every $n \geq 0$,*

$$\sum_{\mathcal{C} \in \mathcal{L}_{\mathrm{im}}(\Lambda_n)} x^{\rho_n(\mathcal{C})} q^{|\,\mathsf{ind}\,\mathcal{C}|} = C_{n+1}(x,q). \tag{8.9}$$

In particular, the rank-generating polynomial of the graded lattice $\mathcal{L}_{\mathrm{im}}(\Lambda_n)$ is

$$\sum_{\mathcal{C} \in \mathcal{L}_{\mathrm{im}}(\Lambda_n)} q^{|\,\mathsf{ind}\,\mathcal{C}|} = C_{n+1}(1,q). \tag{8.10}$$

Proof. We first prove by induction on $d \geq 1$ that

$$Q_{d-1}(1+qx, q) = C_{d+1}(x,q). \tag{8.11}$$

For $d = 1$, both sides equal $1+qx$ since $Q_0(z,q) = z$ and $C_2(x,q) = (1+qx)\Delta_q(x) = 1+qx$.

For the induction step, assume (8.11) for d, and put $U = 1+qx$ and $V = 1+(q-1)x = U - x$. Applying (8.11) at x and at U gives

$$Q_{d-1}(U,q) = C_{d+1}(x,q), \qquad Q_{d-1}(1+qU,q) = C_{d+1}(U,q).$$

Since $1+(q-1)U = qV$, evaluating $\Delta_q Q_{d-1}(z,q)$ at $z = U$ yields

$$q\,\big(\Delta_q Q_{d-1}\big)(U,q) = q\,\frac{Q_{d-1}(1+qU,q) - Q_{d-1}(U,q)}{1+(q-1)U} = \frac{C_{d+1}(U,q) - C_{d+1}(x,q)}{V}.$$

Substituting $z = U$ into (8.7) and using $V - U = -x$, we obtain

$$\begin{aligned} Q_d(U,q) &= UC_{d+1}(x,q) + U^2 \frac{C_{d+1}(U,q) - C_{d+1}(x,q)}{V} \\ &= \frac{U^2 C_{d+1}(U,q) - UxC_{d+1}(x,q)}{V} \\ &= U\Delta_q\big(xC_{d+1}(x,q)\big) = C_{d+2}(x,q), \end{aligned} \tag{8.12}$$

completing the induction for (8.11).

For $n \geq 1$, combining (8.8) with (8.11) at $d = n$ gives (8.9); for $n = 0$, both sides of (8.9) equal 1. Setting $x = 1$ gives (8.10), where $|\mathsf{ind}\,\mathcal{C}|$ is the rank function of $\mathcal{L}_{\mathrm{im}}(\Lambda_n)$ by Corollary 4.14(3). □

Grouping pretorsion theories by their intersection $\mathcal{Z} = \mathcal{T} \cap \mathcal{F}$ relates $B_{n+1}(1,q)$ and $C_{n+1}(1,q)$ through the number $|\mathsf{PT}_{\mathcal{Z}}(\Lambda_n)|$ of realizations of each $\mathcal{Z} \in \mathcal{L}_{\mathrm{im}}(\Lambda_n)$:

$$B_{n+1}(1,q) = \sum_{\mathcal{Z} \in \mathcal{L}_{\mathrm{im}}(\Lambda_n)} |\mathsf{PT}_{\mathcal{Z}}(\Lambda_n)|\, q^{|\operatorname{ind} \mathcal{Z}|}, \qquad C_{n+1}(1,q) = \sum_{\mathcal{Z} \in \mathcal{L}_{\mathrm{im}}(\Lambda_n)} q^{|\operatorname{ind} \mathcal{Z}|}.$$

In Table 2, for instance, the eight image-closed subcategories for $n = 2$ have $5, 3, 1, 3, 1, 2, 1, 1$ realizations.

We also check Theorem 8.5 directly for $n = 2$, recovering the count of Example 5.6.

Example 8.6. By Example 5.6, every subset of the three intervals $[1,1], [1,2], [2,2]$ in $\{1,2\}$ is intersection-closed. Because $[1,2]$ and $[2,2]$ end at vertex 2 while $[1,1]$ does not, choosing each interval independently yields

$$\sum_{\mathcal{C} \in \mathcal{L}_{\mathrm{im}}(\Lambda_2)} x^{\rho_2(\mathcal{C})} q^{|\operatorname{ind} \mathcal{C}|} = (1+q)(1+qx)^2.$$

From the recurrence (1.4), we have $C_2(x,q) = 1 + qx$ and, using $\Delta_q(x^2) = 1 + (1+q)x$,

$$C_3(x,q) = (1+qx)\Delta_q\big(x + qx^2\big) = (1+qx)\big(1 + q + q(1+q)x\big) = (1+q)(1+qx)^2.$$

At $x = 1$, this specializes to $C_3(1,q) = (1+q)^3$.

Using the classical identity $C_m(1,1) = H_{2m+1}$ [15, pp. 63–64, Equation (2)], equation (8.10) at $q = 1$ gives

$$|\mathcal{L}_{\mathrm{im}}(\Lambda_n)| = H_{2n+3}. \tag{8.13}$$

Remark 8.7. Han and Zeng [15, Section 4, Equation (17)] prove that $(1+q)^n$ divides $C_{n+1}(1,q)$ and define the polynomial $\bar{c}_{n+1}(q) = C_{n+1}(1,q)/(1+q)^n$. Feigin [12, Corollary 2.4] identifies $\bar{c}_{n+1}(q) = q^{n(n-1)/2} h_n(q^{-1})$, where h_n is the Poincaré polynomial of the degenerate complete flag variety for $\mathbb{C}^n$ (with cohomological degrees divided by two), and Cerulli Irelli, Feigin, and Reineke [6, Proposition 2.7] realize these degenerate flag varieties as quiver Grassmannians for linearly oriented quivers of type A.

Remark 8.8. The counts of pretorsion theories and image-closed subcategories depend on the orientation of the quiver. The table below was computed over $\mathbb{F}_2$ from the Hom spaces and images of interval modules by testing Theorem 2.12 on every pair of a quotient-closed and a submodule-closed subcategory. By Theorem 4.4, the distinct intersections of these pretorsion theories give the $|\mathcal{L}_{\mathrm{im}}|$ column, while by Proposition 2.5 the pairs with zero intersection give the column of torsion pairs. The resulting counts hold over every field, for the reason given below. Each word in L and R records the orientation of the arrows between consecutive vertices $1, \dots, n$, with R denoting $i \to i+1$ and L denoting $i \leftarrow i+1$; orientations with equal counts are grouped together.

n	Orientations	$\lvert\mathsf{PT}\rvert$	$\lvert\mathcal{L}_{\mathrm{im}}\rvert$	Torsion pairs
1	empty word	3	2	2
2	L, R	17	8	5
3	LL, RR	155	56	14
	LR, RL	163	64	14
4	LLL, RRR	2073	608	42
	LLR, LRR, RLL, RRL	2297	784	42
	LRL, RLR	2441	880	42

For any orientation of A_n and any base field, the indecomposable modules are the thin interval representations, with a one-dimensional vector space at each vertex of their support and identity maps along the arrows of the support. A nonzero morphism between two such representations is a scalar multiple of the identity on the overlap of their supports; whether such a morphism exists depends only on the arrow orientations, and its image and kernel are coordinate subspaces. Consequently, the trace and the reject of any set of indecomposables, evaluated on an interval module, are sums and intersections of fixed coordinate subspaces, and both they and the resulting submodules and quotients (which decompose into thin interval modules) are independent of the field. By

Proposition 2.10(3), membership in the quotient closure of a set is detected by whether its trace is the whole module, and membership in its submodule closure by whether its reject vanishes; moreover, Proposition 2.10(4) reduces the trace–reject equalities on arbitrary modules to those on interval modules. Thus the quotient-closed and submodule-closed subcategories, the pairs satisfying Theorem 2.12, and—by Theorem 4.4—their intersections, the image-closed subcategories, are all independent of the field.

Extending the $\mathbb{F}_2$ computation to all orientations with $n \leq 6$ shows that, in this finite range, the linearly oriented quivers have the smallest numbers of pretorsion theories and image-closed subcategories, whereas the alternating orientations have the largest. In contrast, for every n and every orientation of A_n, the number of torsion pairs is Cat_{n+1}: over an algebraically closed field this follows from the bijection between torsion classes and noncrossing partitions [16, Theorem 1.1], and the argument above showing independence of the field extends the count to arbitrary fields.

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