hexagon:2611.00205v2

LUSZTIG'S CONJECTURES FOR B_n, $\widetilde{B}_n$, AND $\widetilde{C}_n$

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Abstract. We prove Lusztig's conjectures P1–P15 for every positive integer-valued weight function on the finite Coxeter group of type B_n and the nonextended affine Weyl groups of types $\widetilde{B}_n$ and $\widetilde{C}_n$.

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1. Introduction

Kazhdan and Lusztig introduced the canonical basis of the equal-parameter Hecke algebra in their study of Weyl group representations [11]. The coefficients of multiplication in this basis define left, right, and two-sided preorders on the Coxeter group. Passing to an equivalence class for the left preorder gives a subquotient of the left regular module. In type A, the Robinson–Schensted correspondence describes the cells, linking these subquotients to the representation theory of symmetric groups. In his work on left cells in Weyl groups, Lusztig allowed different parameters on the simple reflections and studied the resulting representations in connection with primitive ideals [12].

Date: November 8, 2026.
2020 *Mathematics Subject Classification.* Primary 20C08. Secondary 20F55.
Keywords. Unequal-parameter Hecke algebras, Kazhdan–Lusztig cells, asymptotic algebras, Plancherel measure.

In the study of affine Weyl groups, Lusztig introduced the a-function, which records the highest exponent at which a fixed basis element can occur in a product [13]. The coefficients at those exponents give the multiplication of the asymptotic algebra. Their relation to the Kazhdan–Lusztig preorders governs the decomposition of this algebra by two-sided cells. In the equal-parameter theory, Shi described the cells in affine type A [14], and Lusztig identified the two-sided cells of affine Weyl groups with unipotent conjugacy classes in the associated complex groups [15, Theorem 4.8]. Lusztig organized the required degree bounds, symmetries, and compatibility statements as properties P1–P15 in his monograph [8, §14.2]. Among them, P4 orders the a-values along the two-sided preorder, P12 compares the a-function with that of a parabolic subgroup, and P15 compares left and right multiplication with independent parameters on a fixed degree layer.

In the split case, positivity and boundedness give the properties by [8, §15]. The Hodge theory of Soergel bimodules gives the equal-parameter positivity theorem for Coxeter systems [16]. The quasisplit case in [8, §16] obtains unequal weights from the restriction of the length function to a fixed-point Coxeter group. For general positive weights the multiplication coefficients can have both signs. In particular, arguments using positivity to rule out cancellation require a different justification. Chen and Hu proved boundedness for all positively weighted Coxeter systems with finitely many simple reflections [17, Theorem 1.3]. This gives a uniform upper bound on the degrees.

For finite type B_n, Bonnafé and Iancu described left cells in the asymptotic regime by a signed Robinson–Schensted correspondence [28]. Geck and Iancu computed the a-function in that regime [18], and Geck proved the fifteen properties in the generic asymptotic case [4, Theorem 5.3]. For general unequal weights, Bonnafé, Geck, Iancu, and Lam related cells to domino insertion. They showed that, away from the ratios $1, \dots, n-1$ of the short weight to the long weight, the leading matrix coefficients lie in $\{0, 1, -1\}$, and the indices supporting each leading column lie in a single left cell [27, Theorem 2.1 and Proposition 2.3]. They conjectured that the left cells in these parameter ranges are the recording fibres of domino insertion [27, Conjectures A(a) and A^+]. The complete result for finite B_3 appears in Bonnafé's treatment of rank three and in Gao and Xie's theorem for Coxeter systems with three simple reflections [19, §25.1A] and [20, Theorem 9.1].

Unequal weights also arise in Lusztig's construction of weighted affine Weyl groups attached to unipotent blocks [21, §§5, 7]. For affine Weyl groups with unequal weights, Xie established the properties needed for the based ring of the lowest two-sided cell and determined that ring [22]. Guilhot and Parkinson proved all fifteen properties for all positive weights in types $\widetilde{G}_2$ and $\widetilde{C}_2$, using balanced representations and the affine Hecke Plancherel formula [6, 7]. Dawydiak used the Plancherel formula to bound the denominators of the coefficients of the inverse asymptotic map in the equal-parameter

case [23]. The formula has also been used to determine the asymptotic algebra in equal-parameter extended affine type A [24]. For equal-parameter extended type $\widetilde{B}_n$, Parkinson gave a factorization of elements and computed the asymptotic algebra of the second lowest two-sided cell [25].

Theorem 1. *Lusztig's properties* P1–P15 *hold for every positive integer-valued weight function on the finite Coxeter system B_n and the nonextended affine Weyl systems $\widetilde{B}_n$ and $\widetilde{C}_n$, in the following ranks:*

$$B_n \quad (n \geq 1), \qquad \widetilde{B}_n \quad (n \geq 3), \qquad \widetilde{C}_n \quad (n \geq 2).$$

The Coxeter system $\widetilde{B}_2$ is isomorphic to $\widetilde{C}_2$. In rank one the affine system, also denoted $\widetilde{C}_1$, is infinite dihedral. Its unequal-parameter case is [8, §17], and its equal-parameter case is split.

In finite type B_n the proof uses an induction that carries the off-wall cell lattices together with the fifteen properties. A comparison with standard modules for the rational Cherednik algebra gives a filtration of the original regular module. At integral parameter ratios the filtration is obtained by specialization and an integral projection argument.

In affine type the Plancherel formula assigns a degree to each tempered inducing family. We use finite parabolic subgroups to find a constituent of that degree in every irreducible component of the family. Integral projections onto spectral levels and relative canonical induction then identify the spectral filtration with ideals spanned by the original Kazhdan–Lusztig elements. An integral trace inequality in the resulting leading algebra gives the distinguished elements. For $\widetilde{B}_n$ the finite-parabolic witnesses and the trace inequality come from an auxiliary $\widetilde{C}_n$ algebra with one endpoint weight zero. The filtration is constructed in the $\widetilde{B}_n$ algebra itself.

2. Hecke algebras and the fifteen properties

Let (W, S) be a Coxeter system with length function ℓ and Bruhat order $<$. A weight function $L : W \to \mathbb{Z}$ satisfies $L(xy) = L(x) + L(y)$ whenever $\ell(xy) = \ell(x) + \ell(y)$. We take $L(s) > 0$ for $s \in S$, except in the auxiliary $\widetilde{C}_n$ algebra with affine endpoint weight zero. Put $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. The Hecke algebra H has normalized standard basis $(N_w)_{w \in W}$, with length-additive multiplication and quadratic relations

$$(N_s - v^{L(s)})(N_s + v^{-L(s)}) = 0.$$

The bar involution is the semilinear algebra involution sending v to v^{-1} and N_w to $N_{w^{-1}}^{-1}$. The positive Kazhdan–Lusztig basis consists of its fixed elements

$$C_w = N_w + \sum_{y<w} p_{y,w} N_y, \qquad p_{y,w} \in v^{-1}\mathbb{Z}[v^{-1}].$$

The standard trace has $\tau(N_1) = 1$ and $\tau(N_w) = 0$ for $w \neq 1$, where 1 is the identity in W. The $\mathcal{A}$-linear anti-involution $N_w^* = N_{w^{-1}}$ also satisfies

$C_w^* = C_{w^{-1}}$. Upon specialization at a real $v > 1$ we extend $*$ conjugate-linearly. The form $\langle h, k\rangle = \tau(hk^*)$ then has orthonormal basis (N_w).

Write $h_{x,y,z}$ for the coefficient of C_z in C_xC_y. The left preorder $\leq_L$ is the reflexive transitive closure of the arrows from y to z for which $h_{s,y,z} \neq 0$ for some $s \in S$, with $z \leq_L y$ for such an arrow. The right preorder $\leq_R$ is defined by right multiplication. The two-sided preorder $\leq_{LR}$ is generated by both. The equivalence classes for the symmetric parts of these preorders are the left, right, and two-sided cells, respectively.

For a Laurent polynomial f, write $[v^j]f$ for its coefficient at the integer exponent j and $\deg_v f$ for its largest exponent, with $\deg_v 0 = -\infty$. Set

$$a(z) = \sup_{x,y\in W} \deg_v h_{x,y,z}.$$

Whenever this number is finite the supremum is a maximum, since the degrees are integral and $C_1C_z = C_z$. For strictly positive weights, $\tau(C_z) \neq 0$ by [8, Proposition 5.4(a)]. In this case write $\Delta(z) = -\deg_v \tau(C_z)$, $n_z = [v^{-\Delta(z)}]\tau(C_z)$, and

$$\mathcal{D} = \{z \in W : a(z) = \Delta(z)\}, \qquad \gamma_{x,y,z} = [v^{a(z^{-1})}]h_{x,y,z^{-1}}.$$

Inversion preserves a by the anti-involution. The auxiliary algebra with a zero endpoint is used only for restrictions and trace calculations. The statements to prove, with the labels of [8, §14.2], are as follows.

- P1 $a(z) \leq \Delta(z)$.
- P2 If $d \in \mathcal{D}$ and $\gamma_{x,y,d} \neq 0$, then $x = y^{-1}$.
- P3 For every y there is a unique $d \in \mathcal{D}$ with $\gamma_{y^{-1},y,d} \neq 0$.
- P4 If $z' \leq_{LR} z$, then $a(z') \geq a(z)$.
- P5 The nonzero coefficient in P3 is $n_d = \pm 1$.
- P6 For every $d \in \mathcal{D}$, $d^2 = 1$.
- P7 $\gamma_{x,y,z} = \gamma_{y,z,x}$.
- P8 If $\gamma_{x,y,z} \neq 0$, then $x \sim_L y^{-1}$, $y \sim_L z^{-1}$, and $z \sim_L x^{-1}$.
- P9 If $z' \leq_L z$ and $a(z') = a(z)$, then $z' \sim_L z$.
- P10 The right-hand version of P9 holds.
- P11 The two-sided version of P9 holds.
- P12 For y in a standard parabolic subgroup $W_I = \langle I\rangle$, $I \subseteq S$, its intrinsic a-value is its ambient a-value.
- P13 Each left cell contains a unique $d \in \mathcal{D}$, and $\gamma_{y^{-1},y,d} \neq 0$ for every y in that cell.
- P14 $z \sim_{LR} z^{-1}$ for every z.
- P15 If $a(w) = a(y)$, then for any $x, x' \in W$,

$$\sum_{z\in W} h_{w,x',z}(u)h_{x,z,y}(v) = \sum_{z\in W} h_{x,w,z}(v)h_{z,x',y}(u)$$

in $\mathbb{Z}[u, u^{-1}, v, v^{-1}]$. Here $h(u)$ denotes the same Laurent polynomial with indeterminate u. Both sums have finite support because each product of basis elements has finite support.

When P4 holds, the spans of the C_z with $a(z) \geq A$ and with $a(z) > A$ are two-sided ideals for each integer A. Indeed multiplication by C_s, for $s \in S$, respects the two thresholds by the definition of the preorders, and the C_s generate H. On the quotient of the former ideal by the latter, apply the left action with indeterminate v and the right action with indeterminate u to the common coordinate basis indexed by $a(z) = A$. The equality in P15 says that these actions commute. In either sum a nonzero term between endpoints of value A has intermediate index of value A by P4.

In type B_n the weight on the adjacent transpositions is $a > 0$ and the short endpoint has weight $b > 0$. In affine type $\widetilde{C}_n$ the middle weights are a, the finite endpoint has weight b, and the affine endpoint has weight c. In type $\widetilde{B}_n$ the fork and the chain up to the short endpoint have weight a, and the short endpoint has weight b. The notation $a(w)$ always denotes the function defined by multiplication, whereas a without an argument denotes the common middle weight.

3. Relative canonical induction

We use the following construction for both the finite induction and the affine filtration. The algebra of a two-sided cell is the quotient of its downward-closed span by the span of the strictly smaller cells. These are spans of the original C_w. The finite cell algebra in the next lemma has an integral asymptotic algebra J_I on the same indices. Its product uses the coefficient of degree A in the cell multiplication, where A is the common intrinsic a-value. Its unit is denoted by e_I.

Lemma 3.1. *Let Q be a quotient of H by a two-sided ideal spanned by original Kazhdan–Lusztig elements. Suppose every coefficient of the product of two surviving original basis vectors in Q has degree at most A. Let W_I be a finite standard parabolic with* P1–P15*, and let $\mathfrak{c}_I$ be a two-sided cell of intrinsic degree A. Suppose the strictly smaller source cells have zero image in Q. Let V_I be the image of*

$$H \otimes_{H_I} (\mathcal{A} \otimes_{\mathbb{Z}} J_I) \otimes_{H_I} H \longrightarrow Q, \qquad h \otimes t_p \otimes k \longmapsto hC_pk \quad (p \in \mathfrak{c}_I).$$

Here the source bimodule in the tensor product is the ordinary cell subquotient, with t_p identified as a vector with its original basis element. Then V_I is spanned over $\mathcal{A}$ by constant integral vectors in the surviving C-coordinates. Over $\mathbb{Q}$ their constant span carries commuting left and right actions with independent parameters. It contains the image of the source cell, and is stable under the degree-A multiplication operators in Q.

Proof. Write $\star$ for the product of J_I and μ_v for the ordinary product in the cell subquotient, both on its symbol space. The exact source degree layer is a direct sum of source cell subquotients by P4 and P11 in W_I. On the summand for $\mathfrak{c}_I$, take the coefficient of degree A in the right parameter of P15, with the right input in the source cell. This says that the left action of each C_w in H_I commutes with right $\star$-multiplication. The right version

follows by inversion. A linear map commuting with right multiplication in a unital algebra is left multiplication by its value on the unit. In particular, the left action of $h \in H_I$ is left $\star$-multiplication by $h \cdot e_I$, and the right action is right $\star$-multiplication by $e_I \cdot h$, where the dots denote the ordinary actions. For vectors $p, r \in \mathcal{A} \otimes J_I$, first apply this fact to $\mu_v(p, -)$, then apply its right-handed version to $\mu_v(-, e_I)$. This gives

$$\mu_v(p, r) = p \star d(v) \star r, \qquad d(v) = \mu_v(e_I, e_I).$$

The coefficients of $d(v)$ are bar invariant. Its degree-A coefficient is e_I. Consequently right multiplication by $d(v)$ for $\star$ has matrix $M(v)$ with $[v^A]M(v) = \mathrm{Id}$ and no higher coefficients.

Let X and $Y = X^{-1}$ be the minimal representatives of W/W_I and $W_I\backslash W$, respectively. In left induction write the relative canonical lift of $N_x \otimes t_p$ in the form

$$\sum_{x' \in X} N_{x'} \otimes (a_{x',x}(v) \star t_p).$$

The coefficients $a_{x',x}(v)$ belong to $\mathcal{A} \otimes J_I$. To see this, the induced bar on the coset filtration has blocks obtained from the ordinary left H_I action. Each block commutes with constant right multiplication for $\star$. The solution of the triangular bar equation with strictly negative off-diagonal coefficients commutes with those same constant maps. Indeed applying such a map to the bar equation preserves the diagonal prescription and the strict negativity. Their difference is zero by taking its maximal coset and comparing its negative coefficients with their bar transforms. This proves right J_I-linearity of each correction block, giving the displayed expression. The corresponding right correction blocks are right multiplications by elements $b_{y,y'}(v)$ of $\mathcal{A} \otimes J_I$. The sums are finite for each input because each relative Bruhat interval is finite. The same proof works for an infinite ambient Coxeter group.

Relative canonical induction identifies the multiplication images of these lifts with the images of the original C_{xp} and C_{py} [5, Proposition 3.2, Corollary 3.5, Theorem 3.6]. The omitted source terms lie in the strictly smaller source cells, whose images in Q are zero. For a vector $r \in \mathcal{A} \otimes J_I$, write C_r for its linear expansion in the source C-coordinates. Use the $\mathcal{A}$-linear extension of $\star$ to multiply the correction vectors and source basis vectors in $\mathcal{A} \otimes J_I$. For $x \in X$, $p \in \mathfrak{c}_I$, and $y \in Y$, put

$$Z_{x,p,y} = \sum_{x',y'} N_{x'}\, C_{a_{x',x}(v) \star t_p \star b_{y,y'}(v)}\, N_{y'} \quad \text{in } Q.$$

Write $C_{e_I,y}$ for the right relative canonical lift of e_I at y followed by multiplication. It is the constant integral linear combination of the C_{ry} given by the t_r-coefficients of e_I. The source multiplication formula and associativity of $\star$ give

$$C_{xp}\, C_{e_I,y} = \sum_{r \in \mathfrak{c}_I} M_{r,p}(v) Z_{x,r,y}.$$

All original coefficients on the left have degree at most A. If a coefficient in the finite vector of $Z_{x,r,y}$ at any fixed original coordinate had maximal degree $j > 0$, the leading identity of $M(v)$ would give a nonzero coefficient of degree $A + j$ on the right. Therefore all coordinates of all Z have degree at most zero. The left side and $M(v)$ are bar invariant. The leading identity makes $M(v)$ invertible over $\mathbb{Q}(v)$, so the coordinates of the Z are bar invariant and hence constant integers.

Use the same bimodule construction over $\mathbb{Q}[v^{\pm 1}, u^{\pm 1}]$ with $H(v)$ on the left and $H(u)$ on the right. On the middle space the left action uses $h \cdot e_I$ at parameter v and the right action uses $e_I \cdot h$ at parameter u. These commute by associativity of $\star$. In the basis obtained by the two relative corrections, the left matrices still depend only on v and the right matrices only on u, since the corrections on the opposite side commute with each action. At $u = v$ the map to Q has the constant columns Z. These columns give the image of the double-induced comparison, since the relative corrections on each side are triangular with identity diagonal. The kernel of a constant matrix over $\mathbb{Q}$ extends by scalars to its Laurent kernel. Equivariance on the diagonal makes that kernel invariant coefficient by coefficient under each action, so it is invariant with the parameters independent as well. This proves the commutation on V_I. The column $Z_{1,p,1}$ is C_p. Stability under the degree-A operators follows by taking coefficients in the left and right ordinary actions. □

4. Finite type B

Write B_n as the group of signed permutations of $1, \dots, n$. Use the simple reflection s_0 that changes the sign of the first position and the simple reflections s_i that interchange positions i and $i + 1$, for $1 \le i < n$. The weights are $b = L(s_0)$ and $a = L(s_i)$ for $i > 0$. The trivial group is denoted by B_0.

Theorem 4.1. *For every n and every pair of positive integers a, b, the weighted Coxeter system B_n satisfies* P1–P15.

For a finite Hecke algebra use its trace $\tau(N_w) = \delta_{1,w}$. The irreducible representations are taken over the fraction field of $\mathbb{Z}[v, v^{-1}]$, extended to a splitting field if necessary. The Schur element $s_E(v)$ of an irreducible representation E is normalized by

$$\tau(h) = \sum_E s_E(v)^{-1} \operatorname{tr}_E(h).$$

Here tr_E is the ordinary matrix trace. Write $A(E) = \frac{1}{2} \deg_v s_E(v)$. The automorphism $\dagger$ is given by $N_s \mapsto -N_s^{-1}$ for simple s. On generic labels it is the sign twist, written $E \otimes \mathrm{sgn}$. Write $a_{\mathrm{Sch}}(E) = -\frac{1}{2} \min \deg_v s_E(v)$ for the ordinary Schur valuation, where $\min \deg_v$ means the least exponent with nonzero coefficient. The semilinear automorphism $v \mapsto v^{-1}$, $N_s \mapsto -N_s$ preserves the trace and acts on the labels by the sign twist. Consequently the ordinary Schur valuation of $E \otimes \mathrm{sgn}$ is $A(E)$.

We use three assertions in the induction on n:

(1) P1–P15 for all positive integral a, b.
(2) if $r < b/a < r+1$ with $r \geq 0$ integral, the left and right cells are the recording and insertion fibres of rank-r domino insertion, respectively, with the irreducible characters given by their two-quotient labels. The two-sided cells are the common-shape sets. Left cells with the same character have signed based isomorphic cell modules, with the insertion labels preserved.
(3) in the same open ranges, the original left cell module of character E, completed at any positive logarithmic lift of a root of unity, is the image under the sign-twisted KZ functor of the standard Cherednik module labelled by $E \otimes \mathrm{sgn}$.

Here a signed based isomorphism sends each given basis vector to a signed basis vector. The notions in the last assertion are specified in Section 4.2. The first assertion for finite type A, and its irreducible cell theory, are classical [8, Chapter 15]. We prove its local KZ comparison as well.

4.1. **Off-wall cells and their canonical lattices.** Let b/a lie strictly between the integers r and $r+1$. The rank-r domino correspondence sends a signed permutation w to $(P_r(w), Q_r(w))$, its insertion and recording tableaux. These are standard domino tableaux of the same partition shape, with fixed staircase 2-core $\delta_r = (r, r-1, \ldots, 1)$. The core is filled by zeroes, and the remaining boxes are tiled by dominoes labelled $1, \ldots, n$. The labels weakly increase along rows and columns, and the two boxes with a given positive label form one domino. We use the insertion and inverse insertion conventions of [40, §2]. In particular inversion exchanges the two tableaux, and replacing every letter by its negative transposes both tableaux. For a shape $\lambda = (\lambda_i)_{i \geq 1}$ with core δ_r, padded by zero row lengths, we use the fixed-core two-quotient as its ordinary bipartition label. One way to fix the runner convention is to take ℓ at least the number of rows with $\ell \equiv r+1 \pmod 2$, form beta numbers $\lambda_i + \ell - i$, $1 \leq i \leq \ell$, and read the even runner first. On each runner, divide the even numbers by two and subtract one from the odd numbers before dividing by two. Subtract the runner staircase from the decreasing list to obtain its partition. Changing ℓ by two preserves the label. In these ordinary labels a shape (α, β) corresponds to positive cell character $(\beta^{\mathsf{t}}, \alpha^{\mathsf{t}})$. Removing a last domino removes one box of the two-quotient, and the dimension of the labelled irreducible is the number of standard tableaux of that shape. These facts follow from the fixed-core correspondence with standard bitableaux, or from the two-runner abacus: each domino addition moves a bead up one step on one runner, hence adds one box to that runner's partition. A labelled standard tableau is exactly a chain of such additions. In the asymptotic range $n \leq r+1$ the usual type B insertion description and its character labels are classical [28][19, §26.1.B, p. 320].

Leading matrix units and parabolic transport. The unconditional balanced leading algebra for the present weights is a sum of full matrix algebras with original basis elements t_w represented by signed matrix units. A column lies in an actual left cell and a row lies in an actual right cell. These statements are [27, Theorem 2.1 and Proposition 2.3], in the convention at infinity from Section 4. The leading coefficient of every finite Schur element is one off the walls: in the cancellation-free formula [1, Theorem 3.2], quantum hooks have leading coefficient one and a cross factor can tie only if b/a is integral. The same normalization holds for type A. The leading algebra here is defined by the leading representation matrices before its identification with the γ-product. We call its columns and rows leading columns and leading rows. Its involution is $t_w^* = t_{w^{-1}}$.

We give the parabolic calculation controlling the original indices. For a finite group and for each parabolic define the central element $c = \sum_E v^{-A(E)} e_E$, where e_E is the primitive central idempotent of the generic block E. Use subscripts W and I to distinguish the ambient and parabolic elements, and equip each algebra with the rescaled product $h \star k = chk$. The map

$$h \longmapsto c_W^{-1} c_I h$$

is an injective unital homomorphism for the rescaled products, compatible with the involutions. In a balanced model the rescaled regular product on canonical coordinates is bounded as $v \to +\infty$, and its limit is the leading algebra. Indeed the map from original coordinates to the tuple of scaled representation matrices has an invertible limit by the signed matrix-unit property. In these matrix coordinates the rescaled product is ordinary matrix multiplication.

For h in the source, left rescaled multiplication by its image is ordinary multiplication by $c_I h$. Let Y be the minimal representatives of $W_I \backslash W$ and $X = Y^{-1}$ those of W/W_I. In the standard basis the operator is the direct sum of copies of the source operator on the cosets $W_I y$ with $y \in Y$. The canonical and standard basis transitions tend to the identity in both algebras. Therefore the limiting action has exactly the intrinsic coset blocks in the original canonical coordinates. The image of an element has a limit as well: apply its operator to c_W^{-1}, whose limit is the signed sum of diagonal coordinates, since its scaled matrices are identities. We obtain a unital integral map of leading algebras, with left action on t_{uy}, $u \in W_I$, given by the source action on t_u with y unchanged. Its right counterpart is given by the cosets xW_I and acts on t_{xu} with x unchanged. The maps are transitive for nested parabolics, by the exact generic formula. Ordinary branching multiplicities survive: the image of $v^{A(F)} e_F$ has regular action by e_F, whose orthogonal isotypic projector in an ambient balanced block has constant trace $\dim(F)[\mathrm{Res}_I E : F]$. Taking the limit preserves this trace and the idempotent equation.

In particular a move in a proper intrinsic column or row preserves the ambient leading column or row, respectively. For the parabolics B_j, $j < n$, the left source subword is obtained by keeping the letters of absolute value at most j, and the right source subword by standardizing the first j positions. In the first case the insertion tableau is the part of $P_r(w)$ on labels $\leq j$, by largest-label deletion [40, Lemma 2.18]. The second case records the part of $Q_r(w)$ on these labels. Here standardization replaces absolute values by their ranks while retaining signs. Hence the source shape labels at B_{n-1} give the two predecessor bipartitions of a coordinate. If these predecessors differ, the ambient leading label is their unique common upper cover. Indeed its restriction contains both labels, and in the product of two Young lattices their componentwise union is the only possible common upper cover.

The complete tableau labels at all proper ranks are useful as well. A source diagonal unit acts by a coordinate projection, and the source unit labels a row or column of the ambient block. The labels along the nested B_j determine the full proper tableau by its successive shapes. Thus a map along an ambient leading row preserves the part of the insertion tableau on labels at most $n-1$, by commutation with all left source projections. A map along a column preserves the corresponding recording part by commutation with the right source projections.

A coordinate quotient criterion. The original trace form is Euclidean on the standard basis. Its canonical Gram matrix tends to the identity at infinity, so the induced form on a coordinate ideal subquotient has the same limit, by the Schur complement. In an original coordinate left-ideal subquotient, the leading coordinates form a union of whole leading columns. This follows by taking the limit of the bounded rescaled ambient action, which preserves the original generic subquotient. The number of columns of label E equals its generic multiplicity. In fact the central projector e_E is bounded for the original trace form, and its limiting matrix is the coordinate projector onto that block's basis indices: its (x, y) entry in the standard basis is the limit of $s_E^{-1} \operatorname{tr}_E(N_x N_{y^{-1}})$, the orthonormal matrix pairing of the two leading units. The limiting trace in the subquotient counts these indices. Each column has $\dim E$ indices.

Suppose an irreducible target E has an integral, bar-fixed basis and a positive invariant Gram matrix tending to the identity. Let an irreducible proper cell module V_F of character F embed by signed coordinate vectors, and suppose $\deg_v(s_E/s_F) = 0$ with leading coefficient one. Then the induction counit

$$H \otimes_{H_I} V_F \longrightarrow E, \qquad h \otimes f \longmapsto h\, i(f)$$

sends each relative canonical vector to zero or one signed target coordinate, with every target coordinate occurring exactly once. Here i is the embedding. To prove this, let $P = ii^*$ for the restricted form, and let x run over the minimal representatives of W/W_I. Schur orthogonality, factoring $w = xu$

with $u \in W_I$, gives

$$\sum_x \rho_E(N_x) P \rho_E(N_x)^* = (s_E/s_F) 1_E.$$

Indeed the full Schur sum with P is $s_E \dim(F) 1_E$, and the inner parabolic sum is $s_F \dim(F) P$. In the chosen bases the sum is $\sum_x M_x G_F^{-1} M_x^{\mathsf{t}} = (s_E/s_F) G_E^{-1}$, where M_x is the matrix of $\rho_E(N_x) i$ and G_F, G_E are the two Gram matrices. Positivity makes every entry of every M_x bounded at infinity. Relative canonical correction preserves this bound and gives bar-invariant integral Laurent coordinates, hence constant integers. Their column matrix M satisfies $MM^{\mathsf{t}} = 1$ by taking the limit. Each row thus has one entry ± 1, in distinct columns. This proves the assertion. By relative induction [5, Theorem 3.6], the induced canonical module is an original coordinate left-ideal subquotient. The kernel is coordinate and the irreducible quotient is one actual left cell: a union of more than one strongly connected component of its coefficient graph would have a proper nonempty initial union, giving a nonzero proper submodule.

Pure cells. We use the preceding criterion to construct at least one actual cell for each pure label. In the positive convention let λ be a partition of n, of height d, with bottom row of length l. Remove that row to obtain μ. Let the sign of the short eigenvalue be fixed, so the pure module is the type A Specht module of shape λ, inflated with $N_{s_0} = v^b$ or $-v^{-b}$. In the type A induction from shape μ and the trivial module of $\mathfrak{S}_l$, Pieri gives λ as the unique dominance-minimal constituent, with multiplicity one. All other constituents dominate λ: a horizontal strip adds at most one new row, and the partial row sums above row d already dominate those of μ. The type A two-sided preorder in the positive convention places the more dominant shapes below this constituent. Thus the quotient onto λ is a coordinate quotient. The identity coset of the induction generates it, so that proper irreducible coordinate copy embeds in the quotient, with the original relative basis. This gives the proper based embedding also for the inflated module from $B_{n-l} \times \mathfrak{S}_l$. An actual pure B_{n-l} cell restricts to one type A cell basis: its relative-cell filtration has just one factor because its restriction is irreducible. Therefore it can be used as the proper source after the usual type A based identification.

With the negative short eigenvalue the Schur ratio of this induction is the type A Schur ratio times $\prod_{j=1}^{l}(1+v^{-2b+2a(j-d)})$. The type A ratio has degree zero and leading coefficient one because half the highest degree of a partition Schur element is $a\sum_i \binom{\lambda_i}{2}$. The ratio therefore has the required leading term when $b/a > l - d$. For the positive short eigenvalue the same calculation uses the product with $2b$ in place of $-2b$ and works when $b/a < d - l$. These formulas follow by cancelling the unchanged rows in the pure Schur product of [1, Theorem 3.2]. The cross factor for a box (i, j) in the pure shape is $1 + v^{\pm 2b+2a(j-i)}$ after reversing the box order within the row. For a negative pure target, use the first inequality, or apply the second to its sign twist of

transposed shape and use longest-element duality. Indeed if the largest row has length h and occurs t times, the latter inequality reads $b/a < h - t$. We have $h - t \geq l - d$, with equality only for a rectangle. The nonintegral ratio therefore satisfies one of the two strict inequalities. Both sources have smaller rank. The coordinate quotient criterion proves by induction that every pure label has an actual irreducible cell.

4.1.1. *The three-corner and hook coordinates.* We use the following elementary insertion rules in the anchor calculations. In rank-r Garfinkle insertion a new largest positive letter adds a horizontal domino at the end of row one, and a new largest negative absolute letter adds a vertical domino at the end of column one. In general one first inserts into the tableau on smaller absolute labels and reinserts the larger dominoes in increasing order. A fully covered horizontal domino moves to the end of the next row. If only its northwest square is covered, it turns into the vertical domino immediately to the right of that square. The transposed rules apply to a vertical domino, and a disjoint domino is unchanged. These are the rules H1–H3 and V1–V3 in [40, §2.1]. A decreasing positive block of j letters all larger than the old labels adds a horizontal domino to each of the first j rows. A negative block with decreasing absolute labels, all larger in absolute value than the old labels, adds a vertical domino to each of the first j columns. Indeed each next letter fully covers the first block domino and moves the existing block dominoes one row or column farther. The old smaller tableau is unchanged.

Lemma 4.2. *The three-corner configurations in the off-wall reverse-insertion argument, up to transposition, have ordinary bipartitions*

$$\begin{aligned} &((d^{h-1}, d-1), (1)), \quad && h = d + r, \\ &((1), (h^{d-1}, h-1)), \quad && h = d + r + 1, \end{aligned} \qquad d \geq 2.$$

Here $(d^{h-1}, d-1)$ has $h-1$ rows of length d and a final row of length $d-1$. Each shape has a pair of coordinates in the same leading column, with a common recording tableau whose last domino is the overlapping vertical corner, and with insertion last corners in the two different proper-move components.

Proof. The staircase boundary in the three-corner configuration is

$$(L, L-1, \ldots, t+2, t, t, t-2, t-3, \ldots, 1), \qquad 2 \leq t \leq L-2.$$

Let $j = L - t - 1$. Its removable corners are the upper horizontal corner $\{(j, t+1), (j, t+2)\}$, the vertical corner $\{(j+1, t), (j+2, t)\}$, and the lower horizontal corner $\{(j+2, t-1), (j+2, t)\}$. With beta-set length $L-1$, the even runner divided by two is $\{t-1\} \cup \{t+1, \ldots, L-1\}$, while the odd runner after subtracting one and dividing by two is $\{0, \ldots, t-3\} \cup \{t-1\}$. Their difference in sizes is $L - 2t + 1$. A packed runner pair with size difference D has core δ_{D-1} for $D \geq 1$ and δ_{-D} for $D \leq 0$, as follows by merging the consecutive packed beta sets. Thus the core condition gives $(L, t) = (r + 2d, d)$ or $(r + 2d + 1, d + r + 1)$. The runner partitions, ordered

in the fixed-core convention, are precisely the displayed labels. Removing the vertical corner packs the runner with the singleton and gives a pure rectangle with its southeast box deleted. Removing either horizontal corner leaves the singleton, and the two mixed predecessors are different: the lower removal decreases the even beta rank $t-1$ by one and the upper removal decreases the rank $t+1$ by one.

We now construct an anchor for each configuration. Write the pure rectangle as $R = (W^H)$, $n = WH$, with $(W, H) = (d, d+r)$ in the first case and $(d+r+1, d)$ in the second case. Its mixed companion has one component R^-, obtained by deleting the southeast box from R, and the other (1). Put

$$p = (W-2)(H-1), \qquad m = (W-1)H, \qquad c = p + H.$$

Here c is a letter index in this calculation. Use consecutive absolute intervals for the blocks below. In the first case let P be the concatenation of the first $d-2$ blocks in which block i consists of $r+i$ decreasing positive letters followed by i negative letters of decreasing absolute value. In the second case let P be the concatenation of the first $d-1$ blocks in which block i consists of $i-1$ decreasing positive letters followed by $r+i$ negative letters of decreasing absolute value. In both cases P uses the labels $1, \dots, p$. Set $B = (-m, -(m-1), \dots, -(c+1))$ and $C = (n-1, n-2, \dots, m+1)$. In the first case with $d \geq 3$ set $A = (c, c-1, \dots, p+2)$ and $z = -(p+1)$. In the second case, and in the first case with $d = 2$, set $A = (c-1, c-2, \dots, p+1)$ and $z = -c$. The two words are

$$w = PA\,(-n)\,BCz, \qquad w' = P\,n\,ABCz. \tag{1}$$

Here is the insertion calculation. In the first case the shape of the first i pure blocks is

$$(r+2i, r+2i-1, \dots, i+1, i, i, i-1, \dots, 1).$$

Indeed the positive block adds two boxes to each of the first $r+i$ rows, and the negative block then adds two to each of the first i columns, whose heights before this addition are $r+2i-j$ for $1 \leq j \leq i$. In the second case the first i pure blocks have shape

$$(r+2i-1, r+2i-2, \dots, r+i+1, r+i, r+i, r+i-1, \dots, 1).$$

The additions are to the first $i-1$ rows and then the first $r+i$ columns. Before the column additions the latter heights are $r+2i-1-j$. These statements start with the core at $i = 0$.

For the first case set $k = d-1$ and $h = H-1$. After PA the first $k-1$ column heights are $h+k-j$ and column k has height h. The letters of B occupy pairs in column j at rows $h+k-j+1, h+k-j+2$, $1 \leq j < k$. The largest domino of the first prefix in (1) is vertical at rows $h+1, h+2$ of column k. In the second prefix it is horizontal at columns $k, k+1$ of row $h+1$, since the decreasing A-block has carried it through its h rows. The B-pairs are disjoint from this horizontal pair, and the C-pairs are in

the first h rows. When z is inserted, its initial vertical pair fully covers the first old B-pair. Full covering continues from column to column, so the new portion with n deleted is the vertical pair in column k at rows $h+1, h+2$. This fully covers the old vertical largest domino in w, and covers only the northwest square of the old horizontal largest domino in w'. The two rules give the same final largest domino at rows $h+1, h+2$ of column $k+1$. When $d = 2$ the B-block is empty and $k = 1$. The final letter adds the vertical pair directly at rows $h+1, h+2$ in column one, giving the same conclusion. In all cases the final recording corner for w is the largest final vertical pair. That for w' is the horizontal pair in row $h+2$ at columns $k, k+1$.

For the second case put $k = d-1$ and $t = r+k$. After PA the first t column heights are $k+t-j+1$, and column $t+1$ has height k. The B-pairs are at rows $k+t-j+2, k+t-j+3$ of columns $j = 1, \ldots, t$. The largest domino of the first prefix in (1) is vertical at rows $k+1, k+2$ of column $t+1$. The second is horizontal at columns $t+1, t+2$ of row $k+1$. The C-block lies in the first k rows. The final negative letter begins at rows $k+t+1, k+t+2$ of column one. The same full-cover cascade leaves the vertical pair in column $t+1$ at rows $k+1, k+2$ after erasing n. The largest domino is then in column $t+2$ at rows $k+1, k+2$ in both tableaux. It is the final recording corner for w, while for w' that corner is horizontal at columns $t+1, t+2$ of row $k+2$. In both configurations the smaller labels agree by largest-label deletion, so $P_r(w) = P_r(w')$ with the claimed largest domino. The shape follows from the displayed row and column additions.

It remains to place the inverse coordinates of (1) in one leading column. They lie in one proper induced module from $B_m \times \mathfrak{S}_H$. Indeed reversing the last H entries of the two inverses to make them increasing gives minimal coset representatives with respective last strings

$$(-c, m+1, \ldots, n-1), \qquad (p+1, m+1, \ldots, n-1).$$

Their first m entries are the remaining positive absolute values in order. The common proper right factor is the inverse of $PABz$ in B_m and the longest element of $\mathfrak{S}_H$. In the second case $PABz$ is the next pure block word. In the first case its tableau is obtained by deleting the largest H insertion labels from the full tableau. The positions just calculated leave the pure prefix shape at $i = d-1$. The runner calculation gives proper ordinary label $((d-1)^H, \varnothing)$ in the first case, and $(\varnothing, (W-1)^H)$ in the second. For example the first pure prefix shape at i has, with beta-set length $r+2i+1$, even ranks $i, \ldots, r+2i$ and odd ranks $0, \ldots, i-1$. For the second pure prefix use length $r+2i$, giving even ranks $r+i, \ldots, r+2i-1$ and odd ranks $0, \ldots, r+i-1$. This yields the stated proper labels with the fixed-core runner order.

The induced module has a coordinate counit onto the pure rectangle, by the pure-cell construction: in the positive convention its type A rectangle is transposed, and the proper source removes its last row of length H. In the first case the target short eigenvalue is negative and the criterion is $b/a > H - W = r$. In the second it is positive and the criterion is $b/a <$

$W-H=r+1$. Both hold. The part of the induced module with no negative sign in the coset representative is the type A canonical induction from the corresponding proper A cell and the symmetric factor. Indeed the no-sign representatives lie in $\mathfrak{S}_n$. Their bar transition uses only the type A parabolic action. Its image in the pure target is a nonzero type A submodule, because it contains the embedded proper copy. It is therefore the whole pure target. The coordinate counit has exactly one preimage per target coordinate, so all coset layers with a negative letter are in its kernel. Thus w^{-1} is in the kernel.

For $(w')^{-1}$ it suffices to show that its left $\mathfrak{S}_n$ factor has ordinary insertion shape different from R. Let ρ be the list of positions of P in increasing signed-value order. If P has i blocks, the negative positions occur in block order $i,\dots,1$ and increase within each block. The positive positions follow in block order $1,\dots,i$ and decrease within each block. Every position of block j is smaller than every position of block $j+1$, and within a block the positive positions are smaller than the negative positions.

An increasing subsequence using a negative position of block j uses negatives from that block alone and at most one positive from each later block. A decreasing subsequence whose last negative position belongs to block j uses at most $i-j+1$ negative positions and can use positives from just one block of index at most j. For a subsequence using a single sign, the same block orders apply: an increasing subsequence uses at most one positive per block, and a decreasing one uses at most one negative per block or positives from one block.

In the first case the negative and positive blocks of index j have lengths j and $r+j$. Thus the longest increasing and decreasing subsequences have lengths at most i and $r+i+1$. There are $i(r+i+1)$ positions, so Schensted's theorem forces a rectangle of width i and height $r+i+1$. In the second case the lengths of the negative and positive blocks of index j are $r+j$ and $j-1$. The same bounds give width at most $r+i$ and height at most i, with $i(r+i)$ positions. Both rectangles are $((W-2)^{H-1})$ for the indicated prefix lengths, with the empty shape when P is empty. The two left factors, obtained by listing positions in signed-value order, are

$$(c,c+1,\dots,m,n,\rho,c-1,\dots,p+1,n-1,\dots,m+1),$$
$$(c+1,\dots,m,n,\rho,c,c-1,\dots,p+2,n-1,\dots,m+1,p+1).$$

In the first list every increasing subsequence has length at most W. It either uses the initial increasing block, with at most one subsequent large letter if it omits n, or uses at most $W-2$ from ρ and at most one from each of the remaining two decreasing blocks. A decreasing subsequence uses at most one initial letter and at most $H-1$ from one of the three later bands, which are ordered by increasing values. Thus the first shape is R, since there are WH letters. The second list has the decreasing subsequence $n,n-1,\dots,m+1,p+1$ of length $H+1$. Thus the ordinary shape of the left factor of $(w')^{-1}$ differs from R. Its minimal representative for $B_m\times\mathfrak{S}_H$ has only positive

entries. In the type A induction the corresponding coordinate belongs to a cell of a different character from the pure target, so the coordinate kernel contains it.

The two B_{n-1} predecessors of w^{-1} are both the pure R^-, by the recording corner computation. Among pure children of this label only R has $\mathfrak{S}_n$ type R. The sole mixed child is the claimed companion. Its leading label is therefore that companion, since the pure rectangle has multiplicity zero in the kernel. The two predecessors of $(w')^{-1}$ differ, forcing the same companion label. It occurs with multiplicity one in the proper induction by Pieri, hence once in the kernel. The coordinate quotient criterion now puts both inverse indices in one leading column. Their recording tableau ends vertically and their two insertion last corners are the lower horizontal and vertical corners. Transposition follows from sign reversal and the longest-element leading duality, as explained below. □

We will also use longest-element duality at the level of leading rows and columns. For completeness, write D_w for the trace-dual element specified by $\tau(C_x D_w) = \delta_{x,w}$ for every x. The scaled coordinates of D_w are trace dual to those of C_w, since every scaled Schur trace weight tends to one. The limiting matrix coordinate map for the C_w is invertible. Thus the scaled matrices of D_w have limits, and the supported block is the transpose of the corresponding signed matrix unit. In the identity $(D_w N_{w_0}^{-1})^\dagger = (-1)^{\ell(w^{-1}w_0)} C_{w^{-1}w_0}$, the central factor $N_{w_0}^{-1}$ is scalar in each simple block. The determinant of N_{w_0} on E, computed along a reduced expression using the two generator eigenvalues, shows that this scalar has degree $A(E \otimes \mathrm{sgn}) - A(E)$. Indeed w_0 contains n short and $n(n-1)$ long reflections. In the determinant each generator's exponent is its weighted character value at that reflection divided by $\dim E$ when recovering the central scalar. This yields the weighted reflection character scalar, equal to $A(E) - A(E \otimes \mathrm{sgn})$ by the Schur degree-plus-valuation formula in Lemma 4.11. Its leading scalar is a sign, since its absolute value is determined by the determinant, a signed monomial. Realize the sign-twisted block by composing its matrices with †. This is again a positive model, since † commutes with the adjoint involution. The trace-dual identity sends the transposed unit to the leading unit of $C_{w^{-1}w_0}$ in the sign-twisted block. Inversion transposes the two matrix indices again. Thus right translation by w_0 takes rows to rows and columns to columns with the sign-twisted label. In a signed permutation w, right translation by w_0 reverses all letter signs and therefore transposes the two domino tableaux.

Lemma 4.3. *The two-corner configurations left by the noncritical moves have $n = r + 2$ and ordinary labels $((1^p), (k))$, where $p, k \geq 1$ and $p + k = n$. For each such label there are two full leading anchor columns and two actual irreducible anchor cells, one with each possible last corner.*

Proof. In the overlapping 2×2 configuration without a larger rectangle or another corner, the boundary is a staircase outside the plateau. It has shape

$$(n, n-1, \dots, k+2, k+1, k+1, k-1, \dots, 1), \qquad 1 \le k < n.$$

It is the staircase δ_n with one added box. Its beta set with length n is $\{1, 3, \dots, 2n-1\}$ with $2k-1$ replaced by $2k$. The even runner is $\{k\}$ after division and the odd ranks are $\{0, \dots, n-1\} \setminus \{k-1\}$. The core is δ_{n-2} and the fixed-core label is $((1^p), (k))$, with $p = n - k$.

The two words

$$\begin{aligned} w &= (n, p-1, \dots, 1, -(n-1), \dots, -p), \\ w' &= (-n, p-1, \dots, 1, -(n-1), \dots, -p) \end{aligned} \tag{2}$$

have the same insertion tableau, with last insertion domino vertical at $\{(p, k+1), (p+1, k+1)\}$. After deleting n, the $p-1$ decreasing positive letters extend the first $p-1$ rows of the core and the k negative letters extend its first k columns. In w the largest horizontal domino has first moved through the $p-1$ rows to $\{(p, k), (p, k+1)\}$. The last negative letter covers its northwest square and turns it vertically to the right. In w' the initially vertical largest domino is pushed successively into columns $2, \dots, k+1$ by the negative letters. The positive additions do not meet it. These are respectively the H3 and full-cover V2 rules. The final recording corner of w is the lower horizontal one and that of w' is the vertical one, as follows by comparing the shapes before and after the last insertion. Deleting the absolute label n gives the proper asymptotic label $((1^{p-1}), (k))$. Thus the common insertion tableau ends in the first quotient component.

We show that the inverses of (2) have leading matrices in one column of the positive label $E = ((1^k), (p))$. Write $q = v^a$, $Q = v^b$. Use the n-dimensional orthonormal model with $N_{s_0} = \operatorname{diag}(Q, -Q^{-1}, \dots, -Q^{-1})$. For $1 \le i < n$, let $x_i = Q^2 q^{-2(i-1)}$ and on coordinates $i, i+1$ let N_{s_i} have block

$$\begin{pmatrix} (q - q^{-1})/(1 + x_i) & \beta_i \\ \beta_i & (q - q^{-1})x_i/(1 + x_i) \end{pmatrix},$$

$$\beta_i = \frac{\sqrt{(1 + x_i q^2)(1 + x_i q^{-2})}}{1 + x_i},$$

acting by q elsewhere. Use the positive square root for $v > 1$. The block has trace $q - q^{-1}$ and determinant -1. The adjacent three-term braid follows by multiplication of the 3-by-3 blocks with $x_{i+1} = x_i/q^2$. The only needed off-diagonal squares are $1 + (q - q^{-1})^2 x_i/(1 + x_i)^2$. At the short edge the product of the two 2-by-2 blocks has trace zero and determinant one, so its square is -1 in either order. Disjoint blocks commute. These checks give a representation. Take its k-th exterior power, scaling the long generators by q^{1-k} and the short generator by $(-Q^{-1})^{1-k}$. The braid relations are preserved, and the two eigenvalues on the exterior powers after scaling are the original two Hecke eigenvalues. At $v = 1$ the model has the sign-subgroup weights indexed by k-subsets, with a column on the chosen positions and a

row on the complement. The wreath-product classification gives the label E.

Put $m = n-1$, $S = \{p, \dots, n-1\}$, and let e_S be the associated exterior basis vector. The first inverse is xu, length-additively, where

$$x = s_1 \cdots s_{n-1}, \qquad u = (p-1, \dots, 1, -m, \dots, -p) \in B_m.$$

The other inverse is $s_0 xu$ with length one larger. For the commuting elements $N_{s_{j-1}} \cdots N_{s_1} N_{s_0} N_{s_1} \cdots N_{s_{j-1}}$, the eigenvalue on a subset with j chosen and c earlier chosen positions is Qq^{-2c}. If j is unchosen it is $-Q^{-1}q^{2(j-1-c)}$. This follows by induction using the displayed two-coordinate matrices. The element u is the product of the reversal on the first $p-1$ positions, the sign changes at $p, \dots, m$, and the reversal of that negative block. The sign changes are length additive, whereas the last reversal shortens length. Thus its standard-basis product uses the inverse of the last reversal's standard element. On e_S these factors give the scalar

$$\xi = (-1)^{\binom{k}{2}} Q^k q^{\binom{p-1}{2} - \binom{k}{2}}.$$

In applying the long generators in x from right to left, there is a unique path from S to $\{p+1, \dots, n\}$. It moves each chosen position once to the right, and gives the matrix entry $\xi q^{p-1} \prod_{i=p}^{n-1} \beta_i$. There is also a unique path to $\{1, p+1, \dots, n-1\}$. It keeps $n-1$ in the first step, passes the remaining chosen-chosen pairs, and moves p to 1. Including the short generator, this entry in the other inverse is

$$\xi Q \frac{q - q^{-1}}{1 + x_{n-1}} (-q^{-1})^{k-1} \prod_{i=1}^{p-1} \beta_i.$$

Path uniqueness follows successively from the required membership of $n, n-1, \dots, 1$, as each adjacent position is visited only once.

Put $\kappa = (n-1)a - b$, so $0 < \kappa < a$. For $i \le n-2$ the factors β_i tend to one, while the last is asymptotic to v^κ. The last upper diagonal entry has degree $a - 2(b - (n-2)a)$. Both displayed entries therefore have degree

$$kb + a\big(\tbinom{p}{2} - \tbinom{k}{2}\big) + \kappa = A(E)$$

and absolute leading coefficient one. For the last equality the cross hooks in the column component are $2-i$, $1 \le i \le k$, and those in the row component are $n-1$ at its first box and $p-j$ at the later boxes. In [1, Theorem 3.2], the column cross factors all have positive degree, and only the first row cross factor has positive degree. Together with the two ordinary quantum hook products and the monomial $q^{-k(k+1)}$, they give the displayed value and leading coefficient one. Now the leading trace identity for $N_w N_w^*$ is the sum of squares of all leading matrix entries in the orthonormal models, with each block weighted by its reciprocal leading Schur coefficient. That sum is one. The single entry of absolute leading coefficient one therefore forces each of these two inverse matrices to be a signed unit, with the common input column S. This common-column assertion is independent of the model: it

is equality of their leading products $t^*_{w^{-1}}t_{w^{-1}}$ and $t^*_{(w')^{-1}}t_{(w')^{-1}}$. Together with the proper fixed-corner moves, this identifies one full leading column. Exchanging p, k and applying longest-element duality supplies the other last corner.

We give an actual cell anchor independently of the preceding leading calculation. On a basis $f_1, \dots, f_n$, use the integral model in which $N_{s_i}f_i = f_{i+1}$, $N_{s_i}f_{i+1} = f_i + (q-q^{-1})f_{i+1}$, and the other coordinates have eigenvalue q. Let $N_{s_0}f_1 = Qf_1$ and

$$N_{s_0}f_j = -Q^{-1}f_j + Q^{-1}(q-q^{-1})q^{j-2}f_1 \quad (j \geq 2).$$

These generators obey the quadratic and braid relations: the long generators are the type A permutation model, the upper row commutes with N_{s_i} for $i \geq 2$ because its consecutive coefficients have ratio q, and the short braid is obtained on f_1, f_2 by the trace-zero product and on f_j, $j \geq 3$, by substituting the coefficient $Q^{-1}(q-q^{-1})q^{j-2}$ in both products. This model has the same specialization at $v = 1$ as the orthogonal model above. Put

$$c_j = f_j + q^{-1}f_{j-1} + \cdots + q^{-(j-1)}f_1, \qquad t_j = Qq^{-(j-1)} + Q^{-1}q^{j-1}.$$

Use $c_0 = 0$. Direct substitution gives

$$\begin{aligned} N_{s_i}c_i &= -q^{-1}c_i + c_{i-1} + c_{i+1}, \\ N_{s_i}c_j &= qc_j \quad (j \neq i), \\ (N_{s_0} + Q^{-1})c_j &= t_jc_1. \end{aligned}$$

Since the coefficients of C_s in this basis are fixed by bar, declaring the basis fixed defines a compatible bar. Replace c_n by c_n/t_n, retaining the other c_j. The coefficient into c_n/t_n for $N_{s_{n-1}}$ becomes t_n, and the returning short coefficient becomes one, so the new basis still spans an integral invariant lattice.

To check its form, in the f-basis set $h = (1, q, \dots, q^{n-1})^{\mathsf{t}}$ and $\eta = (q^2 - 1)/(q^2Q^2 + 1)$. The matrix $1 + \eta hh^{\mathsf{t}}$ is positive and makes every generator self-adjoint. The long generators fix the line of h, and the short adjoint equation reduces to $(1+\eta)(q-q^{-1}) = \eta q(Q^2+1)$. In the c_j-basis, $h^{\mathsf{t}}c_j = q^{j-1} + q^{j-3} + \cdots + q^{1-j}$. For $i, j < n$ the rank-one Gram contribution has degree $(i+j-2)a - 2b < 0$. After dividing the last column by t_n the mixed contributions still have negative degree. The final squared norm tends to one, since its numerator and t_n^2 have the same leading monomial $Q^{-2}q^{2(n-1)}$. The Euclidean part tends to the identity on the earlier coordinates. Hence the new Gram tends to the identity.

Take the scaled k-th exterior power of this lattice, as before. Its basis is integral, bar fixed, and has limiting Gram identity. In this basis the exterior vector on the coordinates $1, \dots, k-1, n$ spans the one-dimensional proper source of $B_{k-1} \times \mathfrak{S}_{p+1}$. Its long eigenvalues on the first factor are $-q^{-1}$ and its short eigenvalue is Q, while the symmetric factor is trivial. Indeed in the displayed formulas all neighbouring replacements on the first block duplicate an exterior factor, and every factor is a q-eigenvector on

the symmetric block. This is an actual proper singleton by the asymptotic classification. Its degree is

$$(k-1)b - a\binom{k-1}{2} + a\binom{p+1}{2} = A(E),$$

using $p+k=n$. The Schur ratio has leading coefficient one. The coordinate counit therefore constructs an actual irreducible cell containing the identity coset source index

$$(-(k-1),\dots,-1,n,n-1,\dots,k).$$

This index is an involution. Deleting the absolute letter n gives the proper asymptotic label $((1^p),(k-1))$, so the last domino is in the second quotient component. The sign-dual construction after exchanging p,k gives an actual anchor in the first component. □

Lemma 4.4. *Under the proper-rank induction assumptions, every rank-r recording fibre is one original left cell with its labelled irreducible character. Insertion fibres are the right cells.*

Proof. We first identify the leading fibres. Fix an insertion tableau. In its fibre, classify words by the corner occupied by the last recording domino. For a fixed corner, reverse insertion fixes the last signed letter and the prefix insertion tableau. Thus the words in that corner class are related by proper B_{n-1} moves. The noncritical cases of the reverse-insertion proof of [40, Theorem 3.5] join different corner classes by these moves and ordinary signed-alphabet Knuth moves. We specify the use of its case analysis. When the northeast region of a lower corner and the southwest region of an upper corner contain an intermediate removable domino, reverse insertion at the three corners gives a triple c,b,a, with b greater than c,a by its Lemma 2.28. The appropriate Knuth switch is $cba \leftrightarrow bca$ if $c<a$ and $cba \leftrightarrow cab$ if $a<c$. The recording corner switches by its Corollary 2.32. For overlapping corners in a rectangle larger than 2×2, the two corners exposed by the first removal give the same triple and switch (Case 3.1.1). Cases 3.2.1, 3.2.2 before its final subcase, and 4.1 reduce to these switches through a third or fourth corner. Case 4.2 has $n \le r+1$ and is asymptotic. The two configurations left by this reduction are the overlapping pair with no larger rectangle in Case 3.1.2, and the final three-corner configuration of Case 3.2.2. In the former the shape and the rank are those in Lemma 4.3. In the latter, tracing the staircase boundary gives the shape in Lemma 4.2, up to transposition. The lower and upper horizontal corner classes are already joined by the intermediate-corner switch, since their region contains the plateau square. Every Knuth move just used is a type A move in the proper subgroup on the permutation generators. The parabolic formulas therefore preserve the ambient leading row on all of these moves.

For a nonexceptional shape with at least two removable corners, choose a recording tableau whose last corner differs from that of the fixed insertion tableau. Their proper predecessors differ, so the unique-cover argument

above identifies the ambient leading label. The full fixed-insertion fibre is in its one leading row, and its size is the dimension of that label. The row and the fibre therefore coincide.

For a three-corner shape, Lemma 4.2 gives a full leading column, with its recording tableau ending in the overlapping vertical corner. Indeed its anchor pair joins the two remaining corner components within that recording fibre, and the dimension gives equality. Label the rows by the insertion tableaux on this column, and the columns by the recording tableaux on the inverse row. Their common diagonal element is an involution, since the involution fixes a signed diagonal unit. The two labels agree there. At a matrix position with distinct last corners, the proper tableau labels from parabolic transport give both predecessors. The known block label is their unique common upper cover. A predecessor tableau has only one extension to that shape, so both complete tableaux are forced. For any row ending at a horizontal corner, take a recording tableau ending at the other horizontal corner and one ending at the vertical corner. The two forced coordinates connect the two corner components in that row. For a row ending vertically, use its known coordinate in the anchor column and a coordinate ending horizontally. The proper moves through these coordinates put the whole insertion fibre in the row. Its size equals the row dimension. For a two-corner hook, Lemma 4.3 supplies two full anchor columns, one for each last corner. Label using the first column and its inverse row. The second column is placed by its intersection with a row having the opposite last corner, which is already forced by the two predecessors. Each row has an entry from each last-corner class, one by its known entry in a full anchor column and the other by the distinct-corner argument. Proper moves fill the row and the dimension gives equality. This proves the required row and column statements for the exceptional shapes.

After these blocks are exhausted, the remaining labels and shapes have just one corner. They are pure rectangles. A predecessor of positive size has at most one one-corner parent in ranks $n \geq 3$: deleting the southeast box from a rectangle with both side lengths greater than one determines the rectangle from its two row lengths, while a single row or column determines its extension unless the predecessor is a single box. The remaining coordinate therefore has its labelled block by restriction to B_{n-1}. The fixed-corner move and dimension give its entire row. Inversion gives every corresponding column.

It remains to exclude mergers of leading columns into actual left cells. Relative induction fixes one proper B_{n-1} left cell for all indices in an actual ambient cell, by [5, Theorem 3.6]. If its known leading column has predecessor F, its character is a multiplicity-free sum of parents of F, since B_{n-1} branching is multiplicity-free. A proper based isomorphism between source cell lattices induces an isomorphism preserving the relative canonical basis up to the same signs. It sends $N_x \otimes C_u$ to $N_x \otimes (\pm C_{u'})$ and commutes with bar, so canonical uniqueness gives the claim. Consequently it

transports actual cells and their characters along proper right moves, using the induction hypothesis that the isomorphism preserves the insertion label. This also applies to the proper type A moves, by ordinary same-shape based equivalence.

If a proper-move component of recording tableaux has two distinct predecessors, all the actual characters there coincide and each is a sum of common parents. The only common parent is the given shape. This proves actual irreducibility in all nonexceptional multiple-corner cases. The single-corner pure rectangles already have one actual irreducible cell by the pure-cell construction. Within a fixed last-corner class, the proper B_{n-1} based isomorphisms change the recording prefix arbitrarily while keeping the insertion prefix and final letter fixed. Thus that irreducibility propagates to every recording tableau in the class. For a three-corner shape the two horizontal classes have distinct predecessors and are already irreducible. The vertical class has a pure predecessor, with just one mixed parent. Since all the pure columns are already their own actual cells, no such column can merge with the mixed column. Multiplicity one proves irreducibility there. For a hook, Lemma 4.3 supplies an actual irreducible anchor for each last corner. The same proper transport reaches every filling in either class. This completes the proof. □

Lemma 4.5. *Under the same assumptions, all actual left cells of a given character E have signed based isomorphic lattices, preserving the insertion labels. When the proper source of Lemma 4.11 has E as its unique minimum-degree constituent, its induced canonical lattice surjects onto the original lattice of every actual left cell with character E.*

Proof. The chosen proper irreducible F occurs with multiplicity one in $\mathrm{Res}_I E$ and $A(F) = A(E)$. Restrict an actual left cell of character E. For a minimal representative y of a right coset $W_I y$, the H_I-generator edges within the coset are the intrinsic edges, by [5, Lemma 4.3]. Thus if the ambient left cell meets Dy for an intrinsic left cell D, it contains Dy. Filtering the restricted module by cosets and by intrinsic left cells gives factors of the characters of these D, by the same coefficient formula and [5, Proposition 4.4]. There is exactly one such relative cell of character F. Any edge out of it for a source generator has intrinsic two-sided label below that of F, by the same proposition. By proper P4 its spectral degree is at least $A(F)$. On the other hand a restriction constituent has degree at most $A(E)$: the restriction of the trace gives $s_F^{-1} = \sum_G [\mathrm{Res}_I G : F] s_G^{-1}$ with positive leading terms, and hence the inequality. Proper P11 and the proper two-sided classification now force the same label F. Multiplicity one prevents an edge to another relative cell of that label. Thus the chosen relative cell spans an actual source coordinate submodule.

The target lattice has Gram tending to the identity as an original cell quotient. The Schur ratio is a degree-zero unit at infinity by off-wall normalization. Apply the coordinate quotient criterion above, using the proper

signed based identification of the source. For any two target cells the resulting kernels have the same generic span, since the target constituent has multiplicity one in induction. Both kernels are coordinate submodules, so they are identical, yielding the signed based isomorphism. If the proper source is obtained for $E \otimes \mathrm{sgn}$, exact longest-element cell duality [8, Proposition 11.6 and Corollary 11.7] transfers the based isomorphism to E by the signed transpose of the cell matrices. Finally a constant based H-isomorphism intertwines the action of every $v^{-A(E)}C_x$ and hence of every limiting leading matrix. It takes a leading column to the same-row coordinates of the other column. By Lemma 4.4 these are exactly the corresponding insertion labels. □

4.2. **Category $\mathcal{O}$ and the spectral filtration.** Let V_Γ be the original cell lattice of a finite left cell Γ, with its canonical basis. For a root of unity ζ, choose a positive real λ with $\exp(2\pi\mathrm{i}\lambda) = \zeta$, and use the complete local ring $R = \mathbb{C}[[u]]$ with $v = \exp(2\pi\mathrm{i}(\lambda + u))$. This identifies R with the completion at $v = \zeta$, since the linear term of $v - \zeta$ is nonzero. Use the corresponding Cherednik category $\mathcal{O}_R$, with standard module $\Delta_R(U)$ for an ordinary group label U. This category has the finite split highest-weight realization of [37, Theorem 5.2]. Its KZ functor is exact, represented by the projective P_{KZ}, and sends that projective to the regular Hecke module. These statements follow from [33, Theorems 5.13, 5.15 and 5.16] and [37, Proposition 4.36 and §5.2], with the KZ normalization in [38, §15]. Set

$$F_R(M) = \big(\mathrm{KZ}_R(M)\big)^\dagger,$$

where the superscript denotes twisting the Hecke action by †. For type B the Dunkl–Griffeth parameters on this line are

$$c_0 = -2a(\lambda + u), \qquad d_0 - d_1 = -4b(\lambda + u), \qquad d_0 + d_1 = 0.$$

In the monodromy normalization of [30, (1.18)], the generators T_i attached to s_i have roots $v^b, -v^{-b}$ for $i = 0$ and $v^{2a}, -1$ for $i > 0$. Thus $N_{s_0} = T_0$ and $N_{s_i} = v^{-a}T_i$ for $i > 0$.

For a finite multiset Z with decreasing list $z_1, \dots, z_m$, write $n(Z) = \sum_{i=1}^{m}(i-1)z_i$. For a partition U, $n(U)$ applies this definition to its list of row lengths.

Lemma 4.6. *In the residue category at each positive logarithmic lift, write $\Delta(U)$ for the residue standard and $L(U)$ for its simple head. If $L(U')$ occurs in $\Delta(U)$ and $U' \neq U$, then $a_{\mathrm{Sch}}(U') < a_{\mathrm{Sch}}(U)$. The generic fibre of $\mathcal{O}_R$ is split semisimple. The corresponding assertions hold in type A and for products.*

Proof. For a box (i, j) in colour $k \in \{0, 1\}$, its charged content is $j - i + b/a$ when $k = 0$ and $j - i$ when $k = 1$. The triangular generalized weights in [30, Theorem 1.3(a) and (4.3)–(4.4)] match the boxes of U with the boxes of U',

if $L(U')$ occurs. For a matched pair at positions (i,j), (i',j') with colours k,k' and charged contents η,η', the necessary equations are

$$m = d_k - d_{k'} + 2c_0\big((j-i)-(j'-i')\big), \qquad m \in \mathbb{Z}_{\geq 0}, \qquad m \equiv k-k' \pmod 2.$$

The comparison uses the lowest joint weight of $L(U')$ and the triangular diagonal weights in $\Delta(U)$. Substitution of the residue parameters gives $m = 4a\lambda(\eta'-\eta)$. Thus the contents weakly increase, and equality preserves the colour.

Cancel common boxes by following the matching through any common vertex, using the row, column and colour to identify that vertex. This leaves a matching between the two set differences. On every remaining path the contents strictly increase. Otherwise every edge would preserve both colour and content, whereas boxes on one diagonal of a given partition form an initial interval of that diagonal. Both set differences cannot contain boxes on the same coloured diagonal. Sort the two remaining lists of contents in increasing order. The sorted contents satisfy strict entrywise inequalities, since the first i target entries have i distinct strictly smaller sources.

Starting with the common diagram, add the unmatched boxes in this sorted order in both diagrams. Within a row they are added from left to right. Each matched pair raises two symbol entries, one in each diagram, by one. The old entry is the charged content of the added box plus $N-1$, where N is the common padding length in the symbol of Lemma 4.11. The target old entry is therefore larger. The intermediate row lengths give multisets of entries, which suffice for the comparison. For a real threshold t, raising x to $x+1$ increases the sum of the positive parts $\sum_z \max(z-t,0)$ by $\max(x+1-t,0)-\max(x-t,0)$, a nondecreasing function of x. The increase is strictly larger for some t when the old entries differ. The target symbol therefore strictly majorizes the source symbol when the diagrams differ. Indeed the two multisets have the same sum, and comparison of the positive-part sums is the majorization criterion. For a decreasing list $z_1,\dots,z_m$,

$$n(Z) = (m-1)\sum_i z_i - \sum_{j=1}^{m-1}\sum_{i=1}^{j} z_i.$$

Strict majorization strictly decreases this statistic. The Schur symbol formula from Lemma 4.11 gives the strict inequality.

In the fraction field the same necessary equations use $\lambda+u$ in place of λ. Since m is constant, they force equality of charged contents and then equality of colours. The two partitions in each colour are determined by their diagonal lengths, so $U'=U$. Every standard is then simple and BGG reciprocity makes every projective cover simple. This proves generic semisimplicity. For the symmetric group the weight comparison is the single-colour case of the same formula, with $2a\lambda$ instead of $4a\lambda$ and ordinary contents. Its Schur statistic is $a\,n(U)$ for the ordinary partition U. The same argument applies. Standards and spectral degrees factor on products. □

By BGG reciprocity and equality of the standard and costandard classes [33, Theorem 2.19 and Proposition 3.3], the projective kernel over a standard has a filtration by standards with strictly larger Schur valuation. Thus the same standards give a highest-weight structure ordered by that valuation, with unequal labels of equal valuation incomparable. The residue-field criterion [37, Theorem 4.15] gives the same order over R. In particular,

$$\operatorname{Ext}^1_{\mathcal{O}_R}(\Delta_R(U), \Delta_R(U')) = 0 \quad \text{if} \quad a_{\mathrm{Sch}}(U) \geq a_{\mathrm{Sch}}(U').$$

This includes self extensions. Projective standard filtrations can be ordered with the greater valuations at the bottom, and sections of equal valuation are direct sums. These are the standard filtration properties in [37, Proposition 4.13].

We record two compatibility facts needed for the use of KZ. Its generic labels are the ordinary Tits labels on every such logarithmic line. To see this, use the holomorphic standard-module connection on the complex logarithmic line [39, §3.4]. On the complement of the discrete nonsemisimplicity set the primitive central idempotents of the split seminormal representations vary holomorphically [36, (2.4), Corollary 3.12 and Theorem 3.14]. Each acts on the holomorphic KZ standard by an idempotent of locally constant rank. At logarithm zero the KZ standard is the ordinary irreducible U, so its idempotent acts with full rank and all others with rank zero. The complement is connected. The same ranks and the seminormal dimension formula therefore identify the KZ standard with the irreducible of label U throughout the complement. The same comparison over the completion follows by specializing the holomorphic family and comparing its character germs.

Also, relative parabolic induction and restriction are exact, biadjoint and compatible with KZ over R [34, §2.8]. They preserve filtrations by standards. Here is the integral argument for the filtration claim. In the finite R-algebra realizations, restriction is tensor with a finite bimodule, projective on the input side by exactness and on the output side by preservation of projectives. Its adjoint is Hom from that bimodule, so both functors preserve R-freeness and commute with reduction of the defining bimodule. To identify the reduced restriction, use the completion description in [34, §2.8]. Completion at the parabolic point commutes with reduction modulo u on the finite polynomial modules. The centralizer isomorphism specializes by its formula [26, Theorem 3.2], as its denominators are units in the point completion. The restricted object is characterized by its completion under the equivalence with the completed category. Its reduction is therefore the field restriction, and biadjunction identifies the reduced induction. At the residue field induction and restriction preserve standard filtrations, with ordinary branching multiplicities [35, Lemma 2.24]. To lift a standard filtration of a free R-module, take a maximal standard, which is a projective summand P of the finite algebra, with $\operatorname{End}(P) = R$. Its evaluation map $P \otimes_R \operatorname{Hom}(P, M) \to M$ reduces to the inclusion of the maximal-standard trace: the copies of that projective standard can be placed at the bottom of the residue filtration.

The projective copies move to the bottom because any extension with the maximal standard as quotient splits. Hom from this projective to each remaining standard is zero, since the simple head of the projective does not occur in any remaining standard. Thus evaluation identifies its source modulo the uniformizer with the sum of the bottom copies. The source is free, and a unit minor gives an injective map with free cokernel. This cokernel is a module for the split quasi-hereditary quotient by the corresponding heredity ideal. Induction on the labels lifts the whole filtration. Exactness then gives preservation for all standard-filtered modules.

Lemma 4.7. *Under the proper-rank induction assumptions and the off-wall actual-cell statement of Lemma 4.4, every original left cell of character E satisfies*

$$V_\Gamma \otimes R \simeq F_R \Delta_R(E \otimes \mathrm{sgn}).$$

The analogous assertion holds in equal-parameter finite type A.

Proof. First suppose E is the unique constituent of minimum spectral degree in the proper induction of Lemma 4.11. By Lemma 4.5 its original lattice is a quotient of the corresponding proper canonical induction by an integral surjection. The proper lattice is already recognized by induction. In category $\mathcal{O}_R$, the induced standard has exactly one standard factor labelled $E \otimes \mathrm{sgn}$ at the smallest valuation. The Ext vanishing above moves it to a quotient: an adjacent factor of larger valuation above it splits off from it, so the two positions can be swapped. Applying F_R gives a surjection from the same induced Hecke lattice to $F_R\Delta_R(E \otimes \mathrm{sgn})$. Both quotients are R-free. Over the fraction field both kernels are the sum of all isotypic summands except the single copy of E. Saturated submodules of a free lattice are intersections with their generic spans. Thus the two kernels coincide.

For a label for which this induction is obtained only after the sign twist, use exact longest-element duality on the original lattices and homological duality on the standards. More precisely, write $^\vee$ for the linear contragredient with respect to $N_w \mapsto N_{w^{-1}}$, with the dual basis. The lattice of Γw_0 is $(V_\Gamma^\vee)^\dagger$ up to signs, by [8, Proposition 11.6 and Corollary 11.7]. This formula uses right translation by w_0 and applies also to type A. It follows from the trace-dual identity used in Proposition 4.13. If one uses left translation, the additional diagram automorphism is conjugation by the integral unit N_{w_0}. Homological duality takes $\Delta_R(U)$ to $\Delta_R(U^\vee \otimes \mathrm{sgn})$. The target has the same parameter, since the reflections have order two [33, Remark 4.5 and Proposition 4.10]. Over R the standard computation uses the polynomial Koszul resolution over R and the PBW decomposition, and after localization the KZ identity is duality of the free flat connection [33, §5.4.2]. For a monodromy generator the anti-map takes its inverse times the product of its two eigenvalues. In normalized generators that product is -1, so the anti-map is $N_s \mapsto -N_s^{-1}$. The ordinary labels in these finite Weyl groups are self-dual. Applying these dualities to the already recognized paired cell proves the assertion for E.

For type A induct on the size m of the symmetric group, with the two one-dimensional initial cases. For a partition μ of m, let W_J be the row Young parabolic and w_J its longest element. The left ideal generated by C_{w_J} is the coordinate span of all C_{xw_J} for minimal coset representatives x. In fact the simultaneous right positive-eigenvalue equations select exactly the coordinates having all the descents of J, and the $N_x C_{w_J}$ are unitriangular in those coordinates. The left cell of w_J affords the ordinary Specht character μ, by the usual Robinson–Schensted description. Its coordinate quotient in this ideal is a quotient of induction from the trivial positive representation of W_J. Type A based equivalence supplies the quotient for any cell of the same label. Young's rule makes μ the unique dominance-minimal constituent, with multiplicity one. In positive coordinates its spectral degree is $a\sum_i \binom{\mu_i}{2}$, strictly increasing under strict dominance. Hence the preceding direct argument applies when $\mu \neq (m)$ and W_J is proper. The sign label (1^m) can also be obtained from the empty parabolic as the unique minimum of the regular character. Longest-element duality gives (m). This completes the type A initialization and the required product statements. □

Lemma 4.8. *Off the walls, the recognition of Lemma 4.7 and the irreducibility of the actual left cells imply that for each d the spectral ideal with generic labels $A(E) \geq d$ is spanned by a subset of the original C-basis.*

Proof. Work over the completed ring R. For a simple reflection s, let S_s^+ be its one-dimensional positive Hecke module, on which N_s acts by $v^{L(s)}$. Put

$$T_R = H_R \oplus \bigoplus_s H_R \otimes_{H_{R,s}} S_s^+.$$

The rank-one induced summand is $H_R C_s$, by the standard coset basis. The functor $G_T = \mathrm{Hom}_{H_R}(T_R, -)$ is exact on every coordinate left-ideal filtration and its subquotients. Indeed the positive rank-one eigenvectors in a coordinate subquotient have only left-descent coordinates for s. The ascent coordinate of $(N_s - v^{L(s)})x$ is its coefficient in x times $-v^{L(s)} - v^{-L(s)}$, a nonzero element of the domain R. This follows from the generator formula, whose off-diagonal outputs are left descents. Each surviving descent coordinate lifts as a positive eigenvector, proving the required exactness.

The highest spectral degree standard of rank one has F_R-image S_s^+ and is projective by the order in Lemma 4.6. Denote it by Δ_s^+. The projective

$$Q_R = P_{\mathrm{KZ}} \oplus \bigoplus_s \mathrm{Ind}_s \Delta_s^+$$

maps to T_R under F_R, with the same endomorphism algebra by full faithfulness on projectives. Its additive closure is generated by the projective covers of residue simples whose support has codimension at most one, lifted to R. For rank one $P_{\mathrm{KZ},s}$ together with Δ_s^+ generates the projectives. If the two Hecke roots are different then KZ already detects both simple heads. If they coincide, exactly the high standard has the finite-support simple head and the other head survives KZ. This follows directly from the two rank-one

standard modules, or the rank-one calculation in [34, proof of Lemma 4.4]. In higher rank, restriction to the rank-one stabilizer of a generic point on a reflection hyperplane detects the simple heads with support of codimension one, by parabolic support detection in [34, §2.8 and proof of Lemma 4.4]. Induction and adjunction give the corresponding projective summands. Inducing $P_{\mathrm{KZ},s}$ gives the regular-image projective already represented by P_{KZ}. This verifies the assertion over the residue field, and projective summands lift over R.

Let $\widetilde{F}_R = \mathrm{Hom}_{\mathcal{O}_R}(Q_R, -)$. The enlarged functor is still fully faithful on projectives. A homomorphism between its images of two projectives restricts, via the idempotent for P_{KZ}, to a homomorphism between their KZ images. Lift it by KZ full faithfulness. After subtracting that lift, the homomorphism vanishes after precomposition with every map $P_{\mathrm{KZ}} \to Q_R$. KZ faithfulness on projectives then makes it zero. Its residue quotient is fully faithful on standard-filtered objects by [34, Lemma 4.3], which uses precisely the projective covers for support of codimension at most one. Indeed a standard-filtered object is free over the polynomial ring, since standards are free over the polynomial ring and extensions split over that ring. Objects in the kernel have support of codimension at least two. They admit no map to an object free over the polynomial ring. An extension of one by an object free over the polynomial ring splits: off the support the extension is identified with that free object, and taking sections there extends it back by Hartogs across codimension at least two. The restriction and extension map respects the Cherednik action, which sheafifies over the quotient of the reflection space by the finite group. These Hom and Ext vanishings identify the morphisms between standard-filtered objects with their morphisms in the Serre quotient. Since R is a discrete valuation ring and the generic category is split semisimple, [37, Propositions 4.41–4.42] then identify $\widetilde{F}_R$ as an exact equivalence from standard-filtered objects onto modules filtered by its standard images.

For a standard-filtered object M the natural map $\widetilde{F}_R M \to G_T(F_R M)$ is an isomorphism. On the P_{KZ}-summand this is the defining comparison. On the rank-one summands use relative adjunction and compatibility with restriction. In rank one the comparison $\mathrm{Hom}(\Delta_s^+, M) \to \mathrm{Hom}(S_s^+, F_R M)$ is an isomorphism on both standards, since their generic labels are different, the Hom modules are torsion-free, and the identity maps to the identity. Induct on the standard filtration. A last factor Δ_s^+ splits by projectivity. For the other last factor both Hom spaces into that factor vanish, so both functors have the same value as on the preceding subobject. This proves the natural comparison, including compatibility with the endomorphism actions.

Choose a left-cell filtration of the regular H_R given by a linear extension of its left-cell order. Apply G_T, which is exact on that filtration. Its factors are $\widetilde{F}_R$-images of standards by Lemma 4.7. The preceding equivalence lifts the entire flag to a standard-filtered object whose total image is $\widetilde{F}_R(P_{\mathrm{KZ}})$.

Full faithfulness identifies the total object with P_{KZ}, and the idempotent selecting its summand in Q_R recovers the original Hecke flag. The same argument lifts each two-cell subquotient.

Suppose two adjacent cells of different spectral values are ordered with the smaller value below the larger. Their lifted extension splits by the displayed Ext vanishing. The generic projector onto the lower cell is unique because the generic labels are distinct. Thus this projector is regular at every root of unity, by completing at each of them as above. The factors of the Schur elements vanish at finite nonzero values only at roots of unity. At any other such value the Hecke algebra is split semisimple and its generic block projectors are regular by [36, (2.4) and Corollary 3.12]. The projector matrix is therefore Laurent polynomial. It is fixed by bar, by uniqueness of the projection onto the coordinate submodule. For positive real v the original trace gives the Euclidean form in the standard basis. The canonical Gram matrix tends to the identity at infinity, as does the Schur complement on the two-cell subquotient. The generic projector is orthogonal for this form and hence bounded at infinity. A bar-fixed Laurent polynomial bounded there is constant. The limiting Gram identity makes this constant projector the ordinary coordinate projection. Its coordinate complement can now be placed below the other cell. Repeating these adjacent swaps sorts the original coordinate flag by decreasing spectral value. Its initial segments are precisely the spectral threshold ideals. □

Lemma 4.9. *If every spectral threshold of B_n is a C-coordinate ideal, no generator edge at a fixed spectral degree joins distinct actual two-sided cells.*

Proof. The threshold property identifies the spectral value of each cell and is Geck's E1 [4, Lemma 3.2 and Proposition 3.6]. In the generic regular representation the threshold spans exactly the simple blocks of the indicated degrees. On P_{KZ}, order a standard filtration by Schur valuation using Lemma 4.6. The section at each valuation is a direct sum of standards by the Ext vanishing. Its KZ image is therefore a filtration of H_R with each exact section a direct sum of KZ standards. Twisting by † converts the labels to the positive convention. Both this filtration and the original coordinate spectral filtration are saturated, since their successive quotients are R-free. Their generic spans agree, so the filtrations coincide.

Inside an exact degree layer, filter by actual two-sided cells in a linear extension of their preorder. The generic constituents of distinct two-sided cells are disjoint by [4, Remark 3.3]. Each initial segment is saturated and its generic span consists of complete distinct simple matrix blocks. In the direct sum of KZ standards just obtained, its intersection is exactly the sum of the standard lattices belonging to those labels. Hence every such initial segment splits integrally at every root of unity. The unique spectral projection onto it is Laurent, bar fixed and bounded at infinity, by the same semisimplicity and trace argument as in Lemma 4.8. It is therefore the coordinate projection. Differences of successive projections give the individual

two-sided-cell projections, which commute with both Hecke actions. This proves the claim. □

Lemma 4.10. *Suppose the off-wall spectral thresholds of rank n are C-coordinate ideals. The same holds on each positive integral ratio b/a in that rank.*

Proof. Fix $r = b/a \in \mathbb{Z}_{>0}$. Form the two-variable Hecke algebra over $\mathbb{Z}[v^{\pm 1}, t^{\pm 1}]$ with lexicographic exponent order, primary variable v, and weights $(a, 0)$, $(ra, 1)$. Specializing $v = z^N$ and $t = z$ for sufficiently large integral N preserves all required strict signs in the finite canonical basis and the orders of the Schur degrees. For a fixed primary spectral threshold the selected simple labels thus form a spectral threshold at $(Na, Nra+1)$, which is off the wall. The corresponding coordinate span lifts to a two-variable coordinate ideal. Indeed one chooses N so that the specialization is injective on each finite exponent support of the relevant structure coefficients and the matrix coefficients in integral generic Specht models. Vanishing into excluded coordinates and vanishing in excluded irreducibles lift. Its dimension is the sum of the squares of dimensions of the selected labels, as at the one-variable specialization, so the lifted ideal has precisely their generic support.

At $t = 1$ the specialized basis changes to the wall canonical basis by a constant integral unitriangular matrix. Every lexicographically negative term has nonpositive primary exponent, so the transition and its inverse have coefficients in $\mathbb{Z}[v^{-1}]$. The transition is bar invariant and hence constant. The primary Schur degrees are the wall degrees. In the cancellation-free Schur product [1, Theorem 3.2], use the Ariki–Koike parameters $q = v^{2a}$, $Q_0 = v^{2ra}t^2$, $Q_1 = -1$. Each cross factor with zero primary exponent is $1+t^2$ or $1 + t^{-2}$, which becomes 2, and the highest primary coefficient of every other factor survives. Over $\mathbb{Q}(v)$ the wall Hecke algebra is split semisimple. Its generic primitive idempotents are regular at $t = 1$, since the finite Schur elements are units in that local ring and the trace-dual basis is integral. The lifted coordinate ideal is saturated, so it is the image of the sum of the selected idempotents over that local ring: this equality holds generically and both submodules are saturated. It follows that its specialized constant rational coordinate subspace is exactly the required spectral ideal.

It remains to show that this subspace is coordinate in the wall basis. The Schur maximum degree is constant on Rouquier blocks of the two-colour integral-charge cyclotomic algebra by [29, Proposition 3.18]. Apply this to the Ariki–Koike generators $v^b N_{s_0}$ and $v^a N_{s_i}$ for $i > 0$, with parameters $x, x^r, -1$, where $x = v^{2a}$. The change from normalized generators uses only Laurent units. Substituting $x = v^{2a}$ sends each allowed denominator $x^m - 1$ to $v^{2am} - 1$, so the aggregate block idempotent specializes in the same integral localization. Thus the spectral projector belongs to the Hecke algebra over $\mathbb{Z}[v^{\pm 1}, (v^m - 1)^{-1} : m \geq 1]$. This ring expands in integer Laurent series

in v^{-1}, since every inverted denominator is monic at infinity. The projector is orthogonal for the positive original trace form, whose wall canonical Gram matrix tends to the identity. Its range is the constant subspace just obtained. Therefore its limit is the Euclidean orthogonal projector onto that subspace. Each diagonal entry of the limit is the limit of the trace of $C_w C_w^*$ multiplied by the central idempotent. This trace is in the displayed integral ring and lies between zero and $\tau(C_w C_w^*) \to 1$. Its limit is an integer, hence zero or one. An orthogonal projector with such diagonal entries is coordinate, since the squared norm of its image of a unit coordinate vector equals the corresponding diagonal entry. The wall spectral ideal is therefore a C-coordinate span. □

4.3. **Proper parabolic sources and the finite induction step.** We use ordinary bipartition labels (α, β), where the first component corresponds to the positive short eigenvalue. Tensoring with the full sign character sends this label to $(\beta^{\mathsf{t}}, \alpha^{\mathsf{t}})$, with $^{\mathsf{t}}$ denoting partition transposition.

Lemma 4.11. *Let $n \geq 1$. If $A(E) \leq A(E \otimes \mathrm{sgn})$, there are a proper standard parabolic subgroup and a generic irreducible F of that subgroup such that E occurs in $\operatorname{Ind} F$ and $A(F) = A(E)$. If b/a is nonintegral, F can be chosen so that E has multiplicity one and every other constituent has strictly greater spectral degree.*

Proof. Write $U = E \otimes \mathrm{sgn}$, with ordinary label (α, β). The hypothesis is $a_{\mathrm{Sch}}(U) \leq a_{\mathrm{Sch}}(U \otimes \mathrm{sgn})$. Set $b/a = r + \delta$ with r a nonnegative integer and $0 \leq \delta < 1$. For N at least the lengths of α, β, the charged symbol has upper entries $\alpha_i + N + r - i + \delta$ for $1 \leq i \leq N + r$, and lower entries $\beta_j + N - j$ for $1 \leq j \leq N$. Partitions are padded by zeroes. Let Z be the multiset of these entries, $Z_\varnothing$ the corresponding multiset for two empty partitions, and $m = 2N + r$. The statistic $n(Z)$ is the sum of the pairwise minima over unordered pairs of occurrences, since the i-th entry in decreasing order is the minimum in exactly $i - 1$ pairs with an earlier entry. The cancellation-free Schur formula gives

$$a_{\mathrm{Sch}}(U) = a\big(n(Z) - n(Z_\varnothing)\big);$$

see [1, Theorem 3.2 and §4.3]. The same formula, applied to the minimum and maximum degrees of the Schur element, gives

(*)
$$\frac{a_{\mathrm{Sch}}(U) - a_{\mathrm{Sch}}(U \otimes \mathrm{sgn})}{a} = \sum_{z \in Z} z(m - 1 + \delta - z) - \frac{m(m-1)(m - 2 + 3\delta)}{6}.$$

For this identity the degree-plus-valuation of the Schur product, divided by 2, is $b(|\alpha| - |\beta|) + 2a(n(\alpha^{\mathsf{t}}) - n(\alpha) + n(\beta^{\mathsf{t}}) - n(\beta))$. For the cross factors one

uses

$$\begin{aligned}&\sum_{(i,j)\in\alpha}(\alpha_i - i + \beta^{\mathsf{t}}_j - j + 1)\\ &\quad + \sum_{(i,j)\in\beta}(\beta_i - i + \alpha^{\mathsf{t}}_j - j + 1)\\ &= n(\alpha^{\mathsf{t}}) + n(\beta^{\mathsf{t}}) - n(\alpha) - n(\beta) + 2\sum_j \alpha^{\mathsf{t}}_j\beta^{\mathsf{t}}_j.\end{aligned}$$

The cross term cancels against the contribution of the sorted union of the two partitions in the Schur monomial. In the symbol sum on the right of $(*)$, adjoining a box (i,j) changes the sum by $-2(j-i)-(r+\delta)$ in the upper row and $-2(j-i)+(r+\delta)$ in the lower row. The box sums $\sum_{(i,j)\in\alpha}(j-i) = n(\alpha^{\mathsf{t}}) - n(\alpha)$ and the corresponding sum for β give the negative of the degree-plus-valuation divided by $2a$. The empty-symbol entries are $\delta,\dots,N+r-1+\delta$ and $0,\dots,N-1$. Summing the quadratic over these two lists gives the subtracted constant in $(*)$.

The ordinary Pieri rule for $B_{n-k}\times\mathfrak{S}_k$ with sign on $\mathfrak{S}_k$ raises k selected entries of the source symbol by one, with the condition that each row is a strictly increasing admissible beta set. This is the two-colour Pieri rule [31, Lemma 7.6]. Raising the k largest entries increases $n(Z)$ by $\binom{k}{2}$, the sign degree on $\mathfrak{S}_k$ divided by a. For any selection the increase is at least this much, because a pair of raised entries contributes one and a mixed pair contributes a nonnegative amount. When all entries are distinct, equality occurs only for the k largest: if a raised entry is below an unraised entry, the mixed pair contributes strictly positively. The Pieri multiplicities are one, since each resulting bipartition specifies the vertical strips in both rows. We will find the source by lowering a nonempty top set of target entries. Its rank is $n-k$, so $B_{n-k}\times\mathfrak{S}_k$ is proper, with the trivial-group convention for B_0.

Remove initial pairs $(\delta,0)$ from the two rows and subtract one from the remaining entries for each removal. These operations amount to reducing N without changing the bipartition. If the minimum of the normalized symbol is at least one, lower all entries by one. If a successive gap is greater than one, lower all entries above it by one. Both operations give admissible rows and the required top set. For integral ratios, the remaining case has consecutive levels $0,\dots,t$, for some $t\geq 0$, with zero single. If every level is single, both rows occur at positive rank. Choose adjacent levels in different rows and lower the suffix above their cut. The only new tie is across rows, so this again gives an admissible source.

If there are $k>0$ double levels in the integral case, let D be their set, a subset of $\{1,\dots,t\}$. Substitution in $(*)$ gives

$$\sum_{i\in D} i(t+k-i) - \sum_{j=0}^{k-1} j(t+k-j) \;\geq\; kt > 0.$$

Indeed the k smallest values of $i(t+k-i)$ on $\{1,\dots,t\}$ occur at $1,\dots,k$: the expression is increasing up to k and is at least kt from k to t, by concavity and the endpoint values. Thus this case cannot satisfy the inequality in the lemma.

For $0<\delta<1$, suppose the minimum is less than one and all successive gaps are at most one. In $(*)$ the difference is affine as a function of δ while the integer indices in the rows are held fixed. At $\delta=0$ the integer levels are consecutive and zero is single. A missing level would give a gap at least $2-\delta>1$. Hence the difference at zero is nonnegative, and is positive if there is a double level. At $\delta=1$, shift the upper indices up by one and adjoin an upper zero to obtain the symbol at ratio $r+1$. The expression in $(*)$ at $\delta=1$ is exactly the expression for this enlarged symbol at integral ratio. Its levels are consecutive, since a missing level would give a gap at least $1+\delta$ in the original symbol. If zero is double, normalize it once: the original lower zero must have been followed by a lower one, since the upper δ was absent. The new zero is therefore single. The difference at one is again nonnegative. If the difference at zero vanishes, each original index occurs once. The sequence of rows, from the smallest index upwards, is then an upper prefix followed by a lower suffix, since a lower-to-upper transition would give a gap $1+\delta$. Positive rank forces both rows to occur. The upper-to-lower transition becomes a positive double level at the endpoint one, making that endpoint strictly positive. Affine interpolation now makes the difference strictly positive throughout the remaining nonintegral case. This excludes it under the hypothesis of the lemma. At each cut used in the construction the gap is at least one, so lowering by one leaves the selected entries a top set in the source. This gives the asserted equality. Off the walls all source entries are distinct, giving the strict-minimum and multiplicity claims. Twisting back proves the statement for $A(E)$. □

Lemma 4.12. *Let a finite split rational algebra have an integral basis t_w, an involution permuting it, and a positive symmetric trace for which $\langle x,y\rangle=\tau(xy^*)$ makes this basis orthonormal. Suppose that the algebra over $\mathbb{Z}$ contains its unit, that its irreducible matrix representations can be chosen rational with positive adjoint forms, and that the trace is a positive linear combination of their matrix traces. The nonzero coefficients of the unit in the given basis are signs, and the corresponding signed basis elements are orthogonal self-adjoint idempotents. Their left and right multiplications are coordinate projections.*

Proof. For an integral basis element t_w, its matrix in a rational simple block has integral characteristic polynomial. Its eigenvalues satisfy the monic integer characteristic polynomial of left multiplication on the integral algebra. Hence they are algebraic integers and the block characteristic coefficients, being rational, are integers. For any such matrix M, computed with its positive adjoint form, Schur triangularization gives $\operatorname{tr}(MM^*)\geq\sum_\lambda|\lambda|^2$, summing over its eigenvalues with multiplicity. For the roots of a nonzero

irreducible factor of degree k of its characteristic polynomial, the product has absolute value at least one. Their squared moduli therefore sum to at least k, and Cauchy–Schwarz bounds the modulus of their sum by this sum of squared moduli. Adding the factors gives $\operatorname{tr}(MM^*) \geq |\operatorname{tr} M|$. Equality forces normality and equality in the root inequalities. Each nonzero factor then has identical roots of modulus one, hence its roots are 1 or -1 by rationality and irreducibility. The triangle inequality forces a common sign.

The coefficient of t_w in the unit is $\tau(t_w)$, since $\tau(t_w^*) = \tau(t_w)$. The weighted trace inequalities give

$$1 = \tau(t_w t_w^*) \geq |\tau(t_w)|.$$

An integral nonzero coefficient is therefore a sign, and its signed basis element is a self-adjoint projection in all blocks. These projections sum to the identity. For a vector in the range of one projection, the sum of its squared projection norms is its squared norm, so all other projections annihilate it. Multiplication by each signed basis projection is an integral orthogonal projection in the regular orthonormal basis. Its diagonal entries lie in $\{0, 1\}$, and each equals the sum of squares of the entries in its row. This forces all off-diagonal entries to vanish, on either side. □

Proposition 4.13. *For finite B_n, suppose the spectral thresholds are C-coordinate ideals as in Lemma 4.8, and no generator edge at a fixed spectral degree joins different actual two-sided cells, as in Lemma 4.9. If the proper standard parabolic subgroups satisfy* P1–P15*, then so does B_n.*

Proof. The threshold assertion is Geck's E1. By [4, Proposition 3.6, Lemma 3.7 and Corollary 3.8], it gives P1, P4, P7 and P8, identifies the spectral label with a, and identifies the leading algebra with the γ-product. The leading trace τ_J takes the value n_d on a distinguished element d and zero elsewhere. The leading matrix construction [32, Propositions 2.3 and 3.4–3.6] gives split rational blocks with positive adjoint forms and positive trace weights. The unit is integral, since its coordinates are the values of that trace. Apply Lemma 4.12. The unit is a sum of signed basis projections $n_d t_d$ indexed by distinguished elements. Self-adjointness gives $d = d^{-1}$. The coefficient of t_d in $t_x t_y$ is

$$\gamma_{x,y,d} = \tau_J(t_x t_y t_d) = n_d \, \delta_{x,y^{-1}} \, \mathbf{1}_{\{t_y n_d t_d = t_y\}},$$

where the indicator is one when its condition holds and zero otherwise. The unique right unit of t_y gives P2, P3 and P5. Thus P1–P8 hold. Lusztig's parabolic argument [8, §14.12] gives P12 from P3, P4 and P8 for the ambient and source groups.

Fix an exact a-layer of value A. In its generic Hecke algebra, the simple blocks have $A(E) = A$. If a proper parabolic two-sided cell has value A, its lower boundary has larger intrinsic a by P4 and P11 in that subgroup. By P12 the source has ambient degree A and its boundary has ambient degree greater than A. In the ambient quotient by degrees strictly greater than A,

the source cell quotient therefore embeds on its original coordinates. The ambient threshold ideal of degrees at least A therefore contains every hC_pk from the source. Its image in the quotient is the exact A-layer. The defining bound for the a-function gives degree at most A at all surviving output coordinates. Apply Lemma 3.1 to the quotient and source. The constant comparison image is stable under both actions, which commute on it with independent parameters.

We verify the generic block support of the comparison image and of the leading ideal generated by the source. The generic image is the two-sided ideal generated by the source cell quotient: the two canonical comparison matrices are invertible and multiplication in the middle source is identified with the original source action. Thus the generic blocks in the image are exactly those irreducibles whose restrictions contain a constituent of that source cell. The leading ideal has the same block support. To see this, complete at $v^{-1} = 0$ and scale the source and ambient products by v^{-A}. The resulting finite free algebras reduce to their leading algebras. A lift of a unit of the special fibre has invertible left and right multiplication matrices over the complete ring. If u is this lift, solve $ue = u$ and $e'u = u$. Associativity and invertibility imply e is a left unit and e' a right unit, so they agree. On the ambient layer the map in each block is $h \mapsto v^{-A}\rho_E(h)$, which is multiplicative for the scaled product. It is integral by balancedness, and its reduction is the leading representation. The direct sum of the reduced maps is an isomorphism by the leading matrix construction, so the direct sum over the complete ring is an isomorphism as well. The lifted source unit is therefore an idempotent matrix in each such block, and its generic and special ranks agree. Its generic rank is positive exactly when that block sees the source quotient, and its special rank is positive exactly when the corresponding leading block sees the source leading unit. Hence the block supports, and therefore dimensions, agree. The constant image contains the ordinary leading ideal by stability and inclusion of the source basis. Equality of dimensions identifies them.

For each actual two-sided cell, the spectral degrees of all constituents have one common value A. The longest-element dual cell has constituents obtained by the sign twist, with common value A'. Suppose $A \leq A'$. Lemma 4.11 gives, for each constituent, a proper source constituent of the same spectral degree. The source support just proved puts every corresponding leading block in a constant image. These images give commuting independent actions on the whole cell quotient.

For the other inequality use the dual cell. Here are the duality conventions for the transfer. Identify the coefficient functional $D_w(C_x) = \delta_{w,x}$ with a trace-dual element of the finite Hecke algebra. For the longest element w_0, [8, Proposition 11.5] gives

$$(D_w N_{w_0}^{-1})^\dagger = (-1)^{\ell(w^{-1}w_0)} C_{w^{-1}w_0}.$$

The map $D \mapsto (DN_{w_0}^{-1})^\dagger$ sends hDk to $h^\dagger(DN_{w_0}^{-1})^\dagger k^\dagger$, since w_0 is central in type B. It gives a signed identification of the dual cell quotient with the longest-element dual cell quotient, by the order reversal in [8, Corollary 11.7]. Dualizing interchanges the transpose of a left action with the right action, and $C_s^\dagger = (v^{L(s)} + v^{-L(s)})1 - C_s$. The scalar terms disappear from commutators, so the independent generator commutators vanish on one cell quotient exactly when they vanish on its dual. This proves the cell-quotient assertion in both orientations.

Taking the coefficient of u^A in the right action at an independent parameter u shows that the full left action in a cell quotient commutes with every right multiplication in its leading algebra. In particular it preserves the coordinate projections onto right-unit blocks. P3 and P8 place a basis index in the same left cell as its right unit, and any left edge inside that two-sided cell preserves the right unit. This proves P13 and also proves that a left preorder relation inside a single two-sided cell is a left cell equivalence. P6 and P13 give P14 by [8, §14.14]. The assumed separation of two-sided cells at equal degree, combined with the left equivalence just proved and its inverse, gives P9 and P10. P4 and reversal of each edge of an equal-degree path give P11. Finally the exact a-layer is the direct sum of its cell quotients, because an equal-degree edge cannot join different cells. The commuting independent actions on these quotients give P15 on the entire layer. □

4.4. **Induction on the rank.**

Proof of Theorem 4.1. In rank zero the algebra is $\mathcal{A}$. In rank one the cell lattices and the KZ standards are the two one-dimensional modules, with labels exchanged by †. In rank two the finite dihedral calculation gives P1–P15 [4, Proposition 5.1]. For the off-wall rank-two cell classification, in addition to the singleton cells $\{1\}, \{w_0\}, \{s\}, \{w_0 s\}$, where s is the lighter of the two simple reflections and w_0 is the longest element, the left cells are the two sets $\{t, st\}$ and $\{ts, sts\}$, where t is the heavier reflection. This follows from the generator multiplication formula in the dihedral calculation. Each two-element set affords the unique two-dimensional representation. Rank-r domino insertion on the eight signed permutations gives these recording fibres: if $b < a$ their two-element sets are $\{(-2,1),(-2,-1)\}$ and $\{(2,1),(2,-1)\}$. If $a < b$ they are $\{(-1,2),(-2,1)\}$ and $\{(2,-1),(1,-2)\}$. The singleton characters are determined by their left descent sets, since N_s acts by $v^{L(s)}$ for a descent s and by $-v^{-L(s)}$ otherwise. Inversion gives the right cells. The based isomorphisms and the local recognition in rank two follow from the proper rank-one sources as in Lemmas 4.5 and 4.7. For higher ranks, assume all three assertions in smaller ranks. Lemma 4.4 first identifies the off-wall actual cells and Lemma 4.5 gives their based isomorphisms. The third assertion follows from Lemma 4.7. Lemma 4.8 gives the spectral threshold ideals and Lemma 4.9 separates the two-sided cells in each exact layer. Proposition 4.13 now gives P1–P15 off the walls. By P4, P9 and P10, a two-sided path between elements of one two-sided cell stays

at a common value of a, and each edge preserves the shape. Conversely the recording and insertion fibres of a fixed shape connect all its pairs of tableaux. This proves the two-sided claim in the second induction assertion.

For an integral ratio b/a, use the off-wall result just proved in the same rank in Lemma 4.10. This supplies the spectral threshold ideals on the wall. Lemma 4.9 and Proposition 4.13 complete P1–P15 there, using the wall statements in proper ranks. This proves the induction for all positive weights. □

Every proper standard parabolic of either affine group in the paper is a product of finite groups of type B and equal-parameter finite Weyl groups. The latter include the finite factors of type D at the affine B fork. The following observation gives the finite results for all those products.

Lemma 4.14. *A finite direct product of weighted Coxeter systems satisfying* P1–P15 *also satisfies* P1–P15*.*

Proof. The canonical basis is the tensor product basis, by bar invariance and triangularity. Its structure coefficients are products and its a- and Δ-values are sums of the factor values. The a-formula follows by independently choosing a maximizing product in each factor. P1 therefore identifies distinguished tuples with tuples of distinguished elements, and the signs and leading coefficients are products. Left, right and two-sided preorders are product preorders, since a generator acts in one tensor factor. This proves P1–P8 and P12–P14 componentwise. For P9–P11 use P4 to see that equal sums along a preorder relation give equality of every factor value. On an exact a-layer a projected generator changes at most one factor and preserves its value. The layer thus splits by the tuple of factor values. On each summand the independent left and right actions commute by the factor P15 when they act in the same position, and by tensor factorization otherwise. □

Lemma 4.15. *Let W_I be a finite standard parabolic subgroup of either affine group. For every irreducible E choose a finite balanced model ρ_E for the convention at infinity, and put*

$$b_E(w) = \lim_{v\to+\infty} v^{-A(E)}\rho_E(N_w) \qquad (w \in W_I).$$

The limit is the same with C_w in place of N_w, and $b_E(w) \neq 0$ implies $a_I(w) = A(E)$. For each d the span of the C_w with $a_I(w) \geq d$, over the fraction field, is the sum of the generic simple matrix blocks with $A(E) \geq d$. On the exact d-layer the matrices $b_E(w)$ realize the finite asymptotic algebra, with its unit acting as the identity on each block with $A(E) = d$. If $W_J \subseteq W_I$ and F is a constituent of the restriction of E to the smaller Hecke algebra, then $A(F) \leq A(E)$.

Proof. Use the finite balanced models of [32, Proposition 2.3], with the semilinear sign change described above. The lower coefficients of C_w have strictly negative degree, so the two matrix limits agree. From [4, Lemma 3.2 and

Remark 4.2], P1–P15 imply that the constituents of a left cell have spectral degree equal to its a_I-value, and that $b_E(w) \neq 0$ implies $a_I(w) = A(E)$. P4 makes each threshold span a two-sided ideal. Its left-cell filtration, and that of its quotient, therefore identify its generic blocks. The identification of the leading matrix algebra with the γ-product is [4, Proposition 3.6]. The unit is sent to the identity by the block realizations of [32, Propositions 3.4–3.6].

For the restriction statement let $m_{F,E'}$ be the multiplicity of F in the restriction of E'. The standard traces of the two parabolic algebras satisfy $\tau_I|_{H_J} = \tau_J$, so

$$s_F(v)^{-1} = \sum_{E'} m_{F,E'} s_{E'}(v)^{-1}.$$

At a large positive real v, the central primitive idempotents are self-adjoint for the positive standard trace form. Indeed the adjoint of left multiplication by such an idempotent is left multiplication by its conjugate under $*$, and positivity forbids two different central idempotents from being adjoints. The trace identity at each idempotent gives $s_{E'}(v) > 0$. Thus the terms of highest degree on the right have positive leading coefficients, and $A(F) = \min_{m_{F,E'}>0} A(E')$. □

5. The affine spectral filtration

Write e_i for the ith standard coordinate vector in $\mathbb{R}^n$ and W_0 for the finite group of signed permutations of these coordinates. For the affine Coxeter systems here the translation lattice X is $\mathbb{Z}^n$ in type $\widetilde{C}_n$, with finite root system B_n in the Bernstein presentation, and is $\{x \in \mathbb{Z}^n : \sum_i x_i \in 2\mathbb{Z}\}$ in type $\widetilde{B}_n$, with finite root system C_n in that presentation. In both cases $W = X \rtimes W_0$. The Bernstein lattice elements θ_x for $x \in X$ generate a Laurent group algebra, with $\theta_x\theta_y = \theta_{x+y}$. The W_0-invariants are central. The group ring in these elements and the finite Hecke algebra give the integral Bernstein presentation [3, Theorem 2.1 and §2.0.4].

5.1. **Tempered families and their exponents.** Specialize v to a real number greater than one. We use the Plancherel decomposition of the trace [9, Theorem 4.43 and Corollary 4.45]. Its data are parabolically induced representations $\pi_{\alpha,v,z}$ indexed by finitely many association classes α, with z in a compact torus $T_{\alpha,u}$. More precisely, choose one finite Levi and one of its discrete-series representations for each class and vary its twist by unitary unramified characters. The torus is the compact part of a complex algebraic torus T_α. The finite group G_α of association maps of this family acts by affine monomial automorphisms of T_α, using conjugation by normalized intertwiners on its compact-model matrices. The Plancherel measure on the chosen torus includes the reciprocal finite-group factors. Write $d\mu_{\alpha,v}$ for that measure and Tr for the unnormalized matrix trace. Thus

$$\tau(h) = \sum_{\alpha} \int_{T_{\alpha,u}} \operatorname{Tr}\big(\pi_{\alpha,v,z}(h)\big)\, d\mu_{\alpha,v}(z).$$

The associated Fourier isometry on the Hilbert completion in the standard basis has image the square-integrable equivariant matrix sections. At each unitary point the commutant of the intertwiner action of its stabilizer is the algebra of blocks corresponding to the irreducible tempered constituents [3, Theorem 5.5].

The central character on a family is the W_0-orbit of a character of X with values $\zeta(x)v^{2\langle x,R\rangle}$, where ζ is unitary, $X^\vee = \mathrm{Hom}(X,\mathbb{Z})$, $R \in X^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\langle x, R\rangle = R(x)$. The coordinates R_i are those of R in the dual of the standard coordinate basis of $X \otimes_{\mathbb{Z}} \mathbb{R}$. Its real exponent R is constant on the torus up to W_0. On a fixed integral parameter ray there are finitely many such orbits. Indeed the discrete-series residual equations are root-character equations with signed monomial right sides and the semisimple Levi residual points are isolated. Choosing a linearly independent set of equations expresses their coordinates as roots of unity times fixed rational powers of v. Unramified twists only change the unitary part. We take v sufficiently large in what follows.

For a real number $r \geq 0$ put $K_r(t) = \max(0, r - |t|)$. The Plancherel exponent of a family is the following W_0-invariant function of its real exponent:

(1)
$$A_R = \sum_{i<j} \big(K_a(R_i - R_j) + K_a(R_i + R_j)\big) + \sum_i \big(K_{(b+c)/2}(R_i) + K_{|b-c|/2}(R_i)\big).$$

In the $\widetilde{B}_n$ case put $c = 0$ in this expression. Specifically, with Haar probability measure dz on the compact torus,

$$d\mu_{\alpha,v}(z) = v^{-2A_R} m_{\alpha,v}(z)\, dz, \qquad m_{\alpha,v} \longrightarrow m_{\alpha,0} \tag{2}$$

uniformly, where $m_{\alpha,0}$ is Laurent polynomial, nonnegative on the compact torus, and positive on a nonempty open set.

For the degree calculation use the product of reciprocal c-functions with the inverse finite longest-element parameter as prefactor, as in [9, (3.57)]. The prime product at a residual torus omits identically zero numerator and denominator factors [9, Theorem 3.25]. Inverting the monomial inside any such factor to change conventions has no effect on the degree of the prime product, since that monomial is identically one on the residual torus. The labels in [9, Convention 2.1] allow exponent zero. A factor $1 - \epsilon v^j z^\nu$ with $|\epsilon| = 1$, a torus character z^ν, and $j \neq 0$ has degree $\max(j, 0)$. The same degree rule applies generically on the torus for $j = 0$, and deleted factors have exponent zero. For a root $e_i \pm e_j$ let $t = R_i \pm R_j$. The numerator for the root and its negative contributes $2|t|$. The denominator contributes $2\max(|t| - a, 0)$, and the prefactor contributes $-2a$. This gives $-2K_a(t)$. For an endpoint root in the $\widetilde{C}_n$ datum the numerator contributes $4|R_i|$. The denominator contains the two pairs with parameter exponents $-(b+c)$ and $-(b-c)$, respectively. If $b \geq c$ their degree is $2\max(|R_i| - (b+c)/2, 0) + 2\max(|R_i| - (b-c)/2, 0)$. If $b < c$, replace $b-c$ by $c-b$ and add $2(c-b)$ to that

degree. Combining with the prefactor $-2b$ gives the endpoint term in $-2A_R$. In the reduced $\widetilde{B}_n$ datum the endpoint root $2e_i$ gives $-2K_b(2R_i)$, which is $-4K_{b/2}(R_i)$. These computations include the formal-degree factors of the inducing discrete series, using the same prime product on the Levi roots. The residual-degree constants are positive and independent of both v on the fixed ray and the torus variable [9, Corollary 3.32 and Theorem 5.6]. Each nonstationary factor, after its leading monomial is removed, has uniform limit 1 in absolute square. Stationary binomial denominators cancel on the unitary torus, by real analyticity of the exact density. Each irreducible component of a stationary binomial divisor has a Zariski-dense unitary locus, so cancellation is algebraic. This proves (2).

5.2. **Leading compact matrices.** The compact induction model is the tensor product of the inducing space and the span of the finite minimal coset representatives, with positive form the tensor product of the inducing form and the standard orthonormal coset form [9, Proposition 4.19]. In particular its Gram matrix is independent of z. For each fixed algebra element with Laurent coefficients, its compact matrix is Laurent polynomial in z, with finite support independent of v. This follows by expressing multiplication in the right H^P-coset basis for the finite Levi with full Bernstein lattice. The quotient and twist to its semisimple inducing algebra send a Bernstein element to the product of its projected Bernstein element and a torus character [9, Proposition 4.16].

The coefficients of the inducing matrices can be taken algebraic over $\mathbb{C}(v)$. For completeness, the algebra at a prescribed algebraic residual central character is finite dimensional over the function field since the affine algebra is finite over its Bernstein center. Over a finite extension of the function field its semisimple quotient is a direct sum of split matrix algebras. After localizing the finite algebra on the parameter curve, its radical and its split quotient specialize as such: the former is nilpotent and the latter a sum of matrix algebras. Thus for all sufficiently large v each branch of simple modules is given by an algebraic matrix representation. The strict weight inequalities for a discrete series distinguish its branches and are constant on the ray. The invariant Hermitian form on a unitary simple branch can also be chosen algebraic on the real ray, by solving the linear equations for the adjoint relations and choosing the eventual positive sign. Applying Hermitian elimination and rescaling the positive diagonal pivots gives a torus-independent algebraic basis change with orthonormal form. The matrices have convergent Puiseux expansions at infinity. These arguments can be carried out after choosing a single finite extension for the finitely many branches and taking its real positive determinations where needed.

Write

$$c_{\alpha,w}(z) = \lim_{v \longrightarrow \infty} v^{-A_R} \pi_{\alpha,v,z}(N_w).$$

To justify this limit, an entry of degree d in its Puiseux expansion is bounded from below in absolute value by a positive constant times v^d on a compact

patch with nonempty interior where its leading coefficient is nonzero and $m_{\alpha,0}$ is positive. The Plancherel squared norm of N_w is 1, so (2) forces $d \leq A_R$. The limit is uniform since the torus support is finite. It is unchanged for C_w, by its finite strictly negative triangular transition from N_w.

The finite torus twisting groups depend only on the root datum. To test association of two branches, use the twisting and Weyl isomorphisms of the Levi algebras [3, §3.2] and solve the equations for an intertwiner on a finite generating set. The coefficient matrix of this linear system is algebraic in v, so its rank is constant for all sufficiently large v. For two simple branches a nonzero solution is an isomorphism. Thus the families and their finite association groups can be fixed on that interval. A nonzero algebraic solution can be chosen by elimination, and in the orthonormal Levi models its adjoint times itself is a positive scalar by Schur's lemma. Dividing by the positive square root of this scalar gives a unitary identification. These identifications are independent of the induction variable.

The normalized intertwiners and their inverses have uniform limits $U_g(z)$ in the orthonormal compact model, including at stabilizer points. Indeed the formula [9, Theorem 4.33, (4.74)] uses the inverse of a product of Bernstein factors on the indicated Levi subspace. Its torus denominators are products of binomials $1-\epsilon v^j z^\nu$ on that subspace. Nilpotent parts can increase powers of these denominators, since the inverse on a generalized weight is a finite geometric expression. Each stationary divisor has a Zariski-dense unitary locus. Unitarity at every sufficiently large v therefore cancels the corresponding pole at each such v. This cancellation holds over the parameter field as well: division by a power of a fixed primitive binomial leaves a remainder with algebraic coefficients in v, which vanish identically if they vanish at all sufficiently large v. The remaining denominators, with their leading monomials removed, have uniform nonzero limits. Boundedness removes the positive powers of v in the resulting expansion. The same argument for the inverse shows that U_g is an invertible Laurent matrix. These limiting operators are projective intertwiners, so conjugation gives an ordinary group action. Its invariant polynomial sections form

$$\mathcal{B}_\alpha = \left(\mathrm{Mat}_{d_\alpha}(\mathbb{C}[T_\alpha])\right)^{G_\alpha},$$

where d_α is the compact model dimension and $\mathbb{C}[T_\alpha]$ its Laurent coordinate ring.

5.3. **The polynomial Fourier image.** The Plancherel identity for fixed pairs of standard basis vectors, after taking the uniform limits above, gives an isometry

$$F_0 : \ell^2(W) \longrightarrow \bigoplus_\alpha L^2(T_{\alpha,u}, m_{\alpha,0}dz; \mathrm{Mat}_{d_\alpha})^{G_\alpha}, \qquad N_w \longmapsto (c_{\alpha,w})_\alpha. \tag{3}$$

Here $\ell^2(W)$ is the Hilbert space with orthonormal basis $(N_w)_{w\in W}$, and a matrix M in a fiber has squared norm $\mathrm{Tr}(MM^*)$. We prove that F_0 identifies the finite-support complex vector space on W with $\bigoplus_\alpha \mathcal{B}_\alpha$.

Write an element of W as ut_xu', where $u, u' \in W_0$, $x \in X$ is dominant, and t_x is translation by x. Opdam's formula [9, (2.25)] expresses $N_{ut_xu'}$ as a finite sum of fixed finite factors on both sides of θ_x, with coefficients constant as x varies in the relative interior of one face of the dominant cone. Subdivide the faces into relative interiors of rational simplicial cones. Choose lattice generators on the rays of each cone. These generate a finite-index sublattice of the lattice in its span. Subtracting integral multiples of the ray generators leaves finitely many lattice offsets in a bounded parallelepiped. For a relative interior, choose the fractional coefficients of the offsets in $(0, 1]$, so that the remaining integral coefficients are nonnegative. In the resulting parametrization the coordinates m_i are nonnegative integers, x is a fixed offset plus the sum of m_i times fixed generators, and each entry is a finite sum of terms of the form

$$R_j(v, z)P_j(m)\epsilon_j(m)v^{\kappa_j(m)}z^{\nu_j(m)}. \tag{4}$$

In this formula P_j is polynomial, ϵ_j is a unitary lattice character, κ_j is a real linear form nonpositive on the cone, and ν_j is a linear map to the torus character lattice. The finite factors and the offsets are absorbed into R_j. To obtain the formula, use the simultaneous generalized translation weights. On one generalized weight space the unipotent parts of the commuting translations have commuting nilpotent logarithms. Their powers give a polynomial in m of bounded degree. The projection onto a generalized weight space is a product of univariate Hermite spectral projectors for the members of a lattice basis. Its denominators are powers of differences of distinct monomial eigenvalues. After taking Puiseux coefficients, all possible torus poles of R_j are therefore stationary unitary binomial divisors.

For each term the only point of the nonnegative real cone with $\kappa_j(m) = 0$ and $\nu_j(m) = 0$ is $m = 0$. In the induced model, the Bernstein relations give a triangular action of θ_x on the finite coset basis ordered by Bruhat order. Indeed commuting θ_x past a reduced word for a representative u gives the top term $N_u\theta_{u^{-1}x}$, and each correction omits a simple factor and has lower Bruhat order. Replacing the lower terms by their minimal right coset representatives preserves the triangular order. Thus every joint weight comes from evaluating $\theta_{u^{-1}x}$ on a Levi weight times the unramified character. Since u is minimal in its right coset, it sends positive Levi roots to positive roots, so $u^{-1}x$ is Levi-dominant for dominant x. Its projection to the unramified quotient is measured by ν_j. If that projection vanishes, the pullback lies in the Levi span, where the strict discrete-series Casselman inequality is negative on every nonzero dominant translation. The criterion applies first to lattice points [9, Lemmas 2.20 and 2.22]. The equations $\kappa_j = \nu_j = 0$ and the cone are rational, so a nonzero real solution would give a nonzero rational solution and hence a lattice solution after scaling. Consequently, for $\nu_j(m)$ in a fixed finite set, there are constants $C, c > 0$ with

$$\kappa_j(m) \le C - c\sum_i m_i. \tag{5}$$

Indeed a sequence with bounded frequencies, unbounded $\sum_i m_i$, and $\kappa_j(m)/\sum_i m_i \to 0$ would yield a common zero on the compact simplex $\sum_i m_i = 1$. Choose c smaller than the resulting uniform lower bound for $-\kappa_j(m)/\sum_i m_i$ at large $\sum_i m_i$, and enlarge C for the bounded part.

First consider a polynomial matrix $B(z)$ on one family. Let $B_v(z)$ be its average under the exact conjugation action of G_α and $B_0(z)$ its limiting average. These are equivariant sections. Their convergence is uniform on the compact torus. Choose a scalar invariant Laurent polynomial $f(v,z)$ of degree zero in v, with nonzero limit $f_0(z)$, clearing the torus denominators of all products used in the Fourier coefficients of B_v. Concretely take the product of the finitely many denominators of B_v, the exact density, and the R_j^*, and their G_α-translates, then divide by the leading power of v. The R_j^* here are adjoints for real v and unitary z, which extend rationally to the torus variables. An additional invariant Laurent factor does not affect the argument. The exact packet $v^{A_R} f B_v$ on this family, zero on the other families, has inverse Fourier coefficients given by integrating its matrix pairings with $\pi_{\alpha,v,z}(N_w)$ against $d\mu_{\alpha,v}$. They are finite constant terms in z. In (4) only a fixed finite set of $\nu_j(m)$ can contribute. By (5) their degrees tend linearly to $-\infty$ with $\sum m_i$. Moreover all their coefficient functions in v come from a fixed finite list, up to polynomial factors in m and $v^{\kappa_j(m)}$. Hence each coefficient is bounded by $C'(1+\sum_i m_i)^k v^{D-c\sum_i m_i}$ for some constants C', D, k independent of m and sufficiently large v. Once $c\sum_i m_i > D$, the bound tends to zero and is dominated by its value at any fixed sufficiently large v. The number of nonnegative integer tuples with fixed sum grows at most polynomially in that sum. The squared bounds are therefore summable. The fixed-coordinate limits, which exist by uniform convergence in the Plancherel pairing, give a finite vector x, and dominated convergence gives convergence of the inverse vectors in $\ell^2(W)$.

The norms of the exact packets tend to the norm of $f_0 B_0$ in (3). Also the limiting coordinates are exactly the pairings of $f_0 B_0$ with $F_0(N_w)$, by the fixed-vector limit. Since F_0 is isometric, the equality of norms shows that $f_0 B_0 = F_0(x)$. The same holds for $g f_0 B_0$ for any invariant Laurent polynomial g. The continuous invariant functions $\overline{f_0}/(|f_0|^2+\varepsilon)$ can be uniformly approximated on the compact torus by invariant Laurent polynomials, by trigonometric approximation followed by averaging. Multiplying by $f_0 B_0$ and taking $\varepsilon \downarrow 0$ shows B_0 is in the closed range of F_0. Here the zero set of the nonzero Laurent polynomial f_0 has Haar measure zero. Laurent matrices are dense, and averaging for the limiting unitary conjugation action is a bounded projection. Thus F_0 is surjective on the Hilbert spaces.

We next check that the inverse image of a Laurent matrix in the invariant range has finite support. On each lattice parametrization, the top Puiseux degree of each R_j is fixed. If κ_j is strictly negative on a coordinate direction, choose a cutoff so that the top degree plus $\kappa_j(m)$ is less than A_R once that coordinate is above the cutoff. The nonpositivity of κ_j on the other directions makes the cutoff independent of their values. Subdivide into

finitely many strata where each coordinate is either fixed below the cutoff or free above it. After shifting the free coordinates, on a stratum with r free coordinates the leading matrix has the expression

$$c_m(z) = \sum_j Q_j(m,z)\chi_j(m,z), \qquad m \in \mathbb{N}^r.$$

Here χ_j are distinct unitary lattice characters with torus frequencies $\nu_j(m)$, and the Q_j are polynomials in m with rational torus coefficients. Terms with identical characters have been combined. For every surviving term κ_j vanishes in the free directions, so $\|\nu_j(m)\| \geq c_j \sum_i m_i$ for some $c_j > 0$. Distinct characters have distinct phase vectors for almost all z. If the maximum degree in m of the Q_j is d, the average of $\|c_m(z)\|^2$ over $0 \leq m_i < M$, divided by M^{2d}, tends pointwise almost everywhere to

$$\sum_j \int_{[0,1]^r} \|Q_{j,d}(x,z)\|^2 \, dx.$$

The diagonal terms are Riemann sums. For a fixed phase $\omega \neq 1$, the partial sums of ω^ℓ are bounded, so summation by parts gives $\sum_{\ell=0}^{M-1} \ell^h \omega^\ell = O_\omega(M^h)$ for every integer $h \geq 0$. Applying this in a coordinate with distinct phases makes each off-diagonal term $O(M^{-1})$ after normalization. Since each c_m has squared L^2-norm at most one, Fatou's lemma forces the degree-d terms to vanish if $d > 0$. Thus each amplitude is independent of m, say $Q_j(z)$. Taking the same average with $d = 0$ shows that each Q_j is square-integrable for $m_{\alpha,0}dz$.

A pole of order k of an entry of Q_j along a primitive stationary binomial divisor is detected near a generic unitary point of that divisor. With u a real transverse coordinate, a squared entry times the density is bounded below by a positive multiple of $|u|^{s-2k}$ near a generic point, where s is the vanishing order of the density. Integrability gives $s - 2k > -1$, hence $s \geq 2k$. As these are the only possible poles and the Laurent ring is factorial, $m_{\alpha,0}Q_j^*$ is Laurent polynomial. Pairing c_m with a fixed Laurent matrix therefore gives only a finite set of possible frequencies $\nu_j(m)$. The lower bound for $\|\nu_j(m)\|$ restricts m to a finite set. Strata with no free coordinate are already finite. This proves finite support and the asserted polynomial Fourier identification.

5.4. **Integral level projections and multiplication.** The following construction identifies the original index sets of the spectral levels. For a remaining real exponent orbit choose a representative R of largest norm for a rational W_0-invariant inner product, and take a positive integral multiple $\lambda \in X$ of its metric dual. Write $D = 2\langle \lambda, R \rangle$. The central element

$$z_\lambda = \sum_{\mu \in W_0 \lambda} \theta_\mu$$

sums over distinct orbit elements. On the chosen orbit $v^{-D} z_\lambda$ has one leading monomial of modulus one. On the other remaining orbits its value tends

uniformly to zero. Indeed Cauchy–Schwarz gives $2\langle \mu, R' \rangle \leq D$, with equality only for R' in the orbit of R and μ its chosen multiple of the metric dual. If the last remaining orbit is zero, use $z_\lambda = 1$.

Identify the regular Hilbert spaces at different v by the orthonormal standard basis, and choose $t = v^{-1/e}$ for a positive integer e clearing the finitely many exponents. The left regular operator of a fixed Laurent Hecke element is a Laurent polynomial in t with bounded operator coefficients, integer entries, and finite row and column supports. This follows from the quadratic multiplication formula for the finitely many simple factors in each term. If such a family, or an operator-meromorphic family with a finite-order pole, is bounded as $t \downarrow 0$, its negative coefficients vanish, as can be checked in each matrix entry.

Suppose the projections already constructed sum to $P(t)$ and commute with the regular actions. Begin with $P = 0$. Apply the preceding observation to

$$B(t) = v^{-D}(1 - P(t))L_{z_\lambda},$$

where L_h denotes ordinary left multiplication by h. Its norm is bounded by the supremum of its central spectral values. Put $Y = B^\sharp B$, where $\sharp$ is adjunction of Taylor coefficients, agreeing with Hilbert adjunction for real t. The constant coefficient $Y(0)$ is an orthogonal projection because on the positive real axis $\|Y^2 - Y\| \to 0$. It has integer matrix entries. An integral orthogonal projection is coordinate in an orthonormal basis: its diagonal entries are in $\{0, 1\}$, and the squared column norm equals that diagonal entry.

Set $f(Y) = 3Y^2 - 2Y^3$. The identities

$$f(Y) - Y = -(2Y - 1)(Y^2 - Y), \qquad f(Y)^2 - f(Y) = (Y^2 - Y)^2(4Y^2 - 4Y - 3)$$

give uniform convergence on a small closed disk about $t = 0$. In fact shrink the disk so that the initial norm is at most $5/4$ and the initial defect norm at most $1/100$. While the norm is at most 2, a defect of norm δ gives an increment of norm at most 5δ and a new defect of norm at most $27\delta^2$. The sum of all increment bounds is at most $5/(100 - 27) < 3/4$, so the iterates stay in that norm bound. On the real axis the spectrum of Y is uniformly close to 0 or 1. The scalar iterations stay in the respective neighborhoods and converge to 0 or 1, by the same bounds. Functional calculus therefore identifies the limit with the exact projection $E_R(t)$ onto the cluster near 1. The order of vanishing of the defect at $t = 0$ at least doubles at each step. Each Taylor coefficient therefore stabilizes and has integral entries and finite row and column supports. The projection is zero on $P(t)$, and recursion gives mutually orthogonal projections summing to the identity. By the fixed-pair Plancherel limit their constant coefficients correspond under F_0 to the real-exponent-orbit cutoffs. Summing over equal values of A_R therefore gives a partition of W into coordinate sets S_A, where A runs over the finitely many spectral degrees. The involution $w \mapsto w^{-1}$ preserves these

sets, since the corresponding matrix cutoffs commute with the adjoint on sections.

Proposition 5.1. *The leading matrices give a unital associative $\mathbb{Z}$-algebra J on the basis $(t_w)_{w\in W}$, with an involution $t_w^* = t_{w^{-1}}$. Its complexification is $\bigoplus_\alpha \mathcal{B}_\alpha$. The real-exponent cutoffs decompose J into its spectral levels. The symmetric trace on J, given by integrating matrix traces against the measures $m_{\alpha,0}dz$, makes the original basis orthonormal for $\langle x, y\rangle = \tau_J(xy^*)$.*

Proof. On the regular Hilbert space define the central operators, summing over real exponent orbits

$$P_-(v) = \sum_R v^{-A_R} E_R, \qquad P_+(v) = \sum_R v^{A_R} E_R.$$

Their series at $t = 0$ are meromorphic with finite-order poles, with integral coefficient matrices of finite row and column support. For fixed x, the operator $P_-(v)L_{N_x}$ is bounded because its Fourier matrix is $v^{-A_R}\pi_{\alpha,v,z}(N_x)$. Its constant column at N_y is therefore a finite integral vector. Its coefficient at N_w is the limit of

$$\sum_\alpha \int_{T_{\alpha,u}} \operatorname{Tr}\big(v^{-A_R}\pi_{\alpha,v,z}(N_x)\, v^{-A_R}\pi_{\alpha,v,z}(N_y) \cdot (v^{-A_R}\pi_{\alpha,v,z}(N_w))^*\big) m_{\alpha,v}(z)\, dz.$$

Uniform convergence identifies these limits with the pairings of the pointwise product $(c_{\alpha,x}c_{\alpha,y})_\alpha$ against $F_0(N_w)$. The polynomial image result shows that this pointwise product lies in the image. Thus the constant columns give the stated multiplication, and associativity and the involution follow in the Fourier model.

Likewise $\|P_+(v)N_1\|^2 = \sum_\alpha \int d_\alpha m_{\alpha,v}\, dz$ is bounded. The constant coefficient is a finite integral vector. Pairing with each N_w and taking the limit identifies its Fourier image with the identity sections. This gives the unit. The trace and orthonormality follow from (3). The direct sum by spectral levels is a decomposition into ideals because it is pointwise on the families. □

6. Finite parabolic constituents in type $\widetilde{C}_n$

Fix $n \geq 2$. Use the $\widetilde{C}_n$ realization $W = \mathbb{Z}^n \rtimes W_0$, with W_0 the group of signed permutations. The middle weight is $a > 0$, the finite endpoint weight is $b > 0$, and the affine endpoint weight is $c \geq 0$. For $0 \leq k \leq n$, the finite standard subgroup W_k obtained by deleting node k, with nodes numbered $0, \ldots, n$ from the affine endpoint, has factors of types B_k and B_{n-k} with endpoint weights c, b, respectively. A factor of rank zero is trivial.

For a simple representation σ of a finite parabolic Hecke algebra, its Schur element s_σ is defined by the trace expansion

$$\tau(h) = \sum_\sigma s_\sigma^{-1} \operatorname{Tr}(\sigma(h)),$$

where τ is the coefficient-of-identity trace of that algebra and Tr is the ordinary matrix trace. Its finite spectral degree is half the largest exponent of v in s_σ. For real weights we use the same definition with real exponents in the finite Schur formula.

Proposition 6.1. *Let R be a real central exponent vector of an irreducible tempered representation of the $\widetilde{C}_n$ Hecke algebra with weights a, b, c as above. Its restriction to the finite parabolic Hecke algebra of some W_k contains a simple type of finite spectral degree A_R, the spectral level in (1).*

For the rest of this section use v^a as the base of exponents, so that the middle weight is one. Divide b, c and the coordinates of R by a and retain their notation. To recover degrees in v, multiply both sides of the degree equality by a.

6.1. **Root sums and Schur degrees.** Define the two signed short centers by

$$m = \frac{b+c}{2}, \qquad d = \frac{b-c}{2}. \tag{6}$$

For $r \geq 0$ and $t \in \mathbb{R}$, write $K_r(t) = \max(0, r - |t|)$, as in Section 5, and use $u_+ = \max(u, 0)$ for $u \in \mathbb{R}$. For a list $R = (R_1, \ldots, R_N)$ of real exponents, let

$$A_R = \sum_{i<j} \big(K_1(R_i - R_j) + K_1(R_i + R_j)\big) + \sum_i \big(K_{|m|}(R_i) + K_{|d|}(R_i)\big). \tag{7}$$

Here $1 \leq i < j \leq N$ and $1 \leq i \leq N$ in the respective sums. This is (1) with middle weight one. In the recursions the list may be empty, in which case the sum is zero.

Use the bipartition labels for the simple finite type B modules, with the upper partition α associated with the trivial coordinate character and the lower partition β with the nontrivial one. For endpoint weight $t \geq 0$, let $a_t(\alpha, \beta)$ be minus half the smallest Schur exponent. The finite spectral degree is a_t at the label twisted by the full sign, the one-dimensional character taking every simple reflection to -1. Indeed $v \mapsto v^{-1}$, $N_s \mapsto -N_s$ is semilinear, preserves the symmetrizing trace in the sense of that parameter change, and specializes to the sign twist. We calculate with these twisted labels. A partition $\alpha = (\alpha_i)_{i \geq 1}$ has weakly decreasing nonnegative integer parts and a zero tail. Put $|\alpha| = \sum_i \alpha_i$ and $n(\alpha) = \sum_i (i-1)\alpha_i$. The charged-symbol formula is

$$a_t(\alpha, \beta) = n(\alpha) + n(\beta) + \sum_{i,j \geq 1} \left(\min(\alpha_i - i + t, \beta_j - j) - \min(-i + t, -j)\right). \tag{8}$$

For $t \geq 0$, specialize [1, Section 4.3] with ordinary parameter $q = v^{2a}$ and eigenvalues $q^t, -1$ for the unnormalized endpoint generator. In the shifted symbol, subtract the empty-symbol pair contributions and use common padding in the two rows. Any extra entries from the longer row lie

below all nonzero parts of the opposite row for sufficiently large padding and cancel against the empty symbol. Within either row the minimum for indices $i < j$ is the entry at j, giving $n(\alpha)$ and $n(\beta)$. The cross terms then give the displayed sum, in which only finitely many summands are nonzero. For real $t \geq 0$, the valuation is given by the same continuous piecewise-linear formula, since each cross factor of the Schur product is a sum of two positive monomials. For signed t extend a_t by (8). Interchanging the two rows gives $a_{-t}(\beta, \alpha) = a_t(\alpha, \beta)$. For example, the rank-one trivial type has Schur element $1 + q^t$, so its normalized spectral degree is $t = a_t(\varnothing, (1))$.

Deletion of one row. Suppose a row of length $g \geq 1$ has exponents $m, m+1, \ldots, m+g-1$, and $m+g-1$ is their largest absolute exponent and dominates the absolute values of all remaining exponents. Remove this row and replace b, c by $b-1, c-1$. The remaining exponents stay fixed. For any remaining exponent t, the sum of the difference and sum root contributions from the deleted row is

$$H(m-t) + H(m+t), \qquad H(u) = (1-u)_+ - (-u)_+.$$

Indeed the consecutive functions $K_1(t-m-j)$, $0 \leq j < g$, sum to one between the first and last centers and rise linearly from zero to one on $[m-1, m]$. For $t \leq m+g-1$ this sum is $H(m-t)$. The same calculation at $-t$ gives the sum-root contribution. The identity

$$H(m-t) + H(m+t) + K_{|m|}(t) - K_{|m-1|}(t) = 1$$

follows from $(x+t)_+ + (x-t)_+ = x + \max(|x|, |t|)$. Thus the root degree of the full list is the new root degree of the remaining list plus the number of remaining exponents and the root degree of the removed row.

The root degree of the removed row is

$$\sum_{j=0}^{g-1} \big((-b-j)_+ + (-c-j)_+ \big). \tag{9}$$

To check this, write $r_j = m + j$. The difference-root terms vanish, and

$$(-r_j - d)_+ + (-r_j + d)_+ = K_{|d|}(r_j) + 2(-r_j)_+.$$

It suffices to show that the sum-root terms and the $|m|$-potential add to $2\sum_j (-r_j)_+$. For $m \geq 0$ both are zero. Define $F(u) = \sum_{j \geq 0} (u-j)_+$ for any real u. For $m < 0$, put $t = -m$. Since $g - 1 \geq 2t$, the $|m|$-potential is $F(2t) - 2F(t)$. Write $t = h + \theta$ for $h = \lfloor t \rfloor$, $0 \leq \theta < 1$. For $j < \ell$, only the pairs with $j + \ell = \lfloor 2t \rfloor$ or $\lfloor 2t \rfloor + 1$ can contribute to $K_1(j + \ell - 2t)$. There are $\lfloor (u+1)/2 \rfloor$ such pairs with $j + \ell = u$, and every pair of positive contribution has $\ell < g$. Hence their sum is $h + 2\theta$ if $\theta \leq 1/2$, and $h + 1$ otherwise. By the finite progression formula for F this is $4F(t) - F(2t)$, as required.

For (8), prepend an upper row of integer length $x \geq 0$ to α, writing (x, α) for the resulting partition. Suppose $x \geq \alpha_1$ and $x + t - 1 \geq \beta_j - j$ whenever

$\beta_j > 0$. Its effect is

$$a_t((x,\alpha),\beta) = a_{t-1}(\alpha,\beta) + |\alpha| + |\beta| + \sum_{j=0}^{x-1}(-t-j)_+. \tag{10}$$

The surviving upper entries have charge $t-1$. The new row adds $|\alpha|$ to the within-color term, and its cross term is $|\beta|$ plus the difference of its empty-opposite-row contributions. Increasing x by one changes the latter difference by $(-t-x)_+$, by telescoping the minima against the entries $-j$, $j \geq 1$. This also proves (10) for $x = 0$ when the stated inequalities hold, by using a leading zero in the padded symbol.

A split into upper rows $x, g-x$ at the left and right endpoints realizes (9) provided

$$\min(g, \max(0, \lceil -c \rceil)) \leq x, \qquad \min(g, \max(0, \lceil -b \rceil)) \leq g - x. \tag{11}$$

These bounds keep all positive summands in the two finite progression sums truncated at x and $g-x$, respectively. They apply to the row of positive phase. For a column of positive phase, change both endpoint signs and interchange both colors. For a row of negative phase, change the left endpoint sign and interchange its two colors. For a column of negative phase, change the right endpoint sign and interchange its two colors. These changes leave (7) and (8) invariant after the corresponding relabeling.

Deletion of a balanced string. For an integer $k \geq 1$, a balanced string is the list $R_k = ((k-1)/2 - j)_{0 \leq j < k}$. Let

$$C(k,u) = \max(0, \min(k, \lfloor (k+1)/2 - u \rfloor)),$$
$$e = C(k,m), \quad f = k - e, \quad E = C(k,d), \quad F_0 = k - E.$$

In the generic region $b, c, b+c, b-c \notin \mathbb{Z}$, $C(k,u)$ is the clipped nearest integer to $k/2-u$ at each of the signed centers $u = \pm m, \pm d$, and $C(k,-u) = k - C(k,u)$. We choose four rows, with left upper and lower lengths x, y, and right upper and lower lengths $e-x, f-y$, subject to

$$0 \leq x \leq e, \quad 0 \leq y \leq f, \qquad x - y = e - E. \tag{12}$$

Their upper-upper, lower-lower, lower-upper and upper-lower sums across the two endpoints are e, f, E, F_0. In particular the same four constraints accommodate a negative-phase string by exchanging the two left colors.

Every solution to (12) has

$$a_c((x),(y)) + a_b((e-x),(f-y)) = \lfloor k/2 \rfloor + \sum_{r \in R_k} \big(K_{|m|}(r) + K_{|d|}(r)\big). \tag{13}$$

Here and below a single part of size zero gives the empty partition. Both sides are invariant under exchanging the endpoints, or negating an endpoint weight and exchanging its colors. These operations permute the four row-sum constraints. Thus assume $b \geq c \geq 0$, so $m \geq d \geq 0$ and $e \leq E \leq \lfloor k/2 \rfloor$.

For $t \geq 0$ and $0 \leq z \leq w$, the cancellation-free hook formula [1, Proposition 4.5] gives

$$a_t((z),(w)) = z + F(t) - F(t-w-1) - (t-w+z)_+. \tag{14}$$

In fact the merged row partition contributes z and the upper hooks contribute zero to the negative valuation. The lower generalized hook in column j is $w - j + \mathbf{1}_{j \leq z}$, $1 \leq j \leq w$. Its contribution is $(t - w + j - \mathbf{1}_{j \leq z})_+$, which sums to the last three terms of (14).

For a real u, put $P(u) = (u - \lfloor u \rfloor)(1 - u + \lfloor u \rfloor)$. Direct summation gives $2F(u) = u^2 + u + P(u)$ for $u \geq -1$. Also, for $|U|, |V| \leq 1/2$,

$$\frac{P(U+V) + P(U-V)}{2} = \max(|U|,|V|) - U^2 - V^2. \tag{15}$$

The left side is even in each variable and symmetric in the two variables. For $0 \leq V \leq U \leq 1/2$, both arguments are in $[0,1]$, so the displayed identity is the quadratic formula for P on that interval.

Let $M = m + e - k/2$ and $D = d + E - k/2$. If $e > 0$, then both $|M|, |D| < 1/2$. In (14), the first endpoint has $z = x,\ w = x + E - e$ and the second has $z = e - x,\ w = k - E - x$. The two terms $F(t-w-1)$ vanish because their arguments are $M - D - x - 1$ and $M + D - (e - x) - 1$. The sum is

$$e + F(b) + F(c) - (M+D)_+ - (M-D)_+ = \frac{k}{2} + m^2 - M^2 + d^2 - D^2,$$

Indeed $b \equiv M + D$ and $c \equiv M - D$ modulo $\mathbb{Z}$. Substituting the quadratic expression for F and using $b^2 + c^2 = 2(m^2 + d^2)$ gives

$$F(b) + F(c) = m^2 + d^2 + m + \max(|M|,|D|) - M^2 - D^2.$$

Now $(M+D)_+ + (M-D)_+ = M + \max(|M|,|D|)$ and $e + m - M = k/2$. If $e = 0 < E$, then $x = 0, y = E$, $M > -1/2$, and $|D| < 1/2$. The two lower rows contribute

$$F(c) + F(b) - F(M-D) - F(M+D) = \frac{k}{2} + m^2 - M^2 + d^2 - D^2,$$

since $c = M - D + E$, $b = M + D + k - E$, and $M \pm D > -1$. Subtracting the quadratic expressions for the two pairs eliminates P by periodicity and gives $kM + (k-2E)D + \big(E(E+1) + (k-E)(k-E+1)\big)/2$, which is the right side. Finally, if $e = E = 0$ the contribution is $F(b) - F(b-k) = kb - k(k-1)/2$, since $b = m + d > k - 1$.

For $u \geq (k-1)/2$, every string entry satisfies $|r| \leq u$, so $\sum_{r \in R_k} K_u(r) = ku - (k^2 - \epsilon)/4$, where ϵ is the parity of k. For $0 \leq u \leq (k+1)/2$, when k is odd put $j = \lfloor u \rfloor$. The sum is $(2j+1)u - j(j+1)$, by pairing the terms at the positive and negative integers around zero. When k is even put $j = \lfloor u + 1/2 \rfloor$. The sum is $2ju - j^2$, by pairing the terms at the positive and negative half-integers. Any missing term at the upper endpoint $u = (k+1)/2$ has coefficient zero. In either parity this is $u^2 - \mathrm{dist}(u - k/2, \mathbb{Z})^2 + \epsilon/4$, where the distance is the minimum absolute difference from an integer. Substitution

in the three cases gives (13). The difference-root contributions within R_k vanish, and each unordered pair of distinct opposite entries contributes one sum-root term. There are $\lfloor k/2 \rfloor$ such pairs, proving the standalone root degree formula.

When a balanced string has largest radius among the remaining absolute exponents, both t and $-t$ lie between its extreme centers for each exponent t left after deletion. Hence both sums $\sum_{r \in R_k} K_1(r \pm t)$ are one, giving twice the length of the remaining list. The endpoint weights are unchanged. For (8), prepend an upper and a lower row at one endpoint with remaining partitions α, β. If the new rows give partitions and each new symbol entry is at least every positive-part symbol entry in the opposite tail, each new cross term against a positive opposite part increases by that part over the term against zero. The two new cross sums thus contribute $|\alpha| + |\beta|$ beyond the standalone two-row degree. The within-color terms increase by $|\alpha| + |\beta|$ and the cross terms among the old rows are unchanged because both rows and their empty counterparts shift by one. The increments from the two endpoints add to twice the length of the remaining list, as required.

6.2. **Splitting the rows between the endpoints.** Take two partitions λ, μ, whose box exponents are $m + j - i$ and $d + j - i$, respectively, for a box in row i and column j. Also take a list of balanced strings. At each step delete the object containing an exponent of largest absolute value. For a partition compare its upper-right box exponent with the negative of its lower-left box exponent. Delete the first row for the former extremum and the first column for the latter. For a balanced string delete the entire string. The signs of a strip, in left-right order, are $(+,+)$ for a row of λ, $(-,-)$ for a column of λ, $(-,+)$ for a row of μ, and $(+,-)$ for a column of μ. On deleting a strip with signs (s,t), replace c, b by $c - s, b - t$, respectively. The exponent of each undeleted box stays fixed under this reindexing. Deleting a string leaves the parameters fixed. Throughout the generic region the strip extrema have distinct absolute values across the two partitions and cannot tie a string radius. Strings of equal length can be ordered arbitrarily.

The two endpoint bipartitions are constructed by reversing the deletion order. Number the deletions, called events below, in the order they occur. At each endpoint a strip contributes one entry in its assigned color, and a string contributes an entry in each color. Count entries of each color in event order starting at one, including entries of length zero. Write $r(i,s)$ for the resulting index of an entry of color $s \in \{+,-\}$ at event i. For entries of colors s, t at events $i < j$ and initial endpoint weight u, define

$$\delta((i,s),(j,t)) = \begin{cases} 0, & s = t, \\ \lceil r(i,s) - r(j,t) - u(\mathbf{1}_{s=+} - \mathbf{1}_{t=+}) \rceil, & s \neq t. \end{cases}$$

Let $z(i,s)$ be the length of the entry. We impose

$$z(i,s) \geq z(j,t) + \delta((i,s),(j,t)) \quad \text{when } i < j \text{ and } z(j,t) > 0. \tag{16}$$

These conditions give partition order in each color and the dominance of each newly prepended entry over the later positive entries of the opposite color. They give exactly the hypotheses of the degree recursions in Section 6.1. For a strip of length g with signs (s,t), the two lengths sum to g, and the standalone lower bounds are $\min(g, \max(0, \lceil -sc \rceil))$ and $\min(g, \max(0, \lceil -tb \rceil))$, at its current parameters. For a string use (12) at its current parameters.

Lemma 6.2. *Under $b, c, b+c, b-c \notin \mathbb{Z}$, every choice of lengths on a suffix of the deletion list satisfying the standalone bounds and (16) extends through the preceding event.*

Proof. A positive row of either color in the suffix forces its first row of that color to be positive. Only that first row can give the strongest constraint on a prepended row, since subsequent lengths decrease and the opposite-color shifts in (16) also decrease. Each relevant first row lies at or before the first string event in the suffix, since a string has both colors. In particular, if two such rows come from distinct events, the earlier event is a strip.

At current endpoint weight u, the new row of sign s is bounded below by the maximum of zero, the first same-color length, and the first opposite-color length plus $\lceil -su \rceil$ for a strip. For a string the last shift is $\lceil -su - 1 \rceil$, because the string also adds a row of the opposite color. Terms from empty colors are omitted. These formulas follow by counting earlier upper and lower entries in (16). Equivalently, after removing any initial sequence of events and updating the endpoint weight by their signs, the index differences give the same shifts on the remaining events.

We use the inequality

$$\delta(i,k) \le \delta(i,j) + \delta(j,k) \tag{17}$$

at one endpoint when $i < j < k$ and k is the first entry of its color after i. The entry signs are implicit in this notation. If the signs of i, j agree, the index at i is at most that at j, and if the signs of j, k agree, the index at k is at least that at j. These observations give (17) in both cases. In the alternating case, the two opposite-color shifts sum to $r(i,s) - r(k,s) + 1 = 0$, since the endpoint weight is nonintegral and k is first of its color. This proves (17), including when i is one of the entries of a string.

Independent sign changes of the two endpoint weights with the corresponding color exchanges preserve the inequalities. More explicitly, multiplying the left and right signs and weights by $l, r \in \{1, -1\}$ relabels the phases by multiplication by lr and transposes both residual partitions if $r = -1$. All box exponents are multiplied by r. The signed center $(tb+sc)/2$ associated with a pair of endpoint signs (s,t) is unchanged under the transformed pair. The four string capacities are $C(k, (tb+sc)/2)$, so they are permuted by this operation. Exchanging the endpoints and transposing μ is another symmetry. We may therefore take a current strip to have signs $(+,+)$. For the current string it suffices to bound the sum of the requirements for the two upper rows.

Prepending a strip. Let the current strip be the first row of λ, of length g. Let h be the height of λ, and put $K = \lceil -c \rceil$, $J = \lceil -b \rceil$. The radius of the strip is $M = m + g - 1$. Comparing the other extrema with M bounds h by $g + \lfloor b + c \rfloor$, and bounds the width and height of μ, when nonempty, by $g - K$ and $g - J$. Every later strip of signs $(+,+), (-,+), (+,-), (-,-)$ has length at most

$$g, \quad g - K, \quad g - J, \quad g + \lfloor b + c \rfloor - 1,$$

respectively. Its two shifts from the current rows add to at most $0, K, J, K+J$ in those four cases. Thus its length plus its two shifts is at most g, using $\lfloor b + c \rfloor - \lfloor b \rfloor - \lfloor c \rfloor \leq 1$.

Consider two positive tail requirements at different events, one at each endpoint. If the first comes from the left, its event j is a strip. Write k for the event of the right requirement. By (16) at the right and (17), the right requirement is at most the right row of j plus its shift from the current event. The sum of the two requirements is therefore at most the length of the strip j plus its two shifts, and hence at most g. Exchanging endpoints gives the same conclusion when the first requirement comes from the right. When both requirements come from one strip, the same length bound applies directly.

If both positive requirements come from one string of size k' with selected colors (s,t), every intervening strip has signs $(-s,-t)$. Let p be their number and $M' = (k'-1)/2 < M$ the radius of the later string. Its selected capacity is the clipping to $[0,k']$ of the floor of

$$M' + 1 - \frac{tb + sc}{2} + \frac{s+t}{2} - p.$$

This follows by applying the strip updates to the two weights. The untruncated floors in the four sign cases are bounded respectively by

$$g - p, \quad g - K - p, \quad g - J - p, \quad g + \lfloor b + c \rfloor - 1 - p.$$

The selected capacity is positive, so clipping can only decrease the floor. Adding the shifts K or J for each opposite color gives again a bound of g.

It remains to pair a tail requirement at one endpoint with the standalone requirement at the other. We prove that each retained right requirement B satisfies $B \leq g - \max(0, K)$. Swapping endpoints proves the left version. Suppose its source is a strip. If its right sign is $+$, the preceding strips all have right sign $-$. Let p, q count their $(-,-), (+,-)$ signs. The left weight there is $c - 1 + p - q$ and the shift on the right is zero. If its right sign is $-$, let p, q count the preceding $(+,+), (-,+)$ strips. The left weight there is $c - 1 - p + q$ and its right shift is J. In these two cases, the retained positive right row forces its companion left row to be at least the ceiling of minus its left sign times the left weight there. Indeed a ceiling at least the full strip length would force the right row to vanish. Combining this companion

bound with the length bounds gives the following two bounds in each case:

source signs	upper bounds for B	
$(+,+)$	$g-p$	$g-K-1-q$
$(-,+)$	$g-K-q$	$g-p$
$(+,-)$	$g-q$	$g-K-1-p$
$(-,-)$	$g-K-p$	$g-q$

For the first two rows the source lengths are at most $g-p$ and $g-K-q$, respectively. For the third and fourth they are at most $g-J-q$ and $h-1-p$. In the fourth row use $h-1+K+J \le g$. These bounds give the claimed inequality in every sign case.

Suppose the retained right row is from a string. For a right upper row let p, q count the preceding $(-,-),(+,-)$ strips. The two centers there are $m-1+p, d+q$. Its length is at most both $C(k', m-1+p)$ and $C(k', d+q)$. The radius inequality $M' < M$ gives upper bounds $g-p, g-K-q$ for the untruncated floors. For a right lower row let p, q count the preceding $(+,+),(-,+)$ strips. The centers there are $m-1-p, d-q$. Its length is at most both $C(k', -m+1+p)$ and $C(k', -d+q)$, whose untruncated floors are at most $g+\lfloor b+c \rfloor - 1 - p, g-J-q$. These capacities are positive. Adding the shift J in the lower case gives bounds $g-K-p, g-q$. Thus every retained right requirement is at most $g-\max(0,K)$.

Finally the two standalone bounds sum to at most g. If both $K, J > 0$, then $h \ge 1$ implies $\lfloor b+c \rfloor \ge 1-g$ and hence $K+J \le g$. If one is nonpositive, its standalone bound is zero. The pairs of standalone and tail bounds are compatible by the unary bounds just proved, and every pair of tail bounds is compatible by the two-target argument. Taking the maximum of the bounds at each endpoint thus gives two nonnegative integers with sum at most g. They admit a split of the strip of length g.

Prepending a balanced string. Let k be its length and use e, f, E, F_0 as in (12). Denote the four nonnegative integer requirements from the suffix by l_+, l_-, r_+, r_-, with l and r indicating the two endpoints. On the diagonal $x-y = e-E$, the possible upper-left values are exactly the integers in

$$\big[\max(l_+, l_- + e - E),\ \min(e - r_+, f - r_- + e - E)\big].$$

Thus existence is equivalent to the four inequalities

$$l_+ + r_+ \le e, \quad l_- + r_- \le f, \quad l_- + r_+ \le E, \quad l_+ + r_- \le F_0. \tag{18}$$

The sign changes above reduce them to $l_+ + r_+ \le e$.

Assume $0 < e < k$. For a nonempty residual partition at center u, the first box gives $|u| \le (k-1)/2$. Its upper-right and lower-left boxes bound its width and height by $\lfloor (k+1)/2 - u \rfloor$ and $\lfloor (k+1)/2 + u \rfloor$. These floors are in $[0, k]$ by the first-box bound. Empty partitions have width and height zero. Thus the bounds are e, f for λ and E, F_0 for μ. Let E' and F' be the two unclipped floors $\lfloor (k+1)/2 - d \rfloor, \lfloor (k+1)/2 + d \rfloor$. The floor difference

for two arguments separated by c, respectively b, gives

$$e - E' \in \{K-1, K\}, \qquad e - F' \in \{J-1, J\}, \qquad E' + F' = k.$$

In particular $E + K - 1 \le e$ when $E > 0$, $F_0 + J - 1 \le e$ when $F_0 > 0$, and $f + K + J - 2 \le e$. These bounds show that any later strip length plus its two shifts from the current upper rows is at most e. In fact the four length bounds are e, E, F_0, f, and their shift bounds are $0, K-1, J-1, K+J-2$, respectively. For two positive requirements at distinct events, or from one strip, (17) and the strip argument above now give total at most e. If both come from the same later string, every intervening strip has both colors opposite to its two selected colors. Its corresponding signed center has increased by the number of those strips. Its size has weakly decreased, so its capacity is at most the current capacity for those signs, since $C(k, u)$ increases in k and decreases in u. Adding the shifts bounded by $K-1, J-1$ for the opposite colors gives again a total at most e.

To pair a positive requirement with zero, it suffices by endpoint exchange to bound each left requirement by e. If its source has left color $+$, let p, q count the preceding $(-,+), (-,-)$ strips. The centers at its event are $m+q, d-p$, and the right weight is $b-p+q$. A $(+,+)$ source has length at most $e-q$. A $(+,-)$ source has length at most F_0-p, with right companion at least $\lceil b-p+q \rceil = 1-J-p+q$ by the standalone bound and positivity of the left row. Its left row is therefore at most $F_0 + J - 1 - q \le e$. A string source has left upper length at most $C(k', m+q) \le e$, with $k' \le k$. If the source has left color $-$, let p, q count the preceding $(+,+), (+,-)$ strips. The centers are now $m-p, d+q$ and the right weight is again $b-p+q$. The left shift is $K-1$. A $(-,+)$ strip has length at most $E-q$, giving a shifted bound at most e. A $(-,-)$ strip has length at most $f-p$ and right companion at least $1-J-p+q$, giving a shifted bound $f+K+J-2-q \le e$. A string has left lower length at most $C(k', d+q) \le E$. Its shifted bound is at most $E + K - 1 \le e$. This proves $l_+ + r_+ \le e$ for $0 < e < k$.

If $e = 0$, then $m > (k-1)/2$ and λ is empty. The center m stays fixed. A later row of μ at center d', of length g, satisfies $d' + g - 1 \le (k-1)/2 < m$. Its left lower bound is $\min(g, \max(0, \lceil m - d' \rceil)) = g$, so its right upper row vanishes. A column of μ similarly has vanishing left upper row. Every later string has zero upper-upper capacity. If a first positive left lower row occurs, all preceding strips are columns of μ. If there are q such columns, its center is $d+q$. The radius bound gives length at most $\lceil c \rceil - q$ for a strip. For a string this same bound follows from the unclipped floor of $C(k', d+q)$, using $k' \le k$. Hence its requirement on the new left upper row is at most $\lceil c \rceil + \lceil -c - 1 \rceil = 0$. The right upper requirement also vanishes by endpoint exchange.

If $e = k$, the simultaneous sign and color reversal of the preceding argument shows that λ is empty and all lower tail rows vanish. The first left upper row, if present, follows only rows of μ. If there are p such rows, a column source has length at most the initial height of μ minus p, bounded

by $F_0 - p$. A string source has left upper length $C(k', -d + p) \leq F_0$, since its right lower row is zero. Therefore $l_+ \leq F_0$. Endpoint exchange gives $r_+ \leq E$, so $l_+ + r_+ \leq F_0 + E = k$. This finishes all four inequalities in (18) and the proof. □

Starting with the empty suffix, Lemma 6.2 yields two endpoint bipartitions. The degree recursions show that their Schur valuations add to the root degree of the original list. The signs determine how a row or a column of a phase partition is split. For strings, the four capacities give the prescribed row totals in either phase.

6.3. **Tempered types and their finite restrictions.** Write the central torus character as its unitary phase times its positive real part, and choose a representative $z = (z_1, \ldots, z_n)$ for the unitary phase up to signed permutation. Coordinates with phase $+1$ and with phase -1 give two signed-permutation factors of the stabilizer $(W_0)_z$. Within each other group of phases equivalent under inversion, choose the coordinates all to have the same phase. The stabilizer on that group is the symmetric group permuting its coordinates. These factors form a direct product, and Lusztig's reduction associates a real-central graded tempered factor to each of them [41, Corollary 1.12][2, Section 2.2.1]. In the normalized units take $q = v^{2a}$. In the presentation of [41, Definition 1.1 and Remark 1.3] the parameters are $q_0 = -q^d$, $q_1 = q^m$, and $q_2 = q$, with m, d as in (6). Indeed, $-q_0 q_1 = q^b$ and $q_0 + q_1 = q^{b/2}(q^{c/2} - q^{-c/2})$. In the logarithmic reduction of [41, Theorem 1.10], the middle Bernstein relation gives coefficient $\log_q(q_2) = 1$, twice the exponent of the square-root parameter in Remark 1.3. At an endpoint phase $z_i = \pm 1$, the first-order coefficient of $(-q_0 q_1 - 1) + z_i(q_0 + q_1)$ in $\log q$ is $b + z_i c$. The endpoint denominator uses the square of the torus coordinate, so the short graded parameter is $(b + z_i c)/2$, giving m and d, respectively. These half-sum and half-difference parameters also occur in the discrete-series reduction in [42, Section 4.2, before (4.10)]. The type A factors at the other phases have zero real exponent in their free central direction, by the tempered inequalities. Multiplying each short reflection by -1, leaving the polynomial algebra fixed, changes the sign of the short graded parameter and interchanges the two finite colors.

For the symmetric-group factors, real-central tempered type A induction uses Steinberg factors [41, Theorem 4.5]. A Steinberg factor of size k has the exponents R_k of Section 6.1 and finite character the sign of $\mathfrak{S}_k$. If several such factors share a unitary phase, their joint finite character is the induction of their sign characters. The same strings occur in the real-central tempered signed-permutation factors. Away from half-integer short parameter, those factors are the irreducible inductions of one discrete-series factor and a list of such strings, as in [41, Theorem 4.7 and Corollary 4.23]. The discrete-series exponents for short parameter u are the box exponents $u + j - i$ of a partition, up to signed permutation. This convention also applies to $u < 0$, by transposing the partition and changing the signs of the coordinates.

We describe the canonical finite type in the full-sign-twisted convention of Section 6.1. For a signed-permutation factor with $2u \notin \mathbb{Z}$, delete a first row or first column of the discrete-series partition at a largest absolute box exponent, or delete a balanced string of largest radius, comparing these radii in a single list. Record a deleted row in the upper partition and a deleted column in the lower partition, changing u by -1 or $+1$, respectively. A string of size k adds upper and lower rows of lengths $C(k,u)$ and $k-C(k,u)$, leaving u fixed. This gives the canonical label in the chamber, which occurs in the factor.

For $u > 0$ with $4u$ odd, the real-central tempered representations and their canonical finite types are described by the Spin generalized Springer correspondence [41, Theorems 4.12 and 4.15 and Proposition 4.19]. For a real-central tempered module the central character determines the semisimple middle element of the nilpotent's $\mathfrak{sl}_2$-triple [45, Theorem 1.21]. In the cuspidal Levi of [44, Section 2.13(e)], each relative coordinate acts on a two-dimensional block and its dual. In the coordinate scale of [41, Section 4.4], a real exponent r contributes scalar weights $4r, -4r$ on these two spaces, in addition to their regular $\mathfrak{sl}_2$ weights $1, -1$. Thus the four weights contributed by r are $\pm(4r+1), \pm(4r-1)$, in addition to the fixed cuspidal part of the ambient space. For the discrete-series box exponents this gives the odd-part construction preceding [41, Algorithm 4.17]. For a string R_k, the weights $4r+1, 4r-1$ form the sequence $2k-1, 2k-3, \dots, 1-2k$. Their negatives give the same sequence, so the string contributes two $\mathfrak{sl}_2$ summands of dimension $2k$. Consequently its Jordan contribution consists of two blocks of length $2k$.

The odd-part branch of [41, Algorithm 4.14] is the row and column rule by Claim E there. This computation applies with even pairs present because the odd branch uses only the deleted odd part and the defect, and every even pair contributes zero to the defect. The even-pair branch at size $2k$, with $u = h+1/4$ or $u = h+3/4$ for an integer $h \geq 0$, has upper part $\lfloor k/2 \rfloor - h$ or $\lceil k/2 \rceil - h - 1$, respectively. Indeed the defects in the two cases are $-h$ and $h+1$. These are the two nearest-integer formulas for $k/2 - u$. The Jordan part $2k$ is ordered by its radius $(k-1)/2$, since $2k = 4((k-1)/2) + 2$. Every surviving cuspidal odd part is at most $4u-2$, whereas a nonempty discrete-series remainder gives an odd part of size at least $4u+2$. Thus at a cuspidal deletion the discrete-series remainder is empty, and the odd parts then form the current cuspidal partition, whose largest part is $4u-2$. If a string remains, then $2k < 4u-2$ and $u > 1$. Substituting $4u-2$ and the defect in Algorithm 4.14, the odd branch records a zero and changes u to $u-1 > (k-1)/2$. Its upper length remains zero in the clipped rule. After a sign change of u the corresponding statement uses the color exchange. This proves the rule at the geometric quarter point. The full finite characters and their canonical labels are constant on each open half-integer chamber by [41, Corollary 4.20]. The deletion rule is constant in the chamber. A tie between absolute box exponents with opposite signs requires $2u \in \mathbb{Z}$, and a

tie with a string radius requires u to be a half-integer. The row and column deletions shift u by integers, and the floors in $C(k,u)$ also change only at half-integers. Negative u follows by the short-sign twist, transposition of the discrete-series partition, and exchange of the two finite colors.

Scale the numerical labels along their ray to one with the unitary phase fixed. The finite parabolic Hecke algebras remain semisimple, so the primitive central idempotents indexed by ordinary finite types have continuous coefficients. Their ranks on the scaling family are constant. In the limit, the character is induced from $\mathbb{Z}^n \rtimes (W_0)_z$, with $\mathbb{Z}^n$ acting by z and $(W_0)_z$ acting by the restriction of the graded factors [2, Corollary 2.2][42, (4.10) for discrete series]. Mackey restriction of this induced character therefore computes the finite parabolic multiplicities before scaling.

For the explicit phase factor in Mackey restriction, let p_k be the vector with first k entries $1/2$ and the rest zero. The facet subgroup W_k consists of $(p_k - wp_k, w)$ for w in the two-block signed-permutation subgroup. A coordinate sign change in the first block contributes its translation coordinate one, whereas permutations in either block have translation part zero. Thus, on -1 coordinates in the first block, the restriction twists by the character $(-1)^{\text{number of sign changes}}$. It exchanges the two finite colors in the left block. On $+1$ coordinates and on permutations inside the other phase groups there is no such factor. In particular, each distribution of the coordinates of a phase among the two blocks supplies a Mackey summand with its actual restriction multiplicities. The full sign of a facet is the determinant of its projection to W_0, so the simultaneous full-sign twist used above commutes with these restriction and induction operations.

When $b, c, b+c, b-c \notin \mathbb{Z}$ in normalized units, apply Lemma 6.2 to the two discrete-series partitions and all strings. The subsequence at each phase $+1, -1$ is its canonical deletion order, since deletions outside that phase preserve its short parameter and partition. In each phase color the two allocated partitions add rowwise to the canonical partition. For phase -1 this uses the lower-upper and upper-lower sums in (12), as the first block exchanges the colors. The Littlewood–Richardson coefficient of the rowwise sum is one, by the Cartan constituent. For a string in a phase other than $+1, -1$, let h be the number of entries allocated to one block. After the full-sign twist its $\mathfrak{S}_h$ constituent is trivial. Every bipartition of h with at most one row in each color restricts to a representation containing the trivial $\mathfrak{S}_h$-type: restriction gives the induction of the trivial types of the two Young subgroup factors. Frobenius reciprocity therefore gives each of the allocated two-row pieces in the corresponding B_h induction. Combine the pieces at each endpoint by taking the sorted union of their rows in each color, a Littlewood–Richardson constituent of multiplicity one by transposing the Cartan constituent. Condition (16) already puts those rows in partition order. This gives the required constituent on the facet whose first block size is the sum of the left row lengths. Its spectral degree is A_R by the recursions of Section 6.1.

6.4. **Specialization at critical short parameters.** For a real-central graded signed-permutation factor of short parameter $u > 0$ with $2u$ integral, the finite characters of the irreducible tempered representations, after the common full-sign twist, form a nonnegative unitriangular matrix indexed by the finite types. This is the generalized Springer character matrix. The identification at ratio u is given by [43, proof of Theorem 4.12] and [45, Theorems 1.17 and 1.21]. In the Springer character matrix the same-orbit block is the identity, and the other types lie on strictly larger orbits in the closure order, by [43, Theorems 3.1 and 3.5]. The same statement for a negative short parameter follows by the short-sign twist.

At $u = 0$ in rank $r \geq 2$, the graded algebra is the equal-parameter type D_r graded algebra crossed with the order-two outer action. In the graded Bernstein presentation the short reflection now acts on the polynomial algebra by conjugation without a correction term, and conjugating the long reflection at the end of the chain gives the second fork reflection of D_r. Here $D_2 = A_1 \times A_1$. For the type D_r factor use the tempered classification in the ordinary Springer series, namely the series for the torus cuspidal datum [45, Theorems 1.17 and 1.21]. Its character matrix has the same unitriangularity by [43, Theorems 3.1 and 3.5]. The outer involution preserves the orbit closure order and acts on the canonical labels by transport of Springer fibers and their local systems. Clifford theory for the order-two group transports this matrix to a unitriangular matrix of type B_r. A pair of exchanged type D_r modules induces to one simple module and their canonical types induce to one simple finite type, with multiplicity one. A fixed module has two extensions. Its canonical finite type has multiplicity one on the D_r subgroup, so its isotypic space is preserved by the outside involution and affords one extension of that type. Twisting by the quotient sign interchanges the two extensions of the module and also the two extensions of its canonical type. The other types in either case come from strictly larger orbits. This gives each type B_r label once, including the labels obtained from the very-even pair of type D_r. In rank one the zero-parameter graded algebra is $\mathbb{C}[x] \rtimes \{\pm 1\}$, with the involution sending x to $-x$. Real-central temperedness forces the real eigenvalue of x to be zero, since the two opposite eigenvalues would otherwise violate the tempered inequalities. At zero the two simple modules have the two one-dimensional finite labels. The rank-zero assertion is the one-dimensional trivial case.

Fix a critical short parameter u and one adjacent open chamber. Index the complete lists of irreducible real-central tempered modules in the chamber and at u by their canonical finite types. In the two matrices U, V, an entry is the multiplicity of the column's finite type in the row's module. These matrices are nonnegative and unitriangular in their respective label orders.

For $u \geq 0$, realize the generic graded factors on the positive phase of affine algebras with both endpoint weights equal to the varying short parameter, taking the positive chamber when $u = 0$. Fix the numerical base greater

than one. The normalized affine simple generators are self-adjoint in the tempered representations, and their two possible eigenvalues stay bounded as the weights converge. Thus, for each generic label, the generator matrices in orthonormal bases converge simultaneously along a subsequence. The defining relations persist in the limit. The Bernstein elements are products of standard generators and their inverses, which also converge because the generator eigenvalues stay bounded away from zero. The tempered inequalities on eigenvalue moduli of dominant translations persist, and the limiting central character is real by continuity of the eigenvalues. The limit is unitary and splits into real-central tempered simples at u. Ranks of the finite primitive central idempotents are preserved. By the scaling identification, the same finite-character identity holds on the graded side. For $u < 0$, or the negative chamber adjacent to zero, apply the short-sign twist.

Let T record the multiplicities of the critical irreducibles in these limits, with rows indexed by generic modules and columns by critical modules. Then $U = TV$. In the label order of U, the nonnegative entries of T have triangular support because V has diagonal entries one. Both U and V have determinant one, as simultaneous permutations of their rows and columns preserve the determinant. Therefore T has determinant one and each of its nonnegative integral diagonal entries is one. This shows that from either adjacent chamber the generic factor with the prescribed critical label has a limit containing that critical factor. The generic factor's central character is scalar on every central Bernstein element. The continuity of those elements therefore gives the same limiting central character on all summands. The box exponents and balanced strings of the chosen generic label have the same limits along any approach in that chamber.

Proof of Proposition 6.1. For $b, c > 0$, perturb the two phase centers independently in the region $m > |d|$. At a critical center choose the factor with the prescribed canonical label from an adjacent chamber. At a noncritical center keep its label in its chamber. Keep all other phase factors fixed, and take $b, c, b + c, b - c$ nonintegral along the perturbation. This includes $b = c$, where the negative phase uses the short-parameter-zero argument. At $c = 0$ the two centers coincide at $b/2$. If this is critical, take $(b + \epsilon/2, \epsilon)$ with small $\epsilon > 0$, putting the positive phase in the upper adjacent chamber and the negative phase in the lower one. If it is noncritical, keep both centers in the same chamber. At the target $c = 0$ the affine endpoint label is one, so the scaling and finite restriction formulas still apply.

Pass to a subsequence on which the attaining facet type and the branching choices are fixed. The branching calculation uses the same canonical finite types of the prescribed target factors and the same type A strings, so it gives that facet type in the prescribed irreducible representation. Formula (8) is continuous in the endpoint weights, and (7) is continuous in the weights and exponents. The limiting equality gives the required degree after multiplication by a. □

7. The zero endpoint and type $\widetilde{B}_n$

Fix $n \geq 3$. Let W^0 be the affine Coxeter group of type $\widetilde{C}_n$, with generators $s_0, \ldots, s_n$ in path order. Let H^0 be its Hecke algebra with weight zero at s_0, weight b at s_n, and weight a at all other generators. This is the specialization $c = 0$ of the $\widetilde{C}_n$ datum. Write W and H for the group and algebra of type $\widetilde{B}_n$. The fork generators t_0, t_1 are adjacent to t_2, followed by the chain $t_2, \ldots, t_n$ with an edge of order four at the last pair. The weights are a at $t_0, \ldots, t_{n-1}$ and b at t_n. A superscript 0 distinguishes the standard basis and the coefficient-of-identity trace of H^0.

Lemma 7.1. *The assignments*

$$t_0 = s_0 s_1 s_0, \qquad t_i = s_i \quad (1 \leq i \leq n).$$

embed W in W^0 as an index-two subgroup. Let γ be the diagram automorphism interchanging t_0 and t_1 and fixing $t_2, \ldots, t_n$, acting on H by permuting the normalized generators. Setting $U = N^0_{s_0}$ gives

$$H^0 = H \rtimes \langle U \rangle, \qquad U^2 = 1, \qquad UhU = \gamma(h) \quad (h \in H).$$

Under this embedding, $N_x = N^0_x$ and $N_x U = N^0_{xs_0}$ for $x \in W$. In particular, $\tau^0|_H = \tau$ and $\tau^0(HU) = 0$.

Proof. On $\mathbb{R}^n$ take the $\widetilde{C}_n$ alcove $1/2 > x_1 > \cdots > x_n > 0$. Its walls give s_0 at $x_1 = 1/2$, s_i at $x_i = x_{i+1}$ for $1 \leq i < n$, and s_n at $x_n = 0$. The s_0-walls are the planes $x_i \in \mathbb{Z} + 1/2$. Removing them gives the $\widetilde{B}_n$ arrangement. The union of the two adjacent $\widetilde{C}_n$ alcoves across the s_0 wall, with the intervening open facet included, is the $\widetilde{B}_n$ alcove. Its fork walls are $x_1 = x_2$ and $x_1 + x_2 = 1$, with reflections s_1 and $s_0 s_1 s_0$. Its reflection group is the kernel of the character of W^0 that sends s_0 to -1 and $s_1, \ldots, s_n$ to 1. In the affine permutation realization this character is the parity of the sum of the translation coordinates.

In H^0 the relation for U is $U^2 = 1$. Conjugation by U exchanges $U N^0_{s_1} U$ and $N^0_{s_1}$ and fixes $N^0_{s_i}$ for $2 \leq i \leq n$. The order-four braid relation for s_0, s_1 becomes the commutation of the fork generators. The other braid relations for the fork follow by conjugating those for s_1. Conversely these relations and the action of U give the $\widetilde{C}_n$ relations. The quadratic relations have the stated weights, so the maps on generators yield the crossed product.

To check the basis, expand a reduced W expression by replacing each t_0 by $s_0 s_1 s_0$. Its total weight equals the weighted length of its element in W^0 because the separating positive-weight walls are precisely the separating walls of the larger alcoves. The Coxeter deletion process reduces this expression by braid moves and removal of adjacent equal generators. The braid moves preserve weight and each removal of a positive-weight generator would lower the total weight. Hence all removals have weight zero, for which $U^2 = 1$ holds in the Hecke algebra. Appending s_0 leaves the weighted length unchanged because the two resulting alcoves are adjacent

across a zero-weight wall. The same argument gives $N_x U = N^0_{xs_0}$. The trace assertions follow from these basis identities. □

The common finite Weyl group W_0 is the group of signed permutations on n coordinates. The translation lattice of H is the index-two sublattice of the translation lattice $\mathbb{Z}^n$ of H^0 given by

$$Q(C_n) = \left\{ x \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i \text{ is even} \right\}.$$

The Bernstein elements of H are those of H^0 indexed by this sublattice. For a dominant lattice element the Bernstein element is the standard basis element of its translation, so the equality follows from Lemma 7.1. Every lattice element is a difference of dominant ones, and the Bernstein group law extends the equality. The center of H is therefore contained in the center of H^0.

Proposition 7.2. *Every irreducible tempered representation of H has a constituent of the same spectral degree on a finite standard parabolic Hecke subalgebra. Its spectral level is (1) at $c = 0$, where $R = (R_1, \dots, R_n)$ is obtained from the moduli v^{2R_i} of a lift of a central torus character to the torus of H^0.*

Proof. The two lifts differ by the character $x \mapsto (-1)^{\sum_i x_i}$ of $\mathbb{Z}^n$, which simultaneously changes the signs of the torus coordinates. Thus their real exponent vectors agree. The term for $2e_i$ in the reduced datum is $K_b(2R_i) = 2K_{b/2}(R_i)$, the sum of the two endpoint terms at $c = 0$. The other terms of (1) agree.

By Lemma 7.1, induction and restriction for this crossed product preserve temperedness. Indeed, the regular representation of H^0 restricts to two copies of the regular representation of H, up to the automorphism γ, and is itself induced from the regular representation of H. These identifications extend induction and restriction to the reduced C^*-algebras. For each irreducible tempered representation ρ of H one can therefore choose an irreducible tempered representation π of H^0 above ρ. Its restriction is ρ if ρ is γ-invariant and is $\rho \oplus \gamma(\rho)$ otherwise. Since the center of H lies in the center of H^0, all constituents in the restriction have the same central character on H. The finite-facet attainment statement of Proposition 6.1 at the actual numerical endpoint $v^{2c} = 1$ gives a finite-facet constituent of π of degree A_R.

For $0 \le k \le n$, let F_k be the finite subgroup of W^0 obtained by deleting s_k. For $k = 0$ it is the finite type-B_n subgroup generated by $t_1, \dots, t_n$ and lies in W. For $k = 1$ it is $\langle s_0 \rangle \times B_{n-1}$ and its intersection with W is the standard subgroup generated by $t_2, \dots, t_n$, contained in either fork facet. For $k \ge 2$ it is $B_k \times B_{n-k}$, where B_0 means the trivial group. The first factor acts on the first k coordinates as the signed permutation group centered at $(1/2, \dots, 1/2)$ in that block. Its translation parity is the parity

of the number of sign changes, so its intersection with W is the type-D_k group generated by $t_0, \dots, t_{k-1}$. Thus for $k \geq 2$ the intersection $F_k \cap W$ is the standard parabolic obtained by deleting t_k. Here $D_2 = A_1 \times A_1$ and $D_3 = A_3$. Conjugation by U fixes the intersections for $k > 0$ and sends the intersection for $k = 0$ to the other standard fork facet.

The inclusion at each intersection preserves the standard trace. For $k > 0$ the finite algebra of F_k is a crossed product of the intersection algebra by $\langle U \rangle$. Let σ be a simple type of the larger algebra and η a simple type in its restriction. Over a splitting field of characteristic zero, Clifford theory gives one simple σ above an orbit of two inequivalent types η and $\gamma(\eta)$, with both restriction multiplicities one. Comparing the coefficient of the character of η in the restricted trace gives $s_\sigma^{-1} = s_\eta^{-1}$. For an invariant η there are two extensions, each of restriction multiplicity one. Their reciprocal Schur elements sum to s_η^{-1}. They are equal because the quotient sign interchanges the extensions and preserves the trace. Thus $s_\sigma = 2s_\eta$ in this case. The algebras agree for $k = 0$, so in every case the finite spectral degree is preserved.

Choose σ to be an attaining constituent of π on F_k. At least one constituent of its restriction to $F_k \cap W$ occurs in the restriction of ρ or of $\gamma(\rho)$. In the latter case conjugate this constituent and the intersection by γ. This gives a constituent on a standard finite parabolic of H in the restriction of the prescribed ρ. Its spectral degree is A_R in either case. For integral a, b, this degree is integral by the type B symbol formula for the original constituent on F_k, with endpoint weights $0, b$. □

The $\widetilde{B}_n$ application of Proposition 5.1 uses the induction tori and intertwiners for the reduced datum $R_0 = C_n$, $X = Q(C_n)$. The stabilizer commutant is described by [3, Theorem 5.5], and [3, Proposition 6.8] gives the groupoid comparison for this datum. The projected semisimple lattice can be seen by subtracting each type-A block's mean from its coordinates and retaining the tail coordinates. Applied to $\mathbb{Z}^n$ this gives the product of the type-A weight lattices and the weight lattice $\mathbb{Z}^\ell$ of the type-C_ℓ tail, where ℓ is the tail size. Lifts to $\mathbb{Z}^n$ of a point of this product differ by adding an integer constant to each type-A block. If a block size is odd, one such shift changes the total parity and every point has a lift to $Q(C_n)$. If all block sizes are even, parity is well defined on the product lattice and the projected lattice from $Q(C_n)$ is its index-two kernel. When there is at least one type-A block, the order-two character of this product lattice trivial on the kernel is nontrivial on an even-sized block. On that block it is the nontrivial quadratic twist of the Steinberg torsion label, which acts freely on the set of such labels. The resulting groupoid comparison is precisely that of [3, Proposition 6.8]. For the full Levi there are no type-A blocks, the unramified torus is a point, and the stabilizer description is the discrete-series block itself. In every case the compact model and the generalized Bernstein weights use the

actual projected lattice and the unramified quotient of $Q(C_n)$. The normalized relative density on each torus has a Laurent polynomial limit as used in Proposition 5.1. In fact its root factors, after restricting to the torus, have the form $1-\zeta v^d z^\nu$ and their conjugates, where z^ν is a torus character, d is a real exponent and $|\zeta| = 1$. Nonstationary factors have uniform normalized limits. A stationary denominator is a product of primitive Laurent binomials. Real analyticity of the exact density on the compact torus cancels each such factor because the compact points of every component of its divisor are Zariski dense [9, Theorem 3.25 and Proposition 3.27]. After cancellation the normalized density converges uniformly. The proof of Proposition 5.1, using the finite Bernstein expansion and the actual stabilizer action, now gives the full equivariant Laurent polynomial range for H. In applying its central cutoff construction one chooses the integral multiples of the dual real exponents in $Q(C_n)$. This is a full-rank lattice, so the choice is possible. Proposition 7.2 gives a same-degree finite source in every unitary stabilizer block. Hence the ordinary ideal generation and the canonical filtration in Proposition 8.2 apply to H.

The trace used for the signed-unit argument can be computed by the Plancherel formula of H^0. To see its compatibility with the leading product of H, restrict all the compact families of H^0 to H. The spectral level of every constituent is the level of the family by Proposition 7.2. For a compact family $\pi^0_{\alpha,v,z}$ of H^0 write $\mu^0_{\alpha,v}$ for its Plancherel measure on the family torus. Lemma 7.1 gives the exact identity

$$\tau(h) = \sum_{\alpha} \int \operatorname{Tr}(\pi^0_{\alpha,v,z}(h))\, d\mu^0_{\alpha,v}(z) \qquad (h \in H).$$

After normalizing the matrices by v^{-A_R} and the measures by v^{2A_R}, its two-factor limit gives an isometry on the orthonormal basis indexed by W. The integral spectral projections in Proposition 5.1 act on each restricted fiber by the corresponding scalar cutoff. Indeed the central Bernstein elements act on each fiber by its central character on H, and the polynomial iteration defining the projections then selects the real exponent orbit. We now identify the product under the isometry. For fixed $x, y \in W$ the leading product column is the constant term of the normalized multiplication of N_x on N_y from Proposition 5.1. The three-factor Plancherel formula identifies its image with the orthogonal projection of the pointwise product onto the closed image of the isometry. The norm of the column converges in the standard basis Hilbert space by the operator-norm convergence of that normalized multiplication. The four-factor Plancherel formula identifies the limit of the squared norms with the squared norm of the pointwise product itself. Equality of these norms places the product in the image. For the unit, apply $P_+(v)$ from Proposition 5.1 to N_1. In the restricted families this gives v^{A_R} times the identity matrix. Its squared norm tends to the sum over the families of the integrals of their dimensions against the limiting measures. Pairing with each N_w shows that the image of its constant coefficient

is the orthogonal projection of the family identities onto the closed image of the isometry. Since the squared norms agree, this projection is the family identity itself.

8. The original affine Kazhdan–Lusztig filtration

For a finite standard parabolic let $s_\sigma(v)$ be the Schur element of its simple character σ for the standard trace. Thus the coefficient of that character in the trace is $s_\sigma(v)^{-1}$. Its positive-canonical spectral degree is $A(\sigma) = \frac{1}{2}\deg_v s_\sigma(v)$. The finite spectral description in Lemma 4.15 identifies each finite a-level with the corresponding sum of generic matrix blocks. On a block of degree $A(\sigma)$, convergence of its normalized matrices to zero for an element and its adjoint implies convergence to zero in any unitary model. Indeed the squared Hilbert–Schmidt norm there is the character of the product with its adjoint, computed also as the trace of the product of the two normalized balanced matrices. The required finite parabolic theorems follow from Theorem 4.1 and the equal-parameter theory [8, §15]. Indeed the factors in $\widetilde{C}_n$ are of types A and B, and those in $\widetilde{B}_n$ can also be of type D, with equal parameters on each D factor. The product argument is Lemma 4.14.

Let $\pi_{\alpha,v,z}$ have degree $A = A_R$ and fix a finite standard parabolic W_I. Every constituent σ of its restriction satisfies $A(\sigma) \le A$. To see this, finite Schur orthogonality shows that in an orthonormal basis

$$\sum_{w\in W_I} \|\sigma(N_w)\|_{\mathrm{HS}}^2 = (\dim\sigma)^2 s_\sigma(v),$$

where $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert–Schmidt norm. The compact matrix bound for each of the finitely many N_w gives the inequality. This applies at every unitary point and to every constituent of the induced representation because the representation is unitary and the orthogonal projections onto its constituents have norm one. The finite-facet attainment results show that every prescribed irreducible tempered component has such a finite constituent with $A(\sigma) = A$. In particular all spectral degrees are integers. We use these witnesses to generate the whole degree-A polynomial algebra by finite cell units.

Lemma 8.1. *In the degree-A summand of $J_{\mathbb{C}} = \mathbb{C}\otimes_{\mathbb{Z}} J$, the ordinary two-sided ideals generated by the images of the asymptotic unit elements of all finite source cells of intrinsic degree A sum to the whole summand.*

Proof. For a finite source cell $\mathfrak{c}$ of degree A and a family of degree A, its lower boundary has intrinsic degree $> A$, by finite P4 and P11, and hence acts by zero. Its normalized leading action defines a homomorphism from its finite asymptotic algebra to $\mathcal{B}_\alpha$. On restriction to the finite parabolic, a simple σ with $A(\sigma) < A$ has zero leading action at degree A. The finite asymptotic algebra of degree A is the sum of its two-sided cell ideals by finite P8. Each of its simple matrix blocks belongs to exactly one of those ideals,

so the unit of a finite cell acts as the identity on that cell's blocks and as zero on the others. On a constituent of degree A, the normalized lift of that unit (the same linear combination of $v^{-A}C_w$) converges to the indicated operator in norm. To see this, subtract the indicated scalar operator and apply the character formula for the squared norm to the difference. The lift is self-adjoint by finite P6, and its balanced limit is given by Lemma 4.15. The unit image $E_{\mathfrak{c}}(z)$ is therefore a Laurent polynomial idempotent. Its action is nonzero on an irreducible tempered component containing an attaining σ.

At a unitary point z, let G_z be the subgroup of G_α fixing z. The exact and limiting intertwiners determine continuous projective representations of G_z for large v including the limit. Divide each matrix by a continuous d_α-th root of its determinant, with the roots chosen continuously up to the limit. The resulting cocycle has values in the finite group of d_α-th roots of unity, by taking determinants in the projective multiplication law. It is constant. The normalized matrices thus represent a fixed finite central extension of G_z, with isotypic projectors of constant ranks. The exact stabilizer theorem identifies its active blocks with the irreducible tempered components. The exact isotypic projectors commute with the finite parabolic central idempotents, since the intertwiners at z commute with the Hecke action. These central idempotents are rational in the parameter and continuous on the interval of large v. The ranks of their products with the exact isotypic projectors are constant on this interval. A nonzero product means that the associated tempered constituent contains the corresponding finite simple. Choose an attaining simple for that constituent and use its source cell. The rank of the source unit on that isotypic space is positive in the limit: the normalized lift differs in norm by $o(1)$ from the sum of the finite central idempotents for the simple blocks of the source cell, by the preceding norm calculation. The isotypic projector itself converges to its limiting projector. Their limiting product has positive rank, since the exact orthogonal projection products have positive constant rank. This proves coverage of every simple block of $\mathrm{Mat}_{d_\alpha}(\mathbb{C})^{G_z}$ on the unitary torus.

We extend this statement to all complex points of T_α. For a subgroup $K \leq G_\alpha$, each connected component of its fixed locus is a translate of a subtorus with dense unitary points. In fact write a point as $z_0 \exp(x)$ with z_0 unitary and x a real logarithmic vector. The affine monomial fixed equations separate into equations for z_0 and linear equations for x. Thus $z_0 \exp(tx)$ stays fixed for $0 \leq t \leq 1$, and the unitary points are Zariski dense in the component. If a point has exact stabilizer K, removing the other fixed loci leaves a nonempty Zariski-open set in its component and still leaves dense unitary points.

On such a component adjoin d_α-th roots of the invertible Laurent functions $\det U_g(z)$, $g \in K$. This gives a finite torus cover, possibly disconnected. Each component has dense unitary points since the root equations are monomial and the determinant values at unitary points have modulus one. On a connected component of the cover the determinant-normalized cocycle is

constant. Each finite-group isotypic projector is regular there. Its product with $E_{\mathfrak{c}}$ is again an idempotent because the source image commutes with the stabilizer. Its trace is an integer-valued regular function and hence constant. A source hitting a given block at a unitary point of exact stabilizer K thus hits the same block throughout the component of the cover. This proves the required coverage at every complex point.

For each family, write $R_\alpha = \mathbb{C}[T_\alpha]$ and $Z_\alpha = R_\alpha^{G_\alpha}$. Evaluation at z maps $\mathcal{B}_\alpha$ onto $\mathrm{Mat}_{d_\alpha}(\mathbb{C})^{G_z}$, by interpolation on the finite orbit of z followed by averaging. The algebra $\mathcal{B}_\alpha$ is finite over Z_α. For the maximal ideal $\mathfrak{m}$ of Z_α corresponding to the orbit, averaging commutes with reduction modulo $\mathfrak{m}$: the average of $\sum_i a_i M_i$, with $a_i \in \mathfrak{m}$, is $\sum_i a_i$ times the respective averages of M_i. The fiber is therefore the invariant matrix algebra over $R_\alpha/\mathfrak{m}R_\alpha$. This ring is Artinian and has the orbit as its support, since invariant functions separate finite orbits by interpolation and averaging. Its quotient by its nilradical is the product of the residue fields on the orbit. Evaluation on those residue fields is equivariant. Its kernel in the fiber is nilpotent, since a product of sufficiently many matrices with entries in the nilradical is zero. Every simple fiber module therefore factors through the evaluated stabilizer algebra. If the ideal generated by the source idempotents were proper, the quotient would be a nonzero finite Z_α-module. Localization at a maximal ideal in its support and Nakayama's lemma give a nonzero quotient fiber. That finite-dimensional algebra has a simple module, on which all source idempotents vanish. This contradicts the coverage. The same conclusion holds for the finite direct sum of all the families at degree A, since its central family identities belong to that algebra. □

Proposition 8.2. *For each spectral degree A, the span of C_w with $w \in \bigcup_{B \geq A} S_B$ is an original two-sided ideal. Every family of degree below A annihilates this ideal for sufficiently large real v. On the quotient of consecutive ideals, the left and right actions commute with independent parameters. Moreover $a(w) = A$ for $w \in S_A$, and the actual degree-A product of original basis vectors is the product of the corresponding leading algebra of Proposition 5.1.*

Proof. Descend through the finite list of degrees. Assume the ideal for levels $> A$ has been constructed, with exact annihilation by all families of degrees $\leq A$. At the largest degree this ideal is zero. In the quotient, take the positive form given by the truncated Plancherel integral over the remaining families. On any fixed finite set of original surviving C_w its Gram matrix tends to the identity, because the leading Fourier image of each such vector has support in the remaining levels. This uses the degree-zero triangular transition between leading standard and canonical vectors.

For fixed x, y, on a family of degree $B \leq A$ the product matrix has norm $O(v^{2B})$ uniformly on the compact torus. Its squared norm integrated against the family measure is $O(v^{4B} v^{-2B}) = O(v^{2B})$. Summing gives $O(v^{2A})$ for the squared norm of the quotient product. The product has finite original

support. The Gram limit on that support forces every coefficient of this product in the quotient to have degree at most A. For a surviving index z, the coefficient at degree A is

$$[v^A]h_{x,y,z} = \sum_{\alpha:\, A_R=A} \int_{T_{\alpha,u}} \mathrm{Tr}\big(c_{\alpha,x}(\xi)c_{\alpha,y}(\xi)c_{\alpha,z}(\xi)^*\big)\, m_{\alpha,0}(\xi)\, d\xi.$$

Indeed one divides the original finite-coordinate expansion by v^A before taking the limit, so the Gram off-diagonal terms vanish. In the trace pairing of the product with C_z on a family of degree B, the three matrices contribute at most v^{3B}, the measure contributes v^{-2B}, and the division by v^A leaves a factor v^{B-A}. Thus the families of smaller degree have zero limit. By the coordinate cutoffs the right side can be nonzero only when $x, y, z \in S_A$.

For p in any finite standard parabolic, its leading family matrix can be nonzero only at degree $a_I(p)$, its intrinsic finite a-value. In the finite orthogonal decomposition of a restriction, only a constituent with $A(\sigma) = A_R$ can contribute at degree A_R. In such a constituent the finite leading matrices on N_p, equivalently C_p, vanish unless its degree equals $a_I(p)$, by the finite spectral description. The corresponding leading matrix for the adjoint also vanishes, since $a_I(p^{-1}) = a_I(p)$ by finite P14. The squared-norm character formula above gives the vanishing in the compact unitary model. The total Fourier norm of p is one, so it belongs to $S_{a_I(p)}$.

For a finite source cell of intrinsic degree A, its strictly lower source cells annihilate all remaining families, by their intrinsic degree and the finite restriction bound. Their original indices therefore lie in the higher ideal. The source cell survives in the quotient, by the preceding paragraph. Lemma 3.1 gives a constant source image V_I with commuting independent actions. It annihilates the smaller families exactly. In fact the entire source cell acts by zero there, since the restriction contains only finite characters of smaller degree. A constant vector in the remaining quotient that annihilates the smaller families has no coordinates in their levels, by its leading Fourier image. It follows that V_I is contained in the constant span of S_A.

The source image contains the finite cell unit and is stable under both original actions by Lemma 3.1. Its span is constant in the parameter, so each Laurent coefficient of either action preserves it. In particular the coefficient formula shows stability under multiplication on both sides by the degree-A leading algebra. Lemma 8.1 shows that the sum of these images after complexification is the whole span of S_A. All columns defining these images have integer coordinates. Thus over $\mathbb{Q}$ their sum is already the coordinate space, as follows by extending the quotient by this sum to $\mathbb{C}$. The rational invariance implies integral coordinate invariance because the original Hecke action has integral Laurent coefficients. Independent-parameter commutation holds on the entire layer, since it holds on each summand of a spanning sum. Exact lower annihilation passes to each original C_w in that layer. This proves the induction.

For $w \in S_A$, the quotient bound gives $a(w) \leq A$. The degree-A coefficient formula and multiplication by the unit of that leading level show that some coefficient with output w attains degree A. Thus $a(w) = A$ and the asserted product follows. □

9. The signed units

In the trace calculation for $\widetilde{C}_n$ use its own Plancherel families. For $\widetilde{B}_n$ use the restricted families of H^0 in Section 7. Thus in this section the family algebras used in the trace always come from the $\widetilde{C}_n$ datum, with $c > 0$ or $c = 0$.

9.1. **Integral characteristic polynomials.** Fix one compact induction family $\pi_{v,z}$ with Levi discrete-series representation δ_v, where z varies in its unramified torus. In the $\widetilde{C}_n$ datum a finite standard Levi consists of type-A blocks and a possibly empty type-B tail in the finite root system. The semisimple Hecke algebra is a product of the extended type-A algebras on the weight lattices and the nonextended affine-C tail. By [3, Proposition 6.5], the finite torus twisting group acts freely on the type-A discrete series and trivially on the tail. We choose the real Steinberg representation on each type-A factor and absorb its torsion twists into z. These representatives have rational characters in the parameter. The parameter value $v^{2c} = 1$ is positive and the same Levi description applies.

We first verify rationality for the tail. Work in the field $\mathbb{Q}(v^{1/e})$ for a positive integer e clearing the residual exponents. For a discrete series of the nonextended affine-C tail the residual coordinates are signed powers of $v^{1/e}$, by the signed residual string description of [2, Theorem 1.1 and Section 4.2.1]. This description applies to the discrete series surviving at the specified weights, including $c = 0$. The joint Bernstein weights lie in the Weyl orbit of this central character and hence also in the field. To see directly why the residual phases are signs at a specialized ray, the pole roots of a point residual coset span the full rank [10, Theorem 2.38 and proof of Proposition 2.56]. In the standard torus coordinates x_i, their equations have the forms $x_i x_j^{\pm 1} = v^k$ and $x_i = \pm v^k$ for integer exponents k. For the unitary phases, regard the two-coordinate equations as edges joining equal or inverse phases. A connected component with a one-coordinate equation has all phases equal to signs. In a component without one, every phase is the phase of one chosen coordinate or its inverse. Unless a cycle identifies that phase with its inverse, one complex parameter varies freely, contradicting full rank. Thus in this component also every phase is a sign. Galois conjugation fixes the joint weights. The ranks determining their generalized weight multiplicities are unchanged by conjugation. These ranks and the simplicity of the algebraic branches persist at all sufficiently large regular specializations, by the finite central quotient construction in Section 5. The strict Casselman inequalities therefore hold for the specialized conjugate, so it too is a discrete series.

Every finite parabolic of the tail has split Hecke algebra over $\mathbb{Q}(v^{1/e})$. In the cyclotomic presentation, the split semisimplicity criterion [1, Theorem 4.2] uses quantum integers at v^{2a} and cross factors of the form $1+v^j$, with $j \in \mathbb{Z}$. These factors are nonzero, including when an endpoint weight is zero. The finite simple multiplicities are therefore Galois invariant. For a face f of the closed fundamental alcove, denote by $m_f(\delta,\sigma)$ the multiplicity of the finite simple σ on restriction to the Hecke algebra of the stabilizer of f. For this nonextended datum the finite-face formula for the Euler–Poincaré pairing gives

$$\mathrm{EP}(\delta,\delta') = \sum_f (-1)^{\dim f} \sum_\sigma m_f(\delta,\sigma) m_f(\delta',\sigma).$$

This is the specialization of [10, Equations (57)–(58)] to the nonextended datum. The sum is over all faces, including the alcove itself. The group of length-zero elements is trivial for this datum. Taking δ' to be a Galois conjugate gives the self-pairing of δ, which is one. Distinct discrete series are orthogonal [10, Corollary 2.34], so the conjugate has the same character. Thus δ_v has character over $\mathbb{Q}(v^{1/e})$. Enlarging e will also allow the monomial v^{-A_R}.

Let P denote the finite Levi simple roots and W_P its finite Weyl group. In the Bernstein lattice X the unramified quotient is $X/(X \cap \mathbb{R}P)$, and the projected semisimple lattice is $X/(X \cap (\mathbb{R}P^\vee)^\perp)$. For $x \in X$ write $\nu(x)$ and $\bar{x}$ for its respective images. A torus monomial $z^{\nu(x)}$ means evaluation at the character $\nu(x)$. The twisted quotient defining parabolic induction takes the Bernstein basis element $\theta_x N_p$, for $p \in W_P$, to $z^{\nu(x)}\theta_{\bar{x}} N_p$ [9, Proposition 4.16]. Put $\mathcal{O} = \mathbb{Z}[v^{1/e}, v^{-1/e}]$ for the coefficient ring in this argument. Over $\mathcal{O}$ the Bernstein presentation decomposes the ambient algebra as a right module over the parabolic algebra, with basis N_d for the minimal representatives d of W_0/W_P. The Bernstein cross-relations lie in this ring because the differences of lattice monomials in their numerators are divisible by the corresponding root binomial. For any fixed standard basis element N_w, expand

$$N_w N_d = \sum_{d',\,x,\,p} a_{d',d,x,p} N_{d'} \theta_x N_p, \qquad a_{d',d,x,p} \in \mathcal{O}.$$

This is a finite expansion and its parabolic coefficients give the compact induction matrix. In particular the (d',d) entry of its coefficient at z^ν is

$$\delta_v\left(\sum_{\substack{x,p\\ \nu(x)=\nu}} a_{d',d,x,p}\theta_{\bar{x}} N_p\right).$$

A coefficient matrix is thus the entrywise image under δ_v of a matrix over the integral semisimple Levi algebra.

Impose the residual central character of δ_v on this Levi algebra. The Bernstein lattice algebra is finite over its Weyl invariants: each of its lattice

generators and its inverse satisfy the monic polynomial given by their Weyl orbit. The invariant elements take values in $\mathcal{O}$ at the chosen signed residual monomials. The resulting central quotient is consequently finite over $\mathcal{O}$, and so is its matrix algebra. The determinant trick makes every element of that matrix algebra integral over $\mathcal{O}$. The eigenvalues of its image under δ_v are integral as well. The characteristic polynomial of the image belongs to $\mathbb{Q}(v^{1/e})[T]$: traces of all powers are rational by character rationality of δ_v, and the Newton identities give the coefficients. Integral closedness puts the polynomial in $\mathcal{O}[T]$.

The compact Gram form is independent of z [9, Proposition 4.19]. Passing to the orthonormal compact basis of Section 5 conjugates every torus coefficient by the same matrix. For a finite integral Laurent polynomial $\mathfrak{d}(z)$, multiplication by $\mathfrak{d}$ replaces the coefficient matrices by integral linear combinations. After the conjugation and multiplication by v^{-A_R}, their characteristic polynomials still belong to $\mathcal{O}[T]$. Write

$$c_w(z) = \lim_{v\to\infty} v^{-A_R}\pi_{v,z}(N_w)$$

in this orthonormal compact basis. The limit exists uniformly by Proposition 5.1. The fixed finite torus support also gives coefficientwise convergence. For a Laurent polynomial in z, let $[z^\nu]$ denote the coefficient of its character monomial z^ν, with absent coefficients equal to zero. Each characteristic polynomial coefficient has a finite limit and is a Laurent polynomial in $v^{1/e}$ with integer coefficients. Its limit is its integer constant term. We obtain

$$\det\bigl(TI - [z^\nu](c_w(z)\mathfrak{d}(z))\bigr) \in \mathbb{Z}[T]. \tag{19}$$

For the restricted H^0 families, the standard basis identification of Lemma 7.1 gives the same statement for $w \in W$ of type $\widetilde{B}_n$.

9.2. **The limiting measures.** We compute the stationary factors of the relative Plancherel density. In the $\widetilde{C}_n$ lattice $X = \mathbb{Z}^n$, let the type-A blocks have sizes $k_1, \dots, k_r$ and let the tail have size $n - \sum_j k_j$. The unramified quotient records the sum of the lattice coordinates in each block. It therefore has primitive torus coordinates $z_1, \dots, z_r$. The coordinates of the real Steinberg factor on a block of size k_j are $z_j v^u$, with

$$u \in S_{k_j}, \qquad S_k = \{a(k-1-2i) : 0 \le i < k\}.$$

A tail coordinate is εv^u with $\varepsilon \in \{1, -1\}$ and a fixed real exponent u.

Write x_i for the resulting ambient coordinates. The exact positive Plancherel product, with the Levi roots removed, has long-root factors

$$\frac{|1 - x_i x_j^{\pm 1}|^2}{|1 - v^{2a} x_i x_j^{\pm 1}|^2},$$

and doubled short-root factors

$$\frac{|1 - x_i^2|^2}{|1 - v^{b+c} x_i|^2\, |1 + v^{b-c} x_i|^2},$$

up to the overall monomial and the tail formal-degree factor. In the long-root display each choice of sign denotes one factor, when the corresponding root is outside the Levi. These are the absolute-square expressions on the tempered torus in [9, Theorem 3.25 and Proposition 3.27]. The Weyl element that inverts the residual real part and fixes the unitary part gives the absolute-square expression from the product of opposite roots. When $b = c$ the second short denominator is $|1 + x_i|^2$. At $c = 0$ the two short denominators multiply to $|1 - v^{2b}x_i^2|^2$.

For a unitary monomial Z set $D_\varepsilon(Z) = |1 - \varepsilon Z|^2$ for $\varepsilon = \pm 1$, and write D_+, D_- for the two signs. The normalized limit of $|1 - \varepsilon v^d Z|^2$ is one when $d \neq 0$ after division by $v^{2\max(d,0)}$. For $d = 0$ its factor is $D_\varepsilon(Z)$. We count these stationary factors, with denominator factors counted negatively.

- Between two blocks, the ratio-root multiplicity is $|S_k \cap S_h| - |S_k \cap (S_h - 2a)|$ for block sizes k, h. If their parities differ, both intersections are empty. For equal parity and unequal sizes both have cardinality $\min(k, h)$, and for $k = h$ the cardinalities are k and $k - 1$. The sum-root multiplicity is the same by $S_h = -S_h$. Thus the factor for equal-size blocks i, j is $D_+(z_i/z_j)D_+(z_i z_j)$.
- Within one block only sum roots are outside the type-A Levi. The numerator pairs whose exponents sum to zero number $\lfloor k/2 \rfloor$. The denominator pairs whose exponents sum to $-2a$ number $\lfloor (k-1)/2 \rfloor$. The doubled short numerator adds one factor $|1 - z_j^2|^2$ when k is odd. Hence the net internal numerator is $|1 - z_j^2|^2 = D_+(z_j)D_-(z_j)$. The short denominators subtract one from the positive and negative exponents when $b + c \in S_k$ and $b - c \in S_k$, respectively.
- A tail coordinate εv^u contributes only at $z_j = \varepsilon$. Counting the sum and ratio roots gives the signed multiplicity

$$2\mathbf{1}_{u \in S_k} - \mathbf{1}_{u+2a \in S_k} - \mathbf{1}_{u-2a \in S_k},$$

  where $\mathbf{1}$ denotes the indicator of the specified condition. The two numerator tests are $u, -u \in S_k$ and the two denominator tests are $u - 2a, -u - 2a \in S_k$. The displayed form uses $S_k = -S_k$.

Thus, for each size k, the short factors $D_\varepsilon(z_j)$ have a common net exponent on its size-k blocks. This exponent is nonnegative, since a negative exponent would give a pole of the exact density at a generic point of $z_j = \varepsilon$, contrary to its real analyticity on the compact tempered torus [9, Theorem 3.25]. If the exponent is positive, that divisor lies in the zero set of the density. A short root supported on a coordinate of block j projects to the direction of the primitive block character z_j on the unramified torus. The divisor $z_j = \varepsilon$ is therefore the connected translate of the projected root kernel through any of its points. It is a Plancherel mirror in [3, Definition 4.1]. The relative c-function has a simple pole on a mirror [3, Theorem 4.3(iii)]. This theorem allows zero label exponents, since the labels in [3, §2] are arbitrary positive real numbers. Inverting the unramified character preserves

the divisor $z_j = \varepsilon$. The density is proportional to the reciprocal of the product of the c-function with its value at that inverted character. Its zero has order two, which is the order of D_ε. The exponent is therefore one. At $b = c$ the negative test is $0 \in S_k$. At $c = 0$ the tests for the two signs use the same number b but the divisors $z_j = 1$ and $z_j = -1$ are distinct. The argument on each divisor is unchanged.

On the blocks of a fixed size, take the product

$$\prod_{i<j}(1 - z_i^{-1}z_j)(1 - z_i^{-1}z_j^{-1}),$$

where the indices are ordered within that set of blocks. If there are only positive short factors, multiply by $\prod_i(1 - z_i^{-1})$. If there are only negative short factors, multiply by $\prod_i(1 + z_i^{-1})$. If both occur, multiply by $\prod_i(1 - z_i^{-2})$. With neither short factor use the displayed product alone. These are the Weyl denominators of the corresponding types B, its simultaneous sign translate, C and D. In the last case with one block the product is one, and with two blocks it is the product for $A_1 \times A_1$. The Weyl denominator identity expresses each as the alternating sum with exponents $w\rho - \rho$, where ρ is half the sum of the positive roots. These exponents are integral and distinct by regularity of ρ in the corresponding root system. The sign translation also has coefficients ± 1. The sets of coordinates for different sizes are disjoint, so the product over sizes has coefficients ± 1.

Let $\mathfrak{d}$ be this product. The normalized limiting measure on the family is $\kappa|\mathfrak{d}|^2\,dz$, where dz is probability Haar measure and $\kappa > 0$ includes the finite normalization factors and the leading tail formal degree. If the torus is a point, $\mathfrak{d} = 1$ and dz is the point mass. The same measure is used on restricting a family to H, by Lemma 7.1.

9.3. **The trace inequality.**

Lemma 9.1. *Let D be a complex matrix with characteristic polynomial in $\mathbb{Z}[T]$. For the adjoint D^* with respect to any fixed positive Hermitian form,*

$$\operatorname{Tr}(DD^*) \geq |\operatorname{Tr} D|.$$

In the equality case D is zero or an orthogonal projection multiplied by the sign of its nonzero trace.

Proof. In an orthonormal basis Schur triangularization bounds the squared Hilbert–Schmidt norm below by the sum of the squared absolute eigenvalues, with equality only for a normal matrix. Factor the monic characteristic polynomial into monic irreducibles over $\mathbb{Q}$, counting multiplicities. For a factor other than T, let its roots be $\lambda_1, \ldots, \lambda_m$ and put $S = \sum_i |\lambda_i|^2$. The absolute value of their product is a nonzero integer by Gauss's lemma. The arithmetic–geometric mean inequality gives $S \geq m$, so

$$\left|\sum_i \lambda_i\right| \leq \sqrt{mS} \leq S.$$

The factor T contributes zero. Summing over the factors gives the inequality. Equality makes all roots of each nonzero factor equal and of absolute value one, so that factor is $T-1$ or $T+1$. Triangle equality makes their signs agree. Normality then gives the asserted projection. □

Lemma 9.2. *Let $\mathfrak{d}$ be a nonzero Laurent polynomial with all nonzero coefficients in $\{1,-1\}$, and let $M(z)$ be a matrix Laurent polynomial on the same torus. Suppose every matrix coefficient of $M(z)\mathfrak{d}(z)$ has characteristic polynomial in $\mathbb{Z}[T]$. For a torus-independent positive Hermitian form,*

$$\int \operatorname{Tr}(MM^*)|\mathfrak{d}|^2\,dz \;\geq\; \left|\int \operatorname{Tr}(M)|\mathfrak{d}|^2\,dz\right|.$$

Equality makes M constant, either zero or an orthogonal projection multiplied by the sign of the nonzero integral on the right.

Proof. Write $D_\nu = [z^\nu](M\mathfrak{d})$ and $\varepsilon_\nu = [z^\nu]\mathfrak{d}$. Parseval identifies the left side with $\sum_\nu \operatorname{Tr}(D_\nu D_\nu^*)$ and the integral inside the absolute value with $\sum_\nu \varepsilon_\nu \operatorname{Tr}(D_\nu)$. Apply Lemma 9.1 term by term and use the triangle inequality. In the equality case $D_\nu = 0$ whenever $\varepsilon_\nu = 0$. For any nonzero scalar entry f of M, the convex hull of the support of $f\mathfrak{d}$ is the Minkowski sum of the convex hulls of the supports of f and $\mathfrak{d}$. Indeed, leading terms for each supporting linear functional multiply without vanishing in the Laurent polynomial domain. This sum lies in the convex hull of the support of $\mathfrak{d}$. Evaluating every linear functional shows that f has support $\{0\}$. Hence M is constant. For any ν with $\varepsilon_\nu \neq 0$, the coefficient is $D_\nu = \varepsilon_\nu M$. The equality case of Lemma 9.1 then makes M zero or a signed orthogonal projection, with sign that of $\operatorname{Tr} M$. The integral of $|\mathfrak{d}|^2$ is positive, so the integral on the right has the same sign. □

Proposition 9.3. *The integral leading algebra has a finite unit which is a sum of signed basis elements. The corresponding signed elements are mutually orthogonal self-adjoint idempotents. In the original orthonormal basis both their left and their right multiplication operators are coordinate projections.*

Proof. Use the finite integral unit e from Proposition 5.1 and write $e = \sum_w m_w t_w$, where t_w is its leading basis indexed by $w \in W$. Denote the positive trace of this algebra by τ_J. Its basis satisfies $\tau_J(t_w t_y^*) = \delta_{w,y}$. Since the unit is self-adjoint, $\tau_J(t_w) = \langle t_w, e\rangle = m_w$. The Plancherel realization sends each t_w to the family matrices c_w. In the $\widetilde{B}_n$ case this follows with the restricted matrices by Section 7. For every family Lemma 9.2 applies by (19) and the computed density. Multiplying its inequality by κ and summing over the families yields

$$1 = \tau_J(t_w t_w^*) \geq |\tau_J(t_w)| = |m_w|.$$

The integers m_w are consequently 0 or ± 1. For $m_w \neq 0$, all the inequalities used in the sum are equalities. Since the masses κ are positive, the nonzero

projections in the family matrices have the common sign m_w. Thus $m_w t_w$ is a self-adjoint idempotent. These finitely many positive projections sum to the identity. For a vector ξ in the range of one of them, the inner product of the sum with ξ is $\|\xi\|^2$, which is already the contribution of that projection. Each other term is the squared norm of the corresponding projection of ξ and therefore vanishes. Thus the projections are pairwise orthogonal.

The corresponding left and right multiplication operators are orthogonal projections with integer matrix entries. For such a projection P the diagonal entry P_{ii} is the squared norm of its ith column. This diagonal entry is an integer between zero and one, and hence all off-diagonal entries of that column vanish. This proves the coordinate assertion. □

10. The affine properties

We finish the proof of Theorem 1. The finite case is Theorem 4.1. Let W now be one of the two affine systems.

Proposition 8.2 gives P4 and P15: stability of the threshold ideals gives $a(x) \geq a(y)$ when $x \leq_{LR} y$, and independent-parameter commutation on each layer is P15. It gives P12 as well, since the leading matrices of an original finite parabolic element have support precisely at its intrinsic degree. A proper standard parabolic of the affine system is finite, and the full-system assertion is automatic. For $w \in S_A$, the exact trace of C_w involves only families of degree $B \geq A$. Each contribution is $O(v^{-B})$ from the compact matrix bound and the measure normalization. Hence $\deg_v \tau(C_w) \leq -A$, giving P1. Moreover

$$[v^{-A}]\tau(C_w) = \tau_J(t_w). \tag{20}$$

The higher degrees make no contribution to this coefficient. For the families of degree A it is the leading trace integral. The identification of the leading product and cyclicity of the matrix trace give P7, since

$$\gamma_{x,y,z} = \tau_J(t_x t_y t_z).$$

Here the right side and the left side vanish unless all three indices lie in the same spectral level.

Consider the directed left multiplication graph on the basis of a fixed level of J. It has an arrow from t_y to t_z if some $t_x t_y$ has nonzero t_z-coefficient. This relation is symmetric, since

$$\langle t_x t_y, t_z \rangle = \langle t_y, t_x^* t_z \rangle.$$

Such an arrow gives an original left preorder comparison, by the degree-A coefficient formula. Conversely, every original left generator edge with both endpoints at degree A lies in this graph. In fact, by P15 the ordinary left action on the degree-A quotient commutes with the right leading multiplication, obtained by taking the coefficient of degree A in its independent right parameter. A map on the unital right regular module of this leading algebra which commutes with right multiplications is left multiplication by

its value on the unit. A nonzero original coordinate in its image must thus occur in some leading product. By P4 an original left preorder path between equal-degree endpoints stays at that degree. Each edge can be reversed in the original preorder, giving P9. The right version gives P10, and the same argument for paths consisting of both types gives P11. This also identifies the connected components of the left multiplication graph with the original left cells.

If $\gamma_{x,y,z} \neq 0$, the three indices have the same degree and $z^{-1} \leq_L y$, so P9 gives $y \sim_L z^{-1}$. The other two relations follow by P7, proving P8. To obtain P14, choose a nonzero output in $t_x t_x^*$, whose trace is one. Its index z is in the same left cell as x^{-1} and the same right cell as x by P9 and P10. Thus $x^{-1} \sim_{LR} z \sim_{LR} x$.

Write the unit of J as $\sum_w m_w t_w$. Orthonormality gives $m_w = \tau_J(t_w)$. By (20), $m_w \neq 0$ exactly for $w \in \mathcal{D}$, and then $m_w = n_w$. Proposition 9.3 gives $n_d = \pm 1$ for $d \in \mathcal{D}$ and mutually orthogonal self-adjoint projections $p_d = n_d t_d$. Since $t_d^* = t_{d^{-1}}$, self-adjointness gives $d^{-1} = d$, which is P6. The right multiplications by the p_d are coordinate projections summing to the identity. Each t_y therefore has a unique right unit p_d among them. For $d \in \mathcal{D}$,

$$[t_d](t_x t_y) = \tau_J(t_x t_y t_d) = n_d \, \delta_{x,y^{-1}} \, \mathbf{1}_{\{t_y p_d = t_y\}},$$

where $[t_d]$ denotes its basis coefficient. This proves P2, P3, and P5. Every left multiplication edge preserves the right unit: associativity gives $(t_x t_y)p_d = t_x t_y$ whenever $t_y p_d = t_y$, and right multiplication by p_d is a coordinate projection. Also $t_d p_d = t_d$ since p_d is idempotent. The nonzero coefficient of t_d in $t_{y^{-1}} t_y$ connects t_y to its right-unit index d. Thus its original left cell contains precisely that distinguished element, and the coefficient formula gives $\gamma_{y^{-1},y,d} \neq 0$ for each y in the cell. This proves P13.

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