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COUNTEREXAMPLES IN POSITIVE CHARACTERISTICS

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Abstract. AI found the following counterexamples (following human directions) to 3 important questions: For every prime p we construct a smooth projective rational surface that is globally +-regular, with anticanonical divisor nef but neither big nor semiample. Its completed cone is a three-dimensional F-finite splinter with regular punctured spectrum and embedding dimension six. Both test ideals are the maximal ideal, so the cone is not weakly F-regular. A discrepancy calculation by humans rules out every effective klt boundary. A separate four-dimensional complete normal local domain has distinct big and finitistic test ideals.

Contents

1. Introduction 1
Human statement 3
2. Frobenius, duality, and splitting criteria 4
3. Finite covers of a rational surface with an elliptic anticanonical curve 6
4. The completed cone over the rational surface 13
5. Two test ideals in characteristic two 23
6. The discrepancy in odd characteristic 32
References 41

1. Introduction

Let $R \subseteq S$ be a finite extension of domains of characteristic $p > 0$. An R-linear retraction $S \to R$ gives an injection $M \to S \otimes_R M$ for every R-module M. In particular, extending an ideal to S and contracting it recovers the original ideal. We call R a *splinter* if every finite domain extension has a retraction. A splitting under a finite cover of a proper scheme likewise gives injections on coherent cohomology.

Every strongly F-regular ring is a splinter. The proof comes from the tight-closure methods of Hochster and Huneke [HH90, HH94b, HH94a]. Singh proved that a locally excellent $\mathbb{Q}$-Gorenstein domain of characteristic p is F-regular if and only if it is a splinter [Sin99a, Theorem 1.1]. Our completed cones are splinters with ideals that are not tightly closed. They are necessarily non-$\mathbb{Q}$-Gorenstein.

1.1. Finite covers and anticanonical positivity.

Smith used section rings to prove cohomology vanishing for nef line bundles on globally F-regular projective varieties [Smi00, Corollary 4.3]. Schwede and Smith later proved that such varieties are of log Fano type [SS10, Theorem 1.1].

Now suppose only that every finite cover splits. In characteristic p, Bhatt proved that this also gives derived splittings under proper covers [Bha12, Theorem 1.4]. Global +-regularity gives the corresponding formulation in mixed characteristic [BMP$^+$23]. The conjecture in [BMP$^+$23,

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Conjecture 6.17] predicts an effective boundary that keeps the pair globally +-regular and makes its negative log canonical divisor ample. Krah and Vial ask whether the anticanonical divisor is at least big for a projective $\mathbb{Q}$-Gorenstein splinter [KV25, Conjecture 3.12].

Theorem A. For every prime p there is a smooth projective globally +-regular rational surface X over an algebraically closed field of characteristic p. Its anticanonical divisor $-K_X$ is nef but neither big nor semiample. In particular, X is not of Fano type.

For the smooth varieties here, global +-regularity with zero boundary means that every finite surjective cover splits [BMP+23, Definition 6.1]. Since X is not of Fano type, it is not globally F-regular.

Choose an ordinary plane cubic E and its generic point P over a function field. Blow up $P, 2P, \ldots, 9P$. The strict transform C is anticanonical, has square zero, and has nontorsion normal bundle. These are the classical nine-point examples of nef nonsemiample divisors [Tot09, §2][LT13, Example 5.7]. We chose nine multiples of one point to get an additional property: every integral curve other than C meets C.

Pick a nontrivial étale $\mathbf{F}_p$-torsor on the ordinary curve C. The boundary sequence sends its class to a nonzero element of $H^2(X, \omega_X)$. To split a finite separable cover, we show that this element stays nonzero after pullback. If it vanished, connectedness of the inverse image of C would give a nontrivial étale $\mathbf{F}_p$-cover of X, contradicting rationality. We use the nontorsion normal bundle to control ramification along C.

The same blowups can be made in a family. Krah–Vial's result gives nonsplitting on its geometric closed fibers [KV25, Remark 11.6 and Proposition 11.10], whereas our geometric generic fiber is a splinter. The locus of geometric splinter fibers is therefore a single generic point.

1.2. **The completed cone.** Take $L = H + C$, where H is the pullback of a line on the plane. This line bundle is ample. Form its full section ring and complete at the vertex. The canonical ideal is the ideal of the cone over C. Thus the elliptic cohomology class can also be studied in the local ring.

Theorem B. For every prime p there is a three-dimensional complete F-finite normal Cohen–Macaulay local splinter $(R, \mathfrak{m})$ of characteristic p. Its punctured spectrum is regular, its embedding dimension is six, and

$$\tau_b(R) = \tau_{\mathrm{fg}}(R) = \mathfrak{m}.$$

The ring fails weak F-regularity. There is no effective $\mathbb{Q}$-divisor Δ for which $(\operatorname{Spec} R, \Delta)$ is klt.

The elliptic class survives every finite extension of R. On the other hand, one degree-one multiplier kills all its tensor-Frobenius transforms. We realize the latter relation in a fixed finite-length module by a bounded-degree presentation. This gives an ideal I and an element u with

$$I^F = I^+ = I, \qquad u \in I^* \setminus I.$$

Splitting gives the equality with plus closure. The strict containment in tight closure proves failure of weak F-regularity. Such a witness must use a nonparameter ideal. Indeed, Smith identified tight and plus closure for ideals generated by part of a system of parameters in an excellent local domain [Smi94, Theorem 5.1]. Her work also relates F-rationality to rational singularities [Smi97].

Theorem B answers the local expectation in [BMP+23, Remark 6.16] and the klt-boundary question in [CEMS18, Question 5.10]. Chiecchio, Enescu, Miller and Schwede prove the positive result when the anticanonical algebra is finitely generated [CEMS18, Corollary 5.9]. For our cone that algebra cannot be Noetherian. We also prove this directly.

The splitting argument for the completed cone needs more than splitting of the projective surface. A finite extension of the completed ring need not be graded. We normalize the vertex blowup in the finite extension and prove connectedness by intersecting divisors on a resolution.

1.3. **The two test ideals.** For the complete normal F-finite local domains here, both test ideals are annihilators of closures of zero in the injective hull. Big tight closure tests a relation in the whole hull. Finitistic tight closure requires it to hold in one finite submodule, chosen before the Frobenius iterate varies.

Lyubeznik and Smith studied this comparison and proved equality in several settings, including isolated singularities [LS01]. The conjecture for rings essentially of finite type over a perfect field is stated in [ST12, Conjecture 5.14]. Other equality results use numerical $\mathbb{Q}$-Gorensteinness or bounds on local cohomology [Tak21, AHP24].

Theorem C. For every prime p there is a four-dimensional complete excellent F-finite normal local domain T of characteristic p such that

$$\tau_b(T) \subsetneq \tau_{\mathrm{fg}}(T).$$

We use a family of rank-two bundles on a curve over a complete DVR. The generic bundle is periodic under curve Frobenius. The special bundle is stable, but one of its Frobenius pullbacks is unstable. Families with this variation of Frobenius behavior occur in [Las01, LP04, Duc08, BS13]. In our family the cohomology has cyclic quotients $k[[t]]/(t^{k_n q})$, with $k_n \to \infty$.

The class in the injective hull has denominator t. Localizing in the Čech calculation kills the relative t-torsion. A finite approximation keeps a denominator t^N, which becomes t^{Nq} under Frobenius. We choose the principal part so that its prescribed Frobenius multiples reach cyclic quotients of length $k_n q > Nq$. For each fixed N, this happens at arbitrarily large q. No one finite approximation can therefore support all the relations. The calculation first gives an example for a pair. Adjoining a root absorbs the ideal exponent into a ring equation and gives distinct ordinary test ideals.

The ring T is complete and non-Cohen–Macaulay, with imperfect F-finite residue field. Its singular locus is nonisolated, as required by the isolated-singularity equality in [LS01].

Section 2 fixes the Frobenius and duality conventions. In Sections 3 and 4 we follow the same elliptic class from the surface to its completed cone. Sections 5 and 6 construct the test-ideal discrepancy in characteristic two and in odd characteristic.

Human statement

All main ideas and constructions in this draft are either explicitly suggested or fully developed by AI. The humans (R. Ajit and S. Lee) asked specific questions and gave related directions, papers, and even precise result statements to AI so that it could explore those directions and construct counter-examples. In most cases, AIs (and humans) were successful. However, their mathematical details were quite incomprehensible to the humans. Currently, Ajit and Lee are carefully verifying AI's claims as well as improving the exposition and adding their personal human inputs. What follows is a very preliminary attempt to that. In particular, the humans think the math in this draft is correct. In the near future, with a better understanding, the humans think that they will be able to substantially improve the exposition, add and correct proper citations, and most importantly add precise human contribution (right definitions, for example) to this project. We will also add more precise human contributions to the existing material soon. We make this available to the world purely for mathematical reasons. The humans involved in this project can be contacted at

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2. Frobenius, duality, and splitting criteria

Over a complete local ring, we can test splitting on one socle generator of the injective hull. We use the same hull to compare big and finitistic tight closure.

2.1. **Frobenius and the two test ideals.** All rings are Noetherian and have characteristic $p > 0$. We call R F-finite if the Frobenius map $F : R \to R$ is finite. Fix a domain R and set $q = p^e$. In the restriction of scalars $F^e_* M$, the scalar r acts by $r \cdot F^e_* m = F^e_*(r^q m)$. For extension of scalars we write

$$F^e_R(M) = R \otimes_{F^e,R} M, \qquad m^{[q]} = 1 \otimes m$$

The left factor has its usual left action of R. Its right action is through $r \mapsto r^q$, so $(rm)^{[q]} = r^q m^{[q]}$. Restriction of scalars is exact, but tensor Frobenius is only right exact. Even when $L \subset M$, the map $F^e_R(L) \to F^e_R(M)$ can have a kernel. When M carries a geometric Frobenius action, we write F for its additive map with $F(rm) = r^p F(m)$. Its eth iterate gives the R-linear map $F^e_R(M) \to M$ sending $1 \otimes m$ to $F^e(m)$. On a curve defined over $\mathbf{F}_p$ and extended to $k[[t]]$, we use two Frobenius maps. Curve Frobenius fixes t, while absolute Frobenius sends t to t^p.

Fix a nonzero ideal $\mathfrak{a} \subset R$, a real number $s \geq 0$ and an R-module M. We say that v belongs to $0^{*\mathfrak{a}^s}_M$ if we can pick one $c \in R \setminus \{0\}$ such that

$$c\,\mathfrak{a}^{\lceil sq \rceil} v^{[q]} = 0 \quad \text{in } F^e_R(M) \quad \text{for all } e \gg 0. \tag{2.1}$$

The equality must hold for each $a \in \mathfrak{a}^{\lceil sq \rceil}$ with the same c. At $s = 0$ we get ordinary tight closure 0^*_M. For the finitistic closure $0^{*\mathfrak{a}^s,\mathrm{fg}}_M$, take the union in M of $0^{*\mathfrak{a}^s}_L$ over finite submodules $L \subset M$. Functoriality puts this union in $0^{*\mathfrak{a}^s}_M$. The finite submodule L is chosen once, before e varies [HH94b, LS01, HY03, HT04].

For an ideal $I \subset R$, we have $I^*/I = 0^*_{R/I}$. Write $I^{[q]}$ for the ideal generated by qth powers of elements of I. An element u belongs to the Frobenius closure I^F if $u^q \in I^{[q]}$ for some $q = p^e$. Let R^+ be the integral closure of R in an algebraic closure of its fraction field. The plus closure is $I^+ = IR^+ \cap R$. An element of I^+ therefore lies in IS for one finite domain extension $R \subset S$ [HH90, Smi94, Sin99b].

Now let $(R, \mathfrak{m}, k)$ be complete local and set $E = E_R(k)$. We use

$$\tau_b(R, \mathfrak{a}^s) = \mathrm{Ann}_R(0^{*\mathfrak{a}^s}_E), \qquad \tau_{\mathrm{fg}}(R, \mathfrak{a}^s) = \mathrm{Ann}_R(0^{*\mathfrak{a}^s,\mathrm{fg}}_E). \tag{2.2}$$

These are the big and finitistic test ideals for the complete normal F-finite local domains considered here [LS01, HT04]. We omit $\mathfrak{a}^s$ at $s = 0$ and write $D_R(-) = \mathrm{Hom}_R(-, E)$. Matlis duality makes the annihilators distinguish the two closures.

Lemma 2.1. *Every submodule $N \subset E$ satisfies $N = 0 :_E \mathrm{Ann}_R(N)$. Thus a strict inclusion of submodules of E reverses to a strict inclusion of their annihilators.*

Proof. Completeness gives $D_R(E) = \mathrm{End}_R(E) = R$. Restrict the endomorphisms to N. Since E is injective, restriction maps onto $D_R(N)$, with kernel $\mathrm{Ann}_R(N)$. Dualizing $D_R(N) = R/\mathrm{Ann}_R(N)$ identifies N with $0 :_E \mathrm{Ann}_R(N)$ by Matlis biduality [Sta26, Tag 08Z9]. □

Lemma 2.2. *Let B be a finite R-algebra. Pick $0 \neq u \in \mathrm{Soc}\, E$. The unit $R \to B$ splits as an R-module map if and only if $1 \otimes u \neq 0$ in $B \otimes_R E$.*

Proof. Since the socle is essential, the map $E \to B \otimes_R E$, $v \mapsto 1 \otimes v$, is injective precisely when $1 \otimes u \neq 0$. Its dual is evaluation at 1, under

$$D_R(B \otimes_R E) = \mathrm{Hom}_R(B, D_R(E)) = \mathrm{Hom}_R(B, R).$$

Exactness and faithfulness of D_R make injectivity equivalent to surjectivity of evaluation. A functional taking 1 to 1 is the required retraction. □

2.2. **Top local cohomology and finite stages.** Fix elements $x_1, \dots, x_d$ in a Noetherian ring R. Put $I = (x_1, \dots, x_d)$ and $x = x_1 \cdots x_d$. Write $[v/x^n]$ for the top Čech class with numerator v and denominator x^n. In a d-dimensional local ring we can take the x_i to be a full system of parameters. Then H^d_I is local cohomology with support at the maximal ideal.

Lemma 2.3. *For a map $R \to S$ of Noetherian rings and any R-module M, we have a natural isomorphism*

$$S \otimes_R H^d_I(M) \simeq H^d_{IS}(S \otimes_R M).$$

In particular, $F^e_R(H^d_I(M)) \simeq H^d_I(F^e_R(M))$. Under this isomorphism the tensor Frobenius class of $[v/x^n]$ is $[v^{[q]}/x^{nq}]$.

Proof. Compute the top Čech cohomology as the cokernel of the last differential. Tensoring preserves the cokernel and commutes with each localization. Now take $R \to S$ to be $F^e : R \to R$. This replaces each x_i by x_i^q. The radical stays the same and the denominator becomes x^{nq}. □

We can keep track of a denominator before tensoring by writing

$$H^d_I(M) = \varinjlim_n M/(x_1^n, \dots, x_d^n)M, \qquad [v]_n \longmapsto [xv]_{n+1}. \tag{2.3}$$

The image of $[v]_n$ is $[v/x^n]$. For a finite submodule of the limit, lift the generators to stages. The transition maps can have kernels, so the relations may hold only at a later stage. The following lemma puts all the lifts at one stage.

Lemma 2.4. *Let $M_\infty = \varinjlim_i M_i$ be a filtered direct limit over a Noetherian ring, and let L be finitely generated. Every map $L \to M_\infty$ factors through some M_i. Suppose also that for finitely many elements of L we prescribe representatives of their images in stages of the system. We may take the factorization at a common later stage where it agrees with each prescribed representative.*

Proof. Pick a finite presentation of L and lift its generators to a common stage. Each defining relation is zero in the limit. Move far enough in the system that all the finitely many relations vanish. This gives a map from L. For each prescribed representative the difference from its image under this map is also zero in the limit. Another common stage kills these finitely many differences. □

Let $(R, \mathfrak{m}, k)$ be a complete normal local domain of dimension d with canonical module W. The injective hull is

$$H^d_{\mathfrak{m}}(W) \simeq E. \tag{2.4}$$

Tensor the top Čech cokernel for R with W to get $H^d_{\mathfrak{m}}(W) = W \otimes_R H^d_{\mathfrak{m}}(R)$. Local duality gives $D_R(H^d_{\mathfrak{m}}(R)) = W$, even when the dualizing complex has cohomology in other degrees [Sta26, Tags 0A81, 0DW3]. Hence $D_R(H^d_{\mathfrak{m}}(W)) = \mathrm{Hom}_R(W, W)$. Now W is rank one and reflexive, so an endomorphism is multiplication by a rational function. Since W is invertible in codimension one, this function belongs to every height-one local ring. Their intersection is R by normality. Thus $\mathrm{Hom}_R(W, W) = R$, and dualizing gives (2.4) without a Cohen–Macaulay hypothesis.

For the section-ring calculations, let Y be projective over a field or a complete DVR. Fix an ample invertible sheaf L and set $S = \bigoplus_{n\geq 0} H^0(Y, L^n)$, so $Y = \operatorname{Proj} S$. For any graded S-module M and $i \geq 1$ we have

$$[H^{i+1}_{S_+}(M)]_n = H^i(Y, \widetilde{M(n)}).$$

For a section module of a coherent sheaf $\mathcal{F}$, the right side is $H^i(Y, \mathcal{F} \otimes L^n)$. To see the shift by one, take the degree-n part of the augmented Čech complex on homogeneous elements cutting out the irrelevant locus. Its localization terms compute sections on the standard affine cover of Y [Har77, Ch. III, §5]. Over a DVR this computes local cohomology in S_+. To reach the closed vertex we must also take local cohomology in the uniformizer.

2.3. **Artin–Schreier classes.**

Lemma 2.5. *Let k be algebraically closed of characteristic p.*

(i) *If F is p-semilinear on a finite-dimensional k-vector space, then $F-1$ is onto. In particular this holds for the pth power map on any finite-dimensional k-algebra, even with nilpotents.*

(ii) *If a Henselian local ring of characteristic p has algebraically closed residue field, then $b \mapsto b^p - b$ is onto. The same holds for a finite product of such rings, and for a finite algebra over a complete Noetherian local ring of characteristic p with algebraically closed residue field.*

Proof. Pick coordinates. In (i) the map is $v \mapsto Av^{[p]} - v$, with differential $-I$ everywhere. It is étale, so its image is open. Every nonempty fiber over a k-point has a k-point. The image on k-points is therefore an open subgroup. Two nonempty open cosets would intersect, so the subgroup is the whole vector group. For (ii), solve $T^p - T - b = 0$ first in the residue field. The derivative is -1, so Hensel's lemma lifts the root. A finite algebra over the given complete local ring is a finite product of complete local rings. Their residue fields are finite extensions of the algebraically closed residue field, hence equal to it. We can lift a root on each factor. □

Let T be a scheme of characteristic p. If $F-1$ is onto on $H^0(T, \mathcal{O}_T)$, the Artin–Schreier sequence gives

$$H^1_{\text{ét}}(T, \mathbf{F}_p) \simeq \ker\bigl(F - 1 : H^1(T, \mathcal{O}_T) \longrightarrow H^1(T, \mathcal{O}_T)\bigr)$$

Thus each fixed coherent class determines a unique torsor class [Mil80, Ch. III, §4]. When we have to lift a fixed class along a restriction map, we apply Lemma 2.5(i) to the kernel of restriction.

3. Finite covers of a rational surface with an elliptic anticanonical curve

Let X be a smooth projective rational surface over an algebraically closed field k of characteristic $p > 0$. We will split finite covers by keeping a generator of $H^2(X, \omega_X)$ nonzero under pullback. Here the coefficients upstairs are the pullback of ω_X. A smooth anticanonical curve C lets us work on a curve instead: its boundary sequence identifies $H^1(C, \mathcal{O}_C)$ with $H^2(X, \omega_X)$. When C is ordinary, we can represent a generator by an étale $\mathbf{F}_p$-torsor on C. A cover killing this class will then produce a nontrivial étale cover of the rational surface X.

For this argument we need $\mathcal{O}_C(C)$ to have degree zero and infinite order. This makes every positive multiple of C fixed, and later controls ramification along C. We also need C to meet every other integral curve, so that ramification away from C can be detected on the inverse image of C.

3.1. **The nine points and their anticanonical curve.** Fix a prime p. Pick a smooth ordinary plane cubic E over a finite field k_0 of characteristic p, with a rational flex O as origin. We can make this choice for every p. For $p = 2$ take

$$Y^2Z + XYZ = X^3 + Z^3$$

over $\mathbf{F}_2$. The point at infinity is smooth, and an affine singular point would have $x = y = 0$, which is not on the curve. The Weierstrass coefficient $a_1 = 1$ gives a nonzero Hasse invariant [Sil09, Ch. V, §4]. For odd p set $m = (p-1)/2$ and put

$$H_p(T) = \sum_{j=0}^{m} \binom{m}{j}^2 T^j.$$

For $a \neq 0, 1$, the Legendre cubic $y^2 = x(x-1)(x-a)$ is ordinary if and only if $H_p(a) \neq 0$. Indeed, the coefficient of x^{p-1} in $[x(x-1)(x-a)]^m$ is its Hasse invariant $(-1)^m H_p(a)$ [Sil09, Ch. V, §4]. For $p \geq 5$, the set $\mathbf{F}_p \setminus \{0, 1\}$ has $p - 2 > m$ elements. Since $\deg H_p = m$, some a is not a root of H_p. For $p = 3$ take $a \in \mathbf{F}_9$ with $a^2 = -1$. Then $H_3(a) = 1 + a \neq 0$.

To obtain nine points with a sum of infinite order, take the generic point $P \in E(k_0(E))$ and set $k = \overline{k_0(E)}$. This field is perfect and F-finite. Multiplication by any nonzero integer b is finite and surjective on E, even if p divides b [Sil09, Ch. III, §4]. Thus $[b]P \neq O$. In particular $P, 2P, \ldots, 9P$ are distinct. Blow up these points and let C be the strict transform of E:

$$\pi : X = \mathrm{Bl}_{P,2P,\ldots,9P} \mathbf{P}^2_k \longrightarrow \mathbf{P}^2_k, \qquad C = \widetilde{E}.$$

We identify C with E_k by π. Write H for the pullback of a line and E_j for the exceptional curve over jP. The blowup formula gives

$$C \sim -K_X \sim 3H - \sum_{j=1}^{9} E_j, \qquad C^2 = 0, \qquad C \cdot E_j = 1. \tag{3.1}$$

Now restrict this divisor to C. Since the flex line cuts out $3O$ on E, the normal bundle is

$$N := \mathcal{O}_C(C) \simeq \mathcal{O}_E\Big(9[O] - \sum_{j=1}^{9} [jP]\Big).$$

Under $\mathrm{Pic}^0(E_k) \simeq E_k$ this line bundle is the point $[-45]P$. Thus N has infinite order. For the relation between this normal bundle and semiampleness on a nine-point blowup, see [Tot09, discussion after Theorem 2.1] and [LT13, Example 5.7]. The choice of successive multiples of P also controls intersections with every other curve.

Lemma 3.1. *The divisor C is nef. Every integral curve $A \neq C$ on X satisfies $C \cdot A > 0$, and $h^0(X, \mathcal{O}_X(nC)) = 1$ for every $n \geq 0$.*

Proof. The exceptional curves satisfy $C \cdot E_j = 1$ by (3.1). Every other integral curve $A \neq C$ is the strict transform of an integral plane curve $A_0 \neq E$ of degree $d > 0$. Set $m_j = \mathrm{mult}_{jP} A_0$. Smoothness of E makes the local intersection multiplicity at jP at least m_j. Bézout gives

$$C \cdot A = 3d - \sum_{j=1}^{9} m_j \geq 0.$$

If equality holds, the full intersection divisor on E is $\sum_j m_j[jP] \sim 3d[O]$. The group law gives $[\sum_j jm_j]P = O$. But $\sum_j m_j = 3d > 0$, so $\sum_j jm_j$ is a positive integer. This contradicts the infinite order of P. Hence $C \cdot A > 0$ for $A \neq C$, and $C^2 = 0$ completes the nefness assertion.

Write s_C for the defining section. For $n > 0$ restrict to C to get

$$0 \longrightarrow \mathcal{O}_X((n-1)C) \xrightarrow{s_C} \mathcal{O}_X(nC) \longrightarrow N^n \longrightarrow 0.$$

Since N^n is nontrivial of degree zero, it has no nonzero section. Thus multiplication by s_C identifies the two spaces of global sections. Induction from $H^0(X, \mathcal{O}_X) = k$ gives $h^0(X, \mathcal{O}_X(nC)) = 1$. □

For $n > 0$, the linear system $|nC|$ consists of nC alone, so C is neither big nor semiample. To rule out a Fano boundary, take any effective $\mathbb{Q}$-divisor Δ. Nefness of C gives

$$-(K_X + \Delta) \cdot C = -\Delta \cdot C \leq 0. \tag{3.2}$$

This excludes ampleness of $-(K_X + \Delta)$, since an ample divisor has positive intersection with C.

3.2. **Frobenius and the boundary class.** Write F for absolute Frobenius. We need a Frobenius splitting to handle purely inseparable covers. Ordinarity of C gives this splitting, as in Krah–Vial's cubic blowup criterion [KV25, Remark 11.6]. We include the calculation to fix the boundary map used for separable covers. Ordinarity will also let us choose the torsor on C.

Proposition 3.2. *The surface X is Frobenius split and is not globally F-regular.*

Proof. Take $\omega_X = \mathcal{O}_X(-C)$. Since X is a blowup of the plane at smooth points, $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. The boundary sequence therefore gives an isomorphism

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0, \qquad \delta : H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^2(X, \omega_X). \tag{3.3}$$

Consider pth powers $\mathcal{O}_X(-C) \to F_*\mathcal{O}_X(-pC)$ followed by the inclusion into $F_*\mathcal{O}_X(-C)$. We compute the induced map on H^2 using (3.3). On the quotients, these maps take pth powers $\mathcal{O}_C \to F_*\mathcal{O}_{pC}$ and then restrict to $F_*\mathcal{O}_C$. The kernel of $\mathcal{O}_{pC} \to \mathcal{O}_C$ has successive quotients N^{-j} for $1 \leq j \leq p-1$. These nontrivial degree-zero bundles have vanishing H^0 and H^1 by Serre duality on C. The quotient composite is absolute Frobenius on $\mathcal{O}_C$. Naturality of δ now identifies our map on $H^2(X, \omega_X)$ with Frobenius on $H^1(C, \mathcal{O}_C)$. It is nonzero because C is ordinary.

The inclusion $\mathcal{O}_X(-pC) \to \mathcal{O}_X(-C)$ is multiplication by $s_C^{p-1} : \omega_X^p \to \omega_X$. Finite duality for F associates to this section of ω_X^{1-p} an $\mathcal{O}_X$-linear map $F_*\mathcal{O}_X \to \mathcal{O}_X$. By the trace criterion its value at 1 is nonzero exactly when our map on H^2 is nonzero [BK05, §1.3]. Its value lies in $H^0(X, \mathcal{O}_X) = k$, so scaling gives a splitting.

To exclude global F-regularity, pick an effective ample divisor A. The same duality identifies maps $F_*^e\mathcal{O}_X(A) \to \mathcal{O}_X$ with sections of $\mathcal{O}_X((p^e - 1)C - A)$, with the Frobenius action on scalars on the section side. This divisor has intersection $-A \cdot C < 0$ with the nef curve C, so it has no nonzero sections. Hence no e gives the retraction required for this A. □

3.3. **Finite étale covers of the complement.** Take a finite cover étale off C. Before using Abhyankar's lemma, we must rule out wild ramification along C. We do this with a two-form having a simple pole on C. Adjunction determines its pole order upstairs, and its residue proves tameness. We can then read the ramification index on an exceptional curve.

Lemma 3.3. *Every connected finite étale cover of $U = X \setminus C$ has degree one.*

Proof. Fix a connected finite étale cover $U' \to U$. Since U is regular and connected, U' is integral. Normalize X in $k(U')$ to obtain a finite cover $Z \to X$. Pick a projective resolution $V \to Z$ which is an isomorphism over U' [Lip69]. Write $g : V \to X$ for the composite and set $D = g^*C$. The projection formula gives nefness of D and $D^2 = 0$. It also gives $D \cdot \Gamma = 0$ for each prime component Γ of D, since $g_*\Gamma$ is zero or a multiple of C. Now pick a rational two-form θ with divisor $-C$,

using (3.1). Its pullback is nonzero because $k(U')/k(X)$ is separable. Set $K_V = \operatorname{div}(g^*\theta)$. The map is étale over U', so K_V is supported on D and $K_V \cdot D = 0$.

We first check that each connected component of $\operatorname{Supp} D$ contains a curve dominating C. Every component of the finite inverse image of C on Z dominates C. An exceptional curve above a point of this inverse image lies in a connected fiber, since Z is normal. By properness the fiber meets the strict transform of a curve through that point. This proves the claim. Moreover, every curve dominating C has positive geometric genus. Otherwise its normalization would give a nonconstant map $\mathbf{P}^1 \to C$, contrary to Lüroth's theorem, also for an inseparable map.

We can now contract all smooth rational (-1)-curves in $\operatorname{Supp} D$. None dominates C, so each contraction is over X. Each Castelnuovo contraction leaves a smooth projective surface and decreases its Picard number, so the process stops. Keep the names V, g, D, K_V. The projection formula still gives $D \cdot \Gamma = 0$ on the boundary and $K_V \cdot D = 0$. The contractions have connected fibers and preserve curves dominating C. Thus each connected component of the new boundary still contains such a curve.

We next show $K_V \cdot \Gamma \geq 0$ for each prime component Γ of D. Hodge index gives $\Gamma^2 \leq 0$: the nonzero nef class D has square zero and is orthogonal to Γ. If $\Gamma^2 = 0$, the equality $D \cdot \Gamma = 0$ also forces Γ to miss every other component. Adjunction gives

$$K_V \cdot \Gamma = 2p_a(\Gamma) - 2 - \Gamma^2.$$

This is nonnegative when $p_a(\Gamma) \geq 1$. If $p_a(\Gamma) = 0$, the curve is smooth rational and cannot have square -1, by our contractions. It cannot have square zero either: it would be a whole connected component with no curve dominating C. Thus $\Gamma^2 \leq -2$, and the inequality follows in this case as well.

Since $K_V \cdot D = 0$, summing with the positive coefficients of D now gives $K_V \cdot \Gamma = 0$ for every Γ. In each connected component pick a prime B dominating C. Its positive geometric genus and adjunction force $p_a(B) = 1$ and $B^2 = 0$. The arithmetic and geometric genera agree, so B is smooth. Its square zero makes it miss all other components. We obtain

$$D = \sum_i e_i B_i, \qquad K_V = \sum_i b_i B_i,$$

where the B_i are disjoint smooth elliptic curves mapping finitely onto C by $h_i = g|_{B_i}$. Any curve contracted by g would lie in this boundary, which has no contracted curves. Hence g is proper and quasi-finite, so finite. Thus V is the normalization Z.

To find the pole order along B_i, set $N_i = \mathcal{O}_{B_i}(B_i)$. Restricting D gives $N_i^{e_i} \simeq h_i^* N$. The norm of $h_i^* N$ is $N^{\deg h_i}$, even when h_i is inseparable, so N_i has infinite order. Adjunction gives

$$\mathcal{O}_{B_i} \simeq \omega_{B_i} \simeq \mathcal{O}_{B_i}(K_V + B_i) \simeq N_i^{b_i+1}.$$

The infinite order of N_i forces $b_i = -1$. Thus $g^*\theta$ has a simple pole along every B_i.

Now compute its residue at the generic point of B_i. Pick a local equation x of C. Lift a separating parameter $\bar{y}$ of $k(C)/k$ to an element y of the DVR downstairs. Then $\theta = u\,(dx/x) \wedge dy$ for a unit u. Upstairs write $x = az^{e_i}$, with a a unit and z a uniformizer. We get

$$g^*\theta = g^*u \left(e_i \frac{dz}{z} + \frac{da}{a} \right) \wedge d(g^*y), \qquad \operatorname{res}_{B_i}(g^*\theta) = \overline{g^*u}\, e_i\, d(h_i^*\bar{y}).$$

For a smooth divisor on a smooth surface, residue identifies two-forms with at most a simple pole, modulo regular two-forms, with one-forms on the divisor. Since our pole has order exactly one, the displayed residue is nonzero. Thus $p \nmid e_i$ and $d(h_i^*\bar{y}) \neq 0$ in $\Omega^1_{k(B_i)/k}$. The second condition makes $k(B_i)$ separable over $k(\bar{y})$, hence over $k(C)$. Together these conditions prove tameness at the DVR.

We use an exceptional curve $E_j \simeq \mathbf{P}^1$ to determine the indices. It meets C transversely at a single point Q. Pull the cover back to $E_j \setminus \{Q\}$ and normalize E_j in each connected component. Tameness at the generic point of the regular divisor C lets us apply Abhyankar's lemma [Sta26, Tag 0EYG]. Strictly locally at Q, the cover has sheets $T^e = x$, with e prime to p and x a local equation for C. The restriction of x to E_j is a uniformizer, so each sheet keeps its ramification index e on E_j. The resulting curve covers are tame and branch only at Q. For a connected component of degree n with r points over Q, Riemann–Hurwitz over the algebraically closed field gives

$$2g - 2 = -2n + (n - r) = -n - r.$$

Since $g \geq 0$ and $n, r \geq 1$, this forces $n = r = 1$. Each B_i maps onto C, so its index occurs on a sheet over Q. Thus every $e_i = 1$. The residue extensions are already known to be separable. Purity of the branch locus now makes the normalization étale over all of X [Sta26, Tag 0BMB].

It remains to exclude a nontrivial connected finite étale cover of X. Away from the exceptional curves, X is the plane minus finitely many points. The cover stays integral on this open, and purity extends it to a finite étale cover of $\mathbf{P}^2$ by normalization. Pull back a line. Its inverse image is ample and is a disjoint union of curves étale over $\mathbf{P}^1$, each a copy of $\mathbf{P}^1$. Each component has positive square: it meets the ample pullback positively and misses all the other components. Hodge index allows only one such component, so the cover has degree one. □

3.4. **Connectedness of the boundary on an arbitrary finite cover.** Now allow branching away from C. We want the inverse image of C to remain connected. On a Galois cover, inertia away from C will fix each connected component. The action on components then gives an étale cover of $X \setminus C$, which Lemma 3.3 rules out.

Lemma 3.4. *Let $f : W \to X$ be finite and surjective, with W a normal integral surface. Then the support of $D = f^*C$ is connected.*

Proof. First assume $k(W)/k(X)$ is Galois, with group G. Write $D = D_1 + \cdots + D_s$, where D_i is the part on one connected component of $\operatorname{Supp} D$, retaining all multiplicities. Each D_i is Cartier even if W is singular. Indeed, use the ideal of D off the other parts and the unit ideal off $\operatorname{Supp} D_i$. These ideals agree on their overlap.

To compare the D_i numerically, pick a projective resolution $\rho : V \to W$ and set $\widetilde{D}_i = \rho^* D_i$, $\widetilde{D} = \rho^* D$. The $\widetilde{D}_i$ have disjoint supports. Projection to X shows that $\widetilde{D}$ is nef of square zero and has degree zero on every component of its support. Thus

$$\widetilde{D} \cdot \widetilde{D}_i = \widetilde{D}_i^2 = 0.$$

Hodge index now gives $\widetilde{D}_i \equiv \lambda_i \widetilde{D}$. Indeed, on the orthogonal complement of a nonzero isotropic vector, the intersection form is negative semidefinite with radical its span. Intersecting with an ample class gives $\lambda_i > 0$.

Take a prime curve B on W over a curve $A \neq C$, and write $\widetilde{B}$ for its strict transform. Lemma 3.1 gives

$$D_i \cdot B = \widetilde{D}_i \cdot \widetilde{B} = \lambda_i [k(B) : k(A)](C \cdot A) > 0.$$

Thus B meets every connected component of D; no separability of $k(B)/k(A)$ is needed. The inertia group I_B stabilizes B and acts trivially on $k(B)$, so it fixes the reduced curve B pointwise. It fixes a point of each $B \cap \operatorname{Supp} D_i$, and therefore preserves each connected component.

Let J be the kernel of the action on components. We have $I_B \subset J$ for every prime divisor over U. Inertia in $k(W)^J/k(X)$ is the image of inertia in G [Sta26, Tag 0BU7]. Thus the normalization is étale in codimension one over U, and purity makes it étale over U. It is integral, so Lemma 3.3 gives $J = G$. On the other hand, G acts transitively on primes over the generic point of C [Sta26,

Tag 09E3]. Every connected component contains such a prime, since f is finite and D is Cartier. The action on components is therefore both trivial and transitive, which means $s = 1$.

For a separable extension, take a Galois closure. Its connected boundary maps onto $\operatorname{Supp} D$, proving connectedness there. For the general case, let L_s be the maximal separable subextension of $k(W)/k(X)$ and let W_s be its normalization over X. Pick $q = p^e$ with $k(W)^q \subset L_s$. On an affine open of X, the qth power of an integral element is still integral and belongs to L_s, hence to its normalization. Thus $W \to W_s$ is radicial. It is also finite and surjective, so it is a universal homeomorphism. The connected boundary on W_s therefore has connected inverse image on W. □

3.5. **Persistence of the cohomology class on separable covers.** We need a Frobenius-fixed class on C to obtain an étale torsor. Pick $0 \neq v \in H^1(C, \mathcal{O}_C)$. This space is one-dimensional, and ordinarity gives $Fv = cv$ with $c \neq 0$. Pick $t \in k^\times$ with $t^{p-1} = c^{-1}$; then $F(tv) = tv$. Fix such a nonzero class α and use (3.3) to set

$$\eta = \delta(\alpha) \in H^2(X, \omega_X)$$

The Artin–Schreier sequence $0 \to \mathbf{F}_p \to \mathcal{O} \xrightarrow{F-1} \mathcal{O} \to 0$ in the étale topology gives an étale $\mathbf{F}_p$-torsor $C' \to C$ representing α [Mil80, Ch. III, §4]. Its class is unique because $F - 1$ is onto on $H^0(C, \mathcal{O}_C) = k$, and nontrivial because $\alpha \neq 0$. Since $\mathbf{F}_p$ has prime order, a nontrivial torsor has full monodromy. Thus C' is connected.

Lemma 3.5. *If F is a p-semilinear endomorphism of a finite-dimensional k-vector space M, then $F - 1 : M \to M$ is onto. For any proper k-scheme T, the map $H^1_{\text{ét}}(T, \mathbf{F}_p) \to H^1(T, \mathcal{O}_T)$ identifies its source with the Frobenius-fixed classes. This holds also for nonreduced T.*

Proof. Apply Lemma 2.5(i) for the first assertion. For proper T, it also applies to the finite-dimensional algebra $H^0(T, \mathcal{O}_T)$, including its nilpotents. The Artin–Schreier sequence then gives the second assertion, using the comparison of étale and coherent cohomology for $\mathcal{O}_T$. □

Proposition 3.6. *Let $f : Y \to X$ be a finite generically separable cover with Y normal and integral. The pullback of η to $H^2(Y, \mathcal{O}_Y(-f^*C))$ is nonzero.*

Proof. Suppose $f^*\eta = 0$ and set $D = f^*C$, with its scheme structure. Since Y is integral and dominates X, the divisor D is effective Cartier. Its boundary sequence is

$$0 \longrightarrow \mathcal{O}_Y(-D) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

Naturality sends the pullback α_D of α to $f^*\eta = 0$. Thus α_D lifts to some $\beta \in H^1(Y, \mathcal{O}_Y)$. To obtain a torsor on Y, we must make this lift Frobenius fixed. Set

$$M = \ker\bigl(H^1(Y, \mathcal{O}_Y) \to H^1(D, \mathcal{O}_D)\bigr)$$

The kernel is Frobenius stable and finite-dimensional by properness. Since α_D is fixed, $(F-1)\beta \in M$. Lemma 3.5 gives $v \in M$ with $(F - 1)v = (F - 1)\beta$, so $\beta - v$ is a fixed lift.

Take the corresponding étale $\mathbf{F}_p$-torsor on Y. Its restriction to D and the pullback of C' have the same coherent class α_D. They are isomorphic by Lemma 3.5, applied to the possibly nonreduced divisor D. In particular we use surjectivity of $F - 1$ on its global functions. Pick a connected component of the torsor. It is normal and surjects onto Y; the tautological section trivializes the torsor on this component. Normalize X in a Galois closure of its function field over $k(X)$. These extensions are all separable. Write $f_W : W \to X$ for the resulting Galois cover, set $D_W = f_W^*C$, and let $G = \operatorname{Gal}(k(W)/k(X))$. The section on D_W gives

$$\phi : D_W \longrightarrow C' \quad \text{over } C$$

For $g \in G$, the difference between $\phi \circ g$ and ϕ is a map from D_W to the constant group $\mathbf{F}_p$. Connectedness from Lemma 3.4 makes this difference a constant $\lambda(g)$. Composition in G shows that $\lambda : G \to \mathbf{F}_p$ is a homomorphism. To prove it is nonzero, we examine the reduced generic fiber over C.

Let ξ be the generic point of C and set $A = \mathcal{O}_{X,\xi}$, $\kappa = k(C)$. The integral closure B of A in $k(W)$ is a finite semilocal normal domain with $B^G = A$. Its reduced residue algebra is

$$R = (B \otimes_A \kappa)_{\mathrm{red}} = \prod_{\mathfrak{q}|\mathfrak{m}_A} \kappa(\mathfrak{q}).$$

We claim that every element of R^G is purely inseparable over κ. Pick $b \in R^G$ and a lift $\widetilde{b} \in B$. The orbit polynomial $\prod_{g\in G}(Z - g\widetilde{b})$ has coefficients in A and reduces to $(Z-b)^{|G|}$ in $R[Z]$, since b is fixed. Write $|G| = p^a m$ with $p \nmid m$. Then

$$(Z-b)^{|G|} = (Z^{p^a} - b^{p^a})^m.$$

The coefficient of $Z^{|G|-p^a}$ is $-mb^{p^a}$, so b^{p^a} lies in the diagonal copy of κ. This proves the claim. If $\lambda = 0$, then ϕ is G-invariant. On the reduced generic fiber it gives a unital κ-algebra map $k(C') \to R^G$. This embeds the nontrivial separable extension $k(C')/\kappa$ in R^G, contradicting the claim. Hence $\lambda \neq 0$.

We finish by showing that every divisorial inertia group lies in $\ker\lambda$. Take a prime curve B_0 on W. Its inertia fixes the reduced curve B_0 pointwise. If B_0 lies over C, pick a closed point on $B_0 \subset \operatorname{Supp} D_W$. Otherwise set $A_0 = f_W(B_0)$. We have

$$D_W \cdot B_0 = [k(B_0) : k(A_0)]\, C \cdot A_0 > 0$$

by Lemma 3.1, so $B_0 \cap \operatorname{Supp} D_W$ has a closed point. In either case we have a k-point of D_W fixed by inertia. For an inertia element g, evaluate ϕ and $\phi \circ g$ at this point. They agree, and the free action on the fiber of C' forces $\lambda(g) = 0$. The degree-p extension defined by $\ker\lambda$ therefore has trivial inertia at every divisor. Purity, as in Lemma 3.4, gives a connected finite étale cover of X of degree p. Lemma 3.3 proved that X has no such cover. This contradiction proves $f^*\eta \neq 0$. □

3.6. **Retractions and the purely inseparable part.**

Proposition 3.7. *For every finite surjective morphism $f : Y \to X$, the unit map has an $\mathcal{O}_X$-linear retraction $f_*\mathcal{O}_Y \to \mathcal{O}_X$.*

Proof. First assume Y is normal integral and f is generically separable. Set $M = f_*\mathcal{O}_Y$. For the duality calculation, we first check that M is locally free. Fix a point of X and choose parameters in its regular local ring. Finiteness and equality of local dimensions make them parameters at each point upstairs. Those local rings are normal of dimension at most two, hence Cohen–Macaulay. The parameters therefore form a regular sequence upstairs and on the localized finite module M. Auslander–Buchsbaum gives projective dimension zero, as required.

Now we seek a functional on M taking 1 to 1. Finiteness and the projection formula identify the pullback in Proposition 3.6 with

$$H^2(X, \omega_X) \longrightarrow H^2(X, M \otimes \omega_X)$$

induced by the unit $\mathcal{O}_X \to M$. Serre duality on X gives

$$H^0(X, \mathcal{H}om_X(M, \mathcal{O}_X)) \longrightarrow H^0(X, \mathcal{O}_X) = k, \qquad \psi \longmapsto \psi(1). \tag{3.4}$$

Finite duality gives the same evaluation: it identifies $f_*\omega_Y$ with $\mathcal{H}om_X(M, \omega_X)$ and the counit with evaluation at 1 [Har66]. Proposition 3.6 says that the map on H^2 is nonzero on its one-dimensional source. Thus (3.4) is onto. Pick ψ with $\psi(1) = 1$ to split the cover.

Keep Y normal integral, but allow $L = k(Y)$ to be inseparable over $k(X)$. Let $X_s \to X$ be the normalization in the maximal separable subfield L_s of $L/k(X)$, and write M_s for its algebra on X. Pick $q = p^e$ with $L^q \subset L_s$. The integrality calculation at the end of Lemma 3.4 shows that qth powers lie in this normalization on each affine open of X. These power maps give

$$f_*\mathcal{O}_Y \longrightarrow F^e_* M_s, \qquad b \longmapsto F^e_*(b^q),$$

which is $\mathcal{O}_X$-linear because scalars act by qth powers on the target. Follow it by $F^e_* r$, where $r : M_s \to \mathcal{O}_X$ is the separable retraction, and then by the iterated Frobenius splitting $\sigma_e : F^e_*\mathcal{O}_X \to \mathcal{O}_X$ from Proposition 3.2. This composite takes 1 to 1, so it splits the unit.

Finally, for any finite surjective $Y \to X$, pick an irreducible component dominating X and normalize its reduced structure. This gives a finite normal integral cover $f' : Y' \to X$ factoring through Y. Compose $f_*\mathcal{O}_Y \to f'_*\mathcal{O}_{Y'}$ with the retraction for Y'. □

Proof of Theorem A. The construction works for every p and gives a smooth projective rational surface X. Every finite cover splits by Proposition 3.7, which is global +-regularity with zero boundary [BMP+23, Definition 6.1]. Lemma 3.1 makes $-K_X \sim C$ nef with every positive multiple fixed, hence neither big nor semiample. Equation (3.2) excludes every Fano boundary. □

4. The completed cone over the rational surface

We now use the surface X to construct a local splinter. Write H for the pullback of a line and C for the strict transform of the ordinary cubic. We have $C = -K_X$ and $C^2 = 0$, but C meets every other integral curve. Moreover, $h^0(X, mC) = 1$ for $m \geq 0$ by Lemma 3.1. To form the cone, set $L = H + C$ and take

$$S = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nL)), \qquad \mathfrak{m}_S = S_+, \qquad R = \widehat{S_{\mathfrak{m}_S}}, \qquad \mathfrak{m} = \mathfrak{m}_S R.$$

Thus R is the completion at the vertex of the full section ring. We use additive notation for line bundles and omit $\mathcal{O}_X$ in cohomology. The cone over C defines a canonical ideal J. Its boundary map sends the elliptic class on C to a socle generator of the injective hull of k.

We will follow this generator in two different maps. Under tensor Frobenius, one nonzero element kills all its images, already in a fixed finite module. This proves failure of weak F-regularity. Under a finite extension, the generator must stay nonzero to prove the splinter assertion of Theorem B. For this we use connectedness of the inverse image of C on every finite normal integral cover of X (Lemma 3.4).

4.1. **The section ring and its canonical ideal.** First check that L is ample and globally generated. Both H and C are nef, and $L^2 = 7$, $L \cdot C = 3$. Every other integral curve meets C, so Nakai–Moishezon gives ampleness. Off C, the sections $s_C H^0(X, H)$ generate L, where s_C cuts out C. On C, the restriction $H^0(X, L) \to H^0(C, L|_C)$ is onto: the blowup map gives $H^1(X, H) = 0$. Now $L|_C$ has degree three, so these sections embed C as a plane cubic.

Lemma 4.1. *The ring S is a normal Cohen–Macaulay domain of dimension three. Six elements of degree one generate S. Their relations are generated in degrees at most three. The Hilbert series is*

$$\mathrm{Hilb}_S(z) = \frac{1 + 3z + 3z^2}{(1 - z)^3}.$$

It is Frobenius split and is regular off the vertex.

Proof. We first compute the cohomology of all powers of L. Tensor the splitting of Proposition 3.2 with nL and iterate to get $H^1(X, nL) \hookrightarrow H^1(X, p^e nL)$ for every n and e. For fixed $n > 0$, Serre vanishing kills the target for $e \gg 0$. For $n < 0$, apply Serre vanishing to its dual $H^1(X, K_X - p^e nL)$. Rationality handles $n = 0$, so $H^1(X, nL) = 0$ for every n. Also $H^2(X, nL)$ is dual to $H^0(X, -C - nL) = 0$ for $n \geq 0$. We can therefore use Riemann–Roch:

$$h^0(X, nL) = 1 + \tfrac{1}{2}(nL) \cdot (nL + C) = 1 + \frac{7n^2 + 3n}{2} \qquad (n \geq 0). \tag{4.1}$$

Summing gives the stated Hilbert series. Since S is the full section ring of the normal surface X, it is normal and $H^0_{\mathfrak{m}_S}(S) = H^1_{\mathfrak{m}_S}(S) = 0$. The remaining group below the top has degree-n piece $[H^2_{\mathfrak{m}_S}(S)]_n = H^1(X, nL) = 0$. Thus S is Cohen–Macaulay at the vertex [BH98, §§3.6, 4.4], [Wat81]. We will identify its punctured spectrum below.

To control all Frobenius powers later, we need bounds for the degrees of both generators and relations. We first generate S in degree one. Let $Z \subset \mathbb{P}^2$ be the nine blowup centers. Since $h^0(X, C) = 1$, evaluation of cubics at Z has rank nine. Hence $H^1(\mathbb{P}^2, \mathcal{I}_Z(3)) = 0$. Together with $H^2(\mathbb{P}^2, \mathcal{I}_Z(2)) = 0$, this says that $\mathcal{I}_Z$ is 4-regular. The Castelnuovo–Mumford multiplication lemma gives

$$H^0(\mathbb{P}^2, \mathcal{O}(1))\, H^0(\mathbb{P}^2, \mathcal{I}_Z(4)) = H^0(\mathbb{P}^2, \mathcal{I}_Z(5)).$$

Take a section of $2L$. On C it is a sum of products of sections of $L|_C$, since the plane cubic embedding is normally generated. Lift the factors to $H^0(X, L)$ and subtract their products. The difference belongs to $s_C H^0(X, 2L - C)$, with $2L - C = 5H - \sum E_i$. The displayed multiplication for quintics writes this difference as products too: one factor lies in $s_C H^0(X, H)$, the other in $H^0(X, L)$. For $n \geq 2$, use the vanishings $H^1(X, (n-1)L) = H^2(X, (n-2)L) = 0$. The multiplication lemma on X then gives $H^0(nL)H^0(L) = H^0((n+1)L)$ [Laz04, §1.8]. Thus S_1 generates S and $\dim_k S_1 = 6$ by (4.1).

For the relations, write $S = P/I_S$ with P polynomial in six degree-one variables. Pick three general linear parameters for S. They are regular on P and S; the Koszul complex gives $\mathrm{Tor}_1 = 0$ when we reduce $0 \to I_S \to P \to S \to 0$ modulo the parameters. The Artinian reduction has Hilbert function $(1, 3, 3)$, so its defining ideal contains all cubics. Lift its ideal generators and apply graded Nakayama. Thus I_S is generated in degrees at most three.

The degree-one charts now identify the punctured spectrum with the $\mathbf{G}_m$-bundle of L^{-1} on X, so it is regular. To split Frobenius on S, use the splitting of X and the projection formula in degrees divisible by p. Set the map to zero in the other degrees [BK05, §1.2]. □

Thus we can use six plane quartics through Z as generators. Since k is perfect, S and R are F-finite. Flat completion preserves dimension, embedding dimension, Cohen–Macaulayness and Frobenius splitting; excellence gives normality after completion. It also makes the completion map regular. A nonclosed prime of R contracts away from the vertex since $\mathfrak{m}_S R = \mathfrak{m}$, so the puncture of R is regular.

To represent the canonical module by an ideal, use $K_X = -C$ and multiply its sections by s_C. The section-ring formula [Wat81], [BH98, §3.6] gives

$$J_S = \bigoplus_{n \geq 0} s_C H^0(X, nL - C) \subseteq S, \qquad J = J_S R.$$

Multiplication by s_C embeds $H^0(X, K_X + nL)$ into S_n. For $n < 0$ this canonical component is zero. For $n \geq 0$, Lemma 4.1 gives $H^1(X, nL - C) = H^1(X, -nL)^\vee = 0$. Restriction to C therefore identifies S/J_S with the full section ring of $L|_C$. This is the normal plane cubic cone. Its vertex local ring is excellent, so R/J is normal too. In particular J is a height-one prime.

4.2. **A tight-closure witness in a finite module.** Put $E_S = H^3_{\mathfrak{m}_S}(J_S)$. By graded local duality this is $S^\vee$, where $[S^\vee]_n = \operatorname{Hom}_k(S_{-n}, k)$. Its degree-zero part is the one-dimensional socle; pick a generator η. After localization and completion, E_S identifies with the injective hull E_R of k over R. We keep the notation η for the resulting socle generator.

Proposition 4.2. *Let $N = (E_S)_{\geq -5}$. It is a submodule of length 221. For every $0 \neq h \in S_1$ and every $q = p^e$,*

$$h\eta^{[q]} = 0 \quad \text{in } F^e_S(N).$$

In particular $\eta \in 0^{*\mathrm{fg}}_{E_R}$.

Proof. We first kill $h\eta^{[q]}$ in the whole injective hull, where we can compute by cohomology on X. Give $F^e_S(E_S)$ the grading $\deg(u \otimes v) = \deg u + q \deg v$. Lemma 2.3 identifies it with the top local cohomology of $F^e_S(J_S)$. We may replace this coefficient module by $J_S^{(q)}$. Indeed, multiplication gives $F^e_S(J_S) \twoheadrightarrow J_S^{[q]}$ followed by $J_S^{[q]} \subseteq J_S^{(q)}$. Both maps are isomorphisms at height one because J_S is locally principal. Their kernels and cokernels have dimension at most one, so their second and third local cohomology vanish. In degree one,

$$[H^3_{\mathfrak{m}_S}(J_S^{(q)})]_1 = H^2(X, L - qC) = H^0(X, (q-1)C - L)^\vee = 0.$$

The last vanishing follows from $((q-1)C - L) \cdot C = -3$ and nefness of C. Now $h\eta^{[q]}$ has degree one, so it is zero in $F^e_S(E_S)$.

To find one truncation for all q, we bound the degrees in a right module presentation of Frobenius. Give $P = k[T_1, \dots, T_6]$ the right action $v \cdot r = vr^q$. Since k is perfect, the monomials $T^{\boldsymbol{\nu}}$ with $0 \leq \nu_i < q$ form a right P-basis. Their ordinary degrees are at most $6(q-1)$. Tensor on the right with S to get $P/I_S^{[q]}$, free on the same monomials.

We must quotient further by $I_S/I_S^{[q]}$ to present S itself. Choose generators f of I_S in degrees at most three, as in Lemma 4.1. Expand the coefficient of each polynomial multiple of f in the monomial basis over P^q. It follows that the $T^{\boldsymbol{\nu}} f$ generate the extra right S-module relations. Thus the generator degrees are at most $6(q-1)$ and the relation degrees at most $6(q-1) + 3 = 6q - 3$.

Now tensor the presentation with E_S and take degree one. A shift b uses $(E_S)_{(1-b)/q}$, with zero contribution when $(1-b)/q$ is not an integer. Our bounds give $(1-b)/q > -6$ for every generator and relation. Thus both sides of the presentation use only pieces in $N = (E_S)_{\geq -5}$, and their map is unchanged when we replace E_S by N. Hence

$$[F^e_S(N)]_1 \xrightarrow{\sim} [F^e_S(E_S)]_1$$

for every e. The relation already proved in E_S therefore holds in N, with the same h for all q. Formula (4.1) gives

$$\ell(N) = \sum_{j=0}^{5} \dim_k S_j = 1 + 6 + 18 + 37 + 63 + 96 = 221.$$

Since N and $F^e_S(N)$ have finite length at the vertex, we may localize and complete. Tensor Frobenius commutes with this base change, giving the required relation in $F^e_R(N)$. □

The first cohomology calculation alone puts η in big tight closure. The presentation is needed for finitistic tight closure: it lets us choose N before e varies. See Lyubeznik–Smith [LS99, LS01] for the general graded comparisons.

Corollary 4.3. *The ring R is not weakly F-regular, and $\tau_b(R) = \tau_{\mathrm{fg}}(R) = \mathfrak{m}$.*

Proof. Proposition 4.2 puts the nonzero socle element η in finitistic tight closure. Since its annihilator is $\mathfrak{m}$, we get $\tau_{\mathrm{fg}}(R) \subseteq \mathfrak{m}$ and failure of weak F-regularity. The big test ideal is nonzero and uniformly Frobenius-compatible [ST12, §§3,5]. Frobenius splitting makes it radical: if it contains r^{p^e}, a splitting applied to $F_*^e r^{p^e}$ puts r in the ideal. On the regular puncture the big test ideal is the unit ideal. Consequently, $\tau_b(R) \subseteq \tau_{\mathrm{fg}}(R) \subseteq \mathfrak{m}$ gives both equalities. □

Proposition 4.4. *There are linear forms $x_1, y_2, y_3, w \in S_1$ with $x_1 \in J_S$ having the following properties. Set*

$$I = (x_1^5 J, y_2^6, y_3^6)R, \qquad u = (x_1 y_2 y_3)^5 w^2$$

and $K = I + (u)$. Then $u \in I^* \setminus I$. The ideals $I \subsetneq K$ are $\mathfrak{m}$-primary and have colengths 1368 *and* 1367*, respectively. Also $e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(K) = 0 = s(R)$.*

Proof. We choose the parameters so that the socle is visible on the cubic cone. Pick two plane lines meeting off the cubic. Their forms give parameters y_2, y_3 on S/J_S; complete them to a basis of $(S/J_S)_1$ with w. Modulo these parameters the cubic equation is a nonzero multiple of w^3, so the quotient is $k[w]/(w^3)$. General lifts of y_2, y_3 extend to parameters in S as well.

We need the third parameter in $(J_S)_1 = s_C H^0(H)$. Since H is globally generated, these sections have common zero locus C on X, and their ideal in S has radical J_S. No minimal prime of (y_2, y_3) contains J_S: the ideal $J_S + (y_2, y_3)$ cuts out the vertex. The field k is infinite, so we can choose $x_1 \in (J_S)_1$ outside these minimal primes. This makes x_1, y_2, y_3 parameters.

For $n \geq 1$ set

$$I_n = (x_1^{n-1} J, y_2^n, y_3^n)R, \qquad u_n = (x_1 y_2 y_3)^{n-1} w^2.$$

We will show that u_n generates the socle of R/I_n. The ideal $x_1^{n-1} J \simeq J$ is canonical, and its quotient is two-dimensional Gorenstein. Indeed, the canonical ideal sequence gives depth two. Applying $\mathrm{Hom}_R(-, J)$ and using $\mathrm{End}_R(J) = R$ identifies the canonical module of the quotient with the quotient itself. Now y_2, y_3 cut out the vertex on both $V(J)$ and $V(x_1)$. Thus they are parameters modulo $x_1^{n-1} J$, and R/I_n is Artinian Gorenstein. To follow its socle as n increases, we use

$$(I_{n+1} : x_1 y_2 y_3) = I_n.$$

Modulo $x_1^n J$, the elements y_2, y_3 form a regular sequence. Taking the colon by $y_2 y_3$ therefore drops their exponents from $n+1$ to n. Modulo (y_2^n, y_3^n), the element x_1 is regular, so the colon by x_1 changes $x_1^n J$ to $x_1^{n-1} J$. This proves the equality and gives an injection $R/I_n \to R/I_{n+1}$ by multiplication by $x_1 y_2 y_3$. The socle kw^2 of R/I_1 consequently maps to the nonzero socle ku_n of R/I_n.

To include the module N of Proposition 4.2, take $n = 6$. The Artinian Gorenstein ring R/I_6 is its own Matlis dual, so it embeds in E_R with image $\mathrm{Ann}_{E_R}(I_6)$. Since $I_6 \subseteq \mathfrak{m}^6$, this image contains $\mathrm{Ann}_{E_R}(\mathfrak{m}^6) = N$; the last equality is the dual of $S/\mathfrak{m}_S^6$. The socle ku_6 maps to $k\eta$. Tensor $N \hookrightarrow R/I_6$ with Frobenius. The proposition gives $hu_6^q = 0$ in $F_R^e(R/I_6) = R/I_6^{[q]}$ for every q, so $u_6 \in I_6^* \setminus I_6$.

We can compute the colength before completion. Subtracting the Hilbert series of the cubic hypersurface S/J_S from that of S gives

$$\mathrm{Hilb}_{J_S}(z) = \frac{3z + 3z^2 + z^3}{(1-z)^3}.$$

Multiplication by x_1^{n-1} shifts J_S by $n-1$ degrees. The powers y_2^n, y_3^n are regular on its quotient. Thus the graded quotient defining R/I_n has series

$$(1 - z^n)^2 \left(\mathrm{Hilb}_S(z) - z^{n-1} \mathrm{Hilb}_{J_S}(z) \right).$$

The two numerators have equal value at 1. The derivative of their difference there is $9-12-7(n-1) = 4-7n$. Hence the displayed polynomial has degree $3n-1$ and value $n^2(7n-4)$ at 1. For $n=6$, the socle degree is 17 and the length is 1368. Quotienting further by u kills only the one-dimensional socle, so R/K has length 1367.

For the Hilbert–Kunz difference, fix $q=p^e$ and put $B_q = (I^{[q]} : u^q)$. The same nonzero h lies in every B_q by Proposition 4.2. To bound the colength, we need one constant for all q. Choose d with $\mathfrak{m}^d \subset I$ and set $c = d+6$. Every monomial of degree $(d+6)q$ in the six generators is divisible by the qth power of a degree-d monomial. Thus $\mathfrak{m}^{cq} \subset I^{[q]}$, and

$$\ell_R(K^{[q]}/I^{[q]}) = \ell_R(R/B_q) \le \ell_R(R/(h,\mathfrak{m}^{cq})) = O(q^2).$$

Here the $O(q^2)$ bound comes from Hilbert–Samuel growth on the two-dimensional ring $R/(h)$. Dividing by q^3 proves $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(K)$. Since R fails weak F-regularity, it also fails strong F-regularity. The positivity criterion then gives $s(R)=0$ [AL03, BST12]. □

4.3. **Incidence and connectedness on finite extensions.** A finite extension of R need not be graded, so we cannot apply Lemma 3.4 directly to a cover of X. Instead, blow up $\mathfrak{m}$ and write $g : V \to \operatorname{Spec} R$. We will use X in the exceptional fiber.

Before localization and completion, the blowup is the total space of L^{-1}. On a degree-one chart with frame s, we adjoin ratios t/s of linear generators. They generate the affine chart of X, with s as fiber coordinate. Blowup commutes with flat base change; the localization and completion maps are regular, so V is regular. Write $X_0 \cong X$ for the exceptional surface and C_V for the pullback of C along $V \to X$. Then

$$\mathfrak{m}\mathcal{O}_V = \mathcal{O}_V(-X_0), \quad J\mathcal{O}_V = \mathcal{O}_V(-X_0 - C_V), \quad \mathcal{O}_V(X_0)|_{X_0} = L^{-1}. \tag{4.2}$$

For the middle equality, write a degree-n section of J_S on a line-bundle chart. It contains the nth power of the fiber coordinate and vanishes along C. In degree one, remove the zero section and C_V. The remaining sections generate everywhere since $L - C = H$ is globally generated. This proves the equality before, and hence after, flat base change. Off X_0, the divisor C_V is the punctured prime divisor $V(J)$. Since X_0 is not one of its components, C_V is prime as well.

Fix a finite extension of normal domains $R \subset B$. As a finite algebra over R, the ring B is a product of complete local rings. It is a domain, so there is only one factor. Write $\mathfrak{n}_B$ for its maximal ideal; the residue field is k since k is algebraically closed. Set

$$U_B = \operatorname{Spec} B \setminus \{\mathfrak{n}_B\}, \qquad D_B^\circ = \operatorname{Spec}(B/JB) \setminus \{\mathfrak{n}_B\}.$$

Later we will compare sections of a torsor on D_B°. Connectedness will make their difference one constant. We also need every prime divisor of B to meet D_B°, where we can evaluate the inertia action on those sections.

Lemma 4.5. *If $P \neq J$ is a height-one prime of R, then $V(P) \cap V(J)$ contains a point different from $\mathfrak{m}$. In particular every prime divisor of* $\operatorname{Spec} B$ *meets D_B°.*

Proof. Suppose $V(P) \cap V(J)$ is supported at the vertex. We will use negative definiteness on a surface over $V(P)$. Normalize the strict transform of $V(P)$ in V and resolve it. This excellent surface has a resolution Z proper birational over the normalization of $V(P)$. Let A be the pullback of C_V to Z. Since the generic point of $V(P)$ is outside $V(J)$, the divisor A is effective. Our assumption makes it exceptional. But its line bundle comes from the nef divisor C on X, so $A \cdot T \ge 0$ for every exceptional curve T and hence $A^2 \ge 0$. Negative definiteness of the exceptional intersection matrix forces $A = 0$ [Mum61, Lip69].

We now show that A cannot be zero. From the graph of the restricted generators, the strict transform of $V(P)$ is the blowup of its maximal ideal. The exceptional fiber is the Proj of the associated graded ring of the two-dimensional local domain R/P, so it is one-dimensional. Pick a proper curve in Z mapping onto a curve T_0 in X_0. If $T_0 = C$, our curve lies in the support of A. Otherwise T_0 meets C, again giving nonempty support for A. Both cases contradict $A = 0$.

For a prime divisor of B, finiteness and going down give a height-one contraction in R, and the divisor maps onto it. If the contraction is J, its generic point already lies in D_B°. Otherwise the intersection just proved supplies a point off the vertex, which lifts to the divisor in $\operatorname{Spec} B$. □

Proposition 4.6. *The punctured divisor D_B° is connected for every finite extension of normal domains $R \subset B$.*

Proof. Normalize V in $\operatorname{Frac}(B)$ and write $f : Y \to V$. Excellence makes f finite. The map $Y \to \operatorname{Spec} B$ is proper birational and an isomorphism over U_B. Set

$$E = f^* X_0, \qquad D = f^* C_V.$$

The support of E is the closed fiber over $\mathfrak{n}_B$. It is connected by normality of B and proper birationality. Every prime component of E maps finitely onto X. Thus, for an integral curve T in this fiber, (4.2) gives

$$E \cdot T = -[k(T) : k(f(T))]\, L \cdot f(T) < 0. \tag{4.3}$$

Off E, the support of D is D_B°. Suppose it has $s > 1$ connected components. Take their closures with the multiplicities from D, giving Weil divisors $D_1, \dots, D_s$. Every prime component of D dominates C_V under the finite map f, so $\sum_i D_i = D$. Each D_i is Cartier off E, and distinct D_i can meet only in E.

We will first show that distinct D_i cannot meet along a curve. Their intersections with E should have negative degree by (4.3), but the D_i need not be Cartier. Choose a projective resolution $\pi : Z \to Y$, which exists for this quasi-excellent threefold [CP19, Corollary 1.2 and Proposition 4.4]. We will add exceptional surfaces to the strict transforms and obtain rational divisors A_i whose intersections we can compute.

For a triple product with $\pi^* E$, intersect the other divisors on each surface component of $\pi^* E$. Sum the degrees with the multiplicities of these components. The degrees are defined because the support of $\pi^* E$ is proper.

Write D_i' for the strict transform. For an exceptional surface whose center is a curve T, localize at the generic point of T. We get a resolution of the normal surface $\operatorname{Spec} \mathcal{O}_{Y,T}$. Let $M = (E_a \cdot E_b)$ be its exceptional intersection matrix. We choose c_a so that the corrected divisor is orthogonal to every exceptional curve:

$$\left(D_i' + \sum_a c_a E_a \right) \cdot E_b = 0 \quad \text{for all } b.$$

Negative definiteness of M gives unique coefficients c_a. For later mixed intersections, we also need $c_a \geq 0$. The matrix $-M$ is positive definite with nonpositive off-diagonal entries. Write $-M = tI - N$ with N nonnegative and t larger than its spectral radius. Then $(-M)^{-1} = \sum_{n \geq 0} t^{-n-1} N^n$ has nonnegative entries, strictly positive within each connected block. Since D_i' meets the exceptional curves nonnegatively, this gives $c_a \geq 0$. Use these coefficients on the corresponding exceptional surfaces of Z, for every curve center. Set the coefficients over point centers to zero for now, and call the resulting divisor A_i^0.

It remains to correct along the exceptional surfaces $F_1, \dots, F_r$ with point centers. We use an effective sufficiently ample relative hyperplane H_Z on Z over $\operatorname{Spec} B$. Choose H_Z to contain no

exceptional surface and no curve common to two exceptional surfaces. There are finitely many conditions, so a relatively very ample system over the infinite residue field allows this choice. The matrix

$$M^0_{ab} = H_Z \cdot F_a \cdot F_b$$

is negative definite. To see this, normalize and resolve each integral surface component of H_Z. It maps properly birationally to the normalization of its image in Y. Restriction and pullback send the F_a to effective exceptional curve divisors, on which the surface intersection form is negative definite.

No nonzero combination of the F_a restricts to zero on all these surfaces. Indeed, choose F_a with nonzero coefficient. The ample restriction $H_Z|_{F_a}$ cuts a curve whose generic point lies on no other F_b, by our choice of H_Z. Its strict transform retains the nonzero coefficient on a resolved component. Summing the surface forms therefore gives a negative definite matrix M^0. We can now choose unique rational numbers b_{ia} with

$$A_i = A_i^0 + \sum_a b_{ia} F_a, \qquad A_i H_Z F_a = 0 \quad (1 \le a \le r).$$

The new surfaces do not change the equations at generic curve centers, so A_i satisfies both sets of equations. The ordinary pullback $\pi^* D$ does too: it has degree zero on every contracted curve, and on each localized resolution it is the pullback of the Cartier divisor D. The corrections are linear and unique. Hence

$$\sum_i A_i = \pi^* D. \tag{4.4}$$

We do not need a sign for b_{ia}.

We next show $A_i^2 \pi^* E \le 0$ using Hodge index on each surface in $\pi^* E$. The projection formula allows us to normalize and resolve these surfaces. On the strict transform of a component of E, the pullback of C is nonzero and nef with square zero, since this surface maps generically finitely onto X. Every curve in A_i restricted to this surface maps into C or to a point of X. Indeed, D_i' lies over C_V, and a curve center has a nonzero correction only if it lies in D_i. Since $C^2 = 0$, the restriction of A_i is orthogonal to the pullback of C. Hodge index makes its square nonpositive.

On a surface mapping onto a curve in Y, normalize the base curve and take Stein factorization. Its fiber class is nonzero and nef with square zero. Our equations at the generic curve center give degree zero for A_i on the fiber, so Hodge index again applies. On a surface F_a over a point, use $A_i H_Z F_a = 0$ and ampleness of $H_Z|_{F_a}$. Every surface thus contributes a nonpositive number:

$$A_i^2 \pi^* E \le 0. \tag{4.5}$$

For $i \ne j$, we need the sign of the mixed intersection. We claim that $\pi_*(A_i A_j)$ is an effective rational sum of the curves common to D_i, D_j, with positive coefficient at each one. In computing this pushforward, we can replace A_j by D_j'. A surface of $A_j - D_j'$ over a point contributes zero as a one-cycle. A surface over a curve contributes the degree of A_i on the generic fiber, which is zero by our equations.

The intersections of D_j' with D_i' and with the corrections over curves are proper and effective, since $c_a \ge 0$. The terms over points still push forward to zero. At the generic point of a common curve, compute on the normal local surface [Mum61]. If the strict transforms meet, they have positive intersection. If they do not, both meet the connected exceptional fiber over the closed point. Strict positivity of $-M^{-1}$ on this connected block again gives a positive numerical intersection. This proves the claim. Every common curve lies in E, where its degree against E is negative by

(4.3). By projection,

$$A_iA_j\pi^*E \le 0, \qquad \text{with strict inequality if } D_i, D_j \text{ meet along a curve.} \tag{4.6}$$

The sum of these nonpositive intersections is zero. Indeed, push down first to Y and then to V:

$$A_i\pi^*D\pi^*E = D_iDE = (f_*D_i)C_VX_0 = 0.$$

Here f_*D_i is a multiple of C_V, and $C_V^2X_0 = C^2 = 0$ on X_0. Substitute (4.4) to get

$$0 = A_i^2\pi^*E + \sum_{j\neq i} A_iA_j\pi^*E.$$

By (4.5) and (4.6), every term is nonpositive, so every term is zero. The strict inequality in (4.6) therefore rules out a curve common to distinct D_i.

We must also rule out isolated intersections. If two D_i meet at an isolated point, complete the local ring of Y there. It is a complete normal local domain of dimension three, and D is principal. Since a principal ideal has cohomological dimension at most one, Grothendieck connectedness makes its punctured divisor connected [Var09, Theorem 1.6], [Sta26, Tag 0ECR]. But the parts through the point still have positive-dimensional support after completion and meet only at the closed point. Removing that point disconnects them, a contradiction. Thus the D_i are disjoint on all of Y.

We can now use connectedness on covers of X. Join two components of E when they share a curve. This graph is connected: otherwise, connectedness of E would give isolated intersections between two groups of components. The same principal-divisor argument, applied to E, rules this out.

Normalize a component of E. It maps finitely onto X, so the inverse image of C is nonempty and connected by Lemma 3.4. Hence this component meets exactly one D_i. A curve shared by two components maps finitely onto a curve in X. Its image is C or meets C by Lemma 3.1, so these components meet the same D_i. Following the graph shows that all components meet one D_i. But every D_i meets the closed fiber by properness over the local base. Thus $s = 1$, proving the connectedness of D_B°. □

4.4. **A class which survives every finite cover.** To prove splitting, we return to the elliptic class on C. Since C is ordinary and k is algebraically closed, we can pick $0 \neq \alpha \in H^1(C, \mathcal{O}_C)$ with $F\alpha = \alpha$. Pull the corresponding nontrivial Artin–Schreier $\mathbb{F}_p$-torsor back to the punctured cubic cone, and then to D_R°. Call this torsor $\mathcal{P}$ and its coherent class α. On the graded cone, the class lies in degree zero of $H^2_{\mathfrak{m}_S}(S/J_S)$. Localization at the vertex and faithfully flat completion preserve its nonvanishing.

The boundary for $0 \to J \to R \to R/J \to 0$ sends α to a socle generator η of $H^3_{\mathfrak{m}}(J) = E_R$. Indeed, before completion its degree-zero part is

$$H^1(C, \mathcal{O}_C) \longrightarrow H^2(X, \mathcal{O}_X(-C)),$$

which is an isomorphism since $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. The torsor $\mathcal{P}$ is nontrivial even over $\mathrm{Frac}(R/J)$. Otherwise, finiteness and normality would extend its generic section to D_R°. This would kill the coherent class α, contradicting its nonvanishing after completion.

Proposition 4.7. *Every finite extension of R splits as an R-module map.*

Proof. For a finite R-algebra B, we must show that $1 \otimes \eta \neq 0$ in $B \otimes_R E_R$ by Lemma 2.2. Indeed, the dual map is evaluation at 1,

$$\mathrm{Hom}_R(B, R) \longrightarrow R.$$

and a functional mapping to 1 is the retraction we need.

First suppose B is normal and generically separable over R. If $1 \otimes \eta = 0$, we will lift α from D_B° to U_B. Lemma 2.3 gives

$$B \otimes_R H^3_{\mathfrak{m}}(J) \simeq H^3_{\mathfrak{n}_B}(JB).$$

The lemma initially gives coefficients $B \otimes_R J$. We may replace this module by JB: the surjection has finite-length kernel because J is invertible off the closed point, with a nonzerodivisor as local generator in B. It therefore induces an isomorphism on top cohomology. Since η is the boundary of α, the vanishing of $1 \otimes \eta$ now lifts the pullback of α to $\beta \in H^2_{\mathfrak{n}_B}(B) = H^1(U_B, \mathcal{O}_{U_B})$.

To obtain a torsor, we need the lift to satisfy $F\beta = \beta$. The normal three-dimensional ring B is Cohen–Macaulay on its puncture. Present B as a quotient of a complete regular local ring and apply local duality. The dual of $H^2_{\mathfrak{n}_B}(B)$ is a finite deficiency module. Equidimensionality and Cohen–Macaulayness make it zero away from the closed point. Thus the dual and $H^2_{\mathfrak{n}_B}(B)$ have finite length, and the latter is finite-dimensional over k.

Let W be the kernel of restriction to $H^1(D_B^\circ, \mathcal{O})$. It is Frobenius-stable and contains $F\beta - \beta$, since $F\alpha = \alpha$. Lemma 2.5(i) gives $w \in W$ with $F\beta - \beta = Fw - w$. Replace β by $\beta - w$. This keeps its restriction and makes $F\beta = \beta$. Artin–Schreier on U_B now gives a finite étale $\mathbb{F}_p$-torsor killing β.

We pass to a Galois cover where the coherent class of $\mathcal{P}$ vanishes. Pick a connected component of the torsor just obtained and take the Galois closure of its function field over $\operatorname{Frac}(R)$. Normalize R in this field to get A, finite by excellence, and write G for the Galois group. The coherent class vanishes on D_A°. To obtain a section of $\mathcal{P}$ itself, we still need $F - 1$ onto on $\Gamma(D_A^\circ, \mathcal{O})$.

Put $B' = A/JA$. Off the closed point this quotient is cut out by a Cartier divisor. Each component dominates $V(J)$ by going down and contains the closed point in its closure. Hence B' is equidimensional of dimension two. At a nonclosed point, it is the quotient of a normal local ring of dimension at most two by a nonzerodivisor, so it is Cohen–Macaulay. The local duality argument used for B gives finite length of $H^1_{\mathfrak{n}_A}(B')$. The sequence

$$0 \longrightarrow H^0_{\mathfrak{n}_A}(B') \longrightarrow B' \longrightarrow \Gamma(D_A^\circ, \mathcal{O}) \longrightarrow H^1_{\mathfrak{n}_A}(B') \longrightarrow 0$$

therefore makes $\Gamma(D_A^\circ, \mathcal{O})$ finite over the complete local ring B', whose residue field is k. Lemma 2.5(ii) makes $F - 1$ onto on this finite algebra, even if B' has nilpotents. Artin–Schreier now embeds $H^1_{\text{ét}}(D_A^\circ, \mathbb{F}_p)$ into $H^1(D_A^\circ, \mathcal{O})$. Since the coherent class is zero, $\mathcal{P}$ has a section on D_A°.

Fix a section s. For $\sigma \in G$, connectedness of D_A° from Proposition 4.6 makes $\sigma s - s$ a single constant $\lambda(\sigma) \in \mathbb{F}_p$. The action on constants is trivial, so we get a character $\lambda : G \to \mathbb{F}_p$. We will use it to construct a nontrivial torsor on U_R. First we must show that $\lambda \neq 0$. At the generic point of J, the residue extensions may be inseparable even though $\operatorname{Frac}(A)/\operatorname{Frac}(R)$ is separable.

Take the integral closure of the DVR R_J in $\operatorname{Frac}(A)$ and reduce modulo the radical of its extended maximal ideal. Call the quotient T. The integral closure is semilocal, so T is the product of the residue fields at primes over J. The group G permutes these factors transitively. We claim that T^G is purely inseparable over $\kappa(J)$. Lift an invariant $b \in T$ to the semilocal ring. Its orbit polynomial has coefficients in R_J and reduces to $(Z - b)^{|G|}$. Write $|G| = p^a m$ with $p \nmid m$. The coefficient of $Z^{|G|-p^a}$ gives $b^{p^a} \in \kappa(J)$, proving the claim.

If $\lambda = 0$, the section over T is G-invariant and maps the coordinate field of the generic torsor $\mathcal{P}$ into T^G. This is a separable field of degree p over $\kappa(J)$: the generic torsor is nontrivial, and a nontrivial torsor of prime order over a field is connected. It cannot lie in a purely inseparable extension. Thus $\lambda \neq 0$, giving a quotient of G of order p.

To remove ramification, let σ lie in the inertia group of a prime divisor of A over R. It fixes the residue field at the generic point, so it acts identically on the reduced divisor. By Lemma 4.5, choose a point of this divisor in D_A°. At a geometric point above it, s and σs have equal values.

Thus $\lambda(\sigma) = 0$ for every divisorial inertia group. The normalization in the fixed field of $\ker \lambda$ is consequently étale in codimension one. Indeed, after strict henselization of a height-one DVR, the local Galois group is the inertia group; this includes any inseparable residue contribution.

Purity of the branch locus on the regular puncture U_R now gives a nontrivial étale $\mathbb{F}_p$-torsor [Sta26, Tag 0BMB]. But normality gives $\Gamma(U_R, \mathcal{O}) = R$, and Cohen–Macaulayness gives $H^1(U_R, \mathcal{O}) = H^2_{\mathfrak{m}}(R) = 0$. Also $(F-1)R = R$ by Lemma 2.5(ii). Artin–Schreier then gives $H^1_{\text{ét}}(U_R, \mathbb{F}_p) = 0$, a contradiction. This proves splitting for finite normal generically separable extensions.

It remains to allow inseparable extensions. Given a finite domain extension, pass to its normalization and let B_s be the normalization in its maximal separable subfield. Choose $q = p^e$ so that every qth power in the extension belongs to B_s. The extension is then a subring of $B_s^{1/q}$. Taking qth roots of the retraction $B_s \to R$ gives an $R^{1/q}$-linear retraction $B_s^{1/q} \to R^{1/q}$. Compose with the Frobenius retraction $R^{1/q} \to R$ and restrict to the given extension. For a finite surjective R-algebra, take a prime over zero and compose the quotient map with the retraction of the resulting domain. □

For the ideal of Proposition 4.4, splitting now gives

$$I^F = I^+ = I \subsetneq I^*.$$

Indeed, a split extension contracts the extended ideal to I. Frobenius splitting gives $I^F = I$, while Proposition 4.7 gives $I^+ = I$. Smith's theorem identifies plus and tight closure for parameter ideals [Smi94], so this I cannot be a parameter ideal.

4.5. **The discrepancy over the vertex.** To rule out klt type, we use the same exceptional surface for every boundary. Fix an effective rational Weil divisor Δ on $\operatorname{Spec} R$ with $K_R + \Delta$ rational Cartier. On the regular blowup $g : V \to \operatorname{Spec} R$, write

$$K_V + \Delta_V = g^*(K_R + \Delta) + aX_0,$$

where Δ_V is the strict transform, X_0 is the only exceptional prime, and a is its discrepancy. A Cartier multiple of $K_R + \Delta$ is principal on $\operatorname{Spec} R$, so its pullback is numerically trivial on X_0. The restriction $\Delta_X = \Delta_V|_{X_0}$ is effective since X_0 is not a component of Δ_V. Adjunction and (4.2) now give

$$K_X + \Delta_X \equiv -(a+1)L.$$

Intersect with the nef curve C. Since $K_X \cdot C = 0$, we get $0 \le \Delta_X \cdot C = -3(a+1)$, hence $a \le -1$. No effective boundary can therefore make the pair klt, which requires $a > -1$ [KM98, §2.3]. This proves the last assertion of Theorem B; Lemma 4.1, Corollary 4.3 and Proposition 4.7 give its other assertions.

Corollary 5.9 of [CEMS18] assumes that the anticanonical algebra is finitely generated. We can see its failure here by retaining the internal degree of a section, in addition to its reflexive degree. The sections of internal degree zero cannot generate those of internal degree one from finitely many elements.

Corollary 4.8. *The anticanonical symbolic algebras of S, $S_{\mathfrak{m}_S}$ and R are not finitely generated. The same holds for every positive Veronese with respect to the anticanonical degree.*

Proof. Pull $\mathcal{O}_X(C)$ back to the punctured cone to obtain the anticanonical divisorial module. Its powers extend reflexively across the vertex, with bigraded pieces

$$\mathcal{B}_{m,n} = H^0(X, nL + mC), \qquad m \ge 0, \quad n \in \mathbb{Z}.$$

If $n < 0$, intersection with C shows that the piece is zero. For $n = 0$, the pieces form $k[s_C]$ since $h^0(X, mC) = 1$. To compute internal degree one, start from $h^0(X, H) = 3$ and $H^1(X, H) = 0$. For

$b \geq 1$, the restriction of $H + bC$ to C has degree three. Induction along the restriction sequences gives $H^1(X, H + bC) = 0$ and $h^0(X, L + mC) = 6 + 3m$.

We want the same obstruction after completion, so pass to the closed fiber $\mathcal{T} = \mathcal{B}/\mathfrak{m}_S\mathcal{B}$. Both gradings descend, and internal degree zero remains $k[s_C]$. At reflexive degree m, the part removed from internal degree one is exactly $s_C^m H^0(X, L)$. Hence

$$\dim_k \mathcal{T}_{m,1} = 3m.$$

If $\mathcal{T}$ were finitely generated, choose bihomogeneous generators. All internal degrees are nonnegative, so a monomial of internal degree one uses exactly one degree-one generator. The other factors lie in $k[s_C]$. Thus $\mathcal{T}_{*,1}$ would be finite over $k[s_C]$, with bounded dimensions in the anticanonical grading. This contradicts growth as $3m$. For a Veronese of index d, use $k[s_C^d]$ instead; the dimensions are $3dm$ and remain unbounded.

Finally, localization at the vertex and completion preserve this closed fiber algebra. Indeed, flat base change for finite Hom modules commutes with the duals defining the reflexive powers [Sta26, Tag 087R]. Apply it to the finite divisorial modules and their powers. Finite generation over the local or complete ring would then make the closed fiber algebra finitely generated over k, contrary to what we just proved. □

5. Two test ideals in characteristic two

We prove Theorem C in characteristic two by following a cohomology class through Frobenius. At a fixed Čech stage, its denominator t^N becomes t^{Nq}. We will produce torsion classes requiring more than Nq powers of t, for every fixed N. In the injective hull we instead pass from relative cohomology M to $H^1_{(t)}(M) = M_t/\operatorname{im}(M)$, where this torsion dies.

Take a section ring over $\mathbf{F}_2[[t]]$ and complete at the closed vertex to get R. The principal part of Section 5.3 will give one element of its injective hull E_R whose multiples detect all these torsion classes. This separates the two closures for $(R, J^{10/3})$. A cubic cover will then give the ordinary test ideals.

5.1. **The bundle on a genus-two curve.** Set $k = \mathbf{F}_2$, $A_0 = k[[t]]$, and $K = k((t))$. Let C be the smooth projective model of $y^2 + y = x^5$. Write P for its point at infinity and put $B = \mathcal{O}_C(P)$. Take the charts

$$U = \operatorname{Spec} k[x, y]/(y^2 + y + x^5), \qquad V = \operatorname{Spec} k[u, z]/(z^2 + u^3 z + u), \qquad u = x^{-1}, \quad z = y/x^3.$$

The map $x : C \to \mathbf{P}^1$ has degree two, with only $(u, z) = (0, 0)$ over infinity. There z is a parameter and u has order two, so $y = x^3 z$ has order -5. The differential dx has no zero on U. On V the equation $(1 + u^2 z)du = u^3\, dz$ gives order two at P. Thus $\omega_C = B^2$ and $g(C) = 2$.

Set $C_{A_0} = C \times_k \operatorname{Spec} A_0$. On the overlap we have $w = (y + 1)/x^2 = z^{-1}$. Define the bundle $\mathcal{E}$ on C_{A_0} by

$$\operatorname{frame}_V = \operatorname{frame}_U G_t, \qquad G_t = \begin{pmatrix} z(1 + tx^2 + t^2x^3) & x + tx^2 \\ tx & w \end{pmatrix}. \tag{5.1}$$

Since $zw = 1$, the two copies of $tx^2 + t^2x^3$ cancel and $\det G_t = 1$. Put $E_* = \mathcal{E}|_{t=0}$.

Lemma 5.1. *We have $E_* = F_{C*}B^{-1}$. It is geometrically stable of degree zero. The counit $c : F_C^* E_* \to B^{-1}$ fits in an exact sequence*

$$0 \longrightarrow B \longrightarrow F_C^* E_* \xrightarrow{c} B^{-1} \longrightarrow 0, \qquad F_C^{2*} E_* \simeq B^2 \oplus B^{-2}.$$

Consequently the slopes of every $\operatorname{Sym}^j E_*$ and its dual lie between $-j/2$ and $j/2$.

Proof. Use $1, x$ as a Frobenius basis on U, which is étale over the x-line. On V use $z^2, z+u^3$ in the ideal of P. Their columns in the first basis are $(z,0),(x,w)$, since scalars act by squaring and $x^2+w^2x = z+u^3$. The determinant is $zw=1$ on the overlap. At P their orders are two and one, so they span $\mathfrak{m}_P/\mathfrak{m}_P^3$, the fiber of $F_{C*}B^{-1}$. The transition is therefore G_0, as for the destabilized bundle in [Duc08, Proposition 5.1].

To check geometric stability, extend constants to an algebraic closure and pick a line subbundle $M \subset E_*$. Frobenius adjunction gives a nonzero map from its pullback to B^{-1}, hence $2\deg M \le -1$. The counit is locally multiplication, so it is onto, with kernel B by determinants. Its next pullback splits because $H^1(C, B^4) = 0$. After these two pullbacks, the symmetric power and its dual have line summands of degrees between $-2j$ and $2j$. Divide the slope bounds by four to get the asserted bounds before pullback. □

We use $\varphi = F_C \times \mathrm{id}_{A_0}$ to compare the generic bundles. This curve Frobenius squares x, y and fixes t. For tight closure we need absolute Frobenius, which also squares t. For $q = 2^e$, the coefficient change $t \mapsto t^q$ takes $\varphi^{e*}\mathcal{E}$ to $F_{\mathrm{abs}}^{e*}\mathcal{E}$. It preserves any external twist pulled back from C.

Lemma 5.2. *For $\sigma(t) = t/(1+t)$, the generic bundles satisfy*

$$\varphi^{2*}\mathcal{E}_K \simeq (\mathcal{E}_{\sigma(t)})_K, \qquad \varphi^{4*}\mathcal{E}_K \simeq \mathcal{E}_K.$$

All symmetric powers of $\mathcal{E}_K$, their duals, and their curve Frobenius pullbacks are geometrically semistable of degree zero.

Proof. Section 5.6 gives a map $\mathcal{E}_{\sigma(t)} \to \varphi^{2*}\mathcal{E}$ of determinant $t^2(1+t+t^2+t^7)$, so it is invertible over K. Apply the map twice, using $\sigma^2(t) = t$ and the fact that φ fixes t. This gives periodicity after four pullbacks, also for symmetric powers, duals and curve Frobenius pullbacks. Over an algebraic closure, a positive-slope subbundle would then have pullbacks of unbounded slope inside the same bundle. The maximal slope excludes this. Absolute Frobenius pullbacks have the same conclusion by coefficient change. □

5.2. **Cohomology with growing torsion.** Set $\gamma = \varphi^4$, of degree 16 on C. Pick the isomorphism $\gamma^*\mathcal{E}_K \to \mathcal{E}_K$ of Lemma 5.2. Clear its denominators and divide out common powers of t to get a primitive map $\psi : \gamma^*\mathcal{E} \to \mathcal{E}$. Its special reduction is nonzero, since global Hom is a finite torsion-free A_0-module and a map reducing to zero is divisible by t. By Lemma 5.1, $F_C^{4*}E_* = B^8 \oplus B^{-8}$. Stability forces ψ_0 to vanish on B^8, giving

$$\psi_0 = i\pi, \qquad \pi = F_C^{3*}c : F_C^{4*}E_* \twoheadrightarrow B^{-8}, \qquad 0 \neq i : B^{-8} \longrightarrow E_*.$$

The adjoint of i under $E_* = F_{C*}B^{-1}$ is the nonzero map $cF_C^*i : B^{-16} \to B^{-1}$. Write β for its section of B^{15}. Pull back three more times to get

$$\pi\gamma^*i = \beta^8 : B^{-128} \longrightarrow B^{-8}. \tag{5.2}$$

We will use these powers of β to force vanishing along a fixed divisor while keeping the map nonzero at the generic point of C.

Lemma 5.3. *Fix $\ell \ge 1$ and set $j_0 = \lceil \log_{16} \ell \rceil$. For $n \ge j_0 + 2$, put $q = 16^n$ and $k_n = \lfloor (n - j_0)/2 \rfloor$. There is a surjection of A_0-modules*

$$H^1(C_{A_0}, F_{\mathrm{abs}}^{4n*}\mathcal{E} \otimes B^{15\ell}) \longrightarrow A_0/(t^{k_n q}).$$

The source is a finite torsion module.

Proof. Set $D_\beta = \operatorname{div}(\beta)$ and $P_n = \psi\,\gamma^*\psi\cdots(\gamma^{n-1})^*\psi$. Number the factors starting at zero. The adjacent pair starting at j is divisible by $\beta^{8\cdot 16^j}$ modulo t, by (5.2). For $j \geq j_0$ it vanishes on the whole divisor ℓD_β, including its multiplicities. Take the disjoint pairs starting at $j_0, j_0+2, \ldots$. Each gives a factor of t on $(\ell D_\beta)_{A_0}$, so their product is divisible by t^{k_n} there.

Thus on $C_{A_0/(t^{k_n})}$ the entries of P_n belong to the ideal of ℓD_β. This divisor comes from C and remains Cartier. Dividing by its local equation gives

$$(\gamma^n)^*\mathcal{E} \otimes B^{15\ell} \longrightarrow \mathcal{E}/t^{k_n}\mathcal{E}.$$

The special map is nonzero generically by (5.2). To detect it on H^1, use finite duality $E_*^\vee \otimes \omega_C = F_{C*}B^3$. The sections $1, x$ generate its generic fiber with the Frobenius scalar action. Indeed x has a simple zero at $(0,0)$, so it is not a square in $k(C)$. The dual map sends at least one of these sections to a nonzero section. Serre duality therefore makes the map on special H^1 nonzero.

The target $H^1(C_{A_0}, \mathcal{E})$ is free, since $H^0(C, E_*) = 0$ makes multiplication by t injective. On the relative curve, H^1 of these flat bundles commutes with reduction modulo powers of t, because coherent H^2 vanishes. Lift a special class with nonzero image and project onto a unit coordinate in the target modulo t^{k_n}. This gives a quotient $A_0/(t^{k_n})$.

Now make the finite flat coefficient change $t \mapsto t^q$. It gives the asserted absolute Frobenius bundle and quotient, with the same external twist. Properness gives finiteness of the source. Its generic H^1 vanishes by semistability and Serre duality since $15\ell > 2 = \deg \omega_C$, so the source is torsion. □

5.3. **A section ring and a distinguished principal part.** We now pass from the bundle to a local ring. Set

$$Y = \mathbf{P}_{C_{A_0}}(\mathcal{E}), \qquad L = \mathcal{O}_Y(1) \otimes \pi^* B^3, \qquad S = \bigoplus_{j\geq 0} H^0(Y, L^j), \qquad R = \widehat{S_{(t,S_+)}}.$$

Write $\pi : Y \to C_{A_0}$ for the projection, with $\pi_*\mathcal{O}_Y(j) = \operatorname{Sym}^j \mathcal{E}$ for $j \geq 0$. We omit π^* on bundles from the curve. In the fiber coordinates X, Y_f over U, take

$$s_0 = X, \qquad s_1 = t^2 yX + (1+tx)Y_f,$$
$$s_2 = (1+tx)yX + x^2 Y_f, \qquad s_3 = (x + tx^3 + t^2x^4)X + t(1+y)Y_f.$$

Set $h = s_0$, $a = s_1 + s_3$, $b = s_2$ and $J = (h, a, b)$. We use the same notation after localization and completion.

Lemma 5.4. *The line bundle L is relatively ample over A_0 and is generated by h, a, b. For $d \geq 3$ one has $S_d = JS_{d-1}$; in particular $S_{m+2} \subseteq J^m$ for $m \geq 0$. The ring R is an excellent F-finite normal local domain of dimension four, t, h, a, b form a system of parameters, and R_h is regular.*

Proof. On V, multiply the four columns by $z^3 G_t^{-1}$. We get

$$(z^2, txz^3), \quad (xz^3, (1+tx+tx^2)z^4), \quad (0, x^2z^4), \quad (xz^2, 0).$$

These columns are regular at infinity by the orders of x, z. On the special curve, h, a have columns $(1,0), (x,1)$ over U. At P, where xz^2 has value one, a, b have values $(1,0), (0,1)$. Thus the sections span on the special curve. Properness makes the cokernel support have closed image in the local base, so they span on the whole family and generate L.

For ampleness, pull the special bundle back twice by curve Frobenius. It becomes $B^{14} \oplus B^{10}$ by Lemma 5.1. Its tautological line is ample, so ampleness descends under the finite pullback. Openness then gives relative ampleness of L.

For $j \geq 1$ the minimum slope of $\operatorname{Sym}^j E_* B^{3j}$ is at least $5j/2 > 2$. Serre duality, reduction and Nakayama give $H^1(Y, L^j) = 0$. Projective-bundle cohomology also gives $H^2(Y, L^j) = 0$ for $j \geq 0$. Twist the Koszul complex on h, a, b by L^d. For $d \geq 3$ the obstruction groups $H^1(Y, L^{d-2})$ and $H^2(Y, L^{d-3})$ vanish. Hence $S_{d-1}^{\oplus 3} \to S_d$ is onto, and iteration gives $S_{m+2} \subseteq J^m$.

Relative ampleness gives finite generation and $\operatorname{Proj} S = Y$. After rationally trivializing L, its sections are described by order inequalities at the prime divisors of the regular scheme Y. Together with nonnegative grading, these express S as an intersection of integrally closed rings. Thus S is normal. We also have $\sqrt{J} = S_+$ by the generation just proved. Inverting h, a, b gives Laurent polynomial charts over regular opens of Y, since these sections have degree one. The dimension formula over the DVR gives height three for S_+ and four for (t, S_+), with $S/S_+ = A_0$. Thus t, h, a, b form a system of parameters. Excellence preserves normality and regularity on $D(h)$ after completion. Finally F-finiteness passes from A_0 through finite type, localization and completion. □

Write W_S for the graded canonical module and W for its completed localization. Off $V(S_+)$ the cone is the $\mathbf{G}_m$-bundle of L. Its logarithmic fiber differential identifies the relative canonical line, and changes of fiber coordinate add only a base differential. The vertex section has codimension three, so reflexive extension gives

$$W_S = \bigoplus_{j \in \mathbf{Z}} H^0(Y, \omega_{Y/A_0} L^j), \qquad \omega_{Y/A_0} = \mathcal{O}_Y(-2) B^2.$$

Here $\det \mathcal{E}$ and the canonical module of A_0 are trivial. Put $\mathcal{D}_i = (\operatorname{Sym}^i \mathcal{E})^\vee$, keeping the dual outside the symmetric power. Relative duality gives $R^1 \pi_* \mathcal{O}_Y(-i-2) = \mathcal{D}_i$ for $i \geq 0$. The graded local-cohomology calculation in Section 2 therefore identifies degree -4 in $H^3_{S_+}(W_S)$ with

$$H^2(Y, \omega_{Y/A_0} L^{-4}) = H^1(C_{A_0}, \mathcal{D}_4 B^{-10}).$$

Pick $Q_0 = (0, 0)$, where x is a parameter, and trivialize B there. Let η_0 be the class of the principal part

$$x^{-2} X^{-3} Y_f^{-3}.$$

The fiber monomial is a Laurent basis element of $H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-6))$, and the pole maps by the boundary for $2Q_0$. On the special curve, $\mathcal{D}_4 B^{-10}(2Q_0)$ has maximum slope at most $2 - 10 + 2 = -6$. Its H^0 vanishes, so principal parts inject into H^1 and η_0 has nonzero reduction. The bundle before twisting by $2Q_0$ also has zero special H^0. Thus its integral H^1 is free over A_0, and η_0 is primitive.

Take top Čech cohomology first in h, a, b, then in t. We get

$$H^4_{(t, S_+)}(W_S) = H^1_{(t)}(H^3_{S_+}(W_S)).$$

The class $\eta = \eta_0 / t$ is nonzero in degree -4 by primitivity. It survives localization and completion, since this closed-point local cohomology is the union of its finite-length submodules. For the maximal ideal $\mathfrak{m}$ of R, the duality calculation of (2.4) gives

$$H^4_{\mathfrak{m}}(W)^\vee = \operatorname{Hom}_R(W, W) = R.$$

Hence we regard η as an element of E_R.

5.4. **The distinction between two closure operations.** We use the pair closure of (2.1) [HY03, HT04]. Over a domain D, finitistic closure in a module M requires the same multiplier to work in $F^e_D(L)$ for one finite submodule $L \subset M$ containing the given element. The finite-presentation lemma applies to these relations as follows.

Lemma 5.5. *Let $(D,\mathfrak{n})$ be a Noetherian local ring, let $x_1,\dots,x_d$ generate an $\mathfrak{n}$-primary ideal, and let V be finite. Put $Q_N = V/(x_1^N,\dots,x_d^N)V$, with transition maps multiplication by $x_1\cdots x_d$. Suppose $v\in H^d_{\mathfrak{n}}(V)$ has a representative $v_{N_0}\in Q_{N_0}$. Suppose v is in the tight closure of zero for a pair in one finite submodule of $H^d_{\mathfrak{n}}(V)$. Then the image of v_{N_0} in some single Q_N is in the same pair tight closure. The assertion applies to ordinary closure too.*

Proof. Apply Lemma 2.4 to the system (2.3), with $M=V$, $I=(x_1,\dots,x_d)$ and prescribed representative v_{N_0}. It gives a map from the finite witnessing submodule L to one Q_N sending v to that representative. Applying F^e_D carries every closure relation to Q_N, with the same multiplier and this fixed N. □

Proposition 5.6. *The element constructed above satisfies*

$$\eta\in 0^{*J^{10/3}}_{E_R}\setminus 0^{*\mathrm{fg},J^{10/3}}_{E_R}.$$

In fact $hJ^{\lceil 10q/3\rceil}\eta^{[q]}=0$ for all sufficiently large q.

Proof. Set $q=2^e$ and $m=\lceil 10q/3\rceil$. On each degree-one chart, Frobenius raises the transition functions of W_S to their qth powers and multiplies the generator degrees by q. By Lemma 2.3 we can compute $\eta^{[q]}$ in top Čech cohomology. For a product g of m of the sections h,a,b, the relative class $hg\eta_0^{[q]}$ belongs to the second cohomology of

$$\omega_{Y/A_0}^{\otimes q}L^{-4q+m+1}=\mathcal{O}_Y(-6q+m+1)B^{-10q+3m+3}.$$

The exponent of B is at least three. If the fiber degree is at least -1, the H^2 vanishes. Otherwise it is curve H^1 of a dual symmetric power with this twist. Lemma 5.2 and Serre duality give zero generic H^1. Thus the integral group is t-torsion and its $H^1_{(t)}$ vanishes. Applying this to the generators g of J^m gives $hJ^m\eta^{[q]}=0$, also after completion.

For the finite-witness argument, fix $D\ge 1$ and take $q=16^n$, $e=4n$. Set

$$m_q=\frac{10q+2}{3},\qquad s_q=m_q+2,\qquad r=\frac{8q-14}{3}-D,\qquad M_{q,D}=H^1(C_{A_0},\mathcal{D}_rB^{3D+8}).$$

We take q sufficiently large for all symmetric exponents below to be nonnegative. We first show that the Frobenius multiples of our single class fill $M_{q,D}$. Consider

$$S_{s_q}\longrightarrow M_{q,D},\qquad g\longmapsto h^Dg\eta_0^{[q]}. \tag{5.3}$$

Multiplication gives fiber degree $-6q+D+s_q=-r-2$ and base exponent $-10q+3D+3s_q=3D+8$, as required. At Q_0 we have $h=X$ and Frobenius gives the principal part $x^{-2q}X^{-3q}Y_f^{-3q}$. For $0\le j\le r$ use the degree-s_q coefficient

$$X^{3q-1-D-j}Y_f^{(q+11)/3+D+j}$$

Its exponents are nonnegative for large q, including at $j=0,r$. Multiplying by $h^DX^{-3q}Y_f^{-3q}$ gives $X^{-j-1}Y_f^{-(r-j+1)}$ with coefficient one. These form a Laurent basis for the target fiber cohomology. Hence coefficients modulo x^{2q} prescribe all target principal parts along $2qQ_0$.

To choose these coefficients globally, use $\mathrm{Sym}^{s_q}E_*B^{3s_q}(-2qQ_0)$, whose minimum slope is at least $5s_q/2-2q=(19q+20)/3>2$. On the target, adding the same pole gives $\mathcal{D}_rB^{3D+8}(2qQ_0)$ with minimum slope at least

$$-r/2+3D+8+2q=2q/3+7D/2+31/3>2.$$

Duality makes both special H^1 groups vanish, and reduction and Nakayama give the vanishings over A_0. The source divisor sequence now lifts the coefficient jets to S_{s_q}. The target divisor

sequence sends the principal parts onto $M_{q,D}$. This proves surjectivity of (5.3). Lemma 5.4 also gives $S_{s_q} \subseteq J^{m_q}$, so these coefficients are allowed.

We next give $M_{q,D}$ a quotient whose torsion order grows with q. Set $\ell = D+1$ and $a_D = 15\ell - (3D+8) = 12D+7$ to obtain the twist in Lemma 5.3. Put $b_q = r - q \geq 0$. Riemann–Roch gives

$$\chi(\mathrm{Sym}^{b_q} \mathcal{E}_K \, B^{a_D}) = (b_q+1)(a_D-1) > 0.$$

Choose a nonzero generic section and clear denominators, then divide by its largest common power of t. We get an integral section σ nonzero on both fibers. Its zero scheme is proper with finite fibers, hence finite over A_0. Multiply the Frobenius monomials by σ to obtain

$$F_{\mathrm{abs}}^{e*}\mathcal{E} \otimes B^{-a_D} \longrightarrow \mathrm{Sym}^q \mathcal{E} \otimes B^{-a_D} \longrightarrow \mathrm{Sym}^r \mathcal{E}.$$

Off that zero scheme the images of X^q, Y_f^q are independent in each fiber, so the map is locally a split injection. Dualize, use $\mathcal{E}^\vee \simeq \mathcal{E}$, and twist by B^{3D+8}. The cokernel has finite support over the affine base and zero H^1. The kernel has zero H^2 on the relative curve. We obtain

$$M_{q,D} \twoheadrightarrow H^1(C_{A_0}, F_{\mathrm{abs}}^{e*}\mathcal{E}\, B^{15\ell}) \twoheadrightarrow A_0/(t^{k_n q}), \tag{5.4}$$

where the last map is Lemma 5.3 and $k_n \to \infty$ for this fixed D.

Now suppose η had a finite pair witness. Write $\eta_0 = w/(hab)^{N_0}$ in relative top Čech cohomology, with $N_0 \geq 1$ and $w \in W_S$ homogeneous. Apply Lemma 5.5 with $V = W$ and parameters t, h, a, b. For some fixed $N \geq N_0$ the element

$$v_N = t^{N-1}(hab)^{N-N_0} w \quad \text{in} \quad Q_N = W/(t^N, h^N, a^N, b^N)W$$

has a pair witness. Its numerator comes from writing η_0/t over the denominator $t^N(hab)^N$. Keep N fixed and choose $h^D \in \tau_b(R)$ with $D \geq 1$, using F-finiteness and regularity of R_h [Sch11, Proposition 3.23(ii),(iii)]. Use this D in the two surjections above.

For every $g \in J^{m_q}$, the element $g v_N^{[q]}$ is in $0^*_{F_R^e Q_N}$. Indeed the original pair witness applies at further powers q' because $q'\lceil 10q/3 \rceil \geq \lceil 10qq'/3 \rceil$. The test element h^D therefore kills $g v_N^{[q]}$. The radical of (t, h, a, b) in S is the closed vertex, so these quotients and their Frobenius transforms have finite length. Frobenius base change thus identifies them with their quotients before localization and completion. In particular, for every $g \in S_{s_q}$,

$$t^{(N-1)q} h^D g (hab)^{(N-N_0)q} w^{[q]} \;\in\; (t^{Nq}, h^{Nq}, a^{Nq}, b^{Nq}) F_S^e W_S.$$

Divide by $(hab)^{Nq}$ in relative top Čech cohomology. The terms with h^{Nq}, a^{Nq}, b^{Nq} are coboundaries, leaving

$$t^{(N-1)q}\alpha \in t^{Nq} M_{q,D}, \qquad \alpha = h^D g \eta_0^{[q]}.$$

Since t has degree zero, take the homogeneous summand $M_{q,D}$ on both sides. Surjectivity of (5.3) gives this relation for every α in $M_{q,D}$. Choose n with $k_n > N$, then use (5.4) to pick α mapping to one in $A_0/(t^{k_n q})$. The relation would make $t^{(N-1)q}$ divisible by t^{Nq} in this quotient, a contradiction. □

5.5. **Removing the pair by a cubic cover.** To turn the pair relation into ordinary tight closure, we use a cube root of seven linear factors. Its qth power contributes about $7q/3$ factors, and multiplying by one more linear form contributes q. To distinguish their coefficients, take independent $\lambda_1, \ldots, \lambda_7$ and set

$$k' = k(\lambda_1, \ldots, \lambda_7), \quad A_1 = R[\lambda_1, \ldots, \lambda_7]_{\mathfrak{m}R[\lambda_1,\ldots,\lambda_7]}, \qquad A = \widehat{A_1}.$$

The ring A has dimension four and residue field k'. Set

$$\ell_i = h + \lambda_i a + \lambda_i^3 b, \qquad F = \prod_{i=1}^{7} \ell_i, \qquad T = A[Z]/(Z^3 - F). \tag{5.5}$$

The exponents $0, 1, 3$ distinguish the terms under Frobenius. Their derivatives will also make the branch divisor simple normal crossings on $D(h)$.

Lemma 5.7. *If I is generated by three elements in characteristic p, Q a power of p, and $u, v \geq 0$ integers with $u \geq 2(Q-1)$, then*

$$I^u (I^v)^{[Q]} = I^{u+Qv}.$$

In particular, for $q = 4^j$ the ideal in R generated by all coefficients of $\ell_1^q F^{(q-1)/3}$ as a polynomial in the λ_i is precisely $J^{(10q-7)/3}$.

Proof. Pick a monomial of degree $u + Qv$ in the three generators and reduce its exponents modulo Q. Their residue sum is at most $3(Q-1)$ and congruent to u modulo Q. A sum exceeding u would therefore be at least $u + Q > 3(Q-1)$. Add back multiples of Q until the sum is u, without exceeding the original exponents. What remains has each exponent divisible by Q and total degree Qv, giving the factorization.

For $q = 4^j$ we have $(q-1)/3 = 1 + 4 + \cdots + 4^{j-1}$. A choice in $\ell_i^{4^l}$ gives a digit $0, 1$ or 3 at position l in the exponent of λ_i. The factor ℓ_1^q gives one additional digit at position j for λ_1. Thus every selection is recorded by a distinct parameter monomial, and the coefficient ideal is

$$J^{[q]} \prod_{l=0}^{j-1} (J^7)^{[4^l]} = J^{[q]} J^{7(q-1)/3} = J^{(10q-7)/3}.$$

For the middle equality apply the first assertion repeatedly with $u = 7$, $Q = 4$. For the last, take $u = 7(q-1)/3 \geq 2(q-1)$ and $Q = q$. □

Lemma 5.8. *The ring T is a complete excellent F-finite normal local domain of dimension four with residue field k'. Some h^D, $D \geq 1$, belongs to $\tau_b(T)$.*

Proof. Work first over the normal excellent ring A_1. Over $\operatorname{Frac} R$ the factors of F involve distinct variables and are relatively prime. Each is squarefree, since $\ell_i + \lambda_i \partial_{\lambda_i} \ell_i = h \neq 0$. A height-one prime with nonzero contraction to R is extended from a height-one prime, by flat going down and height. At its DVR one of h, a, b is a unit because $\operatorname{ht} J = 3$. Gauss's lemma then gives valuation zero for F. Thus F has reduced divisor on A_1. As a nonzero nonunit, it has a prime divisor of valuation one. Eisenstein there shows that $T_1 = A_1[Z]/(Z^3 - F)$ is a domain.

Finite freeness preserves S_2, since heights agree and regular sequences remain regular. At height one away from F the derivative $3Z^2$ is a unit, so the extension is étale. Over a divisor of F, the maximal ideal upstairs is generated by Z, giving a DVR. Hence T_1 is normal by Serre's criterion. Its closed fiber $k'[Z]/(Z^3)$ is local, and finiteness identifies its completion with T. Excellence preserves normality. Dimension four, residue field k', excellence and F-finiteness follow from the construction.

We want a test element which comes from R, so consider T_h. The base A_h is regular, since R_h is regular and the maps $R \to A_1 \to A$ are regular. Here the first is essentially smooth and the second is completion of an excellent ring. The derivations ∂_{λ_i} extend continuously to A. On $\ell_i = 0$ the value of $\partial_{\lambda_i} \ell_i$ is h/λ_i, a unit on $D(h)$. For two vanishing factors the derivatives give a diagonal unit minor. Their cotangent classes are therefore independent, since derivations send the square of a maximal ideal into that ideal. Three factors cannot vanish there, since the determinant of their

rows $(1, \lambda_i, \lambda_i^3)$ is a unit in k' and would force $h = a = b = 0$. Thus at most two factors vanish, and they belong to a regular system of parameters.

The cover is étale with no vanishing factor and regular with one. With two, write it as $Z^3 = uxy$ over a regular local ring D, with u a unit and x, y parameters. It is the degree-zero part of the $\mathbf{Z}/3$-graded algebra

$$D[r,s]/(r^3 - ux,\ s^3 - y), \qquad \deg(r) = 1, \quad \deg(s) = -1,$$

which is regular. Indeed it has dimension $\dim D$ and a unique closed point, with parameters r, s and the remaining base parameters. Its degree-zero part has basis $1, rs, (rs)^2$ and identifies with the cover under $Z = rs$. Projecting to degree zero gives a retraction. Strong F-regularity descends along this retraction, so T_h is strongly F-regular. Localization of the big test ideal gives a power h^D in $\tau_b(T)$ [Sch11, Proposition 3.23(ii),(iii)]. □

To extract coefficients from a finite witness over T, we need them all in one finite module over R. We use the following fact.

Lemma 5.9. *For a complete equicharacteristic local ring $(R, \mathfrak{m}, k)$ with coefficient field k, put $A = \widehat{R[\boldsymbol{\lambda}]_{\mathfrak{m}R[\boldsymbol{\lambda}]}}$ and $k' = k(\boldsymbol{\lambda})$. There are identifications*

$$E_A = A \otimes_R E_R = E_R \otimes_k k'.$$

Every finite A-submodule of E_A is contained in $A \otimes_R N$ for some finite R-submodule N of E_R. In characteristic p, for such N one has

$$F_A^e(A \otimes_R N) = A \otimes_R F_R^e N = F_R^e N \otimes_k k'.$$

Proof. For each l we have $A/\mathfrak{m}^l A = (R/\mathfrak{m}^l) \otimes_k k'$. Indeed, after inverting the nonzero polynomials in $k[\boldsymbol{\lambda}]$, all further required denominators are units modulo a nilpotent ideal.

Represent E_R by the continuous dual $\varinjlim_l \operatorname{Hom}_k(R/\mathfrak{m}^l, k)$. Its one-dimensional socle is essential. For injectivity, take a map from an ideal into this dual and evaluate at one. Its generator images are killed by some $\mathfrak{m}^l$. Artin–Rees makes the resulting functional continuous in the induced topology, so it extends over a finite quotient of R. This extends the module map. The same description over k', and finite-dimensional duality at each quotient, give the formula for E_A.

Expand generators of a finite A-submodule in pure tensors. Their finitely many E_R coefficients generate a finite-length module N, whose extension $N \otimes_k k' = A \otimes_R N$ contains the submodule. The first Frobenius equality is tensor associativity along the Frobenius square. The second uses the quotient calculation above, since $F_R^e N$ has finite length. □

Proposition 5.10. *For the ring T in (5.5) one has $\tau_b(T) \subsetneq \tau_{\mathrm{fg}}(T)$.*

Proof. The coefficient of Z^2 in a product gives a perfect pairing on T. Its matrix on $1, Z, Z^2$ is antidiagonal with entries one, so $\operatorname{Hom}_A(T, A) = T$. By adjunction, $\operatorname{Hom}_A(T, E_A)$ is injective over T, supported at the maximal ideal and with one-dimensional socle over k'. It is therefore $E_T = T \otimes_A E_A$. Set

$$\xi = Z\ell_1(\eta \otimes 1) \in E_T.$$

We first check $\xi \neq 0$. If $J\eta = 0$, then $J^{[q]}$ kills $\eta^{[q]}$ in $F_R^e(R\eta)$. But $J^{3q-2} \subseteq J^{[q]}$ and $\lceil 10q/3 \rceil \geq 3q-2$, so $R\eta$ would witness pair closure with multiplier one. This contradicts Proposition 5.6, hence $J\eta \neq 0$. Independence of $1, \lambda_1, \lambda_1^3$ gives $\ell_1(\eta \otimes 1) \neq 0$, and multiplication by Z places it in a free coordinate.

Fix $q = 2^e$ and write $q = 3m + r$, with $r = 1$ or 2. Since $Z^q = Z^r F^m$, the coefficient polynomial is $\ell_1^q F^m$. Its coefficients lie in J^{q+7m}, and

$$\lceil 10q/3 \rceil - (q + 7m) = \lceil 7r/3 \rceil \leq 5.$$

Thus multiplication by h^5 puts the coefficients in $J^{\lceil 10q/3 \rceil}$. One more h applies Proposition 5.6, giving $h^6 \xi^{[q]} = 0$ in $F_T^e E_T$ for all large q, by tensor Frobenius base change. Hence ξ is in big closure.

Suppose it had a finite witness. Apply Lemma 5.9 to the three coordinates of its generators and include η in N. The witness persists in $T \otimes_R N$ for this fixed finite R-submodule $N \subset E_R$. Every $\xi^{[q]}$ is in ordinary tight closure there after tensor Frobenius, because its relations are those at subsequent powers. The test element h^D of Lemma 5.8 kills all these images.

Take $q = 4^j$. Here $Z^q = Z F^{(q-1)/3}$, so reading the Z coordinate in $T \otimes_R F_R^{2j} N$ gives

$$h^D \ell_1^q F^{(q-1)/3} \eta^{[q]} = 0 \quad \text{in } F_R^{2j} N \otimes_k k'.$$

The coefficients lie in the k-space $F_R^{2j} N$, where independence of the parameter monomials forces each coefficient to vanish. By Lemma 5.7 this gives $h^D J^{d_q} \eta^{[q]} = 0$, with $d_q = (10q - 7)/3$. The required pair exponent is $d_q + 3$, so we have the relation for every $q = 4^j$.

For $2q$, apply tensor Frobenius once more and use $J^{2d_q+3} \subseteq (J^{d_q})^{[2]}$ from Lemma 5.7. Since $\lceil 20q/3 \rceil = 2d_q + 5$, the fixed multiplier h^{2D} works for both q and $2q$. Thus the same N witnesses pair closure at every large power of two, contradicting Proposition 5.6. We have $\xi \in 0^*_{E_T} \setminus 0^{*\mathrm{fg}}_{E_T}$. Finally apply Lemma 2.1 and (2.2) over the complete ring T to obtain the strict inclusion of test ideals. □

5.6. **Polynomial certificate for the generic comparison.** We compute the comparison map of Lemma 5.2 on U and V over $k = \mathbf{F}_2$. Set $u' = t/(1 + t)$ and $d = (1 + t)^2$. To form $G_t^{[4]}$ in (5.1), replace x, y by x^4, y^4 and keep t fixed. On U take

$$M = \begin{pmatrix} t & x^2 + t(1 + x^6) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & x \\ 0 & 1 \end{pmatrix} O, \qquad O = \sum_{r=0}^{5} t^r O_r.$$

For each O_r the table gives its entries in the form $A + By$ with $A, B \in k[x]$. A dash denotes zero.

r	ij	A	B	r	ij	A	B
0	11	1	–	3	11	$x^5 + 1$	$x^2 + 1$
0	12	$x^3 + 1$	–	3	12	$x^5 + x^3 + x^2$	$x^3 + x^2 + 1$
0	21	–	–	3	21	$x^4 + x^3 + x^2$	1
0	22	1	–	3	22	$x^4 + x$	$x^3 + x^2 + 1$
1	11	$x^4 + x$	–	4	11	$x^5 + x^3 + 1$	$x^2 + 1$
1	12	x^2	x^2	4	12	given below	
1	21	$x + 1$	–	4	21	$x^5 + x^3 + x + 1$	$x^2 + x + 1$
1	22	x^3	–	4	22	$x^5 + x^4 + x^3 + x^2$	$x^3 + x^2 + 1$
2	11	$x^4 + x^3 + x + 1$	–	5	11	–	–
2	12	$x^3 + x^2 + x + 1$	$x^3 + x^2 + 1$	5	12	–	–
2	21	$x^4 + x + 1$	1	5	21	$x^5 + x^2 + x + 1$	$x^2 + x + 1$
2	22	$x^3 + x^2 + 1$	$x^2 + 1$	5	22	$x^6 + x^3 + x$	$x^4 + x^3 + x$

The remaining entry is

$$(A_{4,12}, B_{4,12}) = (x^6 + x^5 + x^4 + x^3 + x^2 + x,\ x^4 + x^3 + x^2 + x + 1).$$

On V the frame relation gives the matrix N/d with

$$N = \operatorname{adj}(G_t^{[4]})\, M\, (dG_{u'}). \tag{5.6}$$

Indeed $\det(G_t) = 1$ and d clears the denominators of $G_{u'}$. Reduction modulo $y^2 + y + x^5$ gives

$$\det O = 1 + t + t^2 + t^7, \qquad \det M = t^2(1 + t + t^2 + t^7).$$

Here is a direct check of the determinants and the poles of N. Represent $t^i x^j y^b$ by (i, j, b), with $b = 0, 1$, adding by toggling triples. To multiply, add the first two indices to get (I, J) and add the last indices. A last sum of zero or one gives one triple. A last sum of two gives $(I, J, 1)$ and $(I, J + 5, 0)$ by the curve equation. Start with $(0, -3, 1)$ for z and $(0, -2, 1), (0, -2, 0)$ for w. Use z^4, w^4, x^4 for $G_t^{[4]}$, and replace $1, t, t^2$ in G_t by $1 + t^2, t + t^2, t^2$ for $dG_{u'}$. Multiplication in (5.6) gives

(i, j) in N	11	12	21	22
# surviving triples	36	26	62	26
$\max\{\, 2J + 5b : (I, J, b)$ survives $\}$	0	0	0	0

All powers of t are nonnegative. The last row makes every surviving monomial regular at P, since $\operatorname{ord}_P(x) = -2$ and $\operatorname{ord}_P(y) = -5$. Elsewhere on V, x is invertible. As d is a unit, N/d is regular on V. The two matrices agree under the frame change and have determinant invertible over K. This gives $\mathcal{E}_{u',K} \simeq \varphi^{2*}\mathcal{E}_K$.

6. The discrepancy in odd characteristic

Fix an odd prime p. We prove Theorem C using a modular curve to construct the required family of bundles. Full level at J makes one fiber trivial. Added levels at ℓ make another fiber geometrically stable. The resulting nonconstant moduli map will force a stable, Frobenius-unstable boundary fiber, while its generic fiber remains periodic under curve Frobenius.

6.1. **The modular curve and its family.** Put $Q = p^3$ and $F = \mathbf{F}_Q(T)$. Pick distinct elements $0, 1, j, \ell, o \in \mathbf{F}_Q$. Set

$$K = \mathbf{F}_{Q^2}(T), \qquad D = K \oplus Ku, \qquad u^2 = T(T - 1), \quad ua = a^Q u \quad (a \in \mathbf{F}_{Q^2}).$$

The cyclic quaternion algebra D is ramified at $0, 1$ and split elsewhere, including infinity. At a rational finite place the constant quadratic extension is unramified of degree two. The invariant is therefore half the valuation of $T(T - 1)$ modulo $\mathbb{Z}$. This calculation also works in characteristic two. Fix the maximal order $\mathbf{F}_{Q^2}[T] \oplus \mathbf{F}_{Q^2}[T]u$ away from infinity and extend it to an order sheaf on $\mathbf{P}^1$.

Let J be the place $T = j$, and write

$$C = \operatorname{Ell}_{J,o}$$

for the characteristic-o fiber of the fine moduli curve of D-elliptic sheaves with full level J, modulo index shift. It is smooth and projective by [Pap08, Theorem 3.1]. We need the component calculation at other full levels as well. With the level-group action understood, [Pap08, proof of Proposition 3.2] gives

$$\pi_0(\operatorname{Ell}_{I,\bar{o}}) = \mathbf{F}_Q^\times \backslash \mathcal{O}_I^\times. \tag{6.1}$$

For $I = J$ there is just one component, so C is geometrically connected. Keep this level and add full level $n\ell$. With $t = T - \ell$ and $B_n = \mathbf{F}_Q[t]/(t^n)$ we get the tower

$$X_n = \operatorname{Ell}_{J+n\ell,o} \longrightarrow C.$$

Lemma 6.1. *Each $X_n \to C$ is a finite étale $\mathrm{GL}_2(B_n)$-torsor. The compatible tower gives a continuous representation*

$$\rho : \pi_1(C) \longrightarrow \mathrm{GL}_2(\mathbf{F}_Q[[t]])$$

whose geometric image is exactly $\mathrm{SL}_2(\mathbf{F}_Q[[t]])$.

Proof. Since ℓ avoids the characteristic, pole and ramification places, the restriction at $n\ell$ is an étale τ-module. Use $D_{n\ell} \simeq M_2(B_n)$. Morita equivalence identifies a full level with a Frobenius-compatible frame of a rank-two B_n-module. The group $\mathrm{Res}_{B_n/\mathbf{F}_Q} \mathrm{GL}_2$ is smooth and connected, with reduction GL_2 and successive additive congruence quotients. Its Lang map therefore gives a finite étale frame torsor [Lan56]. The group of $\mathbf{F}_Q$-points is $\mathrm{GL}_2(B_n)$. Full level at J kills automorphisms, so this is a torsor of schemes.

By (6.1) its geometric components are

$$\mathbf{F}_Q^\times \backslash (\mathbf{F}_Q^\times \times B_n^\times) \simeq B_n^\times, \qquad [(a,b)] \longmapsto b/a.$$

Changing the $n\ell$-frame acts by determinant on b/a. Thus the component stabilizer, and hence the geometric monodromy, is $\mathrm{SL}_2(B_n)$. Pick compatible frames to define ρ. Its geometric image is closed and surjects onto $\mathrm{SL}_2(B_n)$ for every n, so it is the full inverse limit. □

In particular $g(C) \geq 2$, since geometric étale fundamental groups in genera zero and one are respectively trivial and abelian. The residual representation is absolutely irreducible. Indeed, the upper and lower elementary transvections in $\mathrm{SL}_2(\mathbf{F}_Q)$ have different unique invariant lines. For the genus, apply [Pap09, Theorem 5.3] with one rational full level and two rational ramified places. We obtain

$$2 - 2g(C) = -2Q(Q-1)^2, \qquad g(C) = 1 + Q(Q-1)^2.$$

6.1.1. *The algebraic family and its boundary.* Set $k = \mathbb{F}_{Q^2}$ and $U = \mathbb{P}^1_k \setminus \{0, 1, o, \infty\}$. We now fix the curve C_k and vary T in U.

Lemma 6.2. *There is a rank-two bundle $\mathcal{E}$ on $C_k \times U$ with trivial underlying determinant and with an isomorphism*

$$(F_{C_k}^{Q^2} \times \mathrm{id}_U)^* \mathcal{E} \simeq \mathcal{E}. \tag{6.2}$$

Here $F_{C_k}^{Q^2}$ raises functions on the curve over k to the Q^2th power. The map in (6.2) fixes T. At $T = \ell$ the completed bundle agrees with the rank-two bundle from the full-level tower up to an inner Morita change and an arithmetic scalar Frobenius twist. Every geometric fiber is semistable. The generic fiber is geometrically stable. The moduli map

$$f : U \longrightarrow \mathrm{SU}_{C_k}(2, \mathcal{O})$$

is nonconstant and extends uniquely over $\mathbb{P}^1_k$. The image of the extended map is a projective curve.

Proof. Let $\mathcal{M}_i$ be the rank-four bundles with right D-action on $\mathbb{P}^1_T \times C$ in the elliptic-sheaf diagram. Its maps

$$j_i : \mathcal{M}_i \longrightarrow \mathcal{M}_{i+1}, \qquad t_i : (\mathrm{id}_{\mathbb{P}^1} \times F_C^Q)^* \mathcal{M}_i \longrightarrow \mathcal{M}_{i+1}$$

are D-linear, their squares commute, and their cokernels are supported on $\infty \times C$ and $o \times C$, respectively [Pap08, §3]. Both maps are therefore invertible on $U \times C$. The j_i identify the restrictions of the $\mathcal{M}_i$ and all the maps $j_i^{-1} t_i$. These identifications respect composition of integer shifts. Thus they give a bundle $\mathcal{M}$ with curve-Frobenius isomorphism on the fine moduli problem modulo index shift.

Over k the maximal order on U becomes a matrix algebra by

$$\alpha \longmapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^Q \end{pmatrix}, \qquad u \longmapsto \begin{pmatrix} 0 & T(T-1) \\ 1 & 0 \end{pmatrix}.$$

An element of $k \setminus \mathbf{F}_Q$ gives the diagonal idempotents, and u gives the off-diagonal units since $T(T-1)$ is invertible on U. Apply the Frobenius isomorphism twice so that it fixes k and e_{11}. Morita equivalence then gives the rank-two bundle $\mathcal{E} = \mathcal{M}e_{11}$ and (6.2). At ℓ any two matrix trivializations over the complete DVR differ by an inner automorphism. Their Morita modules differ by a free coefficient line, and an arithmetic scalar twist has trivial geometric monodromy. This proves the completion statement.

The residue field at a closed point of U is finite. Thus some iterate of the curve-Frobenius isomorphism is an absolute Frobenius isomorphism on that fiber. The degree is zero, and a positive-slope subbundle would acquire unbounded slope inside a fixed bundle. This proves geometric strong semistability there. Openness gives semistability on all of U.

Now consider the determinant map $U \to \mathrm{Pic}^0(C_k)$. It extends over $\mathbb{P}^1_k$ by properness and is constant since the Albanese of $\mathbb{P}^1_k$ is zero. Full level at J makes its value at j trivial. Put $D_0 = \det \mathcal{E}$ and let $\pi_U : C_k \times U \to U$. Each geometric restriction of D_0 has a one-dimensional space of sections. Cohomology and base change makes $\pi_{U*}D_0$ a line bundle, with evaluation $\pi_U^*\pi_{U*}D_0 \to D_0$ an isomorphism on every fiber. Hence it is an isomorphism everywhere. Since $\mathrm{Pic}(U) = 0$, the determinant is trivial over k.

We can now compare the moduli points at j and ℓ. The J-level frame respects the index shifts and gives

$$\mathcal{E}_j = (D_J \otimes \mathcal{O}_{C_k})e_{11} \simeq \mathcal{O}_{C_k}^{\oplus 2}.$$

To prove stability at ℓ, pass to a connected proper finite étale cover trivializing the residual local system. A line subbundle of nonnegative degree would pull back to a constant line in the trivial bundle. Descent would make it $\mathrm{SL}_2(\mathbb{F}_Q)$-invariant, contrary to the transvection calculation. Thus $\mathcal{E}_\ell$ is geometrically stable, and so is the generic fiber by openness. Since $\mathcal{E}_j$ is strictly semistable, f is nonconstant. The projectivity results [Ses82, Théorème 17], [Hof12, §2 and Proposition 2.1] extend f uniquely over $\mathbb{P}^1_k$ by the valuative criterion. □

Proposition 6.3. *For at least one*

$$a \in \{0, 1, o, \infty\},$$

the point $f(a)$ is stable and its pullback by $F_{C_k}^{Q^2}$ is unstable. Put $A = \widehat{\mathcal{O}}_{\mathbb{P}^1_k, a}$ and $K = \mathrm{Frac}(A)$. There is a rank-two bundle $\mathcal{E}_A$ on $C_k \times \mathrm{Spec}(A)$ with trivial determinant. Its special fiber represents $f(a)$. Its generic fiber is isomorphic over K to the base change of $\mathcal{E}_{k(T)}$.

Proof. Let $\mathcal{V}$ be pullback by $F_{C_k}^{Q^2}$ on $\mathrm{SU}_{C_k}(2, \mathcal{O})$, defined where the pullback is semistable. For the ample determinant-of-cohomology polarization Θ of [Hof12, Theorem 3.7], we have on this domain

$$\mathcal{V}^*\Theta \simeq \Theta^{\otimes Q^2}.$$

Here is the degree calculation we need. Put $\Phi = F_{C_k}^{Q^2}$, $b = Q^2$, and pick a line M_0 of degree $g-1$ after finite constant extension if necessary. Projection formula identifies the determinants of cohomology of $\Phi^*E \otimes M_0$ and $E \otimes \Phi_* M_0$. The latter coefficient bundle has rank b and degree $b(g-1)$. Thus $[\Phi_* M_0] - b[M_0] = [M] - [\mathcal{O}]$ in $K_0(C)$ for a degree-zero line M. Write M as a divisor and use its point exact sequences. The extra determinant factors involve $\det E$ and cancel in a family with fixed trivial determinant class, since the divisor has degree zero. The remaining constant line contributes degree zero. This works also in characteristic two and requires only the determinant class.

Suppose all four boundary values lay in the domain of $\mathcal{V}$. Periodicity gives $\mathcal{V} \circ f = f$ on U, and then on $\mathbb{P}^1_k$. Taking degrees would give

$$\deg f^*\Theta = Q^2 \deg f^*\Theta,$$

contrary to $\deg f^*\Theta > 0$, since f is nonconstant and Θ is ample. Pick a boundary value with unstable pullback. It is stable, since a strictly semistable rank-two bundle of degree zero is an extension of degree-zero lines. Every Frobenius pullback of that extension is still semistable.

To lift this point over A, pull back the stable moduli gerbe along $\operatorname{Spec}(A) \to \mathrm{SU}_{C_k}(2, \mathcal{O})$. We fix the trivial determinant class. Allowing scalar automorphisms makes it a $\mathbb{G}_m$-gerbe. It is neutral because

$$\operatorname{Br}(A) = \operatorname{Br}(k) = 0$$

for a complete DVR with finite residue field [Mil80, Chapter IV, Corollaries 2.12–2.13]. Choose a lift $\mathcal{E}_A$. Its determinant comes from A and is trivial since $\operatorname{Pic}(A) = 0$. Over K it and the original bundle represent the same stable point. Their isomorphism sheaf is a $\mathbb{G}_m$-torsor, trivial since $H^1(K, \mathbb{G}_m) = 0$. Thus the lift has exactly the required generic and special fibers. □

6.2. **Long Frobenius torsion from one comparison map.** Use the family of proposition 6.3, with special fiber E_0 and uniformizer t of A. Pick a sufficiently high iterate φ of the periodic curve Frobenius to fix its finite field of definition. Write B for its degree. It fixes t. Geometric stability makes the generic endomorphisms scalar, so $H^0(C_A, \mathcal{H}om(\varphi^*\mathcal{E}, \mathcal{E}))$ has rank one over A. It is finite by properness and torsion-free since the Hom sheaf is locally free. Pick a generator ψ of this free rank-one module.

Lemma 6.4. *The reduction ψ_0 has rank one. Let $L \subset E_0$ be its saturated image. Then $\deg L < 0$, and the restriction $\varphi^*L \to L$ is nonzero with a positive zero divisor. In particular $\chi = \psi \circ \varphi^*\psi$ has nonzero reduction and its reduction vanishes at some closed point P of C.*

Proof. The multiplication-by-t sequence injects the reduction of the Hom module into sections of the reduced sheaf. Hence $\psi_0 \neq 0$. Rank two would make its determinant a nonzero constant and identify the unstable φ^*E_0 with E_0. Thus ψ_0 has rank one. Stability gives $\deg L < 0$, while the saturated kernel has positive degree. If φ^*L lay in that kernel, the two saturated lines would be equal, contradicting their degrees. Its map to L is therefore nonzero and has zero divisor of degree $(B-1)(-\deg L) > 0$. The reduction of χ factors through this map, proving the claim. □

Fix a point P of this divisor. Enlarge constants and iterate χ until P and a local parameter are defined over the fixed finite field. Keep the name χ and write b for the new Frobenius degree. The parameter then pulls back to its bth power. At the generic point each new factor acts nontrivially on the relevant line, so the reduction remains nonzero with saturated image L. It still vanishes at P.

Lemma 6.5. *Every nonzero rank-one map from a vector bundle to the stable degree-zero bundle E_0 induces a nonzero map on $H^1(C, -)$.*

Proof. Let L be the saturated image. The maps from the domain to its image and then to L are surjective on H^1. Indeed, the kernel has no H^2 on a curve and the saturation quotient is torsion. Thus the image we want is $\operatorname{im}(H^1(L) \to H^1(E_0))$. Stability and Riemann–Roch give $h^1(E_0) = 2g-2$, where $g = g(C)$. Also $h^1(L^{-1}) = h^0(\omega_C L) \leq g-1$ since $\deg(\omega_C L) \leq 2g-3$, by Clifford's theorem in the special case and Riemann–Roch otherwise. The sequence $0 \to L \to E_0 \to L^{-1} \to 0$ now gives image dimension at least $g - 1 > 0$. □

Theorem 6.6. *Fix $r > 2g-2$ and $K \geq 1$. For arbitrarily large Frobenius powers $q = p^e$ there is a surjection*

$$H^1(C_A, F_{\mathrm{abs}}^{e*}\mathcal{E}(rP)) \twoheadrightarrow A/(t^{Kq}).$$

The module on the left is finite torsion over A. The point P and the twist rP are independent of e.

Proof. Form the length-n product

$$\chi_n = \chi\, \varphi_b^*\chi \cdots \varphi_b^{(n-1)*}\chi,$$

where φ_b fixes t. In a coefficient of t^j, at most j factors contribute positive powers of t. At least $n-j$ remaining factors vanish at P, with distinct Frobenius weights. Every matrix entry of that coefficient therefore vanishes to order at least

$$1 + b + \cdots + b^{n-j-1} = \frac{b^{n-j}-1}{b-1}.$$

For fixed K, r, choose n so that this order is at least r whenever $j < K$. Dividing by the equation of rP modulo t^K gives

$$\varphi_b^{n*}\mathcal{E}(rP) \longrightarrow \mathcal{E}/t^K\mathcal{E}.$$

Its reduction has rank one and acts nontrivially on H^1 by lemma 6.5. Apply the free-target argument of lemma 5.3, now modulo t^K. Since $H^0(E_0) = 0$, $H^1(C_A, \mathcal{E})$ is free. Base change lifts a closed-fiber class whose image has a unit coordinate, giving a quotient $A/(t^K)$.

Set $q = b^n$. The flat coefficient change $t \mapsto t^q$ converts curve Frobenius to absolute Frobenius and takes this quotient to $A/(t^{Kq})$. The external twist rP stays fixed. As in lemma 5.2, generic periodicity gives geometric strong semistability. Over the geometric generic field, the two Frobenius maps differ by an automorphism of the constants, which preserves slopes. Thus $r > 2g-2$ gives vanishing of generic H^1 by duality. The integral H^1 is finite by properness, hence torsion. □

6.3. **A pair discrepancy in every odd characteristic.** Put $\delta = g-1$. After finite constant extension, pick a theta characteristic θ and a rational point Q_0. We use only $\theta^2 \simeq \omega_C$ and $\deg\theta = \delta$. Neither effectivity of θ nor $\omega_C = \mathcal{O}_C(2\delta P)$ is required.

6.3.1. *Uniform slope bounds and the section ring.*

Lemma 6.7. *The slopes of $\mathrm{Sym}^j E_0$ and of its dual lie between $-j\delta/2$ and $j\delta/2$.*

Proof. Let L_+ be the positive Harder–Narasimhan line at the first unstable Frobenius pullback of E_0. Its second fundamental form is nonzero. Otherwise Cartier descent would give a positive-degree line in the preceding semistable bundle [Kat70, Theorem 5.1]. The map $L_+ \to L_+^{-1} \otimes \omega_C$ therefore gives $2\deg L_+ \leq 2\delta$. Further pullbacks preserve this two-line Harder–Narasimhan filtration, since any other line maps to the negative quotient.

At this first unstable stage, $\mathrm{Sym}^j E_0$ pulls back to a bundle with line factors L_+^{j-2i}, $0 \leq i \leq j$. Its maximal slope is at most $j\delta$. Divide by the Frobenius degree, which is at least p, to bound the original slopes by $j\delta/p \leq j\delta/2$. Apply the same argument to the dual for the lower bound. □

Set

$$Y = \mathbf{P}_{C_A}(\mathcal{E}), \qquad L = \mathcal{O}_Y(1) \otimes \pi^*\theta^4, \qquad S = \bigoplus_{d\geq 0} H^0(Y, L^d).$$

With $\pi_*\mathcal{O}_Y(j) = \mathrm{Sym}^j \mathcal{E}$ we have $\omega_{Y/A} = \mathcal{O}_Y(-2)\theta^2$. The twist θ^4 gives relative ampleness, global generation and

$$H^1(Y, L^j) = 0 \ (j \geq 1), \qquad H^2(Y, L^j) = 0 \ (j \geq 0). \tag{6.3}$$

These assertions follow as in lemma 5.4, with the following slope estimates. At the first unstable stage of degree q, both line factors become positive after twisting by θ^{4q}. This proves ampleness, which descends under finite Frobenius. For every geometric point Q we have $\mu_{\min}(E_0\theta^4(-Q)) \geq 7\delta/2 - 1 > 2\delta$. For $j \geq 1$, the minimum slope of $\operatorname{Sym}^j E_0\theta^{4j}$ is at least $7j\delta/2 > 2\delta$. Duality gives the required curve H^1 vanishings. Projective-bundle cohomology gives $H^2(Y, L^j) = 0$ for $j \geq 0$. Base change and Nakayama give generation and vanishing over A.

We also need a generically separable map defined by three sections. First L_0 is very ample. For a length-two subscheme $Z \subset C$, the dual of $H^1(E_0\theta^4(-Z))$ is $H^0(E_0^*\theta^{-2}(Z))$. Stability and slope $2 - 2\delta \leq 0$ make this zero, including in the equality case. Hence $E_0\theta^4$ separates length-two subschemes, and its tautological line separates points and tangents on the ruled surface.

Pick a general triple h, a, b on this surface. It has no common zero. Very ampleness makes first-jet evaluation onto the three-dimensional jet space at each point. The three jets form a matrix, and rank at most one has codimension four. Varying the point on a surface still leaves a proper bad locus of triples. Thus we may require differential rank at least one everywhere. Require a nonzero jet determinant at one fixed point for generic étaleness, and h nonzero over Q_0. These open conditions have a solution after finite constant extension. Lift the sections using $H^1(Y_0, L_0) = 0$. Their basepoint and rank-zero loci are proper over A and miss the closed fiber, hence are empty. The map $Y \to \mathbf{P}^2_A$ is finite because an ample L cannot be trivial on a positive-dimensional fiber. Thus S is finite over $A[h, a, b]$.

Now apply the Koszul argument of lemma 5.4 using (6.3). It gives

$$S_d = (h, a, b)S_{d-1} \quad (d \geq 3), \qquad S_{m+2} \subseteq (h, a, b)^m. \tag{6.4}$$

We will use this bound for the coefficients. Normal generation in degree one is unnecessary. The full section ring is normal by the argument of lemma 5.4. Put

$$R = \widehat{S_{(t,S_+)}}, \qquad \mathfrak{j} = (h, a, b)R.$$

This is a complete excellent F-finite normal domain of dimension four, finite over $k[[t, h, a, b]]$. The regularity argument of lemma 5.4 applies to R_h. Before completion S_h is Laurent polynomial over the regular affine Y_h, and the formal fibers are regular by excellence. Going down gives height two for two independent linear combinations of h, a, b and height three for $\mathfrak{j}$.

6.3.2. *The marked cohomology class.* Write $\mathcal{D}_j = (\operatorname{Sym}^j \mathcal{E})^\vee$. Since h is nonzero over Q_0, take $h = X$ as one local fiber coordinate and Y_f as the other. For a parameter x at Q_0 use the principal part

$$x^{-2\delta} X^{-3} Y_f^{-3}.$$

It gives a class

$$\eta_0 \in H^2(Y, \omega_{Y/A} L^{-4}) = H^1(C_A, \mathcal{D}_4\theta^{-14}).$$

Its reduction is nonzero by the principal-parts sequence. Indeed, after adding $2\delta Q_0$ the special maximum slope is at most $2\delta - 14\delta + 2\delta = -10\delta < 0$, so the boundary is injective. The untwisted-by-the-pole special bundle also has zero H^0. As in section 5.3, its integral H^1 is therefore free and η_0 is primitive.

Write W_S for the graded canonical module and W for its completed localization. The punctured-cone calculation in section 5.3 gives

$$(W_S)_j = H^0(Y, \omega_{Y/A} L^j).$$

Top local duality gives

$$H^4_{\mathfrak{m}}(W)^\vee = \operatorname{Hom}_R(W, W) = R, \qquad H^4_{\mathfrak{m}}(W) = E_R.$$

Indeed the only hyper-Ext term in total degree -4 is $\mathrm{Hom}(W, H^{-4}(D_R))$, and normality gives $\mathrm{End}_R(W) = R$. Thus Cohen–Macaulayness is unnecessary here. Apply $H^1_{(t)}$ to relative top local cohomology. Primitivity gives

$$\eta = \eta_0/t \in E_R.$$

Theorem 6.8. *With this notation,*

$$\eta \in 0_{E_R}^{*\mathfrak{j}^{7/2}} \setminus 0_{E_R}^{*\mathrm{fg},\mathfrak{j}^{7/2}}.$$

Proof. Fix $q = p^e$ and $m_q = \lceil 7q/2 \rceil = (7q+1)/2$. The ratio $14/4$ of the base twists determines this exponent. Multiplying $F^e\eta_0$ by a degree-m_q product and by h gives second cohomology of

$$\mathcal{O}_Y(-6q + m_q + 1)\theta^{-14q+4m_q+4} = \mathcal{O}_Y(-6q + m_q + 1)\theta^6.$$

When the fiber degree is at least -1, this cohomology vanishes. Otherwise it is H^1 of a dual symmetric power twisted by θ^6. Generic periodicity passes to the symmetric power and its dual, so the slope argument of lemma 5.2 gives geometric semistability. Since $6\delta > 2g - 2$, generic H^1 vanishes. The integral module is t-torsion and disappears under $H^1_{(t)}$.

The Čech comparison is the one in proposition 5.6. On the punctured cone, tensor Frobenius of the canonical module is its qth power. The kernel and cokernel of its map to the reflexive hull have support on the one-dimensional vertex section. They contribute no fourth local cohomology. Since top Čech cohomology commutes with tensor Frobenius, we obtain

$$h\mathfrak{j}^{m_q}\eta^{[q]} = 0 \qquad (q \gg 0). \tag{6.5}$$

The degree-m_q products generate $\mathfrak{j}^{m_q}$ also after completion, so the multiplier h works for big pair closure.

For a finite witness, we will test by a power of h. Fix $D \geq 1$ and put

$$d_q = m_q + 2 = \frac{7q+5}{2}, \qquad n_{q,D} = 6q - d_q - D - 2 = \frac{5q-9}{2} - D,$$
$$M_{q,D} = H^1(C_A, \mathcal{D}_{n_{q,D}}\theta^{4D+10}).$$

For large q, the coefficient map

$$S_{d_q} \longrightarrow M_{q,D}, \qquad g \longmapsto h^D g F^e\eta_0 \tag{6.6}$$

is surjective. To see this, use $h = X$ near Q_0. For each $0 \leq j \leq n_{q,D}$ the monomial

$$X^{3q-1-D-j}Y_f^{3q-1-n_{q,D}+j}$$

has degree d_q and takes $X^DX^{-3q}Y_f^{-3q}$ to the jth fiber-cohomology basis vector. Both exponents are nonnegative for large q. To prescribe the coefficients along $2\delta qQ_0$, subtract that divisor from the source bundle. Its special minimum slope is at least

$$\delta\left(\frac{7}{2}d_q - 2q\right) = \delta\frac{41q+35}{4} > 2\delta.$$

Thus global sections give all the coefficient jets. Adding the same pole on the target gives special minimum slope at least

$$\delta\left(-\frac{1}{2}n_{q,D} + 4D + 10 + 2q\right) = \delta\left(\frac{3q+49}{4} + \frac{9}{2}D\right) > 2\delta.$$

Both bounds exceed $2g - 2$. Duality, reduction and Nakayama give the two integral H^1 vanishings, and the divisor sequences prove (6.6). Every such g lies in $\mathfrak{j}^{m_q}$ by (6.4).

We next apply the multiplication argument of proposition 5.6, with $b_q = n_{q,D} - q \geq 0$ and twist θ^2. Riemann–Roch gives generic Euler characteristic $(b_q + 1)\delta > 0$, so pick a nonzero section of

$\operatorname{Sym}^{b_q}\mathcal{E}\otimes\theta^2$. Clear denominators and divide out the largest common power of t. It is now nonzero on both fibers and its zero scheme is finite over A, being proper with finite fibers. Multiply the Frobenius monomials by this section to get

$$F_{\mathrm{abs}}^{e*}\mathcal{E}\otimes\theta^{-2}\longrightarrow\operatorname{Sym}^{n_{q,D}}\mathcal{E}.$$

Off the zero scheme the two multiplied monomials are independent. Dualizing, using $\mathcal{E}^\vee\simeq\mathcal{E}$, and twisting by θ^{4D+10} gives a surjection

$$M_{q,D}\twoheadrightarrow H^1(C_A,F_{\mathrm{abs}}^{e*}\mathcal{E}\otimes\theta^{4D+12}).$$

because the sheaf cokernel has finite support over A and hence zero H^1, while the kernel has zero H^2. For this D, pick $r>2g-2$ so large that $\mathcal{O}_C(rP)\theta^{-(4D+12)}$ has a nonzero section. Multiplication gives another H^1-surjection onto $H^1(F_{\mathrm{abs}}^{e*}\mathcal{E}(rP))$. The twist is fixed before q varies. For every fixed K, theorem 6.6 therefore gives arbitrarily large q with

$$M_{q,D}\twoheadrightarrow A/(t^{Kq}). \tag{6.7}$$

Suppose η had a finite pair witness. Write $\eta_0=w/(hab)^{N_0}$ in relative top cohomology. By lemma 5.5 we can pass to one fixed Čech stage

$$W_S/(t^N,h^N,a^N,b^N)W_S$$

with representative $v_N=t^{N-1}(hab)^{N-N_0}w$, increasing N if needed. These stages and their Frobenius transforms have finite length, so completion and localization preserve their relations. Fix $h^D\in\tau_b(R)$ using regularity of R_h. For $g\in\mathfrak{j}^{m_q}$, the pair relation puts $gv_N^{[q]}$ in ordinary tight closure of zero. Indeed it applies at further powers q' since $q'\lceil(7/2)q\rceil\geq\lceil(7/2)qq'\rceil$. Thus $h^Dgv_N^{[q]}=0$.

As in proposition 5.6, divide by $(hab)^{Nq}$ in relative top Čech cohomology. The h^{Nq},a^{Nq},b^{Nq} terms are boundaries, so we get

$$t^{(N-1)q}\alpha\in t^{Nq}M_{q,D}\qquad\text{for every }\alpha\in M_{q,D},$$

by (6.6). Now choose $K>N$ and then q satisfying (6.7). A class α mapping to 1 would force $t^{(N-1)q}\in t^{Nq}A/(t^{Kq})$, which is impossible. The stage N and test exponent D were fixed before this choice. Hence the finite witness cannot exist. □

6.4. **Transfer to ordinary tight closure.** Write R_0 for the ring of section 6.3 and k_0 for its finite coefficient field. The exponent $7/2$ leads us to a double cover with five linear factors. Adjoin ten independent residue parameters α_i,β_i, $1\leq i\leq5$, and put

$$A'=R_0[\boldsymbol{\alpha},\widehat{\boldsymbol{\beta}]_{\mathfrak{m}R_0[\boldsymbol{\alpha},\boldsymbol{\beta}]}},\qquad k'=k_0(\boldsymbol{\alpha},\boldsymbol{\beta}).$$

Take the double cover

$$\ell_i=h+\alpha_ia+\beta_ib,\qquad F=\prod_{i=1}^{5}\ell_i,\qquad T=A'[z]/(z^2-F). \tag{6.8}$$

These parameters enlarge the coefficient field. The maximal ideal is still the extension of that of R_0.

For $q=p^e$ set $n=(q-1)/2$ and $c=(p-1)/2$. Every base-p digit of n is c. At each digit in a row, the α_i,β_i exponents determine the h-exponent. The multinomial coefficient is nonzero since $c<p$. Thus the coefficient ideal of F^n is

$$\prod_{j=0}^{e-1}(\mathfrak{j}^{5c})^{[p^j]}=\mathfrak{j}^{5(q-1)/2},$$

by repeated application of lemma 5.7. The additional factor ℓ_1^q gives one high digit in the first row. All lower exponents there are less than q, so choices still give distinct parameter monomials. Hence

$$\operatorname{coeff}(\ell_1^q F^{(q-1)/2}) = \mathfrak{j}^{[q]}\mathfrak{j}^{5(q-1)/2} = \mathfrak{j}^{(7q-5)/2} = \mathfrak{j}^{\lceil 7q/2\rceil - 3}. \tag{6.9}$$

For the middle equality, a monomial of degree $(7q-5)/2$ in three generators has an exponent at least q. The resulting exponent is three less than the pair exponent, which we will supply with h^3.

Lemma 6.9. *The ring T is a complete excellent F-finite normal local domain of dimension four. Its localization T_h is strongly F-regular.*

Proof. Before completion, work on $D(h)$. On $\ell_i = 0$ one of a, b is a unit, so solve for α_i or β_i. For two rows we solve in independent parameters. Their quotients are therefore regular localizations of polynomial algebras over $(R_0)_h$. Three rows cannot vanish after coefficient-field localization, since their determinant is nonzero in k' and would force $h = a = b = 0$. Excellence preserves regularity of the base and row quotients under completion. Thus the branch divisor has simple normal crossings on $D(h)$, with at most two rows through a point.

The local cover is étale away from the branch, regular over one row, and of the form $z^2 = xy$ over two rows after an étale unit-root extension. Apply the direct-summand construction of lemma 5.8 with two replacing three. Since p is odd, it proves strong F-regularity on $D(h)$.

Going down from $k'[[t, h, a, b]]$ gives height two for (h, ℓ_i) and for any two distinct rows. Thus the height-one points over h are étale. Each nonunit row has a height-one divisor by the principal ideal theorem. It lies on $D(h)$, has multiplicity one, and occurs in no other row. Hence F has odd valuation and is not a square in $\operatorname{Frac} A'$, so T is a domain. We have checked R_1, and finite flatness over A' gives S_2. This proves normality. The finite monic extension has local closed fiber and gives the stated completeness, dimension, excellence and F-finiteness. □

Theorem 6.10. *For every odd prime p, the ring in (6.8) satisfies*

$$\tau_b(T) \subsetneq \tau_{\mathrm{fg}}(T).$$

It is not F-pure.

Proof. The coefficient-of-z pairing is perfect, with matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on $(1, z)$. Finite coinduction gives $E_T \simeq T \otimes_{A'} E_{A'}$ as in proposition 5.10. Take

$$\xi = z\ell_1\eta \in E_T.$$

For odd q, $\xi^{[q]} = z\ell_1^q F^{(q-1)/2}\eta^{[q]}$. The coefficient identity (6.9) and (6.5) give $h^4\xi^{[q]} = 0$ for all large q. Here h^3 supplies the missing ideal powers and the remaining h is the multiplier from the big pair relation.

Suppose a finite module witnessed this closure. By lemma 5.9 its generators have both coordinates in $N_0 \otimes k'$ for one finite-length R_0-submodule $N_0 \subset E_{R_0}$. Enlarge N_0 to contain η, so the witness persists in $T \otimes_{A'} (N_0 \otimes k')$. By lemma 6.9 and [Sch11, Proposition 3.23(ii),(iii)] a fixed h^D tests ordinary closure over T. Test the Frobenius relation and take its z-coefficient to get

$$h^D\ell_1^q F^{(q-1)/2}\eta^{[q]} = 0 \quad \text{in } F^e_{A'}(N_0 \otimes k').$$

By lemma 5.9 we can write the module on the right as

$$F^e_{A'}(N_0 \otimes k') = F^e_{R_0}(N_0) \otimes_{k_0} k'.$$

also over the imperfect field k'. Its coefficients lie in the finite-dimensional k_0-space $F^e_{R_0}(N_0)$. Independence of the parameter monomials and (6.9) therefore give

$$h^D\mathfrak{j}^{\lceil 7q/2\rceil - 3}\eta^{[q]} = 0 \quad \text{in } F^e_{R_0}(N_0)$$

for all q. The same multiplier then works at exponent $\lceil 7q/2 \rceil$ in the fixed module N_0, contradicting theorem 6.8. Thus $0^{*\mathrm{fg}}_{E_T} \subsetneq 0^{*}_{E_T}$. Matlis duality over the complete ring T recovers submodules of E_T from their annihilators, giving the strict test-ideal inclusion.

Finally, if T were F-pure, purity of the coefficient extension and freeness of the monic cover would descend it to R_0. Descend through completion and localize at the generic homogeneous vertex. The evaluation ideal $\{\phi(F_*1) : \phi \in \mathrm{Hom}(F_*S_K, S_K)\}$ is homogeneous for the fractional grading on F_*S_K. If it is the unit ideal at the vertex, it contains a nonzero constant, yielding a graded splitting. This splits $Y_{k((t))}$ and, after pushing forward along π, its genus-$g \geq 2$ base curve. But $H^0(C, \omega_C^{1-p}) = 0$, which proves the final assertion. □

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