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Odd/Odd Records for a Relative Approximation Error: Three Quadratic Classifications and an Exact Certificate

Elias De Jesús
ORCID: 0009-0007-0190-9143

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Abstract

For a quadratic irrational $\alpha > 1$ and positive *odd* integers m, k, set $H = m\alpha + k$ and $\varepsilon = |m\alpha - k|/(m\alpha + k)$. A state (m, k) is a *strict chronological record* if ε is smaller than at every admissible state of smaller H. We classify the strict records completely for $\alpha = \varphi$, $\alpha = 1 + \sqrt{3}$ and $\alpha = \sqrt{3}$. In each case they are a finite explicit union of orbits of a single unimodular matrix, with signed defect cycles $(-1, +1, -5)$, $(-3, +1, -11)$ and (-2) respectively. The proofs rest on two exact identities that make ε a function of the defect N and the conjugate factor η alone, on a norm cutoff $|N| \leq 18,204,27$ derived from heights, and on a finite exact certificate that decides every remaining comparison. Two phenomena require care and are isolated: at φ the competition between the Fibonacci and Lucas ladders is a genuine tie at leading order, decided by an integer identity $N_B = -5N_A$; and at $\sqrt{3}$ a companion family has *exactly* equal error at larger height, so it is excluded by equality rather than by loss. The results resolve two conjectures of the author's earlier note and correct one assertion there. Nothing here is formally verified or refereed, and the uniform generalisation is left open.

1 Introduction

Let $\alpha > 1$ be irrational and consider pairs (m, k) of positive odd integers. Write

$$H = H(m, k) = m\alpha + k, \qquad \varepsilon = \varepsilon(m, k) = \frac{|m\alpha - k|}{m\alpha + k}. \tag{1}$$

Thus ε is a *relative* error, normalised by the same quantity H that orders the states. Because α is irrational, distinct integer pairs have distinct H, so the states are linearly ordered by height with no ties.

Definition 1.1 (Records)**.** A state (m, k) is a *strict chronological record* if $\varepsilon(m, k) < \varepsilon(m', k')$ for every admissible state (m', k') with $H(m', k') < H(m, k)$. It is a *weak record* if $\varepsilon(m, k) \leq \varepsilon(m', k')$ for every such state. Every strict record is therefore a weak record, and a weak record that is *not* strict is one for which equality holds at some lower state: it ties the running minimum instead of improving it.

Throughout, "records" without qualification means strict records. Section 8 states three complete classifications of them. We also exhibit, at $\alpha = \sqrt{3}$, an explicit infinite family of weak records that are not strict (Theorem 8.3 (ii)); we do *not* claim to classify all weak records, and in particular we do not treat imprimitive weak records.

What this note does. The three classifications are proved, not merely computed to a large height. The architecture is:

1. two exact identities (Lemma 2.1) reduce ε and H to functions of the defect N and the conjugate factor η;
2. imprimitive states are discarded (Lemma 2.3), the admissible unit step is identified (Section 3), and a *height-only* cutoff bounds $|N|$ on every record (Theorem 4.1);
3. the finitely many surviving unit orbits are enumerated exactly inside a bounded lattice rectangle, with no unique-factorisation hypothesis (Proposition 5.1);
4. every cross-orbit comparison of error and of height is reduced to the sign of one algebraic quantity, constant along the orbits (Section 6); and
5. a finite exact check covers the initial segment, beyond which the pattern is proved to repeat (Section 7).

A machine-generated table of every orbit, its classification, and a checkable exclusion witness is given in Appendix A.

Relation to the author's earlier note. This is a companion to [3], pinned at

`commit 10ea1c62c6866990ded0a85a531883841d38d034`
`PDF SHA-256 f36c23ab099a7a0bca1e68dfe699ea02e15ef6e767bbc52428a77b272da247b3`

In that note the $\sqrt{2}$ analogue is proved, while the φ and $1+\sqrt{3}$ chronologies are *computationally verified* with completeness left conjectural, and the note states explicitly that its norm cutoff for those two slopes is assumed rather than derived. Section 9 records precisely which statements are resolved and which one is corrected. (Citations there use the numbering of the pinned PDF: the two conjectures are Conjecture 5.11 and Conjecture 5.13, and the competition remark is Remark 5.10. The repository `README` refers to them as "Conjectures 5.8, 5.10"; that is a `README` error, not a different revision.)

What this note does not do. It does not prove a uniform theorem for general quadratic slopes and congruence classes; Section 12 explains where the present argument uses $t \geq 0$, $\alpha' < 0$ and the odd/odd class specifically, and gives a counterexample to a naive generalisation. It does not determine novelty relative to the literature beyond the limited comparisons in Section 11. Nothing here is machine-checked in a proof assistant, nor refereed.

2 Setting and the two master identities

Let α satisfy $\alpha^2 - t\alpha - s = 0$ with $t, s \in \mathbb{Z}$, let α' be its conjugate, and put

$$D_0 = \alpha - \alpha' > 0, \qquad \xi = k - m\alpha, \qquad \eta = k - m\alpha', \qquad N = \xi\eta = k^2 - tkm - sm^2.$$

N is the integral defect. For the three slopes of this note:

$$\begin{aligned} \varphi: &\quad t = s = 1, &\quad D_0 = \sqrt{5}, &\quad N = k^2 - km - m^2; \\ 1+\sqrt{3}: &\quad t = s = 2, &\quad D_0 = 2\sqrt{3}, &\quad N = k^2 - 2km - 2m^2; \\ \sqrt{3}: &\quad t = 0,\ s = 3, &\quad D_0 = 2\sqrt{3}, &\quad N = k^2 - 3m^2. \end{aligned}$$

In all three, $t \geq 0$ and $\alpha' < 0$, so $\eta > 0$ on every positive state. Set

$$G = 2\alpha\eta^2 - tN. \tag{2}$$

Lemma 2.1 (Master identities)**.** *For every state with $\eta > 0$,*

$$H = \frac{G}{D_0\,\eta}, \qquad \varepsilon = \frac{|N|\,D_0}{G}, \qquad \textit{and in particular} \qquad G > 0. \tag{3}$$

Thus ε and H are determined by the pair (N, η) alone.

Proof. From $\xi = k - m\alpha$ and $\eta = k - m\alpha'$ we get $\eta - \xi = mD_0$ and $\alpha\eta - \alpha'\xi = kD_0$, hence $m = (\eta - \xi)/D_0$ and $k = (\alpha\eta - \alpha'\xi)/D_0$, so

$$H = m\alpha + k = \frac{\alpha(\eta - \xi) + \alpha\eta - \alpha'\xi}{D_0} = \frac{2\alpha\eta - (\alpha + \alpha')\xi}{D_0} = \frac{2\alpha\eta - t\xi}{D_0}.$$

Substituting $\xi = N/\eta$ gives $H = (2\alpha\eta^2 - tN)/(D_0\eta) = G/(D_0\eta)$, and $H > 0$ forces $G > 0$. Then $\varepsilon = |\xi|/H = (|N|/\eta) \cdot D_0\eta/G = |N|D_0/G$. □

Remark 2.2 (The error score, and why height is separate). Since $\varepsilon = |N|D_0/G$, records are the running minima of $|N|/G$, equivalently the running maxima of

$$S := \frac{G}{|N|} = 2\alpha W - t\ \mathrm{sign}(N), \qquad W := \frac{\eta^2}{|N|}.$$

So the *error score* is $2\alpha W$ corrected by a bounded term depending only on the *sign* of N. The height order is a different question and is treated separately (Section 6): by (3), $H = G/(D_0\eta)$ depends on N as well as η, so it is not a function of S, and in particular η-order is *not* height-order across different defects. Example: at $\alpha = \varphi$ the primitive odd/odd states $(3, 1)$ and $(1, 3)$ have $\eta = 3\varphi - 2 \approx 2.8541 < 3.618 \approx \varphi + 2 = \eta$ respectively, while $H = 3\varphi + 1 \approx 5.8541 > 4.618 \approx \varphi + 3$. Any argument that infers one order from the other must be checked.

Lemma 2.3 (Imprimitive states never record)**.** *If $g = \gcd(m, k) > 1$ then $(m/g, k/g)$ is again an odd state, has the same ε, and has strictly smaller H. Consequently no imprimitive state is a strict record, and N may be assumed primitively represented.*

Proof. m, k odd forces g odd, so m/g and k/g are odd and the reduced pair is admissible. Both $|m\alpha - k|$ and $m\alpha + k$ scale by g, so ε is unchanged, while H is divided by $g > 1$. A strict record must beat every state of smaller height, in particular this one, which it cannot. □

The proof of Lemma 2.3 uses that the gcd of two odd numbers is odd. This is special to the odd/odd class; see Section 12.

3 The admissible lattice and the admissible unit

Write $\eta = a + b\omega$ with $\omega = \varphi$ or $\sqrt{D}$ as appropriate, so that the state is recovered by

$$(m, k) = (b,\ a + b) \ \text{ for } \varphi \text{ and } 1 + \sqrt{3}, \qquad (m, k) = (b,\ a) \ \text{ for } \sqrt{3}.$$

Admissibility (m, k both odd) becomes a congruence on (a, b) modulo 2: "a even, b odd" in the first two cases, "a odd, b odd" for $\sqrt{3}$.

Proposition 3.1 (Admissible unit step)**.** *Let u be the fundamental unit of the relevant order and let $U \in GL_2(\mathbb{Z})$ be the matrix of multiplication by u on the basis $(1, \omega)$. Reducing modulo 2 places U in the finite group $GL_2(\mathbb{Z}/2)$ of order 6, so some power U^j fixes the admissibility class; the least such j gives the* admissible unit $\varrho = u^j$*. Multiplication by ϱ preserves admissibility and primitivity, and multiplies η by ϱ while multiplying N by $\delta := \mathrm{N}(\varrho)$. On the column (m, k) it acts by a matrix with positive entries:*

$$\varphi:\ \varrho = \varphi^3 = 1 + 2\varphi,\quad \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix},\ \delta = -1; \qquad 1+\sqrt{3}:\ \varrho = (2+\sqrt{3})^2,\quad \begin{pmatrix} 3 & 4 \\ 8 & 11 \end{pmatrix},\ \delta = +1;$$

$$\sqrt{3}:\ \varrho = 2 + \sqrt{3},\quad \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix},\ \delta = +1.$$

The minimal j can be strictly smaller than the order of U modulo 2, because the class need only be *fixed*, not acted on trivially: at $\sqrt{3}$ the class $(\mathrm{odd}, \mathrm{odd})$ is fixed by the swap $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so $j = 1$ although $U \not\equiv I$. The values are $j = 3, 2, 1$.

Remark 3.2 (φ^3 versus φ^6). The matrix $\left(\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right)$ sends $(1,1) \mapsto (3,5)$ and is the φ^3 step. The matrix $\left(\begin{smallmatrix} 5 & 8 \\ 8 & 13 \end{smallmatrix}\right)$, which appears in [3], sends $(1,1) \mapsto (13,21)$ and is the φ^6 step, i.e. the step *within* a fixed-defect orbit. Both occur below and must not be interchanged.

Lemma 3.3 (Signed transport)**.** *Along an orbit, with ϱ applied p times to a chosen representative,*

$$N_p = \delta^p\, N_{\mathrm{rep}}, \qquad \delta = \mathrm{N}(\varrho). \tag{4}$$

At φ, $\delta = -1$, so the sign *of N alternates along a ϱ-orbit and the fixed-defect orbits are the $\varrho^2 = \varphi^6$ orbits. Taking $|N|$ therefore discards exactly the parity of p relative to the representative. Every comparison below that needs an orientation transports the signed N by* (4)*; magnitudes are used only where the quantity is δ-invariant.*

Also useful later: the ladders have positive entries, so once a state has $m, k > 0$ all its successors do, and their heights increase without bound. Each orbit therefore has a finite initial segment of non-positive states followed by an infinite positive tail.

4 The height-only cutoff

Fix a *comparison ladder*: an admissible orbit of known defect $\pm L$, whose positive members we denote $A, C, \ldots$ in increasing height. For φ and $1+\sqrt{3}$ we take $L = 1$; for $\sqrt{3}$ no admissible state has $|N| = 1$, because odd m, k force $k^2 - 3m^2 \equiv 1 - 3 \equiv 6 \pmod 8$, and we take $L = 2$ with the orbit of $(1,1)$.

Theorem 4.1 (Cutoff)**.** *Let B be a strict record whose height exceeds that of some positive ladder member, and let A, C be consecutive positive ladder members with $H_A < H_B \le H_C$. Put $Y = \eta_A^2$, $n = N_A$, $L = |n|$. Then*

$$|N_B| < \frac{H_C^2\, \varepsilon_A\, (D_0 + t\varepsilon_A)}{2\alpha} =: F(Y, n), \tag{5}$$

where

$$\varepsilon_A = \frac{L\, D_0}{2\alpha Y - tn}, \qquad \frac{H_C}{H_A} = \frac{2\alpha\varrho^2 Y - t\delta n}{\varrho\,(2\alpha Y - tn)}.$$

Consequently $|N_B| \le 18$ for $\alpha = \varphi$, $|N_B| \le 204$ for $\alpha = 1+\sqrt{3}$, and $|N_B| \le 27$ for $\alpha = \sqrt{3}$.

Proof. Write $z = (k - m\alpha)/H$, so $\varepsilon = |z|$. By Lemma 2.1, $\eta = H(D_0 + tz)/(2\alpha)$ and hence

$$|N| = |\xi|\eta = H\varepsilon \cdot \frac{H(D_0 + tz)}{2\alpha} = \frac{H^2\varepsilon(D_0 + tz)}{2\alpha}. \tag{6}$$

The ladder heights increase to infinity, so consecutive members bracket B as stated. Since B is a record, $\varepsilon_B < \varepsilon_A$; also $t \geq 0$ and $|z_B| = \varepsilon_B$, so $D_0 + tz_B \leq D_0 + t\varepsilon_B$, and $x \mapsto x(D_0 + tx)$ is strictly increasing for $x \geq 0$. Applying (6) to B and then $H_B \leq H_C$ gives (5). The displayed formulas for ε_A and H_C/H_A follow from (3) with $\eta_C = \varrho\eta_A$, $N_C = \delta n$. Only heights were compared.

It remains to bound F over the *whole* ladder. Put

$$u = \frac{t}{2\alpha Y} \quad (\text{and } u = \tfrac{1}{2\alpha Y} \text{ at } \varphi, \text{ where } L = t = 1).$$

Two cases. If $t = 0$ then $u = 0$ identically, $\varepsilon_A = LD_0/(2\alpha Y)$ and $H_C/H_A = \varrho$, so (5) collapses to $F = L\varrho^2$, a constant: no bound on u is needed or available, and this is the only case used at $\sqrt{3}$. If $t > 0$ then $0 < u < 1$: at the first ladder member $2\alpha Y > t$ (an exact inequality, checked in the certificate), and Y is multiplied by $\varrho^2 > 1$ at each step while t, α are fixed, so u strictly decreases. The three closed forms with a u in them are used only on $0 < u < 1$. Substituting into (5) and simplifying:

$$\varphi,\ n = -1: \quad F = \varrho^2\Big(1 - \frac{u}{\varrho^2}\Big)^2 \frac{1 + 2u}{(1 + u)^2} < \varrho^2,$$

because $(1 - u/\varrho^2)^2 < 1$ and $(1 + 2u) < (1 + u)^2$ for $u > 0$; and $\varrho^2 = \varphi^6 = 17.9442719\ldots$

$$\varphi,\ n = +1: \quad F = \varrho^2 \frac{(1 + u/\varrho^2)^2}{(1 - u)^2},$$

which is increasing in u, hence decreasing in Y; its maximum over the ladder is attained at the first member with $n = +1$, namely $Y = \varphi^8$, where $F = 18.1960225\ldots < 19$.

$$1 + \sqrt{3}: \quad F = \varrho^2\Big(\frac{1 - u/\varrho^2}{1 - u}\Big)^2,$$

increasing in u, hence maximal at the first ladder member $Y = (2 + \sqrt{3})^2$, where $F = 204.5520518\ldots < 205$.

$$\sqrt{3}: \quad t = 0 \implies F = L\varrho^2 = 2(2 + \sqrt{3})^2 = 27.8564064\ldots < 28.$$

Taking integer parts gives the three bounds. The endpoint inequalities are decided exactly in the certificate; the decimals are display only. □

Remark 4.2 (Why the signs must be separated). At φ the two branches move in *opposite* directions along the ladder: the $n = -1$ values increase $(17.5116\ldots, 17.9435\ldots, 17.9443\ldots)$ while the $n = +1$ values decrease $(18.1960\ldots, 17.9450\ldots, 17.9443\ldots)$, both towards ϱ^2. So F is not monotone in Y, and no finite sample of ladder states bounds the infinite ladder. The separation above does: one branch is bounded uniformly by ϱ^2, the other attains its maximum at an explicit first member.

Combining Theorem 4.1 with Lemma 2.3 and the congruence restrictions on N leaves very little at φ. There $N = k^2 - km - m^2$ is *odd*, since m, k are; $4N = (2k - m)^2 - 5m^2$ forces $4N$ to be a square modulo 5, i.e. $N \equiv 0, 1, 4 \pmod 5$, which among the odd $|N| \leq 18$ leaves $|N| \in \{1, 5, 9, 11, 15\}$; and 3 is inert in $\mathbb{Z}[\varphi]$ (the discriminant 5 is not a square modulo 3), so $3 \mid N$ forces $3 \mid \gcd(m, k)$ and kills $|N| = 9, 15$ by Lemma 2.3. The admissible defects left at φ are therefore $|N| \in \{1, 5, 11\}$, and the four orbits of Appendix A realise exactly these.

5 Exhaustive enumeration of the surviving orbits

Proposition 5.1 (Bounded enumeration, no unique factorisation)**.** *Let $|N| \le B$. Reduce η by powers of ϱ until $1 \le \eta < \varrho$. Then $|\xi| = |N|/\eta \le B$ automatically, so (η, ξ) lies in the rectangle $[1, \varrho) \times [-B, B]$ in the two real embeddings, and since $m = (\eta - \xi)/D_0$ and $k = (\alpha\eta - \alpha'\xi)/D_0$ the integer coordinates of the representative are explicitly bounded. An exhaustive search over that box finds every orbit.*

No class-group or principal-ideal hypothesis is used: this is the classical reduction theory of the generalised Pell equation, see [2], Theorems 1.1 and 2.1. Reduced representatives may have $m \le 0$ or $k \le 0$; they must be retained, since the positive states appear later in the same orbit. The resulting counts are

$$\varphi:\ 4 \text{ orbits at } |N| \le 18, \qquad 1+\sqrt{3}:\ 64 \text{ at } |N| \le 204, \qquad \sqrt{3}:\ 6 \text{ at } |N| \le 27.$$

6 Exact comparisons: error and height

Let A, B be orbits with reduced representatives $r_A, r_B \in [1, \varrho)$ and representative defects n_A, n_B. Consider A at period $p - d$ and B at period p, with offset $d \ge 0$.

Proposition 6.1 (Error comparison)**.** *With $\Gamma := |N_A|\eta_B^2 - |N_B|\eta_A^2$,*

$$\varepsilon_B < \varepsilon_A \iff 2\alpha\Gamma > t\big(|N_A|N_B - |N_B|N_A\big), \tag{7}$$

and $\Gamma = \varrho^{2p-2d}\Gamma_d$ with $\Gamma_d = |N_A|r_B^2\varrho^{2d} - |N_B|r_A^2$ independent of p. Hence $\operatorname{sign}\Gamma = \operatorname{sign}\Gamma_d$ is one exact sign per pair (A, B) and offset d. The right-hand side of (7) is bounded in absolute value by $2t|N_A||N_B|$, so if $\Gamma_d \neq 0$ then (7) is decided by $\operatorname{sign}\Gamma_d$ as soon as

$$2\alpha\,\varrho^{2p-2d}\,|\Gamma_d| \;>\; 2t\,|N_A|\,|N_B|, \tag{8}$$

a strict inequality which first holds at an explicitly computable period $P(A, B, d)$. The threshold starts at $p = d$, where $\varrho^{2p-2d} = 1$.

Proposition 6.2 (Degenerate case $\Gamma_d = 0$)**.** *If $\Gamma_d = 0$ then (7) reduces to the integer condition $0 > t(|N_A|N_B - |N_B|N_A)$ with the defects transported by (4): $N_A = \delta^{p-d}n_A$, $N_B = \delta^p n_B$. Hence*

$$\operatorname{sign}(\varepsilon_B - \varepsilon_A) = \operatorname{sign}\big(t(|N_A|N_B - |N_B|N_A)\big) \in \{-1, 0, +1\}, \tag{9}$$

a three-way outcome. If $\delta = -1$ this depends on the parity of p, so such a pair can give alternating *verdicts; if $t = 0$ the right-hand side vanishes identically and the two states have* exactly equal *error at every period.*

Proposition 6.3 (Height comparison)**.** *With A at period q and B at period $q + d$, put*

$$U = 2\alpha(r_B\varrho^d - r_A)\,r_A\,r_B\varrho^d, \qquad V = \delta^q\,t\big(\delta^d n_B r_A - n_A r_B\varrho^d\big).$$

Then $\operatorname{sign}(H_B - H_A) = \operatorname{sign}(\varrho^{2q}U - V)$. Here U is independent of q, and V depends on q only through δ^q, hence only through the parity of q when $\delta = -1$. If $U \neq 0$ the sign is decided by $\operatorname{sign}U$ once $\varrho^{2q}|U| > |V|$, at an explicitly computable period.

Proof. By (3), $D_0 H = 2\alpha\eta - tN/\eta$, so $D_0(H_B - H_A) = 2\alpha(\eta_B - \eta_A) - t(N_B/\eta_B - N_A/\eta_A)$. Substitute $\eta_A = r_A\varrho^q$, $\eta_B = r_B\varrho^{q+d}$ and the transported defects, then clear the positive factor $\varrho^{-q} r_A r_B \varrho^d$. □

Proposition 6.4 ($U = 0$ is an index shift, not a tie)**.** *$U = 0$ if and only if $r_A = r_B\varrho^d$. With distinct representatives in $[1, \varrho)$ this is impossible: for $d \geq 1$ it forces $r_A \geq \varrho$, and for $d = 0$ it forces $r_A = r_B$, i.e. the same orbit. Moreover $U = 0$ would imply $V = 0$: taking norms of $r_A = r_B\varrho^d$ gives $n_A = \delta^d n_B$, and substituting both identities annihilates V. Hence $U = 0$ denotes the same arithmetic state after an index shift, in contrast to $\Gamma_d = 0$, which does occur between distinct states.*

When the two thresholds are combined one must convert period coordinates: Proposition 6.1 places A at $p - d$ and B at p, while Proposition 6.3 places A at q and B at $q + d$, so $q = p - d$ and the height threshold must be shifted by $+d$ before taking a maximum.

7 From a finite check to the infinite tail

Lemma 7.1 (Error monotonicity)**.** *Along any orbit, ε strictly decreases. Indeed $\varepsilon = |N|D_0/G$ and one unit step gives*

$$G_{\text{next}} - G = 2\alpha\eta^2(\varrho^2 - 1) - tN(\delta - 1). \tag{10}$$

If $\delta = +1$ the second term vanishes and the first is positive, so G increases at every step unconditionally. If $\delta = -1$ the second term is bounded by $2t|N|$ while the first grows like ϱ^{2p}, so once $2\alpha\eta^2(\varrho^2 - 1) > 2t|N|$ it holds at every later step; the finitely many earlier steps are checked directly.

Proposition 7.2 (Only offsets $d \in \{0, 1\}$ matter, in the tail)**.** *Let p_0 be any period such that* (a) *every orbit has a positive state at some period $\leq p_0 - 1$, and* (b) *every pairwise ordering threshold of Proposition 6.3 is already decisive. Then for every orbit B and every $p \geq p_0$, and for each orbit A, the latest A-state strictly below the period-p state of B in height occurs at period p or $p - 1$. If B beats both of those, it beats every earlier state of A.*

Proof. Three ingredients. (i) Normalisation: all representatives lie in $[1, \varrho)$, so for any orbits A, A' one has $r_A\varrho^{p-1} < r_{A'}\varrho^p$, because $r_A/r_{A'} < \varrho$; hence the period-$(p-1)$ state of any orbit precedes the period-p state of any other in the variable η. (ii) Certified height order: the *height* comparison is supplied by Proposition 6.3, pairwise and exactly — not inferred from η (Remark 2.2) — and hypothesis (b) is exactly what makes its sign the constant $\operatorname{sign} U$. (iii) Monotonicity: by Lemma 7.1, a state of orbit A at period $q \leq p - 2$ has larger ε than the period-$(p-1)$ state of A, which by (i) and (ii) lies below B and which B already beats. Hypothesis (a) guarantees that the period-$(p-1)$ state of A exists and is positive, so the window is populated. □

Remark 7.3 (Both hypotheses are needed). Without (a) the conclusion is false, and not only vacuously. At $\alpha = 1 + \sqrt{3}$ take B to be the orbit through $(7, 1)$, of defect -111, at its period $p = 0$, where $H = 8 + 7\sqrt{3} = 20.1244\ldots$. The orbit of defect $+193$ has no positive state at period 0; its first is $(1, 15)$ at period 1, with $H = 16 + \sqrt{3} = 17.7320\ldots < H(7, 1)$. So the latest competitor state below B sits at period $p + 1$, outside the window. There are 13 such pairs at $1 + \sqrt{3}$, all at $p = 0$, and none at φ or $\sqrt{3}$. Every one of them lies inside the region $H \leq 302$ that the unpruned enumeration of Section 7 covers, which is why the global tail start is raised to the least p_0 satisfying (a) and (b) — the values $p_0 = 2, 2, 3$ in the table below.

Consequently the classification of an orbit is determined by finitely many exact signs: for each competitor and each $d \in \{0, 1\}$, the sign of Γ_d, or the three-way rule (9) when $\Gamma_d = 0$, together with the height order of Proposition 6.3. Each orbit receives one of four verdicts: *strict* (beats every earlier competitor at every period parity), *alternating* (beats them at one parity, loses at the other), *equality* (ties exactly with some earlier competitor, hence not a strict record), or *excluded*.

The finite segment. Everything below the period at which all those signs become decisive is covered by *unpruned* enumeration: all odd pairs with H below an exact bound, with no defect restriction. Take for p_0 the least period satisfying both hypotheses of Proposition 7.2: the maximum over all pairs and offsets of the error and ordering thresholds above (hypothesis (b)), raised until every orbit has a positive state at some period $\leq p_0 - 1$ (hypothesis (a)). This gives:

α	tail start p_0	largest pre-tail H	unpruned cap
φ	2	$(29 + 7\sqrt{5})/2 = 22.3262\ldots$	23
$1 + \sqrt{3}$	2	$206 + 55\sqrt{3} = 301.2628\ldots$	302
$\sqrt{3}$	3	$23 + 13\sqrt{3} = 45.5167\ldots$	46

The table is generated by `certificate/tail_table.py`, which also reports the state at which each maximum is attained. The unpruned enumerations below these caps (41, 4173 and 152 admissible states respectively) agree with the claimed families, and the surviving orbits are identified by their actual recurrences, not merely by their defect labels. Larger unpruned brute-force runs were also performed — to $H \leq 3000$ at φ and $1 + \sqrt{3}$ and to $H \leq 2000$ at $\sqrt{3}$, each cross-checked against an independently pruned scan, and the record families compared with their recurrences out to $H \leq 10^{12}$ — but these are supporting verification, not part of the argument.

8 The three classifications

Let $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$.

Theorem 8.1 ($\alpha = \varphi$)**.** *The strict chronological records are exactly the states*

$$(F_{3j+1}, F_{3j+2}), \quad j \geq 0, \qquad \textit{together with} \qquad (L_{6j+4}, L_{6j+5}), \quad j \geq 0,$$

merged in increasing H. The signed defect cycle along the merged sequence is $(-1, +1, -5)$, repeating. The first ladder is the orbit of $(1, 1)$ under φ^3, i.e. under $\left(\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right)$; within it the defect alternates between -1 and $+1$, and the fixed-defect step is $\varphi^6 = \left(\begin{smallmatrix} 5 & 8 \\ 8 & 13 \end{smallmatrix}\right)$.

Theorem 8.2 ($\alpha = 1 + \sqrt{3}$)**.** *The strict chronological records are exactly the three orbits of $\left(\begin{smallmatrix} 3 & 4 \\ 8 & 11 \end{smallmatrix}\right)$ seeded at $(1, 1)$, $(1, 3)$ and $(5, 13)$, merged in increasing H. Their defects are constant along each orbit, equal to -3, $+1$ and -11, and the merged defect cycle is $(-3, +1, -11)$, repeating. Here $\varrho = 7 + 4\sqrt{3}$ and $\delta = +1$.*

Theorem 8.3 ($\alpha = \sqrt{3}$)**.** (i) *The strict chronological records are exactly the orbit of $(1, 1)$ under $\left(\begin{smallmatrix} 2 & 1 \\ 3 & 2 \end{smallmatrix}\right)$, that is*

$$(1, 1), (3, 5), (11, 19), (41, 71), (153, 265), (571, 989), \ldots,$$

all of defect -2. *Here* $t = 0$, *so* $H = \eta$ *and* $h^2\varepsilon = \frac{1}{2}$ *exactly on every strict record, where* $h = H/2$.
(ii) *For each such state* $A = (m, k)$ *the companion* $B = (k, 3m)$ *satisfies*

$$\eta_B = \sqrt{3}\,\eta_A, \qquad N_B = +6, \qquad H_B = \sqrt{3}\,H_A, \qquad \varepsilon_B = \varepsilon_A,$$

and B *is again an admissible primitive state. Moreover* B *lies strictly between* A *and the next strict record* ϱA*:*

$$H_A < \sqrt{3}\,H_A < (2+\sqrt{3})H_A = H_{\varrho A}, \tag{11}$$

so the running minimum of ε *over all states of height* $< H_B$ *is* ε_A *itself. Hence the companion family* $(1,3)$, $(5,9)$, $(19,33)$, $(71,123)$, $(265,459),\ldots$ *consists of* weak *records that are not strict.*

Proof of (ii). $\eta_A = k + m\sqrt{3}$ and $\sqrt{3}\,\eta_A = 3m + k\sqrt{3} = \eta_B$ with $(m_B, k_B) = (k, 3m)$. Norms multiply: $N_B = \mathrm{N}(\sqrt{3})N_A = (-3)(-2) = 6$. Since $t = 0$, $H = \eta$ by (3), so $H_B = \sqrt{3}H_A$; and $\varepsilon = |N|/H^2$ gives $\varepsilon_B = 6/(3H_A^2) = 2/H_A^2 = \varepsilon_A$. Admissibility: k is odd and $3m$ is odd. Primitivity: $k^2 - 3m^2 = -2$ forces $3 \nmid k$ (else $0 \equiv 1 \bmod 3$) and $\gcd(k, m) = 1$, so $\gcd(k, 3m) = 1$. Finally (11) holds because $1 < \sqrt{3} < 2 + \sqrt{3}$, and by part (i) the strict records are exactly the ϱ-orbit of A, so no state of height below H_B has error smaller than ε_A. Since $\varepsilon_B = \varepsilon_A$ and $H_B > H_A$, the state B meets the running minimum without improving it: it is a weak record and not a strict one. □

We emphasise that Theorem 8.3 (ii) exhibits one infinite family of non-strict weak records; it is *not* a classification of all weak records, and imprimitive weak records are not considered.

8.1 The φ tie, resolved by an integer

The $|N| = 1$ admissible states are exactly (F_n, F_{n+1}) with $n \equiv 1 \pmod 3$: η is then a unit $\pm\varphi^n$, the class condition modulo 2 forces $n \equiv 1 \pmod 3$, and F_n and F_{n+1} are both odd precisely when $3 \nmid n(n+1)$. The $|N| = 5$ admissible states are $\eta = \pm\sqrt{5}\,\varphi^n$ with the same congruence on n, since 5 ramifies and $(\sqrt{5}) = (2\varphi - 1)$ is the only ideal of norm 5; their defect is $N = -5(-1)^n$.

For such η we abbreviate $\varepsilon(\eta)$ for the error of the corresponding state, by Lemma 2.1 a function of (N, η).

Theorem 8.4 (The tie and its resolution)**.** *For* $n \equiv 1 \pmod 3$ *put* A *the unit state at* $\eta_A = \varphi^n$ *and* B *the norm-5 state at* $\eta_B = \sqrt{5}\,\varphi^n$. *Then* $\eta_B/\eta_A = \sqrt{5}$, *so* $(\eta_B/\eta_A)^2 = 5 = |N_B|$*: the leading terms cancel exactly and*

$$\varepsilon(\varphi^n) = \frac{\sqrt{5}}{2\varphi^{\,2n+1} - (-1)^n}, \qquad \varepsilon(\sqrt{5}\,\varphi^n) = \frac{\sqrt{5}}{2\varphi^{\,2n+1} + (-1)^n}. \tag{12}$$

Moreover $N_A = (-1)^n$ *and* $N_B = -5N_A$, *so by* (9) B *beats* A *if and only if* $N_A = +1$, *i.e.* n *even. Among* $n \equiv 1 \pmod 3$ *this is* $n \equiv 4 \pmod 6$, *and those states are exactly the Lucas pairs* (L_{6j+4}, L_{6j+5}).

So the competition is decided not by a multiplicative gap but by the bounded term ± 1 in (12), equivalently by the integer identity $N_B = -5N_A$. The normalised representative of the norm-5 orbit has $N_{\mathrm{rep}} = +5$; the states that actually win are those with $N = -5$, reached at the other period parity. This is the "asymptotically degenerate competition" of [3, Remark 5.10].

Remark 8.5 (Method note). Two of the comparisons above are invisible to $|N|$: the alternating verdict of Proposition 6.2 when $\delta = -1$, and the V-term of Proposition 6.3. In the course of this work, asking of each computed quantity which distinctions it discards — and whether a later step needs them — located exactly these two places, and the certificate now transports signed defects by (4) and uses magnitudes only where the quantity is δ-invariant. The mechanism itself is arithmetic: $N_B = -5N_A$, and the exact cancellation in (12).

8.2 Classification summary

α	cutoff	orbits	strict	alternating	equality	excluded
φ	18	4	1	1	0	2
$1+\sqrt{3}$	204	64	3	0	0	61
$\sqrt{3}$	27	6	1	0	1	4

At φ the alternating orbit is the norm-5 (Lucas) family of Theorem 8.4; at $\sqrt{3}$ the equality orbit is the companion family of Theorem 8.3 (ii). Appendix A lists every orbit with its representative, signed defect, verdict, and — for each excluded orbit — a competitor, an offset, the deciding sign and the period from which it is certified.

9 Relation to the earlier note

All references are to the pinned version of [3] identified in Section 1.

Resolved. Conjecture 5.11 (φ-completeness) is Theorem 8.1; Conjecture 5.13 ($(1+\sqrt{3})$-completeness) is Theorem 8.2. The earlier note states that for these two slopes its pruned search range is labelled computational "precisely because the analogous bound is" those conjectures: that is, the norm cutoff was assumed. Theorem 4.1 derives it.

Remark 5.10 of the baseline already identifies the mechanism of the degenerate competition correctly — that $|N| = 5$ is compensated to leading order, that the decision falls to the bounded term in $L_n^2 - 5F_n^2 = 4(-1)^n$, and that its sign selects the odd Lucas pairs with $n \equiv 4 \pmod 6$ — and says explicitly that this is "an observed pattern with an identified finite mechanism, not a proved theorem". Theorem 8.4 proves it, for every admissible n at once, from the closed forms (12); and it places the phenomenon in a general frame, as the case $\Gamma_d = 0$ of Proposition 6.2, where the same three-way integer rule also produces the permanent equality at $\sqrt{3}$.

Corrected. Observation 5.9 of the baseline describes the φ chronology as having "three records per multiplicative period φ^6 in height", and Observation 5.12 says the same of $1+\sqrt{3}$ with period $(2+\sqrt{3})^2$; both are labelled computationally verified. Exact multiplicativity holds for η, not for H: by (3),

$$H_{\text{next}} - \varrho H = \frac{tN\,(\varrho - \delta/\varrho)}{D_0\,\eta}, \tag{13}$$

which vanishes only when $tN = 0$. Concretely, with $\left(\begin{smallmatrix} 3 & 4 \\ 8 & 11 \end{smallmatrix}\right)(1,1) = (7,19)$ and $\varphi^6 \cdot (1,1) = (13,21)$,

$$H(7,19) - \varrho H(1,1) = -8\sqrt{3}, \qquad H(13,21) - \varphi^6 H(1,1) = 4 - 4\sqrt{5},$$

both nonzero. What is true is that $H_{\text{next}}/H \to \varrho$ along each orbit, by (13), since $\eta \to \infty$; the baseline's observed record lists are not in question, only the exactness of the height period. We do *not* assert that every sufficiently large multiplicative height window contains exactly three records; that is a statement about the interleaving of three orbits with different multiplicative phases and would require a separate argument. At $\alpha = \sqrt{3}$, where $t = 0$, one has $H = \eta$ and the height period $2 + \sqrt{3}$ *is* exact (Theorem 8.3).

New. The slope $\sqrt{3}$ does not appear in [3]. It is the cleanest of the three: $t = 0$ collapses the master identities to $H = \eta$ and $\varepsilon = |N|/H^2$, so $h^2\varepsilon$ is exactly $|N|/4$, and it is also the case in which no admissible state has $|N| = 1$, which is why the cutoff is stated for a ladder of arbitrary known defect.

10 The certificate

The finite content of the proofs is isolated in a small standalone bundle : the directory `certificate/` of the companion folder in the repository, documented by its own `README`. It is released under the tag

`congruence-records-v1.0.0`

whose commit and per-file hashes are recorded in `docs/RELEASE_MANIFEST.md`. Two programs are run: `certificate.py` prints the cutoff, the orbit enumeration, the classification and the threshold data; `verify_records.py` runs 31 checks and reports `VERIFY PASSED`. The bundle uses the Python standard library only, reads no data files at runtime, and recomputes everything from the definitions.

All ordering and comparison decisions are exact: arithmetic is carried out in $\mathbb{Z}[\sqrt{D}]$ with integer or rational coordinates, and signs are decided by integer comparisons. Floating point appears only as a presort key — after which the sorted order is re-verified exactly — and for display. The checks include the master identities, the congruence restrictions on N for each slope, the imprimitivity reduction, the cutoff bound against every record found by unpruned search, an independent recount of the orbits by direct rectangle enumeration, the four-way classification against the claimed families, set equality of the record list with the recurrences of Theorems 8.1–8.3, the signed-transport ledger (4) and its invariance under change of representative, and the threshold data of Section 7.

Status of the verification. The certificate is exact arithmetic, not a formal proof: it is not machine-checked in a proof assistant, and this note has not been refereed. Two independent audits of earlier versions of the bundle were carried out with assistance of an automated reviewer; they re-derived the classifications by separate orbit enumeration and tail logic, and they found and prompted the repair of several defects in the written proof and in the implementation, including the incorrect monotonicity justification repaired in Remark 4.2 and a threshold initialisation error. Those audits re-used this bundle's arithmetic module, so they are independent in logic and in control flow but not in the underlying arithmetic implementation. The release history in `docs/CHANGELOG.md` records the versions and what changed.

11 Related work

The objective studied here — a relative error $|m\alpha - k|/(m\alpha + k)$, with the states ordered by the same denominator $m\alpha + k$ — is not one of the standard best-approximation notions, and we make only comparisons that the cited definitions support.

Kim, Lee and Liao [5] introduce the odd–odd continued fraction and prove that the best 1-*rational* approximations of an irrational x — their Definition 1.1: p/q with p, q odd and coprime such that $|qx - p| < |bx - a|$ for every a/b of the same parity type with $0 < b \le q$ — are exactly its principal convergents (their Theorem 1.2). That objective is $|qx - p|$ ordered by q, which is not (1); the two notions select different sets. For instance at $\alpha = \varphi$ the state $(7, 11)$ is a strict record here, while $|3\varphi - 5| = 0.1459\ldots < 0.3262\ldots = |7\varphi - 11|$ with $3 \le 7$, so $11/7$ is not a best 1-rational approximation. A companion paper [6] studies the three parity types and best signed approximations with the same $|qx - p|$ objective. We make no claim that our classifications do or do not follow from that machinery; establishing either would require an argument we have not given.

Hurwitz-type theorems *with* congruence conditions on numerator and denominator are a separate line. Elsner's introduction [4] records it as beginning with Hartman, who first imposed congruence conditions on u and v, with a weaker theorem of Koksma (1951) and with Uchiyama's result the sharpest before his own. Elsner's Theorem 1.1 requires the modulus s to have an odd prime divisor, and he records that the corresponding Corollary 1.1 fails for $s = 2$ with $h = 1$ — that is, for the odd/odd class — giving $\xi = 12 + \sqrt{145}$ as a witness. So the restriction studied here is the acknowledged exceptional case of that line of work. Those results bound an approximation constant; they do not enumerate a record set.

Cha and Kim [1] determine the small values and the smallest accumulation point of the Lagrange spectrum of intrinsic approximation on the unit circle, with height the denominator of a primitive integer vector, and classify the very badly approximable points by Romik digit expansions. That is an asymptotic spectrum on a different space with a different height, and we draw no reduction in either direction. By contrast the accumulation values of $h^2\varepsilon$ along the records here follow immediately from (3): $h^2\varepsilon \to \alpha|N|/(2D_0)$ along a fixed-defect orbit, which at $\sqrt{3}$ is exactly $|N|/4$. These are values *at* records and are not the same quantity as the staircase envelope of $H \mapsto H^2\varepsilon$ between records.

The reduction theory used in Section 5 — finiteness of the orbits of a given norm under the unit group, and the bounded search that exhibits representatives — is classical; see [2], Theorems 1.1 and 2.1. Our contribution is the three exact chronologies for the error and height (1), the two equality mechanisms isolated in Theorems 8.4 and 8.3 (ii), and the effective certificate that makes the passage from a finite computation to the infinite tail explicit.

12 Outlook, with its hypotheses stated

It is natural to ask for a uniform statement: for a quadratic α and a congruence class on (m, k), are the strict records always a finite explicit union of unit orbits beyond an effectively computable height? The present argument does not prove this, and three of its steps use the particular setting.

- **Signs.** Theorem 4.1 uses $t \ge 0$ and $\alpha' < 0$, so that $\eta > 0$ on positive states and $x \mapsto x(D_0 + tx)$ is increasing. Other sign patterns need separate treatment.

- **Imprimitivity.** Lemma 2.3 uses that the gcd of two odd numbers is odd, so dividing it preserves the class. This fails for a general class: at $\alpha = \sqrt{6}$ with the class $(m, k) \equiv (1, 1) \pmod 3$, the state $(4, 10)$ is a strict record within the class although gcd $= 2$, and its reduction $(2, 5)$ leaves the class. A general statement must quantify over primitive states or retain imprimitive bounded-defect orbits.
- **The comparison ladder.** Theorem 4.1 needs an admissible state of *known* defect L. Existence of some primitive seed in an ordinary residue pair $(a, b) \bmod M$ with $\gcd(a, b, M) = 1$ follows from a short construction: choose $m \equiv a$, let P be the product of the primes dividing m but not M, and use the Chinese remainder theorem to choose $k \equiv b \pmod M$ with $k \equiv 1 \pmod P$; common primes of m and M cannot divide b, so $\gcd(m, k) = 1$, and a large enough k makes $\eta > 0$. Its defect is nonzero and computable. Obtaining a *prescribed* or *minimal* $|L|$ is a different and harder question; the values $L = 1, 1, 2$ used here were found by inspection.

Assembling these into a theorem, with the quantifiers in order and the finite region described in general, is left as future work. Three worked slopes do not establish it.

Acknowledgements and AI-assistance disclosure

Anthropic's Claude was used throughout as an assistant: for exact-arithmetic computational design and execution, for drafting and revising this note and the certificate code, and as an adversarial reviewer of both. Several of the mathematical repairs recorded in `docs/CHANGELOG.md` originated in that review process, including the sign-separated cutoff argument of Theorem 4.1, the pairwise height certificate of Proposition 6.3, and the correction of (13). All conceptual choices, and final responsibility for every claim here, belong to the author. No human referee has reviewed this note, and no part of it is formally verified.

References

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[6] D. H. Kim, S. B. Lee and L. Liao. *Diophantine approximation by rational numbers of certain parity types.* Mathematika **71** (2025), e12285. doi:10.1112/mtk.12285. Read in the author version arXiv:2403.12341v1.

A Machine-generated orbit tables

The tables below are generated by `certificate/tables.py` and are reproduced verbatim; the same data is available as CSV in `tables/`.

Orbit identity. Each orbit carries a stable identifier — `P`, `T` or `S` for φ, $1+\sqrt{3}$ or $\sqrt{3}$, followed by its index in the canonical enumeration, which is sorted by the representative η. The identifier is necessary, because N_{rep} is *not* unique: at $1+\sqrt{3}$ thirty values each label two distinct orbits and $N_{\mathrm{rep}} = -143$ labels four, and at $\sqrt{3}$ two values label two each. Here N_{rep} is the defect of the *normalised representative* (reduced into $[1, \varrho)$); by (4) the defect at period p is $\delta^p N_{\mathrm{rep}}$, so at φ the sign alternates. "first positive state" is the first (m, k) with both coordinates positive.

What an exclusion row asserts. A row "B excluded by A at offset d from period P" asserts that for *every* $p \geq P$, the state of A at period $p - d$ lies below the state of B at period p in height and has strictly smaller error. Accordingly P is the maximum of the error threshold (8) in B-period coordinates, the ordering threshold of Proposition 6.3 converted by $+d$, and the first period at which both states are positive. The two states at $p = P$ are printed, so a reader can re-derive any row from Propositions 6.1–6.3 without re-running a search. Every row is re-verified exactly at $p = P, P+1, P+2$ before it is emitted, and check C11 of `verify_records.py` re-derives all 67 rows independently from the orbit each one names and tests them at $p = P, \ldots, P+11$.

Classification of all admissible primitive orbits below the cutoff.

id	α	$\lvert N\rvert$	N_{rep}	η (rep)	first positive state	class
P0	φ	1	-1	1.618034	(1,1)	strict
P1	φ	11	+11	2.381966	(5,7)	excluded
P2	φ	11	-11	2.854102	(3,1)	excluded
P3	φ	5	+5	3.618034	(1,3)	alternating
T0	$1+\sqrt{3}$	143	-143	1.014083	(7,9)	excluded
T1	$1+\sqrt{3}$	39	-39	1.052559	(5,11)	excluded
T2	$1+\sqrt{3}$	73	+73	1.091035	(3,13)	excluded
T3	$1+\sqrt{3}$	193	+193	1.129510	(1,15)	excluded
T4	$1+\sqrt{3}$	131	-131	1.157677	(7,11)	excluded
T5	$1+\sqrt{3}$	11	-11	1.196152	(5,13)	strict
T6	$1+\sqrt{3}$	111	-111	1.301270	(7,13)	excluded
T7	$1+\sqrt{3}$	169	+169	1.378222	(3,17)	excluded
T8	$1+\sqrt{3}$	83	-83	1.444864	(7,15)	excluded
T9	$1+\sqrt{3}$	69	+69	1.483340	(5,17)	excluded
T10	$1+\sqrt{3}$	47	-47	1.588457	(7,17)	excluded
T11	$1+\sqrt{3}$	121	+121	1.626933	(5,19)	excluded
T12	$1+\sqrt{3}$	179	-179	1.693575	(9,17)	excluded
T13	$1+\sqrt{3}$	3	-3	1.732051	(1,1)	strict

continued

id	α	$\lvert N\rvert$	N_{rep}	η (rep)	first positive state	class
T14	$1+\sqrt{3}$	181	+181	1.770527	(5,21)	excluded
T15	$1+\sqrt{3}$	143	-143	1.837169	(9,19)	excluded
T16	$1+\sqrt{3}$	109	+109	2.019238	(7,23)	excluded
T17	$1+\sqrt{3}$	47	-47	2.124356	(9,23)	excluded
T18	$1+\sqrt{3}$	177	+177	2.162831	(7,25)	excluded
T19	$1+\sqrt{3}$	13	+13	2.267949	(9,25)	excluded
T20	$1+\sqrt{3}$	167	-167	2.373067	(11,25)	excluded
T21	$1+\sqrt{3}$	107	-107	2.516660	(11,27)	excluded
T22	$1+\sqrt{3}$	157	+157	2.555136	(9,29)	excluded
T23	$1+\sqrt{3}$	39	-39	2.660254	(11,29)	excluded
T24	$1+\sqrt{3}$	37	+37	2.803848	(11,31)	excluded
T25	$1+\sqrt{3}$	183	-183	2.908965	(13,31)	excluded
T26	$1+\sqrt{3}$	107	-107	3.052559	(13,33)	excluded
T27	$1+\sqrt{3}$	23	-23	3.196152	(3,1)	excluded
T28	$1+\sqrt{3}$	69	+69	3.339746	(13,37)	excluded
T29	$1+\sqrt{3}$	191	-191	3.444864	(15,37)	excluded
T30	$1+\sqrt{3}$	1	+1	3.732051	(1,3)	strict
T31	$1+\sqrt{3}$	109	+109	3.875644	(15,43)	excluded
T32	$1+\sqrt{3}$	191	-191	3.980762	(17,43)	excluded
T33	$1+\sqrt{3}$	83	-83	4.124356	(17,45)	excluded
T34	$1+\sqrt{3}$	33	+33	4.267949	(17,47)	excluded
T35	$1+\sqrt{3}$	157	+157	4.411543	(17,49)	excluded
T36	$1+\sqrt{3}$	183	-183	4.516660	(19,49)	excluded
T37	$1+\sqrt{3}$	59	-59	4.660254	(5,1)	excluded
T38	$1+\sqrt{3}$	73	+73	4.803848	(19,53)	excluded
T39	$1+\sqrt{3}$	167	-167	5.052559	(21,55)	excluded
T40	$1+\sqrt{3}$	121	+121	5.339746	(21,59)	excluded
T41	$1+\sqrt{3}$	143	-143	5.588457	(23,61)	excluded
T42	$1+\sqrt{3}$	13	+13	5.732051	(1,5)	excluded
T43	$1+\sqrt{3}$	177	+177	5.875644	(23,65)	excluded
T44	$1+\sqrt{3}$	111	-111	6.124356	(7,1)	excluded
T45	$1+\sqrt{3}$	61	+61	6.267949	(25,69)	excluded
T46	$1+\sqrt{3}$	71	-71	6.660254	(5,3)	excluded
T47	$1+\sqrt{3}$	23	-23	7.196152	(3,5)	excluded
T48	$1+\sqrt{3}$	181	+181	7.339746	(29,81)	excluded
T49	$1+\sqrt{3}$	179	-179	7.588457	(9,1)	excluded
T50	$1+\sqrt{3}$	33	+33	7.732051	(1,7)	excluded
T51	$1+\sqrt{3}$	131	-131	8.124356	(7,3)	excluded
T52	$1+\sqrt{3}$	97	+97	8.267949	(33,91)	excluded
T53	$1+\sqrt{3}$	169	+169	8.803848	(35,97)	excluded
T54	$1+\sqrt{3}$	11	-11	9.196152	(3,7)	excluded
T55	$1+\sqrt{3}$	61	+61	9.732051	(1,9)	excluded
T56	$1+\sqrt{3}$	143	-143	10.124356	(7,5)	excluded
T57	$1+\sqrt{3}$	141	+141	10.267949	(41,113)	excluded
T58	$1+\sqrt{3}$	71	-71	10.660254	(5,7)	excluded
T59	$1+\sqrt{3}$	97	+97	11.732051	(1,11)	excluded
T60	$1+\sqrt{3}$	193	+193	12.267949	(49,135)	excluded
T61	$1+\sqrt{3}$	59	-59	12.660254	(5,9)	excluded

continued

id	α	$\lvert N\rvert$	N_{rep}	η (rep)	first positive state	class
T62	$1+\sqrt{3}$	37	+37	13.196152	(3,11)	excluded
T63	$1+\sqrt{3}$	141	+141	13.732051	(1,13)	excluded
S0	$\sqrt{3}$	26	-26	1.124356	(5,7)	excluded
S1	$\sqrt{3}$	6	+6	1.267949	(1,3)	equality
S2	$\sqrt{3}$	26	-26	1.660254	(3,1)	excluded
S3	$\sqrt{3}$	22	+22	1.803848	(1,5)	excluded
S4	$\sqrt{3}$	2	-2	2.732051	(1,1)	strict
S5	$\sqrt{3}$	22	+22	3.267949	(3,7)	excluded

Exclusion witnesses.

id	α	N_{rep}	by	N_A	d	P	A at $P-d$	B at P	rule
P1	φ	+11	P0	-1	0	1	(3,5)	(5,7)	$\Gamma_d<0$
P2	φ	-11	P0	-1	0	0	(1,1)	(3,1)	$\Gamma_d<0$
T0	$1+\sqrt{3}$	-143	T30	+1	1	1	(1,3)	(7,9)	$\Gamma_d<0$
T1	$1+\sqrt{3}$	-39	T30	+1	1	1	(1,3)	(5,11)	$\Gamma_d<0$
T2	$1+\sqrt{3}$	+73	T30	+1	1	1	(1,3)	(3,13)	$\Gamma_d<0$
T3	$1+\sqrt{3}$	+193	T30	+1	1	1	(1,3)	(1,15)	$\Gamma_d<0$
T4	$1+\sqrt{3}$	-131	T30	+1	1	1	(1,3)	(7,11)	$\Gamma_d<0$
T6	$1+\sqrt{3}$	-111	T1	-39	0	1	(5,11)	(7,13)	$\Gamma_d<0$
T7	$1+\sqrt{3}$	+169	T1	-39	0	1	(5,11)	(3,17)	$\Gamma_d<0$
T8	$1+\sqrt{3}$	-83	T5	-11	0	1	(5,13)	(7,15)	$\Gamma_d<0$
T9	$1+\sqrt{3}$	+69	T5	-11	0	1	(5,13)	(5,17)	$\Gamma_d<0$
T10	$1+\sqrt{3}$	-47	T5	-11	0	1	(5,13)	(7,17)	$\Gamma_d<0$
T11	$1+\sqrt{3}$	+121	T1	-39	0	1	(5,11)	(5,19)	$\Gamma_d<0$
T12	$1+\sqrt{3}$	-179	T1	-39	0	1	(5,11)	(9,17)	$\Gamma_d<0$
T14	$1+\sqrt{3}$	+181	T1	-39	0	1	(5,11)	(5,21)	$\Gamma_d<0$
T15	$1+\sqrt{3}$	-143	T1	-39	0	1	(5,11)	(9,19)	$\Gamma_d<0$
T16	$1+\sqrt{3}$	+109	T5	-11	0	1	(5,13)	(7,23)	$\Gamma_d<0$
T17	$1+\sqrt{3}$	-47	T5	-11	0	1	(5,13)	(9,23)	$\Gamma_d<0$
T18	$1+\sqrt{3}$	+177	T5	-11	0	1	(5,13)	(7,25)	$\Gamma_d<0$
T19	$1+\sqrt{3}$	+13	T13	-3	0	1	(7,19)	(9,25)	$\Gamma_d<0$
T20	$1+\sqrt{3}$	-167	T5	-11	0	1	(5,13)	(11,25)	$\Gamma_d<0$
T21	$1+\sqrt{3}$	-107	T5	-11	0	1	(5,13)	(11,27)	$\Gamma_d<0$
T22	$1+\sqrt{3}$	+157	T5	-11	0	1	(5,13)	(9,29)	$\Gamma_d<0$
T23	$1+\sqrt{3}$	-39	T13	-3	0	1	(7,19)	(11,29)	$\Gamma_d<0$
T24	$1+\sqrt{3}$	+37	T13	-3	0	1	(7,19)	(11,31)	$\Gamma_d<0$
T25	$1+\sqrt{3}$	-183	T5	-11	0	1	(5,13)	(13,31)	$\Gamma_d<0$
T26	$1+\sqrt{3}$	-107	T5	-11	0	1	(5,13)	(13,33)	$\Gamma_d<0$
T27	$1+\sqrt{3}$	-23	T13	-3	0	1	(7,19)	(13,35)	$\Gamma_d<0$
T28	$1+\sqrt{3}$	+69	T13	-3	0	1	(7,19)	(13,37)	$\Gamma_d<0$
T29	$1+\sqrt{3}$	-191	T5	-11	0	1	(5,13)	(15,37)	$\Gamma_d<0$
T31	$1+\sqrt{3}$	+109	T13	-3	0	1	(7,19)	(15,43)	$\Gamma_d<0$
T32	$1+\sqrt{3}$	-191	T5	-11	0	1	(5,13)	(17,43)	$\Gamma_d<0$
T33	$1+\sqrt{3}$	-83	T13	-3	0	1	(7,19)	(17,45)	$\Gamma_d<0$
T34	$1+\sqrt{3}$	+33	T13	-3	0	1	(7,19)	(17,47)	$\Gamma_d<0$
T35	$1+\sqrt{3}$	+157	T5	-11	0	1	(5,13)	(17,49)	$\Gamma_d<0$
T36	$1+\sqrt{3}$	-183	T5	-11	0	1	(5,13)	(19,49)	$\Gamma_d<0$

continued

id	α	N_{rep}	by	N_A	d	P	A at $P-d$	B at P	rule
T37	$1+\sqrt{3}$	-59	T13	-3	0	1	(7,19)	(19,51)	$\Gamma_d < 0$
T38	$1+\sqrt{3}$	+73	T13	-3	0	1	(7,19)	(19,53)	$\Gamma_d < 0$
T39	$1+\sqrt{3}$	-167	T13	-3	0	1	(7,19)	(21,55)	$\Gamma_d < 0$
T40	$1+\sqrt{3}$	+121	T13	-3	0	1	(7,19)	(21,59)	$\Gamma_d < 0$
T41	$1+\sqrt{3}$	-143	T13	-3	0	1	(7,19)	(23,61)	$\Gamma_d < 0$
T42	$1+\sqrt{3}$	+13	T30	+1	0	0	(1,3)	(1,5)	$\Gamma_d < 0$
T43	$1+\sqrt{3}$	+177	T13	-3	0	1	(7,19)	(23,65)	$\Gamma_d < 0$
T44	$1+\sqrt{3}$	-111	T13	-3	0	1	(7,19)	(25,67)	$\Gamma_d < 0$
T45	$1+\sqrt{3}$	+61	T13	-3	0	1	(7,19)	(25,69)	$\Gamma_d < 0$
T46	$1+\sqrt{3}$	-71	T13	-3	0	1	(7,19)	(27,73)	$\Gamma_d < 0$
T47	$1+\sqrt{3}$	-23	T30	+1	0	0	(1,3)	(3,5)	$\Gamma_d < 0$
T48	$1+\sqrt{3}$	+181	T13	-3	0	1	(7,19)	(29,81)	$\Gamma_d < 0$
T49	$1+\sqrt{3}$	-179	T13	-3	0	1	(7,19)	(31,83)	$\Gamma_d < 0$
T50	$1+\sqrt{3}$	+33	T30	+1	0	0	(1,3)	(1,7)	$\Gamma_d < 0$
T51	$1+\sqrt{3}$	-131	T13	-3	0	1	(7,19)	(33,89)	$\Gamma_d < 0$
T52	$1+\sqrt{3}$	+97	T13	-3	0	1	(7,19)	(33,91)	$\Gamma_d < 0$
T53	$1+\sqrt{3}$	+169	T13	-3	0	1	(7,19)	(35,97)	$\Gamma_d < 0$
T54	$1+\sqrt{3}$	-11	T30	+1	0	0	(1,3)	(3,7)	$\Gamma_d < 0$
T55	$1+\sqrt{3}$	+61	T30	+1	0	0	(1,3)	(1,9)	$\Gamma_d < 0$
T56	$1+\sqrt{3}$	-143	T30	+1	0	0	(1,3)	(7,5)	$\Gamma_d < 0$
T57	$1+\sqrt{3}$	+141	T13	-3	0	1	(7,19)	(41,113)	$\Gamma_d < 0$
T58	$1+\sqrt{3}$	-71	T30	+1	0	0	(1,3)	(5,7)	$\Gamma_d < 0$
T59	$1+\sqrt{3}$	+97	T30	+1	0	0	(1,3)	(1,11)	$\Gamma_d < 0$
T60	$1+\sqrt{3}$	+193	T13	-3	0	1	(7,19)	(49,135)	$\Gamma_d < 0$
T61	$1+\sqrt{3}$	-59	T30	+1	0	0	(1,3)	(5,9)	$\Gamma_d < 0$
T62	$1+\sqrt{3}$	+37	T30	+1	0	0	(1,3)	(3,11)	$\Gamma_d < 0$
T63	$1+\sqrt{3}$	+141	T30	+1	0	0	(1,3)	(1,13)	$\Gamma_d < 0$
S0	$\sqrt{3}$	-26	S4	-2	1	2	(3,5)	(5,7)	$\Gamma_d < 0$
S2	$\sqrt{3}$	-26	S1	+6	0	1	(1,3)	(3,1)	$\Gamma_d < 0$
S3	$\sqrt{3}$	+22	S1	+6	0	1	(1,3)	(1,5)	$\Gamma_d < 0$
S5	$\sqrt{3}$	+22	S4	-2	0	1	(3,5)	(3,7)	$\Gamma_d < 0$