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SUBRINGS OF $\mathbb{Z}^n$ OF PRIME-POWER INDEX AND MODULAR FORMS

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Abstract. Let $f_n(m)$ denote the number of unital subrings of index m in $\mathbb{Z}^n$, with coordinate-wise multiplication, and let $g_n(p^e)$ count those whose reduction modulo p is the diagonal copy of $\mathbb{F}_p$. Voll's uniformity question asks whether the local subring zeta functions can be described by a single rational function in p and p^{-s}. Polynomial formulas for $f_n(p^e)$ are known for $e \leq 9$. We prove that neither $f_{10}(p^{17})$ nor $g_{10}(p^{17})$ agrees with a polynomial in p for all sufficiently large primes. We introduce new geometric and cohomological methods for this problem, moving beyond the earlier combinatorial calculations. We compare the counts over $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$ and, for all sufficiently large primes, express them as point counts on reductions of varieties over $\mathbb{Q}$. The Grothendieck–Lefschetz trace formula relates these counts to compactly supported ℓ-adic étale cohomology. We identify a contribution from Tate twists of the two-dimensional Galois representation attached to the weight-twelve cusp form Δ. We show that this modular contribution does not cancel when all parts of the count are combined. Consequently, there is no polynomial $P \in \mathbb{Q}[X]$ such that $f_{10}(p^{17}) = P(p)$ for all but finitely many primes p. This gives a negative answer to Voll's uniformity question even when finitely many exceptional primes are allowed.

1. Introduction

1.1. **Motivation.** Given a group, module or ring, one may ask how many subgroups, submodules or subrings respectively it has of a fixed finite index. Let A be a ring whose additive group is free of finite rank over $\mathbb{Z}$, and let a_n denote the number of unital subrings S of index $[A : S] = n$. The subring zeta function of A is the Dirichlet series

$$\zeta_A^R(s) = \sum_{n \geq 1} a_n n^{-s}.$$

Its analytic properties can be used to determine the growth of $f(x) := \sum_{n \leq X} a_n$, the number of such subrings of index at most X. Tauberian theorems relate the poles the zeta function to the growth of $f(x)$. Suppose that $\zeta_A^R(s)$ has abscissa of convergence $\alpha > 0$ and extends meromorphically to a neighbourhood of the closed half-plane $\Re(s) \geq \alpha$, with its only pole there at $s = \alpha$, of order b. Let $c = \lim_{s \to \alpha}(s - \alpha)^b \zeta_A^R(s)$ be its leading Laurent coefficient. Then the Ikehara–Delange Tauberian theorem implies that

$$f(x) \sim \frac{c}{\alpha(b-1)!} X^\alpha (\log X)^{b-1}$$

as $X \to \infty$. Analogous zeta functions have been studied to study subgroup growth and submodule growth, cf. [28, 40]. The word growth series of a finitely generated group counts its elements by word length with respect to a fixed finite generating set [16]. Representation zeta functions count irreducible complex representations, up to isomorphism, by their dimension, provided there are only finitely many in each dimension [26]. Such generating functions arise naturally in number

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theory. The prototype is the Dedekind zeta function of a number field K, whose coefficients count ideals I by their norm.

The study of zeta functions associated to algebraic structures has led to many questions. Grunewald, Segal and Smith [17, Section 3] showed that local factors of subgroup zeta functions are p-adic integrals, and hence rational functions of p^{-s} by the work of Denef [9]. On the other hand, du Sautoy and Grunewald [11] studied the analytic properties of these functions, and Voll [39, 41] studied the functional equations they satisfy. A natural question arises from the study of local factors. For a fixed ring A, does there exist a rational function $W(X,Y) \in \mathbb{Q}(X,Y)$ such that $\zeta^R_{A,p}(s) = W(p,p^{-s})$ for every prime p? When such a function exists, the subring zeta function of A is called *uniform*. For $A = \mathbb{Z}^n$, this question was posed explicitly by Voll [41, Question 3.7], and again by Atanasov, Kaplan, Krakoff and Menzel [3, Question 1.12].

Let A be a ring whose additive group is free of finite rank, and let $f_A(m)$ denote the number of unital subrings of index m. Then $\zeta^R_A(s) = \sum_{m\geq 1} f_A(m)m^{-s}$ is the subring zeta function of A; since f_A is multiplicative, it is an Euler product $\prod_p \zeta^R_{A,p}(s)$. For fixed n, uniformity is equivalent to the assertion that, for every $e \geq 0$, there is a polynomial $P_{n,e}$ with rational coefficients for which $f_n(p^e) = P_{n,e}(p)$ for every prime p. Voll's uniformity question can therefore be stated as follows.

Question 1.1. *For each fixed pair (n,e), does there exist a polynomial $P_{n,e} \in \mathbb{Q}[X]$ such that $f_n(p^e) = P_{n,e}(p)$ for every prime p?*

Work on this question has proceeded by establishing polynomial formulas for successively larger exponents e. Liu [27] introduced a decomposition into irreducible subrings and obtained formulas for $e \leq 5$. Atanasov, Kaplan, Krakoff and Menzel [3, Proposition 1.2 and Theorem 1.3] extended these calculations to $e \leq 8$, correcting Liu's formula for $e = 5$. Mishra and Ray [33, Theorem 1.4] subsequently treated $e = 9$. These computations have supported the expectation that the question has an affirmative answer.

These arguments are combinatorial in nature. Liu [27] represents each finite-index additive subgroup of $\mathbb{Z}^n$ by its unique Hermite normal form matrix, whose columns form a basis of the subgroup. The product of the diagonal entries is the index. Thus, for index p^e, the diagonal entries are powers of p whose exponents sum to e. There are finitely many possible diagonals, and for each diagonal the remaining matrix entries lie in prescribed finite ranges. To determine which matrices give subrings, one checks that the subgroup contains the identity and that the coordinatewise product of every pair of columns is an integer linear combination of the columns.

Isham [22] proves that the number of matrices satisfying any one multiplication condition is polynomial in p. Here a multiplication condition requires the coordinatewise product of two columns to belong to the lattice spanned by the columns. Isham's calculations also illustrate a difficulty with imposing several multiplication conditions simultaneously. In [21], Isham obtains explicit lower bounds for $f_n(p^e)$ valid for $e \geq n-1$. Using constructions of Brakenhoff, she improves the earlier general lower bounds for the number of subrings of index at most X when $n \geq 7$, and derives consequences for the convergence of the associated zeta functions.

A subring of $\mathbb{Z}^n$ of p-power index is called *irreducible* if every element has all its coordinates congruent modulo p. Let $g_n(p^e)$ count the irreducible subrings of index p^e. Liu [27, Theorem 3.4] proves that every subring of p-power index decomposes uniquely as a product of irreducible subrings on disjoint blocks of coordinates. Liu also proves the formula

$$f_n(p^e) = \sum_{j=1}^{n} \binom{n-1}{j-1} \sum_{i=0}^{e} g_j(p^i) f_{n-j}(p^{e-i}),$$

where $f_0(1) = 1$ and $f_0(p^e) = 0$ for $e > 0$. Since the term with $j = n$ is $g_n(p^e)$, induction on n shows that polynomiality of all the counts $f_n(p^e)$ is equivalent to polynomiality of all the irreducible counts $g_n(p^e)$.

1.2. **Main results.** Our main result gives a negative answer to Voll's question even when finitely many exceptional primes are allowed.

Theorem 1.2. *The following assertions hold.*

(i) *No polynomial $P \in \mathbb{Q}[T]$ satisfies $g_{10}(p^{17}) = P(p)$ for every sufficiently large prime p.*
(ii) *No polynomial $P \in \mathbb{Q}[T]$ satisfies $f_{10}(p^{17}) = P(p)$ for every sufficiently large prime p.*

The obstruction can stated more precisely. For a composition $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ of e, let $g_\alpha(p)$ count the irreducible subrings whose column Hermite normal form has diagonal $(p^{\alpha_1}, \dots, p^{\alpha_{n-1}}, 1)$. By uniqueness of the normal form, $g_n(p^e)$ is the sum of $g_\alpha(p)$ over all such compositions.

Theorem 1.3. *Let $n = 10$ and $\alpha_0 = (3, 3, 2, 2, 2, 2, 1, 1, 1)$. There is no polynomial $P \in \mathbb{Q}[T]$ for which $g_{\alpha_0}(p) = P(p)$ for every sufficiently large prime p.*

We obtain the following result for subring zeta functions.

Corollary 1.4. *There is no rational function $W(X, Y) \in \mathbb{Q}(X, Y)$ such that $\zeta^R_{\mathbb{Z}^{10},p}(s) = W(p, p^{-s})$ for every sufficiently large prime p.*

1.3. **Methodology.** The key idea is to study the counting problem over $\mathbb{F}_p[[t]]$. For fixed n and e, we prove that, for all sufficiently large primes p, the counts $f_n(p^e)$ and $g_n(p^e)$ agree with the corresponding counts of unital $\mathbb{F}_p[[t]]$-subalgebras of $\mathbb{F}_p[[t]]^n$ (see Section 4). Every such subalgebra contains $t^e\mathbb{F}_p[[t]]^n$, so it is determined by its image in the finite-dimensional algebra $(\mathbb{F}_p[t]/(t^e))^n$. Containment of the identity, stability under multiplication by t, and closure under coordinatewise multiplication give polynomial conditions on this image. Thus there are schemes over $\mathbb{Z}$ whose $\mathbb{F}_p$-points give the counts $f_n(p^e)$ and $g_n(p^e)$. Their reduced generic fibers are varieties over $\mathbb{Q}$, which we study using cohomology. For a variety $X/\mathbb{Q}$ and a prime ℓ, the compactly supported étale cohomology groups $H^i_c(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ are finite-dimensional vector spaces with a continuous action of $G_\mathbb{Q} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. These Galois representations determine the point counts. Indeed, if $\mathscr{X}$ is a model over $\mathbb{Z}[1/N]$, then, after enlarging N, the Grothendieck–Lefschetz trace formula gives

$$\#\mathscr{X}(\mathbb{F}_p) = \sum_i (-1)^i \operatorname{Tr}\Big(\mathrm{Frob}_p \mid H^i_c(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)\Big)$$

for $p \nmid N\ell$, where Frob_p denotes geometric Frobenius; see [31, §29].

Let $\chi_\ell : G_\mathbb{Q} \to \mathbb{Z}_\ell^\times$ be the ℓ-adic cyclotomic character, defined by $\sigma(\zeta) = \zeta^{\chi_\ell(\sigma)}$ for every root of unity ζ of ℓ-power order. The one-dimensional representation $\mathbb{Q}_\ell(r)$ has Galois action given by χ_ℓ^r. Since $\chi_\ell(\mathrm{Frob}_p) = p^{-1}$ for geometric Frobenius at $p \neq \ell$, its action on $\mathbb{Q}_\ell(-r)$ is multiplication by p^r. We call X Tate if each of its compactly supported cohomology groups has a filtration whose successive quotients are Tate representations. Its point count is then polynomial in p. For the counting problem, however, it is the alternating sum of the cohomology groups that matters and contributions in even and odd degrees may cancel. We show that there is a non-Tate contribution that survives this alternating sum.

The relevant representation comes from the weight-twelve cusp form $\Delta(q) = \sum_{m\geq 1} \tau(m) q^m$. Its associated Galois representation $S_{12,\ell}$ has dimension two and geometric-Frobenius trace $\tau(p)$ (see [5]). It is irreducible, so it cannot be built from one-dimensional Tate representations. The same holds for each Tate twist $S_{12,\ell}(-r) = S_{12,\ell} \otimes \mathbb{Q}_\ell(-r)$, whose Frobenius trace is $p^r\tau(p)$.

These representations enter through a locus describing smooth genus-one curves with ten marked points, a degree-four line bundle, and a section nonzero at the markings. Its cohomology

contains the modular contribution associated to Δ, as described by Eichler–Shimura theory and Getzler's calculation [5, 15]. This is the crucial ingredient that shows that the point counts are not eventually polynomial.

1.4. **Organization of the paper.** Section 2 is preliminary in nature and recalls basic combinatorial results and the relationship with the subring zeta function. The lattice reductions and cohomological tools introduced here will be used throughout the article. We discuss Galois representations on the étale cohomology of a variety and its compactly supported Euler classes in the K group of representations. This trace of the geometric Frobenius at p on this class gives the number of $\mathbb{F}_p$-points on the variety. Section 3 expresses the subring conditions as congruences on the entries of Hermite normal form matrices. Separating the matrices according to their diagonals gives the counts $g_\alpha(p)$. We first study the composition $\alpha_0 = (3, 3, 2, 2, 2, 2, 1, 1, 1)$, before returning to the full irreducible count. Section 4 proves that, for each fixed diagonal α, the counts over $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$ agree for all sufficiently large primes. We express both counts as local integrals defined by polynomial valuation conditions and use resolution of singularities to compare them. The passage to $\mathbb{F}_p[[t]]$ is crucial since it allows us to describe $f_n(p^e)$ and $g_n(p^e)$ as $\mathbb{F}_p$-point counts on varieties which depend only on n and e.

Section 5 constructs the parameter space for the Hilbert normal form diagonal α_0. There is a natural $\mathbb{G}_m$-action on this space. We consider the fixed locus of this action. Its points correspond to graded subalgebras, which have the form $k\mathbf{1} \oplus tU \oplus t^2V \oplus t^3k[[t]]^{10}$. Here $\mathbf{1} \in U \subseteq V \subseteq k^{10}$, with $\dim U = 4$ and $\dim V = 8$. The product of two degree-one terms must belong to the degree-two part, so every coordinatewise product of vectors in U must lie in V.

Section 6 explains how elliptic curves arise from these flags. The ten coordinate evaluations on U define ten points in $\mathbb{P}^3$. If the quadratic forms vanishing at the ten points form a two-dimensional space, their common zero locus is the intersection of two quadric surfaces. When smooth, C is a genus-one curve equipped with a degree 4 line bundle. Varying the degree-four line bundle amounts to choosing an additional point on the curve. When this point is distinct from the ten markings, the resulting data are parametrized by $\mathcal{M}_{1,11}$, the space of smooth genus-one curves with eleven ordered distinct marked points. Its cohomology contains a two-dimensional Galois representation associated to the modular form Δ. The trace of geometric Frobenius at p on this representation is the Fourier coefficient $\tau(p)$. This explains how the coefficients of Δ enter our point counts.

Section 7 studies the part of the fixed locus outside the smooth elliptic-curve locus described above. This includes configurations where some of the ten points coincide, where the quadrics fail to cut out a smooth curve, or where the points satisfy more than two independent quadratic relations. We show that the compactly supported Euler class of this boundary has no contribution from any Tate twist of the representation attached to Δ. It therefore cannot cancel the modular contribution from the smooth elliptic-curve locus.

In Section 8, we introduce the linear functional ε_{12}. On the compactly supported Euler class, it counts each copy of a Tate twist of the representation attached to Δ with sign $+1$ in even degree and -1 in odd degree, ignoring all other irreducible Galois constituents. Since every twist is counted equally, its value is unchanged by Tate twisting.

Write X for the parameter variety and F for its $\mathbb{G}_m$-fixed locus. The complement $X \setminus F$ admits a finite partition into locally closed subsets $\mathbb{G}_m \times Y_i$. Since $[\mathbb{G}_m] = \mathbb{L} - 1$, where $\mathbb{L} = [\mathbb{A}^1]$, this gives $[X] = [F] + (\mathbb{L} - 1)[Y]$ for a suitable variety Y. Multiplication by $\mathbb{L}$ becomes Tate twisting in cohomology, so the second term has zero ε_{12} value. Sections 6 and 7 show that the smooth elliptic-curve locus in F contributes 56, while its complement contributes zero. Thus $\varepsilon_{12}(\chi_c^\ell(X)) = \varepsilon_{12}(\chi_c^\ell(F)) = 56$. Since an eventual polynomial point count would force this value to vanish, this proves Theorem 1.3.

To study the full irreducible count, we construct a projective parameter space containing every HNF diagonal of the given rank and index. Its $\mathbb{G}_m$-fixed points are graded subalgebras, described by nested subspaces specifying the coefficients allowed at successive powers of t. In rank ten and colength seventeen, the scalar degree-zero part contributes nine to the colength. The remaining codimensions sum to eight, so their nonincreasing sequences are indexed by the twenty-two partitions of eight. The partition $(6, 2)$ gives the flag incidence variety already studied. Section 9 treats the other twenty-one possibilities and shows that each has zero ε_{12} contribution. The value 56 therefore survives in $g_{10}(p^{17})$, proving Theorem 1.2(i).

Section 10 establishes the corresponding vanishing results in lower ranks, which are needed for Liu's decomposition into irreducible subrings. Finally, Section 11 passes from irreducible subrings to all subrings counted by $f_{10}(p^{17})$. If a subring splits into two or more irreducible factors, each factor involves at most nine coordinates. The exponents of their indices add up to seventeen, so at most one factor can have index p^a with $a \geq 9$. All the other factors fall within the range covered by the known polynomial formulas. We combine these formulas with the lower-rank results to show that the subrings which split contribute zero to ε_{12}. Adding them to the irreducible count proves Theorem 1.2(ii).

1.5. **Further questions.** The failure of polynomiality raises the question of how the functions $f_n(p^e)$ can be described more precisely. For the case considered here, can one find polynomials $P, Q \in \mathbb{Q}[X]$ such that $f_{10}(p^{17}) = P(p) + Q(p)\tau(p)$ for all sufficiently large primes p? Our results do not establish such a formula. The functional ε_{12} records the total signed multiplicity of the Tate twists of $S_{12,\ell}$, but does not determine their individual multiplicities or exclude other non-Tate constituents.

More generally, one can ask which modular forms occur as n and e vary. Can the functions $f_n(p^e)$ be expressed using polynomials in p and Fourier coefficients of a finite number of modular forms? The marked elliptic curves appearing here explain the contribution of Δ. It would be interesting to identify other geometric configurations that produce further modular forms. One should also ask whether modular forms suffice to describe these counts, or whether larger values of n and e involve Galois representations arising from other motives.

Similar questions can be posed for a wider range of algebraic structures. Examples include finite-index subrings of other rings whose additive groups are free of finite rank, orders in number fields, and finite-index subalgebras or ideals of Lie rings. There are already examples relating local normal-subgroup zeta functions of nilpotent groups to point counts on elliptic curves; see [10, Theorem 1.1].

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2. Preliminaries

In this section, we discuss preliminary notions and set up notation which will be used throughout the article.

2.1. **Combinatorial preliminaries on subrings of $\mathbb{Z}^n$.** Given a commutative ring A, multiplication on A^n is coordinatewise:

$$(x_1, \ldots, x_n) \circ (y_1, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n).$$

All subrings contain the identity $\mathbf{1} = (1, \ldots, 1)$. For submodules U, V of A^n, let $U \circ V$ be the submodule spanned by $u \circ v$ where $u \in U$ and $v \in V$, and write $U^{\circ 2} = U \circ U$.

Let $f_n(m)$ count the subrings $R \subseteq \mathbb{Z}^n$ with $\#(\mathbb{Z}^n/R) = m$. Associated with this arithmetical function is the zeta subring zeta function

$$\zeta^R_{\mathbb{Z}^n}(s) = \sum_{m \geq 1} f_n(m) m^{-s}.$$

Given a prime number p, let

$$Z_{n,p}(T) = \sum_{e \geq 0} f_n(p^e) T^e,$$

and set

$$\zeta^R_{\mathbb{Z}^n, p}(s) = Z_{n,p}(p^{-s}).$$

It is easy to see that the function f_n is multiplicative. Hence

$$f_n(m) = \prod_{p \mid m} f_n(p^{v_p(m)})$$

and consequently,

$$\zeta^R_{\mathbb{Z}^n}(s) = \prod_p \zeta^R_{\mathbb{Z}^n, p}(s).$$

For $n \geq 2$, the series and product converge absolutely when $\Re(s) > n - 1$. To see this, each R contains $\mathbb{Z}\mathbf{1}$, and $R/\mathbb{Z}\mathbf{1}$ has index m in $\mathbb{Z}^n/\mathbb{Z}\mathbf{1} \simeq \mathbb{Z}^{n-1}$. If $a_{n-1}(m)$ counts all subgroups of this index, then $f_n(m) \leq a_{n-1}(m)$ and

$$\sum_{m \geq 1} f_n(m) m^{-\sigma} \leq \sum_{m \geq 1} a_{n-1}(m) m^{-\sigma} = \prod_{j=0}^{n-2} \zeta(\sigma - j) < \infty$$

for $\sigma > n - 1$. We refer to [3, Section 1.1, equation (1.2)] for further details. For $n = 1$ the subring zeta function is 1.

Let $\mathcal{O}$ be a discrete valuation ring with uniformizer ϖ and residue field $k = \mathcal{O}/(\varpi)$. We set $q = \#k$. The main examples that we work with in this paper are $\mathbb{Z}_p$ and $k[[t]]$. A full $\mathcal{O}$-lattice $L \subseteq \mathcal{O}^n$ is a free rank-n submodule with finite-length quotient. Its colength is then defined to be $\mathrm{length}_{\mathcal{O}}(\mathcal{O}^n/L)$. A quotient of length e has e composition factors isomorphic to k, its cardinality is q^e. A full unital $\mathcal{O}$-subalgebra of $\mathcal{O}^n$ is a full lattice containing $\mathbf{1}$, which is closed under multiplication. Such an algebra is said to be *irreducible* if its image

$$\overline{R} = (R + \varpi \mathcal{O}^n)/\varpi \mathcal{O}^n \subseteq k^n$$

is $k\mathbf{1}$. Let $g_n(p^e)$ be the number of index-p^e irreducible unital full $\mathbb{Z}_p$-subalgebras of $\mathbb{Z}_p^n$.

Lemma 2.1. *Let $\mathcal{O}$ be a discrete valuation ring with uniformizer ϖ and residue field k. Set*

$$N = \{(x_1, \ldots, x_{n-1}, 0) : x_i \in \mathcal{O}\} \subseteq \mathcal{O}^n.$$

The assignment

$$M \longmapsto R_M = \mathcal{O}\mathbf{1} \oplus \varpi M$$

is a bijection between full lattices $M \subseteq N$ satisfying

$$\varpi(M \circ M) \subseteq M$$

and irreducible full unital $\mathcal{O}$-subalgebras of $\mathcal{O}^n$. Its inverse is

$$R \longmapsto M_R = \{x \in N : \varpi x \in R\}.$$

For corresponding M and R, we have

$$\operatorname{length}_{\mathcal{O}}(\mathcal{O}^n/R) = n - 1 + \operatorname{length}_{\mathcal{O}}(N/M).$$

Proof. The proof of this result is straightforward and is thus ommitted. □

Lemma 2.2. *Let $N = \mathcal{O}^m$ and let M be a lattice with $\varpi^2 N \subseteq M \subseteq N$. Let U and V be the k-vector spaces defined by*

$$U = (M + \varpi N)/\varpi N$$

and

$$V = \{\bar{x} \in N/\varpi N : \varpi x \in M\}.$$

Then $U \subseteq V$, and, for $r = \dim U$ and $s = \dim V$,

$$\operatorname{length}_{\mathcal{O}}(N/M) = 2m - r - s.$$

Fix a flag $U \subseteq V \subseteq k^m$, with $\dim U = r$ and $\dim V = s$. Choose a basis of k^m whose first r vectors span U and whose first s vectors span V, and choose lifts of these basis vectors to N.

These choices give a bijection between the lattices $\varpi^2 N \subseteq M \subseteq N$ inducing this flag and

$$\operatorname{Hom}_k(U, k^m/V).$$

This vector space has dimension $r(m-s)$. Hence, exactly $q^{r(m-s)}$ lattices induce the prescribed flag.

The multiplication condition depends only on the flag. Indeed, we have that $\varpi(M \circ M) \subseteq M$ if and only if $U^{\circ 2} \subseteq V$. Thus all $q^{r(m-s)}$ lattices satisfy this condition when $U^{\circ 2} \subseteq V$, and none do otherwise.

Proof. The containment $\varpi^2 N \subseteq M$ makes V well defined, and $\varpi M \subseteq M$ gives $U \subseteq V$. In the short exact sequence

$$0 \longrightarrow \varpi N/(M \cap \varpi N) \longrightarrow N/M \longrightarrow N/(M + \varpi N) \longrightarrow 0,$$

we have that

$$\dim_k \varpi N/(M \cap \varpi N) = m - s$$

and

$$\dim_k N/(M + \varpi N) = m - r.$$

This proves the length formula.

Choose a basis $u_1, \ldots, u_r$ of U, extend it to bases of V and k^m, and lift this basis to N. Every desired lattice contains $K = \{\varpi x : \bar{x} \in V\}$ and is generated modulo K by $\widetilde{u}_i + \varpi a_i$ for $1 \leq i \leq r$. Because $U \subseteq V$, multiplying these generators by ϖ gives elements of K. Thus the residues of a_i in k^m/V are arbitrary and determine the lattice uniquely: differences of two lifts of the same u_i belong to $M \cap \varpi N$, whose image is ϖV modulo $\varpi^2 N$. These r residues give the stated parameter space.

Finally, $\varpi z \in M$ is equivalent to $\bar{z} \in V$ for $z \in N$. Apply this to products of elements of M, whose reductions fill U, to obtain the closure criterion. □

2.2. **Grothendieck groups and Euler characteristics.** A k-variety is a reduced separated scheme of finite type over k. When a scheme X over k is not reduced, we replace it by its reduced scheme before taking its variety X^{red}. Let $K_{\mathrm{var},k} = K_0(\mathrm{Var}_k) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the Grothendieck $\mathbb{Q}$-algebra of k-varieties. It is generated by isomorphism classes $[X]$, subject to the relation

$$[X] = [Z] + [X \setminus Z]$$

where $Z \subseteq X$ is a closed subvariety. Multiplication is defined by $[X][Y] = [X \times_k Y]$, and the identity $1 = [\operatorname{Spec} k]$. A finite stratification of X is a decomposition $X = \bigsqcup_i X_i$ into locally closed subvarieties. Applying the relation successively gives $[X] = \sum_i [X_i]$.

Write $\mathbb{A}^a_k$ for affine a-space, $\mathbb{P}^a_k$ for projective a-space, and $\mathbb{G}_m = \mathbb{A}^1_k \setminus \{0\}$. We set $\mathbb{L} = [\mathbb{A}^1_k]$. The decomposition of projective space into affine spaces gives

$$[\mathbb{A}^a_k] = \mathbb{L}^a, \qquad [\mathbb{G}_m] = \mathbb{L} - 1 \quad \text{and} \quad [\mathbb{P}^a_k] = 1 + \mathbb{L} + \cdots + \mathbb{L}^a.$$

If $k = \mathbb{F}_q$, the rational point count defines a ring homomorphism $K_{\mathrm{var},\mathbb{F}_q} \to \mathbb{Q}$ defined by $[X] \mapsto \#X(\mathbb{F}_q)$.

Fix a prime ℓ and set $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. A continuous ℓ-adic representation of $G_{\mathbb{Q}}$ is a finite-dimensional $\mathbb{Q}_\ell$-vector space V with a continuous homomorphism $G_{\mathbb{Q}} \to \mathrm{GL}(V)$. We use the rational Grothendieck ring

$$K_\ell = K_0(\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Q}_\ell}(G_{\mathbb{Q}})) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Its defining relations are

$$[V] = [V'] + [V'']$$

for every short exact sequence

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0,$$

and its multiplication is induced by tensor product. A simple constituent of a representation means a simple quotient in a composition series. Its multiplicity is independent of the series, by the Jordan–Hölder theorem. The class of a representation records these multiplicities, and a virtual class records their signed multiplicities.

For a variety $X/\mathbb{Q}$, each compactly supported cohomology group $H^i_c(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is such a representation. We define

$$\chi^\ell_c(X) = \sum_i (-1)^i [H^i_c(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)] \in K_\ell.$$

This induces a ring homomorphism

$$\chi^\ell_c : K_{\mathrm{var},\mathbb{Q}} \longrightarrow K_\ell.$$

Indeed, for $Z \subseteq X$ closed, the compact-support localization long exact sequence [30, Chapter III, (1.30)] gives

$$\chi^\ell_c(X) = \chi^\ell_c(Z) + \chi^\ell_c(X \setminus Z).$$

The compact-support Künneth formula [30, Chapter VI, Section 8] gives

$$\chi^\ell_c(X \times Y) = \chi^\ell_c(X)\chi^\ell_c(Y).$$

The localization sequence for $\mathbb{A}^1_{\mathbb{Q}} = \mathbb{P}^1_{\mathbb{Q}} \setminus \{\infty\}$ gives

$$H^i_c(\mathbb{A}^1_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \simeq \begin{cases} \mathbb{Q}_\ell(-1), & i = 2, \\ 0, & i \neq 2, \end{cases}$$

as $G_{\mathbb{Q}}$-representations; see [30, Chapter III, (1.30)] and [31, Theorem 23.2]. Since degree two contributes with positive sign,

$$\chi^\ell_c(\mathbb{L}) = [\mathbb{Q}_\ell(-1)].$$

For ease of notation, we also set $\mathbb{L} := [\mathbb{Q}_\ell(-1)]$. The one-dimensional objects $\mathbb{Q}_\ell(-r)$ and $\mathbb{Q}(-r)$, for $r \in \mathbb{Z}$, are called Tate objects. A variety is said to be Tate if every simple constituent of every H_c^i is a Tate object. A *virtual Tate class* $P(\mathbb{L})$ is a finite rational linear combination of Tate classes:

$$P(\mathbb{L}) = \sum_{i=0}^{N} a_i \mathbb{L}^i.$$

Here, $P(X)$ denotes the polynomial $\sum_{i=0}^{N} a_i X^i \in \mathbb{Q}[X]$.

The Grothendieck–Lefschetz trace formula relates compactly supported cohomology to point counts. Let X be a variety over $\mathbb{F}_q$, where q is a power of a prime different from ℓ, and write $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. Then

$$\#X(\mathbb{F}_q) = \sum_i (-1)^i \operatorname{Tr}\bigl(\mathrm{Frob}_q \mid H_c^i(\overline{X}, \mathbb{Q}_\ell)\bigr);$$

see [31, Theorem 29.8]. Here Frob_q denotes geometric Frobenius, the inverse of the automorphism $x \mapsto x^q$ of $\overline{\mathbb{F}}_q$. Arithmetic Frobenius acts on $\mathbb{Q}_\ell(1)$ by q, so geometric Frobenius acts on $\mathbb{Q}_\ell(-r)$ by q^r. Consequently, the trace of geometric Frobenius on a virtual Tate class $P(\mathbb{L})$ is $P(q)$. A virtual Euler class is an alternating sum of cohomology groups. Consequently, an equality $\chi_c^\ell(X) = P(\mathbb{L})$ does not by itself say that each $H_c^i(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ has only Tate constituents, since constituents in adjacent degrees may cancel. We distinguish this assertion from the stronger statement that X is Tate.

For a variety $X/\mathbb{Q}$, choose a model $\mathscr{X}$ over $\mathbb{Z}[1/N]$. After increasing N, base change for compactly supported cohomology gives

$$\#\mathscr{X}(\mathbb{F}_p) = \operatorname{Tr}\bigl(\mathrm{Frob}_p \mid \chi_c^\ell(X)\bigr)$$

where $p \nmid N\ell$; see [8]. The trace on a virtual representation is defined by extending the usual trace linearly.

The Hodge Euler class on the other hand is defined as

$$\chi_c(X) = \sum_i (-1)^i [H_c^i(X(\mathbb{C}), \mathbb{Q})]$$

in the Grothendieck ring of rational mixed Hodge structures. Additivity and the Künneth formula make this a ring homomorphism, just as above; see [6, 7]. We refer to this homomorphism and to χ_c^ℓ as the two *realizations*. The terminology "in both realizations" means one assertion about mixed Hodge structures and a second assertion about $G_\mathbb{Q}$-representations. In these rings, $\mathbb{L}$ denotes $[\mathbb{Q}(-1)]$ and $[\mathbb{Q}_\ell(-1)]$, respectively.

3. Hermite normal forms

We abbreviate the Hermite Normal form of a lattice by HNF. We always use the column convention, i.e., the columns form a basis of the lattice, and the entries in row i above the diagonal are reduced modulo the ith diagonal entry.

Lemma 3.1. *Let $\mathcal{O}$ be a discrete valuation ring with uniformizer ϖ. For each $a \geq 0$, fix a set $\mathcal{R}_a \subseteq \mathcal{O}$ of representatives for $\mathcal{O}/\varpi^a\mathcal{O}$, with $\mathcal{R}_0 = \{0\}$. Every full lattice $L \subseteq \mathcal{O}^m$ has a unique basis matrix of the form*

$$A = \begin{pmatrix} \varpi^{e_1} & a_{12} & \cdots & a_{1m} \\ 0 & \varpi^{e_2} & \cdots & a_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \varpi^{e_m} \end{pmatrix},$$

where $e_i \geq 0$ and $a_{ij} \in \mathcal{R}_{e_i}$ for $i < j$. Here the columns of A form an $\mathcal{O}$-basis of L, so $L = A\mathcal{O}^m$. Moreover,

$$\mathrm{length}_{\mathcal{O}}(\mathcal{O}^m/L) = e_1 + \cdots + e_m.$$

Proof. The normal-form assertion follows from the Hermite normal form theorem over a principal ideal domain; see [1, Chapter 5, Theorems 2.9 and 2.13]. The colength formula follows from

$$\mathrm{length}_{\mathcal{O}}(\mathcal{O}^m/A\mathcal{O}^m) = \mathrm{ord}_{\varpi}(\det A) = \sum_{i=1}^{m} e_i.$$

□

The exponents e_i need not be increasing and are not, in general, the Smith exponents of L. The HNF basis is unique after fixing the representative sets $\mathcal{R}_a$; it is this uniqueness that prevents different bases from being counted as different lattices. Over $\mathbb{Z}$, every full lattice $L \subseteq \mathbb{Z}^m$ is represented uniquely by an HNF matrix

$$A = \begin{pmatrix} a_1 & b_{12} & \cdots & b_{1m} \\ 0 & a_2 & \cdots & b_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_m \end{pmatrix},$$

where $a_i > 0$, $0 \leq b_{ij} < a_i$ for $i < j$. Its columns generate L, and

$$[\mathbb{Z}^m : L] = \det A = a_1 \cdots a_m.$$

If this index is p^e, each diagonal entry is a power of p. We may therefore write $a_i = p^{\alpha_i}$, where $\alpha_i \geq 0$ and $\alpha_1 + \cdots + \alpha_m = e$.

Let $R \subseteq \mathbb{Z}^n$ be an irreducible subring of p-power index. In this integral setting, irreducibility means that the image of R in $\mathbb{F}_p^n$ is $\mathbb{F}_p\mathbf{1}$.

Since the identity belongs to R, its projection to the last coordinate is all of $\mathbb{Z}$. Thus $a_n = 1$. Every preceding basis vector has last coordinate zero. By irreducibility, all coordinates of each such vector are divisible by p. Hence $a_i = p^{\alpha_i}$ with $\alpha_i \geq 1$ for $i < n$. The last column is $\mathbf{1}$ and consequently the HNF matrix has the form

$$A = \begin{pmatrix} p^{\alpha_1} & pa_{12} & \cdots & pa_{1,n-1} & 1 \\ 0 & p^{\alpha_2} & \cdots & pa_{2,n-1} & 1 \\ & & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & p^{\alpha_{n-1}} & 1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, \tag{3.1}$$

where $0 \leq a_{ij} < p^{\alpha_i - 1}$. It is a subring precisely when the product of every pair of basis columns belongs to the lattice.

Definition 3.2. Let p be a prime and let $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ be a composition of e into $n-1$ positive parts. Define $g_\alpha(p)$ to be the number of matrices

$$A = \begin{pmatrix} p^{\alpha_1} & pa_{12} & \cdots & pa_{1,n-1} & 1 \\ 0 & p^{\alpha_2} & \cdots & pa_{2,n-1} & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & p^{\alpha_{n-1}} & 1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix},$$

with $a_{ij} \in \mathbb{Z}$ and $0 \leq a_{ij} < p^{\alpha_i - 1}$, whose columns $v_1, \ldots, v_n$ satisfy $v_i \circ v_j \in A\mathbb{Z}^n$ for $1 \leq i \leq j \leq n-1$. Since $v_n = \mathbf{1}$, products involving the last column already belong to $A\mathbb{Z}^n$.

Extension of scalars $L \mapsto L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ gives a bijection between lattices of p-power index in $\mathbb{Z}^n$ and full lattices in $\mathbb{Z}_p^n$.

We now fix $\alpha_0 = (3,3,2,2,2,2,1,1,1)$ and note that $\sum_i \alpha_{0,i} = 17$. The corresponding matrix is

$$A = \begin{pmatrix} p^3 & pa_{12} & pa_{13} & pa_{14} & pa_{15} & pa_{16} & pa_{17} & pa_{18} & pa_{19} & 1 \\ 0 & p^3 & pa_{23} & pa_{24} & pa_{25} & pa_{26} & pa_{27} & pa_{28} & pa_{29} & 1 \\ 0 & 0 & p^2 & pa_{34} & pa_{35} & pa_{36} & pa_{37} & pa_{38} & pa_{39} & 1 \\ 0 & 0 & 0 & p^2 & pa_{45} & pa_{46} & pa_{47} & pa_{48} & pa_{49} & 1 \\ 0 & 0 & 0 & 0 & p^2 & pa_{56} & pa_{57} & pa_{58} & pa_{59} & 1 \\ 0 & 0 & 0 & 0 & 0 & p^2 & pa_{67} & pa_{68} & pa_{69} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & p & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.2}$$

Dividing the first nine columns of A by p and deleting the last row gives

$$B = \begin{pmatrix} p^2 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\ 0 & p^2 & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\ 0 & 0 & p & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\ 0 & 0 & 0 & p & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\ 0 & 0 & 0 & 0 & p & a_{56} & a_{57} & a_{58} & a_{59} \\ 0 & 0 & 0 & 0 & 0 & p & a_{67} & a_{68} & a_{69} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.3}$$

We write $b_1, \dots, b_9$ for the columns of B.

Lemma 3.3. *The matrix (3.2) defines a subring if and only if*

$$p(b_j \circ b_k) \in \mathrm{Col}(B) \tag{3.4}$$

for $1 \le j \le k \le 9$, where $\mathrm{Col}(B)$ denotes the integral column lattice of B.

Proof. The first nine columns of A are the vectors pb_j, with a zero tenth coordinate, and its last column is $\mathbf{1}$. A linear combination of these columns has zero tenth coordinate only when the coefficient of $\mathbf{1}$ is zero. Thus the product of columns j and k belongs to the lattice precisely when $p^2(b_j \circ b_k)$ belongs to $pB\mathbb{Z}^9$, which is equivalent to (3.4). Products with the last column preserve every column. □

Each entry above the diagonal in the first two rows of B is an integer between 0 and $p^2 - 1$. It therefore has a unique base-p expansion

$$a_{ij} = x_{ij} + py_{ij},$$

where $x_{ij}, y_{ij} \in \{0, \dots, p-1\}$. Here x_{ij} is the remainder on division by p, and y_{ij} is the quotient. We regard both digits as elements of $\mathbb{F}_p$. The entries above the diagonal in rows $3, \dots, 6$ already lie between 0 and $p-1$, so each requires only one digit, which we denote by x_{ij}. All entries above the diagonal in the last three rows are zero.

The first two rows have 8 and 7 variable entries, respectively, each requiring two digits. The next four rows have $6, 5, 4, 3$ variable entries, each requiring one digit. Thus the 33 variable entries of B are specified by

$$2 \cdot 8 + 2 \cdot 7 + 6 + 5 + 4 + 3 = 48 \tag{3.5}$$

elements of $\mathbb{F}_p$.

Let K be a non-archimedean local field with residue field of cardinality q, valuation ring $\mathcal{O}_K$, and uniformizer ϖ. Fix nonnegative integers $d = (d_1, \dots, d_m)$. For $d = (d_1, \dots, d_m) \in \mathbb{Z}^m_{\geq 0}$, set

$$\mathscr{B}_d(K) = \left\{ \begin{pmatrix} \varpi^{d_1} & b_{12} & \cdots & b_{1m} \\ 0 & \varpi^{d_2} & \cdots & b_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \varpi^{d_m} \end{pmatrix} : b_{ij} \in \mathcal{O}_K \text{ for } i < j \right\}.$$

The upper-unitriangular group is

$$N_m(\mathcal{O}_K) = \left\{ \begin{pmatrix} 1 & u_{12} & \cdots & u_{1m} \\ 0 & 1 & \cdots & u_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} : u_{ij} \in \mathcal{O}_K \text{ for } i < j \right\}.$$

It acts on $\mathscr{B}_d(K)$ by right multiplication. The product BU is upper triangular and has the same diagonal as B, so it again belongs to $\mathscr{B}_d(K)$.

Lemma 3.4. *Two matrices $B, B' \in \mathscr{B}_d(K)$ have the same column lattice if and only if $B' = BU$ for a unique $U \in N_m(\mathcal{O}_K)$. Consequently the $N_m(\mathcal{O}_K)$-orbits are exactly the lattices admitting a basis with the prescribed upper-triangular diagonal.*

Give the $m(m-1)/2$ entries above the diagonal their product additive Haar measure, normalized so that $\mathcal{O}_K$ has measure one. Every orbit in $\mathscr{B}_d(K)$ has measure

$$q^{-\sum_{i=1}^{m}(m-i)d_i}. \tag{3.6}$$

There are $q^{\sum_i (m-i)d_i}$ such orbits.

Proof. We refer to [23, Definition 5 and equation (6), p. 29]. The same calculation applies over $\mathcal{O}_K$, with $|\varpi|_K = q^{-1}$. The assertion about the number of orbits then follows because $\mathscr{B}_d(K)$ has measure one. □

For our matrix B,

$$d = (2, 2, 1, 1, 1, 1, 0, 0, 0),$$

and the exponent in (3.6) is

$$\sum_{i=1}^{9}(9-i)d_i = 8 \cdot 2 + 7 \cdot 2 + 6 + 5 + 4 + 3 = 48. \tag{3.7}$$

4. Comparison between characteristic zero and characteristic p counts

In this section, we show that irreducible subrings with a fixed HNF diagonal have the same count over $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$ for all sufficiently large primes p. We express these counts as local integrals defined by the same polynomials with integer coefficients.

Let K be a non-archimedean local field with valuation ring $\mathcal{O}_K$, maximal ideal $\mathfrak{m}_K$, and residue field k_K. Set $q = \#k_K$ and write $\mathrm{ord}_K : K^\times \to \mathbb{Z}$ for its normalized valuation. Choose a uniformizer ϖ and normalize the valuation and additive Haar measure by $\mu_K(\mathcal{O}_K) = 1$. Thus $\mu_K(\varpi^a \mathcal{O}_K) = q^{-a}$ for $a \geq 0$. On a Cartesian power of $\mathcal{O}_K$ we use product measure.

Fix a positive composition $\alpha = (\alpha_1, \dots, \alpha_m)$. We set

$$d_i = \alpha_i - 1, \quad r = \sum_{i=1}^{m} d_i, \quad E = \sum_{i=1}^{m}(m-i)d_i \quad \text{and} \quad b = \frac{m(m-1)}{2}.$$

For $\mathbf{b} = (b_{ij})_{i<j} \in \mathcal{O}_K^b$, write

$$B(\mathbf{b}) = \begin{pmatrix} \varpi^{d_1} & b_{12} & \cdots & b_{1m} \\ 0 & \varpi^{d_2} & \cdots & b_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \varpi^{d_m} \end{pmatrix}.$$

Its determinant is ϖ^r. Denote its columns by $b_1, \dots, b_m$.

Define $\mathcal{D}_\alpha(K)$ to be the set of tuples $\mathbf{b}$ such that, for every $1 \le j \le k \le m$, the system

$$B(\mathbf{b}) \begin{pmatrix} c_{1,jk} \\ c_{2,jk} \\ \vdots \\ c_{m,jk} \end{pmatrix} = \varpi(b_j \circ b_k) \tag{4.1}$$

has a solution with all $c_{i,jk} \in \mathcal{O}_K$. Thus this condition requires each such product to be an $\mathcal{O}_K$-linear combination of the columns of B.

Let

$$g_\alpha^K(q) = \#\{B(\mathbf{b})\mathcal{O}_K^m : \mathbf{b} \in \mathcal{D}_\alpha(K)\}.$$

Each lattice is counted only once, regardless of how many matrices represent it. This number is finite by Lemma 3.4. For $K = \mathbb{Q}_p$, the correspondence between integral and p-adic HNF lattices, together with Lemma 2.1, gives $g_\alpha^{\mathbb{Q}_p}(p) = g_\alpha(p)$.

Proposition 4.1. *One has*

$$g_\alpha^K(q) = q^E \mu_K(\mathcal{D}_\alpha(K)). \tag{4.2}$$

Proof. For $M = \mathrm{Col}_{\mathcal{O}_K}(B)$, bilinearity makes (4.1) equivalent to $\varpi(M \circ M) \subseteq M$. It therefore depends only on the lattice. The locus $\mathcal{D}_\alpha(K)$ is a union of orbits, each of measure q^{-E} by Lemma 3.4. Adding their measures gives the formula. □

Let $a \ge 1$ and $s \ge 0$, and let $f_0, \dots, f_s \in \mathbb{Z}[X_1, \dots, X_a]$ be nonzero polynomials. Fix nonnegative integers u, v, integer matrices

$$\mathsf{A} \in \mathrm{Mat}_{u,s+1}(\mathbb{Z}) \quad \text{and} \quad \mathsf{B} \in \mathrm{Mat}_{v,s+1}(\mathbb{Z}),$$

and vectors $\boldsymbol{\beta} \in \mathbb{Z}^u$ and $\boldsymbol{\gamma} \in \mathbb{Z}^v$. We set

$$\Lambda = \left\{ \mathbf{w} \in \mathbb{Z}_{\ge 0}^{s+1} : \mathsf{A}\mathbf{w} = \boldsymbol{\beta}, \quad \mathsf{B}\mathbf{w} \ge \boldsymbol{\gamma} \right\},$$

where the inequality is interpreted coordinatewise. Thus Λ specifies a fixed finite system of integer linear equalities and inequalities. Let μ_K be additive Haar measure on K, normalized by $\mu_K(\mathcal{O}_K) = 1$, and use the product measure $\mu_K^{\otimes a}$ on K^a.

For $x \in \mathcal{O}_K^a$ such that $f_c(x) \ne 0$ for every c, set

$$\mathbf{v}_K(x) = (\mathrm{ord}_K\, f_0(x), \dots, \mathrm{ord}_K\, f_s(x)).$$

Define

$$S_K = \{x \in \mathcal{O}_K^a : f_c(x) \ne 0 \text{ for all } c \quad \text{and} \quad \mathbf{v}_K(x) \in \Lambda\}.$$

Lemma 4.2. *With respect to notation above, there is a positive integer N such that*

$$\mu_K^{\otimes a}(S_K) = \mu_{K'}^{\otimes a}(S_{K'})$$

whenever $k_K \simeq k_{K'}$ and their common characteristic does not divide N.

Proof. We want to compare volumes of sets defined by polynomial divisibility conditions. We use a resolution of the divisor $D = V(f_0 \cdots f_s)$. The ambient affine space is already smooth, but the components of D may be singular or meet nontransversally. The resolution arranges that the components of its inverse image are smooth and meet like coordinate hyperplanes. In suitable local coordinates, each $f_c \circ h$ is then a monomial times an invertible function. For a suitable positive integer N, the resolution and its monomial identities extend over $\mathbb{Z}[1/N]$ with the same smoothness and normal-crossing properties. We then exclude residue characteristics dividing N and check the conditions one variable at a time and calculate the volume explicitly. The volume is shown to depend only on the global data $f_1, \dots, f_c$ and the conditions defining Λ; cf. (4.4). This explains why local fields with the same residue field give the same volume, once finitely many residue characteristics are excluded.

Set $f = \prod_{c=0}^{s} f_c$ and $\omega = dX_1 \wedge \cdots \wedge dX_a$. By the embedded form of Hironaka's theorem [19, 20], we may choose a log resolution $h : Y \to \mathbb{A}^a_{\mathbb{Q}}$ of D. Here, Y is a a-dimensional variety $Y/\mathbb{Q}$ and h is a proper morphism which is an isomorphism over $\mathbb{A}^a_{\mathbb{Q}} \setminus V(f)$. Near each point of Y, there are local étale coordinates $y_1, \dots, y_a$ in which

$$f \circ h = u\, y_1^{N_1} \cdots y_t^{N_t},$$

with $0 \le t \le a$, $N_i \ge 1$, where u is an invertible regular function. Thus the reduced inverse image of $V(f)$ is locally defined by $y_1 \cdots y_t = 0$. Write E_i, for i in a finite set I, for its irreducible components. Write

$$\operatorname{div}(f_c \circ h) = \sum_i N_{ci} E_i, \quad \text{and} \quad \operatorname{div}(h^*\omega) = \sum_i (\nu_i - 1) E_i. \tag{4.3}$$

Here $N_{ci} \ge 0$ and $\nu_i \ge 1$.

The resolution extends to a proper morphism

$$\tilde{h} : \mathscr{Y} \longrightarrow \mathbb{A}^a_R,$$

over $R = \mathbb{Z}[1/N]$ with $\mathscr{Y}/R$ smooth and divisors $\mathscr{E}_i$ having relative strict normal crossings.

We enlarge N so that $\tilde{h}$ is also an isomorphism over the complement of $V(f)$ and all the divisor identities in (4.3) extend to this model. Thus the functions denoted by units in the local expressions, including the Jacobian expression, remain units over R. There are only finitely many equations and local descriptions to retain, so this requires inverting only finitely many further integers.

Fix such a local field K, and write $\mathcal{O} = \mathcal{O}_K$, $k = k_K$ and $q = \#k$. For $J \subseteq I$, set

$$\begin{aligned} \mathscr{E}_J^\circ &= \left(\bigcap_{i \in J} \mathscr{E}_i \right) \setminus \bigcup_{i \notin J} \mathscr{E}_i, \\ N_J &= (N_{ci})_{\substack{0 \le c \le s, \\ i \in J}}, \\ C_J &= \{ \mathbf{n} \in \mathbb{Z}^J_{\ge 1} : N_J \mathbf{n} \in \Lambda \}, \end{aligned}$$

where the empty intersection is $\mathscr{Y}$.

We now comsete the volume of the points of S_K whose lifts reduce to a fixed point of the special fiber. Write $\mathcal{O} = \mathcal{O}_K$, $\mathfrak{m} = \mathfrak{m}_K$, and $q = \#k$. We normalize Haar measure by $\mu_K(\mathcal{O}) = 1$ and use the absolute value $|x|_K = q^{-\operatorname{ord}_K(x)}$. Fix $z \in \mathscr{E}^\circ_{J,k}(k)$ and set $t = |J|$. Choose an open neighborhood U of z that meets no divisor $\mathscr{E}_i$ with $i \notin J$. On U, choose regular functions $(y_i)_{i \in J}, w_1, \dots, w_{a-t}$ giving an étale map

$$\phi : U \longrightarrow \mathbb{A}^a_{\mathcal{O}},$$

such that $y_i = 0$ defines $\mathscr{E}_i \cap U$ for $i \in J$. In particular, $y_i(z) = 0$ in k. For each ℓ, choose a lift $\widetilde{w}_\ell \in \mathcal{O}$ of $w_\ell(z) \in k$.

Let

$$T_z = \{P \in \mathscr{Y}(\mathcal{O}) : \overline{P} = z\}.$$

Given $P \in T_z$, we find that $y_i(P) \in \mathfrak{m}$ and $w_\ell(P) \in \widetilde{w}_\ell + \mathfrak{m}$. Conversely, Hensel's lemma gives a unique point of T_z for every tuple of coordinate values satisfying these congruences. Thus ϕ restricts to a bijection

$$T_z \xrightarrow{\sim} \Omega_z := \mathfrak{m}^J \times \prod_{\ell=1}^{a-t} (\widetilde{w}_\ell + \mathfrak{m}).$$

Write $\psi_z : \Omega_z \to T_z$ for its inverse and set

$$H_z = \tilde{h} \circ \psi_z : \Omega_z \longrightarrow \mathcal{O}^a.$$

We shall calculate the volume of the image of H_z and then sum over all choices of z. We shall then get an expression which only depends on the residue field and the data $f_1, \ldots, f_s$ and the sets C_J (which are defined in terms of Λ).

Fix $\mathbf{n} = (n_i)_{i \in J} \in C_J$. Prescribing $\mathrm{ord}_K(y_i) = n_i$ amounts to restricting the coordinate tuple to

$$\Omega_{z,\mathbf{n}} = \prod_{i \in J} (\mathfrak{m}^{n_i} \setminus \mathfrak{m}^{n_i+1}) \times \prod_{\ell=1}^{a-t} (\widetilde{w}_\ell + \mathfrak{m}).$$

Let $T_{z,\mathbf{n}} = \psi_z(\Omega_{z,\mathbf{n}})$. We first check that

$$H_z(\Omega_{z,\mathbf{n}}) = \tilde{h}(T_{z,\mathbf{n}}) \subseteq S_K.$$

On U, the resolution gives

$$f_c \circ \tilde{h} = u_c \prod_{i \in J} y_i^{N_{ci}}, \qquad 0 \le c \le s,$$

where each u_c is an invertible regular function. Consequently, $u_c(P) \in \mathcal{O}^\times$ for every $P \in T_{z,\mathbf{n}}$, and

$$\mathrm{ord}_K\, f_c(\tilde{h}(P)) = \sum_{i \in J} N_{ci} n_i.$$

These valuations are finite, so none of the $f_c(\tilde{h}(P))$ vanishes. Moreover, $\mathbf{n} \in C_J$ means precisely that this valuation vector satisfies the equalities and inequalities defining S_K. This proves the inclusion.

We next calculate the Haar measure of $\Omega_{z,\mathbf{n}}$ in the coordinate space K^a. The ideal $\mathfrak{m}^r$ has index q^r in $\mathcal{O}$. Its q^r cosets have equal measure and partition $\mathcal{O}$, so

$$\mu_K(\mathfrak{m}^r) = q^{-r}.$$

It follows that

$$\mu_K(\mathfrak{m}^{n_i} \setminus \mathfrak{m}^{n_i+1}) = q^{-n_i} - q^{-n_i-1} = (1 - q^{-1}) q^{-n_i}.$$

Each of the other coordinate sets $\widetilde{w}_\ell + \mathfrak{m}$ has measure q^{-1}. Since $\Omega_{z,\mathbf{n}}$ is a product of these sets, its measure is

$$\mu_K^{\otimes a}(\Omega_{z,\mathbf{n}}) = q^{-(a-t)} \prod_{i \in J} (1 - q^{-1}) q^{-n_i}.$$

To obtain the volume of its image under H_z, we must also calculate the Jacobian of this map. Put $\omega = dX_1 \wedge \cdots \wedge dX_a$ on $\mathbb{A}^a_{\mathcal{O}}$. In our chosen coordinates,

$$\tilde{h}^* \omega = u \prod_{i \in J} y_i^{\nu_i - 1} \left(\bigwedge_{i \in J} dy_i \right) \wedge dw_1 \wedge \cdots \wedge dw_{a-t},$$

where u is invertible. The coefficient of the coordinate differential form is the determinant of the Jacobian matrix of H_z. Since $|u(P)|_K = 1$ and $|y_i(P)|_K = q^{-n_i}$, we obtain

$$|\det DH_z|_K = q^{-\sum_{i\in J}(\nu_i-1)n_i} \qquad \text{on } \Omega_{z,\mathbf{n}}.$$

In particular, this absolute value is constant on the set under consideration.

The map H_z is injective on $\Omega_{z,\mathbf{n}}$. Indeed, its image avoids $V(f)$, and $\tilde{h}$ is an isomorphism over the complement of $V(f)$. The change-of-variables formula therefore gives

$$\begin{aligned}\mu_K^{\otimes a}(\tilde{h}(T_{z,\mathbf{n}})) &= \int_{\Omega_{z,\mathbf{n}}} |\det DH_z(v)|_K \, d\mu_K^{\otimes a}(v) \\ &= q^{-\sum_{i\in J}(\nu_i-1)n_i} q^{-(a-t)} \prod_{i\in J}(1-q^{-1})q^{-n_i} \\ &= q^{-a}(q-1)^t q^{-\sum_{i\in J}\nu_i n_i}.\end{aligned}$$

Thus the last expression is the volume, in the original space $\mathcal{O}^a$, of precisely those points whose lifts reduce to z and satisfy $\mathrm{ord}_K(y_i) = n_i$ for every $i \in J$. When $J = \varnothing$, all products indexed by J are interpreted as 1.

It remains to check that these images account for every point of S_K. Let $x \in S_K$. Since $f(x) \neq 0$, the isomorphism away from $V(f)$ gives a unique lift of the generic point of $\operatorname{Spec}\mathcal{O}$ to $\mathscr{Y}$. The valuative criterion of properness for $\tilde{h}$ extends this lift uniquely to a point $P \in \mathscr{Y}(\mathcal{O})$. Its reduction is a unique point z on a unique stratum $\mathscr{E}^{\circ\circ}_{J,k}$. At P, the coordinates y_i are nonzero, and their positive valuations give a unique tuple $\mathbf{n} \in C_J$. Hence x belongs to exactly one of the images just measured. This proves that their measures may be summed.

Summation over the residue classes gives

$$\mu_K^{\otimes a}(S_K) = q^{-a} \sum_{J\subseteq I} \#\mathscr{E}^{\circ\circ}_{J,k}(k)(q-1)^{|J|} \sum_{\mathbf{n}\in C_J} q^{-\sum_{i\in J}\nu_i n_i}. \tag{4.4}$$

The inner sums converge because $\nu_i \geq 1$. For $J = \varnothing$, the inner sum is 1 if $\mathbf{0} \in \Lambda$, and 0 otherwise.

The right-hand side depends only on the residue field and the fixed integral data. It is therefore unchanged when K is replaced by a local field with an isomorphic residue field. □

To apply Lemma 4.2, we express the lattice closure conditions using polynomials with fixed integer coefficients. Introduce variables T and $\mathbf{b} = (b_{ij})_{1\leq i<j\leq m}$, and set

$$B(T,\mathbf{b}) = \begin{pmatrix} T^{d_1} & b_{12} & b_{13} & \cdots & b_{1m} \\ 0 & T^{d_2} & b_{23} & \cdots & b_{2m} \\ 0 & 0 & T^{d_3} & \cdots & b_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & T^{d_m} \end{pmatrix}.$$

There are $b = m(m-1)/2$ variables in $\mathbf{b}$, and

$$\det B(T,\mathbf{b}) = T^{d_1+\cdots+d_m} = T^r.$$

Write $b_j(T,\mathbf{b})$ for the jth column.

Suppose that $T \in \mathcal{O}_K$ is nonzero and $\mathbf{b} \in \mathcal{O}_K^b$. The columns of $B(T,\mathbf{b})$ form a basis of K^m. Thus there are unique coefficients $c_{jk,1}, \ldots, c_{jk,m} \in K$ such that

$$T(b_j \circ b_k) = c_{jk,1}b_1 + \cdots + c_{jk,m}b_m.$$

The vector on the left belongs to the column lattice $B(T,\mathbf{b})\mathcal{O}_K^m$ precisely when all these coefficients belong to $\mathcal{O}_K$.

We calculate the coefficients by solving

$$B(T,\mathbf{b})\begin{pmatrix} c_{jk,1} \\ \vdots \\ c_{jk,m} \end{pmatrix} = T(b_j \circ b_k).$$

The adjoint matrix satisfies

$$B\,\mathrm{adj}(B) = (\det B) I_m.$$

Its entries are polynomials with integer coefficients in the entries of B. Consequently,

$$F_{jk,l}(T,\mathbf{b}) = \left[\mathrm{adj}(B(T,\mathbf{b}))\, T\big(b_j(T,\mathbf{b}) \circ b_k(T,\mathbf{b})\big)\right]_l \in \mathbb{Z}[T,\mathbf{b}] \tag{4.5}$$

for $1 \le j \le k \le m$ and $1 \le l \le m$, and

$$\begin{pmatrix} c_{jk,1} \\ \vdots \\ c_{jk,m} \end{pmatrix} = T^{-r} \begin{pmatrix} F_{jk,1}(T,\mathbf{b}) \\ \vdots \\ F_{jk,m}(T,\mathbf{b}) \end{pmatrix}.$$

Here $[\cdot]_l$ denotes the lth coordinate of a column vector.

We now require $\mathrm{ord}_K(T) = 1$, so that T is a uniformizer. Since $\mathrm{ord}_K(T^r) = r$, the coefficient $T^{-r}F_{jk,l}(T,\mathbf{b})$ belongs to $\mathcal{O}_K$ if and only if its numerator has valuation at least r. Therefore

$$T(b_j \circ b_k) \in B(T,\mathbf{b})\mathcal{O}_K^m \quad \text{if and only if} \quad \mathrm{ord}_K F_{jk,l}(T,\mathbf{b}) \ge r \quad \text{for every } l, \tag{4.6}$$

where we have the convention $\mathrm{ord}_K(0) = +\infty$. In particular, an identically zero polynomial imposes no condition.

List the polynomials in (4.5) that are not identically zero as $F_1, \dots, F_s$. The choices in Lemma 4.2 are $a = b+1$, $(X_1, \dots, X_a) = (T, (b_{ij})_{i<j})$, $f_0 = T$, $f_h = F_h$ for $1 \le h \le s$. The allowed valuation vectors form the set

$$\Lambda = \left\{(w_0, \dots, w_s) \in \mathbb{Z}_{\ge 0}^{s+1} : w_0 = 1, \quad w_h \ge r \text{ for } 1 \le h \le s\right\}.$$

Thus the equality $w_0 = 1$ requires T to be a uniformizer, and the inequalities $w_h \ge r$ express the lattice closure conditions.

With these choices, the set S_K in that lemma is

$$\left\{(T,\mathbf{b}) \in \mathcal{O}_K^{b+1} : TF_1(T,\mathbf{b}) \cdots F_s(T,\mathbf{b}) \ne 0, \quad \mathrm{ord}_K(T) = 1, \quad \mathrm{ord}_K F_h(T,\mathbf{b}) \ge r \text{ for all } h\right\}.$$

Consider the integral

$$J_K = \int_{\mathcal{O}_K^{b+1}} \mathbf{1}_{\{\mathrm{ord}_K(T)=1\}} \prod_{h=1}^{s} \mathbf{1}_{\{\mathrm{ord}_K F_h(T,\mathbf{b}) \ge r\}}\, dT\, d\mathbf{b}, \tag{4.7}$$

where $\mathbf{1}_A$ is 1 on A and 0 elsewhere. This integral is the volume of the set satisfying the stated valuation conditions. It also includes points where some F_h vanishes, whereas S_K excludes them. After excluding the finitely many residue characteristics in which a polynomial becomes identically zero, these zero sets have measure zero.

Hence

$$J_K = \mu_K^{\otimes(b+1)}(S_K).$$

Lemma 4.2 therefore applies to J_K.

The role of the variable T is to let the uniformizer vary inside the integral. All the polynomials f_h then have integer coefficients, as required by the lemma.

Lemma 4.3. *One has*

$$J_K = (1 - q^{-1}) q^{-E-1} g_\alpha^K(q). \tag{4.8}$$

Proof. Write $\mathcal{O} = \mathcal{O}_K$ and $\mathfrak{m} = \mathfrak{m}_K$. For each uniformizer $\tau \in \mathcal{O}$, define

$$D_\tau = \left\{ \mathbf{b} \in \mathcal{O}^b : \operatorname{ord}_K F_{jk,l}(\tau, \mathbf{b}) \geq r \text{ for all } j,k,l \right\}.$$

These are precisely the inequalities in (4.6), with $T = \tau$.

Put

$$M_{\tau,\mathbf{b}} = B(\tau, \mathbf{b})\mathcal{O}^m.$$

By (4.6), membership in D_τ means that every vector $\tau(b_j \circ b_k)$ belongs to $M_{\tau,\mathbf{b}}$. Since the columns b_j generate this lattice and multiplication is bilinear, this is equivalent to

$$\mathbf{b} \in D_\tau \quad \text{if and only if} \quad \tau(M_{\tau,\mathbf{b}} \circ M_{\tau,\mathbf{b}}) \subseteq M_{\tau,\mathbf{b}}.$$

In particular, when $\tau = \varpi$, this is the condition defining $\mathcal{D}_\alpha(K)$ in (4.1). Hence $D_\varpi = \mathcal{D}_\alpha(K)$,

$$\mu_K^{\otimes b}(D_\varpi) = q^{-E} g_\alpha^K(q)$$

by (4.2).

We next show that D_τ has the same volume for every uniformizer τ. Write $\tau = u\varpi$, where $u \in \mathcal{O}^\times$. For each entry in column j, make the substitution

$$b'_{ij} = u^{-d_j} b_{ij},$$

for $i < j$. Then

$$B(\tau, \mathbf{b}) = B(\varpi, \mathbf{b}') \operatorname{diag}(u^{d_1}, \ldots, u^{d_m}).$$

Indeed, multiplication on the right multiplies column j by u^{d_j}: its diagonal entry becomes τ^{d_j}, and its entries above the diagonal become $u^{d_j} b'_{ij} = b_{ij}$.

The diagonal matrix on the right is invertible over $\mathcal{O}$. The two matrices therefore generate the same lattice, say M. Multiplication by a unit maps M bijectively to itself, so

$$\tau(M \circ M) \subseteq M \quad \text{if and only if} \quad \varpi(M \circ M) \subseteq M.$$

Thus $\mathbf{b} \mapsto \mathbf{b}'$ gives a bijection $D_\tau \to D_\varpi$. Each coordinate is multiplied by a unit, which preserves additive Haar measure. Consequently,

$$\mu_K^{\otimes b}(D_\tau) = \mu_K^{\otimes b}(D_\varpi) = q^{-E} g_\alpha^K(q).$$

Finally, fix $T = \tau$ in the integral (4.7). If τ is a uniformizer, the integral over $\mathbf{b}$ is exactly the volume of D_τ; otherwise the integrand is zero. Fubini's theorem therefore gives

$$J_K = \int_{\mathfrak{m} \setminus \mathfrak{m}^2} \mu_K^{\otimes b}(D_\tau)\, d\mu_K(\tau).$$

The integrand is constant by the preceding calculation. Moreover, the set of uniformizers has measure

$$\mu_K(\mathfrak{m} \setminus \mathfrak{m}^2) = q^{-1} - q^{-2}.$$

It follows that

$$J_K = (q^{-1} - q^{-2}) q^{-E} g_\alpha^K(q) = (1 - q^{-1}) q^{-E-1} g_\alpha^K(q),$$

as claimed. □

Proposition 4.4. *Fix a positive composition α. Outside a finite set of residue characteristics depending on α, the count $g_\alpha^K(q)$ depends only on the residue field of K.*

Proof. Applying Lemma 4.2 with the polynomials $f_0 = T, f_1 = F_1, \ldots, f_s = F_s$ and the set Λ specified above shows that J_K depends only on the residue field outside finitely many residue characteristics depending on α. Lemma 4.3 gives

$$g_\alpha^K(q) = \frac{q^{E+1}}{1 - q^{-1}} J_K.$$

Since E depends only on α and q is the cardinality of the residue field, the assertion follows. □

Corollary 4.5. *For all sufficiently large primes p,*

$$g_{\alpha_0}^{\mathbb{Q}_p}(p) = g_{\alpha_0}^{\mathbb{F}_p((t))}(p). \tag{4.9}$$

Proof. The fields $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ have residue field $\mathbb{F}_p$. Apply Proposition 4.4. □

Remark 4.6. Proposition 4.4 applies to every positive HNF composition. For a fixed rank and colength there are only finitely many such compositions. A single finite set of excluded primes therefore suffices for all of them.

5. The equal-characteristic HNF scheme and loop rotation

The comparison in Section 4 allows us to work over $k[[t]]$. We write the HNF entries as polynomials in t and express closure under multiplication as polynomial equations in their coefficients. The substitution $t \mapsto \lambda t$, called *loop rotation*, induces a $\mathbb{G}_m$-action on this scheme. We use this action to show that, in $K_0(\mathrm{Var}_{\mathbb{Q}})$, the class of the resulting variety differs from that of its fixed locus by a multiple of $\mathbb{L} - 1$.

Recall from Section 3 that

$$\alpha_0 = (3,3,2,2,2,2,1,1,1), \qquad d_i = \alpha_{0,i} - 1.$$

Over $k[[t]]$, the full HNF matrix corresponding to (3.2) is

$$\mathcal{A}(t) = \begin{pmatrix} t^3 & tb_{12} & tb_{13} & tb_{14} & tb_{15} & tb_{16} & tb_{17} & tb_{18} & tb_{19} & 1 \\ 0 & t^3 & tb_{23} & tb_{24} & tb_{25} & tb_{26} & tb_{27} & tb_{28} & tb_{29} & 1 \\ 0 & 0 & t^2 & tb_{34} & tb_{35} & tb_{36} & tb_{37} & tb_{38} & tb_{39} & 1 \\ 0 & 0 & 0 & t^2 & tb_{45} & tb_{46} & tb_{47} & tb_{48} & tb_{49} & 1 \\ 0 & 0 & 0 & 0 & t^2 & tb_{56} & tb_{57} & tb_{58} & tb_{59} & 1 \\ 0 & 0 & 0 & 0 & 0 & t^2 & tb_{67} & tb_{68} & tb_{69} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $b_{ij} = b_{ij}(t)$. Dividing the first nine columns by t and deleting their last coordinate gives the matrix $\mathcal{B}(t)$.

By Lemma 3.1, each entry in row i above the diagonal is represented by a polynomial of degree less than d_i. Thus

$$b_{ij}(t) = \begin{cases} x_{ij} + ty_{ij}, & i = 1,2, \\ x_{ij}, & 3 \le i \le 6, \\ 0, & 7 \le i \le 8, \end{cases} \qquad i < j \le 9,$$

and explicitly

$$\mathcal{B}(t) = \begin{pmatrix} t^2 & x_{12}+ty_{12} & x_{13}+ty_{13} & x_{14}+ty_{14} & x_{15}+ty_{15} & x_{16}+ty_{16} & x_{17}+ty_{17} & x_{18}+ty_{18} & x_{19}+ty_{19} \\ 0 & t^2 & x_{23}+ty_{23} & x_{24}+ty_{24} & x_{25}+ty_{25} & x_{26}+ty_{26} & x_{27}+ty_{27} & x_{28}+ty_{28} & x_{29}+ty_{29} \\ 0 & 0 & t & x_{34} & x_{35} & x_{36} & x_{37} & x_{38} & x_{39} \\ 0 & 0 & 0 & t & x_{45} & x_{46} & x_{47} & x_{48} & x_{49} \\ 0 & 0 & 0 & 0 & t & x_{56} & x_{57} & x_{58} & x_{59} \\ 0 & 0 & 0 & 0 & 0 & t & x_{67} & x_{68} & x_{69} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lemma 5.1. *Every $N_9(k[[t]])$-orbit in $\mathscr{B}_d(k((t)))$ contains a unique matrix whose (i,j) entry, for $i<j$, is a polynomial of degree less than d_i.*

Proof. Apply Lemma 3.1 over $k[[t]]$, using the polynomials of degree less than d_i as representatives modulo t^{d_i} in row i. □

There are 33 variables x_{ij} and 15 variables y_{ij}. Write $\mathbf{z}$ for their collection and set $A=\mathbb{Z}[\mathbf{z}]$. Let t be a separate indeterminate. The matrix $\mathcal{B}(t)$ defined above belongs to $\mathrm{Mat}_9(A[t])$.

The phrase *the α_0 HNF chart* will refer to this affine parameter space of 48 coefficients, with the diagonal entries fixed as above. Its points specify the unique HNF representatives of the corresponding column lattices. Imposing multiplicative closure cuts out the closed subscheme X_{α_0} defined below.

Write b_j for its jth column. For each pair $1\le j\le k\le 9$, define

$$w_{jk}(t)=t(b_j\circ b_k)\in A[t]^9,$$

and write $w_{jk,r}(t)$ for its rth coordinate.

To express w_{jk} as a linear combination of $b_1,\dots,b_9$, we start with the last row and work upwards. At each row, we subtract the terms involving columns whose coefficients have already been comseted, then divide by the diagonal entry.

For each fixed pair (j,k), define the following polynomials successively for $r=9,8,\dots,1$. First set

$$R_{jk,r}(t)=w_{jk,r}(t)-\sum_{s=r+1}^{9}\mathcal{B}_{rs}(t)C_{jk,s}(t). \tag{5.1}$$

Then define the quotient and remainder coefficients by polynomial division:

$$R_{jk,r}(t)=t^{d_r}C_{jk,r}(t)+\sum_{q=0}^{d_r-1}E_{jk;r,q}t^q,\quad \text{where}\quad C_{jk,r}(t)\in A[t]\quad\text{and}\quad E_{jk;r,q}\in A.$$

This identity determines $C_{jk,r}$ and the $E_{jk;r,q}$ uniquely.

At $r=9$, the first sum is empty and $d_9=0$, so

$$R_{jk,9}(t)=C_{jk,9}(t)=w_{jk,9}(t).$$

For a polynomial $P(t)$, write $[t^\nu]P$ for its coefficient of t^ν. We have the following relations

$$C_{jk,r}(t):=\sum_{\nu\ge d_r}\left([t^\nu]R_{jk,r}(t)\right)t^{\nu-d_r} \tag{5.2}$$

and

$$E_{jk;r,q}:=[t^q]R_{jk,r}(t). \tag{5.3}$$

If every remainder vanishes, then $R_{jk,r}=t^{d_r}C_{jk,r}$ for every r. Hence

$$w_{jk,r}=t^{d_r}C_{jk,r}+\sum_{s=r+1}^{9}\mathcal{B}_{rs}C_{jk,s}=\sum_{s=1}^{9}\mathcal{B}_{rs}C_{jk,s},$$

since $\mathcal{B}_{rr}=t^{d_r}$ and $\mathcal{B}_{rs}=0$ for $s<r$. As $\mathcal{B}_{rs}$ is the rth coordinate of b_s, these coordinate identities give

$$w_{jk}=\sum_{s=1}^{9}C_{jk,s}(t)b_s.$$

Conversely, suppose that $w_{jk}=\sum_{s=1}^{9}v_s(t)b_s$ with $v_s(t)\in k[[t]]$. The last row gives $v_9=C_{jk,9}$. If $v_s=C_{jk,s}$ for every $s>r$, the row-r equation gives

$$R_{jk,r}(t)=t^{d_r}v_r(t).$$

The coefficients of degrees less than d_r on the right are zero, so all the remainder coefficients at row r vanish. Dividing by t^{d_r} then gives $v_r = C_{jk,r}$, since the quotient of the polynomial $R_{jk,r}$ is the polynomial already defined above. This completes the descending induction.

In summary, we find that $w_{jk} \in \mathcal{B}k[[t]]^9$ if and only if $E_{jk;r,q} = 0$ for every r, q.

Let $I \subseteq A$ be the ideal generated by the polynomials $E_{jk;r,q}$ for $1 \le j \le k \le 9$, $1 \le r \le 9$, and $0 \le q < d_r$. Define

$$X_{\alpha_0} = \operatorname{Spec}(A/I) \subseteq \mathbb{A}^{48}_{\mathbb{Z}}.$$

Its k-points are precisely the coefficient choices satisfying the lattice closure conditions.

Proposition 5.2. *For every finite field k,*

$$\#X_{\alpha_0}(k) = g_{\alpha_0}^{k((t))}(\#k).$$

Proof. The proof of the result follows from the above discussion. Indeed, a coefficient choice gives one HNF matrix, and the normal-form uniqueness shows that different choices give different lattices. The obstruction equations hold exactly when every product of two basis columns lies in the lattice. Products involving the last column already belong to it because that column is $\mathbf{1}$. This gives the required bijection with the lattices counted by $g_{\alpha_0}^{k((t))}(\#k)$. □

Setting $\mathscr{X} = (X_{\alpha_0})_{\mathrm{red}}$, we note that

$$\#\mathscr{X}(k) = \#X_{\alpha_0}(k) \tag{5.4}$$

for every finite field k.

Corollary 5.3. *For every sufficiently large prime p,*

$$g_{\alpha_0}(p) = \#X_{\alpha_0}(\mathbb{F}_p) = \#\mathscr{X}(\mathbb{F}_p). \tag{5.5}$$

Proof. Corollary 4.5 identifies the fixed-diagonal lattice counts over $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ outside a finite set of primes. Proposition 5.2 identifies the latter count with $\#X_{\alpha_0}(\mathbb{F}_p)$, and (5.4) gives the second equality. □

For $\lambda \in \mathbb{G}_m$, replace t by λt and multiply column j by λ^{-d_j}. This restores its pivot to t^{d_j}.

For $i < j$, write

$$\mathcal{B}_{ij}(t) = \sum_{\nu=0}^{d_i-1} z_{ij}^{(\nu)} t^{\nu}.$$

Thus $z_{ij}^{(\nu)}$ denotes the coefficient of t^{ν} in the (i,j) entry of $\mathcal{B}(t)$. In the notation introduced above, $z_{ij}^{(0)} = x_{ij}$ and $z_{ij}^{(1)} = y_{ij}$. Replacing t by λt and multiplying column j by λ^{-d_j} therefore gives

$$\lambda \cdot z_{ij}^{(\nu)} = \lambda^{\nu - d_j} z_{ij}^{(\nu)}. \tag{5.6}$$

The integer $\nu - d_j$ is the weight of this coordinate.

Lemma 5.4. *Every obstruction polynomial satisfies*

$$E_{jk;r,q}(\lambda \cdot \mathbf{z}) = \lambda^{q - d_j - d_k - 1} E_{jk;r,q}(\mathbf{z}). \tag{5.7}$$

Thus (5.6) defines a $\mathbb{G}_m$-action on X_{α_0} and on $\mathscr{X}$ over $\mathbb{Z}$.

Proof. Use a superscript λ for the transformed expressions. The matrix entries and the product vector satisfy

$$\mathcal{B}_{rs}^{\lambda}(t) = \lambda^{-d_s} \mathcal{B}_{rs}(\lambda t),$$
$$w_{jk,r}^{\lambda}(t) = \lambda^{-d_j - d_k - 1} w_{jk,r}(\lambda t).$$

The extra factor λ^{-1} in the second formula comes from the factor t in $w_{jk} = t(b_j \circ b_k)$.

Suppose that the recursion has already given for $s > r$:

$$C^{\lambda}_{jk,s}(t) = \lambda^{d_s - d_j - d_k - 1} C_{jk,s}(\lambda t).$$

Each product $\mathcal{B}^{\lambda}_{rs} C^{\lambda}_{jk,s}$ then has the common factor $\lambda^{-d_j-d_k-1}$. Equation (5.1) gives

$$R^{\lambda}_{jk,r}(t) = \lambda^{-d_j-d_k-1} R_{jk,r}(\lambda t).$$

Taking the coefficient of t^q proves (5.7). Removing the low-degree terms and dividing by t^{d_r} gives

$$C^{\lambda}_{jk,r}(t) = \lambda^{d_r - d_j - d_k - 1} C_{jk,r}(\lambda t),$$

which is the required formula for this row and completes the descending induction.

The defining ideal is homogeneous for the grading by these weights, so the action restricts to X_{α_0}. Its radical is homogeneous as well, and the action restricts to the reduced scheme $\mathscr{X}$. □

For $i = 1, 2$, write $b_{ij} = x_{ij} + t y_{ij}$. For $3 \le i \le 6$, write $b_{ij} = x_{ij}$. The list of weights is

coordinates	weight	number
x_{12}	-2	1
x_{ij} with $3 \le j \le 6$, and y_{12}	-1	15
x_{ij} with $7 \le j \le 9$, and y_{ij} with $3 \le j \le 6$	0	26
y_{ij} with $7 \le j \le 9$	1	6

Only existing digit variables are included in this table: $i < j$, x_{ij} occurs for $i \le 6$, and y_{ij} occurs for $i \le 2$.

Lemma 5.5. *Let X be a reduced closed subvariety of $\mathbb{A}^M_{\mathbb{Q}}$ or $\mathbb{P}^M_{\mathbb{Q}}$ preserved by a diagonal $\mathbb{G}_m$-action with integer weights. There is a variety $Y/\mathbb{Q}$ such that*

$$[X] = [(X^{\mathbb{G}_m})_{\mathrm{red}}] + (\mathbb{L} - 1)[Y] \quad \text{in } K_0(\mathrm{Var}_{\mathbb{Q}}). \tag{5.8}$$

Proof. First suppose that $X \subseteq \mathbb{A}^M_{\mathbb{Q}}$, with action

$$\lambda \cdot (z_1, \dots, z_M) = (\lambda^{a_1} z_1, \dots, \lambda^{a_M} z_M).$$

For each subset $J \subseteq \{1, \dots, M\}$, set

$$T_J = \{z_i = 0 \text{ for } i \notin J, \quad z_j \neq 0 \text{ for } j \in J\},$$
$$X_J = (X \cap T_J)_{\mathrm{red}}.$$

The X_J form a finite decomposition of X into invariant locally closed subvarieties. Each T_J is a torus with coordinates z_j, $j \in J$.

If $a_j = 0$ for every $j \in J$, the action on T_J is trivial. The corresponding X_J together give precisely $(X^{\mathbb{G}_m})_{\mathrm{red}}$.

Suppose now that some a_j, $j \in J$, is nonzero. Set $m = |J|$ and

$$d = \gcd(a_j : j \in J) > 0.$$

Choose $c = (c_j)_{j \in J} \in \mathbb{Z}^J$ such that $\sum_{j \in J} a_j c_j = d$, and let

$$L = \left\{ u \in \mathbb{Z}^J : \sum_{j \in J} a_j u_j = 0 \right\}.$$

Then $\mathbb{Z}^J = \mathbb{Z}c \oplus L$: for every $n \in \mathbb{Z}^J$,

$$n = \frac{\sum_{j \in J} a_j n_j}{d}\, c + \left(n - \frac{\sum_{j \in J} a_j n_j}{d}\, c \right),$$

where the second term belongs to L, and the decomposition is unique.

Choose a basis $u^{(1)},\dots,u^{(m-1)}$ of L. Define Laurent monomials

$$s=\prod_{j\in J} z_j^{c_j},$$

$$v_\ell=\prod_{j\in J} z_j^{u_j^{(\ell)}}$$

for $1\le \ell\le m-1$. Since $c,u^{(1)},\dots,u^{(m-1)}$ form an integral basis, every z_j is a Laurent monomial in these new coordinates. They therefore identify

$$T_J\simeq \mathbb{G}_m\times(\mathbb{G}_m)^{m-1},$$

with action

$$\lambda\cdot(s,v_1,\dots,v_{m-1})=(\lambda^d s,v_1,\dots,v_{m-1}).$$

Let $Y_J\subseteq(\mathbb{G}_m)^{m-1}$ be the reduced closed subvariety obtained by setting $s=1$ in X_J. For any $s\in\overline{\mathbb{Q}}^\times$, choose $\lambda\in\overline{\mathbb{Q}}^\times$ with $\lambda^d=s^{-1}$. We find that $(s,v)\in X_J(\overline{\mathbb{Q}})$ if and only if $(1,v)\in X_J(\overline{\mathbb{Q}})$ if and only if $v\in Y_J(\overline{\mathbb{Q}})$. Thus X_J and $\mathbb{G}_m\times Y_J$ have the same $\overline{\mathbb{Q}}$-points. Since both are reduced closed subvarieties of T_J, they are equal over $\mathbb{Q}$. It follows that

$$[X_J]=(\mathbb{L}-1)[Y_J].$$

Taking Y to be the finite disjoint union of these Y_J and adding the classes of all the strata proves the affine case.

For the projective case, write the coordinates as $z_0,\dots,z_M$, with weights $a_0,\dots,a_M$. Decompose $\mathbb{P}^M_{\mathbb{Q}}$ into the invariant affine strata

$$U_i=\{z_0=\cdots=z_{i-1}=0,\ z_i\neq 0\},$$

for $0\le i\le M$. The coordinates z_j/z_i, $j>i$, identify U_i with $\mathbb{A}^{M-i}_{\mathbb{Q}}$, and have weights a_j-a_i. Apply the affine case to each reduced closed subvariety $(X\cap U_i)_{\mathrm{red}}\subseteq U_i$. Adding the resulting identities gives (5.8), since these strata also decompose the reduced fixed locus. The required Y is the disjoint union of the varieties obtained from the affine case. □

In what follows, we let $X=\mathscr{X}_{\mathbb{Q}}$ and set $F=(X^{\mathbb{G}_m})_{\mathrm{red}}$.

Proposition 5.6. *There is a variety $Y/\mathbb{Q}$ such that*

$$[X]=[F]+(\mathbb{L}-1)[Y] \tag{5.9}$$

in $K_0(\mathrm{Var}_{\mathbb{Q}})$.

Proof. The affine variety $X\subseteq\mathbb{A}^{48}_{\mathbb{Q}}$ is reduced and is preserved by the diagonal action in (5.6). Lemma 5.5 therefore gives (5.9). □

In the α_0 HNF chart, the loop-rotation weights in (5.6) give the fixed-point equations

$$x_{ij}=0\quad\text{for}\quad 1\le i<j\le 6,\qquad y_{12}=0, \tag{5.10}$$

and

$$y_{17}=y_{18}=y_{19}=y_{27}=y_{28}=y_{29}=0. \tag{5.11}$$

The remaining 26 coordinates have weight zero. The fixed matrix therefore has the block form

$$B_F=\begin{pmatrix} t^2I_2 & tY & P\\ 0 & tI_4 & Q\\ 0 & 0 & I_3\end{pmatrix},$$

where

$$Y=(y_{ij})_{\substack{1\le i\le 2\\ 3\le j\le 6}},\qquad P=(x_{ij})_{\substack{1\le i\le 2\\ 7\le j\le 9}},\quad\text{and}\quad Q=(x_{ij})_{\substack{3\le i\le 6\\ 7\le j\le 9}}.$$

There are eight entries in Y, six in P, and twelve in Q.

For a finite-dimensional vector space W, the projective space $\mathbb{P}(W)$ parametrizes its lines and $\mathrm{Gr}(r, W)$ parametrizes its r-dimensional subspaces. We have that

$$\dim \mathrm{Gr}(r, W) = r(\dim W - r).$$

A flag variety parametrizes nested subspaces of prescribed dimensions. An affine paving of a variety means a finite filtration by closed subvarieties whose successive differences are affine spaces. For a Grassmannian or flag variety, one obtains such a filtration by ordering the Schubert cells compatibly with their closure relations. In particular, the cohomology of these projective varieties is Tate.

Let k be a field, set $\mathcal{O} = k[[t]]$, and write $W = k^{10}$. Let $\mathbf{1} = (1, \ldots, 1) \in W$. For $u, u' \in W$, define their coordinatewise product by

$$u \circ u' = (u_1 u'_1, \ldots, u_{10} u'_{10}).$$

For a linear subspace $U \subseteq W$, set

$$U^{\circ 2} = \mathrm{Span}_k\{u \circ u' : u, u' \in U\}.$$

Consider coordinate projections $\pi_4 : W \longrightarrow k^4$ defined by $(a_1, \ldots, a_{10}) \mapsto (a_7, a_8, a_9, a_{10})$, and $\pi_8 : W \longrightarrow k^8$, given by $(a_1, \ldots, a_{10}) \mapsto (a_3, \ldots, a_{10})$. Let $\mathrm{Fl}(4, 8; W)$ denote the flag variety parametrizing pairs of subspaces $U \subset V \subset W$ with $\dim U = 4$ and $\dim V = 8$. Let $\mathcal{C}_k$ be its reduced closed subvariety defined by

$$\mathbf{1} \in U \quad \text{and} \quad U^{\circ 2} \subseteq V. \tag{5.12}$$

Write $\mathcal{C}_k^\circ$ for the open subvariety on which the restrictions

$$\pi_4|_U : U \longrightarrow k^4 \quad \text{and} \quad \pi_8|_V : V \longrightarrow k^8 \tag{5.13}$$

are linear isomorphisms. Thus, on $\mathcal{C}_k^\circ$, a vector of U is uniquely determined by its last four coordinates, and a vector of V is uniquely determined by its last eight coordinates. Each condition is the nonvanishing of the determinant of the indicated coordinate projection. These are the two *big-cell conditions* used below. Equivalently, U is the graph of a linear map $k^{\{7,8,9,10\}} \to k^{\{1,\ldots,6\}}$, and V is the graph of a linear map $k^{\{3,\ldots,10\}} \to k^{\{1,2\}}$, with the additional incidence and multiplication conditions in (5.12).

Proposition 5.7. *Let F_k be the reduced fixed locus of loop rotation on the α_0 HNF scheme over k. There is an isomorphism of k-varieties*

$$F_k \simeq \mathcal{C}_k^\circ.$$

Under this isomorphism, a pair $(U, V) \in \mathcal{C}_k^\circ$ corresponds to the lattice

$$R_{U,V} = k\mathbf{1} \oplus tU \oplus t^2 V \oplus t^3 \mathcal{O}^{10} \subseteq \mathcal{O}^{10}.$$

Proof. Let $e_1, \ldots, e_{10}$ be the standard basis of k^{10}. For a matrix B_F, define

$$u_j = e_j + \sum_{i=1}^{6} x_{ij} e_i \quad \text{for} \quad 7 \le j \le 9,$$
$$v_j = e_j + y_{1j} e_1 + y_{2j} e_2 \quad \text{for} \quad 3 \le j \le 6.$$

The coordinate matrix of the ordered list $e_1, e_2, v_3, v_4, v_5, v_6, u_7, u_8, u_9, \mathbf{1}$ is

$$H = \begin{pmatrix} I_2 & Y & P & \mathbf{1}_2 \\ 0 & I_4 & Q & \mathbf{1}_4 \\ 0 & 0 & I_3 & \mathbf{1}_3 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\mathbf{1}_a$ denotes the column of a ones. This matrix is upper triangular with diagonal entries equal to 1, so the listed vectors form a basis of k^{10}.

Set

$$U = \mathrm{Span}_k(u_7, u_8, u_9, \mathbf{1}) \quad \text{and} \quad V = \mathrm{Span}_k(v_3, v_4, v_5, v_6, u_7, u_8, u_9, \mathbf{1}).$$

Then $\mathbf{1} \in U \subset V$, with $\dim U = 4$ and $\dim V = 8$. In these bases, the matrices of $\pi_4|_U$ and $\pi_8|_V$ are the lower-right 4×4 and 8×8 blocks of H, respectively. Both are upper triangular with diagonal entries equal to 1. Thus the projection conditions (5.13) hold.

Let $R \subseteq \mathcal{O}^{10}$ be the full HNF lattice. Its generators are $t^3e_1, t^3e_2, t^2v_3, t^2v_4, t^2v_5, t^2v_6, tu_7, tu_8, tu_9, \mathbf{1}$. The constant vectors in this list form the basis given by H. Since each generator is multiplied by a power of t at most three, R contains $t^3\mathcal{O}^{10}$. More explicitly, the coefficient of t^0 in an element of R belongs to $k\mathbf{1}$, the coefficient of t belongs to U, and the coefficient of t^2 belongs to V. Every choice of these three coefficients is obtained from the displayed generators, and all higher coefficients are unrestricted. Hence

$$R = R_{U,V} = k\mathbf{1} \oplus tU \oplus t^2V \oplus t^3\mathcal{O}^{10}.$$

In particular, its colength is

$$\dim_k(\mathcal{O}^{10}/R) = (10-1) + (10-4) + (10-8) = 17.$$

We now check multiplication. The vector $\mathbf{1}$ is the identity for coordinatewise multiplication, so multiplication by an element of $k\mathbf{1}$ preserves R. A product of terms whose degrees add to at least three belongs to $t^3\mathcal{O}^{10} \subseteq R$. The only remaining products are

$$(tu) \circ (tu') = t^2(u \circ u'),$$

for $u, u' \in U$. By the displayed decomposition of R, such a product belongs to R if and only if $u \circ u' \in V$. Thus R is closed under multiplication if and only if $U^{\circ 2} \subseteq V$. A point of F_k therefore determines a point of $\mathcal{C}_k^\circ$.

Conversely, let $(U, V) \in \mathcal{C}_k^\circ$. Using the two projection isomorphisms, define

$$u_j = (\pi_4|_U)^{-1}(\pi_4(e_j)) \quad \text{for} \quad 7 \le j \le 9,$$

and

$$v_j = (\pi_8|_V)^{-1}(\pi_8(e_j)) \quad \text{for} \quad 3 \le j \le 6.$$

The prescribed projections force these vectors to have the forms

$$u_j = e_j + \sum_{i=1}^{6} x_{ij}e_i,$$
$$v_j = e_j + y_{1j}e_1 + y_{2j}e_2.$$

They uniquely determine the eighteen coefficients x_{ij} and the eight coefficients y_{ij}. Together with $\mathbf{1}$, they give the bases of U and V used above: their projected coordinate matrices are the same invertible triangular blocks of H.

These coefficients reconstruct a unique matrix B_F. Its full lattice is $R_{U,V}$, which is closed under multiplication because $U^{\circ 2} \subseteq V$. It therefore defines a point of F_k. The two constructions are inverse.

Finally, these constructions are morphisms. In the forward direction, the basis matrices for U and V have polynomial entries in the coefficients of B_F, and their indicated maximal minors are equal to 1. They therefore define regular maps to the two Grassmannians. In the reverse direction, the projection conditions express U and V as graphs of linear maps to the omitted coordinates. The recovered x_{ij} and y_{ij} are entries of these graph maps, hence regular functions on the stated charts. Thus the constructions give inverse morphisms between the reduced varieties F_k and $\mathcal{C}_k^\circ$. □

Remark 5.8. The Smith exponents of a full lattice $L \subseteq \mathcal{O}^m$ are the integers $a_1 \leq \cdots \leq a_m$ defined by

$$\mathcal{O}^m/L \simeq \bigoplus_{i=1}^{m} \mathcal{O}/(t^{a_i}).$$

A basis adapted to $k\mathbf{1} \subset U \subset V \subset k^{10}$ shows that $R_{U,V}$ has Smith exponents

$$(0, 1, 1, 1, 2, 2, 2, 2, 3, 3).$$

Let S_k be the reduced locus in the α_0 HNF scheme defined by (5.10), without imposing (5.11). Then $S_k \simeq F_k \times \mathbb{A}^6_k$. Indeed, after splitting off the scalar direction as in Lemma 2.1, the residue flag has dimensions 3 and 7 in k^9. Lemma 2.2 parametrizes its lattice lifts by $\mathrm{Hom}_k(U_0, k^9/V_0)$ with $\dim U_0 = 3$ and $\dim V_0 = 7$. In the HNF bases, the six entries of this homomorphism are precisely y_{ij} with $i = 1, 2$ and $j = 7, 8, 9$. Multiplicative closure depends only on the residue flag, so these six coordinates are unrestricted. Setting them equal to zero recovers F_k.

6. Elliptic curves with marked points

In this section, we consider flags $\mathbf{1} \in U \subset V \subset k^{10}$ with $\dim U = 4$, $\dim V = 8$, and $U^{\circ 2} \subseteq V$. For the sake of discussion, first omit the projection conditions (5.13). Choose a basis $\mathbf{1}, x, y, z$ of U. The ten coordinate functionals on k^{10} give ten ordered points

$$P_i = [1 : x_i : y_i : z_i] \in \mathbb{P}^3$$

for $1 \leq i \leq 10$. A quadratic relation among the vectors $\mathbf{1}, x, y, z$, with multiplication taken coordinatewise, is precisely a quadratic form vanishing at all the P_i. Since $\mathrm{Sym}^2 U$ has dimension ten and its multiplicative image $U^{\circ 2}$ lies in the eight-dimensional space V, there are at least two independent quadratic forms vanishing on these points. We study the locus where the ten points are distinct, the space of quadratic relations has dimension exactly two, and the common zero scheme of these quadrics is smooth. Their intersection is then a genus-one curve $E \subset \mathbb{P}^3$ of degree four, and the first marked point supplies an origin. This embedding is called an elliptic normal quartic. On this locus, $\dim U^{\circ 2} = 8$, so $V = U^{\circ 2}$. We show that this is described by elliptic curves with ten ordered distinct markings, a degree-four line bundle L, and a projective section $[s]$ avoiding those markings.

Over finite fields, inclusion–exclusion counts the choices of $(L, [s])$. Summing over the marked elliptic curves then expresses the point count through moments of their Frobenius traces up to degree ten. Birch's tenth-moment formula introduces the coefficient $\tau(p)$ of the weight-twelve cusp form Δ. We subsequently impose the two HNF projection conditions (5.13) and show that the excluded loci contribute no Tate twist of the object attached to Δ.

The multiplication map

$$\mu_U : \mathrm{Sym}^2 U \longrightarrow k^{10} \tag{6.1}$$

has image $U^{\circ 2}$. Its ith coordinate evaluates the corresponding quadratic form at P_i. Hence

$$\begin{aligned} \ker \mu_U &= H^0(\mathbb{P}^3, I_{\{P_1,\ldots,P_{10}\}}(2)), \\ \dim U^{\circ 2} &= 10 - \dim \ker \mu_U. \end{aligned} \tag{6.2}$$

Here I is the ideal sheaf of the reduced finite set, so repeated points impose no additional condition. A two-dimensional space of quadratic forms is called a pencil. The identification in (6.2) uses the chosen basis of U. Intrinsically, the quadrics belong to $\mathrm{Sym}^2 U = H^0(\mathbb{P}(U^\vee), \mathcal{O}(2))$, and P_i is the line in $U^\vee$ generated by the restriction of the ith coordinate functional. This functional is nonzero because it takes the value 1 on $\mathbf{1}$. Changing the basis only changes the projective coordinates; the kernel and its zero scheme are intrinsic to U.

Let $\mathcal{E}_4$ be the locus of these flags for which the P_i are distinct, $\dim\ker\mu_U = 2$, and the common zero scheme of this pencil is a smooth curve. On $\mathcal{E}_4$ we have $\dim U^{\circ 2} = 8$, so $V = U^{\circ 2}$ and there is no further choice of V. We use the characteristic-zero parameter space for cohomology and its models over finite fields for point counts.

A smooth proper geometrically connected genus-one curve E with a point $q_0 \in E(k)$ is an elliptic curve. The map $q \mapsto \mathcal{O}_E(q - q_0)$ identifies E with $\mathrm{Pic}^0(E)$ (cf. [18, Chapter IV, Theorem 4.11]; see also [37, Chapter III, Proposition 3.4(c)–(d)]). The space $\mathrm{Pic}^d(E)$ parametrizes degree-d line bundles; $L \mapsto L(-dq_0)$ identifies it with $\mathrm{Pic}^0(E)$. Riemann–Roch gives $h^0(E, L) = d$, $h^1(E, L) = 0$, where $d = \deg L > 0$, and

$$h^0(E, M) = \begin{cases} 1, & M \simeq \mathcal{O}_E, \\ 0, & M \not\simeq \mathcal{O}_E \end{cases}$$

when $\deg M = 0$ (see [18, Chapter IV, Theorem 1.3 and Example 1.3.6]).

Lemma 6.1. *A smooth complete intersection of two quadrics in $\mathbb{P}^3$ is a geometrically connected genus-one curve of degree four. Conversely, a degree-four line bundle L on a smooth proper geometrically connected genus-one curve E/k embeds E in $\mathbb{P}^3$. The multiplication map*

$$\mathrm{Sym}^2 H^0(E, L) \longrightarrow H^0(E, L^2)$$

is surjective, with two-dimensional kernel, and the two quadrics in this kernel cut out E scheme-theoretically.

Proof. These assertions may be checked after extending k to an algebraic closure. The first assertion follows from [13, Lemma 5.16(ii) and Proposition 5.10(i)], with $n = 4$.

Conversely, Riemann–Roch and [18, Chapter IV, Corollary 3.2(b)] show that L embeds E in $\mathbb{P}^3$. The multiplication map is surjective by [13, Lemma 5.5]. Since $h^0(E, L) = 4$ and $h^0(E, L^2) = 8$, its kernel has dimension two. By [13, Proposition 5.3], these two quadrics generate the homogeneous ideal of the image.

The assertions over k follow by faithfully flat descent and cohomology base change. More explicitly, for an extension K/k, the natural maps $H^0(E, L) \otimes_k K \to H^0(E_K, L_K)$ and $H^0(E, L^2) \otimes_k K \to H^0(E_K, L_K^2)$ are isomorphisms. The multiplication map and its kernel therefore commute with extension of scalars. If its cokernel becomes zero over $\bar{k}$, it was already zero over k, since $\bar{k}/k$ is faithfully flat. The morphism defined by L is a closed immersion for the same reason. Indeed, being a closed immersion can be checked after faithfully flat base change. Finally, let Z be the common zero scheme of the kernel quadrics. The inclusion $E \subseteq Z$ becomes an equality over $\bar{k}$, so its ideal quotient becomes zero after extension of scalars and is already zero over k. This gives the asserted scheme-theoretic equality over k.

□

For $n \geq 1$, let $\mathcal{M}_{1,n}$ denote the moduli stack of smooth proper geometrically connected genus-one curves with n ordered distinct marked points. Choosing the first marking as the origin identifies its objects with elliptic curves carrying $n - 1$ further ordered distinct nonzero points. For $n = 10, 11$, every automorphism is trivial, since a nonidentity elliptic-curve automorphism fixes at most four points.

To obtain these spaces as varieties over k, choose $N \geq 3$ invertible in k. The full-level moduli scheme $Y(N)$ and its universal elliptic curve $\mathcal{E}_N \to Y(N)$ exist by [25, Corollary 4.7.2].

Let $T_n(N)$ be the open subset of the $(n-1)$-fold fiber product of $\mathcal{E}_N$ over $Y(N)$ where the chosen points are pairwise distinct and nonzero. Since $\mathcal{E}_N$ is projective over the affine scheme $Y(N)$, this open subset is quasi-projective. For $n = 10, 11$, changing the level structure gives a

free action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, and

$$\mathcal{M}_{1,n} \simeq T_n(N)/\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

The quotient is a scheme by [38, Tag 07S7]. Thus $\mathcal{M}_{1,10}$ and $\mathcal{M}_{1,11}$ are fine moduli varieties.

Let B be a variety and let E be a locally free $\mathcal{O}_B$-module of rank m. We use the convention

$$\mathbb{P}(E) = \mathrm{Proj}_B \mathrm{Sym}(E^\vee).$$

For a morphism $g : T \to B$, its T-points correspond to line subbundles $\mathcal{L} \subseteq g^*E$. Here a subbundle means a locally free subsheaf whose quotient is locally free. Thus the fiber over $b \in B$ is the projective space of lines in E_b.

For integers $0 < r_1 < \cdots < r_s < m$, the partial flag bundle $\mathrm{Fl}_{r_1,\ldots,r_s}(E)$ represents the functor assigning to $g : T \to B$ the chains of subbundles

$$\mathcal{U}_1 \subset \cdots \subset \mathcal{U}_s \subset g^*E \quad \text{with} \quad \mathrm{rank}\,\mathcal{U}_i = r_i.$$

The Grassmannian bundle $\mathrm{Gr}_r(E)$ is the case $s = 1$ and $r_1 = r$.

A torsor under a vector bundle $E \to B$ is a B-scheme which is Zariski locally isomorphic to E, with E acting by translations. If E has rank a, such a torsor is therefore a Zariski locally trivial bundle with fiber $\mathbb{A}^a$.

The following lemma gives $[X] = [F][B]$ and describes the relative compactly supported cohomology as sums of constant Tate sheaves.

Lemma 6.2. *Let B be a variety over $\mathbb{Q}$. Suppose that $f : X \to B$ is a Zariski locally trivial affine-space bundle, a projective bundle, or a Grassmannian or partial flag bundle associated to a vector bundle. Let F be its standard fiber, and write $P_F(T) = \sum_\lambda T^{d_\lambda}$ for the polynomial given by its usual affine-cell decomposition. Then*

$$[X] = P_F(\mathbb{L})[B].$$

In both realizations, every simple constituent of $H^i_c(X)$ is a Tate twist of a simple constituent of $H^j_c(B)$ for some j.

Proof. Choose a finite locally closed decomposition $B = \coprod_\alpha B_\alpha$ over which the bundle is trivial. The scissor relation gives

$$[X] = \sum_\alpha [B_\alpha][F] = P_F(\mathbb{L})[B].$$

Let $f : X \to B$ be an affine-space bundle of fiber dimension a. For every geometric point $\bar{b} \to B$, compact-support base change identifies

$$(R^q f_! \mathbb{Q}_\ell)_{\bar{b}} \simeq H^q_c(X_{\bar{b}}, \mathbb{Q}_\ell) \simeq H^q_c(\mathbb{A}^a_{\bar{b}}, \mathbb{Q}_\ell).$$

The compactly supported cohomology of affine space is

$$H^q_c(\mathbb{A}^a_{\bar{k}}, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(-a), & q = 2a, \\ 0, & q \neq 2a, \end{cases}$$

where $\ell \neq \mathrm{char}(k)$; see [4, §4, Example (iv)]. Thus every geometric stalk of $R^q f_! \mathbb{Q}_\ell$ vanishes when $q \neq 2a$, and consequently $R^q f_! \mathbb{Q}_\ell = 0$ in these degrees.

Consider the relative trace map

$$\mathrm{Tr}_f : R^{2a} f_! \mathbb{Q}_\ell(a) \longrightarrow \mathbb{Q}_{\ell,B}$$

of [2, Exposé XVIII, Théorème 2.9]. Its compatibility with base change identifies its map on the stalk at $\bar{b}$ with

$$H^{2a}_c(\mathbb{A}^a_{\bar{b}}, \mathbb{Q}_\ell(a)) \xrightarrow{\sim} \mathbb{Q}_\ell,$$

the affine-space trace isomorphism constructed in [2, Exposé XVIII, (2.8.1)]. A morphism of étale sheaves is an isomorphism if it is an isomorphism on every geometric stalk. Hence the relative trace is an isomorphism, giving $R^{2a}f_!\mathbb{Q}_\ell \simeq \mathbb{Q}_{\ell,B}(-a)$.

Now apply the compact-support Leray spectral sequence

$$E_2^{p,q} = H_c^p(B, R^q f_!\mathbb{Q}_\ell) \Longrightarrow H_c^{p+q}(X, \mathbb{Q}_\ell).$$

All rows except $q = 2a$ vanish. In total degree i, the only surviving term is therefore

$$E_2^{i-2a,2a} = H_c^{i-2a}(B, \mathbb{Q}_{\ell,B}(-a)) = H_c^{i-2a}(B, \mathbb{Q}_\ell)(-a).$$

There are no nonzero differentials, and the resulting filtration has only one nonzero graded piece. This proves

$$H_c^i(X, \mathbb{Q}_\ell) \simeq H_c^{i-2a}(B, \mathbb{Q}_\ell)(-a).$$

Since the trace and base-change maps are defined over $\mathbb{Q}$, this isomorphism respects the $G_{\mathbb{Q}}$-action.

For projective and Grassmannian bundles, the cohomology of each fiber has a basis obtained by restricting algebraic classes on X. These are powers of the hyperplane class for projective bundles and Schubert classes for Grassmannian bundles; see [14, Proposition 14.6.5]. Pullback from B, followed by cup product with these classes, gives

$$H_c^i(X) \simeq \bigoplus_r H_c^{i-2r}(B)(-r)^{\oplus c_r},$$

where $P_F(T) = \sum_r c_r T^r$. Over an open set where the bundle is trivial, this is the Künneth formula. The compact-support Mayer–Vietoris sequence then gives the isomorphism over B; compare [31, Theorem 23.2]. Because the classes are algebraic and defined over $\mathbb{Q}$, the resulting maps respect mixed Hodge structures in Betti cohomology and the Galois action in ℓ-adic cohomology.

A partial flag bundle is a tower of Grassmannian bundles, so its formula follows by applying the Grassmannian formula at each stage. □

Proposition 6.3. *The unrestricted locus $\mathcal{E}_4$ parametrizes isomorphism classes of data*

$$(E; q_1, \dots, q_{10}; L, [s]),$$

where $(E; q_1, \dots, q_{10}) \in \mathcal{M}_{1,10}$, $L \in \mathrm{Pic}^4(E)$, and $[s] \in \mathbb{P}H^0(E, L)$ satisfies $s(q_i) \neq 0$ for every i.

Proof. Let $(U, V) \in \mathcal{E}_4$ and choose a basis $\mathbf{1}, x, y, z$ of U. Write X_0, X_1, X_2, X_3 for the corresponding homogeneous coordinates on $\mathbb{P}(U^\vee)$. By definition, the two quadrics in $\ker \mu_U$ cut out a smooth curve E. By Lemma 6.1, this is a geometrically connected genus-one curve of degree four. Set $L = \mathcal{O}_E(1)$, $q_i = P_i$ and $s = X_0|_E$. The points q_i are distinct by the definition of $\mathcal{E}_4$. Since $P_i = [1 : x_i : y_i : z_i]$, the section s does not vanish at any q_i. Changing the chosen basis of U, with $\mathbf{1}$ still first, changes the projective coordinates and carries E, its markings, and the section s to the corresponding objects in the new coordinates. Thus the isomorphism class of $(E; q_1, \dots, q_{10}; L, [s])$ is independent of the chosen basis.

Conversely, suppose that such data are given. By the Riemann–Roch theorem,

$$\dim H^0(E, L) = 4.$$

Consider the map

$$\mathrm{ev}_s : H^0(E, L) \longrightarrow k^{10},$$

defined by

$$\mathrm{ev}_s(\sigma) := \left(\frac{\sigma(q_1)}{s(q_1)}, \dots, \frac{\sigma(q_{10})}{s(q_{10})} \right).$$

If $\mathrm{ev}_s(\sigma) = 0$, then σ vanishes at all ten marked points. A nonzero section of a degree-four line bundle has a zero divisor of degree four, so this is impossible unless $\sigma = 0$. Hence ev_s is injective. Its image U has dimension four and contains $\mathbf{1} = \mathrm{ev}_s(s)$.

To determine $U^{\circ 2}$, also consider

$$\mathrm{ev}_{s^2} : H^0(E, L^2) \longrightarrow k^{10}.$$

This map is injective for the same reason. Indeed, since $\deg L^2 = 8$, we find that a nonzero section cannot vanish at ten distinct points. Moreover,

$$\mathrm{ev}_s(\sigma) \circ \mathrm{ev}_s(\tau) = \mathrm{ev}_{s^2}(\sigma\tau)$$

for all $\sigma, \tau \in H^0(E, L)$. By Lemma 6.1, the multiplication map

$$\mu_L : \mathrm{Sym}^2 H^0(E, L) \longrightarrow H^0(E, L^2)$$

is surjective. Thus every section of L^2 is a linear combination of products of sections of L, and the preceding identity gives

$$U^{\circ 2} = \mathrm{ev}_{s^2}(H^0(E, L^2)).$$

The Riemann–Roch theorem then gives $\dim H^0(E, L^2) = 8$. Since ev_{s^2} is injective, we obtain $\dim U^{\circ 2} = 8$.

We next study the quadratic relations. The isomorphism $\mathrm{ev}_s : H^0(E, L) \xrightarrow{\sim} U$ implies that

$$\mu_U \circ \mathrm{Sym}^2(\mathrm{ev}_s) = \mathrm{ev}_{s^2} \circ \mu_L.$$

For $Q \in \mathrm{Sym}^2 H^0(E, L)$, injectivity of ev_{s^2} therefore implies that $\mu_U(\mathrm{Sym}^2(\mathrm{ev}_s)(Q)) = 0$ if and only if $\mu_L(Q) = 0$. Consequently, $\ker \mu_U$ corresponds exactly to $\ker \mu_L$. The latter is the two-dimensional space of quadrics vanishing on the embedding

$$\iota_L : E \hookrightarrow \mathbb{P}(H^0(E, L)^\vee)$$

defined by the complete linear system of L. By Lemma 6.1, these two quadrics cut out $\iota_L(E)$.

The coordinate points are recovered as well. Choose a basis $s, \sigma_1, \sigma_2, \sigma_3$ of $H^0(E, L)$. The corresponding basis of U is $\mathbf{1}, x, y, z$, where

$$x_i = \frac{\sigma_1(q_i)}{s(q_i)}, \qquad y_i = \frac{\sigma_2(q_i)}{s(q_i)} \quad \text{and} \quad z_i = \frac{\sigma_3(q_i)}{s(q_i)}.$$

Hence

$$P_i = [1 : x_i : y_i : z_i] = [s(q_i) : \sigma_1(q_i) : \sigma_2(q_i) : \sigma_3(q_i)] = \iota_L(q_i).$$

In particular, the P_i are distinct. The first coordinate section restricts to s, and the hyperplane bundle restricts to L, so this construction recovers the markings, L, and $[s]$. Since $\mathbf{1} \in U$, every $u \in U$ equals $\mathbf{1} \circ u$ and therefore belongs to $U^{\circ 2}$. Taking $V = U^{\circ 2}$ gives a flag in $\mathcal{E}_4$.

Finally, start with a flag $(U, V) \in \mathcal{E}_4$. The restrictions of X_0, X_1, X_2, X_3 to its curve E form a basis of $H^0(E, L)$. Indeed, a linear relation among these restrictions would vanish at every P_i, and hence give a linear relation among $\mathbf{1}, x, y, z$ in k^{10}. They are therefore independent, and $\dim H^0(E, L) = 4$. We then find that

$$\mathrm{ev}_s(X_0|_E) = \mathbf{1}, \qquad \mathrm{ev}_s(X_1|_E) = x, \qquad \mathrm{ev}_s(X_2|_E) = y \quad \text{and} \quad \mathrm{ev}_s(X_3|_E) = z.$$

Thus the reconstructed subspace is exactly U. On $\mathcal{E}_4$ we have $V = U^{\circ 2}$, so V is recovered too. The two constructions are therefore inverse. □

Set $B = \mathcal{M}_{1,10}$. Thus B parametrizes smooth genus-one curves with ten ordered distinct marked points $(E; q_1, \ldots, q_{10})$. The universal curve $\mathcal{C} \to B$ has fiber E over this marked curve, and each marking gives a section $q_i : B \to \mathcal{C}$. We use q_1 as the origin on each fiber. The relative degree-four Picard scheme $\mathcal{P}^4 \to B$ parametrizes the additional choice of an isomorphism class

of degree-four line bundle L on E. Its fiber over a marked curve is $\mathrm{Pic}^4(E)$. The superscript records the degree of the line bundle.

The relative Grothendieck ring $K_0(\mathrm{Var}\,/B)$ is generated by classes $[Y \to B]$ of varieties equipped with a morphism to B, and its defining relation is

$$[Y \to B] = [Z \to B] + [Y \setminus Z \to B]$$

for a closed subvariety $Z \subseteq Y$. Multiplication is given by fiber product over B. We suppress the structure maps and write $[Y]$. The class $[B]$ is the multiplicative identity, and $\mathbb{L} = [B \times \mathbb{A}^1 \to B]$. In particular, $\mathbb{L}^r[Y]$ is the class of $\mathbb{A}^r \times Y$ over B. The projective-bundle formula gives $[Y] = (1 + \mathbb{L} + \cdots + \mathbb{L}^a)[S]$ for a projective bundle $Y \to S$ with fiber $\mathbb{P}^a$. Set

$$B_4(T) = T^3 - 9T^2 + 36T - 84.$$

Lemma 6.4. *In $K_0(\mathrm{Var}\,/B)$, we have*

$$[\mathcal{E}_4] = B_4(\mathbb{L})[\mathcal{P}^4] + 210[B]. \tag{6.3}$$

For the compactly supported Euler classes of the total spaces, this gives

$$\chi_c(\mathcal{E}_4) = B_4(\mathbb{L})\chi_c(\mathcal{M}_{1,11}) + (10B_4(\mathbb{L}) + 210)\chi_c(\mathcal{M}_{1,10}). \tag{6.4}$$

Moreover, let $E/\mathbb{F}_p$ have ten ordered distinct rational markings, and set $N_E = \#E(\mathbb{F}_p)$. The number of pairs $(L, [s])$ over this fixed marked curve, with $\deg L = 4$ and $s(q_i) \neq 0$ for every i, is

$$N_E B_4(p) + 210. \tag{6.5}$$

Proof. Put $B = \mathcal{M}_{1,10}$. By Proposition 6.3, $\mathcal{E}_4$ is obtained from the projective bundle of pairs $(L, [s])$ over $\mathcal{P}^4$ by removing the ten loci $s(q_i) = 0$.

For $J \subseteq \{1, \dots, 10\}$, set $D_J = \sum_{j \in J} q_j$. If $|J| \leq 3$, Riemann–Roch and cohomology and base change show that sections vanishing along D_J form a projective bundle with fiber $\mathbb{P}^{3-|J|}$ over $\mathcal{P}^4$.

If $|J| = 4$, such a section exists precisely when $L \simeq \mathcal{O}_E(D_J)$ and is then unique up to scalar. The corresponding locus is a copy of B, given by $\mathcal{O}(D_J)$ and its canonical section. If $|J| \geq 5$, the locus is empty.

Inclusion–exclusion therefore gives

$$[\mathcal{E}_4] = [\mathcal{P}^4]([\mathbb{P}^3] - 10[\mathbb{P}^2] + 45[\mathbb{P}^1] - 120) + 210[B].$$

The expression in parentheses is $B_4(\mathbb{L})$, proving (6.3).

The map $q \mapsto \mathcal{O}_E(q + 3q_1)$ identifies $\mathcal{P}^4$ with the universal curve over B. Its ten marked sections are disjoint copies of B, and their complement is $\mathcal{M}_{1,11}$. Hence

$$[\mathcal{P}^4] = [\mathcal{M}_{1,11}] + 10[B].$$

Substituting this identity, forgetting the maps to B, and applying either compactly supported realization proves (6.4).

Finally, the same inclusion–exclusion calculation over $\mathbb{F}_p$ replaces each projective-space class by its rational point count. Since $\#\,\mathrm{Pic}^4(E)(\mathbb{F}_p) = N_E$, it gives $N_E B_4(p) + 210$, proving (6.5). □

Write

$$\Delta(q) = q \prod_{r \geq 1} (1 - q^r)^{24} = \sum_{r \geq 1} \tau(r) q^r$$

for the normalized cusp form of weight 12 and level one.

For an integer $a \geq 0$, write $(X)_a = X(X-1)\cdots(X-a+1)$, with $(X)_0 = 1$. For $E/\mathbb{F}_p$, set $t_E = p + 1 - \#E(\mathbb{F}_p)$ and

$$M_j(p) = \sum_{[E]/\mathbb{F}_p} \frac{t_E^j}{|\,\mathrm{Aut}_{\mathbb{F}_p}(E, 0)|}.$$

The sum runs over isomorphism classes of elliptic curves with their origins. The denominator is the group of automorphisms defined over $\mathbb{F}_p$ which fix the origin. Let $\delta_{i,j}$ denote the delta function, equal to 1 if $i = j$ and 0 otherwise.

Lemma 6.5. *There are polynomials $m_0, \dots, m_{10} \in \mathbb{Z}[T]$ such that, for every prime $p \geq 5$,*

$$M_j(p) = m_j(p) - \delta_{j,10}\tau(p)$$

for $0 \leq j \leq 10$. For odd j, both M_j and m_j are zero. Thus the only moment containing a cusp-form coefficient is the tenth. In our normalization, $M_{10}(p) = 42p^6 - 90p^4 - 75p^3 - 35p^2 - 9p - 1 - \tau(p)$.

Proof. These are the consequences of Birch's moment formulas in [24, Theorem 1]. Their probability measure assigns mass $1/(p|\operatorname{Aut}_{\mathbb{F}_p}(E,0)|)$ to E, so their expectation satisfies $p\mathbb{E}_p(t_E^j) = M_j(p)$. In this normalization the moments below degree ten are polynomials, and the tenth is a polynomial minus $\tau(p)$. Quadratic twisting changes t_E to $-t_E$ and preserves the automorphism-group order, so the odd moments vanish. □

Proposition 6.6. *There is a polynomial $P_4 \in \mathbb{Z}[T]$ such that*

$$\#\mathcal{E}_4(\mathbb{F}_p) = P_4(p) - B_4(p)\tau(p) \tag{6.6}$$

for every sufficiently large prime p.

Proof. Use q_1 as the origin and set $N_E = \#E(\mathbb{F}_p)$. The remaining markings have $(N_E - 1)_9$ choices, and Lemma 6.4 counts $(L, [s])$ over each choice. For $p \geq 5$, an origin-preserving automorphism has the form $(x, y) \mapsto (u^2x, u^3y)$ on a short Weierstrass model. If it is nonidentity, it fixes at most four points: when $u^2 = 1$ its affine fixed points have $y = 0$, and otherwise they have $x = 0$. Its action on ten distinct markings is therefore free. Proposition 6.3 gives

$$\#\mathcal{E}_4(\mathbb{F}_p) = \sum_{[E]/\mathbb{F}_p} \frac{(N_E - 1)_9(N_E B_4(p) + 210)}{|\operatorname{Aut}_{\mathbb{F}_p}(E,0)|}. \tag{6.7}$$

The correspondence in that proposition holds over $\mathbb{Z}[1/N_0]$ for a suitable N_0, so this formula comsetes the point count of our parameter space for all sufficiently large primes. The use of projective sections and the moduli correspondence already removes section scalars and basis choices.

Set

$$F(T, X) = B_4(T)(X)_{10} + 210(X - 1)_9,$$

$$F(T, T + 1 - Y) = \sum_{j=0}^{10} c_j(T)Y^j.$$

Here $c_j \in \mathbb{Z}[T]$. Since $(X)_{10}$ is monic of degree ten and $(X-1)_9$ has degree nine, $c_{10}(T) = B_4(T)$. Substituting $N_E = p + 1 - t_E$ into (6.7) gives

$$\#\mathcal{E}_4(\mathbb{F}_p) = \sum_{j=0}^{10} c_j(p)M_j(p).$$

By Lemma 6.5, this is $P_4(p) - B_4(p)\tau(p)$, where $P_4(T) = \sum_{j=0}^{10} c_j(T)m_j(T) \in \mathbb{Z}[T]$. □

We use the trace formula as recalled in Section 2:

$$\#X(\mathbb{F}_q) = \sum_i (-1)^i \operatorname{Tr}(\operatorname{Frob}_q \mid H_c^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)), \tag{6.8}$$

for $\ell \nmid q$. Here Frob_q is geometric Frobenius. Its action on $\mathbb{Q}_\ell(-r)$ is q^r, so $P(q)$ is the trace of $P(\mathbb{L})$.

Lemma 6.7. *Let $V \in K_\ell$ be unramified outside finitely many primes. If $\mathrm{Tr}(\mathrm{Frob}_p \mid V) = 0$ for every sufficiently large prime p, then $V = 0$. In particular, let $\mathscr{X}$ be a separated finite-type model over $\mathbb{Z}[1/N]$ of a variety $X/\mathbb{Q}$. If $\#\mathscr{X}(\mathbb{F}_p) = P(p)$ for all sufficiently large primes and some $P \in \mathbb{Q}[T]$, then $\chi_c^\ell(X) = P(\mathbb{L})$ in K_ℓ.*

Proof. We clear denominators and write

$$DV = [V^+] - [V^-]$$

with V^+, V^- semisimple representations. The Chebotarev density theorem implies that the Frobenius conjugacy classes outside any finite set are dense. Continuity extends their trace equality to every element of $G_\mathbb{Q}$. The Brauer–Nesbitt theorem then gives $V^+ \simeq V^-$, hence $V = 0$; see [36, Chapter I, Section 2.2, Corollary 2(a), and Section 2.3].

For the second assertion, enlarge N so that the compactly supported cohomology sheaves of $\mathscr{X} \to \operatorname{Spec}\mathbb{Z}[1/N\ell]$ are lisse. Then (6.8) gives

$$\#\mathscr{X}(\mathbb{F}_p) = \mathrm{Tr}\big(\mathrm{Frob}_p \mid \chi_c^\ell(X)\big)$$

for $p \nmid N\ell$. Subtract the trace of $P(\mathbb{L})$ and apply the first assertion. □

A rational mixed Hodge structure consists of a finite-dimensional $\mathbb{Q}$-vector space H, a finite increasing weight filtration $W_\bullet$ on H, and a finite decreasing Hodge filtration $F^\bullet$ on $H_\mathbb{C} = H \otimes_\mathbb{Q} \mathbb{C}$. Each graded piece $\mathrm{Gr}_m^W H = W_m H / W_{m-1} H$, with its induced Hodge filtration, is a pure Hodge structure of weight m. Explicitly, it has a decomposition

$$(\mathrm{Gr}_m^W H)_\mathbb{C} = \bigoplus_{p+q=m} H_m^{p,q},$$

$\overline{H_m^{p,q}} = H_m^{q,p}$ with

$$F^a(\mathrm{Gr}_m^W H)_\mathbb{C} = \bigoplus_{p \geq a} H_m^{p,m-p}.$$

Thus the weight filtration separates the different weights, while the Hodge filtration describes the Hodge types within each weight; see [6, §2.3].

Deligne's theory equips both $H^i(X, \mathbb{Q})$ and $H_c^i(X, \mathbb{Q})$ with canonical mixed Hodge structures for every complex algebraic variety X. For smooth projective X, these structures are pure of weight i; see [6, 7]. The one-dimensional Tate structure $\mathbb{Q}(-r)$ has Hodge type (r, r) and weight $2r$. Tensoring with $\mathbb{Q}(-r)$ shifts every Hodge type (p, q) to $(p + r, q + r)$, and therefore preserves the differences $p - q$.

The Hodge level of a mixed Hodge structure H is the largest $|p - q|$ among the Hodge types occurring in its weight-graded pieces; we set $\mathrm{lev}(0) = 0$. For a Hodge–Tate representation, its Hodge–Tate spread is the difference between its largest and smallest Hodge–Tate weights. We use the convention that $\mathbb{Q}_\ell(-1)$ has Hodge–Tate weight 1. Both Hodge level and Hodge–Tate spread are unchanged by Tate twists.

Let S_{12} be the rank-two object attached to Δ. The form Δ spans the level-one cusp forms of weight 12; there are none of smaller weight. Its Hodge types are $(11, 0), (0, 11)$, and its ℓ-adic realization, denoted by $S_{12,\ell}$, has Hodge–Tate weights $0, 11$ and geometric-Frobenius polynomial

$$X^2 - \tau(p)X + p^{11}$$

for $p \neq \ell$.

For a virtual realization W, let $m_r(W)$ denote the signed multiplicity of the simple object $S_{12}(r)$ in W. Put

$$[W]_{S_{12}} := \sum_{r \in \mathbb{Z}} m_r(W)\, S_{12}(r).$$

Thus we retain the contributions of S_{12} and its Tate twists, with their multiplicities and signs. In the ℓ-adic realization, we use $S_{12,\ell}$ in place of S_{12}. In particular, $[\chi_c(Z)]_{S_{12}} = 0$ is a statement about the alternating sum of the cohomology groups. For each twist, it says that the sum of its multiplicities in even degrees equals the sum in odd degrees. The stronger assertion that no such twist occurs in any cohomology group will be stated explicitly when it is used.

Proposition 6.8. *In the Grothendieck group of rational mixed Hodge structures, we have*

$$[\chi_c(\mathcal{M}_{1,10})]_{S_{12}} = 0 \quad \textit{and} \quad [\chi_c(\mathcal{M}_{1,11})]_{S_{12}} = -S_{12}. \tag{6.9}$$

Proof. These identities follow from Getzler's calculation of the compactly supported Serre characteristic; see [15, Corollary 5.4 and equation (5.6)]. His formula expresses $\chi_c(\mathcal{M}_{1,10})$ as a polynomial in $\mathbb{L}$, and $\chi_c(\mathcal{M}_{1,11})$ as a polynomial in $\mathbb{L}$ minus S_{12}. Taking the S_{12}-parts gives the result. □

Corollary 6.9. *In the rational Grothendieck group K_ℓ,*

$$\chi_c^\ell(\mathcal{E}_4) = P_4(\mathbb{L}) - B_4(\mathbb{L})S_{12,\ell}. \tag{6.10}$$

In either realization, the unrestricted smooth elliptic-quartic locus satisfies

$$[\chi_c(\mathcal{E}_4)]_{S_{12}} = -B_4(\mathbb{L})S_{12}. \tag{6.11}$$

Proof. The trace formula, (6.6), and the Frobenius trace $\tau(p)$ on $S_{12,\ell}$ show that $\chi_c^\ell(\mathcal{E}_4)$ and $P_4(\mathbb{L}) - B_4(\mathbb{L})S_{12,\ell}$ have equal traces at every sufficiently large prime. Lemma 6.7 proves (6.10), hence its S_{12}-part. In the Hodge realization, apply (6.4) and (6.9): the $\mathcal{M}_{1,10}$ term contributes zero and the $\mathcal{M}_{1,11}$ term contributes $-B_4(\mathbb{L})S_{12}$. □

For a variety $B/\mathbb{Q}$ of dimension at most d, the Hodge indices of $H_c^i(B)$ lie in $[0,d]^2$, and every simple ℓ-adic constituent has Hodge–Tate weights in $\{0,\dots,d\}$. To see this, stratify B into smooth affine varieties and choose smooth projective compactifications with simple normal crossings boundary, using [19, 20]. Localization and the Mayer–Vietoris complex of the boundary reduce the assertion to smooth projective varieties of dimension at most d. The bounds then follow from Hodge theory and the Hodge–Tate comparison theorem; see [6, 7, 34] and [35, Corollary 1.8]. In particular, $\operatorname{lev} H_c^i(B) \le d$.

Lemma 6.10. *Let $\pi : B' \to B$ be finite étale of constant degree $h > 0$. For coefficients $\mathcal{A}$ in either realization, pullback gives a split injection*

$$H_c^i(B,\mathcal{A}) \longrightarrow H_c^i(B',\pi^*\mathcal{A}).$$

Proof. Pullback followed by trace is multiplication by h, so $h^{-1}\operatorname{Tr}_\pi$ is a retraction. These maps respect the Hodge structures and, in the ℓ-adic realization, the $G_{\mathbb{Q}}$-action. □

For $f : X \to B$, the stalk of $R^v f_!$ at a geometric point b is $H_c^v(X_b)$. A constant Tate object on B is pulled back from $\mathbb{Q}(-r)$ or $\mathbb{Q}_\ell(-r)$ on the ground field. In the ℓ-adic case, this includes the arithmetic Galois action.

Lemma 6.11. *Fix either realization, and let $f : X \to B$ be defined over $\mathbb{Q}$, with $\dim B \le d$. Suppose B admits a finite stratification such that, on each stratum, the objects $R^v f_!$ have finite filtrations whose quotients become sums of constant Tate objects after finite étale covers defined over $\mathbb{Q}$.*

Then every simple constituent of $H_c^i(X)$ is a subquotient of a Tate twist of $H_c^j(B')$ for some variety $B'/\mathbb{Q}$ of dimension at most d. Consequently, in the Hodge realization, $\operatorname{lev} H_c^i(X) \le d$. In the ℓ-adic realization, each simple constituent has Hodge–Tate spread at most d.

The same conclusions hold if X has a finite stratification by varieties satisfying these hypotheses.

Proof. Work first over one stratum. Let $\mathcal{A}$ be a filtration quotient, and choose a finite étale cover $\pi : B' \to B$ on which it becomes a sum of constant Tate objects. Lemma 6.10 makes $H^u_c(B, \mathcal{A})$ a direct summand of $H^u_c(B', \pi^*\mathcal{A})$. The latter is a sum of Tate twists of $H^u_c(B')$, and $\dim B' = \dim B \le d$.

The filtrations and the compact-support Leray spectral sequence

$$H^u_c(B, R^v f_!) \Longrightarrow H^{u+v}_c(X)$$

give the asserted constituent bound. Tate twists preserve Hodge level and Hodge–Tate spread, so the dimension bounds above apply. Localization combines the finitely many strata. □

Lemma 6.12. *Suppose a variety $Z/\mathbb{Q}$ satisfies*

$$[Z] = \sum_i P_i(\mathbb{L})[B_i]$$

in $K_0(\mathrm{Var}_{\mathbb{Q}})$, where $P_i \in \mathbb{Z}[T]$, $\dim B_i \le 10$. Then $[\chi_c(Z)]_{S_{12}} = 0$ in either realization. Thus for each integer r, the total multiplicity of $S_{12}(r)$ in even-degree compactly supported cohomology equals its total multiplicity in odd-degree cohomology. In the ℓ-adic realization, we use $S_{12,\ell}(r)$ instead.

Proof. The dimension bounds exclude every Tate twist of S_{12} from the cohomology of each B_i: these twists have Hodge level and Hodge–Tate spread eleven. Apply the realization map. Multiplication by $P_i(\mathbb{L})$ introduces only Tate twists, so every summand has zero S_{12}-part. □

Under the identifications $U \simeq H^0(E, L)$ and $V \simeq H^0(E, L^2)$, the kernels of the two coordinate projections in (5.13) are $H^0(E, L(-q_7 - \cdots - q_{10}))$ and $H^0(E, L^2(-q_3 - \cdots - q_{10}))$, respectively. Dividing evaluations by the nonzero values of s or s^2 does not change these kernels.

The projection conditions in (5.13) require the last four coordinates to determine a vector of U uniquely, and the last eight coordinates to determine a vector of V uniquely.

To express the first condition in coordinates, choose a basis $u_1, \dots, u_4$ of U. Form the 4×4 matrix whose jth column consists of coordinates $7, 8, 9, 10$ of u_j. This is the matrix of the projection $U \to k^{\{7,8,9,10\}}$, so the projection is an isomorphism precisely when its determinant is nonzero. Similarly, choosing a basis of V and retaining coordinates $3, \dots, 10$ gives an 8×8 matrix whose determinant must be nonzero. For the spaces constructed from $(E, L, [s])$, these matrix entries are the normalized evaluations of sections described above.

Changing either basis multiplies the corresponding determinant by a nonzero scalar, so these conditions depend only on U and V. They define open subsets of the two Grassmannians, usually called their standard big cells.

Both line bundles have degree zero, and each projection has source and target of the same dimension. Since a degree-zero line bundle has a nonzero section precisely when it is trivial, the first projection fails to be an isomorphism exactly when

$$L \simeq \mathcal{O}_E(q_7 + q_8 + q_9 + q_{10}), \tag{6.12}$$

and the second fails exactly when

$$L^2 \simeq \mathcal{O}_E(q_3 + \cdots + q_{10}). \tag{6.13}$$

Proposition 6.13. *Let $\mathcal{E}_4^{\mathrm{big}} \subset F$ be the smooth elliptic-quartic locus satisfying both HNF big-cell conditions. In either realization,*

$$[\chi_c(\mathcal{E}_4^{\mathrm{big}})]_{S_{12}} = [\chi_c(\mathcal{E}_4)]_{S_{12}} = -B_4(\mathbb{L})S_{12}. \tag{6.14}$$

Proof. Let $B_U, B_V \subseteq \mathcal{P}^4$ be the loci defined by (6.12) and (6.13), and set $B_{UV} = B_U \cap B_V$. On B_U, the markings determine L, so $B_U \simeq \mathcal{M}_{1,10}$. The map $\mathrm{Pic}^4(E) \to \mathrm{Pic}^8(E)$ given by $L \mapsto L^2$

is, after identifying both Picard schemes with E using the origin, multiplication by two. In characteristic zero it is finite étale of degree four. Hence B_V is finite étale of degree four over $\mathcal{M}_{1,10}$. Thus B_U, B_V, and B_{UV} all have dimension at most ten.

The elliptic curve varies in a one-dimensional moduli space, and the nine further marked points each add one parameter. Removing the diagonals and the origin does not change this dimension and we find that $\dim \mathcal{M}_{1,10} = 10$. A finite étale cover has the same dimension as its base, and a closed subvariety cannot have larger dimension. This explains why both projection conditions fall within Lemma 6.12, even though $\mathcal{P}^4$ itself has dimension 11.

For any of these bases B, let Z_B parametrize the projective sections $[s]$ nonzero at all ten markings. For each four-element subset J, set

$$B_J = \left\{ b \in B : L_b \simeq \mathcal{O}_{E_b}\Big(\sum_{j\in J} q_j(b)\Big) \right\}.$$

The inclusion–exclusion calculation of Lemma 6.4, applied to the family over B, gives

$$[Z_B] = B_4(\mathbb{L})[B] + \sum_{|J|=4} [B_J].$$

Here the four-point intersections occur precisely over B_J. Each B_J is closed in B and has dimension at most ten. Therefore Lemma 6.12 gives $[\chi_c(Z_B)]_{S_{12}} = 0$.

The excluded locus is $Z_{B_U} \cup Z_{B_V}$, with intersection $Z_{B_{UV}}$. Additivity shows that its S_{12}-part is zero. The result follows from (6.11). □

7. The complete boundary of the fixed locus

The preceding section computes the S_{12}-part of the smooth elliptic-quartic locus $\mathcal{E}_4^{\mathrm{big}}$. To obtain the contribution of the complete fixed locus F, we must account for its complement. We show that this complement has zero S_{12}-part in its compactly supported Euler class. In what follows, the *boundary* means the reduced closed complement $F \setminus \mathcal{E}_4^{\mathrm{big}}$.

The description by flags turns this into a question about ten points in projective three-space and the quadratic equations they satisfy. The smooth locus considered above has exactly two independent quadratic relations, cutting out a smooth elliptic quartic, and the ten points are distinct. The remaining configurations have more quadratic relations, a singular intersection, a common plane, or coincident points. We treat these possibilities separately. In each case, the class formulas reduce the calculation to parameter spaces of dimension at most ten. This dimension bound excludes the contribution of S_{12}, whose Hodge level and Hodge–Tate spread are eleven. Dimension estimates and classifications of geometric components may be checked after extending the ground field to an algebraic closure.

Lemma 7.1. *Let $X/\mathbb{Q}$ admit a finite stratification whose strata are affine-space or projective-space bundles over varieties of dimension at most d. Then $\operatorname{lev} H^i_c(X) \le d$, and every simple ℓ-adic constituent has Hodge–Tate spread at most d. If $d \le 10$, these cohomology groups contain no Tate twist of S_{12} in either realization.*

Proof. Apply Lemmas 6.2 and 6.11. Tate twists preserve Hodge level and Hodge–Tate spread, whereas every Tate twist of S_{12} has both invariants equal to eleven. □

Let $e_i^* : k^{10} \to k$ be the ith coordinate functional. For a four-plane U containing $\mathbf{1}$, set $q_i = [e_i^*|_U] \in \mathbb{P}(U^\vee)$. These points are well defined because $e_i^*(\mathbf{1}) = 1$.

Lemma 7.2. *Fix an ordered quadruple $I = (i_0, i_1, i_2, i_3)$ of labels. The four-planes $U \subseteq k^{10}$ containing $\mathbf{1}$ for which these four coordinate evaluations are independent are parametrized by six points in the affine hyperplane*

$$\{(X_0, X_1, X_2, X_3) \in k^4 : X_0 + X_1 + X_2 + X_3 = 1\}.$$

The points with labels in I are the coordinate vertices; the remaining six points may coincide.

Equivalently, these four-planes parametrize labelled projective configurations with the four specified points independent, together with a hyperplane avoiding all ten points, up to projective transformations.

In these coordinates, set

$$H = \mathrm{Span}_k\{X_\alpha X_\beta : 0 \le \alpha < \beta \le 3\}.$$

Set $\mathcal{K}_U = \ker(\mathrm{Sym}^2 U \to k^{10}) \subseteq H$. Then,

$$\dim \mathcal{K}_U = 10 - \dim U^{\circ 2}. \tag{7.1}$$

These identifications are regular and hold in families.

The word *exact* means that $\mathcal{K}_U$ contains all the quadratic relations among the ten evaluation points. We are not choosing a subspace of their relations. In particular, specifying $A = \mathcal{K}_U$ requires both that every member of A vanish at the points and that there be no further independent quadratic relation.

Proof. Choose the basis $f_0, \dots, f_3$ of U dual to the four specified evaluations. The evaluation matrix has identity rows at the labels in I. Since $\mathbf{1} = f_0 + \cdots + f_3$, every row has coordinate sum one.

Conversely, a matrix whose rows indexed by I form the identity matrix has rank four. If every row sums to one, the sum of its columns is $\mathbf{1}$. Its column space therefore contains $\mathbf{1}$, and the projection $\pi_I : U \to k^I$ is an isomorphism.

These constructions identify the matrices above with the open subset of $\mathrm{Gr}(3, k^{10}/k\mathbf{1})$ on which π_I is invertible. Both maps are regular. Taking the column space gives the forward map, while the inverse is obtained by choosing the unique basis of U whose projections are the standard basis of k^I. After fixing the four identity rows, each of the six remaining rows has three free entries, with the fourth determined by its row sum. Thus this open subset is isomorphic to $\mathbb{A}^{18}_k$.

The hyperplane associated to U is $\{\ell : \ell(\mathbf{1}) = 0\}$, which avoids every q_i. Conversely, send the four independent points to the coordinate vertices. The avoided hyperplane then has four nonzero coefficients. A unique diagonal change of coordinates, modulo scalar, makes its equation $X_0 + \cdots + X_3 = 0$. Each remaining point has a unique representative with coordinate sum one. This recovers the preceding matrix. The normalizations divide only by quantities assumed nonzero, so they are regular in families.

The quadratic evaluation map has image $U^{\circ 2}$. A quadric vanishing at the four vertices has no square terms, so its kernel lies in H. Rank–nullity gives (7.1). □

For the HNF chart, take $I = (7, 8, 9, 10)$. The remaining six points give its eighteen affine coordinates. Define

$$\mathcal{K}_{3\dots10} = \{Q \in H : Q(q_3) = \cdots = Q(q_6) = 0\},$$
$$\mathcal{K}_{1\dots10} = \{Q \in \mathcal{K}_{3\dots10} : Q(q_1) = Q(q_2) = 0\} = \mathcal{K}_U.$$

The conditions at $q_7, \dots, q_{10}$ already hold for every $Q \in H$.

Lemma 7.3. *Put $a = \dim \mathcal{K}_{1\dots10} = 10 - \dim U^{\circ 2}$. There exists an eight-dimensional space $V \supseteq U^{\circ 2}$ whose projection onto coordinates $3, \dots, 10$ is an isomorphism if and only if*

$$\mathcal{K}_{3\dots10} = \mathcal{K}_{1\dots10}. \tag{7.2}$$

When this holds, the choices of V form an affine space of dimension

$$2(a-2). \tag{7.3}$$

On a locus where the kernel equality holds and a is constant, these spaces form a Zariski locally trivial affine-space bundle.

Proof. Put $R = U^{\circ 2}$, and let $p : k^{10} = k^2 \oplus k^8 \to k^8$ be the projection onto coordinates $3, \dots, 10$. The full evaluation map $\mu : \operatorname{Sym}^2 U \to k^{10}$ has image R and kernel $\mathcal{K}_{1\dots10}$. The kernel of $p\mu$ is $\mathcal{K}_{3\dots10}$: vanishing at the four vertices first forces a quadric to lie in H, and the remaining conditions are vanishing at $q_3, \dots, q_6$. Hence

$$\ker(p|_R) \simeq \mathcal{K}_{3\dots10}/\mathcal{K}_{1\dots10}.$$

Thus (7.2) is equivalent to injectivity of $p|_R$.

Every eight-plane with the required projection is the graph of a map $\phi : k^8 \to k^2$. If $p|_R$ is injective, R is the graph of a map $\phi_0 : p(R) \to k^2$, and the required choices are exactly its extensions to k^8. They form an affine space under

$$\operatorname{Hom}\left(k^8/p(R), k^2\right),$$

of dimension $2(a-2)$, since $\dim p(R) = 10 - a$.

On a constant-rank locus, $p(R)$ is a vector subbundle. Locally choose a complement and extend ϕ_0 by zero on it. This identifies the family of extensions locally with the displayed Hom bundle. □

For $a \geq 3$, let B_a be the locus of six affine points for which

$$A := \mathcal{K}_{1\dots10} = \mathcal{K}_{3\dots10},$$

and $\dim A = a$. These are locally closed matrix conditions. Indeed, use the six monomials spanning H as a basis. Evaluation at $q_3, \dots, q_6$ gives a matrix with four rows and six columns, and evaluation at $q_1, \dots, q_6$ gives a matrix with six rows and six columns. The two kernels already have the required inclusion. Thus the conditions defining B_a say that both matrices have rank $6 - a$. A specified rank is expressed by the vanishing of all larger minors and the nonvanishing of at least one minor of that rank.

By Lemma 7.3, the choices of V form an affine-space bundle of rank $2(a-2)$ over B_a. For a space A of quadrics, its *curve part* is the union of the one-dimensional components of the reduced common zero locus $Z(A)$. Thus C is a reduced curve and does not include any isolated base points or any multiplicity carried by the equations. Its degree is the sum of the degrees of its irreducible components. The isolated base points will be counted separately.

Lemma 7.4. *Suppose* $\dim A \geq 3$ *and the quadrics in A have no common plane. Let C be their curve part. Then* $\deg C \leq 3$.

If $\deg C = 3$*, then, over an algebraic closure, C is a twisted cubic, a conic together with an incident line outside its plane, a nonplanar chain of three lines, or three noncoplanar concurrent lines. In each case,* $\dim H^0(I_C(2)) = 3$*, $A = H^0(I_C(2))$ and $Z(A) = C$.*

Proof. We work over an algebraic closure. There is no common quadratic factor, since that would make A one-dimensional, and no common linear factor by hypothesis. We may therefore choose $Q_1, Q_2 \in A$ with no common factor: after fixing Q_1, avoid the finitely many proper subspaces of elements divisible by one of its irreducible factors.

Their intersection D is a complete intersection of degree four, so $\deg C \leq 4$. If $\deg C = 4$, every component of D occurs with multiplicity one and there are no other components. Thus D is generically reduced. A complete intersection is Cohen–Macaulay, and a generically reduced

Cohen–Macaulay scheme is reduced; see [38, Tags 02JN and 031R]. Hence $D = C$. The Koszul resolution gives $h^0(I_D(2)) = 2$, contradicting $\dim A \geq 3$. Therefore $\deg C \leq 3$.

Suppose $\deg C = 3$. If C were planar, every containing quadric would contain its plane, contrary to the hypothesis. An irreducible nonplanar cubic is a twisted cubic, whose ideal is generated by three quadrics; see [12, Theorem 1 and §1, p. 6].

For reducible C, first exclude three pairwise skew lines. Using the first two as complementary two-dimensional subspaces, their quadrics are the bilinear forms $x^{\mathsf{t}}My$. The third line is the graph $y = \phi x$ of an invertible map. Vanishing on it requires $M\phi$ to be alternating, leaving only one independent quadric.

Every other reduced nonplanar cubic has the form $C = C_0 \cup l$, where C_0 is a plane conic, possibly a pair of intersecting lines, and l lies outside its plane. To see that these are all the remaining reducible possibilities, note that a reduced reducible cubic has either components of degrees two and one or three line components. In the latter case, unless the three lines are pairwise skew, two of them meet and form a planar conic. The third lies outside that plane because C is nonplanar.

Write the plane as $X_3 = 0$ and the conic equation as $Q = 0$. Its containing quadrics are $X_3L + cQ$.

If l meets the plane outside C_0, vanishing at that intersection forces $c = 0$. Vanishing along l then forces $L|_l = 0$, leaving only two independent quadrics. This case is also excluded.

If l meets C_0, choose coordinates with $l = \{X_1 = X_2 = 0\}$. Then $Q \in (X_1, X_2)$ and

$$I_C = (X_3, Q) \cap (X_1, X_2) = (Q, X_3X_1, X_3X_2).$$

Indeed, modulo (X_1, X_2), multiplication by X_3 is injective. The three displayed quadrics are independent and generate the union ideal. When C_0 is reducible, this gives either a nonplanar chain or three noncoplanar concurrent lines.

In every admissible case, $H^0(I_C(2))$ has dimension three and generates I_C. Since it contains A, whose dimension is at least three, the two spaces are equal. Thus $Z(A) = C$, with no residual points. □

Proposition 7.5. *Fix an ordered quadruple I of labels. Let $X_{a,I}$ parametrize four-planes $U \subset \mathbb{Q}^{10}$ containing* **1** *for which these four coordinate evaluations are independent, the exact quadratic kernel $A = \mathcal{K}_U$ has dimension $a \geq 3$, and the quadrics in A have no common plane. Then* $\dim X_{a,I} \leq 10$.

Over every locally closed sublocus $B \subseteq X_{a,I}$, the choices of an eight-plane $V \supset U^{\circ 2}$ form a Grassmannian bundle with fiber $\mathrm{Gr}(a-2, a)$. *Its compactly supported Euler class has zero S_{12}-part in both realizations. The same holds for the HNF affine bundle over the no-common-plane part of B_a.*

Proof. We make the dimension estimates over $\overline{\mathbb{Q}}$. Choose four coordinate labels $I = \{i_1, i_2, i_3, i_4\}$ such that the coordinate projection $\pi_I : U \to \mathbb{Q}^I$ is an isomorphism. The spaces U satisfying this condition form an open Grassmannian chart. On this chart, choose the unique basis whose coordinates at these four labels form the identity matrix. The corresponding four projective evaluation points are then the coordinate vertices of $\mathbb{P}^3$. Let H be the six-dimensional space of quadrics through the four vertices.

For each configuration, $A \subseteq H$ is the full space of quadrics vanishing at all ten points. Thus a quadric belongs to A precisely when it vanishes at every evaluation point. We shall use this exactness to restrict the possible configurations.

If the common zero set $Z(A)$ is finite, first choose $A \in \mathrm{Gr}(a, H)$. Each of the six remaining points then has only finitely many choices. The resulting incidence has dimension at most $a(6 - a) \leq 9$, and the same bound holds for its image in $X_{a,I}$.

Suppose now that $Z(A)$ has a curve part. Let C be the union of its reduced one-dimensional components. By Lemma 7.4, we need only consider a line, a reduced planar conic, two skew lines, or a curve of degree three.

We first treat a line or planar conic. Fix which of the four coordinate vertices lie on C, and let their number be k. Put

$$W_C = H^0(I_C(2)) \cap H, \qquad m = \dim W_C.$$

Thus W_C consists of the quadrics vanishing on C and at all four fixed vertices. Let D_k be an upper bound for the dimension of the family of possible curves C.

Let $\mathcal{Z}_{k,a}$ denote the incidence locus of choices of C, of an a-plane $A \subseteq W_C$, and of the six remaining evaluation points, with A the exact quadratic kernel and C its entire reduced curve part. Choosing C and then A contributes at most $D_k + a(m-a)$ parameters.

To count the remaining point choices, fix which of the six labels lie outside C, and let their number be b. Once C and A have been chosen, these outside points lie in the finite residual base locus $Z(A) \setminus C$. The other $6-b$ points lie on C, so each contributes at most one parameter.

There is also a lower bound on b. Every quadric in W_C already vanishes at the four vertices and at the variable points on C. Thus the evaluation map $W_C \to \overline{\mathbb{Q}}^b$ at the outside points has kernel exactly A. Its rank is $m-a$, and therefore $b \geq m-a$. Consequently,

$$\dim \mathcal{Z}_{k,a} \leq D_k + a(m-a) + 6 - (m-a) = D_k + 6 + (a-1)(m-a). \tag{7.4}$$

There are finitely many choices of the outside labels. Forgetting the curve and the kernel maps these incidences onto the corresponding configuration loci, so the same dimension bound applies to those loci.

We now compute D_k and m. A line contains at most two coordinate vertices. The family of all lines has dimension four, the lines through one fixed point form $\mathbb{P}^2$, and two points determine a line. Thus $D_k = 4 - 2k$ for $0 \leq k \leq 2$.

Quadrics through a line form a seven-dimensional space. The $4-k$ vertices outside the line impose independent conditions on this space. Indeed, for each such vertex e_i, choose a linear form ℓ_i vanishing on the line but not at e_i. The quadric $X_i\ell_i$ vanishes on the line and at every other coordinate vertex, but not at e_i. It follows that $m = 7 - (4-k) = 3 + k$.

For a reduced planar conic, let $\ell = 0$ be its plane and let Q be a quadratic equation restricting to its equation in that plane. Every quadric through the conic has a unique expression $\ell L + cQ$, where L is linear and c is a scalar. This space has dimension five.

A fixed vertex in the plane but outside the conic would force $c = 0$ for every quadric in A. All members of A would then be divisible by ℓ, giving the common plane excluded in the statement. Hence every fixed vertex in the conic plane lies on the conic. At each of the other $4-k$ vertices, ℓ is nonzero, and the vanishing condition independently determines the corresponding coordinate value of L. Thus $m = 5 - (4-k) = 1 + k$. Since $a \leq m$ and $a \geq 3$, we have $k \geq 2$. The four vertices are noncoplanar, so $k \leq 3$.

The planes through the k specified vertices form a family of dimension $3-k$. Within such a plane, conics through those vertices form a family of dimension $5-k$: the conditions are independent because the specified vertices are noncollinear. Therefore $D_k \leq 8 - 2k$.

Substituting these values into (7.4) gives

C	k	D_k	m	a	dimension bound
line	0	4	3	3	10
line	1	2	4	3, 4	10, 8
line	2	0	5	3, 4, 5	10, 9, 6
planar conic	2	4	3	3	10
planar conic	3	2	4	3, 4	10, 8

In particular, all these loci have dimension at most ten.

Next suppose that the curve part consists of two skew lines L_1, L_2. Choose coordinates $[x : y]$, with x and y each having two coordinates, such that $L_1 = \{y = 0\}$ and $L_2 = \{x = 0\}$. The quadrics vanishing on both lines are exactly the bilinear forms in x, y, and form a four-dimensional space.

There can be no base point outside these two lines. Indeed, at such a point $[x : y]$ both x and y are nonzero. For every quadric $Q \in A$, bilinearity gives $Q(sx, ty) = stQ(x, y) = 0$. Thus the entire joining line $\{[sx : ty] : [s : t] \in \mathbb{P}^1\}$ would lie in $Z(A)$. It is distinct from L_1 and L_2, contradicting the assumption that they form the entire curve part.

All ten evaluation points therefore lie on $L_1 \cup L_2$. Every quadric through these lines vanishes at all ten points, so exactness forces A to be their full four-dimensional quadratic ideal. Hence $a = 4$. Each line contains at most two of the four fixed vertices, so each must contain exactly two and is determined by them. There are finitely many such pairs. The six remaining points contribute at most six parameters.

Finally, suppose $\deg C = 3$. Lemma 7.4 gives $A = H^0(I_C(2))$ and $Z(A) = C$. In particular, all ten evaluation points lie on C.

We first bound the families of possible curves before fixing the four vertices. Twisted cubics form a family of dimension $15 - 3 = 12$: PGL_4 acts transitively, with stabilizer PGL_2. A conic and an incident line outside its plane form a family of dimension at most eleven. Indeed, the conic contributes eight parameters, its intersection point with the line contributes one, and the line through that point contributes two. The remaining cubic types in the lemma are unions of three lines, whose families have dimension at most $3 \dim \mathrm{Gr}(2, 4) = 12$.

Let $\mathcal{F}$ be any one of these PGL_4-invariant families, of dimension $d \leq 12$. Consider the incidence of a curve in $\mathcal{F}$ together with four ordered independent points on it. Its dimension is at most $d+4$. The space of independent ordered quadruples in $\mathbb{P}^3$ is an open subset of $(\mathbb{P}^3)^4$, of dimension twelve.

The projection from the incidence to this space is equivariant for PGL_4. If the incidence is nonempty, transitivity on independent quadruples makes this projection surjective, with isomorphic fibers. Its fiber over our four fixed vertices, which parametrizes the curves through them, therefore has dimension at most $d + 4 - 12 \leq 4$. Choosing the six remaining points on the curve adds at most six parameters. Thus the cubic case also has dimension at most ten.

These estimates cover $X_{a,I}$. There are only finitely many curve types and label assignments, so $\dim X_{a,I} \leq 10$.

It remains to include the choices of V. Let B be a locally closed stratum of $X_{a,I}$. Quadratic evaluation has constant rank $10 - a$, so its image $\mathcal{R}$ is a vector subbundle of the trivial rank-ten bundle on B. The condition $\mathcal{R} \subseteq V$, with $\dim V = 8$, is equivalent to choosing an $(a-2)$-plane in the rank-a quotient. The Grassmannian bundle formula gives

$$[\text{flags over } B] = [\mathrm{Gr}(a-2, a)][B].$$

For the HNF calculation, restrict B to the locus satisfying the kernel-equality condition of Lemma 7.3. That lemma gives an affine bundle of rank $2(a-2)$, and hence the class $\mathbb{L}^{2(a-2)}[B]$. Every such base still has dimension at most ten. Both fiber factors are polynomials in $\mathbb{L}$, so Lemma 6.12 gives the asserted vanishing in both realizations. □

Proposition 7.6. *The compactly supported Euler class of the common-plane part of B_a, together with its choices of V, has zero S_{12}-part in both realizations.*

Proof. Write $A = \ell W$, where $\ell = 0$ defines a plane Π and W is an a-dimensional space of linear forms. Its reduced base is $\Pi \cup Z(W)$, and $a = 3$ or 4. Multiplication by the nonzero linear form ℓ identifies W with A. Since the space of linear forms on $\mathbb{P}^3$ has dimension four, the assumption

$a \geq 3$ leaves only these two possibilities. When $a = 3$, if the point $Z(W)$ lay on Π, the whole reduced base would lie in Π. This is excluded by the four independent evaluation points.

If $a = 4$, the base is Π, which cannot contain the four independent vertices. If $a = 3$, then $Z(W)$ is a point r outside Π. Exactly one vertex equals r; the other three determine Π. Thus there are four possibilities, each with $A = \ell H^0(I_r(1))$. In the fixed affine chart, its base is $\mathbb{A}^2 \sqcup \{r\}$.

Let T_A parametrize $q_3, \ldots, q_6$ in this base for which the quadratic kernel, together with the four vertices, is exactly A. This is locally closed in $(\mathbb{A}^2 \sqcup \{r\})^4$, so $\dim T_A \leq 8$. The remaining points q_1, q_2 may be chosen arbitrarily in the same base: they impose no further condition on the already exact kernel. The choices of V form an affine bundle of rank two. Hence the resulting locus Z_A has class

$$[Z_A] = \mathbb{L}^2(\mathbb{L}^2 + 1)^2[T_A].$$

Apply Lemma 6.12 and sum over the four possibilities. □

Let Q_0, Q_1 be linearly independent quadratic forms on $\mathbb{P}^3$. Their span is called a *pencil of quadrics*: its members, up to scalar, are the forms $aQ_0 + bQ_1$ indexed by $[a : b] \in \mathbb{P}^1$. The *base* of the pencil is the closed subscheme defined by $Q_0 = Q_1 = 0$. Its points lie on every quadric in the pencil, and its definition is independent of the chosen generators.

We call the pencil *proper* if Q_0 and Q_1 have no common surface component, or equivalently, no nonconstant common factor. Their intersection is then a complete intersection curve of degree four. This curve need not be smooth or reduced.

The configurations considered below consist of a pencil together with ten ordered points in its base. Two configurations are projectively equivalent if a projective transformation carries one pencil to the other and carries each marked point to the point with the same label. We want parameter spaces in which equivalent configurations are counted once.

To construct these spaces, first send four specified independent points to the coordinate vertices of $\mathbb{P}^3$. The remaining coordinate changes are diagonal transformations, with three independent scaling parameters, since a common scaling acts trivially on projective space. Under the hypotheses of the following lemma, we can divide the configurations into finitely many locally closed strata and use three nonzero coordinate ratios to fix these remaining scalings uniquely. The resulting normalized locus is called an *algebraic slice*. Each projective equivalence class in the stratum has exactly one representative on this slice. The normalization is regular because all coordinate ratios used in its construction are invertible on the chosen stratum.

Lemma 7.7. *Let Z be a PGL_4-invariant locally closed locus, defined over $\mathbb{Q}$, of pencils in $\mathbb{P}^3$ with ten ordered points in their bases. Require that the pencil be the exact quadratic kernel and that a fixed ordered quadruple of labels be independent.*

If $\dim Z \leq 25$, the configurations modulo projective transformations admit a finite stratification by algebraic slices of dimension at most ten. After adjoining a hyperplane avoiding all ten points, their compactly supported Euler classes have zero S_{12}-part in both realizations. By Lemma 7.2, the same holds for the corresponding U-loci.

These conclusions remain valid after imposing further projectively invariant locally closed conditions defined over $\mathbb{Q}$, including ranks of evaluation maps on subsets of labels.

Proof. Send the four independent points to the coordinate vertices, and let Z_0 be the resulting locus. Independent ordered quadruples form a twelve-dimensional homogeneous space for PGL_4, so

$$\dim Z_0 = \dim Z - 12 \leq 13.$$

The residual coordinate changes form the three-dimensional diagonal torus $D = (\mathbb{G}_m)^4/\mathbb{G}_m$.

Stratify Z_0 by the nonzero coordinate positions of the marked points. On each stratum, form a graph on $\{0,1,2,3\}$, joining two indices when some marked point has both coordinates nonzero. This graph is connected. Otherwise every point would be supported in one connected component, and all cross monomials between different components would vanish at every point. A nontrivial partition of four indices gives at least three such independent quadrics, contradicting the exact kernel dimension two.

Choose a spanning tree of this graph and a marked point witnessing each edge. Orient the tree from a root, and let r_{ij} be the ratio of its jth to its ith coordinate at the chosen point. Under a diagonal transformation,

$$r_{ij} \longmapsto (t_j/t_i)r_{ij}.$$

Set the root coordinate equal to one and recursively choose $t_j = t_i/r_{ij}$. This gives the unique element of D making the three chosen ratios equal to one. It is regular on the stratum, since the ratios are invertible regular functions.

If Y is the locus with these ratios equal to one, this normalization gives an isomorphism from the support stratum to $D \times Y$. Thus $\dim Y \leq 13 - 3 = 10$, and each orbit meets Y exactly once. The finitely many support strata therefore give the required slices, all defined over $\mathbb{Q}$. Indeed, let $t(z) \in D$ be the unique normalizing element just constructed for a point z of a support stratum. The product decomposition sends z to $(t(z)^{-1}, t(z)z) \in D \times Y$; its inverse sends (d, y) to dy. Both maps are regular because the chosen coordinate ratios never vanish on this stratum. This also explains why subtracting the three-dimensional torus orbit is legitimate.

It remains to adjoin the avoided hyperplane. On a slice Y, let $W = \mathbb{Q}^4$ and let $\mathcal{L}_j \subset W \otimes \mathcal{O}_Y$ be the line represented by the jth point. For each subset J, stratify by the rank r_J of

$$W^\vee \otimes \mathcal{O}_Y \longrightarrow \bigoplus_{j \in J} \mathcal{L}_j^\vee.$$

There are finitely many simultaneous rank strata B. On each, hyperplanes containing the points indexed by J form a projective bundle with fiber $\mathbb{P}^{3-r_J}$, interpreted as empty when $r_J = 4$.

Consequently, the locus Z_B of hyperplanes avoiding all ten points satisfies

$$[Z_B] = [B] \sum_{J \subseteq \{1,\dots,10\}} (-1)^{|J|} [\mathbb{P}^{3-r_J}],$$

with rank-four terms zero. The sum is a polynomial in $\mathbb{L}$ and $\dim B \leq 10$. Apply Lemma 6.12. Lemma 7.2 identifies these data with the corresponding U-loci. Further invariant locally closed conditions restrict the same slices and rank strata. □

Proposition 7.8. *Fix an ordered independent quadruple among ten ordered points in $\mathbb{P}^3$, and require their exact quadratic kernel to be a pencil A. Each of the following incidence loci has dimension at most 25:*

(1) A is proper and its base is singular;
(2) A is nonproper;
(3) the base is smooth and proper, and two labels coincide.

After imposing any further projectively invariant locally closed conditions defined over $\mathbb{Q}$, the configurations modulo projective transformations admit finite algebraic slices of dimension at most ten. Adjoining an avoided hyperplane gives zero S_{12}-part in both realizations, as do the corresponding U-loci.

For configurations only required to span $\mathbb{P}^3$, the same conclusions hold after stratifying by the first independent quadruple in a fixed ordering.

Proof. Pencils form $\mathrm{Gr}(2,10)$, of dimension sixteen. The required bounds are

pencil locus	pencil parameters	point parameters	total
proper singular	15	10	25
nonproper	7	18	25
proper smooth with a collision	16	9	25

By Bertini's theorem, smooth complete intersections of two quadrics form a nonempty open locus; see [38, Tag 0FD6]. One can also check nonemptiness directly. Choose distinct rational numbers $b_0,\dots,b_3$ and take the pencil spanned by $\sum_i X_i^2$ and $\sum_i b_i X_i^2$. At a singular point of their intersection, dependence of the two gradients would force all nonzero coordinates to have the same b_i. There would then be at most one nonzero coordinate, which is incompatible with $\sum_i X_i^2 = 0$. Hence this pencil has smooth base over $\overline{\mathbb{Q}}$. Since $\mathrm{Gr}(2,10)$ is irreducible, the complement of the nonempty open locus of smooth complete intersections has dimension at most fifteen.

Thus proper singular pencils have parameter dimension at most fifteen. Their reduced bases are curves, so ten labels add at most ten parameters.

A nonproper pencil has the form $A = \ell M$, where $\ell = 0$ defines a plane Π and the two-dimensional space M of linear forms vanishes on a line l. Its reduced base is $\Pi \cup l$. Choosing the plane and line uses at most $3+4=7$ parameters. Independence of four labels excludes $l \subseteq \Pi$.

There must be two distinct marked points outside Π. Otherwise the four-dimensional space $\ell H^0(\mathbb{P}^3, \mathcal{O}(1))$ would satisfy at most one independent vanishing condition outside Π, leaving at least three quadrics in the exact kernel. Choose two labels at distinct outside points. They contribute at most two parameters on l, and the other eight contribute at most sixteen on $\Pi \cup l$. This gives the bound eighteen for the point choices.

For a smooth proper pencil, a collision between two specified labels leaves at most nine point parameters on the curve. Taking the finite union over pairs of labels gives the third row.

The independence, exact-kernel, and additional rank conditions only restrict these incidence loci. Apply Lemma 7.7. If the points are only required to span, stratify by the first independent quadruple. □

Let Z denote the locus of spanning configurations $(q_1,\dots,q_{10})$. Order the four-element subsets of $\{1,\dots,10\}$ as $I_1,\dots,I_{210}$. Choose nonzero homogeneous representatives $v_i \in k^4$ of the points q_i. For $I = \{i_1 < i_2 < i_3 < i_4\}$, set $\Delta_I = \det(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4})$. Thus $\Delta_I \neq 0$ precisely when the four points indexed by I are independent.

For $1 \le a \le 210$, define

$$Z_a = \{z \in Z : \Delta_{I_a}(z) \neq 0,\ \Delta_{I_b}(z) = 0 \text{ for every } b < a\}.$$

Each Z_a is locally closed. Rescaling the representatives or applying a projective transformation multiplies each determinant by a nonzero scalar, so these conditions are well defined and projectively invariant. Since the configurations span $\mathbb{P}^3$, at least one determinant is nonzero. Each configuration therefore belongs to exactly one Z_a, indexed by its first nonzero determinant. Hence $Z = \bigsqcup_{a=1}^{210} Z_a$.

Proposition 7.9. *The compactly supported Euler class of the part of $F \setminus \mathcal{E}_4^{\mathrm{big}}$ with $\dim \mathcal{K}_U = 2$ has zero S_{12}-part in both realizations.*

Proof. Apply Proposition 7.8 with independent labels $7,8,9,10$. Impose also that quadratic evaluation on labels $3,\dots,10$ have rank eight. Since the full kernel is a pencil, this is equivalent to (7.2). It is a projectively invariant locally closed condition.

Lemma 7.2 recovers U, and Lemma 7.3, with $a = 2$, gives a unique V. Outside $\mathcal{E}_4^{\mathrm{big}}$, the pencil is proper singular, nonproper, or proper smooth with coincident labels. These are precisely the three cases covered by the proposition. □

Corollary 7.10. *The complete fixed locus satisfies*

$$[\chi_c(F)]_{S_{12}} = -B_4(\mathbb{L})S_{12}. \tag{7.5}$$

Proof. Put $a = 10 - \dim U^{\circ 2}$. Since $U^{\circ 2} \subseteq V$ and $\dim V = 8$, we have $a \geq 2$. For $a \geq 3$, Propositions 7.5 and 7.6 give zero S_{12}-part, according to whether the kernel has a common plane. For $a = 2$, Proposition 7.9 gives the same conclusion outside $\mathcal{E}_4^{\mathrm{big}}$. Additivity for these finite stratifications, together with (6.14), proves (7.5). □

8. From the fixed locus to irreducible counts

The preceding section identifies a contribution from the modular form Δ in the fixed locus. In the remainder of this article, we use this contribution to rule out a polynomial counting formula for $f_{10}(p^{17})$. Such a formula would make the compactly supported Euler class a polynomial in $\mathbb{L}$. We define a linear functional which vanishes on these Tate classes but detects the modular contribution. Its value is unchanged by Tate twists, which allows us to pass from a parameter space to its loop-rotation fixed locus.

There are two counts to consider. The variety X records the single HNF composition α_0, and its fixed locus F gives the non-polynomiality of that count. To treat $g_{10}(p^{17})$, we must also include every other HNF composition. We therefore construct one projective parameter space for all irreducible lattices of the given rank and colength. Its fixed points are homogeneous lattices, described by finite filtrations of $\mathbb{Q}^{10}$.

Here the *profile* of such a lattice is the sequence of positive codimensions $10 - \dim F_i$, for $i \geq 1$, with the final zero terms omitted, where $F_0 = \mathbb{Q}\mathbf{1} \subseteq F_1 \subseteq \cdots$ are the spaces of the associated filtration. Since the spaces are nested, the profile is a partition, and its parts sum to $e - (n-1)$, which is 8 in the case at hand. The profile $(6, 2)$ gives the incidence variety $\mathcal{C}$ considered below, without the two projection conditions imposed on F; the later sections show that the other profiles have zero S_{12}-augmentation.

Let $\varepsilon_{12} : K_\ell \to \mathbb{Q}$ be the linear functional which assigns value 1 to each simple class $[\mathbb{L}^r S_{12,\ell}]$, $r \in \mathbb{Z}$, and value 0 to every other simple class. The twists are simple and pairwise nonisomorphic, since their determinants differ. Every virtual class has finite support, so the functional is well defined.

We shall call this sum of signed multiplicities the S_{12}-*augmentation*. Thus an augmentation of 56 is a value of the linear functional ε_{12}; it is not the dimension of a cohomology group. The product identity below concerns multiplication by a Laurent polynomial in $\mathbb{L}$. We do not use multiplicativity of ε_{12} on arbitrary classes.

Tensoring by $\mathbb{L}$ permutes the simple classes and preserves the chosen Tate-twist orbit. Hence, for $V \in K_\ell$ and $P \in \mathbb{Q}[T, T^{-1}]$,

$$\varepsilon_{12}((\mathbb{L} - 1)V) = 0 \quad \text{and} \quad \varepsilon_{12}(P(\mathbb{L})V) = P(1)\varepsilon_{12}(V). \tag{8.1}$$

In particular, ε_{12} vanishes on virtual Tate classes. By Lemma 6.11, it also vanishes on the cohomology covered by that lemma when $d \leq 10$. For a variety Z, we abbreviate $\varepsilon_{12}([Z]) = \varepsilon_{12}(\chi_c^\ell(Z))$.

Apply (8.1) to (5.9). Corollary 7.10 then gives

$$\varepsilon_{12}(\chi_c^\ell(X)) = \varepsilon_{12}(\chi_c^\ell(F)) = -B_4(1) = 56. \tag{8.2}$$

Proof of Theorem 1.3. Suppose that $g_{\alpha_0}(p) = P(p)$ for every sufficiently large prime, where $P \in \mathbb{Q}[T]$. Corollary 5.3 gives $\#\mathscr{X}(\mathbb{F}_p) = P(p)$ for the same primes. By Lemma 6.7, $\chi_c^\ell(X) = P(\mathbb{L})$. Applying ε_{12} gives zero, contrary to (8.2). □

For $n \geq 2$, the uniqueness of the HNF implies that

$$g_n(p^e) = \sum_{\substack{\alpha_1,\ldots,\alpha_{n-1}\geq 1 \\ \alpha_1+\cdots+\alpha_{n-1}=e}} g_\alpha(p). \tag{8.3}$$

The sum is empty when $e < n-1$. In rank one, $g_1(1) = 1$ and $g_1(p^e) = 0$ for $e > 0$.

Fix $e \geq 1$ and set $A_{n,e} = (\mathbb{Z}[t]/(t^e))^n$ and $N = ne$. The algebra $A_{n,e}$ is free over $\mathbb{Z}$ with basis $t^r e_i$, where $1 \leq i \leq n$ and $0 \leq r < e$. Its multiplication is coordinatewise.

Put $A_k = A_{n,e} \otimes_{\mathbb{Z}} k = (k[t]/(t^e))^n$. A k-point of $G = \mathrm{Gr}(N-e, A_{n,e})$ is a subspace $S \subseteq A_{n,e} \otimes_{\mathbb{Z}} k$ of dimension $N-e$. We want to parametrize those subspaces satisfying

$$\mathbf{1} \in S, \qquad tS \subseteq S, \qquad u \circ v \in S \quad \text{for every } u, v \in S. \tag{8.4}$$

Thus S must contain the unit, be preserved by multiplication by t, and be closed under coordinatewise multiplication.

These conditions define a closed subscheme $\mathscr{Q}_{n,e} \subseteq G$, as follows. The universal subbundle $\mathcal{S}$ and quotient bundle $\mathcal{T}$ fit into the exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow A_{n,e} \otimes_{\mathbb{Z}} \mathcal{O}_G \longrightarrow \mathcal{T} \longrightarrow 0.$$

At the point represented by S, their fibers are S and $(A_{n,e} \otimes_{\mathbb{Z}} k)/S$, respectively. An element belongs to S precisely when its image in this quotient is zero.

On an open subset of G where both bundles are trivial, choose a basis $u_1, \ldots, u_{N-e}$ of $\mathcal{S}$. Take the images in $\mathcal{T}$ of $\mathbf{1}$, tu_i for $1 \leq i \leq N-e$ and $u_i \circ u_j \quad (1 \leq i, j \leq N-e)$. Their coordinates in a basis of $\mathcal{T}$ are regular functions. Define $\mathscr{Q}_{n,e}$ by setting all these functions equal to zero. The equations require exactly (8.4); since these requirements are independent of the chosen bases, the local equations define a closed subscheme of G.

To require that an element of A_k belong to S, we require its image in A_k/S to be zero. This allows us to express the algebra conditions by maps to $\mathcal{T}$. Define $\mathscr{Q}_{n,e} \subseteq G$ as the common zero locus of

$$\begin{aligned} \mathcal{O}_G &\longrightarrow \mathcal{T} \quad \text{given by} \quad 1 \longmapsto \mathbf{1} \bmod \mathcal{S}, \\ \mathcal{S} &\xrightarrow{t} A_{n,e} \otimes \mathcal{O}_G \longrightarrow \mathcal{T}, \\ \mathcal{S} \otimes \mathcal{S} &\xrightarrow{\circ} A_{n,e} \otimes \mathcal{O}_G \longrightarrow \mathcal{T}. \end{aligned} \tag{8.5}$$

We impose one further condition to define $\mathscr{I}_{n,e}$. The ambient reduction map $A_k \longrightarrow k^n$ is defined by

$$(f_1(t), \ldots, f_n(t)) \mapsto (f_1(0), \ldots, f_n(0)).$$

We require its image on S to lie in the diagonal line $k\mathbf{1}$. This means that the composite

$$\mathcal{S} \longrightarrow \mathbb{Z}^n \otimes \mathcal{O}_G \longrightarrow (\mathbb{Z}^n/\mathbb{Z}\mathbf{1}) \otimes \mathcal{O}_G \tag{8.6}$$

is zero. We define $\mathscr{I}_{n,e}$ as its zero locus inside $\mathscr{Q}_{n,e}$.

Each map defined is a morphism of vector bundles. After choosing local bases, its vanishing is given by setting all entries of its matrix equal to zero. Hence these conditions define closed subschemes of G. Since the Grassmannian is projective over $\mathbb{Z}$, both $\mathscr{Q}_{n,e}$ and $\mathscr{I}_{n,e}$ are projective over $\mathbb{Z}$.

For $e = 0$, set $\mathscr{Q}_{n,0} = \operatorname{Spec}\mathbb{Z}$, $\mathscr{I}_{n,0} = \varnothing$ for $n > 1$, and $\mathscr{I}_{1,0} = \operatorname{Spec}\mathbb{Z}$.

Lemma 8.1. *Let k be a field and $e \geq 1$. The points of $\mathscr{Q}_{n,e}(k)$ correspond to the unital $k[[t]]$-subalgebras of $k[[t]]^n$ of colength e. The subset $\mathscr{I}_{n,e}(k)$ corresponds to those whose reduction image is $k\mathbf{1}$. For fixed n, e and every sufficiently large prime p,*

$$g_n(p^e) = \#\mathscr{I}_{n,e}(\mathbb{F}_p). \tag{8.7}$$

Proof. The result follows from the above construction. In greater detail, set $\mathcal{O} = k[[t]]$ and let $\pi : \mathcal{O}^n \to (\mathcal{O}/(t^e))^n = A_k$ be the quotient map. If $R \subseteq \mathcal{O}^n$ has colength e, the elementary-divisor decomposition of $\mathcal{O}^n/R$ has the form $\bigoplus_i \mathcal{O}/(t^{a_i})$, where $a_i \geq 0$ and $\sum_i a_i = e$. Since each $a_i \leq e$, this quotient is annihilated by t^e. Hence $t^e\mathcal{O}^n \subseteq R$, and $S = \pi(R)$ is a subspace of A_k of codimension e. The algebra conditions on R descend to the three conditions on S above.

Conversely, let S satisfy those conditions and set $R = \pi^{-1}(S)$. Stability under t makes S a $k[t]/(t^e)$-submodule, so its inverse image is an $\mathcal{O}$-submodule. It contains $t^e\mathcal{O}^n$, and $\mathcal{O}^n/R \simeq A_k/S$ has k-dimension e. Thus R is a full lattice of colength e. The unit and multiplication conditions on S imply the same conditions on R, because π is an algebra homomorphism. These two constructions are inverse. Since $e \geq 1$, the kernel $t^e\mathcal{O}^n$ has zero reduction modulo t; the reduction images of R and S therefore agree.

For fixed n, e, there are only finitely many positive compositions in (8.3). Proposition 4.4 identifies the counts over $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ for all of these compositions outside one finite set of primes. Summing over them proves (8.7). This proof concerns lattices over fields and their rational points. We do not require the same truncation assertion for lattice families over arbitrary nonreduced bases. □

Loop rotation gives an action on $A_{n,e}$ by

$$\rho_\lambda\left(\sum_{r=0}^{e-1} t^r x_r\right) = \sum_{r=0}^{e-1} \lambda^r t^r x_r.$$

It gives an action on both $\mathscr{Q}_{n,e}$ and $\mathscr{I}_{n,e}$.

Replacing t by λt fixes $\mathbf{1}$, respects multiplication, and preserves t-stable subspaces. It also leaves constant terms unchanged, so both $\mathscr{Q}_{n,e}$ and $\mathscr{I}_{n,e}$ are preserved.

To apply Lemma 5.5, use the Plücker embedding of G. Rotation multiplies each basis vector $t^r e_i$ by λ^r. It therefore multiplies a wedge of these vectors by $\lambda^{r_1+\cdots+r_{N-e}}$, where the r_j are their exponents. Thus each Plücker coordinate is scaled separately, giving the required diagonal action on the ambient projective space.

Set $I_{n,e} = (\mathscr{I}_{n,e,\mathbb{Q}})_{\mathrm{red}}$. Lemma 5.5 and (8.1) give

$$\varepsilon_{12}(\chi_c^\ell(I_{n,e})) = \varepsilon_{12}\left(\chi_c^\ell((I_{n,e}^{\mathbb{G}_m})_{\mathrm{red}})\right). \tag{8.8}$$

Proposition 8.2. *For $e \geq 1$, the reduced fixed locus of $I_{n,e}$ is the finite disjoint union of the reduced projective varieties of filtrations*

$$\begin{gathered} F_0 = \mathbb{Q}\mathbf{1} \subseteq F_1 \subseteq \cdots \subseteq F_e = \mathbb{Q}^n, \\ F_i \circ F_j \subseteq F_{i+j} \quad \text{and} \quad \sum_{r=0}^{e-1} (n - \dim F_r) = e, \end{gathered} \tag{8.9}$$

where $F_r = \mathbb{Q}^n$ for $r \geq e$. The positive terms of $c_r = n - \dim F_r$, $r \geq 1$, form a partition of $e - (n-1)$, with each part at most $n-1$.

Proof. The weight decomposition of $A_{n,e,\mathbb{Q}}$ is $\bigoplus_{r=0}^{e-1} t^r\mathbb{Q}^n$. Over any base with trivial $\mathbb{G}_m$-action, an invariant subbundle and its locally free quotient decompose into weight subbundles; see [38,

Tag 0EKL]. Hence every invariant subbundle has the form

$$S = \bigoplus_{r=0}^{e-1} t^r F_r,$$

where each F_r is a subbundle of the trivial rank-n bundle. For fixed ranks f_r, these are parametrized by $\prod_{r=0}^{e-1} \mathrm{Gr}(f_r, \mathbb{Q}^n)$.

The unit and reduction conditions give $F_0 = \mathbb{Q}\mathbf{1}$, while t-stability gives $F_r \subseteq F_{r+1}$. For $i+j < e$, multiplication is $(t^i x) \circ (t^j y) = t^{i+j}(x \circ y)$, so closure is exactly $F_i \circ F_j \subseteq F_{i+j}$. Products of degree at least e vanish. Finally, the codimension of S is $\sum_r (n - f_r)$.

These are closed conditions on tautological bundles over the product of Grassmannians.

The ranks of the weight subbundles are locally constant. Fixing these ranks therefore gives disjoint open and closed pieces of the fixed-point scheme. On such a piece, the inclusions and product conditions just described are precisely the vanishing of maps to the tautological quotient bundles. This explains both why the filtration varieties are projective and why their classes may be added without overlapping pieces.

The constructions are inverse in families, giving the asserted identification after reduction.

The term F_0 contributes $n-1$ to the codimension sum. The remaining codimensions are nonincreasing integers between zero and $n-1$, with sum $e-(n-1)$. Deleting their zero tail gives the stated partition. In particular, the fixed locus is empty when $e < n-1$. □

For $n = 10$ and $e = 17$, these partitions have sum $17-9 = 8$. There are twenty-two possibilities, whose filtration dimensions are listed in Section 9. The partition $(6,2)$ gives dimensions $(1,4,8,10)$. Write $U = F_1$ and $V = F_2$. Its only multiplication condition not already implied by the flag is $U^{\circ 2} \subseteq V$.

Let $\mathscr{C}$ be the closed subscheme of the flag variety of four-planes inside eight-planes in $\mathbb{Z}^{10}$ along with the conditions $\mathbf{1} \in U$ and $U^{\circ 2} \subseteq V$. These are vanishing conditions for the maps from the scalar line to $\mathcal{O}^{10}/U$ and from $\mathrm{Sym}^2 U$ to $\mathcal{O}^{10}/V$, so $\mathscr{C}$ is projective over $\mathbb{Z}$. For a field k, set $\mathcal{C}_k = (\mathscr{C}_k)_{\mathrm{red}}$ and write

$$\mathcal{C}(k) = \mathcal{C}_k(k) = \left\{ (U,V) : \begin{array}{l} \mathbf{1} \in U \subset V \subset k^{10}, \\ \dim U = 4, \quad \dim V = 8, \quad U^{\circ 2} \subseteq V \end{array} \right\}. \tag{8.10}$$

We write $\mathcal{C} = \mathcal{C}_{\mathbb{Q}}$ when taking cohomology. This is the reduced fixed-locus piece with profile $(6,2)$.

Recall that $\mathcal{C}$ parametrizes flags $\mathbf{1} \in U \subset V \subset \mathbb{Q}^{10}$ with $\dim U = 4$, $\dim V = 8$, and $U^{\circ 2} \subseteq V$. The smooth elliptic-quartic locus $\mathcal{E}_4$ is an open subvariety of $\mathcal{C}$. We now show that its contribution accounts for the entire S_{12}-part of $\chi_c(\mathcal{C})$.

Proposition 8.3. *In both realizations one has* $[\chi_c(\mathcal{C})]_{S_{12}} = -B_4(\mathbb{L})S_{12}$, *and consequently* $\varepsilon_{12}(\chi_c^{\ell}(\mathcal{C})) = -B_4(1) = 56$.

Proof. We prove the result in three steps. The flags are stratified according to the number of independent quadratic relations among the ten coordinate evaluations. In the first step we show that the S_{12}-part vanishes on every stratum outside the elliptic locus and outside the common-plane locus. In the second step, the common-plane locus is identified with an explicit variety, built on a smaller ambient space. In the third step, a count of parameters shows that this variety has zero S_{12}-part as well.

Let $R = U^{\circ 2}$ and $r = \dim R$, and let K_U be the kernel of the multiplication map $\mathrm{Sym}^2 U \to \mathbb{Q}^{10}$ which sends uv to $u \circ v$; set $a = \dim K_U = 10 - r$. Since $\mathbf{1} \in U$, every $u \in U$ is of the form $\mathbf{1} \circ u$, and it follows that $U \subseteq R$. The defining condition of the flag gives $R \subseteq V$, hence $r \leq 8$ and $a \geq 2$.

For a fixed U, the spaces V that occur are precisely the eight-planes containing R, that is, the $(8-r)$-planes V/R in $\mathbb{Q}^{10}/R$. On a stratum on which R has constant dimension, and is therefore a vector subbundle, these choices form a Grassmannian bundle with fiber $\mathrm{Gr}(8-r, \mathbb{Q}^{10}/R) \simeq \mathrm{Gr}(a-2, a)$. We may thus study the possible spaces U first, and then account for the choices of V by the bundle formula.

To interpret K_U, let $\ell_i : U \to \mathbb{Q}$ be the i-th coordinate evaluation and put $P_i = [\ell_i] \in \mathbb{P}(U^\vee)$. Each ℓ_i is nonzero because $\ell_i(\mathbf{1}) = 1$, and the ℓ_i span $U^\vee$; hence the ten points P_i span $\mathbb{P}(U^\vee) \simeq \mathbb{P}^3$. The elements of K_U are exactly the quadratic forms vanishing at all of these points, so that a is the number of independent quadratic relations among the coordinate evaluations.

We stratify the U-locus by the first independent quadruple of evaluations, with respect to a fixed ordering of the quadruples. Lemma 7.2 identifies each stratum with the corresponding normalized projective configurations together with an avoided hyperplane, which is the form in which the boundary results proved earlier apply.

Suppose first that $a \geq 3$ and that the quadrics in K_U have no common plane. Proposition 7.5 then gives a zero S_{12}-part, including the Grassmannian choices of V above. If $a = 2$, the quadrics form a pencil, and Proposition 7.8 shows that its contribution vanishes unless the pencil cuts out a smooth elliptic quartic and the ten marked points are distinct. This remaining locus is $\mathcal{E}_4$. Here $r = 8$, so $V = R$ is uniquely determined, and Corollary 6.9 gives $[\chi_c(\mathcal{E}_4)]_{S_{12}} = -B_4(\mathbb{L})S_{12}$. It remains to treat the common-plane locus with $a \geq 3$.

Assume therefore that every quadric in K_U vanishes on a plane $\Pi = \{\lambda = 0\}$. Then every such quadric is divisible by λ, and we may write $K_U = \lambda W$ for an a-dimensional space W of linear forms on $\mathbb{P}(U^\vee)$. Since the space of linear forms has dimension four, $a \leq 4$. If $a = 4$, then W is the whole space of linear forms and the common zero set of K_U is the single plane Π; this is impossible, since the points P_i lie in that zero set and span $\mathbb{P}^3$. Therefore $a = 3$.

The three independent linear forms in W have a unique common zero q, and the reduced common zero set of K_U is $\Pi \cup \{q\}$. Moreover $q \notin \Pi$, since otherwise all the P_i would again lie in Π. Observe also that Π is unique: if all quadrics of K_U vanished on a second plane as well, each of them would be a scalar multiple of the product of the two plane equations, contradicting $\dim K_U = 3$. A pair (Π, q) thus determines a well-defined partition of the labels into $I = \{i : P_i \in \Pi\}$ and $J = \{i : P_i = q\}$, and distinct label partitions give disjoint pieces. Both sets are nonempty, since the points P_i span $\mathbb{P}^3$: if all of them lay in Π, or if all of them were equal to q, this would fail.

The normalization $\ell_i(\mathbf{1}) = 1$ is useful here, since projectively equal evaluations with this normalization are equal as linear functionals. In particular, every vector of U has a common coordinate value on J. Viewed as an element of U, the form λ vanishes on I and takes a common nonzero value c on J; thus $\lambda = c\mathbf{1}_J$, and $\mathbf{1}_J \in U$. Subtracting from a vector its common value on J times $\mathbf{1}_J$ leaves a vector supported on I, and therefore $U = U_I \oplus \mathbb{Q}\mathbf{1}_J$, where $U_I = U \cap \mathbb{Q}^I$ and $\mathbb{Q}^I \subset \mathbb{Q}^{10}$ denotes the subspace of vectors vanishing outside I. We have $\dim U_I = 3$ and $\mathbf{1}_I \in U_I$. Products between the two summands vanish, since their supports are disjoint, and $\mathbf{1}_J \circ \mathbf{1}_J = \mathbf{1}_J$; hence $R = U_I^{\circ 2} \oplus \mathbb{Q}\mathbf{1}_J$. As $a = 3$ means that $\dim R = 7$, it follows that $\dim U_I^{\circ 2} = 6$.

Conversely, let $U_I \subseteq \mathbb{Q}^I$ be a three-plane containing $\mathbf{1}_I$ with $\dim U_I^{\circ 2} = 6$, and set $U = U_I \oplus \mathbb{Q}\mathbf{1}_J$. Choose a basis x_1, x_2, x_3 of U_I and write $y = \mathbf{1}_J$. The three elements yx_i of $\mathrm{Sym}^2 U$ are killed by coordinatewise multiplication, the six products coming from $\mathrm{Sym}^2 U_I$ have independent images, and the image of y^2 has support disjoint from theirs. It follows that K_U is spanned by yx_1, yx_2, yx_3, and that its common zero set is the plane $y = 0$ together with the point $[0:0:0:1]$. This recovers exactly the common-plane configuration with label partition (I, J).

These two constructions identify the U-locus with a variety: the forward map is the coordinate projection to $\mathbb{Q}^I$, of constant rank three, and the inverse is the direct sum with $\mathbb{Q}\mathbf{1}_J$. Both maps are regular.

Let $m = |I|$. The equality $\dim U_I^{\circ 2} = 6$ requires $m \geq 6$, since $U_I^{\circ 2} \subseteq \mathbb{Q}^I$, whereas $J \neq \varnothing$ gives $m \leq 9$. The unital three-planes in $\mathbb{Q}^I$ form the Grassmannian $G_I = \mathrm{Gr}(2, \mathbb{Q}^I/\mathbb{Q}\mathbf{1}_I)$, and the locus required here is $G_I \setminus D_I$, where D_I is the closed subvariety of G_I consisting of those U_I for which $\dim U_I^{\circ 2} \leq 5$; it is closed because it is defined by a rank condition on the multiplication map.

We bound $\dim D_I$ by counting the parameters in a nonzero quadratic relation. Choose three coordinate evaluations that are independent on U_I, and use the unique basis of U_I in which they are the three coordinate vertices of $\mathbb{P}^2$. With respect to this basis, $\mathbf{1}_I$ is the sum of the three basis vectors, and hence every other evaluation point lies in the affine plane $X_1 + X_2 + X_3 = 1$. A quadratic relation must vanish at the three coordinate vertices, and is therefore of the form $\alpha X_1X_2 + \beta X_1X_3 + \gamma X_2X_3$; the nonzero relations up to scalar form a $\mathbb{P}^2$. Once a relation is chosen, each of the remaining $m-3$ evaluation points must lie on its affine conic, and thus contributes at most one parameter; the resulting incidence surjects onto D_I on this chart. Note that a nonzero homogeneous quadratic form cannot vanish identically on the affine plane $X_1 + X_2 + X_3 = 1$: by homogeneity it would then vanish on the dense open set $X_1 + X_2 + X_3 \neq 0$, and hence be zero. The incidence thus has a two-dimensional parameter space for the relation and at most one parameter for each remaining label, and since finitely many choices of the three evaluations cover G_I, we obtain $\dim D_I \leq 2 + (m-3) = m - 1 \leq 8$.

For these spaces U we have $a = 3$, and the fiber of choices of V is a $\mathbb{P}^2$. The class of the piece indexed by (I, J) is therefore $[\mathbb{P}^2]([G_I] - [D_I])$. The Grassmannian class is a polynomial in $\mathbb{L}$, whereas $\dim D_I \leq 8$, and Lemma 6.12 therefore shows that this expression has zero S_{12}-part. Here we argue with classes rather than with dimensions, since the dimension of $G_I \setminus D_I$ itself need not be at most ten. Indeed, the term $[G_I]$ is Tate by its Schubert decomposition, while $[D_I]$ has a base of dimension at most eight, and multiplying either term by $[\mathbb{P}^2] = 1 + \mathbb{L} + \mathbb{L}^2$ introduces only Tate twists. Their difference therefore has zero S_{12}-part in the alternating compactly supported Euler class, which is the assertion needed here.

Summing over the finitely many label partitions proves the same assertion for the whole common-plane locus. We have thus treated the complement of $\mathcal{E}_4$ in $\mathcal{C}$, and its S_{12}-part is zero, so that additivity gives $[\chi_c(\mathcal{C})]_{S_{12}} = -B_4(\mathbb{L})S_{12}$. Finally, $\varepsilon_{12}(\chi_c^{\ell}(\mathcal{C})) = -B_4(1) = 56$. □

9. The remaining 21 profiles

Recall that the *profile* of a homogeneous lattice is the sequence of positive codimensions $10 - \dim F_i$, for $i \geq 1$, with the final zero terms omitted. We call $(6, 2)$ the critical profile because it is the one contributing the elliptic-quartic term. The preceding section identifies a non-Tate contribution coming from a twist of S_{12} from the fixed locus corresponding to the profile $(6, 2)$. The full irreducible count includes twenty-one other profiles, and we must show that their contributions cannot cancel it. We prove that none of these profiles contains a Tate twist of S_{12} in its compactly supported cohomology. Most cases reduce to Grassmannians or spaces of isotropic subspaces for quadratic forms. In the remaining cases, we bound the dimension of the spaces over which these bundles vary. The bounds are at most ten, whereas S_{12} has Hodge level and Hodge–Tate spread eleven. We then apply the loop-rotation identity to deduce that $g_{10}(p^{17})$ is not eventually polynomial in p.

Put $A = \mathbb{Q}^{10}$, with coordinatewise multiplication. For subspaces of $\mathbb{Q}^N$, write

$$BC = \mathrm{Span}_{\mathbb{Q}}\{b \circ c : b \in B,\ c \in C\},$$

and

$$U^j = \mathrm{Span}_{\mathbb{Q}}\{u_1 \circ \cdots \circ u_j : u_i \in U\}.$$

The *profile* of a homogeneous lattice is the sequence of positive codimensions $10 - \dim F_i$, for $i \geq 1$, with the final zero terms omitted. Since the spaces F_i are nested, these codimensions are

nonincreasing. Repeated parts therefore mean repeated spaces in consecutive degrees. We call $(6,2)$ the critical profile because it is the one contributing the elliptic-quartic term; the other profiles are called noncritical.

For a partition $c=(c_1,\dots,c_l)$ of 8, let $\mathcal{T}_c$ parametrize filtrations $F_0=\mathbb{Q}\mathbf{1}\subseteq F_1\subseteq\cdots\subseteq F_l\subseteq A$ with $\dim F_i=10-c_i$ and $F_iF_j\subseteq F_{i+j}$, where $F_i=A$ for $i>l$. Thus only products with $i,j\geq 1$ and $i+j\leq l$ impose conditions.

The parameter space $\mathcal{T}_c$ is the reduced closed subvariety of the corresponding partial flag variety defined by these product inclusions. For example, the inclusion $F_iF_j\subseteq F_{i+j}$ means that the multiplication map from the two tautological subbundles has zero image in the quotient bundle A/F_{i+j}. Repeated dimensions in a flag force equality of the corresponding spaces. This explains why repeated parts of c give the algebra and stability conditions used below.

We first treat conditions of the form $U^2\subseteq T$, where T is a hyperplane. Recall that a subspace P is totally isotropic for a symmetric bilinear form b if $b(x,y)=0$ for every $x,y\in P$.

Lemma 9.1. *Let W be a split nondegenerate quadratic space over $\mathbb{Q}$. The variety $\mathrm{OG}(a,W)$ of totally isotropic a-planes has only Tate cohomology. This includes both components when $\dim W=2a$.*

Here "split" means that W is an orthogonal sum of hyperbolic planes, with one additional nondegenerate line if $\dim W$ is odd. A hyperbolic plane has a basis e,f with $b(e,e)=b(f,f)=0$ and $b(e,f)=1$. The variety $\mathrm{OG}(a,W)$ is a closed subvariety of $\mathrm{Gr}(a,W)$: its defining condition is that the form restrict to zero on the tautological subbundle.

Proof. Each connected component is a projective homogeneous space for the split special orthogonal group. The Bruhat decomposition for its parabolic quotient gives a paving by affine spaces defined over $\mathbb{Q}$; see [29, §21h]. Localization therefore gives the assertion. In the even-dimensional maximal case, apply this to each of the two components. □

Let $E\to B$ be a vector bundle and $L\to B$ a line bundle. A symmetric bilinear form on E with values in L is an $\mathcal{O}_B$-linear map

$$b: E\otimes_{\mathcal{O}_B} E\longrightarrow L,$$

such that $b(u,v)=b(v,u)$.

Lemma 9.2. *Let $E\to B$ be a vector bundle with a symmetric bilinear form taking values in a line bundle, and suppose $\dim B\leq d$. Let $Z\to B$ parametrize its totally isotropic a-planes. Every simple constituent of $H^*_c(Z)$ is a subquotient of a Tate twist of the cohomology of a variety of dimension at most d.*

Consequently its Hodge level and individual Hodge–Tate spread are at most d. If $d\leq 10$, no Tate twist of S_{12} occurs.

Proof. We trivialize the line bundle in which the form takes values, and stratify B by the rank of the resulting map $E\to E^\vee$.

The strata are given their reduced induced structures. On a constant-rank stratum, elementary row operations on an invertible minor identify the kernel and image with locally free summands. Thus the radical used below is a vector bundle. We may replace a finite open cover by a finite disjoint decomposition into locally closed pieces, so all subsequent local choices can be combined by localization. On each rank stratum, its kernel

$$R=\{v\in E: b(v,w)=0 \text{ for every } w\in E\}$$

is a vector subbundle, called the radical. Locally choose a complementary subbundle M, so that $E=R\oplus M$. The restriction of the form to M is nondegenerate, because its entire radical has been removed.

After a further finite decomposition, we may trivialize R and choose an orthogonal basis $m_1, \dots, m_r$ of M, with $b(m_i, m_i) = a_i$ invertible and $b(m_i, m_j) = 0$ for $i \neq j$. This is the usual diagonalization of a symmetric form, carried out locally where each chosen diagonal coefficient is invertible.

For an invertible function a_i, adjoining a square root u_i gives a finite étale cover: the equation is $u_i^2 = a_i$, and its derivative $2u_i$ is invertible. Take the fiber product of these covers and also adjoin v with $v^2 = -1$. Denote the resulting cover by $B' \to B_0$, where B_0 is the piece under consideration. Rescaling m_i by u_i^{-1} makes every diagonal coefficient equal to one. Using v, each pair of these coordinates becomes a hyperbolic plane. Thus, over B', the form is isometric to

$$E_{B'} = R \oplus (M_0 \otimes_{\mathbb{Q}} \mathcal{O}_{B'}),$$

where R is trivial and M_0 is a fixed split nondegenerate quadratic space over $\mathbb{Q}$.

We now describe the isotropic a-planes for this form. Fix a point of B' and an isotropic plane P, and set $D = P \cap R$ and $Q = \operatorname{im}(P \longrightarrow M_0)$. Stratify by $c = \dim D$. Then $\dim Q = a - c$. The form vanishes on R, so P is isotropic exactly when Q is isotropic in M_0. There is no restriction on the c-plane $D \subseteq R$.

For fixed D and Q, the projection $P/D \to Q$ is an isomorphism. Consequently P/D is the graph of a unique linear map $\phi : Q \longrightarrow R/D$. Conversely, every such map gives the isotropic plane

$$P_\phi = \{(r, q) \in R \oplus Q : r \bmod D = \phi(q)\}.$$

It has intersection D with R and image Q in M_0. Thus the choices of P with fixed D, Q form the vector space $\operatorname{Hom}(Q, R/D)$.

Writing $\rho = \operatorname{rank} R$, the stratum with intersection dimension c is a vector bundle of rank $(a - c)(\rho - c)$ over

$$B' \times \operatorname{Gr}(c, \rho) \times \operatorname{OG}(a - c, M_0).$$

The last two factors have Tate cohomology. The vector-bundle formula and Künneth therefore show that every simple constituent of the cohomology of this stratum comes from a Tate twist of $H_c^j(B')$. Localization over the finitely many values of c gives the same conclusion for $Z' = Z \times_{B_0} B'$.

Finally, $p : Z' \to Z|_{B_0}$ is finite étale. Its pullback and trace satisfy $\operatorname{Tr} \circ p^* = (\deg p)\,\mathrm{id}$. With rational coefficients, pullback therefore identifies $H_c^i(Z|_{B_0})$ with a direct summand of $H_c^i(Z')$. These maps respect the Hodge structures and the arithmetic Galois action. Applying localization once more to the pieces B_0 proves the asserted description of the constituents of $H_c^*(Z)$.

Each finite cover B' has dimension at most d. The dimension bounds established earlier therefore give Hodge level and individual Hodge–Tate spread at most d. Since Tate twists preserve these quantities and S_{12} has level and spread eleven, its Tate twists are excluded when $d \leq 10$. □

Lemma 9.3. *Let $N \geq 2$ and $1 \leq d \leq N - 1$. Define*

$$X_{N,d} = \{(U, T) : \mathbf{1} \in U \subseteq T \subseteq \mathbb{Q}^N,\ \dim U = d,\ \dim T = N - 1,\ U^2 \subseteq T\}.$$

Every simple constituent of $H_c^*(X_{N,d})$ is a subquotient of a Tate twist of the cohomology of a variety of dimension at most $N - 2$. In particular, no Tate twist of S_{12} occurs if $N \leq 12$.

Proof. Write $T = \ker \lambda$. The hyperplanes containing $\mathbf{1}$ form

$$B = \mathbb{P}\{\lambda \in (\mathbb{Q}^N)^\vee : \lambda(\mathbf{1}) = 0\} \simeq \mathbb{P}^{N-2}.$$

The form $b_\lambda(x, y) = \lambda(x \circ y)$ takes values in $\mathcal{O}_B(1)$. On T, the vector $\mathbf{1}$ belongs to its radical, so the form descends to $T/\mathbb{Q}\mathbf{1}$. The condition $U^2 \subseteq T$ says exactly that $U/\mathbb{Q}\mathbf{1}$ is a totally isotropic $(d - 1)$-plane in this quotient. Apply Lemma 9.2. □

Let $\pi = \{I_1, \dots, I_h\}$ be a partition of $\{1, \dots, N\}$ into nonempty disjoint blocks. Its coordinate-partition algebra is

$$H_\pi = \{x \in \mathbb{Q}^N : x_i = x_j \text{ whenever } i, j \text{ belong to the same block}\}.$$

Thus a vector in H_π has one common value on I_1, another on I_2, and so on. For example, the partition $\{\{1,3\},\{2,4,5\}\}$ gives $H_\pi = \{(a,b,a,b,b) : a,b \in \mathbb{Q}\}$. Coordinatewise addition and multiplication preserve these equalities, so H_π is a unital subalgebra of $\mathbb{Q}^N$.

For a block I, let $\mathbf{1}_I$ denote the vector whose ith coordinate is 1 if $i \in I$ and 0 otherwise. Every vector in H_π has a unique expression

$$x = a_1\mathbf{1}_{I_1} + \cdots + a_h\mathbf{1}_{I_h},$$

where a_j is its common coordinate value on I_j. Hence the block indicators form a basis of H_π, and $\dim H_\pi = h$. We may identify H_π with $\mathbb{Q}^h$ as an algebra.

For a subspace $U \subseteq H$, write $\mathbb{Q}[U]$ for the smallest unital subalgebra of H containing U; it consists of finite $\mathbb{Q}$-linear combinations of products of elements of U, including the empty product $\mathbf{1}$. A subspace $V \subseteq \mathbb{Q}^N$ is H-stable if $h \circ v \in V$ for every $h \in H$ and $v \in V$.

Lemma 9.4. *Every unital subalgebra of $\mathbb{Q}^N$ is a coordinate-partition algebra. In particular, there are finitely many such algebras. Given an algebra H, set*

$$A_d(H) = \{U \subseteq H : \mathbf{1} \in U,\ \dim U = d,\ \mathbb{Q}[U] = H\}.$$

Then $A_d(H)$ has only Tate cohomology. The same holds for the varieties of H-stable spaces.

Proof. The classification follows from [32, Lemma 2.7]. In particular, H has only finitely many proper unital subalgebras, say $H_1, \dots, H_m$.

To study $A_d(H)$, begin with the Grassmannian $G = \mathrm{Gr}(d-1, H/\mathbb{Q}\mathbf{1})$ of all unital d-planes in H. Such a plane fails to generate H precisely when it is contained in one of the H_i. Thus, setting $Y_i = \mathrm{Gr}(d-1, H_i/\mathbb{Q}\mathbf{1})$, we have $A_d(H) = G \setminus Y$, where $Y = \bigcup_i Y_i$. For every nonempty set of indices J,

$$\bigcap_{j\in J} Y_j = \mathrm{Gr}\left(d-1, \left(\bigcap_{j\in J} H_j\right)/\mathbb{Q}\mathbf{1}\right),$$

or the intersection is empty. Every such intersection therefore has Tate cohomology, by the Schubert decomposition.

We first show that Y has Tate cohomology. For two closed subvarieties, the compact-support Mayer–Vietoris sequence contains

$$\cdots \longrightarrow H_c^{i-1}(Y_1 \cap Y_2) \longrightarrow H_c^i(Y_1 \cup Y_2) \longrightarrow H_c^i(Y_1) \oplus H_c^i(Y_2) \longrightarrow \cdots.$$

By exactness, every simple constituent of the middle group occurs in one of the adjacent groups. Hence the union has only Tate constituents whenever both pieces and their intersection do. Applying this argument inductively proves the assertion for Y. At each step, the intersection of the new piece with the preceding union is itself a union of intersections of the Y_i, so the same induction applies.

Now use the localization sequence

$$\cdots \longrightarrow H_c^{i-1}(Y) \longrightarrow H_c^i(A_d(H)) \longrightarrow H_c^i(G) \longrightarrow \cdots.$$

The adjacent groups have only Tate constituents, and exactness gives the same conclusion for $H_c^i(A_d(H))$.

For the assertion about H-stable spaces, the block idempotents separate the choices on the individual coordinate blocks. Indeed, if the blocks of H are $I_1, \dots, I_h$, stability gives $V = \bigoplus_{j=1}^h V_j$, where $V_j = \mathbf{1}_{I_j} V \subseteq \mathbb{Q}^{I_j}$. The condition $H \subseteq V$ is equivalent to $\mathbf{1}_{I_j} \in V_j$ for every j. Thus, for

prescribed dimensions $v_j = \dim V_j$, the choices form $\prod_j \mathrm{Gr}(v_j - 1, |I_j| - 1)$. These are products of Grassmannians and therefore have Tate cohomology.

If an H-stable space V is not required to contain H, the same decomposition gives $\prod_j \mathrm{Gr}(v_j, |I_j|)$ instead. Taking the finite union over the possible block dimensions proves the assertion for all H-stable spaces. The version with $\mathrm{Gr}(v_j - 1, |I_j| - 1)$ is the one used when $H \subseteq V$.

□

Lemma 9.5. *Let $\mathbf{1} \in B \subseteq C \subseteq \mathbb{Q}^N$ and set $H = \mathbb{Q}[B]$. Then $BC \subseteq C$ if and only if C is H-stable. In this case $H \subseteq C$ and $B^2 \subseteq C$.*

For fixed H and prescribed dimensions, the choices of B generating H and of H-stable C containing H are independent and have only Tate cohomology.

Proof. If multiplication by B preserves C, so does multiplication by every sum of products of elements of B. These elements form H. Since $\mathbf{1} \in C$, this also gives $H \subseteq C$. The converse is immediate. For fixed H, the parameter space is the product of $A_{\dim B}(H)$ with the H-stable-space variety in Lemma 9.4. □

For a subspace $R \subseteq \mathbb{Q}^N$, set $\mathrm{Stab}(R) = \{a \in \mathbb{Q}^N : aR \subseteq R\}$. This is a unital subalgebra. Thus $\mathrm{Stab}(R)$ consists of the vectors whose coordinatewise multiplication maps preserve R. The notation refers to this algebra of multiplication operators, with the vector $\mathbf{1}$ acting as the identity.

Lemma 9.6. *Let $T = \ker \lambda$ contain $\mathbf{1}$, and set $S = \{i : \lambda_i \neq 0\}$ and $s = |S|$. Then*

$$H_S := \mathrm{Stab}(T) = \{x \in \mathbb{Q}^N : x_i \text{ is constant for } i \in S\}$$

and $\dim H_S = N + 1 - s$. Further, $H_S \subseteq T$ and $\mathrm{rad}(b_\lambda|_T) = H_S$. The hyperplanes with support S form

$$B_S = \{[\lambda_i]_{i \in S} : \textstyle\sum_{i \in S} \lambda_i = 0,\ \lambda_i \neq 0 \text{ for } i \in S\}.$$

Here $s \geq 2$, and B_S is a hyperplane-arrangement complement in $\mathbb{P}^{s-2}$ with only Tate cohomology. A totally isotropic d-plane for b_λ satisfies

$$d \leq N - s + \lfloor s/2 \rfloor. \tag{9.1}$$

Proof. The condition $xT \subseteq T$ says that $y \mapsto \lambda(x \circ y)$ is a scalar multiple of λ. Thus $\lambda_i x_i = c\lambda_i$ for every i. This gives the description of H_S. It also gives $\lambda(x) = c \sum_{i \in S} \lambda_i = 0$, so $H_S \subseteq T$. The same calculation describes the radical of the restricted form.

The assertion about B_S follows by removing coordinate hyperplanes from $\sum_{i \in S} \lambda_i = 0$ in $\mathbb{P}^{s-1}$. All their intersections are projective linear spaces, so localization gives Tate cohomology.

Finally, the radical of b_λ on $\mathbb{Q}^N$ has dimension $N - s$. Its nondegenerate quotient has dimension s, and an isotropic subspace of that quotient has dimension at most $\lfloor s/2 \rfloor$. This proves (9.1). □

Lemma 9.7. *Let $N \geq 2$, $1 \leq d \leq N - 1$, and $r \geq 1$. Consider multiplicative filtrations with $F_1 = U$, $F_2 = \cdots = F_{r+1} = T$, $\dim U = d$, and $\dim T = N - 1$. Their cohomology has Hodge level and individual Hodge–Tate spread at most $N - 2$. If $r \geq 2$, it has only Tate cohomology.*

Proof. Recall that $F_1 = U$, that $F_2, \ldots, F_{r+1}$ are all equal to T, and that every later term is $\mathbb{Q}^N$. Thus only products whose indices sum to at most $r + 1$ impose a condition.

For $r = 1$, the proper terms are U, T. The only required product inclusion is $F_1 F_1 \subseteq F_2$, namely $U^2 \subseteq T$. The parameter space is therefore $X_{N,d}$, and Lemma 9.3 applies.

For $r = 2$, the proper terms are U, T, T. Besides $U^2 \subseteq T$, we require $F_1 F_2 \subseteq F_3$, or $UT \subseteq T$. This says that multiplication by every element of U preserves T.

Write $T = \ker \lambda$ and fix the support S of λ. By Lemma 9.6, the condition $UT \subseteq T$ is equivalent to $U \subseteq H_S$. Since H_S is an algebra contained in T, this also gives $U^2 \subseteq H_S \subseteq T$. Thus no separate square condition remains.

For fixed S, the algebra H_S is fixed and has dimension $N+1-s$, where $s=|S|$. We may therefore choose $T \in B_S$ and, independently, a unital d-plane in H_S. The resulting parameter space is

$$B_S \times \mathrm{Gr}(d-1, H_S/\mathbb{Q}\mathbf{1}) \simeq B_S \times \mathrm{Gr}(d-1, N-s).$$

It is empty if $d > N+1-s$. Both factors have Tate cohomology.

For $r \geq 3$, the term F_4 is still T. Hence $F_2F_2 \subseteq F_4$ requires $T^2 \subseteq T$. Since T contains $\mathbf{1}$, it is a unital subalgebra, and therefore one of finitely many coordinate-partition hyperplanes. For each such T, any unital d-plane $U \subseteq T$ is allowed: all products among U and T already lie in T. Its choices form $\mathrm{Gr}(d-1, N-2)$. Thus these cases also have Tate cohomology. □

Lemma 9.8. *Let $N \geq 5$ and $1 \leq d \leq N-2$. Consider multiplicative filtrations with proper terms $U, V, T, \dots, T$, where T occurs $k \geq 2$ times, $\mathbf{1} \in U \subseteq V \subseteq T \subseteq \mathbb{Q}^N$, and $(\dim U, \dim V, \dim T) = (d, N-2, N-1)$. Every simple cohomology constituent is a subquotient of a Tate twist of the cohomology of a variety of dimension at most $N-3$. In particular, no Tate twist of S_{12} occurs if $N \leq 13$.*

Proof. For $k=2$, the conditions reduce to $U^2 \subseteq V$, $UT \subseteq T$ and $V^2 \subseteq T$. Write $T = \ker \lambda$ and fix its support S. The last condition makes V an isotropic $(N-2)$-plane for b_λ. By (9.1), $N-2 \leq N-s+\lfloor s/2 \rfloor$, so $2 \leq s \leq 4$. We have that $U \subseteq H_S$.

If $s=2$, then B_S is a point and $T = H_S \simeq \mathbb{Q}^{N-1}$ as an algebra. The remaining condition is $U^2 \subseteq V$, giving $X_{N-1,d}$. Lemma 9.3 gives the bound $N-3$.

If $s=3$, the form on T/H_S is a nondegenerate form on a line. It has no nonzero isotropic vectors, so $V = H_S$. The choices of U form $\mathrm{Gr}(d-1, N-3)$ over the one-dimensional base B_S.

If $s=4$, the form on T/H_S is nondegenerate of dimension two. An isotropic $(N-2)$-plane V must contain H_S and choose an isotropic line in this quotient. These lines form a finite étale double cover of B_S, of dimension two. Over this cover, U is any unital d-plane in H_S, with choices $\mathrm{Gr}(d-1, N-4)$. Its square lies in $H_S \subseteq V$. Thus the required constituent bound follows from the bundle formula. The double cover need not itself have Tate cohomology.

For $k=3$, the additional condition is $VT \subseteq T$. It forces $V \subseteq H_S$, so only $s=2,3$ remain. For $k \geq 4$, the condition $T^2 \subseteq T$ also appears, forcing $s=2$. The preceding descriptions remain valid. Localization over the finite support decomposition proves the assertion. □

Lemma 9.9. *Let $1 \leq u \leq q \leq N$, and let $\mathbf{1} \in U \subseteq \mathbb{Q}^N$ satisfy $\dim U = u$ and $r = \dim U^2 \leq q$. Set $H = \mathrm{Stab}(U^2)$, $h = \dim H$, $W = HU$, and $d = \dim W$. Then*

$$r \geq 2d-h \quad \text{and} \quad \max(u,h) \leq d \leq r \leq q. \tag{9.2}$$

Fix the coordinate blocks of H, of sizes $n_1, \dots, n_h$, and the dimensions w_j of the corresponding summands of W. The locus of these U has dimension at most

$$\sum_{j=1}^{h} (w_j - 1)(n_j - w_j) + (u-1)(d-u) \leq \Phi_{N,u}(h,d), \tag{9.3}$$

where

$$\Phi_{N,u}(h,d) = (d-h)(N-d) + (u-1)(d-u).$$

Fixing these ranks and algebras gives a finite locally closed stratification.

Proof. Put $R = U^2$. Since $\mathbf{1} \in U$ and $HR = R$, we have $U, H \subseteq W \subseteq R$ and $W^2 = H^2U^2 = R$. Apply [32, Theorem 3.3] to W, W to obtain $r \geq 2d-h$. This proves (9.2).

The reason for enlarging U to $W = HU$ is that multiplication by H then preserves W, while the square remains R. Indeed, H contains $\mathbf{1}$, so $H^2 = H$ and $W^2 = H^2U^2 = HR = R$. The stabilizer in the cited inequality is therefore precisely the algebra H already chosen.

For fixed H, the choices of W form $\prod_j \mathrm{Gr}(w_j - 1, n_j - 1)$. For fixed W, the unital u-planes form $\mathrm{Gr}(u-1, d-1)$. The dimension of this incidence is the first expression in (9.3).

Here every w_j is at least one, since W contains the block indicator $\mathbf{1}_{I_j}$. The factors have dimensions $(w_j - 1)(n_j - w_j)$ and $(u-1)(d-u)$, respectively. We count all such pairs (U, W), allowing pairs that do not have the prescribed square or stabilizer. This can only increase the dimension and hence gives an upper bound for the required locus.

Since $\sum_j (w_j - 1) = d - h$ and $\sum_j (n_j - w_j) = N - d$, it is at most $\Phi_{N,u}(h, d)$.

To apply this bound to the actual U-locus, first fix the rank of $\mathrm{Sym}^2 U \to \mathbb{Q}^N$. Its image R then varies as a vector subbundle. The stabilizer is the kernel of $\mathbb{Q}^N \to \mathrm{Hom}(R, \mathbb{Q}^N/R)$ sending a to the linear map $x \mapsto a \circ x \bmod R$. On constant-rank strata it varies regularly and takes values among finitely many coordinate-partition algebras. Fixing H and the ranks of its block projections on U makes $W = HU$ vary regularly as well. The graph of $U \mapsto W$ lies in the incidence just counted. This proves both the dimension bound and the stratification assertion. □

Lemma 9.10. *The loci*

$$D_5 = \{U \subseteq A : \mathbf{1} \in U,\ \dim U = 5,\ \dim U^2 \leq 7\},$$
$$D_6 = \{U \subseteq A : \mathbf{1} \in U,\ \dim U = 6,\ \dim U^2 \leq 7\}$$

satisfy $\dim D_5 \leq 10$ *and* $\dim D_6 \leq 5$.

Proof. In Lemma 9.9, take $N = 10$ and $q = 7$. Then $u \leq d \leq 7$ and $h \geq 2d - 7$. For fixed d, the bound $\Phi_{10,u}(h, d)$ decreases with h, giving

u	d	$\Phi_{10,u}(2d-7, d)$
5	5, 6, 7	10, 8, 8
6	6, 7	4, 5.

Take the maximum over the finitely many strata. □

Proposition 9.11. *The profiles* $(5, 3)$ *and* $(4, 3, 1)$ *contain no Tate twist of* S_{12} *or* $S_{12,\ell}$ *in compactly supported cohomology.*

Proof. For $(5, 3)$, the proper terms have dimensions $(5, 7)$ and the only condition is $U^2 \subseteq V$. On the stratum where $r = \dim U^2 \leq 7$, the choices of V form $\mathrm{Gr}(7 - r, A/U^2)$. Its base lies in D_5 and has dimension at most ten.

For $(4, 3, 1)$, the proper dimensions are $(6, 7, 9)$, with conditions $U^2 \subseteq V$ and $UV \subseteq T$. If $\dim U^2 = 7$, then $V = U^2$ and the base of choices of U lies in D_6, of dimension at most five. If $\dim U^2 = 6$, then $U^2 = U$, so U is one of finitely many partition algebras; the choices of V form $\mathbb{P}(A/U) = \mathbb{P}^3$.

In either case, stratify by $s = \dim UV$. The hyperplanes T containing UV form $\mathbb{P}((A/UV)^\vee)$, or the empty set if $s = 10$. Since $\mathbf{1} \in U$, we have $V \subseteq UV$, so this also imposes $V \subseteq T$. The bundle formula and Lemma 6.11 give the required exclusions. □

The next argument treats three-step flags. When the first space is two-dimensional, its products are controlled by four elementary moment equations.

Lemma 9.12. *For* $s \geq 3$, *let* M_s *parametrize pairs* $([\lambda], P)$, *where* $[\lambda] \in \mathbb{P}((\mathbb{Q}^s)^\vee)$, P *is a two-plane containing* $\mathbf{1}$, *and* $\lambda(P^3) = 0$. *Then*

$$\dim M_s \leq \max\{s - 3, 2s - 7\}.$$

Proof. Write $P = \langle \mathbf{1}, z \rangle$. Stratify by the partition of the coordinate labels on which the values of z agree, and let $v \geq 2$ be the number of blocks. Replacing z by $az + b\mathbf{1}$ does not change P. Normalize its values on two specified blocks to 0 and 1.

More precisely, two choices z, z' spanning P with $\mathbf{1}$ satisfy $z' = az + b\mathbf{1}$ with $a \neq 0$. On a fixed equality partition, choose two of its blocks in a fixed order. Their values are distinct, so there is a unique such change giving values 0 and 1 on those two blocks. The remaining block values are then genuine coordinates for P on this stratum.

The remaining distinct block values then vary in an open subset of $\mathbb{A}^{v-2}$.

The condition on λ is

$$\sum_{i=1}^{s} \lambda_i z_i^j = 0$$

for $0 \leq j \leq 3$. The coefficient matrix has rank $\min(4, v)$, by the Vandermonde determinant. Thus the projective space of possible λ is empty or has dimension $s - \min(4, v) - 1$. The stratum has dimension at most

$$v - 2 + s - \min(4, v) - 1 = \begin{cases} s - 3, & 2 \leq v \leq 3, \\ s + v - 7 \leq 2s - 7, & v \geq 4. \end{cases}$$

There are finitely many equality partitions. □

Lemma 9.13. *Let $N \geq 4$ and $2 \leq d \leq N - 2$. Define*

$$Y_{N,d} = \left\{ \begin{array}{c} \mathbf{1} \in U \subseteq V \subseteq T \subseteq \mathbb{Q}^N, \\ (U, V, T) : (\dim U, \dim V, \dim T) = (d, N-2, N-1), \\ U^2 \subseteq V, \qquad UV \subseteq T \end{array} \right\}.$$

Every simple constituent of $H_c^*(Y_{N,d})$ is a subquotient of a Tate twist of the cohomology of a variety of dimension at most

$$L(N, d) = \max\{N - 3, 2N - 2d - 3\}.$$

In particular, no Tate twist of S_{12} occurs if $L(N, d) \leq 10$.

Proof. Write $T = \ker \lambda$ and fix its support S, of size s. Put $P = \mathrm{pr}_S(U)$ and $t = \dim P$. Under the ordinary coordinate pairing, $UV \subseteq T$ means

$$\mathrm{diag}(\lambda) U \subseteq V^{\perp}.$$

The two sides have dimensions t and two, respectively. Since $\mathbf{1}_S \in P$, we have $t = 1$ or 2. These are locally closed rank conditions.

Suppose first that $t = 2$. The inclusion is then an equality, so

$$V = (\mathrm{diag}(\lambda) P)^{\perp}$$

is uniquely determined. The condition $U^2 \subseteq V$ says exactly $\lambda(P^3) = 0$. If $P = \langle \mathbf{1}_S, z \rangle$, these are

$$\sum_{i \in S} \lambda_i z_i^j = 0. \tag{9.4}$$

They also ensure $U \subseteq V \subseteq T$.

To see the equivalence, a vector $u_1 \circ u_2$ lies in V exactly when $\lambda(u_1 \circ u_2 \circ u_3) = 0$ for every $u_3 \in U$. The restrictions of the u_i to S fill P, so these are precisely the cubic moment conditions. Taking one of the u_i equal to $\mathbf{1}$ also gives $U \subseteq V$; taking $u_3 = \mathbf{1}$ in the definition of V gives $V \subseteq T$.

The case $s = 2$ is empty, since then $P = \mathbb{Q}^2$ and $\lambda(P) = 0$ forces $\lambda = 0$.

For $s \geq 3$, the pair $([\lambda], P)$ lies in the open part of M_s where every supported coefficient is nonzero. Also $D = U \cap \mathbb{Q}^{S^c}$ has dimension $d - 2$, so $s \leq N - d + 2$. Lemma 9.12 gives the bound

$$\dim\{([\lambda], P)\} \leq \max\{s - 3, 2s - 7\} \leq 2N - 2d - 3.$$

Over this pair base, choose $D \in \mathrm{Gr}(d-2, \mathbb{Q}^{S^c})$. Modulo D, a lift U is the graph of a map $\phi : P \to \mathbb{Q}^{S^c}/D$ satisfying

$$\phi(\mathbf{1}_S) = \mathbf{1}_{S^c} \bmod D.$$

These maps form an affine torsor of rank $N - s - d + 2$: a prescribed map on the line $\mathbb{Q}\mathbf{1}_S$ extends locally to P.

The difference of two extensions vanishes on $\mathbb{Q}\mathbf{1}_S$, so these extensions form a torsor under $\mathrm{Hom}(P/\mathbb{Q}\mathbf{1}_S, \mathbb{Q}^{S^c}/D)$. The first bundle has rank one and the second has rank $N - s - d + 2$. This gives the stated rank and also proves Zariski local triviality of the choices of lifts.

There are no further conditions, since the multiplication equations depend only on the supported coordinates. Grassmannian and affine bundles therefore add only Tate twists of the cohomology of the pair base.

Now suppose $t = 1$. Put $E = \mathbb{Q}\mathbf{1}_S \oplus \mathbb{Q}^{S^c}$ and $M = N + 1 - s$. Then $E = H_S$ is a fixed coordinate-partition algebra and $U \subseteq E$. Since E stabilizes T, the condition $UV \subseteq T$ is automatic.

If $E \subseteq V$, the condition $U^2 \subseteq V$ is automatic too. The choices are $U \in \mathrm{Gr}(d-1, M-1)$ and a hyperplane V/E in T/E. These give Grassmannian and projective bundles over B_S, and hence Tate cohomology.

This case is empty when $s = 2$, because then $E = T$ and an $(N-2)$-plane cannot contain E. For $s \geq 3$, the quotient T/E has rank $s-2$, so its hyperplanes form a projective bundle with fiber $\mathbb{P}^{s-3}$. This is the projective factor in the preceding description.

Otherwise, set $W = V \cap E$. It is a hyperplane in E, and (U, W) belongs to $X_{M,d}$ after identifying E with $\mathbb{Q}^M$. Conversely, every such pair extends to a flag: the choices of V/W are the complements to the line E/W in T/W. They form an affine torsor of rank $\dim(T/E) = s-2$. Thus this stratum is an affine bundle over $B_S \times X_{M,d}$.

The variety B_S has only Tate cohomology. Lemma 9.3 therefore gives the constituent bound $M - 2 \leq N - 3$. If $d = M$, this second stratum is empty. Combining the two projection-rank cases and the finitely many supports proves the lemma. □

Lemma 9.14. *Let $N \geq 3$ and $2 \leq v \leq N-1$. The flags $\mathbf{1} \in U \subseteq V \subseteq T \subseteq \mathbb{Q}^N$ with*

$$(\dim U, \dim V, \dim T) = (2, v, N-1),$$

$U^2 \subseteq V$, and $UV \subseteq T$ have the constituent bound of Lemma 9.13 with base dimension $\max\{N-3, 2N-7\}$. In particular, no Tate twist of S_{12} occurs if $N \leq 8$.

Proof. Write $T = \ker \lambda$. The product conditions imply $\lambda(U^3) = 0$, so $(U, [\lambda])$ lies in M_N. Over this base, set $K = (\mathrm{diag}(\lambda)U)^\perp$. The moment condition gives $U^2 \subseteq K$, and $\mathbf{1} \in U$ gives $K \subseteq T$.

Stratify by $q = \dim U^2$ and $t = \dim \mathrm{diag}(\lambda)U$. On each stratum these spaces vary as vector bundles, and the admissible V form $\mathrm{Gr}(v-q, K/U^2)$, or the empty set. Apply the bundle formula and Lemma 9.12. □

Proposition 9.15. *The profile $(5,2,1)$ contains no Tate twist of S_{12} or $S_{12,\ell}$ in compactly supported cohomology.*

Proof. Its proper dimensions are $(5,8,9)$, and its conditions are $U^2 \subseteq V$ and $UV \subseteq T$. Thus it is $Y_{10,5}$. Lemma 9.13 gives $L(10,5) = 7$. □

It remains to treat the profiles with repeated terms. Equalities among consecutive spaces often force a space to be an algebra, or force the next space to be stable under that algebra.

Proposition 9.16. *For every partition c of 8 other than (8), $(7,1)$, $(6,2)$, $(5,3)$, $(5,2,1)$, and $(4,3,1)$, the compactly supported cohomology of $\mathcal{T}_c$ contains no Tate twist of S_{12} or $S_{12,\ell}$.*

Proof. First consider the seven profiles $(4,4)$, $(3,3,2)$, $(3,3,1,1)$, $(2,2,2,2)$, $(2,2,2,1,1)$, $(2,2,1,1,1,1)$, and (1^8). Here 1^r denotes r successive parts equal to one. In each case $F_1 = F_2 = U$, so $U^2 \subseteq U$ and U is one of finitely many coordinate-partition algebras.

For $(4,4)$, $(2,2,2,2)$, and (1^8) there are no further choices. For $(3,3,2)$, the final space V is an arbitrary eight-plane containing the seven-plane U, with choices $\mathbb{P}^2$. For $(3,3,1,1)$ and $(2,2,2,1,1)$, the remaining condition is $UV \subseteq V$. Lemma 9.4 gives products of block Grassmannians. For $(2,2,1,1,1,1)$, the condition is $V^2 \subseteq V$, so both spaces are partition algebras. All seven profiles therefore have Tate cohomology.

The five profiles $(8-r,1^r)$, with $2 \le r \le 6$, have a first space followed by a hyperplane repeated at least twice. Lemma 9.7 gives Tate cohomology for these as well.

For $(4,2,2)$, the proper terms are U, V, V, of dimensions $6, 8, 8$. The conditions reduce to $UV \subseteq V$, which also implies $U^2 \subseteq V$. Fix $H = \mathbb{Q}[U]$. By Lemma 9.5, the choices of U generating H and of H-stable V containing H are independent and have Tate cohomology. There are finitely many possible H.

For $(3,2,2,1)$, the proper terms are U, V, V, W, of dimensions $7, 8, 8, 9$. The conditions reduce to $UV \subseteq V$ and $V^2 \subseteq W$. Again set $H = \mathbb{Q}[U]$. Then $H \subseteq V$, so $h = \dim H$ is seven or eight. For fixed H, omit the remaining product equations. The choices are contained in the incidence with $U \in \mathrm{Gr}(6, h-1)$, $V/H \in \mathrm{Gr}(8-h, 10-h)$ and $W/V \in \mathbb{P}^1$. Its dimension is

$$6(h-7) + 2(8-h) + 1 = \begin{cases} 3, & h = 7, \\ 7, & h = 8. \end{cases}$$

The actual locus has dimension at most seven, so Lemma 7.1 applies.

Finally, $(4,2,1,1)$ and $(3,2,1,1,1)$ have proper dimensions $(6,8,9,9)$ and $(7,8,9,9,9)$. Apply Lemma 9.8 with $(N,d,k) = (10,6,2)$ and $(10,7,3)$. Both have constituent bound seven.

These groups account for all sixteen profiles in the statement. Localization over their finite decompositions proves the assertion.

In each decomposition used here, the ranks of the multiplication maps are fixed before taking a Grassmannian or projective bundle. Their images and quotients are therefore vector bundles. The bundle formulas apply on each such stratum. Localization then shows that every simple constituent for the whole profile occurs among those of its strata; it cannot introduce a new simple constituent. Thus the argument excludes the twists in each cohomological degree, as asserted. □

Proposition 9.17. *For every partition c of 8 other than $(6,2)$, the compactly supported cohomology of $\mathcal{T}_c$ contains no Tate twist of S_{12} or $S_{12,\ell}$.*

Proof. For (8), the only proper term is a unital two-plane, with parameter space $\mathbb{P}^8$. For $(7,1)$, the proper dimensions are $(3,9)$ and the only condition is $F_1^2 \subseteq F_2$. This is $X_{10,3}$, so Lemma 9.3 applies.

Proposition 9.11 treats $(5,3)$ and $(4,3,1)$, and Proposition 9.15 treats $(5,2,1)$. The sixteen remaining noncritical profiles are covered by Proposition 9.16. These are all twenty-one partitions other than $(6,2)$. □

Theorem 9.18. *Let $I_{10,17}$ be the projective parameter variety over $\mathbb{Q}$ defined in the preceding section. Then*

$$\varepsilon_{12}(\chi_c^{\ell}(I_{10,17})) = 56.$$

Consequently no polynomial in $\mathbb{Q}[T]$ agrees with $g_{10}(p^{17})$ for every sufficiently large prime p.

Proof. The loop-rotation identity (8.8) and Proposition 8.2 give

$$\varepsilon_{12}(\chi_c^{\ell}(I_{10,17})) = \sum_{c \vdash 8} \varepsilon_{12}(\chi_c^{\ell}(\mathcal{T}_c)).$$

By Proposition 9.17, only $c = (6, 2)$ can contribute. Proposition 8.3 gives its contribution as $-B_4(1) = 56$.

For sufficiently large primes, Lemma 8.1 gives $g_{10}(p^{17}) = \#\mathscr{I}_{10,17}(\mathbb{F}_p)$. If these counts agreed with a polynomial $P(p)$, the trace criterion of Lemma 6.7 would give $\chi_c^\ell(I_{10,17}) = P(\mathbb{L})$. The right side is a virtual Tate class and has ε_{12}-value zero, contradicting the value 56. □

10. Lower-rank irreducible counts

To pass from irreducible subrings to arbitrary subrings, we must also consider irreducible factors of ranks less than ten. We show that, up to colength seventeen, these lower-rank spaces have zero ε_{12}-value. This will ensure that the contribution found in rank ten survives when we sum over all decompositions into irreducible factors.

As before, we work with the graded fixed loci. Most of them have dimension at most ten, which excludes every Tate twist of S_{12}. Some have larger dimension but are Grassmannian bundles over smaller bases, or are covered by the hyperplane and moment calculations of the preceding section. The estimates below keep track of both the choices of the first space U and the choices of the spaces containing its square.

Proposition 10.1. *For $1 \le n \le 9$ and $0 \le e \le 17$, we have*

$$\varepsilon_{12}\big(\chi_c^\ell(I_{n,e})\big) = 0.$$

For $n \le 6$, the same assertion holds for every $e \ge 0$.

We first record the dimension estimates used in the proof. Write $U = F_1$, $u = \dim U$, $R = U^2$, and $q = \dim R$. Since $\mathbf{1} \in U$, we have $U \subseteq R$. Every term from F_2 onward contains R. On a constant-square-rank stratum, omitting the remaining multiplication conditions therefore gives a partial flag bundle in $\mathbb{Q}^n/R$.

If the distinct dimensions in this quotient are $d_1 < \cdots < d_l$, its relative dimension is

$$\sum_{j=1}^{l}(d_j - d_{j-1})(n - q - d_j) \le \binom{n-q}{2}, \qquad d_0 = 0. \tag{10.1}$$

Indeed, choose the first subspace and then choose each successive quotient subspace. The summands are the corresponding Grassmannian dimensions. The upper bound is the dimension of a complete flag variety in $\mathbb{Q}^{n-q}$. Further multiplication conditions define closed loci, so these are also upper bounds for the dimensions of the multiplicative loci.

For unital three-planes, we shall use

$$\dim\{U \subseteq \mathbb{Q}^n : \mathbf{1} \in U,\ \dim U = 3,\ \dim U^2 \le 5\} \le n - 1. \tag{10.2}$$

This is the conic incidence argument used in the proof of Proposition 8.3. Choose three independent coordinate evaluations and normalize them to the coordinate vertices of $\mathbb{P}^2$. A nonzero quadratic relation through these vertices is parametrized by $\mathbb{P}^2$. Each of the remaining $n - 3$ evaluation points lies on its affine conic and contributes at most one parameter. The resulting incidence covers the locus in (10.2), giving the bound $2 + (n - 3) = n - 1$.

We also use the Kneser estimate of Lemma 9.9. For square rank at most q, its integers h, d satisfy $\max(u, h) \le d \le q$ and $h \ge 2d - q$. For fixed d, the bound $\Phi_{n,u}(h, d)$ decreases with h. Thus we may take $h = \max\{1, 2d - q\}$ and check the finitely many integers $u \le d \le q$. Only two of the bounds needed below require an additional geometric argument.

Lemma 10.2. *Let $U \subseteq \mathbb{Q}^9$ be a unital four-plane. The loci with $\dim U^2 = 6$ and $\dim U^2 = 7$ have dimension at most eight and ten, respectively.*

Proof. Choose four independent coordinate evaluations and normalize them as in Lemma 7.2. Four evaluation points are then the coordinate vertices of $\mathbb{P}^3$, and the other five lie in the affine chart where the coordinate sum is one. Put $K = \ker(\mathrm{Sym}^2 U \to \mathbb{Q}^9)$ and $a = \dim K$. For square ranks six and seven, we have $a = 4$ and $a = 3$, respectively. The kernel lies in the six-dimensional space of quadrics through the four vertices.

If its common zero set is finite, choose $K \in \mathrm{Gr}(a, 6)$ and then choose the five remaining points in that finite set. This gives the bound $a(6 - a)$, namely eight for $a = 4$ and nine for $a = 3$.

Suppose next that the quadrics have a curve in their common zero set but no common plane. The possibilities and their incidences were treated in Lemma 7.4 and Proposition 7.5. We recall how the number of additional labels enters that calculation.

For a line or planar conic C, let D_k be the dimension of its family through k of the designated vertices, and let m be the dimension of the space of quadrics through all four vertices and C. Choosing K contributes $a(m - a)$ parameters. At least $m - a$ of the five additional labels must lie off C to make K the exact kernel. Those points lie in the finite residual base; each remaining point contributes at most one parameter. The resulting bound is

$$D_k + 5 + (a - 1)(m - a).$$

For a line, $(D_k, m) = (4 - 2k, 3 + k)$ with $0 \le k \le 2$; for a planar conic, $(D_k, m) = (8 - 2k, 1 + k)$ with $2 \le k \le 3$. Substitution gives at most eight when $a = 4$ and at most nine when $a = 3$.

A cubic base occurs only for $a = 3$. Its family through the four vertices has dimension at most four, and the five additional points give a total bound of nine. A pair of skew lines occurs only for $a = 4$. Each line contains two of the four vertices, so there are finitely many such pairs; the remaining points contribute at most five parameters.

Finally, suppose that the quadrics have a common plane. The argument in the proof of Proposition 8.3 applies with nine labels. It forces $a = 3$ and gives $U = U_I \oplus \mathbb{Q}\mathbf{1}_J$, where I, J are nonempty and U_I is a unital three-plane in $\mathbb{Q}^I$. For each label partition, this locus is contained in $\mathrm{Gr}(2, |I| - 1)$. Since $|I| \le 8$, its dimension is at most $2(|I| - 3) \le 10$.

Taking the maximum over these possibilities gives the two asserted bounds.

□

Together with the Kneser estimate, this gives the following bounds for the square-rank loci needed below. Each column specifies (u, q).

n	$(4,5)$	$(4,6)$	$(4,7)$	$(5,6)$	$(5,7)$	$(6,7)$
8	4	8	–	4	–	–
9	5	8	10	4	8	5

(10.3)

The dashes mark cases that will not be needed. For example, when $(u, q) = (4, 5)$, the Kneser inequalities allow $(h, d) = (3, 4), (4, 4), (5, 5)$, whose bounds are $n - 4, 0, 3$. This gives four in rank eight and five in rank nine. The other Kneser entries are obtained in the same way. In rank eight the maxima for $(u, q) = (4, 6)$ and $(5, 6)$ are $\max\{8, 6, 6\} = 8$ and $\max\{3, 4\} = 4$, respectively. In rank nine, the maxima for $(5, 6)$, $(5, 7)$, and $(6, 7)$ are $\max\{4, 4\} = 4$, $\max\{8, 7, 8\} = 8$, and $\max\{3, 5\} = 5$. The entries for the four-planes of square ranks six and seven in rank nine use Lemma 10.2, rather than the weaker Kneser estimates.

Proof of Proposition 10.1. We prove that every graded fixed locus under consideration contains no Tate twist of S_{12} in its individual cohomology groups, in either realization. Equation (8.8) then gives the asserted Euler-class vanishing for $I_{n,e}$.

In rank one, $I_{1,0}$ is a point and $I_{1,e}$ is empty for $e > 0$. For $n \ge 2$, scalar reduction gives a surjection $\mathcal{O}^n/L \to k^n/k\mathbf{1}$ for every irreducible lattice L. Hence its colength is at least $n - 1$, and $I_{n,e}$ is empty if $e < n - 1$.

For $n \leq 6$, every graded profile is a closed subvariety of a partial flag variety in $\mathbb{Q}^n/\mathbb{Q}\mathbf{1}$. Its dimension is at most $(n-1)(n-2)/2 \leq 10$. Lemma 7.1 therefore applies for every colength.

Assume now that $7 \leq n \leq 9$ and $e \leq 17$. The positive codimensions $c_i = n - \dim F_i$, for $i \geq 1$, form a partition of $e-(n-1)$. We call $(c_2, c_3, \ldots)$ its tail. Since $c_1 = n-u$, the tail has sum at most $18-2n+u$. Indeed, its sum is $e-(n-1)-(n-u) = e-2n+1+u$. The bound follows from $e \leq 17$. The parts in the tail are positive and nonincreasing, and each is at most c_1. Thus the lists below are lists of partitions subject to this sum bound. Equal successive parts represent the same subspace, so they do not add choices to the corresponding flag variety.

Two cases can be treated uniformly. If $F_2 = \mathbb{Q}^n$, the only choice is the unital space U, so the profile is a Grassmannian. This includes the empty profile. If F_2 is a hyperplane, every later proper term equals that hyperplane. Lemma 9.7 applies. We may therefore restrict the remaining case analysis to $q \leq \dim F_2 \leq n-2$.

We shall also repeatedly use the case $q = u$. Then $U^2 = U$, so U is one of finitely many coordinate-partition algebras. If $n-u \leq 5$, the remaining flag lies in a quotient of dimension at most five. Its multiplicative locus has dimension at most $\binom{5}{2} = 10$ by (10.1).

Consider first rank seven. If $u = 1$, then $U = \mathbb{Q}\mathbf{1}$ and the tail has sum at most five. A tail of length at most two gives an unconstrained partial flag variety: products involving F_1 impose only the existing inclusions, and products of two later terms have degree at least four, where the filtration is already full.

After the hyperplane cases treated above, the longer tails are $(3,1,1)$, $(2,2,1)$, $(2,1,1)$, and $(2,1,1,1)$. For the first three, the only additional condition is $F_2^2 \subseteq F_4$. They give $X_{7,4}$ or $X_{7,5}$, so Lemma 9.3 applies. For $(2,1,1,1)$, put $A = F_2$ and $T = F_3 = F_4 = F_5$. The conditions are $A^2 \subseteq T$ and $AT \subseteq T$. This is the incidence of Lemma 9.7 with $(N,d,r) = (7,5,2)$.

Suppose $u = 2$. Writing $U = \langle \mathbf{1}, x \rangle$ gives $U^2 = \langle \mathbf{1}, x, x^2 \rangle$, so $q = 2$ or 3. The case $q = 2$ was covered by the finite-algebra observation. For $q = 3$, the U-base is $\mathbb{P}^5$ with finitely many algebra points removed. The quotient $\mathbb{Q}^7/R$ has dimension four. Its complete flag variety has dimension six, while every proper partial flag variety has dimension at most five.

A complete flag in this quotient requires tail codimensions $3, 2, 1$. Their sum exhausts the tail budget of six, so the only possible eleven-dimensional ambient incidence has profile $c = (5,3,2,1)$, at colength seventeen. Its base is geometrically irreducible of dimension five and its flag fiber is geometrically irreducible of dimension six. Here the base is the complement of finitely many points in $\mathbb{P}^5$, namely the unital two-planes which are already algebras. On the square-rank-three locus, R is a rank-three vector subbundle. The incidence is the complete flag bundle of its rank-four quotient. This bundle is Zariski locally trivial with irreducible fiber, so its total space is geometrically irreducible and has dimension $5+6 = 11$.

The condition $UF_2 \subseteq F_3$ is a proper closed condition on this incidence. To see this, choose distinct rational numbers $x_1, \ldots, x_7$ and put $x = (x_1, \ldots, x_7)$. The Vandermonde determinant shows that $\mathbf{1}, x, \ldots, x^6$ are independent. Take

$$\begin{aligned} U &= \langle \mathbf{1}, x \rangle, \\ F_2 &= \langle \mathbf{1}, x, x^2, x^3 \rangle, \\ F_3 &= \langle \mathbf{1}, x, x^2, x^3, x^5 \rangle, \end{aligned} \tag{10.4}$$

and extend F_3 to any six-plane F_4. This is a flag in the ambient incidence, but $x^4 \in UF_2$ does not belong to F_3. Thus the multiplicative locus is contained in a proper closed subset of an irreducible eleven-dimensional variety, and has dimension at most ten. Every other profile with $u = 2, q = 3$ already has ambient dimension at most $5+5 = 10$.

For $u \geq 3$, the cases $q = u$ have been treated. The remaining pairs are $(u,q) = (3,4), (3,5)$, and $(4,5)$. For the first two, (10.2) bounds the U-base by six; the extension flags have dimensions

at most three and one, respectively. For $(4,5)$, the Kneser bound is three and the extension flag has dimension at most one. All these loci have dimension at most nine. This completes rank seven.

Now take rank eight. For $u \geq 4$, the cases $q = u$ again have dimension at most ten. The remaining pairs are $(4,5),(4,6),(5,6)$. The bounds in (10.3), together with (10.1), give total dimensions at most seven, nine, and five, respectively.

Let $u = 3$. The tail has sum at most five. For the one-term tail (2), the choices of F_2 are Grassmannian bundles over square-rank strata in $\mathrm{Gr}(2,7)$. Each base has dimension at most ten, so Lemma 6.11 applies. The tail $(2,1)$ gives $Y_{8,3}$, with constituent bound seven by Lemma 9.13. For $(2,2)$, the equation $UF_2 \subseteq F_2$ allows us to apply Lemma 9.5. The tails $(2,1,1)$ and $(2,1,1,1)$ are covered by Lemma 9.8, with bound five.

The remaining tail beginning with two is $(2,2,1)$. Its proper terms are U, V, V, T, of dimensions $3,6,6,7$, and $UV \subseteq V$. Put $H = \mathbb{Q}[U]$. Then $H \subseteq V$, and there are finitely many possible H. For fixed $h = \dim H$, with $3 \leq h \leq 6$, omit the remaining product conditions. Choosing a unital three-plane in H, a six-plane containing H, and then a hyperplane containing that six-plane gives the dimension bound $2(h-3) + 2(6-h) + 1 = 7$.

If the tail begins with at least three, then $\dim F_2 \leq 5$ and $q \leq 5$. The case $q = 3$ is a finite-algebra case. For $q = 4$, the Kneser estimate bounds the U-base by five. The tail budget permits at most one further distinct space after F_2, and (10.1) bounds the extension choices by five. In greater detail, a tail beginning with at least three and having sum at most five is one of (3), $(3,1)$, $(3,1,1)$, $(3,2)$, (4), $(4,1)$, and (5). When $q = 4$, the last case is empty, since it would require $R \subseteq F_2$ with $\dim F_2 = 3$. In the rank-four quotient, the largest remaining flag has subspace dimensions $(1,3)$ or $(1,2)$, both of dimension five. This gives the stated bound.

For $q = 5$, the conic estimate bounds the base by seven. Here $F_2 = R$, and the remaining choices have dimension at most two. Thus these loci have dimensions at most ten and nine.

If $u = 2$, the tail has sum at most four. The square rank is two or three. The rank-two locus consists of finitely many partition algebras, and its complement in $\mathbb{P}^6$ has Tate cohomology by localization. Consequently, tails of length one give Grassmannian bundles with Tate cohomology. After the hyperplane cases, the longer tails are $(2,1),(3,1),(2,2)$, and $(2,1,1)$. Lemma 9.14 treats the first two, with bound nine. Lemma 9.5 treats $(2,2)$, and Lemma 9.8 treats $(2,1,1)$.

Finally, if $u = 1$, the tail has sum at most three. Outside the hyperplane cases it has length at most two. As in rank seven, all remaining product conditions are automatic, so the varieties are partial flag varieties. This completes rank eight.

It remains to consider rank nine. For $u \geq 5$, the finite-algebra cases have already been treated. The remaining pairs are $(5,6),(5,7),(6,7)$. Adding the complete-flag dimensions $\binom{9-q}{2}$ to the bounds in (10.3) gives total dimensions seven, nine, and six.

For $u = 4$, the tail has sum at most four. Outside the hyperplane cases, the possible profiles are

$$(5,2),\ (5,3),\ (5,4),\ (5,2,1),\ (5,3,1),\ (5,2,2),\ (5,2,1,1).$$

For a second space of dimension v, its choices over a square-rank-q base have dimension $(v-q)(9-v)$. Using the bounds above, the three two-step profiles have dimensions at most ten, eight, and five, respectively. Repeating the second space, as in $(5,2,2)$, adds no parameter; the extra product conditions define a closed locus in the same dimension-ten incidence. For $(5,3,1)$, a hyperplane containing the six-plane F_2 adds two parameters to the dimension-eight incidence. Thus this profile also has dimension at most ten.

The profile $(5,2,1)$ is $Y_{9,4}$ and has constituent bound seven by Lemma 9.13. The profile $(5,2,1,1)$ has proper dimensions $(4,7,8,8)$ and is covered by Lemma 9.8, with bound six.

For $u = 3$, the remaining profiles are $(6, 2), (6, 3)$, and $(6, 2, 1)$. To treat the first two, put $G = \mathrm{Gr}(2, 8)$ and let $D \subseteq G$ be the closed locus with $\dim U^2 \leq 5$. By (10.2), $\dim D \leq 8$. The Grassmannian G has Tate cohomology, so localization excludes every Tate twist of S_{12} from the cohomology of $G \setminus D$, the square-rank-six locus. The lower-square-rank strata lie in D and have dimension at most eight. We stratify D by the exact value of the square rank before forming these bundles. On each stratum, the image of the multiplication map is a vector subbundle, so the quotient by this image is a vector bundle. Its Grassmannian parametrizes precisely the second spaces containing U^2. These are two-step profiles, so there are no further multiplication conditions to cut down these Grassmannian bundles.

Over each of these strata, the second-space choices are Grassmannian bundles. The bundle formula therefore gives the required exclusion for both two-step profiles. The third profile is $Y_{9,3}$, with constituent bound nine.

For $u = 2$, the only remaining profile is $(7, 2)$. Its U-base is $\mathbb{P}^7$, with square rank two at finitely many partition algebras and square rank three on their complement. Both strata have Tate cohomology, and the second-space choices are Grassmannian bundles. For $u = 1$, the tail has sum at most one, so the profile was already treated among the Grassmannian and hyperplane cases.

This exhausts the profiles in ranks seven, eight, and nine. The dimension criterion and the cited bundle and incidence lemmas exclude every Tate twist of S_{12} from their cohomology. Applying (8.8) proves the proposition. □

11. Ordinary total counts

An arbitrary subring decomposes uniquely into irreducible subrings supported on disjoint sets of coordinates. We express this decomposition as an identity of compactly supported Euler classes. The preceding lower-rank vanishing result will then remove the contributions from proper decompositions in rank ten and colength seventeen.

One further input is needed. The known formulas for indices up to p^8 show that the corresponding Euler classes are virtual Tate. Since two factors of colength at least nine would already have total colength at least eighteen, every decomposition at colength seventeen has at most one factor outside this range. This is what allows us to use the Tate-twist identity for ε_{12}.

We work in K_ℓ and write $\mathsf{F}_{n,e} = \chi_c^\ell(Q_{n,e})$ and $\mathsf{G}_{n,e} = \chi_c^\ell(I_{n,e})$, where $Q_{n,e} = (\mathscr{Q}_{n,e,\mathbb{Q}})_{\mathrm{red}}$. Thus $Q_{n,e}$ parametrizes all unital equal-characteristic subalgebras, while $I_{n,e}$ imposes scalar reduction. Put $a_n(e) = \varepsilon_{12}(\mathsf{G}_{n,e})$. Recall from (8.1) that

$$\varepsilon_{12}\big(P(\mathbb{L})W\big) = P(1)\varepsilon_{12}(W) \tag{11.1}$$

for every Laurent polynomial P and $W \in K_\ell$.

Let Π_n denote the set of partitions of the labelled set $\{1, \dots, n\}$.

Proposition 11.1. *For $n \geq 1$ and $e \geq 0$,*

$$f_n(p^e) = \sum_{\pi \in \Pi_n} \sum_{\substack{e_B \geq 0 \ (B \in \pi) \\ \sum_B e_B = e}} \prod_{B \in \pi} g_{|B|}(p^{e_B}). \tag{11.2}$$

For every sufficiently large prime p,

$$f_n(p^e) = \#\mathscr{Q}_{n,e}(\mathbb{F}_p). \tag{11.3}$$

The corresponding identity in K_ℓ is

$$\mathsf{F}_{n,e} = \sum_{\pi \in \Pi_n} \sum_{\substack{e_B \geq 0 \ (B \in \pi) \\ \sum_B e_B = e}} \bigotimes_{B \in \pi} \mathsf{G}_{|B|,e_B}. \tag{11.4}$$

Proof. The decomposition over $\mathbb{Z}_p$ is Liu's irreducible decomposition [27, Theorem 3.4]; see also [3, Theorem 2.6]. We explain why the same decomposition works over $k[[t]]$.

Let $\mathcal{O}$ be either valuation ring, with residue field k and uniformizer ϖ, and let $R \subseteq \mathcal{O}^n$ be a unital subalgebra of finite colength. Reducing coordinates modulo ϖ carries R to a k-subalgebra $\overline{R} \subseteq k^n$, which contains $\mathbf{1}$ because R does.

The subalgebras of k^n which arise in this way have a very restricted shape. As a subring of a product of copies of k, the algebra $\overline{R}$ is reduced, and since it is finite-dimensional over k it is a product of finite field extensions of k. Projection to a single coordinate is a k-algebra homomorphism to k, hence injective on each of these factors, so that each factor is k itself. It follows that $\overline{R}$ consists precisely of the vectors which are constant on the blocks of a partition π of the set $\{1, \dots, n\}$; such an algebra is called a coordinate-partition algebra.

The partition is determined by $\overline{R}$, since two indices i and j lie in the same block exactly when $x_i = x_j$ for every $x \in \overline{R}$. This is the precise sense in which the blocks record which coordinates agree on all of $\overline{R}$, and it is the uniqueness asserted in [32, Lemma 2.7]. The irreducible case, in which $\overline{R} = k\mathbf{1}$, is the partition with a single block.

Choose b with $\varpi^b \mathcal{O}^n \subseteq R$. The reduction map $R/\varpi^b\mathcal{O}^n \to \overline{R}$ has nilpotent kernel. Its kernel consists of the classes of elements of $R \cap \varpi\mathcal{O}^n$. A product of b such elements lies in $\varpi^b\mathcal{O}^n$, so the bth power of this kernel is zero. This is the nilpotence needed for the idempotent-lifting statement.

Each block indicator therefore lifts uniquely to an idempotent, by [38, Tag 00J9]. In the ambient ring $(\mathcal{O}/\varpi^b)^n$, this lift is the constant block indicator. Since R contains $\varpi^b\mathcal{O}^n$, the indicator $\mathbf{1}_B$ itself belongs to R.

We find that R decomposes as a product $R = \prod_{B\in\pi} R_B$ of subalgebras $R_B \subseteq \mathcal{O}^B$, one for each block B, and correspondingly the quotient splits as a direct sum, $\mathcal{O}^n/R \simeq \bigoplus_{B\in\pi} \mathcal{O}^B/R_B$. Each factor has scalar reduction, and the colengths add. Conversely, the partition and its irreducible factors recover R. The factor R_B is $\mathbf{1}_B R$, viewed on the coordinates in B; its identity is $\mathbf{1}_B$. The labels determine its embedding in $\mathcal{O}^n$. Thus no ordering of the blocks is chosen, and no extra factorial occurs in the partition formula.

Counting proves (11.2) over $\mathbb{Z}_p$, using the integral–p-adic correspondence, and gives the same formula over $\mathbb{F}_p[[t]]$.

For fixed n, e, there are finitely many partitions and colength assignments. Lemma 8.1 identifies the mixed- and equal-characteristic irreducible counts in every term outside a finite set of primes. Excluding the union of these finite sets identifies the total counts as well. The equal-characteristic total is $\#\mathscr{Q}_{n,e}(\mathbb{F}_p)$, proving (11.3).

Finally, spread the finitely many varieties in (11.4) over a common $\mathbb{Z}[1/N]$. After enlarging N, their Euler-class Frobenius traces agree with their point counts at every prime outside $N\ell$. The trace of a tensor product is the product of the traces. The equal-characteristic partition formula therefore equates the traces of the two sides of (11.4). Lemma 6.7 gives the identity in K_ℓ.

This last argument requires the equality of counts over the finite fields for all sufficiently large primes. It does not require an isomorphism between the parameter schemes for families over arbitrary base schemes. The resulting identity is an identity of virtual Galois representations, which is the form used below. □

Lemma 11.2. *For every $n \geq 1$ and $0 \leq e \leq 8$, the classes $\mathsf{F}_{n,e}$ and $\mathsf{G}_{n,e}$ are virtual Tate. In particular, $a_n(e) = 0$ in this range.*

Proof. For fixed n, the formulas in [3, Proposition 1.2 and Theorem 1.3] give a polynomial $P_{n,e}$ satisfying $f_n(p^e) = P_{n,e}(p)$ for $0 \leq e \leq 8$. By (11.3) and the trace criterion, $\mathsf{F}_{n,e} = P_{n,e}(\mathbb{L})$.

We recover $\mathsf{G}_{n,e}$ by induction on n. In rank one, the only nonempty space is a point at colength zero. For larger n, isolate the one-block term $\mathsf{G}_{n,e}$ in (11.4). Every other term is a tensor product

of irreducible classes of smaller ranks and colengths at most eight. These are virtual Tate by induction. Subtracting them from $\mathsf{F}_{n,e}$ proves the assertion for $\mathsf{G}_{n,e}$. The final claim follows because ε_{12} vanishes on virtual Tate classes.

Only the Euler classes are asserted to be virtual Tate here. Contributions of non-Tate constituents in different cohomological degrees could cancel in an Euler class; the polynomial point-count formulas alone do not exclude this. The induction and the final argument use the virtual classes throughout.

□

Proof of Theorem 1.2. Part (i) is Theorem 9.18, which gives $a_{10}(17) = 56$.

Apply (11.4) with $(n, e) = (10, 17)$. In a proper-partition term, every block has rank at most nine. If all its colengths are at most eight, the term is virtual Tate by Lemma 11.2 and has zero ε_{12}-value.

Otherwise, exactly one factor has colength at least nine. The product of all the other factors is a virtual Tate class $P(\mathbb{L})$. The remaining factor has rank at most nine and colength at most seventeen, so its ε_{12}-value is zero by Proposition 10.1. Equation (11.1) therefore makes the whole term vanish under ε_{12}.

The restriction to colength seventeen is used at this step: it leaves at most one factor whose Euler class is not known to be virtual Tate. Therefore the other factors may be collected into the single polynomial $P(\mathbb{L})$ before applying (11.1). The argument does not require ε_{12} to preserve arbitrary products.

Only the one-block term remains, giving

$$\varepsilon_{12}(\mathsf{F}_{10,17}) = \varepsilon_{12}(\mathsf{G}_{10,17}) = a_{10}(17) = 56. \tag{11.5}$$

Suppose that a polynomial $P \in \mathbb{Q}[T]$ satisfies $f_{10}(p^{17}) = P(p)$ for every sufficiently large prime p. By (11.3) and Lemma 6.7, we would have $\mathsf{F}_{10,17} = P(\mathbb{L})$. Its ε_{12}-value would then be zero, contradicting (11.5). This proves part (ii). □

References

1. William A. Adkins and Steven H. Weintraub, *Algebra: An approach via module theory*, Graduate Texts in Mathematics, vol. 136, Springer-Verlag, New York, 1992.
2. Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier (eds.), *Théorie des topos et cohomologie étale des schémas. tome 3*, Lecture Notes in Mathematics, vol. 305, Springer-Verlag, Berlin, 1973.
3. Stanislav Atanasov, Nathan Kaplan, Benjamin Krakoff, and Julia H. Menzel, *Counting finite index subrings of* $\mathbb{Z}^n$, Acta Arith. **197** (2021), no. 3, 221–246.
4. Anders Björner and Torsten Ekedahl, *Subspace arrangements over finite fields: cohomological and enumerative aspects*, Advances in Mathematics **129** (1997), no. 2, 159–187.
5. Pierre Deligne, *Formes modulaires et représentations ℓ-adiques*, Séminaire Bourbaki 1968/69, Lecture Notes in Math., vol. 179, Springer, Berlin, 1971, pp. 139–172.
6. ———, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. **40** (1971), 5–57.
7. ———, *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–77.
8. ———, *Cohomologie étale: Séminaire de Géométrie Algébrique du Bois-Marie (SGA* $4\frac{1}{2}$*)*, Lecture Notes in Mathematics, vol. 569, Springer-Verlag, Berlin, 1977.
9. Jan Denef, *Report on Igusa's local zeta function*, Séminaire Bourbaki, Vol. 1990/91, Astérisque, no. 201–203, Société mathématique de France, 1991, Exposé No. 741, pp. 359–386.
10. Marcus du Sautoy, *Counting subgroups in nilpotent groups and points on elliptic curves*, J. Reine Angew. Math. **549** (2002), 1–21.
11. Marcus du Sautoy and Fritz Grunewald, *Analytic properties of zeta functions and subgroup growth*, Ann. of Math. (2) **152** (2000), no. 3, 793–833.
12. David Eisenbud and Joe Harris, *On varieties of minimal degree (a centennial account)*, Algebraic Geometry, Bowdoin, 1985, Part 1, Proceedings of Symposia in Pure Mathematics, vol. 46, American Mathematical Society, Providence, RI, 1987, pp. 3–13.
13. Tom Fisher, *The invariants of a genus one curve*, Proc. Lond. Math. Soc. (3) **97** (2008), no. 3, 753–782.

14. William Fulton, *Intersection theory*, 2 ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 2, Springer-Verlag, Berlin, 1998.
15. Ezra Getzler, *Resolving mixed Hodge modules on configuration spaces*, Duke Math. J. **96** (1999), no. 1, 175–203.
16. Rostislav Grigorchuk and Tatiana Nagnibeda, *Complete growth functions of hyperbolic groups*, Invent. Math. **130** (1997), no. 1, 159–188.
17. F. J. Grunewald, D. Segal, and G. C. Smith, *Subgroups of finite index in nilpotent groups*, Invent. Math. **93** (1988), no. 1, 185–223.
18. Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York, 1977.
19. Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I*, Ann. of Math. (2) **79** (1964), no. 1, 109–203.
20. ———, *Resolution of singularities of an algebraic variety over a field of characteristic zero. II*, Ann. of Math. (2) **79** (1964), no. 2, 205–326.
21. Kelly Isham, *Lower bounds for the number of subrings in* $\mathbb{Z}^n$, Journal of Number Theory **234** (2022), 363–390.
22. ———, *Is the number of subrings of index* p^e *in* $\mathbb{Z}^n$ *polynomial in* p*?*, Comm. Algebra **51** (2023), no. 12, 5127–5156.
23. Nathan Kaplan, Jake Marcinek, and Ramin Takloo-Bighash, *Distribution of orders in number fields*, Research in the Mathematical Sciences **2** (2015), Article 6, 57 pp.
24. Nathan Kaplan and Ian Petrow, *Elliptic curves over a finite field and the trace formula*, Proc. Lond. Math. Soc. (3) **115** (2017), no. 6, 1317–1372.
25. Nicholas M. Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, vol. 108, Princeton University Press, Princeton, NJ, 1985.
26. Michael Larsen and Alexander Lubotzky, *Representation growth of linear groups*, J. Eur. Math. Soc. **10** (2008), no. 2, 351–390.
27. Ricky Ini Liu, *Counting subrings of* $\mathbb{Z}^n$ *of index* k, J. Combin. Theory Ser. A **114** (2007), no. 2, 278–299.
28. Alexander Lubotzky and Dan Segal, *Subgroup growth*, Progress in Mathematics, vol. 212, Birkhäuser, Basel, 2003.
29. J. S. Milne, *Algebraic groups: The theory of group schemes of finite type over a field*, Cambridge Studies in Advanced Mathematics, vol. 170, Cambridge University Press, 2017.
30. James S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, NJ, 1980.
31. ———, *Lectures on Étale cohomology*, 2013, Version 2.21.
32. Diego Mirandola and Gilles Zémor, *Critical pairs for the Product Singleton Bound*, IEEE Trans. Inform. Theory **61** (2015), no. 9, 4928–4937.
33. Hrishabh Mishra and Anwesh Ray, *On the number of subrings of* $\mathbb{Z}^n$ *of prime power index*, arXiv:2211.16595, 2026, Version 2.
34. Chris A. M. Peters and Joseph H. M. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 52, Springer-Verlag, Berlin, 2008.
35. Peter Scholze, *p-adic Hodge theory for rigid-analytic varieties*, Forum Math. Pi **1** (2013), e1, 77.
36. Jean-Pierre Serre, *Abelian* ℓ*-adic representations and elliptic curves*, Research Notes in Mathematics, vol. 7, A K Peters, Wellesley, MA, 1998, Revised reprint of the 1968 original.
37. Joseph H. Silverman, *The arithmetic of elliptic curves*, 2 ed., Graduate Texts in Mathematics, vol. 106, Springer, New York, 2009.
38. The Stacks Project Authors, *The stacks project*, https://stacks.math.columbia.edu, Lemma 10.32.6, Tag 00J9.
39. Christopher Voll, *Functional equations for zeta functions of groups and rings*, Ann. of Math. (2) **172** (2010), no. 2, 1185–1222.
40. ———, *A newcomer's guide to zeta functions of groups and rings*, Lectures on Profinite Topics in Group Theory (Dan Segal, ed.), London Mathematical Society Student Texts, vol. 77, Cambridge University Press, Cambridge, 2011, pp. 99–144.
41. ———, *Zeta functions of groups and rings – recent developments*, Groups St Andrews 2013, Cambridge Univ. Press, 2015, pp. 469–492.

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