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Complex and real counterexamples to an inequality arising from Drury's conjecture

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Abstract

We give counterexamples to the following inequality, stated by Zhang as a consequence of Drury's 2016 permanent conjecture: if $A = (a_{jk})_{j,k=1}^{n}$ is a Hermitian positive semidefinite matrix of order $n \geq 2$, then

$$\sum_{j=1}^{n-1} |a_{jn}|^2 \operatorname{per} A[\,\hat{j},\hat{n}\,] \leq \operatorname{per} A$$

where $A[\,\hat{j},\hat{n}\,]$ is the principal submatrix obtained by deleting indices j and n. Our complex example has order twelve and rank two; our real example has order twenty-one and rank three.

1 Introduction and main results

For a square matrix $A = (a_{jk})_{j,k=1}^{n}$, its permanent is

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{j=1}^{n} a_{j,\sigma(j)},$$

where S_n is the set of permutations of $\{1, \ldots, n\}$; the permanent of the empty matrix is defined to be one. For distinct indices j, r, write $A[\,\hat{j},\hat{r}\,]$ for the principal submatrix obtained by deleting indices j, r, and write $A[\,\hat{r}\,]$ for deletion of index r alone. Set

$$q_r(A) = \sum_{j \neq r} |a_{jr}|^2 \operatorname{per} A[\,\hat{j},\hat{r}\,], \qquad q(A) = q_n(A). \tag{1.1}$$

In his 2016 survey, Zhang [1] recorded a conjecture attributed to Drury (private communication, 2016): for every Hermitian positive semidefinite matrix A of order $n \geq 2$,

$$\left(a_{nn} \operatorname{per} A[\,\hat{n}\,]\right)^2 + q(A)^2 \leq (\operatorname{per} A)^2. \tag{1.2}$$

Zhang stated that this would imply the following inequality, equation (21) of his survey:

$$q(A) \leq \operatorname{per} A. \tag{1.3}$$

This implication uses the nonnegativity of $q(A)$ and $\operatorname{per} A$ for positive semidefinite A, which follows from Lemma 2.1 below. We put the distinguished index last; simultaneous permutation gives the source's first-index formulation, or the equivalent assertion $q_r(A) \leq \operatorname{per} A$ for every r.

Hutchinson [2] disproved (1.2) with a complex matrix of order four. Chalmers [3] subsequently gave a real counterexample of order eight. Both displayed matrices nevertheless satisfy (1.3) at

every distinguished index: their largest ratios $q_r(A)/\operatorname{per} A$ are, respectively, approximately 0.37547 and 0.47963.[1] These examples therefore did not settle (1.3).

Our main result disproves (1.3) over both fields.

Theorem 1.1. *There is a complex positive semidefinite matrix of order twelve and rank two for which $q(A) > \operatorname{per} A$. There is also a real positive semidefinite matrix of order twenty-one and rank three for which $q(A) > \operatorname{per} A$.*

The constructions are explicit. Their gaps $\operatorname{per} A - q(A)$ are $900(5183 - 4608\sqrt{2})$ in the complex case and $-1\,152\,963\,189\,145\,600$ in the real case. These orders are upper bounds on the least orders of counterexamples; we do not assert that they are minimal.

The strict violations also give the following consequence.

Corollary 1.2. *For every $n \geq 12$, there is a complex positive definite matrix of order n, with every diagonal entry one, for which $q(A) > \operatorname{per} A$. For every $n \geq 21$, such a matrix can be chosen real.*

For vectors $v_1, \ldots, v_n \in \mathbb{C}^d$, we use the Gram convention $a_{jk} = \sum_r v_{j,r}\overline{v_{k,r}}$. Thus $A = VV^*$ is positive semidefinite, where V has rows v_j. The proof expresses the permanent of A as a sum of squared polynomial coefficients with factorial weights. In the complex construction, two regular pentagons leave only three monomials in the full product. In the real construction, two hexagons and eight equatorial directions give sparse products, and rotation cancels the cross terms in the sums of deleted products.

Section 2 establishes the polynomial identity. Sections 3 and 4 give the two constructions and their exact evaluations. Section 5 completes the proofs of Theorem 1.1 and Corollary 1.2. Section 6 proves that inequality (1.3) at every distinguished index implies the standard-representation instance of the author's quadratic conjecture [4], and shows that the two counterexamples do not disprove the conjecture in the standard block.

2 A polynomial formula for Gram permanents

For multiindices $\alpha \in \mathbb{Z}^d_{\geq 0}$ and a polynomial $F(z) = \sum_\alpha f_\alpha z^\alpha$, define

$$\|F\|^2 = \sum_\alpha \alpha!\,|f_\alpha|^2, \qquad z^\alpha = \prod_{r=1}^{d} z_r^{\alpha_r}, \quad \alpha! = \prod_{r=1}^{d} \alpha_r!.$$

The associated inner product is $\langle F, G\rangle = \sum_\alpha \alpha! f_\alpha \overline{g_\alpha}$; in particular, distinct monomials are orthogonal. Put $\ell_v(z) = \sum_r v_r z_r$. This is the coefficient form of a standard Gaussian moment norm: if $Z = (Z_1, \ldots, Z_d)$ has independent coordinates with density $\pi^{-1}e^{-|z|^2}$ on $\mathbb{C}$, then the mixed moments $\mathbb{E}[Z^\alpha \overline{Z^\beta}]$ vanish unless $\alpha = \beta$, when they equal $\alpha!$. Thus $\|F\|^2 = \mathbb{E}|F(Z)|^2$. We use the coefficient form for the exact evaluations below.

Lemma 2.1. *For vectors $v_1, \ldots, v_m, w_1, \ldots, w_m \in \mathbb{C}^d$, let $B = (b_{ij})_{i,j=1}^m$ with $b_{ij} = \sum_{r=1}^d v_{i,r}\overline{w_{j,r}}$. Then*

$$\left\langle \prod_{i=1}^{m} \ell_{v_i}, \prod_{j=1}^{m} \ell_{w_j} \right\rangle = \operatorname{per} B.$$

[1] Exact evaluation of the defining permanent sums gives minimum gaps $\min_r(\operatorname{per} A - q_r(A))$ of exactly 3.124761868597335 and 626908608, respectively, for the matrices as printed.

In particular, for the Gram matrix A of $v_1, \dots, v_n$,

$$\operatorname{per} A = \left\| \prod_{j=1}^{n} \ell_{v_j} \right\|^2 .$$

This polynomial norm is invariant under substitutions $z \mapsto Uz$ with $U^*U = I$ (unitary substitutions). If $v_n = e_1 = (1, 0, \dots, 0)$ and $P = \prod_{j<n} \ell_{v_j}$, then

$$\operatorname{per} A = \|z_1 P\|^2, \qquad q(A) = \sum_{j<n} |v_{j,1}|^2 \|P/\ell_{v_j}\|^2. \tag{2.1}$$

Here division means omission of the specified factor.

Proof. The mixed identity is Wick's formula for the complex Gaussian linear forms, written in coefficient form. We verify it directly. Let $F = \prod_{j=1}^{m} \ell_{v_j}$ and $G = \prod_{j=1}^{m} \ell_{w_j}$. An assignment $\phi : \{1, \dots, m\} \to \{1, \dots, d\}$ chooses one coordinate from each factor; write $\operatorname{mult}(\phi) = (|\phi^{-1}(1)|, \dots, |\phi^{-1}(d)|)$ for its coordinate multiplicities. Expanding the coefficients gives

$$\langle F, G \rangle = \sum_{\alpha} \alpha! \sum_{\substack{\phi,\psi \\ \operatorname{mult}(\phi)=\operatorname{mult}(\psi)=\alpha}} \prod_{j=1}^{m} v_{j,\phi(j)} \overline{w_{j,\psi(j)}}.$$

For two assignments with common multiplicities α, the relation $\psi \circ \sigma = \phi$ requires $\sigma \in S_m$ to map each fibre $\phi^{-1}(r)$ bijectively onto $\psi^{-1}(r)$. These bijections can be chosen independently in $\alpha_r!$ ways for each r, giving exactly $\alpha! = \prod_r \alpha_r!$ such permutations. Replacing $\alpha!$ by this count and reindexing the conjugated factors therefore gives

$$\begin{aligned} \langle F, G \rangle &= \sum_{\sigma \in S_m} \sum_{\phi} \prod_{j=1}^{m} v_{j,\phi(j)} \overline{w_{\sigma(j),\phi(j)}} \\ &= \sum_{\sigma \in S_m} \prod_{j=1}^{m} b_{j,\sigma(j)} = \operatorname{per} B. \end{aligned}$$

Taking $w_j = v_j$ gives the Gram norm identity. For unitary U, the random vectors UZ and Z have the same law, so the Gaussian representation above gives $\|F(Uz)\| = \|F(z)\|$. Applying the Gram norm identity to the full product and to each double deletion proves (2.1); the weights are $|a_{jn}|^2 = |v_{j,1}|^2$. □

3 The complex counterexample

In this section write $(x, y) = (z_1, z_2)$ in the polynomial norm.

Proposition 3.1. *Let $\zeta = e^{2\pi \mathrm{i}/5}$. Take the ten vectors $(1, -\rho)$ with*

$$\rho \in \{\zeta^j/\sqrt{2}, -2\zeta^j : 0 \le j < 5\},$$

one further vector $(0, 1)$, and distinguished vector $(1, 0)$. Their Gram matrix A is positive semidefinite of order twelve and rank two, and

$$\operatorname{per} A - q(A) = 900(5183 - 4608\sqrt{2}) < 0.$$

Proof. The assertion can also be checked by exact symbolic computation from the listed vectors. We give a direct derivation that makes the sparse product and the sign of the gap transparent. The two coordinate vectors establish the rank. The two regular pentagons make the product especially sparse. Let P be the product of the linear forms associated with the eleven non-distinguished vectors. The identity

$$\prod_{j=0}^{4}(x - c\zeta^j y) = x^5 - c^5 y^5,$$

applied with $c = 1/\sqrt{2}$ and $c = -2$, gives

$$P = y(x^5 - \frac{\sqrt{2}}{8}y^5)(x^5 + 32y^5).$$

The three monomials in xP are orthogonal, so

$$\begin{aligned}\operatorname{per} A &= 11! + (32 - \sqrt{2}/8)^2(6!)^2 + 32 \cdot 11! \\ &= 33 \cdot 11! + (1024 + 1/32 - 8\sqrt{2})(6!)^2 \\ &= 1\,848\,112\,200 - 4\,147\,200\sqrt{2}.\end{aligned}$$

The extra vector $(0, 1)$ contributes zero to $q(A)$. Deleting a root ρ from its fivefold factor leaves

$$\frac{x^5 - \rho^5 y^5}{x - \rho y} = \sum_{i=0}^{4} \rho^{4-i} x^i y^{4-i}.$$

After multiplication by the other fivefold factor and by y, the two resulting ranges of x-degrees, $0, \ldots, 4$ and $5, \ldots, 9$, are disjoint. The ten resulting monomials are distinct and hence orthogonal, so each deletion norm depends on ρ only through $|\rho|^2$. Set $s = 1/2$ and $t = 4$, the squared root radii on the first and second pentagons. The squared constants in their fivefold factors are s^5 and t^5, respectively. Formula (2.1) now gives

$$\begin{aligned}q(A) = 5\sum_{i=0}^{4} \Big[&(s^{4-i} + t^{4-i})(5+i)!(5-i)! \\ &+ (s^{4-i}t^5 + t^{4-i}s^5)i!(10-i)!\Big] \\ = 5(6\,375\,420 + 362\,314\,080) &= 1\,843\,447\,500.\end{aligned}$$

The two numbers in parentheses are the sums of the first and second terms inside the brackets, respectively. Hence

$$\begin{aligned}\operatorname{per} A - q(A) &= (1\,848\,112\,200 - 1\,843\,447\,500) - 4\,147\,200\sqrt{2} \\ &= 4\,664\,700 - 4\,147\,200\sqrt{2} \\ &= 900(5183 - 4608\sqrt{2}).\end{aligned}$$

Since $(9/8)^2 = 81/64 < 2$, we have $\sqrt{2} > 9/8$, and therefore $4608\sqrt{2} > 4608 \cdot 9/8 = 5184 > 5183$. The gap is negative. □

4 The real counterexample

In this section write $(x, y, z) = (z_1, z_2, z_3)$ in the polynomial norm.

Proposition 4.1. *For $0 \le j < 6$, put*

$$a_j = (1, \sqrt{2/3}\cos(j\pi/3), \sqrt{2/3}\sin(j\pi/3)),$$
$$b_j = (1, 2\cos(j\pi/3 + \pi/6), 2\sin(j\pi/3 + \pi/6)).$$

For $0 \le j < 8$, put

$$c_j = (0, \sqrt{2}\cos(j\pi/8), \sqrt{2}\sin(j\pi/8)).$$

The Gram matrix A of these twenty real vectors and the distinguished vector $w = (1, 0, 0)$ is positive semidefinite of rank three and satisfies

$$\operatorname{per} A - q(A) = -1\,152\,963\,189\,145\,600 < 0.$$

Proof. The assertion can also be checked by exact symbolic computation from the displayed vectors. We give a direct derivation: rotations make the full product sparse and cancel cross terms in the sums of deleted products, leaving finite coefficient sums. Positive semidefiniteness follows from the Gram construction, and w, c_0, c_4 establish rank three. Use the unitary coordinates

$$u = (y + \mathrm{i}z)/\sqrt{2}, \qquad v = (y - \mathrm{i}z)/\sqrt{2}.$$

Set $s = 1/3$, $t = 2$, and for $r > 0$ define

$$C_r = x^6 - 6rx^4uv + 9r^2x^2u^2v^2 - r^3(u^3 + v^3)^2, \tag{4.1}$$
$$D_r = x^6 - 6rx^4uv + 9r^2x^2u^2v^2 + r^3(u^3 - v^3)^2. \tag{4.2}$$

For $L_\theta = e^{-\mathrm{i}\theta}u + e^{\mathrm{i}\theta}v$, let $\theta_j = j\pi/3$, $\phi_j = \theta_j + \pi/6$ and $\psi_j = j\pi/8$. Since $L_\theta = \sqrt{2}(\cos\theta\, y + \sin\theta\, z)$, the linear forms of the listed vectors are

$$\ell_{a_j} = x + \sqrt{s}L_{\theta_j}, \qquad \ell_{b_j} = x + \sqrt{t}L_{\phi_j}, \qquad \ell_{c_j} = L_{\psi_j}.$$

Opposite factors pair to $x^2 - rL_\theta^2$. For the first hexagon, the phases $e^{-2\mathrm{i}\theta_k}$, $0 \le k < 3$, run through the cube roots of unity. With $\omega = e^{2\pi\mathrm{i}/3}$, the identity

$$\prod_{k=0}^{2}(h - \omega^{-k}p - \omega^k q) = h^3 - p^3 - q^3 - 3hpq,$$

applied with $h = x^2 - 2suv$, $p = su^2$, and $q = sv^2$, gives $\prod_j \ell_{a_j} = C_s$. The unitary substitution

$$\tau : (u, v) \mapsto (e^{-\mathrm{i}\pi/6}u, e^{\mathrm{i}\pi/6}v)$$

sends $(u^3 + v^3)^2$ to $-(u^3 - v^3)^2$, and hence sends C_r to D_r. Thus $\prod_j \ell_{b_j} = D_t$. Also

$$\prod_{j=0}^{7}\ell_{c_j} = \prod_{j=0}^{7}(e^{-\mathrm{i}j\pi/8}u + e^{\mathrm{i}j\pi/8}v) = \mathrm{i}(u^8 - v^8).$$

Indeed, $e^{2\mathrm{i}j\pi/8}$ runs through the eighth roots of unity and the product of the scalar phases is $e^{-7\pi\mathrm{i}/2} = \mathrm{i}$. The scalar i can be dropped from every norm, so the product of the twenty non-distinguished factors is represented by $P = (u^8 - v^8)C_sD_t$.

For $F = \sum f_{ijk} x^i u^j v^k$, define

$$H(F) = \sum_{i,j,k} |f_{ijk}|^2 i!\,[(j+8)!k! + j!(k+8)!] = \|u^8 F\|^2 + \|v^8 F\|^2. \tag{4.3}$$

Every monomial of xC_sD_t has $j-k$ divisible by six. A cross term between u^8F and v^8F would require two such differences to differ by sixteen, which is impossible. Thus

$$\operatorname{per} A = H(xC_sD_t). \tag{4.4}$$

To treat the deletions, put $L = u + v$, $M = uv$ and

$$R_r = (x - \sqrt{r}L)[x^4 + r(L^2 - 6M)x^2 + r^2(L^2 - 3M)^2]. \tag{4.5}$$

Multiplication gives $C_r = (x + \sqrt{r}L)R_r$. Here $x + \sqrt{s}L = \ell_{a_0}$, so R_sD_t is exactly the hexagon product with a_0 deleted. By rotation through θ we mean the substitution

$$(u, v) \mapsto (e^{-\mathrm{i}\theta}u, e^{\mathrm{i}\theta}v).$$

Both C_r and D_r are invariant under rotations through $\pi/3$. The six products obtained by deleting one factor from C_sD_t on the first hexagon are obtained from R_sD_t by the substitutions $(u, v) \mapsto (e^{-\mathrm{i}j\pi/3}u, e^{\mathrm{i}j\pi/3}v)$. These substitutions preserve H. Rotation through $j\pi/3$ multiplies a monomial $x^a u^b v^c$ by $e^{-\mathrm{i}(b-c)j\pi/3}$. A cross term from multiplication by $u^8 - v^8$ pairs monomials whose values of $b - c$ differ by sixteen, so it acquires a factor $e^{\pm 16\mathrm{i}j\pi/3}$. Its sum over $j = 0, \dots, 5$ vanishes, since

$$\sum_{j=0}^{5} e^{16\mathrm{i}j\pi/3} = 0.$$

Hence the six squared deleted norms sum to $6H(R_sD_t)$. For the second hexagon, apply τ. It preserves H, since it changes each coefficient only by a unit phase. It sends the b_j form to $x + \sqrt{t}L_{\theta_{j+1}}$ and the a_j form to $x + \sqrt{s}L_{\phi_j}$. Thus, after τ, the hexagon product with b_j deleted is a rotation of R_tD_s through $(j+1)\pi/3$. The equatorial factor becomes

$$\tau(u^8 - v^8) = e^{-4\pi\mathrm{i}/3}(u^8 - e^{2\pi\mathrm{i}/3}v^8).$$

Its cross terms carry a fixed unit phase and one of the factors $e^{\pm 16\mathrm{i}(j+1)\pi/3}$, which again sum to zero over the six values of j. The second deletion sum is therefore $6H(R_tD_s)$. The twelve hexagon vectors have first coordinate one, while the equatorial vectors have first coordinate zero. Therefore

$$q(A) = 6H(R_sD_t) + 6H(R_tD_s). \tag{4.6}$$

The explicit polynomials (4.1)–(4.5) now give

$$\begin{aligned} H(xC_sD_t) &= 647\,582\,321\,598\,464\,000, \\ H(R_sD_t) &= 94\,164\,692\,010\,188\,800, \\ H(R_tD_s) &= 13\,957\,855\,454\,412\,800. \end{aligned}$$

Appendix A supplies the grouped coefficient sums for these finite evaluations. Equations (4.4) and (4.6) yield

$$q(A) = 648\,735\,284\,787\,609\,600,$$

and hence the asserted strict negative gap. □

5 Completion of the proofs

Proof of Theorem 1.1. Proposition 3.1 gives the complex matrix of order twelve and rank two. Proposition 4.1 gives the real matrix of order twenty-one and rank three. Both are positive semidefinite Gram matrices, and both have strictly negative gaps $\operatorname{per} A - q(A)$. □

Proof of Corollary 1.2. Start with the appropriate matrix in Theorem 1.1 and adjoin an identity block of the required size, possibly empty. Retain the original distinguished coordinate, and simultaneously permute that coordinate to the last position. This preserves the gap: terms involving the added indices have zero weight, and identity blocks have permanent one. Since the gap is a continuous polynomial in the real and imaginary parts of the matrix entries, adding a sufficiently small positive multiple of the identity preserves its strict negativity and makes the matrix positive definite.

For a positive diagonal matrix D, both $\operatorname{per}(DAD)$ and $q(DAD)$ equal the corresponding quantity for A multiplied by $\prod_j D_{jj}^2$. Indeed, in a term of q, the squared entry supplies the factors for the two deleted indices, and the permanent supplies all the remaining factors. Choosing $D_{jj} = a_{jj}^{-1/2}$ makes the diagonal one and preserves the sign. Each operation preserves real entries when the initial matrix is real. □

6 Relation to the author's quadratic conjecture

In this section we show that inequality (1.3), imposed at every distinguished index, implies the standard-representation instance of the author's quadratic conjecture [4]. We therefore test the two counterexamples of Theorem 1.1 against that instance; the bounds below show that they do not disprove the author's quadratic conjecture for the standard representation.

Let $A = (a_{ij})_{i,j=1}^n$ be Hermitian positive semidefinite of order $n \geq 1$, and, for $\sigma \in S_n$, write

$$a_\sigma(A) = \prod_{k=1}^{n} a_{k,\sigma(k)}.$$

For a partition $\lambda \vdash n$, let χ_λ be the corresponding irreducible character of S_n, with degree $d_\lambda = \chi_\lambda(e)$. The author's quadratic permanental dominance conjecture [4] is

$$\sum_{\sigma,\tau \in S_n} \chi_\lambda(\sigma\tau) a_\sigma(A) a_\tau(A) \leq d_\lambda (\operatorname{per} A)^2 \qquad (\lambda \vdash n). \tag{6.1}$$

The author proved the conjecture for $n \leq 5$ and also proved that it implies both the S_n immanant case of Lieb's permanental dominance conjecture and Chollet's permanent conjecture [4].

For $n = 1$, (6.1) is an equality. For $n \geq 2$, we restrict attention to the standard representation. Let $\rho_{\mathrm{nat}}(\sigma) e_j = e_{\sigma(j)}$ be the natural permutation representation on $\mathbb{C}^n$, and let ρ_{std} be its restriction to the subspace $\mathbf{1}^\perp$ of vectors whose coordinates sum to zero. This is the irreducible representation of shape $(n-1,1)$ and dimension $n-1$. Put $F_{\mathrm{std}}(A) = \sum_\sigma a_\sigma(A) \rho_{\mathrm{std}}(\sigma)$, and let $\operatorname{fix}(\pi)$ denote the number of fixed points of $\pi \in S_n$. Since $\chi_{(n-1,1)}(\pi) = \operatorname{fix}(\pi) - 1$, the standard-instance form of (6.1) is

$$\operatorname{tr} F_{\mathrm{std}}(A)^2 = \sum_{\sigma,\tau \in S_n} (\operatorname{fix}(\sigma\tau) - 1) a_\sigma(A) a_\tau(A) \leq (n-1)(\operatorname{per} A)^2. \tag{6.2}$$

We call the following equivalent cofactor inequality the *natural representation form*. Let $A(i|j)$ be the matrix obtained from A by deleting row i and column j, and set

$$\Gamma = (\gamma_{ij})_{i,j=1}^n, \qquad \gamma_{ij} = a_{ij} \operatorname{per} A(i|j).$$

Expansion of the permanent along each row and each column gives

$$\Gamma\mathbf{1} = \Gamma^{\mathsf{T}}\mathbf{1} = (\operatorname{per} A)\mathbf{1}.$$

Write $N(A) = \sum_{\sigma\in S_n} a_\sigma(A)\rho_{\mathrm{nat}}(\sigma)$. Its (i,j) entry is

$$\sum_{\sigma:\,\sigma(j)=i} a_\sigma(A) = a_{ji}\operatorname{per} A(j|i) = \gamma_{ji},$$

so $N(A) = \Gamma^{\mathsf{T}}$. The natural representation splits into the trivial representation and the standard one. Its trivial block is $\operatorname{per} A$, and Γ is Hermitian because A is Hermitian. Consequently

$$\sum_{i,j=1}^{n} |\gamma_{ij}|^2 = \operatorname{tr}\Gamma^2 = (\operatorname{per} A)^2 + \operatorname{tr} F_{\mathrm{std}}(A)^2.$$

Thus (6.2) is equivalent to the natural form

$$\boxed{\operatorname{tr}\Gamma^2 = \sum_{i,j=1}^{n} |a_{ij}|^2\,|\operatorname{per} A(i|j)|^2 \le n(\operatorname{per} A)^2.} \tag{6.3}$$

If A is the Gram matrix of $v_1,\dots,v_n$ and $f_i = \prod_{k\neq i}\ell_{v_k}$, the mixed version of Lemma 2.1 gives

$$\gamma_{ij} = \langle v_i, v_j\rangle\langle f_i, f_j\rangle = \langle v_i\otimes f_i, v_j\otimes f_j\rangle.$$

Thus Γ is the Gram matrix of these tensors with the product inner product, and is positive semidefinite. The matrix $F_{\mathrm{std}}(A)$ is the restriction of $N(A) = \Gamma^{\mathsf{T}} = \overline{\Gamma} \succeq 0$ to $\mathbf{1}^{\perp}$. Its eigenvalues are therefore nonnegative and sum to $\operatorname{tr}\Gamma - \operatorname{per} A$. Their squared sum is at most the square of their sum, giving

$$\operatorname{tr} F_{\mathrm{std}}(A)^2 \le \left(\operatorname{tr}\Gamma - \operatorname{per} A\right)^2. \tag{6.4}$$

Proposition 6.1. *Let A be Hermitian positive semidefinite of order $n \ge 2$. If $q_i(A) \le \operatorname{per} A$ for every distinguished index i, then* (6.3) *holds.*

Proof. Fix an index and permute it to the last position. Write

$$A = \begin{pmatrix} B & c \\ c^* & a \end{pmatrix}, \qquad \Pi_{jk} = \operatorname{per} B(k|j) \quad (1 \le j,k < n).$$

Here $B(k|j)$ deletes row k and column j from B. If B is the Gram matrix of $v_1,\dots,v_{n-1}$, set $f_j = \prod_{\ell\neq j}\ell_{v_\ell}$. The assignment count in the proof of Lemma 2.1, applied to the two products with factors k and j omitted, gives $\Pi_{jk} = \langle f_k, f_j\rangle$. Hence Π is positive semidefinite. Put $d = a\operatorname{per} B$ and $\eta = c^*\Pi c$. Expanding along the last row and then the last column, and using the definitions, gives

$$\operatorname{per} A = d + \eta, \qquad q_n(A) = \sum_{j<n} |c_j|^2\Pi_{jj}, \qquad \gamma_{nj} = \overline{c_j}(\Pi c)_j \ (j<n), \quad \gamma_{nn} = d.$$

By Cauchy–Schwarz for the positive semidefinite matrix Π, $|(\Pi c)_j|^2 \le \Pi_{jj}\eta$. Hence

$$\sum_{j=1}^{n} |\gamma_{nj}|^2 \le d^2 + \eta q_n(A) \le d^2 + \eta\operatorname{per} A = (\operatorname{per} A)^2 - d\eta \le (\operatorname{per} A)^2.$$

Here $d, \eta \ge 0$. The same bound holds at every index, and summing its rows proves (6.3). □

The two matrices of Theorem 1.1 violate (1.3) at the distinguished index, but neither disproves (6.2). To evaluate their traces, we use Lemma 2.1 on each principal deletion:

$$\operatorname{tr}\Gamma = \sum_{i=1}^{n} \|v_i\|^2 \left\| \prod_{k\neq i} \ell_{v_k} \right\|^2 .$$

Exact coefficient evaluation of the principal-deletion products gives the following trace values. For the complex matrix of order twelve,

$$\begin{aligned} \operatorname{per} A &= 1\,848\,112\,200 - 4\,147\,200\sqrt{2}, \quad q(A) = 1\,843\,447\,500, \\ \operatorname{tr}\Gamma &= 7\,447\,658\,400 - 1\,382\,400\sqrt{2}. \end{aligned}$$

In particular, $\frac{81}{20}\operatorname{per} A - \operatorname{tr}\Gamma = 37\,196\,010 - 15\,413\,760\sqrt{2} > 0$, since $\sqrt{2} < 3/2$. By (6.4),

$$\frac{\operatorname{tr} F_{\mathrm{std}}(A)^2}{11(\operatorname{per} A)^2} \leq \frac{(\operatorname{tr}\Gamma - \operatorname{per} A)^2}{11(\operatorname{per} A)^2} < \frac{3721}{4400} < 1.$$

For the real matrix of order twenty-one, the corresponding evaluation gives

$$\begin{aligned} \operatorname{per} A &= 647\,582\,321\,598\,464\,000, \qquad q(A) = 648\,735\,284\,787\,609\,600, \\ \operatorname{tr}\Gamma &= \frac{10\,242\,155\,794\,291\,097\,600}{3}. \end{aligned}$$

Again by (6.4),

$$\frac{\operatorname{tr} F_{\mathrm{std}}(A)^2}{20(\operatorname{per} A)^2} \leq \frac{(\operatorname{tr}\Gamma - \operatorname{per} A)^2}{20(\operatorname{per} A)^2} = \frac{523\,670\,800\,175\,527\,514\,454\,169}{573\,887\,169\,546\,878\,472\,124\,500} < 1.$$

Hence neither counterexample to (1.3) disproves the author's conjecture in the standard block.

Acknowledgments

The proofs in this paper were carried out by GPT-5.6-sol, GPT-6 Astra, Claude Fable 5, and Claude Fable 5.1, under the guidance of the author. The author has reviewed the resulting proof arguments. Responsibility for the final text rests with the author.

Appendix A Exact coefficient sums for the real construction

Here $s = 1/3$, $t = 2$ and $g = 2\,867\,200 = 2^{14} \cdot 5^2 \cdot 7$ is a convenient common factor. Each row collects monomials of x-degree i in the polynomial at the top of its column. For the first column, the factorial weight uses $i + 1$ because of the additional factor x in $H(xC_sD_t)$. The two deletion columns use the weight in (4.3) directly. All entries are divided by g. The factors in (4.5) contain $\sqrt{r}$ only in the even x-degree part; D_r has only even x-degrees. Hence their products have rational squared coefficients. Some entries divided by g retain a denominator of three.

i	C_sD_t (append x)	R_sD_t	R_tD_s
12	175134960	0	0
11	0	1122660	1122660
10	1980372240	340200	2041200
9	0	16433928	12941208
8	6835774680	6941088	24712128
7	0	72538935	69766515
6	27872814620	40765522	161003052
5	0	579632760	247708930/3
4	31090176000	562398336	762812608/3
3	0	1364065056	97793008/3
2	8900720640	1868892480	150162880
1	0	4640822784	128911744
0	149003796480	23688084480	3948014080
Sum	225858789620	32842038229	4868113649

For example, the leading monomial x^{12} of C_sD_t contributes $2 \cdot 13! \cdot 8!/g = 175134960$ to the first column. The table follows by expanding the displayed degree-six and degree-five polynomials and applying (4.3); it includes every x-degree. It gives the compact comparison

$$\frac{\operatorname{per} A}{4g} = 56\,464\,697\,405 \quad < \quad \frac{q(A)}{4g} = 56\,565\,227\,817.$$

References

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