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A Dittert-like inequality for homogeneous stable polynomials

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Abstract

We prove a coefficient inequality for every homogeneous real stable polynomial $P(s, x_1, \dots, x_n)$ of degree $n \geq 1$ with nonnegative coefficients. If $P(1, 0, \dots, 0) = 1$, $r_i = \partial_{x_i} P(1, 0, \dots, 0)$, and $\sum_i r_i = n$, the following inequality holds, where brackets denote coefficient extraction:

$$[x_1 \cdots x_n] P(0, x_1, \dots, x_n) + 2 - \prod_i r_i - P(0, 1, \dots, 1) \geq \frac{n!}{n^n}.$$

Equality holds exactly for $P = (s + \sum_i x_i/n)^n$. The normalizations can be removed by rescaling P and the x_i. We give applications to permanents, mixed discriminants, subpermanents, and stable combinatorial measures. In particular, the theorem gives an independent proof of Dittert's conjecture for nonnegative matrices in every dimension.

Keywords: real stable polynomials; permanents; Dittert conjecture; mixed discriminants.
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1 Introduction

For an n-by-n nonnegative matrix $A = (a_{ij})$ with $n \geq 1$ and $\sum_{i,j} a_{ij} = n$, write

$$M = \operatorname{per} A = \sum_{\pi \in S_n} \prod_i a_{i,\pi(i)}, \qquad R = \prod_i \left(\sum_j a_{ij} \right), \qquad C = \prod_j \left(\sum_i a_{ij} \right), \qquad \gamma_n = \frac{n!}{n^n}.$$

Thus R is the product of row sums and C the product of column sums. Dittert's conjecture asserts that

$$M + 2 - R - C \geq \gamma_n,$$

with equality exactly when $a_{ij} = 1/n$ for all i, j. A nonnegative matrix is **doubly stochastic** if every row and column sums to 1. For such a matrix A, we have $R = C = 1$, so this reduces to $\operatorname{per} A \geq \gamma_n$, the van der Waerden theorem proved by Egorychev [E81] and Falikman [F81].

Minc recorded the conjecture in 1983, noting that E. T. H. Wang had communicated it to him and had said it was "apparently due to Eric Dittert" [M83, p. 249]. Sinkhorn proved the case $n = 2$ and showed that every global minimizer of $M + 2 - R - C$ has $M > 0$ [S84]. Hwang showed that any such minimizer with all entries strictly positive is uniform [H86], and proved $n = 3$ [H87]. Cheon and Wanless [CW12] showed that any global minimizer has no $r \times (n - r)$ zero submatrix for $1 \leq r < n$, and that, up to row and column permutations, its zero entries cannot form a single rectangular block.

Pang proved $n \geq 17$ [P26], and Kafidov proved $n = 16$ [K26]. In July 2026, Nascimento's repository assembled a working-proof chain covering every dimension, described there as under

review [N26]. Dai et al. submitted a proof for every dimension to arXiv on 30 July 2026; it became public in October [D26]. Our stable-polynomial theorem gives an independent proof of the matrix conjecture.

For a matrix of arbitrary positive total sum S, the same homogeneous inequality has the scale-free form

$$M + (2 - \gamma_n)(S/n)^n \geq R + C,$$

with equality exactly when every entry is S/n^2.

A nonzero real polynomial is **real stable** if it has no zero when all its arguments have positive imaginary parts. A polynomial $q(x_1, \ldots, x_n)$ is **doubly stochastic** if it is homogeneous of degree n, has nonnegative coefficients, and satisfies $\partial_i q(\mathbf{1}) = 1$ for every i, where $\mathbf{1}$ is the vector of ones. Euler's identity then gives $q(\mathbf{1}) = 1$. Write $[x^\nu]q$ for the coefficient of x^ν in q.

Gurvits [G08, Fact 2.2, Corollary 2.5 and Theorem 5.7] generalized the van der Waerden theorem to a sharp coefficient bound for doubly stochastic real stable polynomials:

Gurvits's theorem. Let $n \geq 1$ and let q be a real stable doubly stochastic polynomial in n variables. Then

$$[x_1 \cdots x_n]q \geq \gamma_n.$$

Equality holds exactly for $q = ((x_1 + \cdots + x_n)/n)^n$.

For a doubly stochastic matrix $A = (a_{ij})$, form

$$q_A(x) = \prod_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} x_i \right).$$

This polynomial is homogeneous of degree n with nonnegative coefficients. It is real stable because every factor has positive imaginary part when all x_i do. The column sums give $q_A(\mathbf{1}) = 1$, and the row sums give $\partial_i q_A(\mathbf{1}) = \sum_j a_{ij} = 1$. Moreover, $[x_1 \cdots x_n]q_A = \operatorname{per} A$. Gurvits's theorem thus gives the van der Waerden bound. At equality, $q_A = (\sum_i x_i/n)^n$; uniqueness of the normalized linear factors gives $a_{ij} = 1/n$ for all i, j.

Our goal is to extend Dittert's conjecture to the same polynomial setting, retaining its two deficit terms. Theorem 1.1 does so for every $n \geq 1$, with the sharp constant and equality case.

The resulting bounds cover permanents, mixed discriminants, sums of subpermanents, and stable combinatorial measures.

Theorem 1.1. Let $n \geq 1$, and let $P(s, x_1, \ldots, x_n)$ be homogeneous of degree n, real stable, and have nonnegative coefficients. Suppose

$$P(1, 0) = 1, \qquad r_i = \partial_{x_i} P(1, 0), \qquad \sum_{i=1}^{n} r_i = n.$$

Define

$$q(x) = P(0, x), \quad M = [x_1 \cdots x_n]q, \quad R = \prod_i r_i, \quad C = q(\mathbf{1}), \quad \Delta = 2 - R - C.$$

Then $\Delta \geq 0$ and

$$M + \Delta \geq \gamma_n. \tag{1.1}$$

Equality holds if and only if

$$P(s, x) = \left(s + \frac{x_1 + \cdots + x_n}{n} \right)^n. \tag{1.2}$$

When $R = C = 1$, the deficit Δ vanishes and (1.1) becomes $M \geq \gamma_n$. For a polynomial q satisfying Gurvits's hypotheses, i.e., a homogeneous real stable polynomial of degree n in n variables with nonnegative coefficients and $\partial_i q(\mathbf{1}) = 1$ for every i, apply Theorem 1.1 to $P(s, x) = q(x + s\mathbf{1})$. This substitution preserves stability because every $x_i + s$ lies in the upper half-plane whenever all arguments do, and it preserves nonnegative coefficients. Euler's identity gives $q(\mathbf{1}) = 1$, so $P(1, 0) = 1$, $r_i = 1$ for every i, and $C = P(0, \mathbf{1}) = 1$; hence $R = C = 1$. Moreover, $M = [x_1 \cdots x_n]q$. Thus (1.1) gives Gurvits's bound, and the equality case (1.2) gives his equality characterization.

The distinguished variable s specifies the reference point at which the derivatives r_i are measured. We call it the anchor variable. It is part of the data: the restriction $q = P(0, \cdot)$ alone need not determine the anchor derivatives. For $P(s, x) = \prod_j (s + \sum_i a_{ij} x_i)$, the theorem's M, R, C are precisely the permanent, row-sum product, and column-sum product defined above.

Corollary 1.2 (without an anchor). Let q be a nonzero homogeneous real stable polynomial of degree $n \geq 1$ in n variables with nonnegative coefficients. Let $T = q(\mathbf{1}) > 0$ and $d_i = \partial_i q(\mathbf{1})$.

(a) **Normalized form.** If $T = 1$, put $M = [x_1 \cdots x_n]q$, $R = \prod_i d_i$, and $C = 1$. Then $M + 2 - R - C \geq \gamma_n$, with equality exactly for $q = (\sum_i x_i/n)^n$.

(b) **Scale-free form.** For arbitrary $T > 0$,

$$[x_1 \cdots x_n]q + T - \frac{\prod_i d_i}{T^{n-1}} \geq \gamma_n T, \tag{1.3}$$

with equality exactly when $q = T(\sum_i x_i/n)^n$.

Proof. For the normalized polynomial q/T, the quantities in Theorem 1.1 are $R = \prod_i d_i/T^n$ and $C = 1$. The polynomial $P(s, x) = q(x+s\mathbf{1})/T$ satisfies the hypotheses of Theorem 1.1. Stability follows because each $x_i + s$ has positive imaginary part when all arguments do. Euler's identity gives $\sum_i d_i = nT$, so the anchor derivatives sum to n, while $P(0, \mathbf{1}) = 1$. Multiplying (1.1) by T proves (1.3); the case $T = 1$ gives (a), and restricting (1.2) to $s = 0$ gives the equality cases. □

The conclusion also applies to a degree-n stable polynomial $f(z_1, \ldots, z_m)$ with nonnegative coefficients and any nonnegative m-by-n matrix B satisfying $B\mathbf{1}_n = \mathbf{1}_m$: use $q(x) = f(Bx)$ in (1.3). Every component of Bx has positive imaginary part when all x_i do. Thus this substitution preserves stability, and $q(\mathbf{1}_n) = f(\mathbf{1}_m) = T$. After normalizing by T, the corresponding R is $T^{-n} \prod_i \sum_a B_{ai} \partial_a f(\mathbf{1}_m)$, and $C = 1$. Homogeneity and nonnegative coefficients remain hypotheses throughout.

The marked-variable theorem has a scale-free form as well.

Corollary 1.3. Let $Q(s, x_1, \ldots, x_n)$ satisfy the stability, homogeneity, and coefficient hypotheses of Theorem 1.1. Suppose $a = Q(1, 0) > 0$, put $d_i = \partial_{x_i} Q(1, 0)$, and assume $t = (\sum_i d_i)/(na) > 0$. Here put $M = [x_1 \cdots x_n]Q(0, x)$, $R = \prod_i d_i/a^{n-1}$, and $C = Q(0, \mathbf{1})$. Then

$$M + (2 - \gamma_n)at^n - R - C \geq 0. \tag{1.4}$$

Equality holds exactly when $Q(s, x) = a(s + (t/n) \sum_i x_i)^n$.

Proof. Apply Theorem 1.1 to $Q(s, x/t)/a$ and multiply its inequality by at^n. The equality case in the matrix statement at the start of this section follows from uniqueness of the monic linear factors in $Q(s, x) = \prod_j (s + \sum_i a_{ij} x_i)$. □

Proof strategy. Write $F_1 = M + \Delta$ and choose a global minimizer. Section 2 proves that a global minimizer exists; Section 3 turns minimality into equations for the anchor and input directions. For $n \geq 4$, the spectral-residue construction in Section 4 gives a capacity bound and, crucially, $M > 0$ at the minimizer. The stationarity equations then give an upper bound for

one covariance, while replacement monomials give a lower bound for the same quantity (Section 5). Section 6 shows that these bounds are incompatible unless every $r_i = 1$: exact rational comparisons handle $4 \le n \le 8$, and one analytic comparison handles $n \ge 9$. When the r_i are all one, the anchor equation and Gurvits's coefficient inequality finish the argument. Degree two is direct; degree three uses a separate determinantal reduction (Sections 7–8). The cited hyperbolicity, representation, and coefficient theorems are identified where used; the remaining inequalities and finite comparisons are proved in the paper.

2 Hyperbolic coordinates and compactness

For $n = 1$, the hypotheses of Theorem 1.1 force $P(s, x) = s + x$, which satisfies (1.1) with equality. For the remainder of its proof, assume $n \ge 2$.

Definition 2.1 (admissible class and objective). A **normalized admissible polynomial** $P(s, x_1, \dots, x_n)$ is homogeneous of degree n, real stable, has nonnegative coefficients, and satisfies $P(1, 0) = 1$ and $\sum_i \partial_{x_i} P(1, 0) = n$. Its quantities M, R, C, Δ retain the meanings in Theorem 1.1. For $\alpha > 0$, set $F_\alpha(P) = M + \alpha\Delta$.

Let V be a finite-dimensional real vector space and $h : V \to \mathbb{R}$ a homogeneous polynomial of degree n. A vector e with $h(e) > 0$ is a **hyperbolicity direction** if $t \mapsto h(te + y)$ has only real zeros for every real vector y. Normalize $h(e) = 1$, and define the e-eigenvalues by

$$h(te + y) = \prod_{j=1}^{n} (t + \lambda_j(y)). \tag{2.1}$$

For a direction v, write $D_v h(y) = \frac{d}{dt} h(y + tv)\big|_{t=0}$. The mixed polarization in n directions is the constant derivative $D_{v_1} \cdots D_{v_n} h$; no factor $1/n!$ is included. The closed hyperbolicity cone is $K = \{y : \lambda_j(y) \ge 0 \text{ for all } j\}$; its interior consists of vectors with all eigenvalues positive. We use the standard cone facts: K is convex, every interior vector is a hyperbolicity direction with the same cone, and directional derivatives in cone directions have nonnegative mixed polarizations. The first two facts are due to Gårding [Ga59]; the mixed-polarization and stable-polynomial forms are discussed in [G04, G08]. Nonnegativity of mixed polarizations extends to directions in ∂K by approximating them with interior cone directions.

Construction 2.2 (hyperbolic coordinates). For a normalized admissible polynomial we may take $h = P$, $e = (1, 0, \dots, 0)$, and v_i the coordinate vector for x_i. Here e lies on the boundary of the nonnegative orthant, so its hyperbolicity does not follow directly from stability. To justify this choice, stability and homogeneity first give hyperbolicity in every strictly positive coordinate direction: a nonreal zero on a real affine line would contradict stability or its complex conjugate. Letting such directions tend to e preserves real-rootedness, because the leading coefficient tends to $P(e) = 1$. The nonnegative orthant lies in the closed cone, and $P(e) > 0$ places e in its interior. Thus, more generally, we work with

$$\begin{aligned} P(s, x) &= h\left(se + \sum_i x_i v_i\right), \quad e \in \operatorname{int} K, \quad v_i \in K, \\ S &= \sum_i v_i, \qquad\qquad r_i = D_{v_i} \log h(e), \quad \sum_i r_i = n. \end{aligned} \tag{2.2}$$

Conversely, (2.2) is stable and has nonnegative coefficients. For stability, the imaginary part of its argument lies in the interior of K. If b is interior, hyperbolicity in direction b implies $h(a + ib) \ne 0$ for real a. Nonnegative coefficients follow from the nonnegative mixed polarizations

in cone directions. Consequently, (2.2) gives an equivalent description of the normalized admissible class; the anchor e need only lie in $\operatorname{int} K$, even when it is outside the original coordinate orthant.

Lemma 2.3 (compactness and nonnegative deficit). For each fixed $n \geq 2$, the normalized admissible class is compact. Every member satisfies $R, C \leq 1$, hence $\Delta \geq 0$, and $P(1, \mathbf{1}) \leq 2^n$.

Proof. The trace $\sum_j \lambda_j(y) = D_y \log h(e)$ is linear in y. In particular the eigenvalues of S are nonnegative and sum to n, so AM–GM gives $C = h(S) \leq 1$. Likewise $R \leq 1$, proving $\Delta \geq 0$. Moreover $P(1, \mathbf{1}) = \prod_j (1 + \lambda_j(S)) \leq 2^n$. All coefficients of a normalized admissible polynomial are therefore bounded. The class is closed: a coefficientwise limit of stable polynomials is stable or zero, by Hurwitz's theorem, and the coefficient of s^n remains one. The class is consequently compact in its finite-dimensional coefficient space. □

Definition 2.4 (Hessian form and lineality). For any interior a, write

$$H_a(u, w) = -D_u D_w \log h(a).$$

The **lineality space** is $K \cap (-K)$. We read vector identities below modulo this space.

Lemma 2.5 (Hessian kernel). The form H_e is positive semidefinite, with kernel equal to the lineality space. If u lies in that space, then $h(a + tu) = h(a)$ for every a and real t; in particular, directional derivatives of h in u vanish.

Proof. At the normalized anchor, (2.1) gives $H_e(u, u) = \sum_j \lambda_j(u)^2 \geq 0$. Zero norm implies $u, -u \in K$. At any interior a, these two inclusions force all eigenvalues of u relative to a to vanish, so $h(a + tu) = h(a)$ for every real t. Polynomial identity extends this equality to every a. The converse follows by differentiation. □

Lemma 2.6 (bounds at a minimizing polynomial). If $\alpha > \gamma_n$ and P minimizes F_α over the normalized admissible class, then

$$\Delta \leq \gamma_n/\alpha < 1, \quad R, C > 0, \quad r_i > 0, \qquad \alpha C - M \geq \alpha - \gamma_n > 0. \tag{2.3}$$

In particular, S is interior. Individual v_i may lie on the boundary, and M need not yet be positive.

Proof. The uniform power (1.2) has value γ_n, so $M + \alpha\Delta \leq \gamma_n$ at a global minimizer. By Lemma 2.3, $\Delta \geq 0$ and $R, C \leq 1$. Thus $C, R \geq 1 - \Delta$ and $M \leq \gamma_n - \alpha\Delta$, giving (2.3). Since $C = h(S) > 0$ and $S \in K$, the vector S is interior. □

The proof now needs information that holds at a minimizer even when some input directions lie on the boundary. The two admissible variations in Section 3 provide it.

3 Variations at a minimizing polynomial

3.1 Varying the anchor

Lemma 3.1 (anchor stationarity). Let $\alpha > \gamma_n$, and let P be a global minimizer of F_α. Set $u_i = (r_i - 1)/r_i$ and $a_\alpha = (\alpha C - M)/(\alpha R) > 0$. Then

$$\sum_i u_i v_i = a_\alpha (e - S), \qquad e = \sum_i \left(1 + \frac{u_i}{a_\alpha}\right) v_i \quad \text{modulo lineality.} \tag{3.1}$$

No positivity of M or interiority of the individual v_i is required.

Proof. Fix a representation (2.2). For an arbitrary real direction w, put

$$e_t = e + tw, \qquad \beta(t) = \frac{n}{D_S \log h(e_t)}, \qquad P_t(s, x) = \frac{h(se_t + \beta(t) \sum_i x_i v_i)}{h(e_t)}. \tag{3.2}$$

For sufficiently small positive and negative t, e_t is interior, $h(e_t) > 0$, and $\beta(t) > 0$. Section 2 proves stability and coefficient nonnegativity. The anchor value is one and the anchor derivatives sum to n, so this is a two-sided admissible variation. At zero, $\beta = 1$ and $\beta'/\beta = H_e(S, w)/n$. Homogeneity gives $D_y \log h(y) = n$. Differentiating this Euler identity at $y = e$ in direction w yields $D_w \log h(e) = H_e(e, w)$. The quantities M, C acquire the factor $\beta^n/h(e_t)$, whereas $R(t) = \beta^n \prod_i D_{v_i} \log h(e_t)$. Thus, at zero,

$$M' = M[H_e(S, w) - H_e(e, w)], \quad C' = C[H_e(S, w) - H_e(e, w)],$$
$$R' = R\left[H_e(S, w) - \sum_i \frac{H_e(v_i, w)}{r_i}\right].$$

These undivided formulas also apply when $M = 0$. Consequently, writing $T_\alpha = \alpha(R + C) - M$,

$$\frac{d}{dt} F_\alpha(P_t)\Big|_0 = H_e\left(\alpha R \sum_i \frac{v_i}{r_i} - T_\alpha S + (\alpha C - M)e, w\right). \tag{3.3}$$

Since (3.3) vanishes for every w, its first argument belongs to lineality by Lemma 2.5. Rearrangement gives (3.1). The coefficients in the last sum need not be nonnegative. The vector represented by their combination is nevertheless the interior anchor. □

3.2 Scaling and adding input directions

Lemma 3.2 (input stationarity and replacement). Let $n \geq 2$, and let P minimize F_1. Set

$$T = R + C - M, \qquad a = (C - M)/R, \qquad b = (R - M)/C.$$

For every i, the input derivative $\mu_i = D_{v_i} \log h(S)$ satisfies

$$C\mu_i = M - R + Tr_i, \qquad \mu_i = 1 + (1 + b)(r_i - 1). \tag{3.4}$$

For $i \neq j$, define the constant mixed derivative

$$A_{ij} = D_{v_j} \prod_{\ell \neq i} D_{v_\ell} h.$$

Then

$$A_{ij} \geq L_{ij} := M + R(r_j/r_i - 1), \qquad A_{ij} \geq 0. \tag{3.5}$$

These conclusions hold for boundary input directions.

Proof. Since $1 > \gamma_n$, Lemma 2.6 gives $R, C > 0$. Scale one input to $(1 + t)v_i$ and then all inputs by $n/(n + tr_i)$. This is admissible for both signs of small t, including boundary inputs. The derivatives of M, R, C are respectively $M(1 - r_i), R(1 - r_i), C(\mu_i - r_i)$. Stationarity of F_1 gives (3.4).

For $i \neq j$, instead replace v_i by $v_i + tv_j$, $t \geq 0$, and scale all inputs by $n/(n + tr_j)$. The first derivative of the objective is $A_{ij} - Rr_j/r_i + Tr_j - C\mu_j$. It is nonnegative at a minimum, so (3.4) gives the first inequality in (3.5). The second follows from nonnegative polarization. Only addition of a cone direction was used, so no interiority of v_i is needed. In the polynomial q, A_{ij} is twice the coefficient of $x^{\mathbf{1} - e_i + e_j}$, where now e_i denotes a coordinate vector in the exponent space. The factor two comes from differentiating twice in variable j. □

Lemmas 3.1–3.2 are the variational input to the rest of the proof. Their quantitative use requires $M > 0$, which will follow from the capacity estimate in Section 4.

4 Spectral residues and the capacity estimate

We need a lower bound for M that remains valid when the capacity infimum is not attained. Spectral residues give, for each positive argument of q, a nonnegative array with controlled row and column margins. A flow argument will turn that array into a uniform capacity bound.

4.1 Intrinsic positive linear functionals

Lemma 4.1 (positive spectral residues). For each real direction y, there are linear functionals w_j^y, one for each e-eigenvalue $\lambda_j(y)$, which are nonnegative on K and satisfy

$$w_j^y(e) = 1, \qquad w_j^y(y) = \lambda_j(y), \qquad \sum_j w_j^y(z) = D_z \log h(e). \tag{4.1}$$

For $S = \sum_i v_i$, they also satisfy

$$\sum_j (w_j^y(S) - 1)_+ \le \sum_j (\lambda_j(S) - 1)_+, \qquad t_+ = \max(t, 0). \tag{4.2}$$

Repeated eigenvalues are allowed.

Proof. For a Hermitian matrix Z, write $Z \succeq 0$ and call it positive semidefinite if $v^* Z v \ge 0$ for every complex vector v, where $*$ denotes conjugate transpose. The notation $Z \succeq Y$ means $Z - Y \succeq 0$. We use the ternary hyperbolic determinantal representation in [LPR05], equivalent to the real-zero form in [HV07]; see also [G04, Proposition 1.2]. For real directions y, z there are real symmetric n-by-n matrices Y, Z with

$$h(te + uy + vz) = \det(tI + uY + vZ). \tag{4.3}$$

Their eigenvalues are the e-eigenvalues of y, z; in particular $z \in K$ implies $Z \succeq 0$. This follows by applying the three-variable theorem to the restriction on the left. There is a separate representation for each pair; a representation of the full polynomial by one matrix pencil is not assumed.

Fix y, and let λ run through its distinct eigenvalues, with multiplicities m_λ. Equation (4.3) gives a unique partial-fraction expansion

$$\frac{D_z h(te + y)}{h(te + y)} = \sum_\lambda \frac{\ell_\lambda(z)}{t + \lambda}. \tag{4.4}$$

Indeed the determinant derivative is $\operatorname{tr}((tI + Y)^{-1} Z)$. Thus the poles are simple and $\ell_\lambda(z) \ge 0$ for $z \in K$. Uniqueness in (4.4) shows that these residues are linear functionals of z, independently of the representation chosen to prove positivity.

Give each of the m_λ labels the functional $w_j^y = \ell_\lambda / m_\lambda$. This gives (4.1). For the pair (y, S), these values at S are the diagonal entries of a matrix with spectrum $\lambda(S)$, averaged within equal-eigenvalue blocks of Y. A matrix is doubly stochastic if its entries are nonnegative and every row and column sums to one. Diagonal entries are doubly stochastic averages of eigenvalues, as follows by squaring the entries of an orthogonal eigenvector matrix. Averaging within blocks is again doubly stochastic. Convexity of $t \mapsto (t-1)_+$ gives (4.2). This construction handles repeated eigenvalues without making choices of differentiable eigenvalue branches. □

4.2 A capacity floor without an attaining point

Definition 4.2 (margin imbalance and capacity). Whenever S is interior, write $c_j = \lambda_j(S) > 0$ and

$$e_r = \tfrac{1}{2}\sum_i |r_i - 1|, \quad e_c = \tfrac{1}{2}\sum_j |c_j - 1|, \quad E = e_r + e_c.$$

For a nonzero homogeneous polynomial q of degree n in n variables with nonnegative coefficients, define

$$\mathrm{Cap}(q) = \inf_{x_i>0} \frac{q(x)}{\prod_i x_i}.$$

Lemma 4.3 (capacity floor). Let P be normalized admissible with S interior. If $E < 1$, then

$$\mathrm{Cap}(q) \geq (1-E)^n, \qquad M \geq \gamma_n (1-E)^n. \tag{4.5}$$

The bound is uniform in the positive arguments of q; it does not require the capacity infimum to be attained.

Proof. For each fixed $x > 0$, use the functionals at $y = \sum_i x_i v_i$ to form $a_{ij} = w_j^y(v_i) \geq 0$. Its row sums are r_i, its column sums $\sigma_j = w_j^y(S)$ have total mass n, and (4.2) bounds their positive surplus by e_c. Equation (4.1) also gives

$$q(x) = \prod_j \sum_i x_i a_{ij}. \tag{4.6}$$

For row and column sets I, J, rows in I carry at least $|I| - e_r$ mass, since $\sum_i (1-r_i)_+ = e_r$, while (4.2) bounds the mass absorbed by columns in J by $|J| + e_c$. If $d = |I| - |J| \geq 1$, then

$$\sum_{i\in I, j\notin J} a_{ij} \geq |I| - e_r - |J| - e_c = d - E \geq d(1-E).$$

For $d \leq 0$, the bound $\sum_{i\in I, j\notin J} a_{ij} \geq d(1-E)$ follows from nonnegativity. If $E < 1$, give the usual network source-to-row and column-to-sink capacities $1-E$, and row-to-column capacities a_{ij}. A cut with rows I and columns J on the source side has capacity

$$(n - |I| + |J|)(1-E) + \sum_{i\in I, j\notin J} a_{ij} \geq n(1-E).$$

Max-flow/min-cut therefore gives an array $0 \leq b_{ij} \leq a_{ij}$ with every row and column sum $1-E$, without an extra support assumption. Applying weighted AM–GM to each column,

$$\sum_i x_i a_{ij} \geq (1-E) \prod_i x_i^{b_{ij}/(1-E)}.$$

The row sums of b show that multiplying gives $q(x) \geq (1-E)^n \prod_i x_i$. Although the array depends on x, the bound is uniform. Taking the infimum proves the capacity bound in (4.5). The second inequality is Gurvits's theorem [G08, Corollary 2.5]: $[x_1 \cdots x_n]q \geq \gamma_n \mathrm{Cap}(q)$ for a homogeneous nonnegative stable polynomial of degree n in n variables. His right-half-plane convention agrees with ours by homogeneity, using $q(iz) = i^n q(z)$. The real specialization defining q is stable or zero; $C > 0$ excludes zero. No capacity minimizer is used. □

Corollary 4.4 (positivity at a minimizing polynomial). If $n \geq 4$ and P minimizes F_1, then $E < 1$, $M > 0$, $M < \min(R, C)$, and the parameters a, b of Lemma 3.2 are positive.

Proof. Pinsker's inequality applied to the probability vectors $(1/n)_i$ and $(r_i/n)_i$, with natural logarithms, gives $e_r^2 \leq -(n/2)\log R$. Their total variation distance is e_r/n and their relative

entropy in that order is $-\log R/n$. Applying the same argument to $(1/n)_j$ and $(c_j/n)_j$ gives $e_c^2 \le -(n/2)\log C$. Since $RC \ge 1-\Delta$,

$$E^2 \le n[-\log(RC)] \le \frac{n\Delta}{1-\Delta}. \tag{4.7}$$

The last step uses $-\log(1-t) \le t/(1-t)$. For $n \ge 4$,

$$\frac{n\gamma_n}{1-\gamma_n} \le \frac{12}{29} < 1;$$

the value at four is $12/29$, and $(n+1)\gamma_{n+1}/(n\gamma_n) = (n/(n+1))^{n-1} < 1$, while γ_n decreases. At this minimum, (4.5) therefore gives $M > 0$. Also $M \le \gamma_n < 1-\gamma_n \le \min(R, C)$. In particular $a, b > 0$. □

This positivity makes the stationary identities quantitative. Section 5 uses them to estimate the same covariance from opposite sides.

5 Two estimates for the same covariance

Throughout Sections 5–6, let $n \ge 4$ and let P minimize F_1. Corollary 4.4 ensures $M > 0$ and $a, b > 0$.

Definition 5.1 (deviation and covariance data). Use the residues at S and set $w_{ij} = w_j^S(v_i)$, $z_j = (c_j - 1)/c_j$, and

$$B = \sum_i r_i u_i^2 = \sum_i \frac{(r_i-1)^2}{r_i}, \quad W = \sum_j c_j z_j^2, \quad E_2 = \sum_i u_i^2, \quad T_k = \sum_i r_i u_i^k \quad (k = 3, 4).$$

Thus $B = 0$ exactly when every anchor margin $r_i = 1$. Give an exponent vector N the probability $\mathbb{P}(N = \nu) = [x^\nu]q/C$; write $\mu = \mathbb{E}N$, $\mathcal{K} = \mathrm{Cov}(N)$, and $g = \mu - \mathbf{1}$. The quadratic form $u^{\mathsf{T}}\mathcal{K}u$ is estimated from both sides below.

5.1 The stationary square identity

Lemma 5.2 (stationary covariance upper bound). We have $bB = aW$ and, with $c_{\min} = \min_j c_j$,

$$u^{\mathsf{T}}\mathcal{K}u \le \frac{(1-ab)B}{c_{\min}}, \qquad 1 - ab = \frac{M}{C}(1+a). \tag{5.1}$$

Proof. The array w has row sums r_i and column sums c_j. From (3.1), (3.4), and (4.4),

$$\sum_j w_{ij} z_j = -b r_i u_i, \qquad \sum_i w_{ij} u_i = -a c_j z_j. \tag{5.2}$$

For the first equality, $\sum_j w_{ij}/c_j = \mu_i$, so the left side is $r_i - \mu_i$. The second is obtained by applying w_j^S to (3.1). This is valid because w_j^S is linear and nonnegative on K, so it vanishes on the lineality space $K \cap (-K)$. Multiplying (5.2) by u_i and z_j, respectively, and summing gives $bB = aW$.

Differentiating $\log q(e^{t_1}, \dots, e^{t_n})$ yields

$$u^{\mathsf{T}}\mathcal{K}u = \sum_i \mu_i u_i^2 - H_S\left(\sum_i u_i v_i, \sum_i u_i v_i\right).$$

Indeed, let $y(\tau) = \sum_i e^{\tau u_i} v_i$. Then $u^{\mathsf{T}}\mathcal{K}u$ is the second derivative of $\log h(y(\tau))$ at zero. Since $y'(0) = \sum_i u_i v_i$ and $y''(0) = \sum_i u_i^2 v_i$, the chain rule gives the displayed formula. Here and below this u is the specific vector from (3.1). By (3.1), $\sum_i u_i v_i = a(e - S)$ modulo lineality. Moreover,

$$h(S + \tau(e - S)) = \prod_j (c_j + \tau(1 - c_j)), \qquad H_S(e - S, e - S) = \sum_j \frac{(c_j - 1)^2}{c_j^2} = \sum_j z_j^2.$$

Thus the second term is $a^2 \sum_j z_j^2$. Expanding squares and using (5.2) now gives

$$u^{\mathsf{T}}\mathcal{K}u = \sum_{ij} \frac{w_{ij}}{c_j}(u_i + a z_j)^2, \qquad \sum_{ij} w_{ij}(u_i + a z_j)^2 = (1 - ab)B. \tag{5.3}$$

Since $c_j \geq c_{\min}$, the two identities in (5.3) give the upper bound in (5.1); the expression for $1 - ab$ follows from the definitions of a, b. □

5.2 Replacement monomials and moment squares

Lemma 5.3 (replacement covariance lower bound). If $B > 0$, then, with $r_{\max} = \max_i r_i$,

$$\frac{n}{r_{\max}} \leq \frac{1 + a}{c_{\min}} + [C(1 + b^2) - M]\frac{B}{M}. \tag{5.4}$$

This is a necessary condition for a nonuniform global minimizer.

Proof. The ordered count vectors $\mathbf{1} - e_i + e_j$, $i \neq j$, are distinct, and their probabilities are $A_{ij}/(2C)$. Retaining them in the second moment about $\mathbf{1}$, (3.5) gives

$$u^{\mathsf{T}}\mathcal{K}u + (g \cdot u)^2 \geq \sum_{i<j} \frac{(L_{ij})_+ + (L_{ji})_+}{2C}(u_i - u_j)^2. \tag{5.5}$$

We have

$$\sum_i r_i u_i = 0, \quad \sum_i u_i = -B, \quad E_2 = B - T_3, \quad g \cdot u = (1 + b)B.$$

Direct expansion, using $\sum_i r_i = n$, gives

$$\sum_{i<j}(u_i - u_j)^2 = nE_2 - B^2, \qquad \sum_{i<j} r_i r_j (u_i - u_j)^4 = nT_4 + 3B^2. \tag{5.6}$$

Also $nT_4 \geq B^2$ by Cauchy–Schwarz. Since $L_{ij} + L_{ji} = 2M + R r_i r_j (u_i - u_j)^2$, using just this sum in (5.5) and (5.6) gives

$$u^{\mathsf{T}}\mathcal{K}u \geq \frac{M}{C}(nE_2 - B^2) + \frac{R}{2C}(nT_4 + 3B^2) - (1 + b)^2 B^2.$$

Here $(L_{ij})_+ + (L_{ji})_+ \geq L_{ij} + L_{ji}$. Since $nT_4 \geq B^2$ and $2R/C = 2b + 2M/C$, this simplifies to

$$u^{\mathsf{T}}\mathcal{K}u \geq \frac{nM}{C}E_2 - (1 + b^2 - M/C)B^2.$$

Combining with (5.1) and $E_2 \geq B/r_{\max}$, then dividing by $MB/C > 0$, gives (5.4). □

The coarse inequality (5.4) is enough for $n \geq 9$ and for small deficits in the finite degrees. The middle deficit intervals need the following retained positive terms.

Lemma 5.4 (refined replacement bound). Define

$$Q_{\mathrm{sq}} = \frac{nR}{2C}\sum_i r_i\left(u_i^2 - \frac{B}{n} - \frac{M}{R}u_i\right)^2, \quad H = \frac{1}{2C}\sum_{i<j}[(-L_{ij})_+ + (-L_{ji})_+](u_i-u_j)^2.$$

Both are nonnegative, and

$$u^{\mathsf{T}}\mathcal{K}u \geq \frac{nM}{C}\left(1-\frac{M}{2R}\right)B - (1+b^2-M/C)B^2 + Q_{\mathrm{sq}} + H. \tag{5.7}$$

If $\eta \geq 0$ and $r_i \leq 1+\eta$ for every i, then, with $J_3 = \sum_{i<j}\min(r_i,r_j)|u_i-u_j|^3$,

$$\begin{aligned} H &\geq \frac{R}{2C}J_3 - \frac{M}{2C}(nE_2 - B^2), \\ J_3 &\geq \kappa B^{3/2}, \qquad \kappa^2 = \frac{2n^2}{(n-1)(1+\eta)^2}. \end{aligned} \tag{5.8}$$

Proof. Expanding the square gives

$$\frac{2C}{nR}Q_{\mathrm{sq}} = T_4 - \frac{B^2}{n} - 2\frac{M}{R}T_3 + \frac{M^2}{R^2}B.$$

Substitution into the baseline of (5.5), using $E_2 = B - T_3$, yields (5.7). The square expansion above reorganizes the baseline terms from (5.5) into Q_{sq}; H is the additional nonnegative term gained by keeping the positive parts.

Suppose $r_i \leq 1+\eta$. If $r_i \leq r_j$, then $-L_{ji} = Rr_i(u_j-u_i) - M$, while $L_{ij} > 0$. Hence the first inequality in (5.8) follows from $t_+ \geq t$, even when its right side is negative. Now $\min(r_i,r_j) \geq r_ir_j/(1+\eta)$, and

$$\sum_{i<j} r_ir_j(u_i-u_j)^2 = nB, \qquad \sum_{i<j} r_ir_j \leq n(n-1)/2.$$

Weighted Jensen, or the monotonicity of power means, gives the second inequality in (5.8). □

5.3 The zero-variance case and equality

Proposition 5.5 (rigidity at zero variance). If $B = 0$ at a global minimizer of F_1 with $n \geq 4$, then $F_1 = \gamma_n$ and $P(s,x) = (s + \sum_i x_i/n)^n$.

Proof. If $B = 0$, all $r_i = 1$, and (3.1) gives $e = S$ modulo lineality because $a > 0$. Thus $C = 1$ and $\mu_i = 1$. The function $t \mapsto \log q(e^{t_1},\dots,e^{t_n}) - \sum_i t_i$ is convex, as a logarithm of a sum of exponentials minus a linear function. At zero it has value and gradient zero. Therefore $\mathrm{Cap}(q) = 1$ and $M \geq \gamma_n$. Since we are at a minimum with value at most γ_n, equality holds.

Gurvits's equality theorem [G08, Theorem 5.7] states that positive-capacity equality in the coefficient bound occurs exactly for $q = (\sum_i d_ix_i)^n$, with $d_i > 0$. The conditions $q(\mathbf{1}) = 1$ and $\partial_i q(\mathbf{1}) = 1$ give $d_i = 1/n$. Finally $e = S$ modulo lineality implies $P(s,x) = q(x + s\mathbf{1})$, proving the full anchored equality (1.2). □

5.4 Remaining cases

Proposition 5.5 settles the case $B = 0$. To prove Theorem 1.1 for $n \geq 4$, it remains to rule out $B > 0$ at a global minimizer. Section 6 does this by combining the necessary covariance conditions with bounds derived solely from the deficit Δ.

6 Excluding positive variance in every degree at least four

There are three deficit ranges. A coarse covariance comparison excludes small deficits, a refined comparison excludes the intervening intervals in degrees four through seven, and the capacity floor excludes large deficits. The same coarse comparison covers all deficits for $n \geq 9$.

6.1 Scalar bounds and a margin refinement

Lemma 6.1 (deficit-to-margin bounds). Let $n \geq 4$ and let P minimize F_1. Suppose $\Delta \leq d \leq \gamma_n$. Choose $0 \leq e_0 < 1$ with $e_0^2 \geq nd/(1-d)$, and put

$$q_0 = 1 - d, \qquad \rho = \frac{q_0(q_0 - \gamma_n)}{1 - \gamma_n} > 0, \qquad \mathcal{B}(t) = \frac{2(3-2t)d}{3(1-t)(1+\rho)q_0}. \tag{6.1}$$

Then $E \leq e_0$ and $a/b, b/a \geq \rho$. If every deviation $|r_i - 1|, |c_j - 1|$ is at most $\eta \leq e_0$, then $B, W \leq \mathcal{B}(\eta)$. The choice $\eta = e_0$ is always valid; a smaller η is certified by

$$0 \leq \eta \leq e_0, \qquad \frac{\eta^2}{1+\eta} \geq \frac{n-1}{n}\mathcal{B}(e_0). \tag{6.2}$$

Proof. Equation (4.7) gives $E \leq e_0$. Both a/b and b/a are at least ρ. Indeed, $M \leq \gamma_n < 1 - \gamma_n \leq q_0 \leq \min(R, C) \leq 1$, so the denominators below are positive and $\rho > 0$. For example,

$$\frac{a}{b} = \frac{C(C-M)}{R(R-M)} \geq \frac{q_0(q_0 - M)}{1 - M} \geq \frac{q_0(q_0 - \gamma_n)}{1 - \gamma_n}.$$

The last fraction before the inequality decreases with $M \leq \gamma_n$.

For $|t| \leq \eta < 1$, Jensen applied to $t - \log(1+t) = t^2 \int_0^1 s/(1+st)\, ds$ gives

$$t - \log(1+t) \geq \frac{3t^2}{2(3+2t)} \geq \frac{3(1-\eta)}{2(3-2\eta)} \frac{t^2}{1+t}. \tag{6.3}$$

In the first step the probability density is $2s$, of mean $2/3$; in the second, $(1+t)/(3+2t)$ is increasing. Put $k_\eta = 3(1-\eta)/[2(3-2\eta)]$. Since $bB = aW$ and both $a/b, b/a \geq \rho$, we have $B + W \geq (1+\rho)\max(B, W)$. Summing (6.3) over the deviations, whose sums vanish, gives

$$\begin{aligned} \frac{d}{q_0} &\geq -\log(RC) \\ &= \sum_i [(r_i - 1) - \log r_i] + \sum_j [(c_j - 1) - \log c_j] \\ &\geq k_\eta (B + W) \geq k_\eta (1+\rho) \max(B, W). \end{aligned}$$

Thus $B, W \leq \mathcal{B}(\eta)$.

To verify the refinement, let θ be the largest absolute deviation in either margin. For the margin containing it, zero sum and Cauchy–Schwarz give sum of squared deviations at least $n\theta^2/(n-1)$. Each reciprocal margin is at least $1/(1+\theta)$, so $\theta^2/(1+\theta) \leq (n-1)\max(B, W)/n$. The left side increases with $\theta \geq 0$, proving (6.2). □

Corollary 6.2 (small-deficit exclusion). Under the hypotheses of Lemma 6.1, choose a certified η and set $m = \gamma_n(1 - e_0)^n$, $A = (1-m)/q_0$, and

$$Q := \frac{n}{1+\eta} - \frac{1+A}{1-\eta} - \frac{(1 + A^2 - m)\mathcal{B}(\eta)}{m}. \tag{6.4}$$

If $Q > 0$, there is no global minimizer with $B > 0$ and $\Delta \in [0, d]$.

Proof. The capacity floor gives $M \geq m$; also $a, b \leq A$, $r_{\max} \leq 1 + \eta$, and $c_{\min} \geq 1 - \eta$. Lemma 5.3 then requires $Q \leq 0$, a contradiction. □

6.2 A stronger comparison on a middle interval

Fix $n \geq 4$ and $0 < d_0 \leq d_1 < \gamma_n$, and choose e_0, η certified by Lemma 6.1 at $d = d_1$. Use $q_0, \rho, \mathcal{B}$ from (6.1) with $d = d_1$, and κ from (5.8). Put $m_* = \gamma_n - d_0$ and

$$L = \frac{2(1-\eta)d_0\rho}{(1+\eta)(1+\rho)}, \qquad U = \mathcal{B}(\eta). \tag{6.5}$$

Define the rational constants

$$\begin{aligned} T_* &= \frac{1 + q_0^{-1} + n/2}{1-\eta} - n + \frac{nm_*}{2q_0}, \\ A_* &= \frac{2m_*}{q_0}(T_*)_+, \qquad D_* = \frac{2(1+q_0^{-2})}{q_0}, \end{aligned} \tag{6.6}$$

and the polynomial

$$\mathcal{P}(t) = \kappa^2 t - (A_* + D_* t)^2. \tag{6.7}$$

Lemma 6.3 (middle-interval exclusion). If $0 < L \leq U$ and $\mathcal{P}(L), \mathcal{P}(U) > 0$, there is no global minimizer with $B > 0$ and $\Delta \in [d_0, d_1]$.

Proof. Suppose such a minimizer exists. The elementary upper bound

$$t - \log(1+t) \leq \frac{1+\eta}{2(1-\eta)} \frac{t^2}{1+t} \quad (|t| \leq \eta)$$

follows from the same integral formula. Since $-\log(RC) \geq (1-R)+(1-C) = \Delta$, and $W/B \leq 1/\rho$, we obtain $L \leq B \leq U$. Also $M \leq m_*$. Combining (5.1), (5.7), and both bounds in (5.8), dropping Q_{sq}, and dividing by $B > 0$, gives

$$\begin{aligned} \kappa\sqrt{B} \leq \frac{2M}{R}\left(\frac{1+a}{c_{\min}} - n + \frac{nM}{2R} + \frac{n}{2}\frac{E_2}{B}\right) \\ + \frac{2C}{R}\left(1 + b^2 - \frac{3M}{2C}\right) B. \end{aligned} \tag{6.8}$$

Using $E_2/B \leq 1/(1-\eta)$, the bracket in (6.8) is at most T_*. If it is negative, its contribution is at most zero; otherwise $M/R \leq m_*/q_0$ applies. This explains the positive part in A_*. Dropping the favorable negative term in the second line proves $\kappa\sqrt{B} \leq A_* + D_* B$. Both sides are nonnegative, so $\mathcal{P}(B) \leq 0$. The polynomial $\mathcal{P}$ is concave. Strict positivity at both endpoints L, U therefore excludes the whole closed interval, contradicting $\mathcal{P}(B) \leq 0$. $\square$

6.3 Complete rational comparisons for degrees four through eight

Proposition 6.4 (finite-degree exclusion). For each $4 \leq n \leq 8$, no global minimizer of F_1 has $B > 0$.

Proof. Lemma 2.6 restricts a global minimizer to $0 \leq \Delta \leq \gamma_n$. The small-deficit rows in Table B.1 satisfy $Q > 1/10$, so Corollary 6.2 excludes the first deficit interval in each degree. The middle-interval rows in Table B.2 satisfy $0 < L \leq U$ and $\mathcal{P}(L), \mathcal{P}(U) > 1/40$, so Lemma 6.3 excludes their closed intervals.

For an upper interval beginning at d, choose e_0 so that

$$e_0^2 \geq \frac{n\gamma_n}{1-\gamma_n}, \qquad \gamma_n(1-e_0)^n + d - \gamma_n > 0. \tag{6.9}$$

Then (4.7) and Lemma 4.3 give $M + \Delta \geq \gamma_n(1 - e_0)^n + d > \gamma_n$, contrary to minimality. Table B.3 verifies (6.9). The intervals in Tables B.1–B.3 meet at their endpoints and cover $[0, \gamma_n]$ in each degree: degrees four through seven use all three types, while degree eight needs no middle interval. Every table entry and comparison is exact rational arithmetic, with the verification procedure in Appendix B. □

6.4 One comparison for every degree at least nine

Proposition 6.5 (high-degree exclusion). For every $n \geq 9$, no global minimizer of F_1 has $B > 0$.

Proof. Take $d = \gamma_n$ and $e_0 = \eta = 9/(10n)$. The identities and base comparison

$$\frac{(n+1)^3\gamma_{n+1}}{n^3\gamma_n} = \left(\frac{n}{n+1}\right)^{n-3} < 1, \qquad \frac{9^3\gamma_9}{1-\gamma_9} < \frac{81}{100}$$

show that $e_0^2 \geq n\gamma_n/(1-\gamma_n)$ for all $n \geq 9$. Put $q_* = 999/1000$, $c_* = (9/10)^9$. Then $1 - \gamma_n \geq q_*$, $\eta \leq 1/10$, and $(1-9/(10n))^n \geq c_*$. For the last assertion, the logarithmic derivative of $(1-9/(10x))^x$ is $\log(1-t) + t/(1-t) > 0$, $t = 9/(10x)$; this expression vanishes at zero and has derivative $t/(1-t)^2 > 0$.

Here $\rho = 1 - 2\gamma_n$, so (6.1) gives

$$\mathcal{B}(\eta) \leq \frac{28\gamma_n}{27q_*^2}, \qquad m \geq c_*\gamma_n, \qquad A \leq q_*^{-1}.$$

In (6.4), discard the favorable term $+\mathcal{B}(\eta)$ and bound the remaining terms to obtain

$$\begin{aligned} Q &\geq \frac{90}{11} - \frac{10}{9}(1 + q_*^{-1}) - \frac{28(1+q_*^{-2})}{27q_*^2c_*} \\ &= \frac{67483666532717526796900}{11460431961523892095323 3} > \frac{1}{2}. \end{aligned} \tag{6.10}$$

For example, $(3-2\eta)/(3(1-\eta)) \leq 28/27$ gives the variance bound, and $n/(1+\eta) \geq 90/11$. This is an analytic proof for the infinite range, using only rational base comparisons. By Corollary 6.2, $Q > 0$ excludes $B > 0$. □

Corollary 6.6 (Theorem 1.1 in degrees at least four). For $n \geq 4$, every normalized admissible polynomial satisfies (1.1), with equality if and only if (1.2) holds.

Proof. By Propositions 6.4–6.5, every global minimizer has $B = 0$. Proposition 5.5 identifies that minimizer and its value. Every equality polynomial is itself a global minimizer. The variations used above permit boundary input directions. □

7 Degree two

Proposition 7.1 (Theorem 1.1 in degree two). Every normalized admissible quadratic satisfies (1.1), with equality if and only if (1.2) holds.

Proof. Write

$$P(s,x,y) = s^2 + s(ax+by) + cx^2 + dxy + ey^2, \qquad a+b=2, \quad a,b,c,d,e \geq 0.$$

Section 2 shows that P is hyperbolic with respect to $e = (1,0,0)$; hence $s \mapsto P(s,x,y)$ is real-rooted for every real x, y, including the three specializations used below. Real-rootedness of $P(s,1,0)$ and

$P(s,0,1)$ gives $c \le a^2/4$ and $e \le b^2/4$. As $M = d, R = ab, C = c + d + e$,

$$M + 2 - R - C - \tfrac{1}{2} = \frac{(a-b)^2}{8} + (a^2/4 - c) + (b^2/4 - e) \ge 0.$$

Equality forces $a = b = 1, c = e = 1/4$. The bound $C \le 1$ gives $d \le 1/2$, whereas real-rootedness of $P(s,1,-1) = s^2 + 1/2 - d$ gives $d \ge 1/2$. Thus $d = 1/2$ and $P = (s + (x+y)/2)^2$, as required. □

8 Degree three and completion of the proof

The cubic proof first treats matrices and then uses the anchor identity to reduce a minimizing stable polynomial to that case. We prove the slightly stronger coefficient $\alpha = 32/45$, because its strict improvement over one makes equality transparent. Lemma 8.1 handles diagonal boundary matrices; Lemma 8.2 reduces singular matrix inputs to them; Proposition 8.3 covers all matrix triples. Proposition 8.4 then transfers the result to stable cubics.

8.1 The diagonal boundary inequality

For a nonnegative 3-by-3 matrix A, let R_A, C_A be the products of its row and column sums and let $p_A = \operatorname{per} A$.

Lemma 8.1 (diagonal boundary inequality). If a nonnegative 3-by-3 matrix A has total sum three and at least one zero entry, then

$$R_A + C_A - \tfrac{5}{4} p_A \le 27/16. \tag{8.1}$$

For $\alpha = 32/45$, it follows that $p_A + \alpha(2 - R_A - C_A) \ge 2/9$.

Proof. Appendix A proves (8.1). Consequently

$$\alpha(R_A + C_A) - p_A \le 6/5 - p_A/9 \le 6/5 = 2\alpha - 2/9. \tag{8.2}$$

Equation (8.2) settles the case of a nonnegative 3-by-3 matrix with a zero entry. Section 8.2 reduces singular matrix triples to this case. □

8.2 Reduction of singular matrix inputs

For Hermitian positive semidefinite matrices X_1, X_2, X_3 with total trace three, use

$$S = \sum_i X_i, \quad M = [x_1 x_2 x_3] \det \sum_i x_i X_i, \quad R = \prod_i \operatorname{tr} X_i, \quad C = \det \sum_i X_i, \qquad \Delta = 2 - R - C.$$

Lemma 8.2 (singular matrix reduction). Let X_1, X_2, X_3 be Hermitian positive semidefinite matrices of order three with total trace three, and suppose at least one is singular. For every $\alpha \ge 1/2$, the value $F_\alpha = M + \alpha(2 - R - C)$ is at least that of a tuple of diagonal positive semidefinite matrices with the same total trace three and a zero diagonal entry. In particular, $F_{32/45} \ge 2/9$.

Proof. Suppose X_1 is singular. Choose an orthonormal basis whose last vector lies in $\ker X_1$. In this basis, write

$$X_1 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} B & u \\ u^* & b \end{pmatrix}, \quad X_3 = \begin{pmatrix} E & v \\ v^* & c \end{pmatrix}.$$

The upper blocks are positive semidefinite of order two. Replace u, v by zero, denoting the resulting quantities by a superscript zero. The inputs remain positive semidefinite and R is unchanged. Let $J = \operatorname{adj} A$, where adj is the classical adjugate. Block expansion gives

$$M - M^0 = -2\operatorname{Re}(u^*Jv), \qquad C^0 - C = (u+v)^* \operatorname{adj}(A+B+E)(u+v).$$

For order two, $\operatorname{adj} T = (\operatorname{tr} T)I - T$ is linear and preserves positive semidefiniteness. Thus $\operatorname{adj}(A + B + E) \succeq J \succeq 0$, and

$$2\operatorname{Re}(u^*Jv) \le \tfrac{1}{2}(u+v)^*J(u+v) \le \tfrac{1}{2}(C^0 - C).$$

It follows that $F_\alpha = M + \alpha(2 - R - C) \ge F_\alpha^0$ for $\alpha \ge 1/2$.

Next diagonalize A and remove the off-diagonal entries of B, E in that basis. For the already block-diagonal tuple,

$$M = b\,[\operatorname{tr} A \operatorname{tr} E - \operatorname{tr}(AE)] + c\,[\operatorname{tr} A \operatorname{tr} B - \operatorname{tr}(AB)].$$

This expression and all traces are unchanged. Hadamard's determinant inequality increases $\det(A+B+E)$, hence increases $C = (b+c)\det(A+B+E)$. Therefore the second replacement also does not increase F_α. The resulting diagonal tuple has a zero entry and is covered by Lemma 8.1. □

8.3 Positive definite minima

Proposition 8.3 (cubic matrix inequality and equality). For any Hermitian positive semidefinite triple of order three with total trace three,

$$M + (32/45)\Delta \ge 2/9. \tag{8.3}$$

If $M + \Delta = 2/9$, then every input is $I/3$.

Proof. Set $\alpha = 32/45$. The matrix domain of total trace three is compact. Consider a global maximum of $G_\alpha = \alpha(R+C) - M$. If some input is singular, Lemma 8.2 gives $F_\alpha \ge 2/9$, equivalently $G_\alpha \le 2\alpha - 2/9$. Otherwise every input is positive definite. For a pair $A + H, A - H$, with the third input fixed, the sum and hence C are fixed, while the trace product and the symmetric mixed derivative are quadratic and even in t. Therefore

$$G_\alpha(A + tH, A - tH, X_3) = K - t^2 Q(H)$$

for a constant K and a real number $Q(H)$. Since $t = 1$ is an interior feasible point, differentiation at a maximum gives $Q(H) = 0$. Replacing the pair by its average thus preserves the global maximum and positive definiteness.

Repeatedly average the three pairs cyclically. To justify convergence, let $\bar{X} = (X_1 + X_2 + X_3)/3$, which is fixed, and consider $V = \sum_i \|X_i - \bar{X}\|_F^2$, where $\|H\|_F^2 = \operatorname{tr}(H^*H)$. Averaging a pair decreases V by half the squared norm of its difference. Each full cycle is a linear map on the finite-dimensional space of triples with sum zero, and strictly decreases its squared norm away from zero: equality throughout a cycle would force all pairs equal. Compactness of the unit sphere gives a contraction factor strictly below one. Thus the iterates converge to $X_i = S/3$. Every iterate is a global maximizer, so the limit is also a global maximizer by continuity of G_α.

At the limit, $R = 1$, $M = (2/9)C$, and $C \le 1$, so $G_\alpha = \alpha + (\alpha - 2/9)C \le 2\alpha - 2/9$. Together with the singular case, this proves (8.3) for every Hermitian positive semidefinite triple. If $M + \Delta = 2/9$, (8.3) forces $\Delta = 0$, hence each trace is one and $S = I$. The determinant polynomial $q(x) = \det \sum_i x_i X_i$ is stable: when every x_i is in the upper half-plane, the imaginary part of $\sum_i x_i X_i$ is positive definite because $S = I$. Its coefficients are nonnegative by decomposing

each X_i into rank-one matrices and expanding with Cauchy–Binet. It satisfies $q(\mathbf{1}) = \partial_i q(\mathbf{1}) = 1$. The log-sum-exp convexity argument in the proof of Proposition 5.5 gives capacity one, so [G08, Theorem 5.7] gives $\det \sum_i x_i X_i = (\sum_i x_i/3)^3$. In particular $\det X_i = 1/27$ and $\operatorname{tr} X_i = 1$. Equality in eigenvalue AM–GM forces every $X_i = I/3$. □

8.4 Reduction of a stable cubic at its minimum

Proposition 8.4 (stable cubic reduction). Every normalized admissible cubic satisfies $F_{32/45} \geq 2/9$. Moreover, each global minimizer of $F_{32/45}$ has a representation $P(s, x) = \det(sI + \sum_i x_i X_i)$ with $X_i \succeq 0$ and $\sum_i \operatorname{tr} X_i = 3$.

Proof. Minimize F_α in the polynomial class with $n = 3$ and $\alpha = 32/45 > \gamma_3$. Lemmas 2.3, 2.6, and 3.1 apply without requiring $M > 0$. With $\theta_i = 1 + u_i/a_\alpha$, (3.1) gives

$$P(s, x) = q(x + s\theta).$$

By (3.1), $e - \sum_i \theta_i v_i$ lies in the lineality space of h. Hence, for every real t and x,

$$q(t\theta + x) = h\left(te + \sum_i x_i v_i\right).$$

The right side is real-rooted as a polynomial in t, and $q(\theta) = h(e) = 1$. Thus the homogeneous ternary cubic q is hyperbolic in direction θ. Apply the ternary representation theorem after a real invertible change of coordinates taking that direction to the first coordinate. We obtain real symmetric matrices X_i with

$$q(x) = \det \sum_i x_i X_i, \qquad \sum_i \theta_i X_i = I.$$

For each i, $\det(tI + X_i) = q(t\theta + e_i) = h(te + v_i)$, so all eigenvalues of X_i are nonnegative. Also $\operatorname{tr} X_i = r_i$. Thus the entire anchored polynomial at the minimum is $\det(sI + \sum_i x_i X_i)$, with total trace three, and (8.3) applies. Therefore $F_{32/45} \geq 2/9$ at the global minimum and hence for every admissible stable cubic. □

The coefficient in (8.3) is sharp already for products of linear forms: the matrix with rows $(3/4, 0, 0), (3/4, 0, 0), (0, 3/4, 3/4)$ has permanent zero, $R = C = 27/32$, and $\Delta = 5/16$.

8.5 Proof of Theorem 1.1

Lemma 2.3 proves $\Delta \geq 0$ for $n \geq 2$; in degree one, Section 2 gives $P(s, x) = s + x$ and $\Delta = 0$. Proposition 7.1 settles degree two, and Corollary 6.6 settles every degree at least four.

For $n = 3$, Proposition 8.4 gives $F_{32/45} \geq \gamma_3 = 2/9$. Since $\Delta \geq 0$,

$$F_1 = F_{32/45} + \frac{13}{45}\Delta \geq \gamma_3.$$

Equality forces $\Delta = 0$ and $F_{32/45} = \gamma_3$. An equality polynomial is therefore itself a global minimum of $F_{32/45}$ and has the matrix representation of Proposition 8.4. For its triple, $M + \Delta = \gamma_3$, so Proposition 8.3 forces $X_i = I/3$, hence (1.2). The polynomial (1.2) attains equality in every degree, completing the proof. □

9 Matrix and combinatorial consequences

9.1 Hermitian mixed discriminants and the matrix inequality

For Hermitian positive semidefinite matrices $X_1, \dots, X_n$ of order n with $\sum_i \operatorname{tr} X_i = n$, set $S = \sum_i X_i$ and put

$$M = [x_1 \cdots x_n] \det \sum_i x_i X_i, \quad R = \prod_i \operatorname{tr} X_i, \quad C = \det \sum_i X_i.$$

This is the unnormalized mixed discriminant: the mixed discriminant of n copies of I is $n!$. The polynomial $\det(sI + \sum_i x_i X_i)$ is real stable because its matrix imaginary part is positive definite when all scalar imaginary parts are positive. Its coefficients are real and nonnegative, as is seen by decomposing each positive semidefinite matrix into rank-one matrices and expanding by Cauchy–Binet. Its anchor derivatives are the traces. Theorem 1.1 therefore gives the normalized inequality

$$M + 2 - R - C \geq \gamma_n. \tag{9.1}$$

When $S = I$ and every $\operatorname{tr} X_i = 1$, this reduces to $M \geq \gamma_n$, the mixed-discriminant bound conjectured by Bapat [B89] and proved by Gurvits [G06]. The deficit terms in (9.1) give its extension to arbitrary normalized positive semidefinite tuples. Equality holds exactly when $X_i = I/n$: restricting equality of the marked-variable polynomial to s, x_i makes every eigenvalue of X_i equal to $1/n$. For arbitrary positive total trace $T = \sum_i \operatorname{tr} X_i$, the scale-free form, obtained by applying (9.1) to nX_i/T, is

$$M + (2 - \gamma_n)(T/n)^n \geq R + C,$$

with equality exactly when $X_i = (T/n^2)I$. Here M, R, C retain the definitions above for the unscaled matrices.

Taking $X_i = \operatorname{diag}(a_{i1}, \dots, a_{in})$ gives the two matrix inequalities in the introduction: M is the permanent, R the product of row sums, and C the product of column sums.

9.2 The elementary symmetric hierarchy

Let $n \geq 1$, and let $X_1, \dots, X_n$ be Hermitian positive semidefinite matrices of order n with $\sum_i \operatorname{tr} X_i = n$. For a Hermitian matrix B of order n, define

$$E_k(B) = [t^{n-k}] \det(tI + B), \qquad 0 \leq k \leq n.$$

Thus E_k is the sum of principal minors of order k, or the elementary symmetric function of its eigenvalues. Let $S = \sum_i X_i$, $r_i = \operatorname{tr} X_i$, and $L(x) = \sum_i x_i/n$. When $n = 1$, $X_1 = I$ and $E_1(X_1) = 1$, so the sole case is trivial. For $k = 1$, $E_1(B) = \operatorname{tr} B$, and the resulting inequality is the AM–GM bound $\prod_i (1 + (r_i - 1)/n) \leq 1$. Equality holds when every $r_i = 1$, even if the X_i are not uniform. Henceforth assume $2 \leq k \leq n$, and form

$$H_{k,X}(s, x) = \frac{E_k(sI + \sum_i x_i X_i)}{\binom{n}{k}} (s + L(x))^{n-k}. \tag{9.2}$$

The first factor is stable: differentiate $\det((t + s)I + \sum_i x_i X_i)$ exactly $n - k$ times in t, divide by $(n - k)!$, and set $t = 0$. Differentiation and real specialization preserve stability or zero [BBL09]; the nonzero s^k coefficient excludes zero here. Coefficients are nonnegative by the same rank-one expansion. Multiplication by the second factor preserves these properties.

The anchor value in (9.2) is one, and its derivatives are

$$\rho_i = 1 + \frac{k}{n}(r_i - 1), \qquad \sum_i \rho_i = n.$$

This follows from $DE_k(I)[Y] = \binom{n-1}{k-1} \operatorname{tr} Y$. Let $\Sigma_k(X)$ be the sum of the coefficients of all square-free degree-k monomials in $E_k(\sum_i x_i X_i)$. Then

$$M_k = \frac{(n-k)!\,\Sigma_k(X)}{n^{n-k}\binom{n}{k}}, \quad R_k = \prod_i \rho_i, \quad C_k = \frac{E_k(S)}{\binom{n}{k}}.$$

The coefficient formula holds because each square-free monomial must be completed by its complementary variables from L^{n-k}, whose coefficient is $(n-k)!/n^{n-k}$. Theorem 1.1 gives

$$M_k + 2 - R_k - C_k \geq \gamma_n. \tag{9.3}$$

Here R_k and C_k are exactly the quantities R and C for the lifted polynomial $H_{k,X}$. For arbitrary $T = \sum_i \operatorname{tr} X_i > 0$, write $t = T/n$. Applying (9.3) to X_i/t gives the scale-free form

$$M_k + (2 - \gamma_n)t^k \geq t^k \prod_i \left(1 + \frac{k}{n}\left(\frac{\operatorname{tr} X_i}{t} - 1\right)\right) + \frac{E_k(S)}{\binom{n}{k}},$$

where M_k and S are computed from the unscaled X_i. For the normalized tuple X_i/t, its R_k is the product in this display and its $C_k = E_k(S)/(\binom{n}{k}\, t^k)$. Equality holds exactly when $X_i = (t/n)I$.

For the normalized inequality, equality forces $X_i = I/n$. To check this without losing matrix information, cancel $(s + L)^{n-k}$ from the polynomial equality in (9.2). The coefficient of $s^{k-1}x_i$ gives $\operatorname{tr} X_i = 1$; the coefficient of $s^{k-2}x_i^2$ then gives $E_2(X_i) = (n-1)/(2n)$. Since $2E_2(X_i) = (\operatorname{tr} X_i)^2 - \operatorname{tr}(X_i^2)$, we obtain $\operatorname{tr}(X_i^2) = 1/n$. Cauchy–Schwarz for the real eigenvalues forces $X_i = I/n$. The converse follows by substitution.

On the balanced slice $S = I, r_i = 1$, both R_k and C_k equal one, so (9.3) becomes

$$\Sigma_k(X) \geq \frac{k!\binom{n}{k}^2}{n^k}, \tag{9.4}$$

again with uniform-only equality for $2 \leq k \leq n$. This balanced Hermitian bound follows from Gurvits's stable-polynomial coefficient theorem [G08]. Its diagonal specialization is the subpermanent bound conjectured by Tverberg [T63] and proved by Friedland [F82]. The deficit terms in (9.3) extend these balanced statements to arbitrary normalized tuples.

For $y = (y_1, \ldots, y_n)$, write $e_k(y) = \sum_{|J|=k} \prod_{j \in J} y_j$. Now let $A = (a_{ij})$ be a nonnegative n-by-n matrix with $\sum_{i,j} a_{ij} = n$, and take $X_i = \operatorname{diag}(a_{i1}, \ldots, a_{in})$. Then

$$E_k\left(\sum_i x_i X_i\right) = e_k(A^{\mathsf{T}}x), \qquad \Sigma_k(X) = \sum_{|I|=|J|=k} \operatorname{per} A[I, J],$$

where $A[I, J]$ is the submatrix with row set I and column set J.

Let $r_i = \sum_j a_{ij}$ and $c_j = \sum_i a_{ij}$ be the row and column sums. For these diagonal inputs,

$$R_k = \prod_i \left(1 + \frac{k}{n}(r_i - 1)\right), \qquad C_k = \frac{e_k(c_1, \ldots, c_n)}{\binom{n}{k}}.$$

Thus (9.3) gives the subpermanent inequality

$$\frac{(n-k)!}{n^{n-k}\binom{n}{k}} \sum_{|I|=|J|=k} \operatorname{per} A[I, J] + 2 - R_k - C_k \geq \gamma_n.$$

At $k = n$, this is Dittert's matrix inequality from Section 1, obtained here through Theorem 1.1.

9.3 Fixed-size stable measures, matroid bases, and spanning trees

Let E be a finite set and let μ be a probability measure on its n-element subsets, where $n \geq 1$. Say that μ is **strongly Rayleigh** if its generating polynomial $f(z) = \sum_{B \subseteq E} \mu(B) \prod_{e \in B} z_e$ is real stable; this terminology and its examples are discussed in [BBL09]. Partition $E = E_1 \sqcup \cdots \sqcup E_n$, and write $N_i(B) = |B \cap E_i|$. Identifying all variables z_e with x_i for $e \in E_i$ gives the stable polynomial $g(x) = \mathbb{E}_\mu \prod_i x_i^{N_i}$, homogeneous of degree n, with $g(\mathbf{1}) = 1$: when all x_i lie in the upper half-plane, so do all identified z_e. Here the square-free coefficient $M = \Pr_\mu(N_1 = \cdots = N_n = 1)$, while $R = \prod_i \mathbb{E}_\mu N_i$ and $C = g(\mathbf{1}) = 1$. The normalized inequality of Corollary 1.2 reads

$$M + 2 - R - C = \Pr_\mu(N_1 = \cdots = N_n = 1) + 1 - \prod_{i=1}^{n} \mathbb{E}_\mu N_i \geq \gamma_n. \tag{9.5}$$

In particular, if every class has expected occupancy one, the probability of selecting one element from every class is at least γ_n. Equality in (9.5) holds exactly when $g(x) = (\sum_i x_i/n)^n$.

For unnormalized stable weights w_B of positive sum $W = \sum_B w_B$, let U be the total weight of sets with one element in each class and $D_i = \sum_B w_B N_i(B)$. The scale-free form of (9.5) is

$$U + (1 - \gamma_n)W \geq \frac{\prod_i D_i}{W^{n-1}}.$$

After division by W, its R and C are $\prod_i D_i/W^n$ and 1, respectively.

One explicit family is a fixed-size determinantal measure. Let V be a complex n-by-$|E|$ matrix of full row rank, and set $\mu(B) = |\det V_B|^2 / \det(VV^*)$ for $|B| = n$. Cauchy–Binet normalizes these weights, while $f(z) = \det(V \operatorname{diag}(z_e) V^*)/\det(VV^*)$ is stable by the positive semidefinite pencil argument of Section 9.1. For a partition into n classes, $C = 1$ and

$$R = \prod_i \operatorname{tr}\left((VV^*)^{-1} V_{E_i} V_{E_i}^*\right),$$

where V_{E_i} is the submatrix of columns indexed by E_i. Thus (9.5) applies to the number of selected points in those classes.

For a connected graph on $n+1$ vertices with positive edge weights, take μ to be the weighted spanning-tree measure and partition the edges into n colors. The matrix-tree determinant gives its stable generating polynomial [COSW04], so (9.5) bounds the probability of a rainbow spanning tree. Here R is the product of the expected numbers of edges in the respective colors, and $C = 1$. The same conclusion holds for weighted bases of any rank-n matroid whose basis-generating polynomial is stable: positive weights preserve stability. For these bases, R is the product of expected class occupancies and $C = 1$. Known examples include uniform and regular matroids [COSW04] and the Vámos cube [WW09]. The qualification is needed: not every matroid has a stable basis-generating polynomial [COSW04].

A The complete cubic boundary proof

We prove (8.1). More generally, for a nonnegative matrix of order three and total sum s with a zero entry, homogeneity gives $R + C - 5p/4 \leq s^3/16$. It suffices to take $s = 3$.

A.1 Two scalar estimates

Put $f(t) = t^2(3-t)/4$ on $[0,3]$. If $x, y \geq 0$ and $x + y \leq 3$, then $f(x) + f(y) \leq 27/16$. In fact, with $r = x + y, p = xy$, $4(f(x) + f(y)) = 3r^2 - r^3 + 3p(r-2)$. For $r \leq 2$ this is at most $r^2(3-r) \leq 4$; for $2 \leq r \leq 3$, use $p \leq r^2/4$ to get $4(f(x) + f(y)) \leq r^2(6-r)/4 \leq 27/4$.

For a nonnegative matrix B of order two, write z for its total sum, h, w for its row-product and column-product, and $q = \operatorname{per} B$. Then

$$q \geq \max\{0, (h+w)/2 - z^2/8\}. \tag{A.1}$$

Indeed $h + w - 2q = (b_{11} + b_{22})(b_{12} + b_{21}) \leq z^2/4$.

A.2 Two zeros in one row or column

After permutations, the first row is $(a, 0, 0)$. Let the other full row sums be $M \geq m$, so $a+M+m = 3$, and use z, h, w, q for the lower-right block as in Section A.1. Then $R = aMm, C = (3-z)w, p = aq$. Using (A.1) and maximizing the resulting piecewise linear expression over $0 \leq w \leq z^2/4$ gives

$$R + C - \tfrac{5}{4}p \leq aMm + \frac{(3-z)z^2}{4} - h\min\{3-z, 5a/8\}. \tag{A.2}$$

The breakpoint is $w = z^2/4 - h$, which lies in the interval because $0 \leq h \leq z^2/4$. The two slopes are $3 - z$ and $3 - z - 5a/8$, so the maximum is at the breakpoint or the right endpoint.

If $z \leq M$, discard the nonpositive last term and use $aMm \leq M(3-M)^2/4 = f(3-M)$. Then $(3-M) + z \leq 3$, and the first scalar estimate in Section A.1 proves the result. If $z = M + d > M$, then $0 < d \leq m$ and $h \geq Md$. For the latter bound, the two block row sums add to $M + d$, are bounded by M, m, and their product is minimized at an endpoint; the smaller endpoint product is Md, since $m(M + d - m) - Md = (m - d)(M - m) \geq 0$.

If the minimum in (A.2) is $3 - z$, the right side is at most

$$aMm + \frac{(a+m-d)(M-d)^2}{4} \leq f(3-M) + f(M-d) \leq 27/16.$$

If it is $5a/8$, it suffices to bound $T = aMm + (a+m-d)(M+d)^2/4 - 5aMd/8$. Put $\xi = 2M - 3$, $\nu = a - m$. Direct expansion gives

$$\frac{27}{16} - T = \frac{\xi+3}{8}\left(\nu + \frac{5d}{8}\right)^2 + \frac{N_d(\xi)}{512}, \tag{A.3}$$

where

$$N_d(\xi) = (56d + 96)\xi^2 + (167d^2 + 192d)\xi + 128d^3 + 117d^2 + 72d.$$

Its discriminant as a quadratic in ξ is $-27d(29d^3 + 416d^2 + 896d + 1024)$. Thus $N_d > 0$ for $d > 0$; at $d = 0$ it is $96\xi^2$. Since $\xi + 3 = 2M \geq 0$, (A.3) completes this case. Transposition handles two zeros in a column.

A.3 Zeros in distinct rows and columns

If no two zeros share a row or column, they form a partial matching of size $k = 1, 2, 3$, which permutations place on the diagonal. On a face with specified zeros maximize $H(A) = R + C - 5p/4$. If two maximizers A, B are positive at all allowed positions, the cubic polynomial $t \mapsto H(tA + (1-t)B)$ is constant: at both zero and one its value is the maximum and its derivative vanishes, because the segment extends slightly beyond its endpoints within the face. A polynomial of degree at most three cannot have two distinct double roots unless it vanishes. Thus convex averages of these maximizers are still maximizers.

Treat the matching faces in the order $k = 3, 2, 1$. A boundary point of each face either has two zeros in one row or column, already handled, or belongs to a larger matching face. It remains to consider relative interior maxima, for which the preceding averaging is justified.

For $k = 3$, average over simultaneous row and column permutations. All six allowed entries become $1/2$. Here $R = C = 1, p = 1/4$, and $H = 27/16$.

For $k = 2$, average with the transpose and with the simultaneous swap of indices one and two. The matrix becomes

$$A = \begin{pmatrix} 0 & x & y \\ x & 0 & y \\ y & y & u \end{pmatrix}, \qquad 2x + 4y + u = 3.$$

Put $t = x + y$. Then $R = C = t^2(3 - 2t)$ and $p = x^2u + 2xy^2$. If $t \le 3/4$, $H \le 2t^2(3 - 2t) \le 27/16$. If $t \ge 3/4$, put $x_0 = 2t - 3/2$. The constraint $u \ge 0$ gives $x \ge x_0$, and expansion at fixed t gives

$$\begin{aligned} p - p(x_0) &= \frac{x - x_0}{2}\left[\left(6y + \frac{3}{2}u - 3\right)^2 + \frac{u}{4}(3u + 8y)\right] \ge 0, \\ p(x_0) &= \frac{(2t-3)^2(4t-3)}{4}, \\ \frac{27}{16} - \left(2t^2(3-2t) - \frac{5}{4}p(x_0)\right) &= \frac{9}{4}(t-1)^2(4t-3) \ge 0. \end{aligned} \tag{A.4}$$

For $k = 1$, place the zero at $(3, 3)$, and average under independent swaps of the first two rows and first two columns. The matrix becomes

$$A = \begin{pmatrix} x & x & y \\ x & x & y \\ z & z & 0 \end{pmatrix}.$$

With $U = y + z, V = yz, s = 4x + 2U$, direct expansion gives

$$\frac{s^3 - 16H(A)}{2} = 32x^3 - 16x^2U + 24xU^2 + 4U^3 - (88x + 16U)V \ge 2x(4x - U)^2 \ge 0,$$

where $V \le U^2/4$ was used. These cases exhaust every zero pattern and prove (8.1).

B Exact verification of the finite comparisons

The three tables below contain the exact rational input for Proposition 6.4. Their comparisons use only integer factorials and rational arithmetic; the continuum deductions are Sections 4–6.

B.1 Small-deficit rows (Table B.1)

Each row satisfies (6.1), (6.2), and $Q > 1/10$ in (6.4). When $\eta = e_0$, the refinement test (6.2) is unnecessary.

n	d	e_0	η
4	1/64	13/50	1/8
5	6/625	9/40	1/10
6	1/162	1/5	2/25

n	d	e_0	η
7	432/117649	13/80	1/16
8	945/524288	1/8	1/8

B.2 Middle-interval rows (Table B.2)

Each row satisfies the hypotheses of Lemma 6.3 and $\mathcal{P}(L), \mathcal{P}(U) > 1/40$.

n	d_0	d_1	e_0	η
4	1/64	1/32	3593/10000	939/5000
4	1/32	3/64	1109/2500	1237/5000
4	3/64	1/16	1291/2500	3077/10000
4	1/16	5/64	5823/10000	149/400
4	5/64	189/2048	3189/5000	4379/10000
5	6/625	114/3125	4351/10000	439/2000
6	1/162	1/72	727/2500	1233/10000
7	432/117649	585/117649	1871/10000	141/2000

B.3 Upper-interval rows (Table B.3)

Each row satisfies (6.9).

n	d	e_0
4	189/2048	129/200
5	114/3125	9/20
6	1/72	31/100
7	585/117649	21/100
8	945/524288	1/7

B.4 Verification procedure

For each small-deficit row, compute $\gamma_n, q_0, \rho, \mathcal{B}$ by (6.1). Check $0 \le \eta \le e_0 < 1$, $e_0^2 \ge nd/(1-d)$, and (6.2) when $\eta < e_0$. Form m, A, Q from (6.4) and check $Q > 1/10$. For each middle-interval row, perform the same margin checks at $d = d_1$, compute L, U, A_*, D_*, κ^2, and check $0 < L \le U$ and both values in (6.7) exceed $1/40$. For each upper-interval row check the two inequalities (6.9). Finally sort the middle intervals in each dimension and check that the first begins at the lower endpoint, adjoining endpoints agree, and the last ends at the upper endpoint. At degree eight check equality of the lower and upper endpoints directly. The infinite-range checks are $\gamma_9 < 1/1000$, the base inequality of Section 6.4, and the exact rational evaluation (6.10).

For example the small-deficit row at $n = 4$ uses $\gamma_4 = 3/32, q_0 = 63/64, \rho = 3591/3712$, $m = (3/32)(37/50)^4$, and $\eta = 1/8$. Every subsequent operation is substitution into the displayed rational formulas; no tolerance or floating comparison enters.

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