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# THE SINGULAR-SUPPORT CHARACTERIZATION OF LUSZTIG SHEAVES

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Abstract. We prove that for every finite loop-free quiver and every dimension vector, a simple equivariant perverse sheaf with complex coefficients on the quiver representation space is a Lusztig sheaf if and only if its singular support lies in Lusztig's nilpotent variety.

## Introduction

Ringel's Hall algebra construction relates the representation theory of Dynkin quivers to one half of a quantum group [15]. Multiplication counts subrepresentations with prescribed subobject and quotient. Lusztig combined this approach with the geometric construction of character sheaves. He used induction of constructible complexes along correspondences of quiver representations with invariant subspaces [14, §0.3 and §9]. Iterating induction from the one-dimensional representations at the vertices gives proper maps from varieties of representations with invariant graded flags. The simple perverse constituents of their direct images realize the canonical basis. This construction applies to the symmetric Cartan datum of any finite loop-free quiver, with parallel arrows counted in its Cartan matrix.

There is also a cotangent description of these sheaves. The cotangent bundle of a quiver representation space is itself a space of representations, obtained by adjoining a reverse to each arrow. The equations of the zero moment fiber are the preprojective relations. Imposing nilpotence on the doubled representation cuts out the Lagrangian variety used by Lusztig. In his 1990 paper, Lusztig proposed attaching to each irreducible component of this variety a simple equivariant perverse sheaf whose characteristic variety contains that component and is contained in the nilpotent variety. He proposed that this correspondence be a bijection for a bipartite graph [13, §10.1(a)]. He also asked for a convolution realization of the quantum group and for orientation independence of the resulting basis, and illustrated the proposal with affine examples [13, §§10.3, 10.5–10.6]. His 1991 construction proves that every flag constituent has singular support in the nilpotent variety [14, Corollary 13.6]. His example of the two-arrow Kronecker quiver in dimension $(2, 2)$ has two flag constituents with full support but different

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local systems on a dense open subset, and their singular supports differ [14, §13.8]. Kashiwara and Saito identify the irreducible components with the crystal of the same canonical basis [9, Theorem 5.3.2 and §6.1]. Thus the number of components is known in each dimension vector. The recognition question asks whether the singular-support condition characterizes the flag constituents among all equivariant simple perverse sheaves.

In Dynkin type, the representation space has finitely many orbits, each with connected stabilizer. The stabilizer of a representation is the group of units in its endomorphism algebra, an open subset of a vector space. The simple equivariant perverse sheaves are the intersection cohomology extensions of the constant local systems on those orbits. Gabriel's classification and the canonical-basis count give the same number of orbits and flag constituents [17, Proposition 4.8]. In affine type the regular representations occur in families. Lusztig's description of the flag constituents uses strata of regular representations and local systems obtained by permuting distinct homogeneous summands [11, §6]. In the cyclic case he also characterized the sheaves attached to aperiodic nilpotent orbits by a simultaneous support condition on a sheaf and its Fourier transform [12, §2.13]. Hennecart proved the microlocal converse for affine quivers. His proof uses the stratification by preprojective, regular and preinjective summands and cyclic-quiver models near the nonhomogeneous tubes [7, Theorem 1.1 and §§6–9]. He formulates the recognition statement for all loop-free quivers in [7, Conjecture 1.2].

Webster gives a general reduction to a large framed quiver variety [17, Proposition 4.8]. The nilpotent singular-support condition becomes support on the fiber over the cone point. His proof then invokes a characteristic-cycle independence theorem attributed to Baranovsky and Ginzburg [17, Theorem 4.9]. Hennecart explains how results of Bezrukavnikov and Losev and of Losev supply the characteristic-cycle input in the affine case [7, §1.6]. Baranovsky and Ginzburg subsequently stated the general independence result for finite-dimensional simple modules under derived localization in [2, Corollary 11.4], where the details of the underlying Chern character theorem are deferred to a separate paper. A different general proof by Fang was withdrawn [5].

Let $Q$ be a finite quiver with vertex set $I$ and arrow set $Q_1$. Write $s(a)$ and $t(a)$ for the tail and head of $a \in Q_1$, and assume $s(a) \neq t(a)$. For a dimension vector $\mathbf{v} = (v_i)_{i \in I}$ of nonnegative integers, choose complex vector spaces $V_i$ with $\dim V_i = v_i$. The representation space and its base-change group are

$$E = \bigoplus_{a \in Q_1} \operatorname{Hom}(V_{s(a)}, V_{t(a)}), \qquad G = \prod_{i \in I} \mathrm{GL}(V_i).$$

We take $\mathrm{GL}(0) = \{1\}$. For $g = (g_i)$ and $x = (x_a)$, the coordinate of $gx$ at $a$ is $g_{t(a)} x_a g_{s(a)}^{-1}$. Let $\overline{Q}$ be the doubled quiver, with one additional arrow $a^* : t(a) \to s(a)$ for every $a \in Q_1$. Write $y_a : V_{t(a)} \to V_{s(a)}$ for the map on $a^*$. The pairing $\sum_a \operatorname{Tr}(y_a x_a)$, with trace on $V_{s(a)}$, identifies $y = (y_a)$ with

the cotangent coordinate. Identify the dual of the Lie algebra of $G$ with $\bigoplus_i \operatorname{End}(V_i)$ by trace. The cotangent moment map $\mu_Q : T^*E \to \bigoplus_i \operatorname{End}(V_i)$ is given by

$$(\mu_Q)_i(x,y) = \sum_{t(a)=i} x_a y_a - \sum_{s(a)=i} y_a x_a.$$

Indeed, the infinitesimal action of $(z_i)_{i\in I}$ at $x$ has coordinates $z_{t(a)}x_a - x_a z_{s(a)}$. Pairing with $y$ and cycling the trace gives this formula.

Let $\Lambda_{\mathbf{v}}$ be the subvariety of $\mu_Q^{-1}(0)$ on which all sufficiently long doubled paths act by zero. Here a path acts by the composition of its arrow maps in traversal order. A complete graded flag $L_\bullet$ in $\bigoplus_i V_i$ is an increasing chain $0 = L_0 \subset L_1 \subset \cdots \subset L_{\sum_i v_i} = \bigoplus_i V_i$ of graded subspaces with one-dimensional successive quotients. The path condition is equivalent to the existence of such a flag with each doubled arrow carrying $L_j$ into $L_{j-1}$. To see this, let the term of index $k \geq 0$ of a decreasing filtration be the graded span of images of paths of length $k$, including the identity paths at $k = 0$. These terms decrease, since each path image lies in the image of its terminal subpath of length one less. Every arrow maps the term of index $k$ to that of index $k+1$. A refinement of the filtration to a complete graded flag, read in increasing order, has the asserted property. Conversely a composition of $\sum_i v_i$ arrows strictly lowers through the entire flag.

For each sequence of vertices of the successive quotients, let the corresponding flag variety parametrize all complete graded flags of that type. The incidence variety of pairs $(x, L_\bullet)$ with $x$ preserving the flag is a vector bundle over that flag variety. Its projection to $E$ is proper because it is closed in the product of $E$ with the projective flag variety. A *Lusztig sheaf* is a simple $G$-equivariant perverse sheaf that occurs, up to cohomological shift, as a direct summand of the direct image of the constant complex under one of these projections. Allowing flag quotients of larger dimension at a single vertex gives the same simple constituents. In fact, every refinement within a single-vertex quotient is preserved, since the quiver has no loops. The forgetful map between the incidence varieties is therefore a relative product of full flag varieties. Its fibers are connected, and its direct image contains the constant complex as a summand by the decomposition theorem. All perverse sheaves below are algebraically constructible, of middle perversity, with complex coefficients. For such a sheaf $\mathcal{F}$ we write $SS(\mathcal{F}) \subseteq T^*E$ for its singular support, equivalently the characteristic variety of its regular holonomic module over the sheaf $\mathcal{D}_E$ of algebraic differential operators under Riemann–Hilbert.

**Theorem 1.** *For every finite loop-free quiver $Q$ and dimension vector $\mathbf{v}$, a simple $G$-equivariant perverse sheaf $\mathcal{F}$ on $E$ is a Lusztig sheaf if and only if*

$$SS(\mathcal{F}) \subseteq \Lambda_{\mathbf{v}}.$$

We use the large-framing construction to compare the number of simple sheaves with nilpotent singular support to the number of components of

the central fiber. Etingof and Schedler use zeroth Hochschild homology to bound the number of finite-dimensional simple modules of quantizations [4, Corollary 1.11]. Here the bound comes from a symplectic action of weight one. Its attracting strata reduce to cotangent bundles of smooth projective fixed components. We use the categorical cell decomposition of Bellamy, Dodd, McGerty and Nevins [1] to bound the zeroth Hochschild homology by the sum of the contributions from these components. Each contribution is one-dimensional by a calculation for twisted differential operators. The resulting bound equals the number of Lusztig sheaves.

## 1. Changing the orientation

**Lemma 1.1.** *Reversing one arrow gives a bijection of isomorphism classes of simple equivariant perverse sheaves with singular support in the respective nilpotent varieties. It carries Lusztig sheaves to Lusztig sheaves.*

*Proof.* Fix $a \in Q_1$. The cotangent lift of dilation of the $a$-coordinate is $(x_a, y_a) \mapsto (ux_a, u^{-1}y_a)$ for $u \in \mathbb{C}^*$. It preserves the moment equations. Each doubled path is rescaled by a nonzero scalar, so nilpotence is preserved. Since the acting torus is connected, it preserves each irreducible component of $\Lambda_{\mathbf{v}}$. Each component is conic and Lagrangian [9, Proposition 5.3.3]. For a conic Lagrangian in a cotangent bundle, contraction of the symplectic form with the fiber Euler field gives the tautological one-form, up to the sign convention for the symplectic form. The Euler field is tangent to the conic subvariety, so this one-form vanishes on its smooth locus. At a general smooth point where projection to the base is smooth onto its image, the cotangent coordinate thus annihilates the tangent space to that image. Equality of dimensions then identifies the component with the closure of this conormal bundle. Its projection is closed, since for a closed fiberwise conic subset the projection equals its intersection with the zero section, identified with $E$. It is invariant under dilation of the $a$-coordinate.

A perverse sheaf with singular support in $\Lambda_{\mathbf{v}}$ has Lagrangian singular support, hence its singular support is a union of these components. Simplicity for a $G$-equivariant perverse sheaf implies simplicity of the underlying perverse sheaf. On an invariant smooth dense stratum of the support, an equivariant local system is a local system on the quotient stack, and the fundamental group of the stratum surjects onto that of the quotient stack because $G$ is connected. Surjectivity also makes the forgetful functor on equivariant local systems fully faithful. On a common invariant smooth dense stratum of the support, an isomorphism of the underlying local systems therefore lifts to an equivariant isomorphism, which extends by intermediate extension. Thus forgetting equivariance distinguishes isomorphism classes of simple equivariant perverse sheaves. Now [7, Corollary D.2] gives monodromicity for the $a$-dilation, meaning that the cohomology sheaves restrict to local systems on its orbits.

Use the space of the other arrow maps as a base and the $a$-arrow space as the fiber of a trivial vector bundle. The relative Fourier–Sato transform is the pullback to the fiber product with the dual bundle, restricted to the closed half-space where the real part of the pairing is nonpositive, followed by direct image with proper supports to the dual bundle. We shift it by the complex rank of the bundle. It is a $G$-equivariant perverse equivalence on monodromic sheaves [7, Appendix D]. On singular supports its action is $(x_a, y_a) \mapsto (y_a, -x_a)$, with the other coordinates unchanged. The moment contributions at $t(a)$ and $s(a)$ after reversal are $x_a y_a$ and $-y_a x_a$, respectively. The correspondence also preserves vanishing of long paths. Thus the transformed sheaf has singular support in the new nilpotent variety. Any simple equivariant perverse subobject has singular support contained in that of the transform by additivity and effectivity of characteristic cycles, so it too is monodromic. Applying the inverse perverse equivalence shows that the transform is simple. Applying the same argument to the inverse gives the bijection.

Over a flag, the $a$-arrow maps preserving the flag have matrix entries only from later source positions to earlier target positions, in bases ordered by the graded flag. The two endpoints are distinct, so the source and target positions cannot coincide. The reverse entries paired with these entries by trace are exactly the entries forbidden by flag preservation for the reverse arrow. Thus the trace annihilator is the subspace of flag-preserving reverse maps. For a complex subspace of dimension $d$ in a fiber of rank $r$, the Fourier transform of its constant complex extended by zero is the constant complex on its annihilator, shifted by $r - 2d$. Indeed, in the half-space description the fiber over a point of the annihilator is the full real $2d$-dimensional vector space, while elsewhere it is a closed real half-space in that vector space, whose compactly supported cohomology is zero. The relative formula holds for the subbundles over the flag variety. Fourier transformation commutes with the proper projection forgetting the flag by base change and composition for direct images with proper supports. It therefore carries the flag complexes to the flag complexes for the reversed orientation, up to shifts. The equivalence on simple constituents gives the last assertion. □

Choose a total order on the vertices and reverse each arrow pointing from a larger to a smaller vertex. Successive applications of Lemma 1.1 reduce Theorem 1 to the resulting acyclic orientation.

## 2. Hochschild homology of twisted differential operators

Let $F$ be a smooth complex variety, with structure sheaf $\mathcal{O}_F$ and tangent sheaf $T_F$. A Picard algebroid is a Lie algebroid $\mathcal{P}$ with an exact sequence $0 \to \mathcal{O}_F \to \mathcal{P} \to T_F \to 0$ of $\mathcal{O}_F$-modules. The projection to $T_F$ is the anchor, the kernel is abelian, and the bracket with a kernel section $f$ is the derivative of $f$ by the anchor. Its enveloping algebra, with the section 1 of the kernel identified with the algebra unit, is a sheaf of twisted differential operators.

Write $\mathcal{D}_F^\tau$ for this sheaf, where $\tau$ records the class of the Picard algebroid. Its order filtration gives degree zero to functions and degree one to lifts of vector fields. The Poincaré–Birkhoff–Witt theorem identifies the associated graded sheaf with $\mathrm{Sym}_{\mathcal{O}_F} T_F$. We write $\mathrm{Perf}(\mathcal{D}_F^\tau)$ for the dg category of perfect left modules, that is, complexes locally quasi-isomorphic to bounded complexes of finitely generated projective modules over $\mathcal{D}_F^\tau$. Write $HH(\mathcal{C})$ for the ordinary Hochschild chain complex, with tensor products over $\mathbb{C}$, of a differential graded category $\mathcal{C}$, and $HH_k(\mathcal{C})$ for its homology in degree $k$. For an algebra $D$, the same notation denotes Hochschild homology of $D$, identified by Morita invariance with that of $\mathrm{Perf}(D)$.

**Lemma 2.1.** *If $F$ is smooth, connected, and projective of dimension $n$, then*

$$HH_0\big(\mathrm{Perf}(\mathcal{D}_F^\tau)\big) \cong H^{2n}_{\mathrm{dR}}(F) \cong \mathbb{C}$$

*for every $\mathcal{D}_F^\tau$. Here $H^j_{\mathrm{dR}}$ denotes the degree-$j$ hypercohomology of the algebraic de Rham complex.*

*Proof.* Let $U \subset F$ be affine and set $D = \Gamma(U, \mathcal{D}_F^\tau)$. Quasi-coherent left $\mathcal{D}_F^\tau|_U$-modules are equivalent by sections to left $D$-modules. The inverse localizes a $D$-module as an $\mathcal{O}(U)$-module. For $f \in \mathcal{O}(U)$, a module element $m$, and $r \geq 0$, a lift $v$ of a vector field acts on the section $m/f^r$ on the principal open $f \neq 0$ by $v(m/f^r) = v(m)/f^r - rv(f)m/f^{r+1}$, where $v(f)$ means the derivative of $f$ by the image of $v$ in $T_F$. These actions satisfy the Picard-algebroid relations and glue, since the relations hold before localizing and the displayed rule determines their extensions. The order filtration $F_pD$ is exhaustive, and we put $F_pD = 0$ for $p < 0$. Affine acyclicity and the Poincaré–Birkhoff–Witt theorem give $\mathrm{gr}\, D = \mathcal{O}(T^*U)$. The symbol of the commutator in order one below the sum of the two orders is the cotangent Poisson bracket.

On $C_k(D) = D^{\otimes_{\mathbb{C}}(k+1)}$, use the increasing filtration spanned in degree $p$ by tensors whose factors have total order at most $p$. With $q = k - p$, Hochschild–Kostant–Rosenberg identifies $E^1_{p,q}$ with the regular $k$-forms on $T^*U$ of weight $p$ under cotangent fiber dilation. Let $\varpi$ denote its Poisson bivector, and let $\iota_\varpi$ be insertion of the bivector in a differential form, and $d$ the de Rham differential. The first differential is $\delta = \iota_\varpi d - d\iota_\varpi$. Choose filtered lifts $\widetilde{f}_i \in D$ of homogeneous functions $f_0, \ldots, f_k$ on $T^*U$. Write $S_k$ for the symmetric group on $1, \ldots, k$, and sgn for the permutation sign. The antisymmetrized chain is

$$\sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \widetilde{f}_0 \otimes \widetilde{f}_{\sigma(1)} \otimes \cdots \otimes \widetilde{f}_{\sigma(k)},$$

Under the Hochschild–Kostant–Rosenberg map, normalized by dividing the wedge of differentials of a tensor by $k!$, this chain represents $f_0\, df_1 \wedge \cdots \wedge df_k$. In its Hochschild boundary, multiplication terms cancel in pairs up to commutators. The terms pairing the coefficient with factor $i$ have sign $(-1)^{i-1}$, and those pairing factors $i < j$ have sign $(-1)^{i+j}$ with the bracket

in the first differential slot. Taking the symbols of the commutators gives

$$\sum_{i=1}^{k}(-1)^{i-1}\{f_0,f_i\}\,df_1\wedge\cdots\wedge\widehat{df_i}\wedge\cdots\wedge df_k \\ +\sum_{1\le i<j\le k}(-1)^{i+j}f_0\,d\{f_i,f_j\}\wedge\bigwedge_{\substack{1\le l\le k\\ l\neq i,j}}df_l,$$

where a hat omits a factor and the wedge in the second sum is in increasing order. Expanding the two contractions in $\delta$ gives the same expression [18, Proposition 9].

Write $\omega$ for the cotangent symplectic form. For a decomposable $k$-form $\alpha_1\wedge\cdots\wedge\alpha_k$, its symplectic star is

$$\iota_{\varpi^\sharp\alpha_k}\cdots\iota_{\varpi^\sharp\alpha_1}\big(\omega^n/n!\big),\qquad \beta(\varpi^\sharp\alpha)=\varpi(\alpha,\beta).$$

It is an isomorphism from $k$-forms to $(2n-k)$-forms. In local cotangent coordinates, let $\alpha$ be a wedge of coordinate differentials. Then $\delta(f\alpha)=\iota_{\varpi^\sharp df}\alpha$. Moving $df$ through the $k$ contractions in the star formula shows $\star\delta(f\alpha)=(-1)^k d\star(f\alpha)$. These identities apply in étale cotangent coordinates and hence hold on $T^*U$. The volume form has weight $n$ and the bivector has weight $-1$, so the star sends a weight-$p$ form to weight $p+n-k$. If $e$ is the vector field generating fiber dilations, Cartan's identity $d\iota_e+\iota_e d=\mathcal{L}_e$ acts on weight $w$ by $w$ times the identity. Thus the de Rham complexes of nonzero weight are contractible. The forms of weight zero are precisely the pullbacks of forms on $U$ because a fiber coordinate and its differential each have weight one. Since $p+n-k=n-q$, this gives

$$E^2_{p,q}=\begin{cases}H^{2n-p-q}_{\mathrm{dR}}(U), & q=n,\\ 0, & q\neq n.\end{cases}$$

For $r\ge 2$, the differential of bidegree $(-r,r-1)$ therefore vanishes. A representative at filtration $p$ surviving to page $r>p$ is a cycle, since its boundary has filtration at most $p-r$. Every chain, including any primitive of a boundary, belongs to some finite filtration level. Hence this stable page computes the associated graded of the homology for the exhaustive filtration. There is only one possible filtration degree in each homological degree, and $H^j_{\mathrm{dR}}(U)=0$ for $j>n$ by affine acyclicity of differential forms. Thus

$$\begin{aligned}&HH_k(D)\cong H^{2n-k}_{\mathrm{dR}}(U) && (n\le k\le 2n),\\ &HH_k(D)=0 && (k<n \text{ or } k>2n).\end{aligned}\tag{1}$$

All the maps in this calculation commute with restriction between affine opens, so the identifications in (1) do as well.

For descent, write $D_{\mathrm{qc}}(\mathcal{D}^\tau_F|_V)$ for the derived category of left modules on an open $V\subset F$ whose cohomology sheaves are quasi-coherent over $\mathcal{O}_V$. For a quasi-compact open inclusion $j:V'\hookrightarrow V$, restriction has fully faithful right adjoint $\mathbf{R}j_*$. Indeed, affine covers of the inverse images of affine opens

under $j$ compute the underlying direct image and preserve quasi-coherence, and $j^*\mathbf{R}j_*$ is the identity. Thus restriction is a localization with kernel the complexes supported on $V \setminus V'$. The forgetful functor to $D_{\mathrm{qc}}(\mathcal{O}_V)$ preserves coproducts, is conservative, and has left adjoint $\mathcal{D}_F^\tau|_V \otimes^{\mathbf{L}}_{\mathcal{O}_V} (-)$. By adjunction, induction of perfect generators for $D_{\mathrm{qc}}(\mathcal{O}_V)$ gives compact generators for $D_{\mathrm{qc}}(\mathcal{D}_F^\tau|_V)$. Induction of perfect generators with support in $V \setminus V'$ likewise generates the kernel and consists of objects compact in the ambient category [16]. The induced objects are locally perfect, and local perfectness is preserved by finite cones and retracts. Conversely, a locally perfect complex is compact by computing its derived Hom using a finite affine cover. Thus restriction of the perfect categories gives an exact dg sequence up to idempotent completion.

If $V = V_1 \cup V_2$, the kernels of restriction from $V$ to $V_1$ and from $V_2$ to $V_1 \cap V_2$ identify. Complexes supported on $V \setminus V_1$ extend uniquely from $V_2$, since they vanish on $V_1 \cap V_2$. The compact objects of the kernels identify as well. Applying Hochschild localization [10, Theorem 2.4(c)] to these two dg sequences gives a homotopy Cartesian square for Hochschild complexes of the four opens. Iterating this square for a finite open cover gives descent by the finite alternating Čech diagram of its nonempty intersections.

Embed $F$ in projective space. Choose $n+1$ hyperplanes with no common intersection on $F$, by successively avoiding all components of the preceding intersection. Their complements $U_0, \dots, U_n$ cover $F$. Each intersection is affine, since it is the principal projective open where the product of the corresponding linear forms is nonzero. In the Hochschild Čech spectral sequence the first page is

$$E_1^{a,-b} = \bigoplus_{0 \le i_0 < \cdots < i_a \le n} HH_b\big(\Gamma(U_{i_0} \cap \cdots \cap U_{i_a}, \mathcal{D}_F^\tau)\big).$$

The differential on this page is the alternating restriction map, and the total homological degree is $b - a$. By (1), $0 \le a \le n$ and $n \le b \le 2n$. The only term of degree zero has $a = b = n$. A differential on page $r \ge 2$ leaving it would have Čech degree $n + r$. An incoming differential would start in local homological degree $n - r + 1 < n$. Therefore the degree-zero Hochschild group is the degree-$n$ cohomology of the row of local $HH_n$'s.

The de Rham spectral sequence for this cover has local terms $H^c_{\mathrm{dR}}(U_{i_0} \cap \cdots \cap U_{i_a})$ in total cohomological degree $a + c$, again with alternating restriction. Here $0 \le c \le n$ since the intersections are affine. The term with $a = c = n$ is the only term in total degree $2n$. After the first differential it survives because both indices are bounded above by $n$. The restriction-compatible isomorphisms (1) identify the row of local $HH_n$'s with the row of local $H^n_{\mathrm{dR}}$'s, giving the first isomorphism in the lemma. De Rham comparison and the complex orientation of the connected compact manifold $F(\mathbb{C})$ give the second. □

## 3. Simple modules supported on the central fiber

Let $\pi : X \to X_0$ be a projective symplectic resolution, with $X_0$ affine. Suppose a $\mathbb{C}^*$-action makes $\pi$ equivariant, contracts $X_0$ to $0 \in X_0$ as $t \to 0$, and satisfies $m_t^*\omega = t\omega$ for the symplectic form $\omega$ on $X$, where $m_t(x) = t \cdot x$. Put $Y = \pi^{-1}(0)$, viewed with its reduced structure. A quantization here is a flat $\hbar$-adically complete sheaf of $\mathbb{C}[[\hbar]]$-algebras $\mathcal{A}$, with $\mathcal{A}/\hbar\mathcal{A} = \mathcal{O}_X$ and commutator divided by $\hbar$ reducing to the Poisson bracket. We use the algebraic equivariance of [1, §3], for the pullback action on functions, with $\hbar$ of weight one and rational actions modulo $\hbar^k$ for every $k \geq 1$. Write $\mathcal{W} = \mathcal{A}[\hbar^{-1}]$. A good module $M$ is an equivariant left $\mathcal{W}$-module with a coherent equivariant $\mathcal{A}$-lattice $P$, meaning $P$ is complete, $P \subset M$, and $P[\hbar^{-1}] = M$. Its support is the support of $P/\hbar P$. On an open set where this quotient vanishes, completeness implies $P = 0$ and hence $M = 0$. Any lattice in $M$ vanishes there, so the support is independent of the lattice.

**Theorem 3.1.** *In the setting above, suppose $H^j(X, \mathcal{O}_X) = 0$ for $j > 0$ and $X$ has a $\mathbb{C}^*$-linearized relatively ample line bundle. Every good module supported on $Y$ has finite length. The number of isomorphism classes of simple good modules supported on $Y$ is at most $\#\operatorname{Irr}(Y)$, the number of irreducible components of $Y$.*

*Proof.* For $x \in X$, its projected orbit extends across $0 \in \mathbb{A}^1$ with value $0 \in X_0$. The valuative criterion for $\pi$ extends the orbit of $x$ across 0, and uniqueness makes the limit fixed. The connected components $F_i$ of $X^{\mathbb{C}^*}$ are smooth and projective, since they lie in $Y$. Each $F_i$ is irreducible. At a point of $F_i$, weight $a$ of the tangent representation pairs nondegenerately with weight $1 - a$. Thus weight zero, the tangent space to $F_i$, pairs with weight one. The negative weights pair with weights at least two. The rank of the negative tangent bundle at $F_i$ is consequently $\frac{1}{2}\dim X - \dim F_i$.

The Białynicki–Birula strata for the zero limit are $C_i^+ = \{x : \lim_{t\to 0} t\cdot x \in F_i\}$. They are filtrable by closed unions of strata [1, §2]. The opposite strata $C_i^- = \{x : \lim_{t\to\infty} t \cdot x \in F_i\}$ are affine bundles over $F_i$ of rank $\frac{1}{2}\dim X - \dim F_i$. For the opposite action this follows by applying the projective Białynicki–Birula theorem in a smooth equivariant projective compactification. Such a compactification is obtained from an equivariant projective embedding using a power of the linearized ample line bundle and resolving the boundary equivariantly. Each proper $F_i$ is a full fixed component in the compactification, and the invariant closed boundary cannot contain a point whose limit lies in $F_i$. The opposite stratum is therefore already in $X$. If $x \in X$ admits an infinity limit, then $t^d f(\pi(x))$ has a finite limit for each homogeneous function $f$ of positive weight $d$ on $X_0$. Hence $f(\pi(x)) = 0$, so $\pi(x) = 0$. Conversely, every point of $Y$ has an infinity limit since $Y$ is proper. Hence $Y$ is the disjoint union of the $C_i^-$. These are irreducible of the common dimension $\frac{1}{2}\dim X$. Their closures are the irreducible components of $Y$. Two distinct strata cannot be dense in the same irreducible closure

because each is open in its own closure. In particular,

$$\#\operatorname{Irr}(Y) = \#\{F_i\}.$$

For the cell decomposition of [1, §2.2], use the point action $x \mapsto m_{t^{-1}}(x)$, so its attracting limit at infinity is our limit at zero. On sections and forms the corresponding action is pullback along the inverse point map, hence along $m_t$. The positive form weight in [1, Definition 1.1] is the weight for this representation on forms, as in the formula in their §2.2. For tangent weights $a, b$ of the inverse point action, the value of $t \cdot \omega$ on the two vectors is $t^{-a-b}$ times that of $\omega$. Thus $a + b = -1$ when they pair nontrivially. The weight of $\hbar$ in the section algebra is $+1$, in agreement with the Poisson bracket having degree $-1$.

Let $S_i$ be the coisotropic reduction of $C_i^+$, so the restricted two-form is the pullback of the symplectic form on $S_i$. In this point-action convention, the tangent weights of $S_i$ along $F_i$ are the weights in $[-1, 0]$ of $TC_i^+|_{F_i}$, by [1, Lemma 2.5]. They have total dimension $2 \dim F_i$ since weights 0 and $-1$ pair nondegenerately. By [1, Theorem 2.21 and Remark 2.26], the reduced variety is $T^*F_i$. Its good quantized modules identify with coherent modules over a sheaf of twisted differential operators $\mathcal{D}_{F_i}^{\tau_i}$ [1, Proposition 4.33 and Lemma 4.34].

Let $U$ be the open union of the remaining attracting strata, with $C_i^+$ closed in $U$, and put $U' = U \setminus C_i^+$. The limit of a point of $U$ remains in its stratum's fixed component, so the attracting action is still defined on $U$. The good modules on the quantization of $S_i$ are equivalent to good modules on $U$ supported on $C_i^+$ [1, Theorem 4.28]. We compare the bounded derived categories. On an adapted invariant affine chart of $U$, write $A$ for the sections of $\mathcal{A}$, $J$ for the left ideal quantizing the stratum, and $B$ for the sections of the reduced quantization on the corresponding affine chart of $S_i$, before inverting $\hbar$. The tensor functor of the abelian equivalence followed by inclusion acts on a good module $N$ on the reduced chart by

$$N \longmapsto (A/J)[\hbar^{-1}] \otimes_{B[\hbar^{-1}]} N.$$

Here $(A/J)[\hbar^{-1}]$ is the left $A[\hbar^{-1}]$, right $B[\hbar^{-1}]$ bimodule of [1, Theorem 4.17 and Corollary 4.18]. The algebraic right adjoint is Hom from this bimodule as a left module. The unit identifies

$$\operatorname{Hom}_{A[\hbar^{-1}]}\big((A/J)[\hbar^{-1}], (A/J)[\hbar^{-1}] \otimes_{B[\hbar^{-1}]} N\big) \cong N,$$

and the same expression with $\operatorname{Ext}^j$ in place of Hom vanishes for $j > 0$, by [1, Claim 4.23]. One obtains these localized statements from the lattice statements using a finite projective resolution of $A/J$ and exactness of central localization. Finite global dimension and finite homogeneous generators on the affine charts are supplied by [1, Lemmas 3.2 and 3.21]. The tensor functor is exact by faithful flatness of $A/J$ over $B$ [1, Theorem 4.17]. To compute maps between two images, resolve the first input by finitely generated projectives over $B[\hbar^{-1}]$. The terms after tensoring are direct summands

of finite sums of the bimodule, so the Ext vanishing makes them acyclic for Hom to the second image. Applying the displayed unit term by term gives the derived Hom comparison. For equivariant modules one uses resolutions by finite sums of character shifts of the free modules and their projective summands. One takes weight zero in the Hom complexes. This is exact on rational representations and on the pro-rational lattices used here. Weight projections on the rational quotients modulo $\hbar^j$ commute with transition maps, so they define projections on the complete lattices. Applying such a projection to a lift of a weight vector produces a lift of the same weight. A weight vector in a localized module is first multiplied by a power of $\hbar$ to put it in a lattice.

The charts in [1, Lemmas 4.25–4.27] have the following compatibility. Their intersections are affine since $U$ is separated. For the adapted cover, Lemma 4.25 identifies the part of each chart in $C_i^+$ with the inverse image of its intersection with $F_i$ under the limit map. The same equality holds for intersections of charts. The corresponding charts of $S_i$ are cotangent bundles over these affine opens of $F_i$, including on intersections. The bimodules and their Hom adjunctions restrict to those on the intersections, since they come from the sheaf quotient $\mathcal{A}/\mathcal{J}$, where $\mathcal{J}$ is the glued left ideal of [1, Lemma 4.26]. For bounded good complexes the global derived Hom is computed by the finite alternating Čech total complex of local derived Homs on these charts. Indeed, on affine charts a finite projective resolution computes the internal derived Hom. Each term of the resulting Hom complex is a direct summand of a finite sum of copies of the target lattice and is acyclic for affine sections. Central localization preserves the calculation [1, Proposition 3.11, Corollary 3.13, and Lemma 5.7]. The equivariant calculation takes weight zero on the same invariant chart intersections. Charts disjoint from $C_i^+$ contribute zero to derived Hom between supported complexes. Thus the local comparisons give a quasi-isomorphism on the Čech total complexes. The functor is fully faithful on bounded derived categories. Every bounded complex with supported cohomology belongs to its image by the finite truncation triangles and the abelian equivalence. This is the single-stratum case of [1, Lemma 5.3].

Let $j : U' \hookrightarrow U$ be the open inclusion. Its restriction $j^*$ on good modules is exact and has a fully faithful right adjoint $j_*$ on Ind-categories [1, Theorem 3.27 and Corollary 3.28]. A bounded good complex on $U'$ lifts to a bounded good subcomplex of its termwise image under $j_*$ by choosing good extensions of the terms and including their differential images [1, Lemma 5.6]. A roof on $U'$ between restrictions of two bounded good complexes on $U$ lifts by this construction on its middle complex. For each endpoint take the sum of the image of the original endpoint and the image from the lifted middle, inside $j_*$ of the restricted endpoint. The endpoint maps are isomorphisms after restriction, and the lifted denominator has supported cone.

To check faithfulness, if the restriction of a roof is zero in the derived category, refine its restricted middle by a bounded quasi-isomorphism on which the restricted numerator is nullhomotopic. Lift its source as a bounded good subcomplex of its termwise right adjoint image. Enlarge the image of the original middle in the right adjoint image of its restriction by the image of this lift, then enlarge the endpoints by the images of the enlarged middle. Each enlargement restricts to an isomorphism, and the lifted refinement has supported cone. If $f$ is the resulting numerator with domain $K$, its restriction factors through $\mathrm{Cone}(\mathrm{id}_{j^*K})$ by nullhomotopy. The map from that cone to the restricted endpoint adjoints to a chain map from $\mathrm{Cone}(\mathrm{id}_K)$ to $j_*$ of the restricted endpoint. Include its image in the endpoint extension. The map $f$ now factors through the contractible cone. This last enlargement is again an isomorphism after restriction. Thus the bounded derived quotient by complexes with supported cohomology is the bounded good derived category on $U'$.

Write $\mathrm{Perf}(\mathcal{W}|_U)$ for the dg category of perfect equivariant quantization complexes on $U$. These are the bounded good complexes on each open [1, Lemma 5.5]. The quotient and the derived supported comparison give a dg exact sequence, up to idempotent completion,

$$\mathrm{Perf}(\mathcal{D}^{\tau_i}_{F_i}) \longrightarrow \mathrm{Perf}(\mathcal{W}|_U) \longrightarrow \mathrm{Perf}(\mathcal{W}|_{U'}).$$

Apply the Hochschild long exact sequence [10, Theorem 1.5(c)] and remove the closed strata successively. Lemma 2.1 yields

$$\dim_{\mathbb{C}} HH_0(\mathrm{Perf}(\mathcal{W})) \;\leq\; \sum_i \dim_{\mathbb{C}} HH_0(\mathrm{Perf}(\mathcal{D}^{\tau_i}_{F_i})) \;=\; \#\,\mathrm{Irr}(Y). \tag{2}$$

Replace a linearized relatively ample line bundle $L$ by a relatively very ample power. The quantized line-bundle bimodule of [3, Proposition 5.2] gives a Morita equivalence from $\mathcal{W}$ to the quantization whose period is translated by $\hbar k c_1(L)$, for a positive integer $k$. Here $c_1(L)$ is the de Rham first Chern class of $L$, and the period is the formal de Rham class classifying the quantization as in [3, §3.1 and Proposition 3.5]. The bimodule lattice reduces modulo $\hbar$ to $L^k$ with the ordinary left and right $\mathcal{O}_X$-actions. Tensoring a module lattice by it therefore tensors the reduction with $L^k$, preserving support. For all sufficiently large $k$, invariant global sections give an abelian equivalence from good modules for the translated quantization $\mathcal{W}'$ to finitely generated left modules over

$$R = \Gamma(X, \mathcal{W}')^{\mathbb{C}^*}$$

by [3, Corollary 5.17]. Let $\mathcal{A}'$ be the translated quantization lattice. For $j \in \mathbb{Z}$, filter $R$ by the subspaces $\Gamma(X, \hbar^{-j}\mathcal{A}')^{\mathbb{C}^*}$. The quotient in order $j$ is the weight-$j$ part of $\Gamma(X, \mathcal{O}_X)$ by coherent-cohomology vanishing and the invariant section calculation of [3, §3.3]. Thus the associated graded ring is $\mathcal{O}(X_0)$, since a proper birational map to a normal variety satisfies $\pi_*\mathcal{O}_X = \mathcal{O}_{X_0}$. Negative orders vanish because the lattice is separated and

the graded ring has no negative weights. Consequently $R$ is Noetherian by lifting generators of graded ideals.

Every $R$-module is a filtered union of finitely generated submodules, and those submodules are finitely presented by Noetherianity. Hence the abelian equivalence on good modules extends to an equivalence of the Ind-category with all $R$-modules. Its derived equivalence identifies $\mathrm{Perf}(\mathcal{W}')$ with $\mathrm{Perf}(R)$. In one direction, perfect quantization complexes are compact by [1, Lemma 5.7]. In the other direction, a compact $R$-complex is represented by a bounded complex of finitely generated projectives, whose inverse image is bounded and good, hence perfect by [1, Lemma 5.5]. Thus $HH_0(R) = HH_0(\mathrm{Perf}(\mathcal{W}'))$, and (2) applies to $\mathcal{W}'$.

Let $M$ be a good $\mathcal{W}'$-module supported on $Y$ and choose an equivariant lattice $P \subset M$. Write $H = \Gamma(X, P)$. Multiplication by $\hbar^j$ identifies $P$ with $\hbar^j P$, so the sections exact sequence gives

$$H/\hbar^j H \hookrightarrow \Gamma(X, P/\hbar^j P) \qquad (j \geq 1).$$

The target is finite-dimensional and rational for $\mathbb{C}^*$. In fact $P/\hbar^j P$ has successive quotients $\hbar^l P/\hbar^{l+1} P$ for $0 \leq l < j$, each isomorphic to the coherent sheaf $P/\hbar P$ with a character shift. That sheaf has proper support. The finite-dimensionality of its sections and left exactness give the assertion by induction on $j$. The rational actions are those of the equivariant quotients of the lattice.

Taking sections commutes with the inverse limit defining $P$. In particular a series with its $j$-th term in $\hbar^j H$ converges in $H$. Factoring out $\hbar^k$ from its tail shows that this tail is in $\hbar^k H$. Also $\bigcap_j \hbar^j H = 0$ by separatedness of $P$. Choose a weight basis of $H/\hbar H$. The surjections $H/\hbar^{j+1} H \to H/\hbar^j H$ are surjective on each weight, so we can lift this basis to homogeneous elements of $H$, using the completeness just proved. Successive subtraction of their linear combinations and division by $\hbar$ express every element of $H$ as a convergent linear combination of these lifts over $\mathbb{C}[[\hbar]]$. If a nonzero relation held, divide by the smallest $\hbar$-valuation of its coefficients, using torsion-freeness, and reduce modulo $\hbar$. At least one coefficient then survives, contradicting independence of the chosen basis. Thus the lifts form a homogeneous $\mathbb{C}[[\hbar]]$-basis of $H$.

Clearing denominators on a finite open cover gives $\Gamma(X, M) = H[\hbar^{-1}]$. A homogeneous basis element of weight $l$ becomes invariant upon multiplication by $\hbar^{-l}$. This produces an invariant basis over $\mathbb{C}((\hbar))$. The invariant scalars of this field are $\mathbb{C}$, since a Laurent series fixed by $\hbar \mapsto t\hbar$ has only its constant coefficient. Consequently $\Gamma(X, M)^{\mathbb{C}^*}$ is finite-dimensional. Every strict chain of its submodules has length bounded by its vector-space dimension. Abelian localization and the line-bundle Morita equivalence therefore give finite length for each good module supported on $Y$.

We use the character bound of [4]. Abelian localization takes pairwise nonisomorphic simple objects supported on $Y$ to pairwise nonisomorphic finite-dimensional simple $R$-modules. If $V_1, \ldots, V_s$ are any such modules,

their characters $a \mapsto \mathrm{Tr}(a|V_j)$ vanish on the linear span of commutators, and hence are linear forms on $HH_0(R) = R/[R,R]$. By the density theorem, the quotient by the annihilator of $V_j$ is $\mathrm{End}_{\mathbb{C}}(V_j)$. The annihilators are distinct maximal two-sided ideals, since a full matrix algebra has a unique simple module. For a fixed index $j$, each other annihilator contains an element congruent to 1 modulo the annihilator of $V_j$, by pairwise comaximality. The product acts as the identity on $V_j$ and as zero on every other module, since the ideals are two-sided. Evaluation on these products gives linear independence of the characters. Hence

$$s \leq \dim HH_0(R) \leq \#\,\mathrm{Irr}(Y). \qquad \square$$

## 4. The framed symplectic resolution

Assume that $Q$ is acyclic. At each vertex $i$ with $v_i > 0$ choose a complex vector space $W_i$ of dimension $w_i \geq 2v_i + 1$, and take $W_i = 0$ when $v_i = 0$. Put

$$E^w = E \oplus \bigoplus_{i \in I} \mathrm{Hom}(V_i, W_i).$$

The base coordinates of $T^*E^w$ are $(x, q)$, where $q_i : V_i \to W_i$. Its dual coordinates are $(y, p)$, where $p_i : W_i \to V_i$ is trace dual to $q_i$. The action of $g \in G$ sends $q_i$ to $q_i g_i^{-1}$ and $p_i$ to $g_i p_i$, and acts on $(x, y)$ by changes of bases at the vertices. Let $\mathfrak{g} = \bigoplus_{i \in I} \mathrm{End}(V_i)$ be the Lie algebra of $G$. Identify $\mathfrak{g}^*$ with this direct sum via the trace pairings. The cotangent moment map $\mu : T^*E^w \to \mathfrak{g}^*$ then has components

$$\mu_i(x, y, q, p) = (\mu_Q)_i(x, y) - p_i q_i.$$

The framing term follows by pairing the infinitesimal change $-q_i\zeta_i$, for $\zeta_i \in \mathrm{End}(V_i)$, with $p_i$.

Call $(x, y, q, p)$ *costable* if the only graded subspace preserved by all doubled arrows and contained in $\bigoplus_i \ker q_i$ is zero. Write $P$ for the costable locus in $T^*E^w$. Let

$$X = (P \cap \mu^{-1}(0))/G, \qquad X_0 = \mathrm{Spec}\, \mathbb{C}[\mu^{-1}(0)]^G,$$

and let $\pi : X \to X_0$ be the map to the affine quotient.

**Lemma 4.1.** *The map $\mu$ is flat, and $\pi$ is a projective symplectic resolution. The action*

$$t \cdot (x, y, q, p) = (x, ty, q, tp), \qquad t \in \mathbb{C}^*,$$

*contracts $X_0$ to the image of the origin and has symplectic weight one on $X$. Also $H^j(X, \mathcal{O}_X) = 0$ for $j > 0$, and $X$ has a relatively ample line bundle linearized for this action.*

*Proof.* For $n \geq 0$, intersect at each vertex $i$ the kernels of the maps $q_j b : V_i \to W_j$ over doubled paths $b$ from $i$ to $j$ of length at most $n$. Here $b$ acts by composition of its arrows, and the length-zero path acts by the identity of $V_i$. The direct sums of these kernels form a decreasing sequence. The term for $n + 1$ consists of the vectors in the term for $n$ whose images under

every doubled arrow lie in the term for $n$. Equality of consecutive terms thus implies invariance under every doubled arrow. The sequence stabilizes by $n = \sum_i v_i$ and its limit is the largest invariant graded subspace killed by all $q_i$.

Use the character $\chi(g) = \prod_i \det(g_i)^{-1}$, with semistability defined by nonvanishing of a polynomial $f$ satisfying $f(gz) = \chi(g)^m f(z)$ for some integer $m > 0$. Costability says that the matrix obtained by stacking the finitely many maps $q_j b$ at each $i$ has full column rank. Choosing a nonzero maximal minor at each vertex gives, by taking their product, a nonvanishing semi-invariant of weight $\chi$. A determinant on a zero-dimensional space is 1. Conversely, let $L = \bigoplus_i L_i$ be a nonzero invariant subspace killed by $q$. The one-parameter subgroup acting by $t$ on $L$ and by 1 on a graded complement has a limit on the point as $t \to 0$, since arrow blocks from $L$ to the complement and framing blocks of $q$ on $L$ vanish, and the blocks of $p$ have weights 0 or 1. Its value under $\chi$ is $t^{-\dim L}$, so any positive-weight semi-invariant vanishes at the point. This identifies $P$ with the semistable open. The same criterion applies to the zero fiber, since restriction of polynomials to this invariant closed subscheme is surjective on each semi-invariant weight by reductivity.

A stabilizer element $g$ has $\operatorname{im}(g-1)$ invariant and killed by $q$, hence $g = 1$. No orbit in $P$ has a smaller-dimensional orbit in its closure inside $P$, so semistability equals stability. In addition, the open subsets obtained by choosing the nonvanishing minors above trivialize the quotient as a principal $G$-bundle in the Zariski topology. On such an open the selected rows at $i$ give an isomorphism $V_i \to \mathbb{C}^{v_i}$ that is multiplied on the right by $g_i^{-1}$ under the action. After choosing an identification $V_i \cong \mathbb{C}^{v_i}$, every orbit meets the locus where all these isomorphisms are identities in a unique point. The change of bases carrying a point to this locus is regular because the determinants are invertible on the open. The geometric invariant theory quotient $P/G$ is separated, so $G \times P \cong P \times_{P/G} P$ is closed in $P \times P$. In particular the action on $P$ is proper.

Fix a value of $\mu$ and fix the doubled-arrow coordinates. At vertex $i$, the remaining equation is $p_i q_i = B_i$, with $B_i \in \operatorname{End}(V_i)$ determined by the fixed data. If $b = \operatorname{rank} B_i$ and $r = \operatorname{rank} q_i$, then $r \geq b$ and $\ker q_i \subseteq \ker B_i$. The possible kernels form a Grassmannian of dimension $(v_i - r)(r - b)$. Given the kernel, injections $V_i/\ker q_i \to W_i$ form an open set of dimension $w_i r$. The allowed maps $p_i$ extend a fixed map on $\operatorname{im} q_i$, so they form an affine space of dimension $v_i(w_i - r)$. The deficit from the full-rank dimension $2v_i w_i - v_i^2$ is therefore

$$(2v_i w_i - v_i^2) - \big((v_i - r)(r - b) + w_i r + v_i(w_i - r)\big) = (v_i - r)(w_i - v_i - r + b),$$

which is at least 2 when $r < v_i$. In the all-injective open, over the space of doubled arrows and injective framings, the equations for $p_i$ form a torsor under the vector bundle of maps $W_i/\operatorname{im} q_i \to V_i$. It is locally an affine bundle by choosing complements to $\operatorname{im} q_i$. Thus the all-injective open of

the moment fiber is smooth irreducible of dimension $\dim T^*E^w - \dim G$. On every remaining rank stratum at least one vertex has the deficit just computed. Varying the doubled arrows does not increase the fiber-dimension bound, so the complementary locus has dimension at most $\dim T^*E^w - \dim G - 2$.

Each moment fiber is cut out by $\dim G$ equations, so every irreducible component has at least the expected dimension. Consequently the all-injective open meets every component, and its irreducibility gives a single component. The equations have the expected codimension in the smooth ambient space, so the fibers are complete intersections. They are generically smooth and Cohen–Macaulay, hence reduced, and the codimension estimate gives regularity in codimension one. Serre's criterion yields normality. These dimension computations work over every algebraically closed extension of $\mathbb{C}$. Miracle flatness now applies to the map from the smooth cotangent space to the smooth vector space $\mathfrak{g}^*$.

The invariant subring of the normal domain $\mathbb{C}[\mu^{-1}(0)]$ is normal. The projective map $\pi$ is given by the relative Proj of the ring of nonnegative $\chi$-semi-invariants on the zero fiber. A relatively ample power of the character line bundle descends to $X$ and has a linearization for the cotangent dilation, which commutes with $G$. The moment-map identity identifies the dual of $d\mu$ with the infinitesimal action paired against the symplectic form. This gives surjectivity of $d\mu$ on $P$. On the zero fiber the kernel of the restricted symplectic form is the tangent space to the orbit, so the form descends along the principal bundle to a symplectic form on the smooth quotient.

For birationality, consider the open condition $\operatorname{rank}(q_i p_i) = v_i$ at every vertex. The minors defining it are invariant, so it is pulled back from an open of $X_0$. To see that this open is nonempty, take two summands isomorphic to $V_i$ in $W_i$, let $q_i$ be the inclusion of the first and $p_i$ the projection from the second, and set all doubled arrows to zero. On the open, the matrix $q_i p_i$ determines $\operatorname{im} q_i$. After identifying the injections by a change of basis in $V_i$, it also determines $p_i$. For each doubled arrow with map $b_a : V_i \to V_j$, the invariant matrix $q_j b_a p_i$ determines $b_a$, since $p_i$ is surjective and $q_j$ injective. Each affine quotient fiber on this open thus consists of one costable orbit. The restriction of $\pi$ there is proper and quasi-finite, hence finite, and has generic degree one in characteristic zero. It is an isomorphism by normality.

Grauert–Riemenschneider vanishing [6] gives $R^j\pi_*\mathcal{O}_X = 0$ for $j > 0$ because the symplectic volume form trivializes $\omega_X$. Normality gives $\pi_*\mathcal{O}_X = \mathcal{O}_{X_0}$. Since $X_0$ is affine, $H^j(X, \mathcal{O}_X) = 0$ for $j > 0$. These direct images give rational singularities of $X_0$, so duality identifies its canonical sheaf with $\pi_*\omega_X$. The descended generator pulls back to the symplectic volume form, hence $\pi$ is crepant. Over the smooth locus of $X_0$, a divisorial exception would have positive discrepancy by the Jacobian formula. If a positive-dimensional fiber remained over that locus, it would contain a curve. A relatively ample divisor agrees there with the pullback of its pushforward Cartier divisor, since there are no exceptional divisors. Its degree on the

curve would therefore be zero. This contradicts relative ampleness and gives an isomorphism over the smooth locus.

The cotangent dilation scales the symplectic form by $t$. Its weights on the coordinate ring are nonnegative, and the degree-zero invariants on the moment fiber are a quotient of $\mathbb{C}[E^w]^G$ by reductivity. Choose negative integer weights for the vertices that increase strictly along every original arrow, using acyclicity. The central one-parameter subgroup of $G$ with those weights contracts all of $E^w$ to zero. On $x_a$ the weight is the head weight minus the tail weight, and on $q_i$ it is the negative of the weight at $i$. Thus $\mathbb{C}[E^w]^G = \mathbb{C}$, and the induced grading on $X_0$ has only constants in degree zero. The corresponding limit is the image of the origin. □

Let $Y = \pi^{-1}(0)$ with its reduced structure, where 0 denotes the image of the origin.

**Lemma 4.2.** *The inverse image of $Y$ in $P \cap \mu^{-1}(0)$, as a closed set, is*

$$P \cap \left( \Lambda_{\mathbf{v}} \times \bigoplus_{i \in I} \operatorname{Hom}(V_i, W_i) \times \{p = 0\} \right).$$

*Its quotient has $\#\operatorname{Irr}(Y) = \#\operatorname{Irr}(\Lambda_{\mathbf{v}})$, where* Irr *denotes the set of irreducible components.*

*Proof.* A point over 0 has the origin in its orbit closure. The affine Hilbert–Mumford criterion gives a one-parameter subgroup of $G$ contracting the point to zero. Relative to the induced weight decomposition of the $V_i$, a nonzero block of a doubled arrow must go from a smaller weight to a larger weight. The sum of the nonnegative weight spaces is invariant under the doubled arrows, and $q$ kills it because the weight of a framing block of $q$ is the negative of its source weight. Costability therefore makes all nonnegative weight spaces zero. A block of $p_i$ has the weight of its target in $V_i$, so $p = 0$. Strict increase of weights along every nonzero arrow block implies that all sufficiently long doubled paths vanish. The moment equations now reduce to $\mu_Q = 0$.

Conversely, nilpotency gives a complete graded flag strictly lowered by every doubled arrow. Split the flag and give its successive graded lines strictly decreasing negative integer weights in the order of an increasing flag. A nonzero arrow block then raises weight, and the negativity of all weights contracts $q$. With $p = 0$, the point contracts to the origin.

The locus with all $q_i$ injective meets the product of every component of $\Lambda_{\mathbf{v}}$ with the framing space. Thus intersecting with $P$ leaves one dense irreducible open in each such product. Their closures in the inverse image of $Y$ are its distinct irreducible components. Each component is $G$-invariant because $G$ is connected. The principal bundle quotient is open with orbit fibers, so saturated closed subsets have closed images and pull back from their images. In particular the component images are closed irreducible and preserve noncontainment. They are precisely the components of $Y$. □

## 5. Reduction on the injective framing locus

For a smooth complex variety $B$, write $\mathcal{D}_B$ for its sheaf of algebraic differential operators. Its order filtration places functions in order zero and vector fields in order one. On $E^w$ let $\eta : \mathfrak{g} \to \Gamma(E^w, \mathcal{D}_{E^w})$ be the infinitesimal action on functions, obtained by differentiating $(gf)(z) = f(g^{-1}z)$. Thus the principal symbol of $\eta(\zeta)$ is $-\langle \mu, \zeta \rangle$. A coherent left $\mathcal{D}_{E^w}$-module is strongly $G$-equivariant if the derivative of its compatible algebraic $G$-action at $\zeta \in \mathfrak{g}$ is left multiplication by $\eta(\zeta)$. We use this condition also on microlocal modules with the same quantum moment map.

Take the order Rees algebra $\bigoplus_{m \geq 0} \hbar^m F_m \mathcal{D}_{E^w}$, where $F_m$ is the order-$\leq m$ subsheaf, microlocalize on $T^*E^w$, and complete in $\hbar$. Denote the resulting deformation quantization by $\mathcal{A}$. Its reduction modulo $\hbar$ is $\mathcal{O}_{T^*E^w}$, and $\hbar$ has weight one for the cotangent dilation. A good order filtration of a coherent $\mathcal{D}_{E^w}$-module similarly gives a coherent $\mathcal{A}$-lattice whose reduction modulo $\hbar$ is the sheaf associated to its graded module on $T^*E^w$. Inverting $\hbar$ gives its microlocalization. This construction and taking dilation invariants over the cotangent projection are inverse equivalences between coherent differential-operator modules and good equivariant microlocal modules [1, Proposition 4.33]. The same equivalence applies over every smooth base $B$ to the order quantization of $T^*B$. In particular microlocalization is exact, and its support is the characteristic variety. For a strongly $G$-equivariant module on the affine space $E^w$, use a finite-dimensional $G$-invariant subspace of generators for its good filtration. The resulting microlocalization is strongly $G$-equivariant, since the infinitesimal relation holds on the Rees module after inverting $\hbar$.

Write $C = P \cap \mu^{-1}(0)$ and $\psi : C \to X$ for the quotient map. The transfer lattice for Hamiltonian reduction at zero is the left-module quotient

$$\mathcal{E}_0 = \mathcal{A}|_P \Big/ \sum_{\zeta \in \mathfrak{g}} \mathcal{A}|_P \, \hbar\eta(\zeta).$$

The quotient is supported on $C$. In expressions involving $\psi$, we view it and the Hom sheaves from it as sheaves on $C$ via the closed inclusion $C \hookrightarrow P$. The reduced quantization $\mathcal{A}_X$ is the pushforward along $\psi$ of the opposite endomorphism algebra of this lattice, which acts on the lattice from the right [3, §3.4]. Its special fiber is $\mathcal{O}_X$ by [8, Proposition 2.8(i)]. Put $\mathcal{E} = \mathcal{E}_0[\hbar^{-1}]$ and $\mathcal{W}_X = \mathcal{A}_X[\hbar^{-1}]$. The free reduction theorem [8, Proposition 2.8], in the algebraic form of [3, §§3.4 and 5.5], identifies good $\mathcal{W}_X$-modules with strongly $G$-equivariant good modules over $\mathcal{A}|_P[\hbar^{-1}]$. The two functors are

$$\mathcal{M} \longmapsto \psi_* \mathcal{H}om_{\mathcal{A}|_P[\hbar^{-1}]}(\mathcal{E}, \mathcal{M}), \qquad \mathcal{N} \longmapsto \mathcal{E} \otimes_{\psi^{-1}\mathcal{W}_X} \psi^{-1}\mathcal{N}.$$

The Hom in this formula consists of sections killed by every $\eta(\zeta)$, hence invariant under the connected group $G$. The equivalence is compatible with cotangent dilation and with restriction to invariant opens. The action on $P$ is free and proper by Lemma 4.1. The cotangent dilation preserves $P$,

and its degree-zero $G$-invariants on $T^*E^w$ are constants. The global-section hypothesis in [3, §3.4] holds because $\pi_*\mathcal{O}_X = \mathcal{O}_{X_0}$. The complement of $C$ in the normal zero fiber has codimension at least two, by the rank estimate in Lemma 4.1. Thus regular semi-invariants on $C$ extend uniquely to the zero fiber, giving the section comparison for powers of the character line in [3, §5.5].

We verify the support formula for this equivalence. The symbols of the $\hbar\eta(\zeta)$ for a basis $\zeta$ of $\mathfrak{g}$ form a regular sequence along $C$ because $d\mu$ is surjective there. They generate the unit ideal off $C$. Use the complex with term $\mathcal{A}|_P \otimes_{\mathbb{C}} \bigwedge^r \mathfrak{g}$ in homological degree $r$. For $\zeta_1, \dots, \zeta_r \in \mathfrak{g}$, its differential on $a \otimes \zeta_1 \wedge \cdots \wedge \zeta_r$ is

$$\sum_i (-1)^{i+1} a\hbar\eta(\zeta_i) \otimes \zeta_1 \wedge \cdots \wedge \widehat{\zeta_i} \wedge \cdots \wedge \zeta_r \\ + \sum_{i<j} (-1)^{i+j} a\hbar \otimes [\zeta_i, \zeta_j] \wedge \zeta_1 \wedge \cdots \wedge \widehat{\zeta_i} \wedge \cdots \wedge \widehat{\zeta_j} \wedge \cdots \wedge \zeta_r,$$

where a hat omits the factor. The Lie algebra identities give $d^2 = 0$, since the commutator of $\hbar\eta(\zeta)$ and $\hbar\eta(\zeta')$ is $\hbar^2\eta([\zeta, \zeta'])$. The zeroth homology is $\mathcal{E}_0$. Modulo $\hbar$ it is the ordinary Koszul complex of the symbols, resolving $\mathcal{O}_C$. The long homology sequence for multiplication by $\hbar$ thus makes multiplication by $\hbar$ injective on degree-zero homology $\mathcal{E}_0$ and identifies $\mathcal{E}_0/\hbar\mathcal{E}_0$ with $\mathcal{O}_C$. Under this identification the right action of $\psi^{-1}\mathcal{O}_X$ is multiplication by pullback of functions, by evaluation of the transfer endomorphisms on the class of 1.

Let $L$ be a coherent $\hbar$-torsion-free $\mathcal{A}_X$-lattice in a good $\mathcal{W}_X$-module $\mathcal{N}$. Consider the derived tensor product of $\mathcal{E}_0$ and $\psi^{-1}L$ over $\psi^{-1}\mathcal{A}_X$, with homological indexing. Its derived reduction modulo $\hbar$ is

$$\mathcal{O}_C \otimes^{\mathbf{L}}_{\psi^{-1}\mathcal{O}_X} \psi^{-1}(L/\hbar L).$$

Indeed one can take a free resolution on the $\psi^{-1}L$ side at each stalk. The terms of the resulting tensor complex are $\hbar$-torsion-free, and the free resolution remains a resolution upon reduction because $\mathcal{A}_X$ and $L$ are $\hbar$-torsion-free. The displayed derived tensor product is concentrated in degree zero since $\psi$ is flat. The multiplication-by-$\hbar$ long homology sequence shows that its degree-zero lift $\mathcal{E}_0 \otimes_{\psi^{-1}\mathcal{A}_X} \psi^{-1}L$ is $\hbar$-torsion-free. It is locally finitely presented as a left module by a finite presentation of $L$ and coherence of $\mathcal{E}_0$. It is therefore coherent and complete by [1, Lemma 3.14]. After inverting $\hbar$ it is the inverse image of $\mathcal{N}$ under the reduction equivalence. Its special fiber is the ordinary pullback of $L/\hbar L$. Thus the support of the inverse image is exactly $\psi^{-1}(\operatorname{Supp}\mathcal{N})$.

Let $U \subset E^w$ be the open locus where all framings $q_i$ are injective. Its cotangent bundle is contained in $P$, and $(T^*U \cap \mu^{-1}(0))/G$ is an invariant open of $X$. To describe it, the quotient $U/G$ has as base $\prod_i \operatorname{Gr}(v_i, W_i)$, where $\operatorname{Gr}(v_i, W_i)$ parametrizes the image subspaces $S_i = q_i(V_i)$. Its fiber at $(S_i)_i$ is $\bigoplus_{a \in Q_1} \operatorname{Hom}(S_{s(a)}, S_{t(a)})$. In fact transport each $x_a$ using the isomorphisms

$q_i : V_i \to S_i$. This realizes $U \to U/G$ as the principal bundle of bases of the tautological subspaces pulled back to that vector bundle. The zero moment condition says that the cotangent vector annihilates the tangent to the $G$-orbit in $U$. Such covectors are precisely pullbacks of covectors on $U/G$, so

$$(T^*U \cap \mu^{-1}(0))/G = T^*(U/G).$$

On this open, the quantization $\mathcal{W}_X$ and the reduction functor are the order microlocalizations of $\mathcal{D}_{U/G}$ and ordinary strong descent of $\mathcal{D}_U$-modules, respectively. To check the identification, trivialize $U \to U/G$ as a product with $G$. On the product, the fields $\eta(\zeta)$ span the vertical tangent bundle. The quotient of differential operators by the left ideal they generate is the exterior product of the differential operators on the first factor and $\mathcal{O}_G$ with its natural differential-operator action. Evaluation of a vertical operator on 1 gives this quotient. The symbols of the vertical fields generate all positive degrees of the vertical order-graded algebra as an ideal. Subtracting elements of the left ideal to lower the order shows that an operator killing 1 belongs to that ideal. Microlocalization and completion identify the transfer lattice on this product with the completed exterior product of the horizontal order lattice and $\mathcal{O}_G$ in order zero. In this transfer lattice an endomorphism is determined by the image of the class of 1, which is killed by the quantized vertical fields. Torsion-freeness and connectedness of $G$ make this image constant in the $G$-direction. Such endomorphisms are precisely right multiplication by horizontal operators, so the opposite endomorphism algebra gives the horizontal quantization. In the product, the equivariant underlying quasi-coherent sheaf descends to the first factor. The vertical vector fields act by differentiation on the $G$-factor by strong equivariance. A horizontal vector field commutes with the $G$-action and induces a vector field action on invariant sections. This gives a differential-operator module on the first factor. Its inverse is exterior product with the trivial connection $\mathcal{O}_G$. Coherence is preserved in this equivalence: on an affine product, write each of a finite set of differential-operator generators of the exterior product as a finite sum of tensors. Their first factors generate the descended differential-operator module, since their span under horizontal operators pulls back to a module containing all original generators. For a good order filtration on the horizontal module, tensor each filtration piece with $\mathcal{O}_G$, given its filtration in order zero. The Rees transfer lattice sends the microlocalization of the horizontal module to the microlocalization of this exterior product. This can be checked on each finite presentation: the transfer quotient imposes precisely the vertical fields, and its horizontal operators are unchanged. The special fiber is the horizontal graded module pulled back along the zero section in the vertical cotangent factor. Changing trivialization acts by a base-dependent $G$-translation in the vertical factor, which acts trivially on descended sections and changes a lift of a horizontal vector field only by a vertical field. Thus the identifications glue.

*Proof of Theorem 1.* For a flag incidence map, proper pushforward bounds the singular support of the constant-complex pushforward by those pairs $(x, y)$ for which $y$ pulls back to zero at some point $(x, L_\bullet)$ of the incidence variety. The incidence variety is smooth, since it is a vector bundle over the flag variety. Varying $x$ in its vector-bundle fiber shows that $y$ belongs to the trace annihilator of the subspace of flag-preserving arrows. The triangularity calculation in Lemma 1.1 then shows that both $x$ and $y$ strictly lower the complete graded flag. Varying $(x, L_\bullet)$ along a $G$-orbit gives $\langle \mu_Q(x, y), \zeta \rangle = 0$ for every infinitesimal change of bases $\zeta$. Hence this cotangent bound is contained in $\Lambda_{\mathbf{v}}$, and the inclusion holds for each shifted simple summand.

For $\mathbf{v} = 0$, the group is trivial and $E = T^*E = \Lambda_{\mathbf{v}}$ is a point. The unique simple sheaf is the constant sheaf, which is the direct image for the empty flag.

For the converse use the acyclic orientation of Lemma 1.1. Let $\mathcal{F}$ be a simple equivariant perverse sheaf on $E$ with $SS(\mathcal{F}) \subseteq \Lambda_{\mathbf{v}}$. It is the intersection cohomology extension of an irreducible local system on a smooth invariant dense stratum of its support. The local system remains irreducible after forgetting equivariance by the connectedness argument of Section 1. Pull back $\mathcal{F}$ to $E^w$ and shift by the dimension of the framing space. Smooth pullback with this shift commutes with intermediate extension. The pullback is simple since the projection is a product with an affine space. Its singular support is

$$SS(\mathcal{F}) \times \bigoplus_{i \in I} \operatorname{Hom}(V_i, W_i) \times \{p = 0\},$$

by the smooth pullback formula. The corresponding regular holonomic $\mathcal{D}_{E^w}$-module is strongly $G$-equivariant by Riemann–Hilbert. Microlocalize it, restrict to $P$, and apply the reduction equivalence. The resulting good module on $X$ is supported on $Y$ by Lemma 4.2 and the support formula.

Its restriction to $T^*(U/G)$ is simple and nonzero. Indeed $U$ is the product of $E$ with the nonempty smooth connected space $\prod_i \operatorname{Inj}(V_i, W_i)$ of injective framings. The shifted pullback to $U$ of the irreducible local system on the stratum is irreducible since projection from this product surjects on fundamental groups. Thus the $\mathcal{D}_U$-module is simple by intermediate extension and Riemann–Hilbert. Its simplicity also holds in the coherent category, since regular holonomic modules are closed under coherent submodules. Ordinary strong descent and the cotangent order equivalence preserve this simplicity. Distinct original sheaves give nonisomorphic modules on $T^*(U/G)$, since their strongly equivariant pullbacks to $U$ are recovered by these equivalences, and restriction of the corresponding sheaves along any fixed injective framing recovers the original underlying sheaves up to the common shift. The injectivity of forgetting equivariance on simple isomorphism classes, proved in Section 1, distinguishes the original equivariant sheaves.

By Theorem 3.1, the reduced module on $X$ has finite length. Restriction of a composition series to $T^*(U/G)$ is exact. Because the restriction of

the module is simple and nonzero, precisely one simple factor in the series has nonzero restriction, and that restriction is the given simple module on $T^*(U/G)$. For distinct original perverse sheaves these global simple factors are nonisomorphic, since their restrictions are nonisomorphic. Every factor is supported on $Y$. Therefore the number of isomorphism classes under consideration is at most $\#\operatorname{Irr}(Y) = \#\operatorname{Irr}(\Lambda_{\mathbf{v}})$ by Theorem 3.1 and Lemma 4.2. The set of components and the set of Lusztig sheaves both identify with the weight $-\sum_i v_i\alpha_i$ part of the crystal of the negative half of the quantum group, where $\alpha_i$ is the simple root at $i$ [9, Theorem 5.3.2 and §6.1]. By the forward implication every Lusztig sheaf belongs to the set being counted. This proves the converse. □

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