Self-injective dimensions under stable equivalences of Morita type

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Abstract

Stable equivalences of Morita type preserve the left and right self-injective dimensions of Artin algebras. The defining tensor functors can send injective modules to noninjective modules.

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1 Introduction

Broué formulated stable equivalence of Morita type in the study of blocks of finite group algebras [2, §5.A]. Over a field, the defining bimodules induce inverse equivalences of the stable module categories, whose morphisms are taken modulo maps factoring through projective modules. For block algebras, Broué recalled that a Rickard equivalence induces a stable equivalence of Morita type [2, Proposition 5.2].

Liu and Xi constructed stable equivalences of Morita type between endomorphism algebras of generators [6, Theorems 1.1 and 3.6]. Their construction can start with self-injective algebras and produce algebras that are no longer self-injective. The example in [6, §5] yields two such endomorphism algebras each having left and right self-injective dimension two. By contrast, the example in [6, p. 581] shows that an equivalence of stable module categories can relate algebras whose left self-injective dimensions are two and infinity.

When the two defining tensor functors are adjoint in both orders, Liu and Xi proved equality of left self-injective dimensions [6, Proposition 4.1]. Their argument uses that a right adjoint to an exact functor preserves injectives [6, Lemma 4.2]. The note added in proof cites Dugas and Martínez-Villa [4, Corollary 3.1], who obtained this adjunction for indecomposable defining bimodules between finite-dimensional algebras with separable semisimple quotients and no semisimple blocks. Chen, Pan and Xi [3, Lemma 4.1(2)] obtained this adjunction over an arbitrary field for finite-dimensional algebras with no semisimple direct summands and defining bimodules with no projective bimodule summands. Liu and Xi's Question 1 [6, §5, p. 584] asks whether an arbitrary stable equivalence of Morita type between Artin algebras preserves finiteness of the left self-injective dimension.

Beligiannis [1, Theorem 8.12(iii)] stated the invariance of the Gorenstein property, meaning simultaneous finiteness of the left and right self-injective dimensions. Sun and Zhao proved equality of the dimensions on each side under symmetric separable equivalence, which requires the two tensor functors to be adjoint in both orders [8, Corollary 4.3 in the preprint].

We use the bimodule splittings to split the maps on Ext induced by the tensor functors. The argument uses the same direct-summand principle as Kadison's transfer of Tor [5, Proposition 6.3].

2 Invariance of self-injective dimensions

Let k be a commutative Artinian ring. An Artin k-algebra is a unital k-algebra finitely generated as a k-module. All bimodules considered have coincident left and right k-actions. For an algebra A, write ${}_AA$ and A_A for the left and right regular modules. For a left A-module X, let $\operatorname{inj.dim}({}_AX)$ denote the least nonnegative length of an injective resolution in the category of all left A-modules, or ∞ if no finite injective resolution exists. We use the same convention on the right.

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For a k-algebra R, write R^{op} for the opposite algebra, with reversed multiplication, and b^{op} for $b \in R$ viewed in R^{op}. An R-bimodule is projective if it is projective over $R \otimes_k R^{\mathrm{op}}$ for the action $(a \otimes b^{\mathrm{op}})x = axb$ on its elements x, where $a, b \in R$.

Artin k-algebras A and B are *stably equivalent of Morita type* if there are bimodules ${}_AM_B$ and ${}_BN_A$, finitely generated and projective on each side, and projective bimodules ${}_AP_A$ and ${}_BQ_B$ with bimodule isomorphisms

$$M \otimes_B N \simeq A \oplus P, \qquad N \otimes_A M \simeq B \oplus Q.$$

Theorem 1. *Let A and B be Artin algebras stably equivalent of Morita type. Then*

$$\operatorname{inj.dim}({}_AA) = \operatorname{inj.dim}({}_BB), \qquad \operatorname{inj.dim}(A_A) = \operatorname{inj.dim}(B_B).$$

Remark 2. Sun and Zhao [7, Proposition 3.2] stated both equalities for finite-dimensional algebras over an algebraically closed field. Their proof uses the assertion in Lemma 3.1 that the defining tensor functors preserve injective modules. Section 3 gives a counterexample to this assertion and exhibits the two different projections identified in its proof.

Proof of Theorem 1. Let $F = N \otimes_A -$ be the functor from all left A-modules to left B-modules, and $G = M \otimes_B -$ the functor in the reverse direction. The right projectivity of N and M makes F and G exact. A projective left A-module is a direct summand of a sum of copies of ${}_AA$. Since F commutes with direct sums and $F(A) \simeq {}_BN$ is projective, F preserves projectives. The same reasoning applies to G.

Fix the two bimodule isomorphisms in the definition. For a left A-module X, associativity of tensor products gives

$$GF(X) \simeq (M \otimes_B N) \otimes_A X \simeq X \oplus (P \otimes_A X).$$

The inclusion $\iota_X \colon X \to GF(X)$ and projection $\pi_X \colon GF(X) \to X$ onto the first summand are natural in X, with $\pi_X \iota_X = 1_X$. The second bimodule isomorphism gives $FG(Y) \simeq Y \oplus (Q \otimes_B Y)$ naturally in left B-modules Y.

For a ring R and left R-modules X, Y, take a projective resolution $L_\bullet \to X$. Write $d_{i+1} \colon L_{i+1} \to L_i$ for its differential in degree $i+1$, where $i \geq 0$. Write $\operatorname{Hom}_R(L_i, Y)$ for the group of R-linear maps. Then $\operatorname{Ext}^n_R(X, Y)$, for $n \geq 0$, is the degree-n cohomology of the complex with terms $\operatorname{Hom}_R(L_i, Y)$ and differential $f \mapsto f \circ d_{i+1}$.

For $R = A$, the complex $F(L_\bullet)$ with differentials $F(d_{i+1})$ resolves $F(X)$ by projectives. Applying F to each homomorphism $L_i \to Y$ gives a cochain map because F preserves composition. Define its degree-i retraction by

$$\operatorname{Hom}_B(F(L_i), F(Y)) \longrightarrow \operatorname{Hom}_A(L_i, Y), \qquad h \longmapsto \pi_Y G(h) \iota_{L_i}.$$

Naturality of ι gives

$$\pi_Y G\big(h \circ F(d_{i+1})\big)\iota_{L_{i+1}} = \pi_Y G(h) \iota_{L_i} d_{i+1},$$

so the retraction commutes with the cochain differentials. For $f \colon L_i \to Y$, naturality of π gives

$$\pi_Y GF(f) \iota_{L_i} = f \pi_{L_i} \iota_{L_i} = f.$$

Taking cohomology yields a split injection

$$\operatorname{Ext}^n_A(X, Y) \longrightarrow \operatorname{Ext}^n_B(F(X), F(Y)) \qquad (n \geq 0). \tag{1}$$

Suppose $d = \operatorname{inj.dim}({}_BB)$ is finite. The finitely generated projective module ${}_BN \simeq F(A)$ is a summand of a finite direct sum of copies of ${}_BB$. For each X and $n > d$, the group $\operatorname{Ext}^n_B(F(X), F(A))$ is therefore a summand of a finite sum of copies of the zero group $\operatorname{Ext}^n_B(F(X), B)$. By (1) with $Y = A$, we have $\operatorname{Ext}^{d+1}_A(X, A) = 0$ for every left A-module X.

Start with $C^0 = {}_AA$ and choose short exact sequences $0 \to C^j \to J^j \to C^{j+1} \to 0$ with J^j injective for $0 \leq j < d$. Since $\operatorname{Ext}^m_A(X, J^j) = 0$ for $m \geq 1$, the long exact sequences give

$$\operatorname{Ext}^m_A(X, C^{j+1}) \simeq \operatorname{Ext}^{m+1}_A(X, C^j) \qquad (m \geq 1,\ 0 \leq j < d).$$

Iterating yields $\operatorname{Ext}^1_A(X, C^d) = 0$ for every X, including $d = 0$ by the original vanishing. If C^d is embedded in an injective module J, then $\operatorname{Ext}^1_A(J/C^d, C^d) = 0$ makes $\operatorname{Hom}_A(J, C^d) \to \operatorname{Hom}_A(C^d, C^d)$ surjective. A preimage of the identity is a retraction, so C^d is injective. Splicing the short exact sequences gives an

injective resolution of ${}_AA$ of length d. Exchanging A and B and using the natural splitting of FG gives the reverse bound whenever $\operatorname{inj.dim}({}_AA)$ is finite. Hence the two left dimensions are equal, whether finite or infinite.

For right modules use $-\otimes_A M$ and $-\otimes_B N$ in place of F and G. Left projectivity makes these functors exact, and right projectivity makes them preserve projectives. For right modules X over A and Y over B, the two composites split naturally as

$$(X \otimes_A M) \otimes_B N \simeq X \oplus (X \otimes_A P), \qquad (Y \otimes_B N) \otimes_A M \simeq Y \oplus (Y \otimes_B Q).$$

The same cochain formulas give split injections on right-module Ext groups. The functors take the regular modules to M_B and N_A, respectively, which are finitely generated projective. Applying the dimension argument to these splittings proves the right equality. □

3 Tensor functors and injective modules

Let k be an algebraically closed field and let A be the algebra of upper triangular 2×2 matrices over k. Write e_{ij} for the matrix unit with a 1 in position (i, j), for $1 \le i \le j \le 2$. Right multiplication by e_{12} embeds the left ideal $Ae_{11} = ke_{11}$ in Ae_{22}, sending e_{11} to e_{12}. If $r\colon Ae_{22} \to Ae_{11}$ is A-linear, then

$$r(e_{12}) = r(e_{12}e_{22}) = e_{12}r(e_{22}) = 0$$

because $e_{12}Ae_{11} = 0$. Thus the embedding has no retraction. It follows that Ae_{11} is a noninjective direct summand of ${}_AA = Ae_{11} \oplus Ae_{22}$.

Take the regular bimodule $M = A$ and the bimodule $N = A \oplus (A \otimes_k A)$, with multiplication on the outside factors of $A \otimes_k A$. The map $a \otimes b^{\mathrm{op}} \mapsto a \otimes b$ identifies this tensor summand with a free module of rank one over $A \otimes_k A^{\mathrm{op}}$. It is also free on each side, by choosing a k-basis of the factor on which that side does not act. Both $M \otimes_A N$ and $N \otimes_A M$ identify with N by multiplication. Thus M, N define a stable self-equivalence of Morita type.

Let $I = \operatorname{Hom}_k(e_{22}A, k)$, with left action $(a\varphi)(x) = \varphi(xa)$ for $a \in A$, $\varphi \in I$ and $x \in e_{22}A$. This module is one-dimensional because $e_{22}A = ke_{22}$. The right ideal $e_{22}A$ is projective as a direct summand of A_A. For any left A-module X, the correspondence

$$\operatorname{Hom}_A(X, I) \longrightarrow \operatorname{Hom}_k(e_{22}A \otimes_A X, k), \qquad f \longmapsto [x \otimes v \mapsto f(v)(x)]$$

is balanced since $f(av)(x) = f(v)(xa)$ for $a \in A$, $v \in X$ and $x \in e_{22}A$. Its inverse sends a functional λ to the map $v \mapsto [x \mapsto \lambda(x \otimes v)]$ from X to I. This map is A-linear by the tensor relation $x \otimes av = xa \otimes v$. Thus $\operatorname{Hom}_A(-, I)$ is the composite of the exact functor $e_{22}A \otimes_A -$ and the exact k-linear dual, so I is injective. The tensor functor $F = N \otimes_A -$ instead gives

$$F(I) \simeq I \oplus (A \otimes_k I) \simeq I \oplus A$$

as left A-modules. Here the first identification uses $(a \otimes b) \otimes v \mapsto a \otimes bv$, with inverse $a \otimes v \mapsto (a \otimes 1) \otimes v$ on the tensor summand. The second uses $\dim_k I = 1$. The summand Ae_{11} shows that $F(I)$ is not injective. This refutes the preservation direction of [7, Lemma 3.1(1)] and the injective-dimension equality for arbitrary left modules in [7, Proposition 3.2(1)]. For the right-module assertions in parts (2), pass to A^{op} and use N^{op} and M^{op} as the first and second bimodules. Here an opposite bimodule has its left and right actions interchanged. Left A-modules are right A^{op}-modules with $va^{\mathrm{op}} = av$. The map $v \otimes n^{\mathrm{op}} \mapsto n \otimes v$ identifies $I \otimes_{A^{\mathrm{op}}} N^{\mathrm{op}}$ with $F(I)$ under this convention. Indeed $n \otimes av = na \otimes v$ verifies balancing, and reversing the factors again gives its inverse. Thus the right tensor functor also sends an injective module to a noninjective module.

In the argument of [7, Lemma 3.1(1), p. 2593], let F and G be the left tensor functors of the main proof. For a left B-module Z, let $\rho_Z\colon FG(Z) \to Z$ be projection onto the first summand of $FG(Z) \simeq Z \oplus (Q \otimes_B Z)$. The proof takes a monomorphism $f\colon X \to Y$ of left B-modules and a map $g\colon X \to F(I)$, with I injective over A. Since $G(f)$ is a monomorphism, $\pi_I \circ G(g)$ extends along $G(f)$ to a map $h\colon G(Y) \to I$. Applying F to $\pi_I \circ G(g) = h \circ G(f)$ gives

$$F(\pi_I) \circ FG(g) = F(h) \circ FG(f).$$

The published proof replaces $F(\pi_I)$ by $\rho_{F(I)}$ in this equality. The same substitution appears in [9, Theorem 4.1, step (1)].

For the triangular algebra above, $G = A \otimes_A -$ identifies with the identity by multiplication. Under $F(Z) \simeq Z \oplus (A \otimes_k Z)$, both π_Z and ρ_Z are projection onto Z. On the summand $A \otimes_k F(I)$ of $F(F(I))$, the projection $\rho_{F(I)}$ is zero, whereas $F(\pi_I)$ restricts to $\mathrm{id}_A \otimes \pi_I$ onto the nonzero summand $A \otimes_k I$ of $F(I)$.

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