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THE ORPHAN THEOREM: DEGENERATE PERPENDICULAR ORBITS IN RATIONAL RIGHT TRIANGLES

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Abstract. Let T be a right triangle with acute angle α, legs L_1, L_2 and hypotenuse H, and suppose T is rational: $2\alpha/\pi = P/Q$ in lowest terms. Shoot a beam perpendicular to the leg L_1 from each of its points. The leg is partitioned into *branches* — maximal intervals of launch heights sharing a combinatorial type — and we prove that the partition is finite, with at most $2Q+1$ branches, which cover the leg but for finitely many points. The first-return map J of the beam to L_1 is an involution, and on each branch it does one of two things: it carries the branch onto a distinct branch of equal width by a reflection, or it fixes the branch pointwise. A pointwise-fixed branch — a whole interval of trajectories that hit a side head-on and retrace — is an **orphan**.

Theorem A. *There is no orphan branch when P is even and at most one when P is odd, with no hypothesis on the direction; when P is odd and Q is a prime power, or twice or four times a prime, the orphan exists.*

Non-existence is a parity count on the velocity set of the unfolded beam. Uniqueness is an Euler count on the genus-zero quotient of the unfolded surface, which admits minimal components. Existence is arithmetic: on a folded disc, an absent orphan forces 1 to be a signed sum of doubled heights of images of the right-angle vertex, which is impossible at the centres named. Running the direction count and the Euler count for each side shows that at most two of the three sides carry a retracing cell, at most one each, the two ends of one band of double normals. On the unfolded surface — the holonomy double cover of a genus-zero quadratic differential $Q(P-2, Q-P-2, -1^Q)$ — the images of the right-angle vertex are Weierstrass points, the leg is the quotient by the hyperelliptic involution of a geodesic segment through one of them, and the first-return map is read in that quotient.

1. Introduction

1.1. **The question.** A billiard trajectory that meets a side of a polygon perpendicularly reverses and retraces its path. Fixing a side and launching perpendicularly from each of its points therefore organises the dynamics: two launch points behave the same way exactly when their trajectories strike the same sequence of sides, and the side decomposes into intervals — *branches* — on which that sequence is constant.

Concretely, place the right triangle T with vertices $O = (0,0)$, $R = (\cot\alpha, 0)$ and $A = (\cot\alpha, 1)$, so that $L_1 = RA$ is the vertical launch leg,

$L_2 = OR$ the horizontal leg and $H = OA$ the hypotenuse, and the acute angle at O is α. Launch the beam from $(\cot\alpha, s)$, $s \in (0,1)$, with velocity $(-1,0)$. Write

$$n(P/Q) := \text{the number of branches the perpendicular beam cuts on } L_1,$$

when the triangle is rational, $2\alpha/\pi = P/Q$ with $\gcd(P,Q) = 1$ and $0 < P < Q$. We call the rational number P/Q — equivalently the pair (P,Q), equivalently the triangle itself — the **centre**, and the word is used in that sense throughout. The generic branch is a strip of parallel periodic trajectories. What we study is the exception.

Why the exception matters. The partition at P/Q is one stage of a hierarchy: moving α toward the next continued-fraction approximant changes the partition, and the change is not uniform. A mirror pair's perpendicular return ends on a copy of the launch leg, and that copy keeps its direction as α moves (Corollary 4.14: the pair turns at R, where the two leg reflections compose to -1 for every α). The orphan's turnaround lies on a wall copy carried a nonzero number of rhombus steps (Definition 4.4), whose direction moves with α. We therefore expect a pair to persist under small detuning and the orphan to break into new branches, so that the orphan, not the generic branch, carries the growth of $n(P/Q)$ along the approximants of an irrational angle, and its existence and uniqueness are where that growth starts. We prove nothing about detuning here.

1.2. **The orphan, and the main theorem.** Unfolding keeps the beam on a fixed horizontal line, and the copy of L_1 on which it returns perpendicularly is, like the launch copy, a vertical segment. The first perpendicular-return map J to the launch leg is therefore an isometry of the launch height, affine with $|\text{slope}| = 1$, and an involution (Theorem 2.2 below). This forces a dichotomy on each branch, and the dichotomy is the subject of the paper:

- a branch is carried isometrically onto a *distinct* branch of equal width — generic branches occur in **mirror pairs**, and J reverses orientation on them, with slope -1 (Lemma 4.9); or
- a branch is fixed *pointwise*: every trajectory launched in it meets a side head-on and retraces to its own launch height. We call such a branch an **orphan**, since it has no partner.

Theorem A says when an orphan can exist, that there is never more than one, and where there is one:

Theorem A (orphan / degeneracy)**.** *Let $2\alpha/\pi = P/Q$ in lowest terms. The first perpendicular-return map J acts on the branches of L_1 as a width-pairing involution, and a pointwise-fixed (orphan) branch*

(a) does not exist when P is even;

*(b) is **unique** when it exists;*

(c) exists when P is odd and Q is a prime power, or twice or four times a prime.

Every orphan orbit turns around at a single head-on collision, on a side of type H when Q is odd and of type L_2 when Q is even. Consequently $n(P/Q)$ is even for even P, and for odd P it is odd exactly when the orphan exists — in particular whenever Q is a prime power, or twice or four times a prime.

We conjecture that (c) holds without the restriction on Q (Conjecture 4.12); the companion paper [Meth] proves it when $\min(P, Q-P) \geq 2$ and either $\gcd(P+2, Q) = \gcd(P-2, Q) = 1$ (its Theorem 2) or Q has at most one odd prime factor (its Theorem 3); Theorem 4.18 below covers $\min(P, Q-P) = 1$. The three parts stand on different footings, and we keep them apart throughout.

Non-existence and the turnaround type are elementary and hold at every Q. A head-on hit on a side requires the beam direction to equal that side's normal, and along a perpendicular orbit the direction takes only the Q values $k\pi/Q$ modulo π (Lemma 4.1). Coprimality makes the required direction unattainable when P is even, and when P is odd attainable on exactly one side type — the hypotenuse at odd Q, the far leg at even Q — which types the turnaround.

Uniqueness is not elementary. Counting vertical *directions* does not bound the vertical copies a single return passage meets, since the return word can be far longer than one turn of the direction cycle. What settles uniqueness is the unfolded surface S_α of §6 and its genus-zero quotient of §7. Each orphan branch lies in a cylinder that the deck involution moves to a different cylinder, and an Euler count on the quotient, which allows minimal components, leaves room for at most one exchanged pair. §4.3 states the theorem and §7.6 proves it.

Existence is not elementary either. Folding the development by its symmetries of the horizontal direction gives a flat disc on which every horizontal leaf from a non-singular point of the launch leg ends either back on the leg or on the one vertical copy of the far side (at odd P), and an orphan is a leaf of the second kind. If there is none, Gauss–Bonnet sorts the leaves into fans, each turning about one image of the right-angle vertex; the fans tile the leg, so 1 is a signed sum of twice the heights of those images. When Q is a power of a prime p every doubled height lies in the prime of $\mathbb{Q}(e^{i\pi/Q})$ over p, and 1 does not (Theorem 4.11). When Q has two distinct prime factors that membership no longer excludes 1 (Remark 4.13), and §4.5 uses more of the configuration. Modulo 2 the identity becomes linear over $\mathbb{F}_2$, indexed by the images whose horizontal rays reach the leg, and near the two acute corners some of those rays are forced onto the leg and others onto the far wall. When enough are forced the identity has no solution (Theorem 4.18); when fewer are it has one (Proposition 4.19). This settles the edges of every row, and every odd P at $Q = 2p$ (Corollary 4.20); at $Q = 4p$ three more rays, followed by hand, cover the row (Theorem 4.24).

Theorem A fixes the *parity* of $n(P/Q)$, not its value, which is arithmetically irregular (Remark 5.2) and is the subject of the companion paper [Neck].

The orphan is a property of the triangle, not of the launch leg. Each of the three sides carries its own perpendicular beam and its own retracing criterion — L_1 requires P odd, L_2 requires $P+Q$ odd, H requires Q odd — so at most two of the three sides carry a **retracing cell**, a cell of that side's own perpendicular partition whose orbits retrace. When they do, the two cells are the two ends of one band of **double normals**, billiard segments meeting the boundary perpendicularly at both of their ends. §5 states this precisely and proves it (Theorem 5.3).

1.3. **The surface reading.** The second half of the paper reads Theorem A on the unfolded surface, and proves its uniqueness half there. The Katok–Zemlyakov unfolding S_α of T is a closed translation surface glued from $4Q$ copies of T, of genus $\lfloor Q/2 \rfloor$ (Theorem 6.1), and its rotation through π exhibits it as the holonomy double cover of the genus-zero quadratic differential $Q(P-2, Q-P-2, -1^Q)$ on $\mathbb{CP}^1$ (Proposition 7.1), as Apisa observes [Api21, §1.2, §7]. The deck involution τ is then the hyperelliptic involution, and its fixed points — the Weierstrass points — are the preimages of the odd-order singularities of the base: the Q points over the right angle, which are regular points of the translation surface but branch points of the cover, together with O when P is odd and A when $Q-P$ is odd, $2g+2$ in all (Proposition 7.2). The points over the right angle are the ones the orphan problem uses. τ does not preserve the launch leg, so J is not a restriction of τ; instead the two copies of the leg in each direction meet at a point over R and form one geodesic segment, τ reflects that segment about the point, and J is the first return to it read in the quotient (§7.4). The same surface carries the proof of uniqueness: each orphan uses up a τ-swapped pair of cylinders, and an Euler count on the genus-zero base allows at most one such pair (§7.6). That O is a Weierstrass point exactly when P is odd, the parity condition of Theorem A, is a match of conditions with a one-line explanation in the reflection group, not a map from the orphan to that point (Remark 7.3).

1.4. **What is proved, what is classical, and what is cited.** Every numbered statement is proved here except Conjecture 4.12, which is labelled as one (it is proved on large families in [Meth]; §4.4), and none rests on computation: the exact branch counts the paper quotes (Figure 3, Remark 5.2, the census behind Conjecture 4.12) carry the completeness certificate of Appendix A and illustrate proved statements, their limits or the conjecture. Where a remark quotes a value proved elsewhere it cites the proof, and no proof depends on it. Three proofs are completed away from their statements: Theorem 4.7's at §7.6, which also carries the one-cell clauses of Theorem 5.3, and Lemmas 4.22 and 4.23, whose trajectories are followed in Appendix B. No proof takes anything from the companion paper [Neck], which imports

results of this one; Remark 5.2 quotes one of its counts for comparison. The external inputs are classical: Boshernitzan's theorem that a perpendicular orbit in a rational polygon avoiding the vertices is periodic [Bos92, Prop. 1.8] (also [GSV92, §5]), the first step of Proposition 3.4; the Katok–Zemlyakov unfolding [ZK75] and its cone-angle bookkeeping; the decomposition of a direction on a flat surface into cylinders and minimal components [Str84] §11.4, [MT02], used in Lemma 7.8; the Gauss–Bonnet formula for a flat disc with cone points [Tro86], used in §4.4; the classical fact that the fixed points of the hyperelliptic involution are the Weierstrass points [FK92] (Ch. III, §7); and the arithmetic of cyclotomic fields [Was97]. Apisa's identification of the genus-zero base [Api21], the description of the branch points of its double cover in [AEZ16], the classification of the lattice locus [Api21, KS00] and McMullen's regular-polygon models [McM25] are cited for comparison and are not inputs: Proposition 7.1 derives what we use. No proof in §§2–5 uses surface theory except §7.6's, on which Theorem 4.7 and the one-cell clauses of Theorem 5.3 rest; a reader can stop after §5 with non-existence, the turnaround type and existence (Proposition 4.3, Theorems 4.6, 4.11, 4.18 and 4.24, Corollary 4.20) and their proofs intact.

1.5. **Related work. The perpendicular problem.** The perpendicular class, and the partition of a side by it, were studied numerically before they were studied as theorems. Ruijgrok [Rui91] singles out "all orbits starting in a direction perpendicular to a side", presents evidence that "almost all orbits of this kind are either periodic or end in a corner", and observes that "the starting points with the same period form intervals" — the partition of this paper, for arbitrary and mainly irrational triangles; he attributes the original observation, for a right triangle and a hypotenuse launch, to Rabouw and Ruijgrok [RR81]. What he offers is numerical evidence and, in his own words, "plausibility arguments"; the statements he draws from it for a right triangle are the elementary ones, and they are consistent with the partition as we describe it. Cipra, Hanson and Kolan [CHK95] proved that in a right triangle almost every trajectory launched perpendicular to a leg returns to that leg in the same direction — the form in which [Tab05] Theorem 7.5 states it — which is what makes the branch partition well-defined in the irrational case. Troubetzkoy [Tro05] treated irrational right triangles whose smaller angle lies in $(\pi/6, \pi/4)$: every perpendicular periodic beam is symmetric (Theorem 1, whose rational counterpart is Corollary 4.14), there is at most one non-singular escape orbit in the perpendicular direction (Theorem 2), and every other non-singular orbit on the invariant surface of that direction is periodic (Theorem 3); the sequel [Tro05b] proves periodic orbits dense in the phase space of every irrational right triangle. All of these concern measure, density or symmetry, and outside the rational case except where [CHK95] recovers the known rational result. Our setting is the rational one, where the partition is finite and we determine its degenerate element exactly. Galperin and Zvonkine [GZ03] treat the same trajectories

by elementary means — the rhombus code word, and (§4B) a proof that every perpendicular periodic trajectory in a right triangle is unstable, the number of its links never being divisible by 4 as [GSV92]'s criterion requires of a stable one — and report the conjecture of [CHK95]. (Their *mirror trajectories*, after [GSV92], are a different object from our mirror pairs.) In [CHK95] that conjecture is a speculation drawn from computation: in every triangle the authors examined, the launch heights resolved into periodic beams but for a single "mystery" interval whose periods exceeded their computation, which inclines them to "speculate that there may be at most one nonperiodic trajectory left over". The qualifier *per direction* is [GZ03]'s and the qualifier *nonsingular* is [Tro05]'s; neither is in [CHK95]. The orphan is not that object: it is periodic, and by Lemma 2.1 its turnaround lies in the relative interior of a side, so it is nonsingular in the sense [GZ03] gives the word (a trajectory that meets a vertex; the right angle is exempted there as a removable singularity). The conjecture is the irrational question, and [Tro05] Theorems 2 and 3 together prove it for irrational right triangles whose smaller angle lies in $(\pi/6, \pi/4)$. Hooper [Hoo07] extended the instability to every periodic billiard path in a right triangle.

Perpendicular families by elementary construction. Holt [Hol93] constructs periodic paths in an arbitrary right triangle — no rationality assumed — by two partial tilings: the rhombus of four copies about the right angle, and a **fan** of $2N+2$ copies reflected about the smallest-angle vertex, where $N = \lceil \pi/2\beta \rceil - 1$ for β the smallest angle; with $\beta = \alpha$ and $\alpha/\pi = P/2Q$ one has $N + 1 = \lceil Q/P \rceil$. His paths are perpendicular and retracing in the sense of §1.1 — each "leaves either the hypotenuse or the longer leg at a right angle ... and retraces the directed segments in the opposite direction" — so they are members of the perpendicular class this paper partitions, reached from the other two sides. What he proves is existence, one family at a time: for each $k = 1, \ldots, N-1$ every interior point of the shorter leg lies on a path of $4k+2$ reflections per cycle, and for $k = N$ the points of a subsegment do. That is N families each sweeping the leg, not a partition of one beam, and the terminal subsegment is where the last family survives rather than a cell of a partition; neither the count $n(P/Q)$ nor the orphan is visible from it. A second result of his is a genuine negative, and of a kind we do not reach: there are no periodic paths of 3, 5 or 7 reflections per cycle in any right triangle, so Birkhoff's theorem for smooth convex bodies has no polygonal analogue.

Counting, the lattice locus, and genus-zero strata. The counting literature for rational billiards — Veech [Vee89], Eskin–Masur [EM01], Apisa [Api21], Gutkin–Rams [GR09] — counts orbits in *all* directions up to length L, and the work on the stratum $Q(P-2, Q-P-2, -1^Q)$ our base inhabits — Athreya–Eskin–Zorich [AEZ16], Boissy [Boi09], Delecroix–Goujard–Zograf–Zorich [DGZZ] — is asymptotic, statistical or generic over the stratum. Our statements are about one direction, at every admissible

(P, Q), and the only count this paper fixes is the parity of $n(P/Q)$; the comparison with that literature belongs to the count itself and is made in [Neck] §1.4. Kenyon–Smillie's criterion [KS00] for a lattice triangle, "smallest angle π/n", is $\min(P, Q-P) \in \{1, 2\}$ for a right triangle; on that locus McMullen [McM25] shows that the per-direction combinatorics of the regular polygons is controlled by arithmetic, and where his results meet ours — the Weierstrass points — the comparison is §7.5. We do not present the genus-zero construction itself as new.

1.6. **Organisation.** §2 sets up the partition and proves Theorem 2.2. §3 gives the unfolding dictionary and the fixed development, and counts the vertex images to prove the partition finite (Proposition 3.4). §4 proves Theorem A's non-existence half and turnaround type (Proposition 4.3, Theorem 4.6), states its uniqueness half (Theorem 4.7), proves existence when Q is a prime power (§4.4, Theorem 4.11) — reading off from the same disc that mirror pairs have slope -1 and are adjacent (Lemma 4.9, Corollary 4.14) — and proves it at the edges of every row, at $Q = 2p$ and at $Q = 4p$ (§4.5). §5 collects the consequences: the parity, what it leaves undetermined, and the three-sided refinement. §6 computes the surface data of S_α; §7 derives the genus-zero base, identifies the Weierstrass points, proves the fold (§7.4) and, at §7.6, proves Theorem 4.7. §8 states what is not settled. Appendix A describes the exact computation behind every count the paper quotes, and Appendix B follows the two trajectories of §4.5 by hand.

2. The perpendicular partition

§2.2 uses the unfolding apparatus of §3.1 — the developed copies and the isometries placing them (Lemma 3.1) — which is a construction and uses nothing from §2.

2.1. **Branches.** Fix the triangle and the beam as in §1.1: T has vertices $O = (0, 0)$, $R = (\cot\alpha, 0)$ and $A = (\cot\alpha, 1)$, with $L_1 = RA$, $L_2 = OR$, $H = OA$, and the beam leaves $(\cot\alpha, s)$, $s \in (0, 1)$, with velocity $(-1, 0)$. Let γ_s denote the resulting billiard trajectory.

Follow γ_s until it next meets L_1 perpendicularly, and record the sequence of sides it strikes in between. Call that finite sequence the **word** $W(s) \in \{L_1, L_2, H\}^*$ of the launch height, and call a maximal open interval of heights on which the word is constant a **branch**. Write $n(P/Q)$ for the number of branches. The map that sends a launch height to the height of that first perpendicular return is the **first-return map** J; it is defined at every s whose trajectory avoids the vertices, that is, at every s interior to a branch.

Two heights lie in the same branch exactly when their trajectories are combinatorially identical, so a branch is an equivalence class of the dynamics rather than an artefact of how the count is organised. Everything in §§2–5 is a statement about this partition.

That the partition is well defined. Three statements are easily run together here. The first two hold unconditionally and are ours to use; the third is stronger than both and is false in general. We separate them once, and keep them separate throughout.

- *The number of branches is **finite**, and the branches cover L_1 but for finitely many points, so their widths sum to* 1. This is **Proposition 3.4**, proved in §3.3 with $n(P/Q) \leq 2Q + 1$, and it is what rationality buys: a perpendicular launch reaching no vertex is periodic by Boshernitzan's theorem [Bos92, Prop. 1.8], and the launches that do reach a vertex number at most $2Q$, by a count of the horizontal rays arriving at vertex images.
- *The branches cover L_1 in the sense of measure, in **every** right triangle, rational or not* — almost every trajectory launched perpendicular to a leg returns to it in the same direction [CHK95], in the form of [Tab05] Theorem 7.5. On the rational locus this is weaker than Proposition 3.4, and **this paper does not rest on it**.
- *The perpendicular direction on the unfolded surface is **completely periodic*** — the surface is, in that direction, a finite union of closed cylinders, so that every leaf is closed or a saddle connection. This is strictly stronger than either of the above, and it is **false in general**. It is part of Boshernitzan's Conjecture 2.2(e) [Bos92]; a remark at the end of [Tro05] §3 states that its method verifies it on an infinite family of rational right triangles and the companion paper [Meth] on larger ones ([Neck] supplies a proof of Troubetzkoy's remark), and on the lattice locus it follows from Veech's dichotomy [Vee89, Propositions (2.10)–(2.11)]. But [DR22] certify a minimal component in a direction perpendicular to a side of the triangle of 8/15. It is a property of a *direction*: each centre carries two perpendicular-beam direction classes, which decompose S_α independently — at odd Q the L_1 beam and the L_2 beam lie in different classes, at even Q in the same one, the other being the class of the H beam — and every direction perpendicular to a side lies in one of them, so at 8/15 one of the two classes is not completely periodic. **No statement of this paper assumes complete periodicity.**

The third deserves a word, because the gap between it and the first two is not an artefact of what we have managed to prove. At 8/15 one of the two perpendicular-beam directions carries a minimal component of positive area, and the branch partition of the leg it is perpendicular to is nonetheless finite, complete, and of total width 1. There is no contradiction, and Proposition 3.4 — which holds for either leg, relabelling the legs replacing P/Q by $(Q - P)/Q$ — says something sharper than an appeal to measure would: the leg misses the minimal component **entirely**. A launch interior to a minimal component is not periodic, so by Proposition 3.4 it is one of the finitely many singular launches; but the leg meets an open subsurface

in a relatively open set, and an open subset of the leg is either empty or infinite. Hence it is empty, and the leg can meet the component's closure in at most finitely many points, on its boundary saddle connections. Complete periodicity is a statement about the whole surface; finiteness and covering are statements about the leg, and the leg can miss what the surface contains.

2.2. **The involution.** The first-return map is rigid, and this is the structural fact the paper turns on.

Before the involution itself we isolate the criterion that decides which branches retrace. It is elementary — time-reversibility and uniqueness of billiard orbits — and it is used in the proof of Theorem 2.2 below and again in §4.

Lemma 2.1 (retrace criterion)**.** *Let s be interior to a branch B, and let t_* be its first-return time. The following are equivalent.*

1. *$J(s) = s$.*
2. *γ_s meets a side head-on at some time $t_1 \in (0, t_*)$.*
3. *In the development of §3, some exit edge of the developed orbit strictly before the return copy is vertical.*

*When they hold, the head-on collision is **unique**, it occurs at $t_1 = t_*/2$ on a side of type L_2 or H, the orbit satisfies $\gamma_s(t_* - t) = \gamma_s(t)$, and $W(s)$ is a palindrome about it. Moreover each of (1)–(3) is a condition on the word $W(s)$ alone, so it holds at **every** point of B or at none.*

Proof. Write $\gamma = \gamma_s$ and let v be its velocity, so $v(0) = (-1, 0)$ and $v(t_*) = (+1, 0)$. Since s is interior to a branch, γ avoids the vertices of T (§2.1), so every collision occurs in the relative interior of a side, where the reflection law $v^+ = v^- - 2(v^- \cdot n)n$ gives $v^+ = -v^-$ precisely when $v^- \parallel n$ — that is, precisely at a head-on hit.

(2) $\implies$ (1). Let $t_1 \in (0, t_*)$ be a head-on collision. It is not a perpendicular meeting of L_1, which would force $t_* \le t_1$; so it lies on L_2 or H. Because $v(t_1^+) = -v(t_1^-)$, the two curves $u \mapsto \gamma(t_1 + u)$ and $u \mapsto \gamma(t_1 - u)$ are billiard orbits agreeing in position and velocity at $u = 0$, hence are equal: $\gamma(t_1 + u) = \gamma(t_1 - u)$ and $v(t_1 - u) = -v(t_1 + u)$ for all u. Taking $u = t_1$ gives $\gamma(2t_1) = \gamma(0) = (\cot\alpha, s)$ with velocity $-v(0) = (+1, 0)$, a perpendicular meeting of L_1; hence $t_* \le 2t_1$. Were $t_* < 2t_1$, then $u = t_* - t_1$ would give $\gamma(2t_1 - t_*) = \gamma(t_*)$ with velocity $-v(t_*) = (-1, 0)$, a perpendicular meeting of L_1 at the time $2t_1 - t_*$, which lies in $(0, t_*)$ because $t_1 < t_*$ — contradicting minimality of t_*. So $t_* = 2t_1$, and $\gamma(t_*) = \gamma(0)$ gives $J(s) = s$. The same computation applied to any interior head-on time shows it equals $t_*/2$, so the collision is unique.

(1) $\implies$ (2). Suppose $J(s) = s$, so that $\gamma(t_*) = (\cot\alpha, s) = \gamma(0)$ with velocity $(+1, 0) = -v(0)$. Billiard flow is time-reversible, so $\tilde\gamma(t) \coloneqq \gamma(t_* - t)$ is again an orbit; it begins at $\gamma(t_*) = \gamma(0)$ with velocity $-v(t_*) = (-1, 0) = v(0)$, so by uniqueness of the orbit through a given position and direction $\tilde\gamma = \gamma$, that is $\gamma(t_* - t) = \gamma(t)$ for all t. At $t = t_*/2$ this forces $v(t_*/2^+) =$

$-v(t_*/2^-)$, a velocity reversal. Velocity is piecewise constant along a billiard orbit, so the reversal occurs at a collision, and by the opening paragraph that collision is head-on; it is interior because $0 < t_*/2 < t_*$. Finally $\gamma(t_* - t) = \gamma(t)$ makes the sides struck on $(0, t_*/2)$ the reverse of those struck on $(t_*/2, t_*)$, so $W(s)$ is a palindrome about the collision.

(2) $\iff$ (3). By Lemma 3.1(2) the folded orbit meets a side perpendicularly exactly when the developed ray meets the corresponding wall-copy perpendicularly, and — the ray being horizontal — exactly when that wall-copy is vertical. The collisions at times in $(0, t_*)$ are exactly the exit edges strictly before the return copy.

Constancy on the branch. By Lemma 3.1(3) the developed copies T_j, hence their exit edges, are determined by the word alone. So (3) is a condition on $W(s)$, which is constant on B. □

Theorem 2.2 (the first-return involution)**.** *The first perpendicular-return map J is an involution, $J^2 = \mathrm{id}$, and is affine with $|slope| = 1$ on each branch. Hence exactly one of two things happens on each branch:*

- *J carries the branch isometrically onto a **distinct** branch of equal width: generic branches occur in equal-width **mirror pairs**.*
- *$J = \mathrm{id}$ on the whole branch. Every trajectory launched in it meets a side head-on and retraces to its own launch height. We call such a branch an **orphan**.*

Proof. Fix a branch B with word W, and unfold the beam, keeping it the horizontal line $y = s$ and reflecting T across each side the beam exits (Lemma 3.1). By Lemma 3.1(3) the developed copies are the same for every $s \in B$.

J is affine with $|slope| = 1$, and the slope is a property of the branch. The launch copy of L_1 is the vertical segment $\{\cot\alpha\} \times [0, 1]$, and the copy of L_1 at which the beam makes its first perpendicular return is vertical as well, by Lemma 3.1(2); write it as $g(L_1)$, where in the notation of Lemma 3.1 $g = g_{|W|} = R_{e_0} \circ \cdots \circ R_{e_{|W|-1}}$ is the product of the side reflections named by W, determined by W alone. (The *intervening* copies of the launch leg that the beam re-meets are crossed obliquely and are not in general vertical; the argument uses only the return copy, which is the one the definition of J selects.) The ray $\{y = s\}$ meets $g(L_1)$ at the point of height s, and since L_1 and $g(L_1)$ are both vertical, height along each is arclength. Folding back by g^{-1} therefore gives

$$J(s) = \varepsilon \cdot s + d, \qquad \text{where} \qquad \varepsilon \in \{+1, -1\}$$

is the sign with which the linear part of g acts on the vertical direction, and d is fixed by the height of $g(R)$. Both are determined by W alone, so both are constant on B.

J is an involution. Let $\gamma = \gamma_s$ have first-return time t_*. Time-reversibility makes $\tilde\gamma(t) \coloneqq \gamma(t_* - t)$ a billiard orbit; it begins at $(\cot\alpha, J(s))$ with velocity $(-1, 0)$, so it is the perpendicular launch from height $J(s)$, and at time t_* it

is at $(\cot\alpha, s)$ with velocity $(+1, 0)$. That is its *first* perpendicular return, since a perpendicular meeting of L_1 by $\tilde{\gamma}$ at some $t' \in (0, t_*)$ would be one by γ at $t_* - t' \in (0, t_*)$. Hence $J(J(s)) = s$.

$J(B)$ is a branch. The time-reversed orbit $\tilde{\gamma}$ above is the perpendicular launch from $J(s)$ and strikes the sides γ strikes in the reverse order, so $W(J(s))$ is the reverse of $W(s)$ and is constant on B; hence $J(B)$ lies in a single branch B'. The same applied to B' gives $B \subseteq J(B')$ with $J(B')$ inside a single branch, which by maximality of B is B itself. So $J(B') = B$, and applying J once more, $J(B) = B'$ — a branch of equal width, J being an isometry.

The dichotomy. Suppose $B' = B$, so that J is an affine isometry of the interval B onto itself. If $\varepsilon = +1$ it is a translation of an interval onto itself, so $d = 0$ and $J = \mathrm{id}$ on the whole branch: every trajectory launched in it returns to its own launch height, which by Lemma 2.1 is exactly to say that it meets a side head-on at an interior time and retraces. If $\varepsilon = -1$ it is the reflection of B about its midpoint s_*, so $J(s_*) = s_*$, and by Lemma 2.1 — each of whose conditions holds at every point of B or at none — $J = \mathrm{id}$ on B, contradicting $\varepsilon = -1$. So either $J = \mathrm{id}$ on B, or $B' \neq B$ and $J(B) = B'$ is a distinct branch of equal width. □

On a mirror pair J is in fact the mirror map $s \mapsto 2s_* - s$, of slope -1, but this argument does not show it: $\varepsilon = +1$ would give a translation $s \mapsto s + d$, $d \neq 0$, carrying B onto a branch on which J is $s \mapsto s - d$. Translations do occur in other rational polygons — in the triangle with angles $(\pi/6, \pi/6, 2\pi/3)$ the beam perpendicular to the base $[0, 1]$ has $J(s) = s + \frac{1}{2}$ for $s \in (0, \frac{1}{2})$ — but not here. That is Lemma 4.9, and it needs the folded disc of §4.4, which is where the right angle enters.

The orphan is a purely rational object: the proof of Lemma 4.1 confines the direction of a perpendicular orbit to the values $2m\alpha$ modulo π, and at irrational α none of them is the normal of L_2 or of H, so no perpendicular orbit turns around head-on on a side.

Which branches retrace, and whether any does, is the content of §4: Lemma 2.1 supplies the criterion and Theorem 2.2 makes it a property of the branch. J also has a topological reading — the launch leg is one half of a geodesic segment folded by the hyperelliptic involution, and J is the perpendicular flow's first return to that segment read in the fold (Propositions 7.5 and 7.6, §7.4) — which is never an input to §§2–5.

3. The unfolding dictionary and the fixed development

The elementary half of the main theorem in §4 — a parity argument and a congruence count — is short provided two facts about the *development* are available. We isolate them here; neither uses the first-return map, so §2's forward references to them are not circular. §3.3 then counts the vertex images and proves the partition finite.

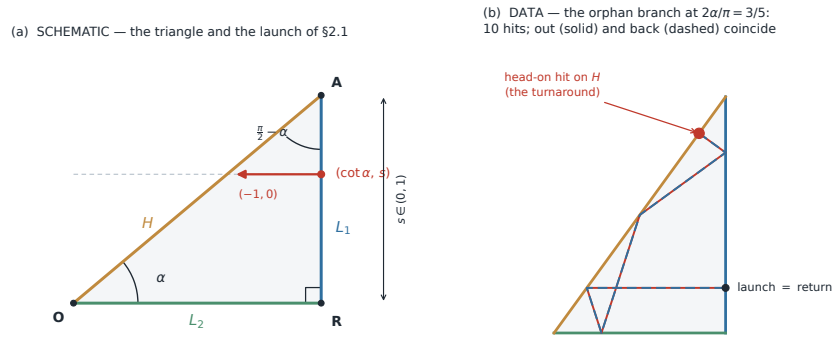


FIGURE 1. (a) The triangle T, its three labelled sides and the launch parametrisation of §2.1: the beam leaves $(\cot\alpha, s)$ with velocity $(-1, 0)$. Schematic — the drawing angle is not a centre. (b) An orphan branch, at $2\alpha/\pi = 3/5$, traced. The orbit strikes the hypotenuse head-on at its temporal midpoint and retraces its own path, so the outbound and return legs coincide and it returns to the height it left from: $J = \mathrm{id}$ on this branch.

3.1. **The dictionary.**

Lemma 3.1 (unfolding dictionary)**.** *Develop the beam by keeping it the horizontal ray $\{y = s\}$ and reflecting the triangle across each side the ray exits, producing copies $T_i = g_i(T_0)$ with $g_i = R_{e_0} \circ \cdots \circ R_{e_{i-1}}$, each R_e the reflection of the plane in the line of the side e of T_0. Then:*

1. *(beam $\leftrightarrow$ ray) the folded billiard orbit in T_0 is exactly g_i^{-1} applied to the straight ray inside T_i;*
2. *(head-on $\leftrightarrow$ vertical) the folded orbit meets a side perpendicularly if and only if the ray meets the corresponding wall-copy perpendicularly, and since the ray is horizontal this holds if and only if that wall-copy is **vertical**;*
3. *(orientations do not move) the copy reached after a word W has orientation $g_W(T_0)$ with g_W determined by W alone. Whether a copy of a given side is vertical is therefore a condition on g_W, independent of the launch height s.*

Proof. (1) is one step plus induction: if the ray hits an edge e, the billiard continuation is R_e applied to the incoming direction inside T_0, while the developed continuation is the straight ray inside $R_e(T_0) = T_1$, and the isometry R_e carries one to the other. (2) is (1) plus the fact that g_i is an isometry, so perpendicularity is preserved; the ray being horizontal turns "perpendicular to the ray" into "vertical". (3) is immediate from the definition of g_W, which involves only the letters of W. □

A consequence of (1) worth recording: every side of T_0 becomes an *interior* edge of the development, crossed straight. There are no residual walls in the unfolded picture.

3.2. **The development is fixed.** Here Σ denotes the closed surface obtained by gluing copies of T along matching edges, not a planar development; it is the surface §6 studies as S_α. The second fact is the one that does the real work in §4: it makes the vertical copies features of a single complex, the *same* copies for every launch height, so that a count of them is a statement about every orbit at once. It does not say which of them a given orbit reaches (§4.4).

Lemma 3.2 (fixed development)**.** *All perpendicular beams share the horizontal direction, so they develop on one fixed complex Σ depending only on (P, Q). In particular the vertical wall-copies of Σ form a fixed, finite, launch-height-independent set.*

Proof. The group $\mathcal{G}$ generated by the side reflections is finite: its rotation subgroup is generated by the vertex rotations through $2\cdot(P/2Q)\pi$, $2\cdot(1/2)\pi$ and $2\cdot((Q-P)/2Q)\pi$, all rational multiples of π, so $\mathcal{G}$ is a finite dihedral group. Only finitely many copy-orientations therefore occur. Glue the triangle copies along matching edges. Around every vertex the development closes with no monodromy defect, because each angle of T is a rational multiple of π: an angle $(a/b)\pi$ in lowest terms is surrounded by $2b$ copies, giving cone angle $2b\cdot(a/b)\pi = 2a\pi$, a multiple of 2π. (The three angles are $(P/2Q)\pi$ at O, $(1/2)\pi$ at R and $((Q-P)/2Q)\pi$ at A.) Hence Σ is a well-defined closed complex determined by (P, Q) alone, every perpendicular beam is a horizontal geodesic on it, and the launch height only selects which cells that geodesic threads. The vertical wall-copies are features of Σ, so they do not depend on s. □

Figure 2 shows a development, with its vertical wall-copy, at two centres of opposite parity. Note that Lemma 3.2 is proved from a cone-angle computation and invokes no translation-surface machinery. The surface Σ is of course the object §6 studies as S_α, but §§3–5 use only its finiteness and its closing-up.

3.3. **The partition is finite.** Lemma 3.2's cone-angle computation, pushed one step further, counts the vertex images and the horizontal rays that can reach them. That count is what makes the branch partition finite.

Lemma 3.3 (vertex images and their prongs)**.** *On Σ the images of the vertex O number $e \coloneqq \gcd(P, 2Q)$, each of cone angle $2\pi P/e$; the images of A number $e' \coloneqq \gcd(Q-P, 2Q)$, each of cone angle $2\pi(Q-P)/e'$; and the images of R number Q, each of cone angle 2π. Consequently the horizontal rays arriving at a vertex image — the arriving **prongs** — number exactly*

$$P + (Q-P) + Q = 2Q.$$

Proof. Continue the computation in the proof of Lemma 3.2. The angle at O is $(a/b)\pi$ with $a = P/e$ and $b = 2Q/e$ in lowest terms, so that vertex is surrounded by $2b = 4Q/e$ copies; $\mathcal{G}$ has order $4Q$ and every copy carries one O-corner, so the images of O number $4Q/(4Q/e) = e$ and the cone angle at each is $2a\pi = 2\pi P/e$. A cone point of angle $2\pi k$ is approached by exactly k rays of any one fixed direction, so the images of O are reached by $e \cdot (P/e) = P$ prongs. The same computation at A gives $Q - P$. At R the angle is $\pi/2$, so $b = 2$, four copies surround each image, the cone angle is 2π — a regular point of the flat structure, marked only because it is a vertex of T — and there are $4Q/4 = Q$ of them, each reached by a single prong. □

The two parities are both inside this and neither is a special case: at odd P one has $e = 1$, a single image of O of cone angle $2\pi P$, while at even P (which forces Q odd) $e = 2$ and there are two, of half that angle. The product is P either way, which is why nothing below splits on the parity of P.

Proposition 3.4 (finiteness, and the covering)**.** *At most $2Q$ launch heights on L_1 have an orbit that reaches a vertex of T; every other launch is periodic. Hence the branch partition of L_1 is **finite**, with*

$$n(P/Q) \leq 2Q + 1,$$

and the branches cover L_1 but for finitely many points, so $\Sigma w = 1$. No hypothesis on the direction is needed.

Proof. Two steps. The first is [Bos92]; the second is the argument of [BGKT98] made exact.

Every launch that reaches no vertex is periodic. This is Boshernitzan [Bos92], Proposition 1.8: in a rational polygon every *regular symmetric* orbit is periodic, where *symmetric* means it meets a side or an axis of symmetry perpendicularly and *regular* means it avoids the vertices. A perpendicular launch from L_1 is symmetric, so the statement applies verbatim. Vorobets, Gal'perin and Stepin [GSV92] (§5, proof of Theorem 2) give an argument, through Poincaré recurrence, for the case of a side, the one used here.

The launches that do reach a vertex number at most $2Q$. On Σ a launch has two lifts, one in T itself and one in its mirror image across L_2; we use the lift in T throughout, so that a launch is a point of the copy of L_1 in T and its orbit a leftward horizontal leaf. Call such a launch *singular*, and send it to the pair consisting of the **first** vertex image its orbit reaches and the prong on which it arrives — first, because an image of R is a regular point of the flat structure, so an orbit passes through it and may reach a further image later. This map is injective. Suppose two singular launches $s \neq s'$ share a pair (V, p). Both lie on the backward continuation of p, so one of them — say s' — is the farther from V, and then the orbit of s' passes through the launch height s on L_1, meeting L_1 perpendicularly there, at some time $t_1 > 0$. A perpendicular meeting reverses the velocity — $v^+ = -v^-$, the mechanism spelled out in the proof of Lemma 2.1 — so the orbit retraces

the path it came by: $\gamma_{s'}(t_1+t) = \gamma_{s'}(t_1-t)$. It was launched perpendicular from L_1, so it retraces at $t = 0$ as well. Now V is the *first* vertex image its orbit reaches and it reaches it after t_1, so $[0, t_1]$ carries no vertex image; the retrace copies that stretch onto $[t_1, 2t_1]$, which therefore carries none either; and at $2t_1$ the orbit is back at s' on L_1, perpendicular. So it closes up with period $2t_1$ and is vertex-free for all time, and it never reaches V at all. Contradiction. The pairs therefore number at most the prongs, which is $2Q$ by Lemma 3.3. The last step needs the *first*-arrival convention and not merely periodicity: an image of R is removable, so a periodic orbit may perfectly well pass through one.

The two steps combine. A non-singular launch is periodic, so its orbit is a closed path of finitely many segments meeting the sides transversally and away from the vertices; a small change of launch height reproduces that path segment by segment, so the word is locally constant off the singular set. The singular set has at most $2Q$ elements, so L_1 minus it has at most $2Q+1$ components, the word is constant on each, and each is contained in a branch: hence $n(P/Q) \le 2Q+1$. Those components exhaust L_1 but for at most $2Q$ points, so the branch widths sum to 1. □

The first step is [Bos92]'s, and [Bos92] prices the singular set only as *at most countable*. The second is proved for every rational polygon in [BGKT98, proof of Theorem 4], by the same mechanism — a singular perpendicular orbit retraces, so it is a generalized diagonal in a fixed direction — with a count by the floors of the invariant surface; a finiteness argument by non-overlapping beams is sketched in [CHK95] (p. 2069), and a smaller count over all three sides together, $2Q-3$, is stated with a one-line argument in [TGMM18] (Section 11, Fact II). What is added here is a complete proof of an explicit bound for one side, by the prong count (Lemma 3.3), from which the covering $\Sigma w = 1$ follows.

4. The orphan theorem

The elementary argument has two steps. The first is a parity obstruction and decides *non-existence*: for even P the direction a head-on hit would require is simply not attainable. The second is a congruence count and types the turnaround: for odd P exactly one side type is ever perpendicular to the beam, for exactly one residue. Neither step decides how many orphans there are, and neither produces one. §4.3 states uniqueness, whose proof needs the surface of §§6–7; §4.4 proves existence when Q is a prime power, by an argument of a different kind, and states the general case as a conjecture; §4.5 carries that argument to $Q = 2p$, $Q = 4p$ and the edges of every row.

4.1. Non-existence at even P.

Lemma 4.1 (velocity set)**.** *Along any perpendicular orbit the velocity direction, taken modulo π, lies in the finite set $\mathcal{D} = \{k\pi/Q : k = 0, \dots, Q-1\}$ — the orbit of the launch direction under rotation by $2\alpha = P\pi/Q$.*

Proof. Reflection in a side of direction θ sends a velocity direction ψ to $2\theta-\psi$ modulo π. The three side directions are $\theta_{L_1} = \pi/2$, $\theta_{L_2} = 0$ and $\theta_H = \alpha$, so modulo π the three reflections act on directions as $\psi \mapsto -\psi$, $\psi \mapsto -\psi$ and $\psi \mapsto 2\alpha-\psi$. Every composite of them is therefore of the form $\psi \mapsto \pm\psi+2m\alpha$ with m an integer, so from the launch direction $\psi = 0$ the attainable set is $\{2m\alpha\} = \{mP\pi/Q\}$ modulo π; and since $\gcd(P, Q) = 1$, mP runs over every residue modulo Q, making that set $\{k\pi/Q : k = 0, \dots, Q-1\} = \mathcal{D}$. □

We stress what Lemma 4.1 does and does not say. The *sequence* of directions realised along an orbit depends on the launch height — two nearby heights diverge after a few hits. Only the *set* of attainable values is height-independent, and that is all the argument uses; in particular no common development is assumed.

Lemma 4.2 (retrace $\iff$ head-on)**.** *A branch is an orphan if and only if its orbits meet some side head-on at an interior time.*

Proof. By Theorem 2.2 a branch is an orphan exactly when $J = \mathrm{id}$ on it. By Lemma 2.1, $J(s) = s$ holds at a point s interior to a branch exactly when γ_s meets a side head-on at an interior time, and each of the two conditions holds at every point of the branch or at none. So the orphan branches are exactly the branches with an interior head-on hit. □

Proposition 4.3 (parity and non-existence)**.** *An orphan branch exists only if P is odd. For P even there is none, and consequently $n(P/Q)$ is even.*

Proof. A hit on a side of type $X \in \{L_1, L_2, H\}$ is head-on exactly when the velocity equals the normal direction v^*_X modulo π, and these are $v^*_{L_1} = 0$, $v^*_{L_2} = \pi/2$, and $v^*_H = \alpha + \pi/2 = (P + Q)\pi/(2Q)$. By Lemma 4.1 a head-on hit requires $v^*_X \in \mathcal{D}$. Now:

- $v^*_{L_1} = 0 \in \mathcal{D}$ always, but $k = 0$ is the launch and return direction; an L_1 head-on is the perpendicular return itself, not an interior retrace.
- $v^*_{L_2} = \pi/2 = (Q/2)(\pi/Q) \in \mathcal{D}$ if and only if Q is even.
- $v^*_H = ((P + Q)/2)(\pi/Q) \in \mathcal{D}$ if and only if $P + Q$ is even.

Since $\gcd(P, Q) = 1$, if P is even then Q is odd, so neither L_2 nor H is enabled and no interior head-on hit is even directionally possible. By Lemma 4.2 no branch retraces, so by Theorem 2.2 every branch is paired with a distinct branch of equal width, and $n(P/Q)$ is even. If P is odd, exactly one interior head-on type is enabled — L_2 when Q is even, H when Q is odd. □

4.2. **The orphan, its turnaround, and how many there are.** Proposition 4.3 leaves open how *many* orphans there are when P is odd. The following congruence count types the turnaround of any orphan there is. It does not produce one, and it does not bound their number; §4.3 takes up the second question and §4.4 the first.

Proposition 4.5 indexes wall-copies by a count that the rest of §4.2 turns on, so we fix it first.

Definition 4.4 (rhombus steps)**.** In the development, every copy of the side X has direction $\theta_X + k \cdot (P\pi/Q)$ modulo π for a unique $k \in \mathbb{Z}/Q$, where $\theta_{L_1} = \pi/2$, $\theta_{L_2} = 0$, $\theta_H = \alpha$. We say such a copy is carried k **rhombus steps**.

Both halves are immediate from the proof of Lemma 4.1. The directions available to a copy of X are $\{\mathrm{Lin}(g) \cdot \theta_X : g \in \mathcal{G}\}$, and the linear parts act on directions as $\psi \mapsto \pm\psi + 2m\alpha$; for each of the three side types the minus branch is already in the plus branch — at L_1 and L_2 because $2\theta_X \equiv 0$, and at H because $-\alpha \equiv \alpha + 2(Q-1)\alpha$ modulo π — so the available directions are exactly $\theta_X + 2m\alpha = \theta_X + m \cdot (P\pi/Q)$. Uniqueness of k is that $(k-k') \cdot (P\pi/Q) \equiv 0$ modulo π forces $Q \mid k-k'$, again by $\gcd(P, Q) = 1$. The step of 2α is the rotation at O, that is, the product of the reflections in the two sides meeting there, and the name echoes the rhombus decomposition of [CHK95], with rhombi labelled by rotations through 2α, as taken up in [GZ03] (§1.5).

Note that k indexes the copy's **direction**: it is an element of $\mathbb{Z}/Q$, so the range $k = 0, \ldots, Q-1$ below runs over one full cycle of directions.

Proposition 4.5 (one vertical direction class per cycle of directions)**.** *In the development, a side of type X carried k rhombus steps has orientation $\theta_X + k \cdot (P\pi/Q)$ modulo π, with $\theta_{L_1} = \pi/2$, $\theta_{L_2} = 0$ and $\theta_H = \alpha$. Over one cycle of directions $k = 0, \ldots, Q-1$, the number of classes $(X, k) \neq (L_1, 0)$ whose copies are vertical is* ***exactly*** *1* ***when*** *P* ***is odd*** *—* ***with*** *$X = H$* ***when*** *Q* ***is odd and*** *$X = L_2$* ***when*** *Q* ***is even*** *—* ***and*** *0* ***when*** *P* ***is even.*** *(This counts direction classes, not copies: one return passage can meet many copies in the vertical class — §4.3 — and need not meet any — §4.4.)*

Proof. Verticality means orientation $\equiv \pi/2$ modulo π. Writing the three conditions out: L_1 gives $kP \equiv 0 \pmod{Q}$, hence $k = 0$ only, which is the excluded launch copy; L_2 gives $2kP \equiv Q \pmod{2Q}$; and H gives $(2k+1)P \equiv Q \pmod{2Q}$. Count the interior solutions in each parity class.

Q odd, P odd. Then $\gcd(P, 2Q) = 1$, so multiplication by P permutes the residues modulo $2Q$ and preserves parity. It therefore carries the odd residues $\{2k+1\}$ bijectively onto the odd residues, and Q is odd, so $(2k+1)P \equiv Q$ has exactly one solution. The L_2 congruence has none, its left side being even and Q odd.

Q even, P odd. The L_2 congruence $2kP \equiv Q \pmod{2Q}$ is $kP \equiv Q/2 \pmod{Q}$, which has exactly one solution since $\gcd(P, Q) = 1$. The H congruence has none, its left side being odd and Q even.

P even, so Q odd. The H congruence is odd $\equiv$ even and the L_2 congruence is even $\equiv$ odd. Neither is solvable. □

Lemma 2.1 places the turnaround of an orphan orbit at its temporal midpoint and makes it unique, and its hypothesis that s is interior to a branch is not decoration. A ray driven into the right-angle vertex R reverses

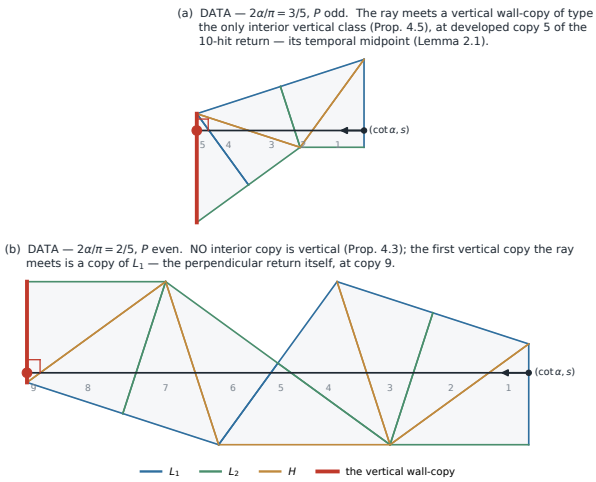


FIGURE 2. The development of Lemma 3.1, with the ray kept horizontal and the triangle reflected across each side the ray exits. Copies are numbered in the order the ray enters them, and the side types are coloured. (a) $2\alpha/\pi = 3/5$, P odd: the fifth copy presents a vertical copy of H, the ray meets it perpendicularly, and by Lemma 3.1(2) that is a head-on hit in the folded picture — the same turnaround drawn folded in Figure 1(b). Its class is the only interior vertical class Proposition 4.5 allows, and it sits at the midpoint of a ten-hit return, counting the final perpendicular hit on L_1, as Lemma 2.1 requires. (b) $2\alpha/\pi = 2/5$, P even: no interior copy is vertical at all, and the first vertical copy the ray meets is a copy of the launch leg L_1 — the perpendicular return itself, which is not an interior retrace.

without meeting any side perpendicularly, because the two reflections there commute and compose to $-\mathrm{id}$; R is the only vertex of T with that property, the products at O and A being rotations by 2α and $\pi - 2\alpha$. But a height whose ray grazes a vertex is a branch *boundary*, where J is undefined, so this case does not arise at any point interior to a branch — which is the hypothesis Lemma 2.1 carries.

Theorem 4.6 (Theorem A, non-existence and turnaround)**.** *Let $2\alpha/\pi = P/Q$ in lowest terms. If P is even there is no orphan branch. Every orphan orbit turns around at a* ***single*** *head-on collision, at its temporal midpoint, and that collision lies on a side of type H when Q is odd and of type L_2 when Q is even.*

(No hypothesis on the direction is needed. Neither clause asserts that an orphan exists: that is Theorem 4.11 when Q is a prime power, Theorems 4.18

and 4.24 at further centres, and Conjecture 4.12 in general, which [Meth] *proves on large families.)*

Proof. If P is even, Proposition 4.3 gives no orphan. Turnaround: let s be any orphan point, so $J(s) = s$. By Lemma 2.1 its orbit has exactly one head-on collision, it occurs at $t_*/2$, and it lies on a side of type L_2 or H. Which of the two is Proposition 4.5: by Lemma 3.1(2) the collision is the ray meeting a vertical wall-copy, and of the three congruences for verticality only one is solvable — the H congruence when Q is odd, the L_2 congruence when Q is even. □

4.3. **Uniqueness.** Theorem 4.6 does not say how many orphan branches there are, and the elementary count does not reach it. Proposition 4.5 counts vertical *directions*, not the vertical copies a single return passage meets, and there can be many more of those: the orphan's return word at 5/24 has 311 letters, against $2Q = 48$, so the vertical direction class recurs many times along it, at distinct copies and distinct heights. *Every orphan orbit turns around on a vertical copy* is proved; *every orphan orbit turns around on the same one* is not.

Theorem 4.7 (Theorem A, uniqueness)**.** *Let P be odd. Then there is* ***at most one*** *orphan branch.*

The proof is an Euler count on the genus-zero quotient of the unfolded surface, which admits minimal components, so no hypothesis on the direction is needed; it is given at §7.6, once the surface of §§6–7 is available. Theorems 4.6 and 4.7, with Theorems 4.11 and 4.24 and Corollary 4.20 below, make up Theorem A's parts (a)–(c) and its turnaround clause; the width-pairing involution is Theorem 2.2, and the parity of $n(P/Q)$ is Corollary 5.1. What is unique is the **branch** — an interval of launch heights of positive width, every one of which retraces — and not a single trajectory.

4.4. **Existence when Q is a prime power.** The congruence count of §4.2 does not produce an orphan. At odd P it finds a vertical copy of the far side, which a ray meets head-on; it does not find a launch height that reaches that copy before returning to L_1, and Lemma 3.2 makes the vertical copies independent of the launch height, not reachable. Here we fold the development onto a disc in which the launch leg and that copy are the only two vertical boundary sides, so that the question becomes where the horizontal leaves from the leg end. If no leaf ends on the copy, the leaves organise themselves into fans, and the fans impose an arithmetic identity on the heights of the images of R. When Q is a prime power the identity is impossible.

Index the copies of T in Σ by the elements of $\mathcal{G}$ — the linear part of the developing isometry — so that copy g is glued across its side s to copy gr_s, where $r_s \in \mathcal{G}$ is the reflection in the direction of s. This is the presentation §7.6 uses again. Let $V_4 \subset \mathcal{G}$ be the stabiliser of the horizontal direction: the

identity, the rotation -1 through π (a rotation of $\mathcal{G}$, the rotations being the multiples of π/Q), and the reflections r_h, r_v in the horizontal and vertical directions, which are the reflections in L_2 and in L_1. V_4 acts on Σ by $h \cdot (x, g) = (x, hg)$, compatibly with the gluing, by isometries that carry the horizontal line field to itself.

Lemma 4.8 (the folded disc)**.** *Let $D \coloneqq \Sigma/V_4$. Then D is a flat closed disc glued from Q copies of T, and the horizontal foliation of Σ descends to it. Its boundary consists of the sides of copies that are vertical or horizontal: the launch leg L_1 of T itself, written $\bar{\Lambda}$; when P is odd, exactly one vertical copy $\bar{W}$ of the far side — of H when Q is odd and of L_2 when Q is even — and when P is even, no vertical side but $\bar{\Lambda}$; the side L_2 of T; when Q is even, one horizontal copy of L_1; and when P is even, one horizontal copy of H. The images of O form a single point $\bar{O}$ and the images of A a single point $\bar{A}$, both on ∂D, and $\bar{A}$ and the right angle $\bar{R}$ of T are the two ends of $\bar{\Lambda}$. Every other singular point of D — apart, when Q is even, from the boundary corner $\bar{R}'$ where $\bar{W}$ meets the horizontal copy of L_1 — is an interior image of R, a cone point of angle π — a **pole** — from which exactly one horizontal ray, its **prong**, issues.*

Proof. V_4 acts freely on the labels, so D has $4Q/4 = Q$ copies, indexed by the cosets $V_4 g$, and copy $V_4 g$ is glued across s to copy $V_4 g r_s$. The side s of that copy is glued to itself, and so becomes a boundary side, exactly when $g r_s g^{-1} \in V_4$. That element is the reflection in the line $g(s)$, so this happens exactly when $g(s)$ is vertical or horizontal. Measure directions in units of $\pi/2Q$, as in Theorem 5.3: the sides of T have indices Q, 0, P for L_1, L_2, H, the elements of $\mathcal{G}$ preserve the parity of an index, and vertical and horizontal are the indices Q and 0. Since the rotation through π/Q moves an index by 2, a side reaches a given direction exactly when its index has the parity of that direction, and then the g carrying it there form a single coset of the stabiliser of that direction — V_4, for the vertical and for the horizontal alike — so a single copy of D. Hence L_1 is vertical in exactly one copy, T itself, and horizontal in one copy when Q is even and in none when Q is odd; L_2 is horizontal in T and vertical in one copy when Q is even; and H, of index P, is vertical in one copy when P and Q are both odd, horizontal in one copy when P is even, and otherwise neither. This is the list of boundary sides.

By Lemma 3.3 the images of O in Σ number $\gcd(P, 2Q)$ — one at odd P, two at even P — and the images of A number $\gcd(Q - P, 2Q)$ — one when $Q - P$ is odd, two when it is even. Where there are two, -1 exchanges them. The copies around an image of O form a left coset gK of the dihedral group K generated by the reflections in L_2 and H, whose rotations are the multiples of $2\alpha = P\pi/Q$; the rotation -1 is central, so it fixes the coset exactly when π is such a multiple, that is, when $mP \equiv Q \pmod{2Q}$ is solvable, which needs P odd. At A the same holds with the reflections in L_1 and H, whose rotations are the multiples of $(Q-P)\pi/Q$, and $Q-P$ in place of P. So D has one point $\bar{O}$ and one point $\bar{A}$. The copies around an image

of R form a left coset gV_4, and its stabiliser in V_4 is $V_4 \cap gV_4g^{-1}$. That is all of V_4 when the sides of copy g at R are boundary sides, which happens for $\bar{R}$ and, when Q is even, for one further boundary point $\bar{R}'$ — the copy of the rotation through $\pi/2$, which turns L_2 vertical and L_1 horizontal, so that its sides at R are $\bar{W}$ and the horizontal copy of L_1; otherwise it is $\{\pm 1\}$, and the image is an interior cone point of angle $2\pi/2 = \pi$. A cone point of angle π has exactly one horizontal direction issuing from it. Finally, with f boundary sides and b boundary images of R ($f = 3$, $b = 1$ at odd Q; $f = 4$, $b = 2$ at even Q), $\chi(D) = (2+b+(Q-b)/2)-(3Q+f)/2+Q = 1$. A compact connected surface with non-empty boundary and Euler characteristic 1 is a disc. □

Lemma 4.9 (mirror pairs have slope -1)**.** *Let s be a non-singular launch height with $J(s) \neq s$. Then J has slope -1 on the branch of s. So on every mirror pair of Theorem 2.2 J is a mirror map $t \mapsto d - t$, and no branch is carried onto another by a translation.*

Proof. Follow the horizontal leaf of D from the point of $\bar{\Lambda}$ at height s. It is the beam orbit of s: it crosses each side that is not a boundary side as the billiard orbit reflects there (Lemma 3.1(1)), it cannot run into a horizontal boundary side, being parallel to it, and it ends at its first head-on collision, after finite length (Proposition 3.4). A head-on hit is exactly an arrival at a vertical boundary side: the elements carrying a side to vertical form a single V_4-coset (proof of Lemma 4.8), so every vertical copy in the development lies in the copy of D whose side is $\bar{\Lambda}$ or $\bar{W}$ (Lemma 3.1(2)). That collision is not on $\bar{W}$, for it would be an interior head-on hit and Lemma 2.1 would give $J(s) = s$; so it is the perpendicular return, at height $J(s)$ on $\bar{\Lambda}$. Call this leaf γ_s. It is simple: two leaves of the horizontal line field do not cross, a return of γ_s to one of its points in the same direction would make it a closed leaf, which cannot end on $\bar{\Lambda}$, and a return in the opposite direction would need a reversal, which away from ∂D only a pole can produce — and the orbit of s meets no vertex image. So γ_s and the segment of $\bar{\Lambda}$ between s and $J(s)$ form a simple closed curve in the disc D, which bounds a closed disc $U_s \subset D$ meeting ∂D only in that segment.

Let I be the open interval between s and $J(s)$. The leaf from a non-singular height $t \in I$ enters U_s, cannot cross γ_s, and ends on a vertical boundary side, of which U_s contains only the closure of I. So J maps the non-singular heights of I into I.

Now let B be the branch of s and suppose that J has slope $+1$ on it: $J(t) = t + d$ with $d = J(s) - s \neq 0$. By Theorem 2.2 J carries B onto a branch on which $J(u) = u - d$, so $J(s)$ is a non-singular height of the same kind with d replaced by $-d$; replacing s by $J(s)$ if necessary, we may take $d > 0$. Then every $t \in B$ slightly above s is a non-singular height in $I = (s, s + d)$, while $J(t) = t + d > J(s)$ lies outside I. This contradicts the previous paragraph. □

The lemma uses neither the slope clause it proves nor the hypothesis of Lemma 4.10 below; what it uses is that D is a disc, which is where the right angle enters. In other rational polygons the corresponding quotient need not be a disc, and translations occur (§2.2).

Lemma 4.10 (the fan tiling)**.** *Suppose there is no orphan branch. Then there are poles $q_1, \dots, q_m$ of D and heights $0 = e_0 < c_1 < e_1 < c_2 < \cdots < c_m < e_m = 1$ such that the prong of q_k first meets ∂D on $\bar{\Lambda}$, at height c_k, and $e_{k-1} + e_k = 2c_k$ for every k. In particular*

$$1 = \sum_{k=1}^{m} (-1)^{m-k} \cdot 2c_k. \tag{4.1}$$

Proof. (i) *The leaves.* Let $s \in (0,1)$ be a non-singular launch height in the sense of Proposition 3.4. As in the proof of Lemma 4.9, the horizontal leaf of D from the point of $\bar{\Lambda}$ at height s is the beam orbit of s and ends at its first head-on collision, after finite length. If that collision is on $\bar{W}$ — which exists only at odd P — it is an interior head-on hit and s is an orphan point (Lemma 4.2). So under the hypothesis it is on $\bar{\Lambda}$: it is the perpendicular return, at height $J(s) \neq s$ (Theorem 2.2). Each such leaf is therefore an arc γ_s from s to $J(s)$ on $\bar{\Lambda}$, meeting ∂D nowhere else.

(ii) *One pole per arc.* As in the proof of Lemma 4.9, γ_s is simple, and with the segment $[s, J(s)]$ of $\bar{\Lambda}$ it bounds a closed disc $U_s \subset D$ meeting ∂D only in that segment. The segment avoids the ends $\bar{R}$ and $\bar{A}$ of $\bar{\Lambda}$, so U_s contains none of $\bar{O}$, $\bar{A}$ and the boundary images of R. Its two corners, at s and $J(s)$, are right angles, and its sides are geodesics. The Gauss–Bonnet formula for a flat disc, $\sum_{\text{corners}}(\pi - \varphi) + \sum_{\text{cone points}}(2\pi - \theta) = 2\pi$, then reads $\pi + \Sigma(2\pi - \theta) = 2\pi$. The cone points in U_s are poles, each contributing π, so U_s **contains exactly one pole**, q say.

(iii) *The arc is centred on the prong.* A leaf from a non-singular height in $(s, J(s))$ enters U_s and can leave it neither across γ_s nor across ∂D, so it ends in $(s, J(s))$: J maps the non-singular heights of $(s, J(s))$ into $(s, J(s))$. On each branch J is affine of slope -1, by Lemma 4.9, since no branch is an orphan. If $(s, J(s))$ held no singular height it would lie in one branch, and J would be an affine map of slope -1 of the interval into itself, with a fixed point at its midpoint — an orphan point. So $(s, J(s))$ holds a singular height c. Its leaf enters U_s and runs into a vertex image before it can reach $\bar{\Lambda}$ again — otherwise its orbit would return to L_1 perpendicularly first, retrace, and be periodic and vertex-free, as in the second step of Proposition 3.4 — and the only vertex image in U_s is q; so the leaf from c is the prong of q. Distinct singular heights have distinct prongs (the injectivity in the second step of Proposition 3.4), so $c = c_q$ is the only singular height in $(s, J(s))$. Now (s, c_q) contains no singular height, so it lies in the branch of s (Proposition 3.4), and there $J(t) = s + J(s) - t$. Its image $(s + J(s) - c_q, J(s))$ consists of non-singular heights, so it does not contain c_q, and $c_q \leq (s + J(s))/2$. The same

argument on $(c_q, J(s))$, which lies in the branch of $J(s)$, gives the reverse inequality. So $[s, J(s)]$ is centred at c_q.

(iv) *The fans.* For a pole q, let I_q be the union of the segments $[s, J(s)]$ with $q \in U_s$. By (iii) these segments share the centre c_q, so they are nested and I_q is an interval centred at c_q. Two arcs γ_s, $\gamma_{s'}$ that are not the same leaf are disjoint. Their segments cannot interleave, or $\gamma_{s'}$ would start inside U_s and end outside it. If the segments were nested with $q \in U_s$ and $q' \in U_{s'}$, then $U_s \supset U_{s'}$ would contain both q and q', against (ii). So distinct poles have disjoint intervals. Every non-singular height lies in some I_q, so the intervals I_q cover $(0, 1)$ but for finitely many points (Proposition 3.4), and there are finitely many of them. Number them $q_1, \dots, q_m$ from the bottom. Their closures $[e_{k-1}, e_k]$ then tile $[0, 1]$, with $e_0 = 0$, $e_m = 1$ and $e_{k-1} + e_k = 2c_k$. Eliminating $e_1, \dots, e_{m-1}$ gives (4.1). □

So an absent orphan forces a rigid configuration: the leg is cut into fans, each turning about one pole, and the alternating sum of the doubled prong heights is 1. The heights are arithmetic. Develop the prong of q_k from $\bar{\Lambda}$ into the plane, starting in T itself. It is a horizontal segment at height c_k that ends at the developed image $h_k(R)$ of the right-angle vertex, where h_k lies in the group generated by the reflections in the three side lines. So $c_k = \Im h_k(R)$.

Theorem 4.11 (Theorem A, existence at prime-power Q)**.** *Let $2\alpha/\pi = P/Q$ in lowest terms, with P odd and Q a prime power. Then an orphan branch exists.*

Proof. Use the coordinates of §1.1, $z = x + iy$ with $O = 0$, $R = L$ and $A = L + i$, where $L := \cot\alpha$, and put $w := e^{i\pi/Q}$ and $u := w^P = e^{2i\alpha}$. The reflections in L_2, H and L_1 are $z \mapsto \bar{z}$, $z \mapsto u\bar{z}$ and $z \mapsto 2L - \bar{z}$. The maps $z \mapsto \varepsilon z + b$ and $z \mapsto \varepsilon\bar{z} + b$, with ε a root of unity in $\mathbb{Z}[w]$ and $b \in 2L \cdot \mathbb{Z}[w]$, form a group, because L is real and $\mathbb{Z}[w]$ is closed under conjugation, and the group contains the three reflections. So $h_k(R) = L \cdot \lambda_k$ for some $\lambda_k \in \mathbb{Z}[w]$, and

$$2c_k = 2L \cdot \Im\lambda_k = (-iL)(\lambda_k - \bar{\lambda}_k), \quad -iL = (u+1)/(u-1),$$

the last because $\cot\alpha = i(u+1)/(u-1)$.

Suppose there is no orphan, so that Lemma 4.10 applies. Let $Q = p^a$, and let $\mathfrak{P}$ be the prime of $\mathbb{Z}[w]$ over p. It is unique and totally ramified, with residue field $\mathbb{F}_p$; and for a root of unity ζ the element $1-\zeta$ generates $\mathfrak{P}$ when ζ is primitive of the largest p-power order in $\mathbb{Z}[w]$ — order Q for odd p, $2Q$ for $p = 2$ — and is a unit when the order of ζ is not a prime power ([Was97], Ch. 1–2). Complex conjugation preserves $\mathfrak{P}$, being the only prime over p, and induces the identity on $\mathbb{F}_p$, so $\lambda - \bar{\lambda} \in \mathfrak{P}$ **for every** $\lambda \in \mathbb{Z}[w]$. Next, u is a primitive $2Q$-th root of unity, P being odd and prime to Q. If p is odd, $-u = w^{P+Q}$ is a primitive Q-th root of unity, so $1 + u$ generates $\mathfrak{P}$ while $1 - u$ is a unit. If $p = 2$, $-u$ is again a primitive $2Q$-th root, and $1 \pm u$ both

generate $\mathfrak{P}$. Either way $-iL$ is $\mathfrak{P}$-integral. So every $2c_k$ lies in $\mathfrak{P}\mathbb{Z}[w]_{\mathfrak{P}}$, and by (4.1) so does 1, which is absurd. □

Only one step of the proof uses P odd, and it is the valuation. Lemmas 4.8 and 4.10 hold at every P; what the oddness of P changes there is that D acquires the wall $\bar{W}$, the one place an orphan leaf could end, so that *no orphan* is a genuine hypothesis rather than automatic — at even P the leg is the only vertical side of D and the fan tiling holds outright. The valuation fails at even P: with $Q = p^a$ odd, u is a primitive Q-th root of unity, so $1 - u \in \mathfrak{P}$ and $-iL$ has a pole at $\mathfrak{P}$, and the argument says nothing — as it must not, there being no orphan there.

Conjecture 4.12. *Let P be odd. Then the orphan branch exists for every Q prime to P.*

The orphan is present at each of the 394 certified centres with P odd in the record of §A.3, 245 of them at a Q that is not a prime power. It is proved in [Meth] under the hypotheses quoted in §1.2 (with Theorem 4.18 at $\min(P, Q - P) = 1$); what remains is Q with two or more odd prime factors and $P \equiv \pm 2$ modulo one of them, outside the range of Theorem 4.18, apart from further families and a finite computation reported in [Meth] (its Remark 6.3).

Remark 4.13 (why the proof stops at prime powers). The proof of Theorem 4.11 uses only that every $2c_k$ lies in the additive group $(-iL) \cdot \mathcal{A}$, where $\mathcal{A} \coloneqq \{\beta - \bar{\beta} : \beta \in \mathbb{Z}[w]\}$, and that 1 does not. When Q has two distinct prime factors, 1 does.

Indeed u, w, and $-u$ are then primitive roots of unity of orders that are not prime powers, so $1 - u$, $1 + u$ and $1 - w$ are units. The element $x \coloneqq (u-1)/(u+1) = 1/(-iL)$ is purely imaginary, and $\beta \coloneqq x/(1-w) \in \mathbb{Z}[w]$ satisfies $\beta - \bar{\beta} = x \cdot (1/(1 - w) + 1/(1 - \bar{w})) = x$, the bracket being 1 for $|w| = 1$. So $1 = (-iL)(\beta - \bar{\beta})$, and no argument that reads only the membership $2c_k \in (-iL) \cdot \mathcal{A}$ can exclude (4.1). A finer argument must use more of the configuration — that the poles are distinct, which classes of λ_k occur, and ultimately which poles' prongs reach $\bar{\Lambda}$, and in which developed copy. §4.5 does this near the acute corners, which settles $Q = 2p$, $4p$ and the edges of every row.

The disc gives one more fact, and this one needs no hypothesis: it says where each branch's mirror partner is.

Corollary 4.14 (mirror partners are adjacent)**.** *At every centre, the two branches of each mirror pair of Theorem 2.2 are adjacent. Their common endpoint c is a launch height whose orbit reaches the right-angle vertex R before any other vertex, and J is the reflection $t \mapsto 2c - t$ on their union. Conversely, every $c \in (0, 1)$ whose orbit reaches R before any other vertex is the common endpoint of a mirror pair; in particular no endpoint of the orphan is such a height. No hypothesis on P, Q or the direction is needed.*

Proof. Work on the disc D of Lemma 4.8, where an image of R reached first by a leaf from $\bar{\Lambda}$ is a pole: $\bar{R}$ and, at even Q, $\bar{R}'$ are corners at which a vertical boundary side meets a horizontal one, and a horizontal leaf can reach such a corner only along the horizontal side, whose other end — $\bar{O}$ for L_2, an image of A for the horizontal copy of L_1 — it reaches first.

Let B be a mirror branch and $s \in B$ a non-singular height. Its leaf ends at its first head-on collision, which is not on $\bar{W}$, since s is not an orphan point (Lemma 4.2); so it is an arc γ_s from s to $J(s)$ on $\bar{\Lambda}$. Steps (ii) and (iii) of the proof of Lemma 4.10 then apply to γ_s as they stand: the hypothesis of that lemma enters them only to rule out orphan points in $(s, J(s))$, and here none can occur, since the leaf of such a point would end on $\bar{W}$, outside U_s (at even P there is no $\bar{W}$ and no orphan point at all); the slope -1 that (iii) uses is Lemma 4.9, which has no hypothesis. So U_s contains exactly one pole q, its prong meets $\bar{\Lambda}$ at the only singular height c_q in $(s, J(s))$, and $[s, J(s)]$ is centred at c_q, with (s, c_q) in the branch of s and $(c_q, J(s))$ in the branch of $J(s)$. Hence c_q is an endpoint of both B and $J(B)$, $J(t) = 2c_q - t$ on their union, and the orbit from c_q — the prong of q, read backwards — reaches an image of R before any other vertex image.

Conversely, let the leaf from c reach an image of R before any other vertex image. By the first paragraph that image is a pole q and the leaf is its prong. A neighbourhood of q is a cone of angle π, in which the horizontal leaves near the prong run beside it at distance δ and turn round q to the other side. Between c and q the prong meets no vertex image, by hypothesis; no vertical boundary side, since that would be a head-on hit, after which the orbit retraces to c and never reaches q; and no horizontal one, since a leaf meeting a horizontal side runs along it through a vertex image first. It is embedded, by the argument of step (ii) of Lemma 4.10, so for small δ it has a flat strip of width δ on either side, away from a small disc about q. So for small $\delta > 0$ the leaf from $c + \delta$ runs beside it to q, turns, and runs back beside it to $\bar{\Lambda}$, arriving at height $c - \delta$. Its first head-on collision is that return, so $J(c+\delta) = c-\delta$, and the branches on the two sides of c are carried onto each other by J: they form a mirror pair, and neither is the orphan, which J fixes. □

In the billiard the turn round the pole is the reversal at R noted in §4.2: the reflections in the two legs commute and compose to $-\mathrm{id}$, so an orbit driven into R comes straight back, and the orbits on either side of it are reflected into each other. Corollary 4.14 is the rational counterpart of [Tro05] Theorem 1(1), that for irrational right triangles with smaller angle in $(\pi/6, \pi/4)$ the middle orbit of every perpendicular periodic beam runs into the right-angle vertex; here it holds at every rational centre, and the orphan is the one branch it leaves out. The observation itself is older: Cipra, Hanson and Kolan [CHK95] (p. 2070) found from computation that the middle trajectory of each beam hits the right-angle vertex, noted the converse, and

left the forward direction open, and [Tro05] credits the observation also to Ruijgrok [Rui91].

4.5. **Existence beyond prime powers.** Remark 4.13 says what is missing at composite Q: the doubled prong heights all lie in a group that contains 1, so membership cannot exclude (4.1), and an argument must use which poles' prongs actually reach $\bar{\Lambda}$. This section does that. Modulo 2, the doubled height of a pole depends only on the copy of D that carries it, so (4.1) becomes a linear identity over $\mathbb{F}_2$, indexed by the set of poles whose prongs reach $\bar{\Lambda}$ (Lemma 4.15). Near the two acute corners of D some prongs are forced to reach $\bar{\Lambda}$, and others are forced to reach $\bar{W}$ (Lemma 4.16). When enough poles are forced, the identity has no solution (Theorem 4.18). At $Q = 4p$ three explicit prongs finish the job across the whole row (Theorem 4.24). Throughout, P is odd and Q is not a power of 2, powers of 2 being covered by Theorem 4.11.

Cells and multipliers. Every $g \in \mathcal{G}$ has the form $z \mapsto w^e z$ or $z \mapsto w^e \bar{z}$ with $e \in \mathbb{Z}/2Q$. Write $\langle g \rangle$ for the class of e in $\mathbb{Z}/Q$ modulo sign. Left multiplication by an element of V_4 changes e to $e + Q$ or to $-e$, so $\langle g \rangle$ depends only on the copy $V_4 g$ of D. The four copies of Σ at an image of R are g, gr_{L_1}, gr_{L_2} and $-g$, and the linear parts of r_{L_1}, r_{L_2} are $z \mapsto -\bar{z}$, $z \mapsto \bar{z}$, so all four have the same class. Call it the **cell** of that image of R, represented in $\{0, 1, \dots, \lfloor Q/2 \rfloor\}$. For $0 < j < Q/2$ exactly four rotations have class j ($e \in \{\pm j, \pm j + Q\}$), each copy of D contains two rotations, and so exactly two copies of D have class j. These two copies are $V_4(w^j z)$ and $V_4(w^j \bar{z}) = V_4(w^{-j} z)$, which meet at the image of R they share across L_2. So each j with $0 < j < Q/2$ is the cell of exactly one image of R, and that image is a pole. We write it q_j. Cell 0 is $\bar{R}$, and at even Q cell $Q/2$ is $\bar{R}'$. This gives the $\lceil Q/2 \rceil - 1$ poles of Lemma 4.8. Since $\gcd(P, Q) = 1$, each $0 < j < Q/2$ has a unique **multiplier** $t \in [1, M]$, $M := \lceil Q/2 \rceil - 1$, with $j \equiv \pm tP \pmod{Q}$.

The prong, read in T. For $0 < j < Q/2$ the prong of q_j issues into the copy $V_4(w^j z)$. Pulled back to T it is the billiard trajectory leaving R in the direction $\pi - j\pi/Q$, where directions are measured as in §1.1. The horizontal direction pulls back through $w^j z$ to the angles $-j\pi/Q$ and $\pi - j\pi/Q$. Through $w^{-j} z$ it pulls back to $j\pi/Q$ and $\pi + j\pi/Q$. The corner of T at R spans the directions $[\pi/2, \pi]$, and only $\pi - j\pi/Q$ lies in it. By Lemma 3.1(1) the prong then runs as the billiard orbit. It ends at the first vertex image it meets, or at its first head-on collision, which by Lemma 3.1(2) is a vertical boundary side of D. A head-on collision with L_1 is a collision with $\bar{\Lambda}$, because by Lemma 4.8 L_1 is vertical in no copy but T. We say the prong **lands on** $\bar{\Lambda}$ (resp. $\bar{W}$) when it meets that side at an interior point before it meets any vertex image or any other point of ∂D.

Lemma 4.15 (the landing set)**.** *Let P be odd and suppose there is no orphan branch.*

(i) The poles $q_1, \dots, q_m$ of Lemma 4.10 are exactly the poles whose prong lands on $\bar{\Lambda}$. Let $\mathcal{J}$ be their set of cells.

(ii) If Q is not a power of 2, then

$$\sum_{j \in \mathcal{J}} (w^j + w^{-j}) \equiv 1 \pmod{2\mathbb{Z}[w]}. \tag{4.2}$$

Proof. (i) By step (iii) of the proof of Lemma 4.10, the leaf from c_k enters U_s and runs into q_k, the only vertex image in U_s, before it meets ∂D again. Read backwards, the prong of q_k lands on $\bar{\Lambda}$ at c_k. Conversely, suppose the prong of a pole q lands on $\bar{\Lambda}$ at c. The leaf from c then reaches q before any other vertex image. The last paragraph of the proof of Corollary 4.14 shows that, for small $\delta > 0$, the arc $\gamma_{c+\delta}$ runs beside the prong to q, turns round q and returns to $c - \delta$. So the disc $U_{c+\delta}$ contains q. By step (ii) of Lemma 4.10, q is the one pole in $U_{c+\delta}$, and by step (iv) it is one of the q_k.

(ii) As in the proof of Theorem 4.11, $h_k(z) = \varepsilon z + b$ or $\varepsilon \bar{z} + b$ with $\varepsilon = \pm w^e$ and $b \in 2L \cdot \mathbb{Z}[w]$. Since $R = L$ is real, $h_k(R) = L(\varepsilon + 2a)$ with $a \in \mathbb{Z}[w]$. The prong of q_k, developed from $\bar{\Lambda}$, ends at the R of the copy $h_k(T)$, whose label is the linear part of h_k. So the cell of q_k is the class of e. Then

$$2c_k = (-iL)(\lambda_k - \bar{\lambda}_k), \quad \lambda_k - \bar{\lambda}_k = (\varepsilon - \bar{\varepsilon}) + 2(a - \bar{a}) \equiv w^e + w^{-e} \pmod{2\mathbb{Z}[w]},$$

because $-1 \equiv 1$ and $w^{e+Q} = -w^e$. Here u is a primitive $2Q$-th root of unity and $2Q$ is not a prime power, so $u - 1$ is a unit ([Was97], as in Theorem 4.11). Hence $-iL = (u+1)/(u-1) = 1 + 2/(u-1) \equiv 1$. Signs do not matter modulo 2, so (4.1) becomes (4.2). □

Lemma 4.16 (the corner grazes)**.** *Let P be odd and put $\mu := Q - P$.*

(i) For every $k \geq 1$ with $2k\mu < Q$, the prong of the pole of multiplier k lands on $\bar{\Lambda}$.

(ii) Let Q be odd. For every $k \geq 0$ with $(2k+1)P < Q$ or $(2k+1)\mu < Q$, the prong of the pole of multiplier $M - k$ lands on $\bar{W}$.

(iii) Let Q be even. For every $m \geq 1$ with $2mP < Q$, the prong of the pole of multiplier $Q/2 - m$ lands on $\bar{W}$.

Proof. We first prove a general claim, and then apply it four times.

The claim. Let $V \in \{\bar{O}, \bar{A}\}$ be an end of a vertical boundary side σ of D. Let v be an image of R joined to V by a side of a copy, that side lying at angular offset $\varphi \in (0, \pi/2)$ from σ about V, with v at distance r from V and $r \cos\varphi < |\sigma|$. If v is a pole, its prong lands on σ.

To prove the claim, note that the two boundary sides at the corner V are each vertical or horizontal, so the corner angle is a positive multiple of $\pi/2$. The copies of T with apex V and with angle at V inside $[0, \varphi]$, measured from σ, therefore lie in the corner. The sides of these copies through V at offsets strictly between 0 and φ are interior sides of D. Develop these copies into the plane with V at the origin. They form a polygon Π: V, then the far ends of the successive sides through V, ending at v. The interior angles of Π are:

- $\varphi < \pi$ at V;
- π at each intermediate image of R, being two right angles;
- twice an acute angle of T at each intermediate image of O or A;
- less than π at the first and last far vertex.

So Π is convex. The foot f of the perpendicular from v to σ lies at distance $r\cos\varphi \in (0, |\sigma|)$ from V, so f is an interior point of σ. The open segment (f, v) lies in the interior of Π, because f and v lie on different edges. So it meets no vertex of a copy and crosses only interior sides. That segment is the horizontal leaf from f, and it arrives at v. A leaf that reaches a pole arrives along the pole's prong, so the prong of v lands on σ at f.

The four applications. Copies around an image of O (resp. A) succeed one another across their sides through that vertex. So, starting from the copy g_0 adjacent to σ, they are g_0, $g_0 r_1$, $g_0 r_1 r_2$, ..., where the reflections r_1, r_2, ... alternate between the two sides of T at that vertex. We use the following linear parts: $r_{L_2} r_H$ is $z \mapsto w^{-P} z$, $r_H r_{L_2}$ is $z \mapsto w^{P} z$, $r_{L_1} r_H$ is $z \mapsto w^{Q-P} z$, and $r_H r_{L_1}$ is $z \mapsto w^{Q+P} z$. The lengths are $|L_1| = 1$, $|L_2| = L$ and $|H| = 1/\sin\alpha$, as in §1.1.

- (i) $V = \bar{A}$, $\sigma = \bar{\Lambda}$, $|\sigma| = 1$, $g_0 = \mathrm{id}$. Write $\beta \coloneqq \pi/2 - \alpha$ for the angle at A. The L_1-sides through $\bar{A}$ lie at the offsets $2k\beta$. The image of R at offset $2k\beta$ is the R of $(r_H r_{L_1})^k$, whose class is $\pm k(Q+P) \equiv \pm kP$. So it has multiplier k. Here $r = 1$, $\cos 2k\beta < 1$, and $2k\beta < \pi/2$ exactly when $2k\mu < Q$.
- (ii), at $\bar{O}$: $\sigma = \bar{W}$ is a copy of H, $|\sigma| = 1/\sin\alpha$. $g_0 = w^{j_W} z$, where $j_W \coloneqq (Q-P)/2$ makes H vertical. The image of R at offset $(2k+1)\alpha$ is the R of $g_0 (r_{L_2} r_H)^k$, of class $\pm(j_W - kP)$. Since $2j_W \equiv -P \equiv 2MP \pmod{Q}$, we have $j_W \equiv MP$, so the multiplier is $M-k$. Here $r = L < 1/\sin\alpha$, and $(2k+1)\alpha < \pi/2$ exactly when $(2k+1)P < Q$.
- (ii), at $\bar{A}$: the same σ and g_0. The image of R at offset $(2k+1)\beta$ is the R of $g_0 (r_{L_1} r_H)^k$, of class $\pm(j_W + k(Q-P)) \equiv \pm(M-k)P$. Here $r = 1 < 1/\sin\alpha$, and $(2k+1)\beta < \pi/2$ exactly when $(2k+1)\mu < Q$.
- (iii) $V = \bar{O}$, $\sigma = \bar{W}$ is a copy of L_2, $|\sigma| = L$, $g_0 = w^{Q/2} z$. The image of R at offset $2m\alpha$ is the R of $g_0 (r_H r_{L_2})^m$, of class $\pm(Q/2 + mP)$. Since P is odd, $(Q/2)P \equiv Q/2$, so this class is $\pm(Q/2 + m)P$, which has multiplier $Q/2 - m$. Here $r = L$, $\cos 2m\alpha < 1$, and $2m\alpha < \pi/2$ exactly when $2mP < Q$.

In each case the multiplier lies in $[1, M]$, so the class is neither 0 nor $Q/2$. The image of R is therefore a pole, and the claim applies. □

The poles in (ii) and (iii) are R-images that the grazes reach from the wall $\bar{W}$. Their prongs end on $\bar{W}$, so they never land on $\bar{\Lambda}$. The poles in (i) land on $\bar{\Lambda}$. Under the hypothesis of Lemma 4.15, then, the multipliers of $\mathcal{J}$ contain the run $Y \coloneqq [1, b]$ and avoid the run $X \coloneqq [M - \ell + 1, M]$. Here b is the number of k in (i), and ℓ is the number of k in (ii), or of m in (iii).

What remains is algebra. For $n \geq 1$ let $D_n \in \mathbb{Z}[y]$ be the Dickson polynomial, defined by $D_0 = 2$, $D_1 = y$, $D_{n+1} = yD_n - D_{n-1}$. It is monic of

degree n, satisfies $D_n(\zeta+\zeta^{-1}) = \zeta^n + \zeta^{-n}$ and $D_n(-y) = (-1)^n D_n(y)$, and has $D_n(0) \in \{0, \pm 2\}$.

Lemma 4.17. *Let $N \geq 3$, let ζ be a primitive N-th root of unity, put $z \coloneqq \zeta + \zeta^{-1}$, and let V be a finite set of positive integers.*

(a) If $\sum_{n\in V} D_n(z) \equiv 1 \pmod{2\mathbb{Z}[\zeta]}$, then $V \neq \emptyset$ and $\max V \geq \varphi(N)/2$.

(b) If N is odd, $V \neq \emptyset$ consists of odd integers, and $\sum_{n\in V} D_n(z) \equiv 0 \pmod{2\mathbb{Z}[\zeta]}$, then $\max V \geq \varphi(N) + 1$.

Proof. From $\zeta^2 = z\zeta - 1$ and $[\mathbb{Q}(\zeta) : \mathbb{Q}(z)] = 2$, $\mathbb{Z}[\zeta]$ is free over $\mathbb{Z}[z]$ on $\{1, \zeta\}$, so $\mathbb{Z}[z] \cap 2\mathbb{Z}[\zeta] = 2\mathbb{Z}[z]$. Let $\psi \in \mathbb{Z}[y]$ be the minimal polynomial of z. It is monic of degree $\varphi(N)/2$, and $\mathbb{Z}[z] = \mathbb{Z}[y]/(\psi)$. So whenever $G \in \mathbb{Z}[y]$ has $G(z) \in 2\mathbb{Z}[\zeta]$, the reduction $\bar{G}$ is divisible by $\bar{\psi}$ in $\mathbb{F}_2[y]$.

(a) Take $G \coloneqq \sum_V D_n - 1$. Then $\bar{G}(0) = 1$, so $\bar{G} \neq 0$, and $\bar{\psi} | \bar{G}$ gives $\deg \bar{G} \geq \varphi(N)/2 \geq 1$. That rules out $V = \emptyset$, where $\bar{G} = 1$. Otherwise $\deg \bar{G} = \max V$.

(b) Take $G \coloneqq \sum_V D_n$. It is non-zero modulo 2, of degree $\max V$, since the degrees are distinct and the leading terms monic. Each D_n with n odd is an odd polynomial, so $\bar{G} \in y\mathbb{F}_2[y^2] = y \cdot \mathbb{F}_2[y]^2$, that is, $\bar{G} = yH^2$. Since N is odd, $x^N - 1$ is separable over $\mathbb{F}_2$. So $\mathbb{Z}[\zeta]/2$ is reduced, and so is its subring $\mathbb{Z}[z]/2 = \mathbb{F}_2[y]/(\bar{\psi})$; hence $\bar{\psi}$ is squarefree. Next, $z = \zeta^{-1}(1+\zeta^2)$, and $1 + \zeta^2 = (1-\zeta^4)/(1-\zeta^2)$ is an algebraic integer whose norm is $N(1-\zeta^4)/N(1-\zeta^2) = 1$, the two being Galois conjugates since $\zeta \mapsto \zeta^2$ is an automorphism. So z is a unit and $\bar{\psi}(0) = 1$. Every irreducible factor of $\bar{\psi}$ divides yH^2 and is prime to y, so it divides H. These factors are distinct, so $\bar{\psi} | H$, and $\max V = 1 + 2\deg H \geq \varphi(N) + 1$. □

Theorem 4.18 (existence from the forced landings)**.** *Let P be odd and Q not a power of 2. Put $\mu \coloneqq Q - P$, $M \coloneqq \lceil Q/2 \rceil - 1$, $h \coloneqq \varphi(2Q)/2$ and $d \coloneqq M - h$. The orphan branch exists in each of the following cases:*

(a) Q odd, $P > Q/2$, and $\lfloor Q/\mu \rfloor \geq d + 1$;

(b) Q odd, $P < Q/2$, and $\min(M, \lceil \lfloor Q/P \rfloor / 2 \rceil) \geq d + 1$;

(c) Q even, and $\min(M, \lfloor Q/2\kappa \rfloor) \geq d + 1$, where $\kappa \coloneqq \min(P, \mu)$.

Here $d = (Q - 1 - \varphi(Q))/2$ at odd Q and $d = Q/2 - 1 - \varphi(Q)$ at even Q.

Proof. Suppose there is no orphan, and let $S \subseteq [1, M]$ be the set of multipliers of the cells in $\mathcal{J}$. If $j \equiv \pm tP \pmod{Q}$, then $w^j + w^{-j} \equiv u^t + u^{-t} = D_t(z)$ modulo 2, where $z \coloneqq u + u^{-1}$. So (4.2) reads

$$\sum_{t\in S} D_t(z) \equiv 1 \pmod{2\mathbb{Z}[w]},$$

$$S \supseteq Y = [1, b], \qquad S \cap X = \emptyset, \qquad X = [M - \ell + 1, M].$$

Here $\mathbb{Z}[u] = \mathbb{Z}[w]$, since P is prime to $2Q$. For $n \geq 1$ the number of $m \geq 1$ with $mn < Q$ is $\lfloor (Q-1)/n \rfloor$, which equals $\lfloor Q/n \rfloor$ unless $n | Q$. For $n \in \{P, \mu, 2P, 2\mu\}$ that happens only when $P = 1$ or $\mu = 1$.

The far run alone covers (b), and the case $P < Q/2$ of (c). In both, $2\mu > Q$, so $Y = \emptyset$. In (b), ℓ counts the $k \geq 0$ with $(2k+1)P < Q$,

because $(2k+1)\mu < Q$ forces $k = 0$. That gives $\ell = \min(M, \lceil \lfloor Q/P \rfloor / 2 \rceil)$; the minimum is needed only at $P = 1$. In (c), $\ell = \min(M, \lfloor Q/2P \rfloor)$. Either way, $S \subseteq [1, M-\ell]$. Lemma 4.17(a) with $\zeta = u$, $N = 2Q$ gives $M - \ell \geq h$, that is, $\ell \leq d$.

(c) with $P > Q/2$. Here $X = \emptyset$ and $b = \min(M, \lfloor Q/2\mu \rfloor)$. As t runs over $[1, M]$, the exponents $\pm t$ run once over $\mathbb{Z}/Q \setminus \{0, Q/2\}$. Also $u^{e+Q} = -u^e$, and $\sum_{e=0}^{Q-1} u^e = -2/(u-1) \in 2\mathbb{Z}[w]$, while $u^{Q/2} = \pm i$. Together these give $\sum_{t=1}^{M} D_t(z) \equiv 1 + i$. So $U \coloneqq [1, M] \setminus S \subseteq [b+1, M]$ satisfies $\sum_U D_t(z) \equiv i$. For $t \in U$ put $k \coloneqq Q/2 - t$. Then $u^t = i^P u^{-k}$ and $u^{-t} = -i^P u^k$, so $D_t(z) \equiv i^P D_k(z)$. Hence $\sum_{k \in Q/2-U} D_k(z) \equiv i^{1-P} \equiv 1$. Lemma 4.17(a) gives $Q/2 - b - 1 \geq h$, that is, $b \leq d$.

(a). Here $\mu < Q/2$, and μ is even, so $\mu \geq 2$ and $\mu \nmid Q$. Every k with $(2k+1)P < Q$ also has $(2k+1)\mu < Q$, so ℓ counts the odd n with $n\mu < Q$ and b counts the even ones. Hence $b + \ell = \lfloor Q/\mu \rfloor \leq M$ and $\ell \in \{b, b+1\}$. Write $S = [1, b] \sqcup S'$ with $S' \subseteq [b+1, M-\ell]$.

Let $\eta \coloneqq e^{i\alpha}$, so that $\eta^2 = u$ and $\mathbb{Z}[\eta] = \mathbb{Z}[w][i]$ is free over $\mathbb{Z}[w]$ on $\{1, i\}$, and put $v \coloneqq \eta + \eta^{-1}$. Since $D_t(z) = \eta^{2t} + \eta^{-2t}$, telescoping gives

$$v \cdot \Big(1 + \sum_{t \leq b} D_t(z)\Big) \equiv D_{2b+1}(v) \pmod{2\mathbb{Z}[\eta]},$$

$$v \cdot D_t(z) = D_{2t+1}(v) + D_{2t-1}(v).$$

Next, $\eta^Q = i^P$ and $2m+1 = Q - 2(M-m)$. So $D_{2m+1}(v) = i^P(\eta^{-2(M-m)} - \eta^{2(M-m)}) \equiv i^P D_{M-m}(z)$. Multiplying the identity by v therefore turns it into $i^P \sum_{n \in \mathcal{B}} D_n(z) \equiv 0$ modulo $2\mathbb{Z}[\eta]$, hence, i^P being a unit and $\mathbb{Z}[\eta]$ free over $\mathbb{Z}[w]$ on $\{1, i\}$, into $\sum_{n \in \mathcal{B}} D_n(z) \equiv 0$ modulo $2\mathbb{Z}[w]$, where

$$\mathcal{B} \coloneqq \{M - b\} \triangle \mathop{\triangle}_{t \in S'} \{M - t, M - t + 1\} \subseteq [\ell, M - b]$$

has odd size. Squaring is additive modulo 2, and $D_n(z)^2 = D_{2n}(z) + 2$, so $\sum_{n \in \mathcal{B}} D_{2n}(z) \equiv 0$. Since $u^Q = -1$, $D_m(z)$ mod 2 depends only on $\pm m$ mod Q. Let $f(n)$ be the representative of $\pm 2n$ in $[1, M]$; f is injective on $[1, M]$. Then $V \coloneqq f(\mathcal{B})$ is a non-empty set with $\sum_V D_m(z) \equiv 0$, and $V \subseteq [b+\ell, M]$. Indeed, for $n \in [\ell, M-b]$, either $f(n) = 2n \geq 2\ell \geq b + \ell$, or $f(n) = Q - 2n \geq 2b + 1 \geq b + \ell$. Under (a), $V \subseteq [d+1, M]$.

Now read from the far end. $\omega \coloneqq -u$ is a primitive Q-th root of unity, and so is $\omega_1 \coloneqq \omega^M$, which satisfies $\omega_1^{-2} = \omega$. For $m = M - k$, $\omega^m = \omega_1^{2k+1}$. So, modulo 2, $D_m(z) \equiv \omega^m + \omega^{-m} = D_{2k+1}(\omega_1 + \omega_1^{-1})$. Here $k \leq M - d - 1 = h - 1$, so $2k + 1 \leq \varphi(Q) - 1$. Since $\mathbb{Z}[\omega_1] = \mathbb{Z}[w]$, this contradicts Lemma 4.17(b) with $N = Q$. □

The proof computed $b + \ell$ in each case, and the three conditions of Theorem 4.18 say exactly that $b + \ell \geq d + 1$. Below that, the landings of Lemma 4.16 decide nothing.

Proposition 4.19 (Theorem 4.18 is sharp)**.** *Let P be odd and Q not a power of 2, with b, ℓ, X, Y as after Lemma 4.16, and M, h, d and $z \coloneqq u + u^{-1}$*

as in Theorem 4.18 and its proof. If $b+\ell \le d$, there is a set $S \subseteq [1, M]$ with $S \supseteq Y$, $S \cap X = \emptyset$ and $\sum_{t\in S} D_t(z) \equiv 1 \pmod{2\mathbb{Z}[w]}$.

So outside Theorem 4.18 the identity (4.2) has a solution compatible with every landing that Lemma 4.16 forces. Reduced modulo 2, those landings exclude nothing more, so an existence proof through (4.2) must force further cells. Lemmas 4.22 and 4.23 do this at $Q = 4p$.

Proof. Let $\mathcal{Z} \coloneqq \mathbb{Z}[z]/2\mathbb{Z}[z]$. As $\mathbb{Z}[w] = \mathbb{Z}[u]$ is free over $\mathbb{Z}[z]$ on $\{1, u\}$, $\mathcal{Z}$ embeds in $\mathbb{Z}[w]/2$, as in the proof of Lemma 4.17. It is an $\mathbb{F}_2$-space of dimension h, and it contains 1 and every $D_t(z)$. We find h positions avoiding $X \cup Y$ whose classes are independent. They are then a basis of $\mathcal{Z}$, so $1 + \sum_{t\in Y} D_t(z)$ is the sum over some set S'' of them, and $S \coloneqq Y \sqcup S''$ works.

The window $[1, h]$ is independent. Let ψ be the minimal polynomial of z, as in Lemma 4.17. First, $\bar\psi(0) = 1$. Indeed $z = u^{-1}(1 + u^2)$, and u^2 is a primitive Q-th root of unity. Since $1 + u^2 \equiv 1 - u^2$ modulo 2, their norms to $\mathbb{Q}$ are congruent modulo 2. The norm of $1 - u^2$ is a power of $\Phi_Q(1)$, which is 1 or an odd prime because Q is not a power of 2. So z has odd norm. Now suppose $\sum_{t\in U} D_t(z) \equiv 0$ with $\emptyset \ne U \subseteq [1, h]$. Then $\bar\psi$ divides $\bar G \coloneqq \sum_{t\in U} \bar D_t$, which is non-zero of degree $\max U \le h = \deg \bar\psi$. So $\bar G = \bar\psi$. But $\bar G(0) = 0$, since $D_t(0) \in \{0, \pm 2\}$, while $\bar\psi(0) = 1$. That is a contradiction.

The far run alone ($P < Q/2$, Q odd or even). Here $Y = \emptyset$ and $X = [M - \ell + 1, M]$. Since $\ell \le d = M - h$, the window $[1, h]$ avoids X.

Q even, $P > Q/2$. Here $X = \emptyset$ and $Y = [1, b]$. The proof of Theorem 4.18 gives $D_t(z) \equiv i^P D_{Q/2-t}(z)$. As t runs over $[M - h + 1, M]$, $Q/2 - t$ runs over $[1, h]$. So the classes of these positions are i^P times a basis, and are independent. They avoid Y, because $b \le d = M - h$.

Q odd, $P > Q/2$. Here $\ell \in \{b, b + 1\}$, as in the proof of Theorem 4.18(a). Put $\beta \coloneqq \lfloor d/2 \rfloor$, $\lambda \coloneqq \lceil d/2 \rceil$ and $W \coloneqq [\beta+1, M-\lambda]$, so that $|W| = M-d = h$. If $\ell = b$, then $2b \le d$. If $\ell = b + 1$, then $2b + 1 \le d$. Either way $b \le \beta$ and $\ell \le \lambda$, so W avoids $X \cup Y$. Suppose $\sum_{n\in B} D_n(z) \equiv 0$ with $\emptyset \ne B \subseteq W$. As in the proof of Theorem 4.18, squaring gives $\sum_{n\in B} D_{f(n)}(z) \equiv 0$, with f injective. For $n \in W$, either $f(n) = 2n \ge 2\beta + 2 \ge d + 1$ or $f(n) = Q - 2n \ge 2\lambda + 1 \ge d + 1$. So $f(B)$ is a non-empty subset of $[d + 1, M]$ whose sum vanishes, which the last paragraph of that proof excludes. □

Corollary 4.20. *Let $Q = 2^k p$ with p an odd prime and $k \ge 1$, and let P be odd with $\min(P, Q - P) < p$. Then the orphan branch exists. In particular it exists at every odd P when $Q = 2p$.*

Proof. $M = 2^{k-1}p - 1$ and $h = \varphi(Q) = 2^{k-1}(p - 1)$, so $d + 1 = 2^{k-1}$. If $\kappa < p$, then $\lfloor Q/2\kappa \rfloor = \lfloor 2^{k-1}p/\kappa \rfloor \ge 2^{k-1}$, and $M \ge 2^{k-1}$. Theorem 4.18(c) applies. At $Q = 2p$, $\kappa \le p$ and $\kappa \ne p$. □

At an odd prime Q, $d = 0$, and Theorem 4.18 recovers Theorem 4.11 there. Elsewhere it is a statement about the edges of a row. At most about Q/κ poles are forced, against $d = M - \varphi(2Q)/2$. So κ must be at most of the order of $Q/(Q - \varphi(Q))$ at odd Q, roughly the least prime factor of Q, and of $Q/(Q - 2\varphi(Q))$ at even Q, which is p at $Q = 2^k p$. In the middle of a row $X \cup Y$ is too small, and by Proposition 4.19 the corner landings, reduced modulo 2, cannot exclude a missing orphan there. At $Q = 4p$ three such prongs can be followed by hand.

Lemma 4.21 (the classes at $Q = 4p$)**.** *Let $Q = 4p$ with p an odd prime, let P be odd, and suppose there is no orphan branch. Then, in the notation of Lemma 4.15:*

(I) every cell $0 < j < Q/2$ with $j \equiv 0 \pmod 4$ is in $\mathcal{J}$;
(II) no cell $0 < j < Q/2$ with $j \equiv 2 \pmod 4$ is in $\mathcal{J}$;
(III) either every odd cell $0 < j < Q/2$ is in $\mathcal{J}$, or none is.

Proof. The map $\mathbb{F}_2[\mathbb{Z}/Q] \to \mathbb{Z}[w]/2$ sending x^a to w^a is well defined, because $w^Q = -1 \equiv 1$, and it is onto. We have $\Phi_{8p}(x) = \Phi_{2p}(x^4) = \Phi_p(-x^4) \equiv \Phi_p(x^4)$, and modulo 2 this is $E \coloneqq \sum_{i<p} x^{4i}$, the sum over the subgroup $K_4 \coloneqq 4\mathbb{Z}/Q\mathbb{Z}$. So the kernel is the ideal (E). Since $x^a E$ is the indicator function of $a + K_4$, that ideal consists of the functions constant on the classes mod 4. Let $f \coloneqq \delta_0 + \sum_{j \in \mathcal{J}} (\delta_j + \delta_{-j})$. By (4.2), f lies in the kernel, so $f(a)$ depends only on $a \bmod 4$. For interior j we have $f(j) = [j \in \mathcal{J}]$, since $-j \notin [1, Q/2 - 1]$. Also $f(0) = 1$ and $f(Q/2) = 0$, with $Q/2 = 2p \equiv 2 \pmod 4$; these give (I) and (II). Finally f is even, so $f(1) = f(-1) = f(3)$, which gives (III). □

Lemma 4.22 (the strip)**.** *Let $Q = 4p$, p an odd prime, and P odd with $Q/3 < P < Q/2$. Then the prong of q_{Q-2P} lands on $\bar{\Lambda}$.*

Idea. The prong leaves R in the direction 4α. Billiards in T is billiards in the rhombus of four copies about R, folded by the two legs, and unfolding the rhombus across its two sides parallel to OA gives a strip in which the prong is a straight line, crossing the walls at the angle $\delta = 3\alpha - \pi/2 \in (0, \pi/4)$ to their normal. It crosses an odd number of walls in their interiors and passes the next beyond its end, so it leaves the last rhombus through a side whose reflection turns it exactly horizontal, and it then meets L_1 head-on at an interior point. A unit argument in $\mathbb{Z}[w]$ keeps every crossing off the vertices. The trajectory is followed in Appendix B.

Lemma 4.23 (the band ladder)**.** *Let $Q = 4p$, p an odd prime, and P odd with $Q/4 < P < Q/3$. Then the prong of q_P or the prong of q_{Q-3P} lands on $\bar{\Lambda}$.*

Idea. In a rotated frame the two prongs leave R at the angles $\pm\theta$, $\theta = 4\alpha - \pi/2$. A periodic chain of copies — a fan of four about one image of O, then a fan of six about the next — has every vertex at height 0, $\pm a$ or $\pm c$, so a strip of height a below the axis contains none, and its bottom edge

is cut alternately into walls and gates. A prong leaving the strip through a gate meets the L_1 of a definite copy perpendicularly. For $\alpha \le \pi/7$ the gate windows of the two prongs, of relative length greater than $1/2$ and staggered by half a period, together cover every exit position; for $\pi/7 < \alpha < \pi/6$ the prong of q_{Q-3P} leaves through the first gate. The trajectories are followed in Appendix B.

Theorem 4.24 (existence at $Q = 4p$)**.** *Let $Q = 4p$ with p an odd prime, and let P be odd and prime to Q. Then the orphan branch exists.*

Proof. Swapping the two legs, the L_1-beam at $(Q-P)/Q$ is the L_2-beam at P/Q. By Theorem 4.6 (Q even) an orphan orbit of it turns around head-on at an interior point of a side of type L_2 of the swapped triangle, that is, of L_1 of T. Read from that point, it is an orbit of the L_1-beam at P/Q that meets L_2 head-on and retraces, and it meets no vertex. It meets L_1 perpendicularly nowhere before that L_2 hit, since by Lemma 2.1 the turnaround is the swapped orbit's only head-on collision. By Lemma 2.1 its launch height is an orphan point. So the orphan exists at P if it exists at $Q-P$, and symmetrically, and we may take $P < Q/2 = 2p$. Also $P \neq p$, and $P \neq 4p/3$, since that is either not an integer or even.

- $P < p$*:* this is Corollary 4.20 with $k = 2$.
- $p < P < 4p/3$*:* suppose there is no orphan. By Lemma 4.16(iii) with $m = 1$, the pole of cell $\pm(Q/2 + P)$ lands on $\bar{W}$, so this odd cell is not in $\mathcal{J}$. By (III) of Lemma 4.21, no odd cell is. But by Lemma 4.23 the odd cell P or the odd cell $Q - 3P$ is in $\mathcal{J}$. This is a contradiction.
- $4p/3 < P < 2p$*:* suppose there is no orphan. By Lemma 4.22 the cell $Q - 2P \equiv 2 \pmod 4$ is in $\mathcal{J}$, against (II).

□

So Theorem A(c) holds at $Q = p^a$, $2p$ and $4p$, and at every centre satisfying the inequality of Theorem 4.18. At $Q = 2^k p$ with $k \ge 3$ the same argument gives the analogue of Lemma 4.21, with classes mod 2^k. There the classes $\equiv 0$ and $\equiv 2^{k-1}$ are pinned as in (I) and (II), but $2^{k-1} - 1$ pairs of classes stay free, and we have no uniform landing for the poles in them. (The orphan exists at every odd P of such Q, by a different argument: [Meth] Theorem 3 when $\min(P, Q-P) \ge 2$, and Theorem 4.18 otherwise.)

5. What Theorem A fixes, and what it does not

Theorem A determines the parity of the branch count and identifies the degenerate branch. It does not determine the value of $n(P/Q)$, and this section is about the gap between those two things.

Corollary 5.1 (parity)**.** *If P is even then $n(P/Q)$ is even. If P is odd then $n(P/Q)$ is odd if and only if the orphan exists — in particular whenever Q is a prime power, or twice or four times a prime. Wherever P is even or the orphan exists, $n(P/Q) \equiv P \pmod 2$.*

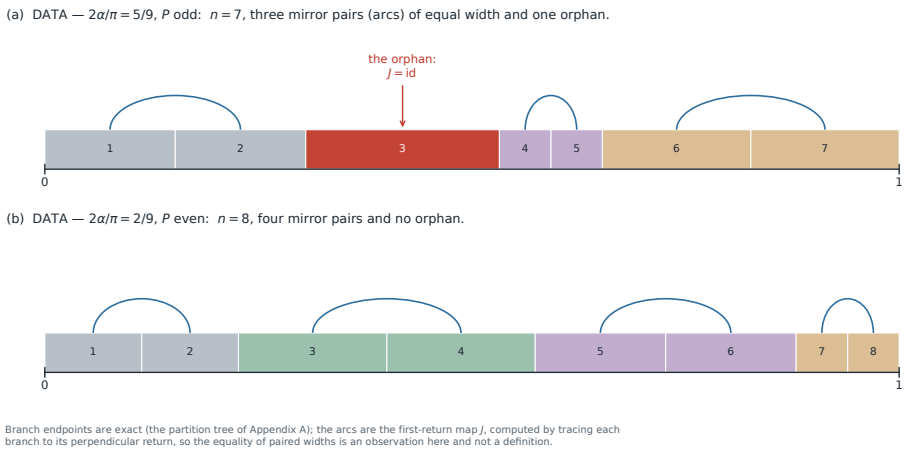


FIGURE 3. The branch partition of the launch leg at two centres of the same denominator, with each mirror pair of Theorem 2.2 joined by an arc. Endpoints are exact. The arcs are the first-return map J, obtained by tracing each branch to its perpendicular return, so the equality of paired widths is read off the picture rather than built into it. (a) $2\alpha/\pi = 5/9$: P is odd, the count is the odd number 7, and the branch drawn in red is the orphan — its own partner under J. (b) $2\alpha/\pi = 2/9$: P is even, every branch has a distinct partner, and the count is the even number 8. This is Theorem A and Corollary 5.1 at two centres.

The even-P statement is Proposition 4.3 and the involution of Theorem 2.2 and needs nothing further. The odd-P statement counts the orphan as the one unpaired branch, so it needs to know there is at most one, which is Theorem 4.7, and the cases named are Theorem 4.11, Corollary 4.20 and Theorem 4.24. All of it needs to know that $n(P/Q)$ is a number at all — a parity is a property of a number — and that is Proposition 3.4, which is unconditional. The qualification at odd P is also met wherever [Meth] proves the orphan exists (under the hypotheses quoted in §1.2, with Theorem 4.18 at $\min(P, Q-P) = 1$), and by Conjecture 4.12 it would be empty.

5.1. **What the parity does not fix.**

Remark 5.2. The parity does not determine the value. At even P the count need not be $Q-1$, its value at $P=2$ for odd $Q \geq 5$ ([Neck] Theorem 10.5): $n(8/15) = 6$, against 14. At odd $P \geq 5$ it is arithmetically irregular: $n(5/22) = 5$ while $n(5/24) = 17$.

All four numbers are exact and carry the completeness certificate of Appendix A. The count itself is the subject of the companion paper [Neck].

5.2. **The orphan belongs to the triangle, not to the leg.** Theorem 4.6 is stated for the beam perpendicular to L_1, but each of the three sides carries its own perpendicular beam, and the retracing criterion for each is read off Proposition 4.3's list of normals in the same way.

Theorem 5.3 (three-sided refinement)**.** *A side σ of T admits a retracing cell only when some **other** side σ' has $\theta_{\sigma'} \equiv \theta_\sigma$ modulo 2, the side orientations being $(\theta_{L_1}, \theta_{L_2}, \theta_H) = (Q, 0, P)$ in side-orientation units of $\pi/2Q$. Equivalently: L_1 only if P is odd, L_2 only if $P+Q$ is odd, and H only if Q is odd. Since $\gcd(P,Q) = 1$, these three conditions are the three two-element subsets of $\{L_1, L_2, H\}$: at most two of the three sides carry a retracing cell, and the third carries none. Each of the two carries at most **one** such cell; the two cells are present or absent together, as the two ends of one band of double normals; and they are present when Q is a prime power, or twice or four times a prime, or when Theorem 4.18 applies to P/Q or to $(Q-P)/Q$. No hypothesis on the direction is needed. The three cases are*

	retracing pair
P odd, Q even	L_1–L_2
P odd, Q odd	L_1–H
P even, Q odd	L_2–H

The criterion is about the σ-beam's *own* direction set, not the L_1-beam's $\mathcal{D}$ of Lemma 4.1, and the two must not be run together: at P even and Q odd the table allows H a retracing cell while $v_H^* \notin \mathcal{D}$, because there the relevant beam is the one launched from L_2. What is uniform across the three sides is the parity condition above; the proof below derives it and the three displayed cases from it.

Proof. Proposition 4.5 is not a fact about L_1: its congruence count runs for any launch side, and running it so gives the whole proposition at once.

Develop the beam perpendicular to σ_0 as in Lemma 3.1, keeping the ray horizontal, so that σ_0 itself is vertical in T_0. Lemma 3.1(2) again makes a head-on collision the same thing as a vertical wall-copy, and Lemma 3.2 — which is stated for every perpendicular beam and not only for the one launched from L_1 — makes the set of vertical copies fixed and independent of the launch height. Measure orientations in units of $\pi/2Q$, so that the sides of T_0 have orientations

$$\theta_{L_1} = Q, \quad \theta_{L_2} = 0, \quad \theta_H = P,$$

and a copy of type X carried k rhombus steps (Definition 4.4) has orientation $\theta_X + 2kP$ modulo $2Q$. Such a copy is perpendicular to this beam exactly when its orientation agrees with that of σ_0:

$$(*) \qquad \theta_X + 2kP \equiv \theta_{\sigma_0} \pmod{2Q}.$$

At $\sigma_0 = L_1$ these are the three congruences of Proposition 4.5 verbatim.

Because $\gcd(P,Q)=1$, the residue of $2kP$ modulo $2Q$ is $2\cdot(kP \bmod Q)$, and $kP \bmod Q$ runs over $0,\dots,Q-1$ bijectively as k does. So $2kP$ takes each even residue modulo $2Q$ exactly once over $k=0,\dots,Q-1$, and no odd one. Hence $(*)$ has exactly one solution when $\theta_{\sigma_0}-\theta_X$ is even and none when it is odd; and the orientation pattern is Q-periodic in k, so that range is one cycle of directions, for every launch side and not only for L_1.

Take $X=\sigma_0$. The unique solution is then $k=0$, the launch copy, which is not an interior wall; and two distinct sides of a nondegenerate triangle have distinct orientations, so no other type solves $(*)$ at $k=0$. The interior vertical classes of the σ_0-beam over one cycle of directions therefore number exactly one for each side $\sigma'\neq\sigma_0$ with $\theta_{\sigma'}\equiv\theta_{\sigma_0}$ modulo 2, and there are none of any other type.

The parities of $(\theta_{L_1},\theta_{L_2},\theta_H)$ are those of $(Q,0,P)$, and $\gcd(P,Q)=1$ keeps P and Q from both being even. Exactly one of the three pairs of sides therefore has matching parity: L_1 and L_2 when Q is even, L_2 and H when P is even, and L_1 and H when P and Q are both odd. Those are the three rows of the table, and they are the stated criterion as well, θ_σ and the index of v^*_σ differing by the constant Q.

So for each side of the matching pair the beam has exactly one interior vertical class, of the partner's type, and for the third side it has none. Every retracing orbit turns around at a head-on collision by Lemma 2.1 — whose proof uses only time-reversibility, the reflection law and Lemma 3.1, and so is indifferent to which side the beam was launched from — and that collision lies on a vertical copy by Lemma 3.1(2). So the third side carries no retracing cell, and a retracing cell on a side of the pair turns around on a copy of its partner. That is the criterion, and it is unconditional. As §4.4 explains, it does not produce a cell.

That there is at most **one** does not follow from the count of vertical *directions*, for the reason set out in §4.3: the passage meets the vertical class many times, and Proposition 4.5 bounds the directions rather than the copies. The one-cell clause is proved by §7.6's argument run with σ_0 in place of L_1, and like Theorem 4.7 it needs no hypothesis on the direction. There σ_v becomes the reflection in σ_0's direction, and the conjugacy count of §7.6's cut-locus paragraph makes its fixed set C two copies of σ_0 and two of the partner σ' — the side types whose orientation has the parity of θ_{σ_0}, which by the count above is the matching pair. The junction of the two σ_0-copies lies on a cut leaf: for $\sigma_0=H$ it is an image of O or A, and for $\sigma_0=L_2$ it is an image of R on the leaf carrying an L_1-copy, which ends at an image of A. A retracing leaf meets C at its launch point and at its turnaround on a σ'-copy, so by Lemma 7.7 nowhere else, and the rest of §7.6 reads no side. Lemma 7.8 applies because the σ_0-direction contains a side copy: L_1 itself for the L_2-beam, and for the H-beam, which has a retracing cell only at odd Q and to whose direction no side of T itself is parallel, a copy of L_2 when P is odd and of L_1 when P is even.

For the second, the turnaround of the σ_0 cell lies on a copy of the partner σ'. Read that orbit from its turnaround point: it leaves σ' perpendicularly and returns to it perpendicularly, so it is a retracing orbit of the σ'-beam, and by the previous paragraph that beam has only one. The two cells are the two ends of one band of double normals, and in particular one is present exactly when the other is.

For the last clause, let Q be a prime power, or twice or four times a prime. When P is odd the pair contains L_1, and its cell is the orphan of Theorem 4.11, Corollary 4.20 or Theorem 4.24. When P is even, Q is odd, hence a prime power, and the pair is L_2–H. Now the beam perpendicular to L_2 is the L_1-beam of the same triangle with the roles of its two acute vertices exchanged, that is, of the centre $(Q-P)/Q$. Since $Q-P$ is odd and prime to Q, Theorem 4.11 gives it a retracing cell.

If Theorem 4.18 applies to P/Q, then P is odd, the pair contains L_1, and its cell is that theorem's orphan. If it applies to $(Q-P)/Q$, then $Q-P$ is odd, so $P+Q$ is odd and the pair contains L_2; the beam perpendicular to L_2 is the L_1-beam of the centre $(Q-P)/Q$, whose orphan is the L_2 cell. □

By [Meth] Theorems 2–3 the two cells are present also at every centre with $\min(P, Q-P) \geq 2$ and either $\gcd(P+2, Q) = \gcd(P-2, Q) = 1$ or Q with at most one odd prime factor; both hypotheses are symmetric under $P \mapsto Q-P$.

Theorem 4.6 is this picture read from L_1, and the parity condition on P is what the L_1 launch happens to see.

6. The unfolded surface

From here to the end of the paper the beam is replaced by a straight line. Instead of reflecting the trajectory in a side of T, reflect the triangle and let the trajectory continue: the Katok–Zemlyakov construction [ZK75] glues finitely many copies of T into a closed translation surface on which the billiard flow becomes the straight-line flow in a fixed direction. Because the angles of T are rational multiples of π, the group generated by the three side reflections has finite image in $O(2)$, and the construction terminates.

The data of the resulting surface is standard bookkeeping. What is worth having is the answer for this family, uniformly in (P, Q) and in both parities of P, because §7 reads the order of the cone point at O off it, and that order changes type with the parity of P.

Theorem 6.1 (the unfolded surface)**.** *Let $2\alpha/\pi = P/Q$ in lowest terms, and write $g = \lfloor Q/2 \rfloor$. The Katok–Zemlyakov unfolding of T is a closed translation surface S_α, glued from $4Q$ copies of T, of genus g, carrying Q regular points over the right-angle vertex R and the cone points below over the two acute vertices. (P and Q are coprime, so they are not both even; the three rows are exhaustive.)*

	over O	*over A*	*stratum*
P odd, Q even	*one, of order $P-1$*	*one, of order $Q-P-1$*	$H(P-1, Q-P-1)$
P odd, Q odd	*one, of order $P-1$*	*two, each of order $(Q-P)/2-1$*	$H(P-1, (Q-P)/2-1, (Q-P)/2-1)$
P even, Q odd	*two, each of order $P/2-1$*	*one, of order $Q-P-1$*	$H(P/2-1, P/2-1, Q-P-1)$

In every case the orders sum to $2g-2$.

Proof. As fractions of π the three angles of T are $P/(2Q)$ at O, $1/2$ at R and $(Q-P)/(2Q)$ at A. Reduce each to lowest terms; write the reduced form as m/q.

At R it is already $1/2$, so $q=2$, $m=1$.

At O: if P is odd then $\gcd(P,2Q)=1$, so $q=2Q$ and $m=P$. If P is even then $P/(2Q)=(P/2)/Q$, and $\gcd(P/2,Q)$ divides $\gcd(P,Q)=1$, so $q=Q$ and $m=P/2$.

At A: if Q is even then P is odd, so $Q-P$ is odd and $\gcd(Q-P,2Q)=1$, giving $q=2Q$, $m=Q-P$. If Q is odd and P is odd then $Q-P$ is even and $(Q-P)/(2Q)=((Q-P)/2)/Q$, giving $q=Q$, $m=(Q-P)/2$. If Q is odd and P is even then $Q-P$ is odd, giving $q=2Q$, $m=Q-P$.

The denominators are $\{2Q,2,2Q\}$, $\{2Q,2,Q\}$ and $\{Q,2,2Q\}$ in the three rows, so in all three $N \coloneqq \operatorname{lcm} = 2Q$, and the unfolding uses $2N=4Q$ copies of T.

A vertex of reduced angle m/q becomes N/q cone points of cone angle $2\pi m$, that is of order $m-1$. Counting corners gives the multiplicity: $2q$ triangle-corners meet at each such point, and each of the $4Q$ copies contributes one corner of the class, so there are $4Q/(2q)=N/q$ of them. Applying this to the reduced data above yields the table: R gives $N/2=Q$ points of cone angle 2π, that is regular points; O gives one point of order $P-1$ when P is odd and two of order $P/2-1$ when P is even; and A gives one point of order $Q-P-1$ except in the middle row, where it gives two of order $(Q-P)/2-1$.

For the genus, the copies of T form a CW structure with $F=4Q$ faces and $E=6Q$ edges (each of the $4Q$ triangles has three sides, each shared by two triangles). The vertices are the Q points over R together with the cone points above, so $V=Q+2$ in the first row and $V=Q+3$ in the other two. Hence $\chi=F-E+V=V-2Q$ equals $2-Q$ in the first row and $3-Q$ in the other two, giving $g=Q/2$ when Q is even and $g=(Q-1)/2$ when Q is odd — in both cases $g=\lfloor Q/2\rfloor$. Summing the orders in each row gives $Q-2$, $Q-3$ and $Q-3$ respectively, which is $2g-2$. □

The third row is not a relabelling of the first: at even P the vertex O has **two** preimages rather than one, and carrying the odd-P shape $H(P-1,Q-P-1)$ across the parity line gives the wrong stratum at every even-P centre.

The extreme numerators are worth isolating, since they are the lattice families.

Corollary 6.2 (the extremes are hyperelliptic)**.** *At $P = 1$ the cone point at O has order 0, so it is regular and S_α has a single acute-vertex singularity: $S_\alpha \in H(2g-2)$ when Q is even and $H(g-1, g-1)$ when Q is odd. At $P = Q-2$ — which forces Q odd — the two points over A have order 0, so $S_\alpha \in H(2g-2)$, the minimal stratum. At $P = 2$ the two points over O have order 0, so again $S_\alpha \in H(2g-2)$. At $P = Q-1$ the point over A has order 0, and $S_\alpha \in H(2g-2)$ when Q is even, $H(g-1, g-1)$ when Q is odd — the $P = 1$ case read from the other leg. In every case the stratum is $H(2g-2)$ or $H(g-1, g-1)$.*

Proof. Read the orders off Theorem 6.1. $P = 1$: the O-order is $P - 1 = 0$; the A-order is $Q-P-1 = Q-2 = 2g-2$ when Q is even, and $(Q-P)/2-1 = (Q-1)/2-1 = g-1$, twice, when Q is odd. $P = Q-2$ with P odd forces Q odd, so the middle row applies and the A-order is $(Q-P)/2-1 = 0$, leaving O of order $P-1 = Q-3 = 2g-2$. $P = 2$ is the third row, where the O-order is $P/2-1 = 0$, leaving A of order $Q-P-1 = Q-3 = 2g-2$. $P = Q-1$ gives $Q-P-1 = 0$, so the A-point is regular: for P odd, $Q = P+1$ is even and the first row leaves O of order $P-1 = Q-2 = 2g-2$; for P even, Q is odd and the third row leaves two O-points of order $P/2-1 = (Q-3)/2 = g-1$. □

On the word *hyperelliptic*, which the stratum does not supply. Membership of $H(2g-2)$ does not by itself put a surface on a hyperelliptic locus — that stratum has non-hyperelliptic components. What is true, and is a forward reference to §7 rather than a consequence of Theorem 6.1, is that *every* S_α is hyperelliptic: by Proposition 7.1 it is a double cover of a genus-zero surface, so its deck involution τ is a hyperelliptic involution, and Proposition 7.2 counts its $2g+2$ fixed points. What the corollary adds is that at these four families — and only there — the stratum is one of the two, $H(2g-2)$ and $H(g-1, g-1)$, that *possess* a hyperelliptic component. We use the degeneration of the cone data and not the component label, so we do not pursue which component.

These four families are exactly the lattice locus $\min(P, Q-P) \leq 2$ [Api21, KS00], and from the point of view of the surface they are the degenerate ones: the cone data collapses onto a single singularity.

7. The surface reading, and the proof of Theorem 4.7

This section is the topological reading of Theorem A, and it is where the surface built in §6 earns its place in the paper: §7.6 proves Theorem 4.7, the uniqueness half, which §4.3 could only state. Along the way the parity of P, which entered §4 as a congruence count over the directions the beam can take, reappears as a ramification condition on S_α, for a reason that lies in the reflection group (Remark 7.3).

7.1. **The genus-zero base.** S_α is not primitive: it double-covers a quadratic differential on the sphere, and every structural statement below is really a statement downstairs. The quotient can be read off the labelled copies of §4.4 — copy $g \in \mathcal{G}$ glued across its side s to copy gr_s — and we read it off rather than cite it, so that the proof of Theorem 4.7 in §7.6 imports nothing. Write ω for the abelian differential of S_α, equal to dz in each copy.

Proposition 7.1 (the genus-zero base)**.** *Let τ act on S_α by $(x, g) \mapsto (x, -g)$, where -1 is the rotation of $\mathcal{G}$ through π. Then τ is an isometric involution with $\tau^*\omega = -\omega$; its fixed points are exactly the Q points over R, the point over O when P is odd and the point over A when $Q - P$ is odd; and the quotient is the genus-zero quadratic differential*

$$(S, q) = S_\alpha/\tau \in Q(P - 2, Q - P - 2, -1^Q)$$

on $\mathbb{CP}^1$ — cone angles $P\pi$ at the image Z_O of O and $(Q-P)\pi$ at the image Z_A of A, and Q simple poles at the images of R. So S_α is the holonomy double cover of (S, q), with deck involution τ.

Proof. -1 is central, so τ respects the gluing: copy $-g$ is glued across s to $-gr_s = \tau(gr_s)$. In the chart of copy g it reads $z \mapsto -z+b$, so it is an isometry with derivative $-\mathrm{id}$, and $\tau^*\omega = -\omega$. It acts freely on the labels, so it fixes no interior point of a copy; and it fixes no interior point of an edge $\{g, gr_s\}$, since $-g = gr_s$ would make the rotation -1 equal to the reflection gr_sg^{-1}. So its fixed points are vertex images. The copies around an image of a vertex form a left coset gH of the dihedral group H generated by the reflections in the two sides through that vertex, and $\tau(gH) = -gH = g(-1)H$ is the same coset exactly when $-1 \in H$. At R the group H is generated by the reflections in the two legs and contains -1. At O its rotations are the multiples of $2\alpha = P\pi/Q$, and π is one of them exactly when $mP \equiv Q \pmod{2Q}$ is solvable, that is, when P is odd. At A the rotations are the multiples of $(Q - P)\pi/Q$, and π is one of them exactly when $Q - P$ is odd.

The quotient is glued from $4Q/2 = 2Q$ copies of T, with $3Q$ edges. By Lemma 3.3 there are Q images of R, each fixed; one image of O when P is odd, fixed, and two when P is even, which are not fixed and so are exchanged; and likewise one or two images of A, according as $Q - P$ is odd or even. So the quotient has $V = Q + 2$ vertices and Euler characteristic $2Q-3Q+(Q+2) = 2$: it is a sphere. The cone angles halve at a fixed point and are unchanged on an exchanged pair, so by Lemma 3.3 the images of R have angle $2\pi/2 = \pi$, Z_O has angle $P\pi$ ($2\pi P/2$ from one fixed point, or πP from each of two exchanged ones), and Z_A has angle $(Q-P)\pi$. A cone angle $k\pi$ is a singularity of order $k - 2$. Finally ω^2 is τ-invariant and descends to q; since q has simple poles it is not a square, and the double cover on which it becomes one, branched exactly at its odd-order points, is S_α. □

This is the identification Apisa gives for rational right triangles [Api21], where the genus-zero base appears as the pull-back under $z \mapsto z^Q$ of the

pillowcase double of T, and S_α as its holonomy double cover — in [Api21] §7 the pillowcase double is the $(2Q)$-differential on the sphere with cone angles $(P/Q, (Q-P)/Q, 1)\cdot\pi$ at $0, \infty, 1$, not the quadratic pillowcase $Q(-1^4)$. We cite it for comparison; the proof above is what §7.6 uses, and it holds in both parities of P with no choice of labelling.

The base is a genuinely independent check on §6 rather than a restatement of it. Its Gauss–Bonnet sum is $(P-2)+(Q-P-2)+Q\cdot(-1) = -4$, as a genus-zero quadratic differential requires, and the standard lift rule — a zero of *odd* order d is a branch point and has a single preimage, a zero of order $d+1$; a zero of *even* order d is not, and has two preimages of order $d/2$; a simple pole has a single preimage, a regular point — recovers Theorem 6.1's three rows exactly, including the doubling at O when P is even and at A when $Q-P$ is even.

7.2. **The Weierstrass points.** Since S_α/τ has genus zero, τ is a **hyperelliptic** involution, and for genus $g \geq 2$ it is the hyperelliptic involution, which is unique; its fixed points are the $2g+2$ Weierstrass points [FK92] (Ch. III, §7). That they are the preimages of the odd-order singularities of the base is the branching description of the canonical double cover, stated for a $\mathbb{CP}^1$ base with simple poles by Athreya, Eskin and Zorich [AEZ16] §3.1; Proposition 7.1 has verified it here directly. At $Q \in \{2, 3\}$ the genus is 1, where the hyperelliptic involution is not unique and the classical notion is empty; there, and throughout, *Weierstrass point* means a fixed point of τ, and Proposition 7.2 and Remark 7.3 hold verbatim in that sense.

Proposition 7.2 (the Weierstrass points of S_α)**.** *The fixed points of τ — the Weierstrass points of S_α — are exactly*

- *the Q points over the right-angle vertex R, always;*
- *the point over O, when P is odd;*
- *the point over A, when $Q-P$ is odd.*

Their number is $Q + [P \text{ odd}] + [Q-P \text{ odd}] = 2g+2$.

Proof. The list is Proposition 7.1's. It agrees with the classical description: the odd-order singularities of (S, q) are the Q simple poles under R, the point of order $P-2 \equiv P$ under O when P is odd, and the point of order $Q-P-2 \equiv Q-P$ under A when $Q-P$ is odd.

For the count, if Q is even then P and $Q-P$ are both odd and the total is $Q+2 = 2(Q/2)+2$. If Q is odd then $P+(Q-P) = Q$ is odd, so exactly one of P, $Q-P$ is odd and the total is $Q+1 = 2\cdot(Q-1)/2+2$. In both cases this is $2g+2$. □

The count is the check rather than the content: a hyperelliptic surface of genus g has exactly $2g+2$ Weierstrass points, so the agreement confirms that τ is the hyperelliptic involution and that Theorem 6.1's cone data and Proposition 7.1's base are mutually consistent. It is not built into either computation, and it is not automatic — it fails immediately if the parity bookkeeping at O or A is wrong.

It is worth naming the first item, since it is the one a reader is least likely to anticipate: **the Q points over the right angle are Weierstrass points**, even though they are *regular* points of the translation surface. Their being unramified upstairs as cone points and ramified as covering points is exactly the distinction between the two structures, and it is the reason the right angle, which is invisible in the cone data of §6, is not invisible here.

7.3. The parity match at O.

Remark 7.3 (the orphan's parity and the Weierstrass point at O). By Proposition 7.2 the point over the acute vertex O is a Weierstrass point exactly when P is odd; when P is even, O has two preimages (Theorem 6.1), exchanged by τ. The orphan exists only when P is odd (Theorem 4.6). So wherever an orphan exists the point over O is a Weierstrass point, and conversely wherever existence is proved: when Q is a prime power, or twice or four times a prime (Theorem 4.11, Corollary 4.20, Theorem 4.24), at the centres of Theorem 4.18, and by [Meth] Theorems 2–3 at every centre with $\min(P, Q-P) \geq 2$ and either $\gcd(P+2, Q) = \gcd(P-2, Q) = 1$ or Q with at most one odd prime factor.

Both conditions are the parity of P, and the match has a one-line explanation in the reflection group, not in the orbits. Use the index notation fixed in §7.6: directions of lines in units of $\pi/2Q$, with L_2, L_1, H at indices 0, Q, P; $r_\varphi := \rho^\varphi r_0$ the reflection in the line of index φ; and $\rho^Q = \tau$. Since $\rho^k r_a \rho^{-k} = r_{a+2k}$ and $r_0 r_a r_0 = r_{-a}$, two reflections r_a, r_b are conjugate in D_{2Q} exactly when $a \equiv b$ modulo 2. The subgroup they generate is $\langle \rho^{a-b}, r_a \rangle$, of order $4Q/d$ with $d := \gcd(a-b, 2Q)$, and it contains ρ^Q exactly when d divides Q. At O the two sides have indices 0 and P, and P is prime to Q, so $d = 1$ when P is odd and $d = 2$, which does not divide the odd Q, when P is even. Hence

$$r_0, r_P \text{ not conjugate} \iff P \text{ odd} \iff \langle r_0, r_P \rangle = D_{2Q} \iff \tau \text{ fixes the point over } O.$$

The first condition is the one Theorem 5.3 reads. Its criterion $\theta_{\sigma'} \equiv \theta_\sigma$ modulo 2 is conjugacy of the two reflections, so L_1 can carry a retracing cell only if r_Q is conjugate to r_0 or to r_P. Three reflections fall into two classes, so that happens exactly when r_0 and r_P are not conjugate to each other. The last condition is O being a Weierstrass point, because the stabiliser of a point over O is a conjugate of $\langle r_0, r_P \rangle$ and τ is central.

Read for every side, the same count gives a statement uniform in the three: *a side σ can carry a retracing cell only if the number of Weierstrass points over the vertex V opposite σ is odd.* Indeed, if a, b are the indices of the two sides at V, then V has $d = \gcd(a-b, 2Q)$ preimages, and these are all τ-fixed when d divides Q and none are otherwise. The count is therefore d or 0, and it is odd exactly when $a - b$ is odd. That is, the two sides at V are not conjugate, which by Theorem 5.3 is the condition for σ to belong to

the retracing pair. Over O and A the count is 1 or 0, and for the legs this is the match above. Over R it is Q. The naive form, *only if V is a Weierstrass point*, is sharp for the legs and vacuous for the hypotenuse. The Q points over R are Weierstrass points at every Q, since $\langle r_0, r_Q\rangle = \{1, \rho^Q, r_0, r_Q\}$ contains τ whether or not r_0 and r_Q are conjugate, while H can carry a retracing cell only when Q is odd.

None of this bears on existence. Conjugacy decides whether the development contains a copy of the partner side perpendicular to the beam, which is the direction count of Proposition 4.5. It does not decide whether some orbit reaches that copy without grazing a vertex; that is the arithmetic of §§4.4–4.5 and Conjecture 4.12. Nor do we know a map carrying the orphan to the point over O, and the geometry is against one. By Lemma 2.1 the orphan's turnaround lies in the relative *interior* of a side. And §7.6 finds the orphan's cylinder τ-**swapped**, not τ-fixed.

Remark 7.4 (not the spin invariant). The $Z/2$ invariant of Remark 7.3 is Weierstrass — τ-fixed — parity, not the Kontsevich–Zorich spin (Arf) invariant [KZ03]: that is defined only when every zero of the abelian differential has even order, which on S_α cuts across the parity of P (Theorem 6.1: 5/8 and 3/5 give $H(4,2)$ and $H(2,0,0)$, where it is defined, 3/7 gives $H(2,1,1)$, where it is not, and the orphan exists at all three).

What the surface does give is the shape of the count. Under τ the closed geodesics of the perpendicular direction fall into τ-swapped pairs and τ-fixed ones. Writing $n = N_{\text{fix}} + 2 \cdot N_{\text{pair}}$ for the splitting of the branches induced by J, Theorems 4.6 and 4.7 give $N_{\text{fix}} = 0$ at even P and $N_{\text{fix}} \le 1$ at odd P, with equality wherever the orphan exists — a statement about branches, not about Weierstrass points — and all of the arithmetic difficulty of the count sits in N_{pair}, about which the surface picture says nothing. We stop short of a correspondence between the τ-orbits of *cylinders* and the pairs of branches: it is a statement about how often the transversal crosses each cylinder, so it belongs with the count in [Neck]. The branch-level fact is Corollary 4.14: the two branches of a pair are adjacent, across a launch height whose orbit runs into R.

7.4. **The fold.** J is not the hyperelliptic involution restricted to a transversal: τ does not preserve the launch leg. What is true is a fold, and it takes one count to state. The right angle unfolds to a *regular* point of S_α (Theorem 6.1), so the copy of L_1 continued straight through such a point is a geodesic segment made of leg-copies. How many is the count.

Proposition 7.5 (two leg-copies per direction)**.** *Each of the Q directions of L_1-copies in S_α carries exactly two of them, and they are the two L_1-edges at a single regular point over R, opposite each other there. Writing Λ for the union of those two edges — a single straight segment through that regular point, twice the length of the leg — Λ therefore consists of exactly two copies of the leg, and contains every copy of the leg parallel to it.*

Proof. The angles of T are $P\pi/2Q$, $Q\pi/2Q$ and $(Q-P)\pi/2Q$, so the linear parts of the unfolding group form a dihedral group D acting on directions; measure directions in units of $\pi/2Q$, so that D acts on $Z/4Q$ by $\psi \mapsto \pm\psi+m$. Composing reflections in the sides gives the rotations by $2P$ and by $2Q$, and $\gcd(P,Q)=1$ makes those generate every even residue, so D has $2Q$ rotations and $2Q$ reflections — order $4Q$, matching the $4Q$ copies of T in Theorem 6.1 — and its reflections are $\psi \mapsto 2\varphi-\psi$ for **every** integer φ. The leg's line carries the pair of opposite directions $\psi = Q$ and $\psi = 3Q$; its stabiliser in D — the elements preserving that pair — consists of the rotations by 0 and π and the reflections in the horizontal and vertical lines, so it has order 4 and is exactly the group generated by the reflections in L_1 and L_2 (written $\langle r_Q, r_0\rangle$ in the index notation §7.6 fixes). The orbit therefore has $4Q/4 = Q$ directions and each is the L_1-direction of exactly four copies of T; those four are glued in pairs along their L_1-sides, leaving two leg-copies per direction, $2Q$ in all, which is the $4Q$ L_1-sides divided by two.

That the stabiliser is generated by the reflections in L_1 and L_2 — the two sides meeting at R — says which two. It identifies the four copies as the star of one R-image, and around a regular point the four edges of that star alternate L_1, L_2, L_1, L_2, so its two L_1-edges are opposite and their union is a straight segment, which is the Λ of the statement. □

The argument used nothing about L_1 beyond its being a side, so two-per-direction holds for L_2 and H as well; what is special to the leg is that its two copies meet at a *regular* point and so form a single geodesic segment.

One point of precision, since it is easy to call Λ a saddle connection and at two families it is not one. Its two endpoints are images of A (one and the same point when $Q-P$ is odd, where A has a single preimage), and Theorem 6.1 gives those order 0 exactly when $P = Q-1$, or $P = Q-2$ at odd Q — the two lattice families of Corollary 6.2 with a regular acute image. There Λ is a proper sub-segment of a longer geodesic rather than a saddle connection. Nothing below turns on the difference: what is used is only that Λ is the segment of twice the leg's length centred at the point over R.

Proposition 7.6 (the fold)**.** *Let Λ be the geodesic segment of Proposition 7.5. Then τ carries Λ to itself and acts on it as the reflection about the regular point over R, exchanging the two leg-copies; so the leg is the quotient Λ/τ, and J is the first return of the perpendicular flow to Λ read in that quotient.*

Proof. By Proposition 7.2 the point over R is a Weierstrass point, so τ fixes it, and $\tau^*\omega = -\omega$ gives $d\tau = -\mathrm{id}$ there — indeed everywhere, τ being an isometry of the flat metric with that derivative. A geodesic through a fixed point of such a map is carried to itself with its two directions interchanged, so $\tau(\Lambda) = \Lambda$ and $\tau|_\Lambda$ is the reflection about that point; each leg-copy has it as an endpoint, so the two are exchanged and Λ/τ is a single copy of the

leg. For the last clause, the perpendicular flow is by definition orthogonal to L_1, so a trajectory returns to L_1 *perpendicularly* exactly when it crosses a copy of the leg parallel to the one it left — and by Proposition 7.5 those copies are precisely the two inside Λ. Hence the perpendicular returns to the leg are the crossings of Λ, and reading them in Λ/τ returns them to the leg as J. □

Since $d\tau = -\mathrm{id}$ reverses the flow, τ conjugates the first return to Λ to its inverse. That is the surface-side reason for the involutivity Theorem 2.2 proves by hand, though we do not derive the one from the other.

7.5. **Relation to the literature.** Proposition 7.2 and Remark 7.3 claim no novelty, and their ingredients are in the literature. On the lattice locus $\min(P, Q-P) \le 2$ McMullen [McM25] §3 determines the Weierstrass points of the regular n-gon surface: the cone point z_0 lies in $W(X)$ if and only if $n \neq 4g+2$. On the slice $\min(P, Q-P) = 1$, where the regular $2Q$-gon model has $n = 2Q$ and its cone point is O when $P = Q-1$ and A when $P = 1$, this is Proposition 7.2 vertex by vertex, in both parities of Q (§A.4); perpendicular beams and the orphan do not appear in [McM25]. And the parity split of Theorem 6.1 is the specialisation $a = P-2$, $b = Q-P-2$ of Apisa and Wright's holonomy double cover of a surface in $Q(a, b, -1^{a+b+4})$ [AW24, §9.1], whose orders over the zero of order a are $\{a+1\}$ when a is odd and $\{a/2, a/2\}$ when a is even; we give the Katok–Zemlyakov computation in §6 because it is self-contained. One notational caution: in [AW24] §9.1, and in [Api21], J names involutions of the cover; our J is the billiard return map of §2, and none of them.

7.6. **The proof of Theorem 4.7.** We can now prove the uniqueness half of Theorem A. The argument is the one sketched in §1.2: every orphan branch uses up a pair of cylinders exchanged by the deck involution, and an Euler count on the genus-zero base allows at most one such pair. The ingredients are the base (§7.1), the Weierstrass points (§7.2) and the vertical copies (Proposition 7.5). Throughout, P is odd and **no hypothesis is placed on the direction**: it may have minimal components. τ is the deck involution of Proposition 7.1, $(S, q) = S_\alpha/\tau$ the base of Proposition 7.1, σ_v the reflection of S_α fixing the launch leg's direction, and ν the number of orphan branches (not ω, which is this paper's abelian differential, §7.1).

The reflection σ_v. The unfolding of §6 presents S_α as the $4Q$ copies of T of Theorem 6.1, indexed by the dihedral group $D_{2Q} =< \rho, r_0 >$ — of order $4Q$, ρ having order $2Q$, as Proposition 7.5's proof records — glued so that the group acts on the labels alone: a point is a pair (x, h) with $x \in T$ and $h \in D_{2Q}$, and $h' \cdot (x, h) = (x, h'h)$. Fix the generators geometrically: measure the directions of unoriented lines in units of $\pi/2Q$, place L_2 at index 0 — so that L_1 sits at Q and the hypotenuse at P, the angle at A being $(Q-P)\pi/2Q$ — and write ρ for the rotation by π/Q, which advances a direction index by two and has order $2Q$, and r_0 for the reflection of index

0. Then $r_\varphi \coloneqq \rho^\varphi r_0$ is the reflection in the line of index φ, since composing two reflections at angles ψ_1, ψ_2 rotates by $2(\psi_2 - \psi_1)$; and ρ^Q is the rotation by π, which is τ. So the reflections in L_2, L_1 and H are r_0, r_Q and r_P respectively, and we name them by index throughout — the launch height s of §§2–5 plays no part in this section. Put

$$\sigma_v \coloneqq r_Q, \qquad \text{the reflection of } D_{2Q} \text{ in the launch leg } L_1,$$

acting in that way. This is an isometry of the half-translation structure; it fixes the L_1 direction and therefore reverses the beam direction, which is why it is orientation-reversing and why nothing in §6 supplies it — it is a symmetry of the *unfolding*, not an automorphism of the translation surface.

The cut locus. Write $C \coloneqq \mathrm{Fix}(\sigma_v)$. Since the action is on labels, a side copy — an edge $\{h, h \cdot r_j\}$ of the development, $j \in \{0, Q, P\}$ — is σ_v-fixed exactly when $hr_jh^{-1} = \sigma_v$, and is then fixed *pointwise*. So the fixed copies are those whose reflection r_j is conjugate to σ_v **in** D_{2Q}, and for such an r_j the solutions h of $hr_jh^{-1} = \sigma_v$ form a **right** coset of the centraliser of σ_v — if h and h_0 both solve it then hh_0^{-1} centralises σ_v — and $\mathrm{Cent}(\sigma_v) = \{1, \rho^Q, \sigma_v, \rho^Q\sigma_v\}$ has order four, ρ^Q being central and $2Q$ even. Those four solutions name only **two** copies, since h and hr_j are the two labels of the same edge and both solve the equation. Exactly two side types are conjugate to σ_v — L_1 itself, and one of L_2, H by the parity below — so there are $2 \times 2 = 4$ fixed copies. Hence C is the union of the **four** side copies perpendicular to the beam: the two L_1-copies λ_1, λ_2 of Proposition 7.5, and two further copies W_a, W_b, of the hypotenuse when Q is odd and of the far leg when Q is even. σ_v fixes C pointwise and reverses the beam direction, so C is transverse to the perpendicular foliation.

*(Which two is decided by a conjugacy class. In D_{2Q} the reflections form two classes, according as the reflection is $\rho^{even}r_0$ or $\rho^{odd}r_0$; call these the **even** and **odd** classes. $\sigma_v = \rho^Q r_0$, so σ_v is even exactly when Q is even, while r_0 is even and r_P is odd always, the hypotenuse's index P being odd here.)*

A convention, and it is load-bearing. Throughout §7.6, *cone point* means an image of O or of A of order $k \geq 0$, and *cylinder* means a maximal cylinder of the perpendicular direction in the complement of the leaves through the cone points. The cut locus C is **not** cut. At the families where an acute image has order 0 — by Corollary 6.2 the odd-P cases are $P = 1$, $P = Q - 1$ at even Q and $P = Q - 2$ at odd Q — such an image is a regular *marked* point of S_α, and we cut along its leaf anyway. The points over R are not marked: they are regular and are not cut. So a leaf of a cylinder is a closed regular leaf meeting no cone point, and a branch, being an interval of such leaves with a constant word, lies in a single cylinder: a leaf through a cone point grazes it and bounds the branch.

Lemma 7.7 (two arcs or none)**.** *Every cylinder of the perpendicular direction meets C in exactly **two** arcs or in none at all.*

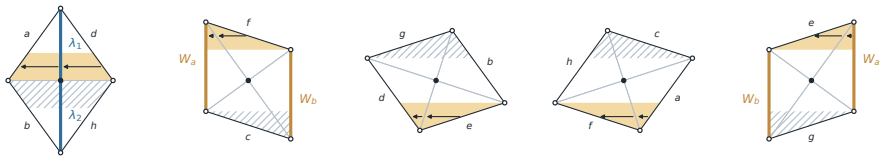


FIGURE 4. The cut locus at $2\alpha/\pi = 3/5$, on S_α drawn as its $Q = 5$ rhombi — each the four copies of T around an image of R (filled dot), the j-th turned through $j\pi/5$ — with edges glued in pairs by translation as lettered; hollow dots are the images of O and A, and the perpendicular direction is horizontal. C is drawn heavy: $\lambda_1 \cup \lambda_2$, the vertical diagonal of the first rhombus, and the two vertical copies W_a, W_b of the hypotenuse, each appearing as two glued edges. The orphan's cylinder Z (shaded) is swept by the leaves through the orphan branch $(0, (3-\sqrt{5})/2)$ of λ_1; one leaf is drawn, through a, f, W_a, e and d back to λ_1. It meets C in two arcs, on λ_1 and on W_a, as Lemma 7.7 requires, and misses λ_2. The deck involution τ is the half-turn of each rhombus about its centre, and τZ (hatched) meets λ_2.

Proof. Let Z be a cylinder meeting C and let γ be any leaf of Z. Some component of $C \cap Z$ exists; it carries no cone point, those lying on ∂Z, so it is an arc with both ends on ∂Z, and C being perpendicular to the beam its transverse coordinate is strictly monotone along it — so the arc runs from one boundary level to the other and crosses every leaf of Z, γ included, exactly once. Pick p in $\gamma \cap C$. Then σ_v fixes p and carries the horizontal geodesic through p to itself, so $\sigma_v(\gamma) = \gamma$; and σ_v reverses the beam direction, so it acts on the circle γ as an orientation-reversing isometry, that is as a reflection, with exactly two fixed points. Those fixed points are $\gamma \cap \mathrm{Fix}(\sigma_v) = \gamma \cap C$, so $|\gamma \cap C| = 2$; and since every arc of $C \cap Z$ meets γ exactly once, Z has exactly two arcs. □

The orphan's cylinder. Let an orphan branch be given, and Z its cylinder. The interior point where λ_1 and λ_2 meet is an image of R lying on the leaf that carries the side L_2, and that leaf ends at an image of O, a cone point; so the junction lies on ∂Z, and each arc of $C \cap Z$ lies in a single one of the four copies. An orphan leaf meets C at its launch point, on λ_1, and at its head-on turnaround, which is on a vertical copy of type H or L_2 (Theorem 4.6) — call it W_a. By Lemma 7.7 those are its only two meetings with C, so it misses λ_2; and since each arc of $C \cap Z$ crosses every leaf of Z, Z **misses** λ_2. As $\lambda_2 = \tau(\lambda_1)$, $\tau Z \neq Z$.

Since $\tau Z \neq Z$ and distinct cylinders are disjoint, Z contains no τ-fixed point. By Proposition 7.2 every one of the Q points over R is τ-fixed, so no

image of R lies in the interior of Z, while the images of O and A are cone points and lie on ∂Z. So the interior of Z contains no vertex image at all, no trajectory through Z grazes a vertex, and no branch boundary is interior to $\lambda_1 \cap Z$. **The single arc $\lambda_1 \cap Z$ is therefore one branch.** Consequently distinct orphan branches lie in distinct cylinders Z_1, Z_2, and $Z_1 \neq \tau Z_2$ as well, because τZ_2 meets λ_2 in $\tau(\lambda_1 \cap Z_2)$ while Z_1 misses it. So:

> *the ν orphan branches give ν distinct pairs $\{Z, \tau Z\}$ of cylinders exchanged by τ.*

What remains is to count such pairs.

Lemma 7.8 (at most one exchanged pair)**.** *Take any direction on S_α containing a side copy — a copy of L_1, L_2 or H parallel to it — and cut along the leaves through the cone points as in the convention above. Then at most one pair of its cylinders is exchanged by τ; every other cylinder is τ-invariant. No parity of P is assumed.*

Proof. Work on the base. Its vertices are the images of the Q points over R, which are simple poles, and the images Z_O, Z_A of O and A, of orders $P-2$ and $Q-P-2$ (Proposition 7.1). So Z_O is a simple pole at $P=1$, a marked regular point at $P=2$, and a zero otherwise; likewise Z_A at $Q-P=1$, $Q-P=2$ and otherwise. A vertex of order k emits $k+2$ horizontal prongs, and the orders sum to -4, so there are $2V-4$ prongs, $V = Q+2$.

Let G be the graph whose vertices are these V points and whose edges are the E horizontal saddle connections among them, and let N be the number of prongs lying on no saddle connection, so that $2E + N = 2V - 4$. The complement of G is a finite union of cylinders and minimal components ([Str84] §11.4; [MT02]); let there be C_B cylinders and M minimal components, the j-th having k_j boundary circles. A regular neighbourhood of G is planar, so if it has c components and b boundary circles its Euler characteristic is $V - E = 2c - b$. Every boundary circle bounds a cylinder or a minimal component, so $b = 2C_B + \Sigma k_j$, and the minimal components have genus 0. Additivity of the Euler characteristic over the sphere gives $2 = (V-E) + \sum_j (2 - k_j)$. With $2E + N = 2V - 4$ this becomes

$$\sum_j k_j = 2M + N/2, \quad C_B = c - 1 - M - N/2.$$

Now sort the components of G. A simple pole emits one prong, so a component with an edge whose vertices are all poles is a single pole–pole edge. The components are therefore of three kinds: z isolated vertices; a single pole–pole edges, of which a_R join two images of R; and κ'' components with an edge and a vertex emitting at least two prongs, which is Z_O or Z_A. So $c = z + a + \kappa''$. The circle around an isolated vertex bounds a minimal component, since a cylinder's boundary is made of saddle connections; write $\Sigma k_j = z + B$, where B counts the circles around the other components that bound minimal components. G has an edge — the image of the side copy the direction contains, which joins two vertices — so no minimal component

is bounded by isolated-vertex circles alone: with those discs filled in it would be closed, hence the whole sphere, leaving no room for the edge. So $B \geq M$, and

$$C_B - a_R = (a - a_R) + \kappa'' - 1 + (M - B) \leq (a - a_R) + \kappa'' - 1.$$

Each of Z_O, Z_A contributes at most 1 to $(a - a_R) + \kappa''$: as a simple pole it emits one prong and so ends at most one edge, and otherwise it lies in at most one component. The images of R contribute to neither term. Hence $C_B - a_R \leq 1$.

Finally, lift. The core of a base cylinder is a closed geodesic, whose holonomy is trivial, so the open base cylinder lifts to two annuli exchanged by τ. They lie in a single cylinder of S_α exactly when a boundary circle of the base cylinder runs along a pole–pole edge whose lift is a leaf that is not cut. An edge between two images of R lifts to a closed regular leaf through two unmarked points over R, which is not cut, and the two annuli then form one τ-invariant cylinder with those two points on its core. An edge ending at Z_O or Z_A as a pole lifts to a leaf through a marked point, which is cut, so the two annuli remain distinct cylinders. A pole–pole edge is a whole component of G with a single boundary circle, and the piece of the complement on that circle is always a cylinder, never a minimal component: at each end the cone angle is π and the edge is the only prong, so a leaf parallel to the edge at small distance turns around each pole and closes up after length about twice the edge's. So every pole–pole edge bounds exactly one base cylinder. And no base cylinder is bounded by two edges between images of R: the sphere would then be that cylinder together with its two slits, carrying four simple poles over R and no other singularity, which happens at no admissible P/Q. So the cylinders of S_α are a_R τ-invariant ones together with $C_B - a_R \leq 1$ pairs exchanged by τ. □

Proof of Theorem 4.7. By the paragraph before Lemma 7.8 the orphan branches give ν distinct exchanged pairs. The perpendicular direction contains the side L_2, so Lemma 7.8 applies and there is at most one. So $\nu \leq 1$. □

The count allows minimal components — the orphan's cylinder is a cylinder whatever the rest of the direction does — so the proof needs no hypothesis on the direction. It holds at every odd P, lattice or not, with no restriction on Q: the marked-point convention absorbs the order-0 families, and the genus-one centres 1/2 and 1/3 are covered with *Weierstrass point* read as in §7.2. The inequality is attained wherever the orphan exists, in particular at the centres of Theorems 4.11, 4.18 and 4.24, and there the one exchanged pair is the orphan's. At even P, where Theorem 4.6 gives no orphan, the same count gives none, because the hypotenuse's direction index P is even, the parity of the beam's index 0, so a hypotenuse copy parallel to the beam joins Z_O to Z_A and $\kappa'' = 1$.

8. What is not settled

Every numbered statement of this paper is proved, except Conjecture 4.12. Two questions are left open.

1. Existence at every odd P (Conjecture 4.12). Lemma 4.10 reduces it to excluding one configuration, a fan tiling of the leg whose doubled prong heights sum alternately to 1. Once Q has two distinct prime factors the membership of those heights cannot exclude it (Remark 4.13), and below Theorem 4.18's threshold the corner landings of §4.5, reduced modulo 2, exclude nothing more (Proposition 4.19). What the method of §4.5 lacks is control of which prongs reach the leg in the middle of a row — at $Q = 2^k p$, $k \geq 3$, of the classes Lemma 4.21's analogue leaves free. By a different argument the conjecture is proved in [Meth] under the hypotheses quoted in §1.2 (with Theorem 4.18 at $\min(P, Q-P) = 1$). Where $P+2$ and $P-2$ both share a factor with Q, [Meth] Theorem 4 shows that no certificate built from its window and signs can settle it.

2. An elementary proof of uniqueness. Theorem 4.7 is proved by an Euler count on the genus-zero base (§7.6). We know no argument that stays inside the development of §3, for the reason given in §4.3.

The value of $n(P/Q)$ is not a question about the orphan; it is the subject of [Neck], which uses results of this paper.

Appendix A. Exact computation

Every **branch count** stated in this paper is an exact algebraic computation, not a numerical estimate, and every one comes with a certificate that the partition it counts is complete. §A.1 shows that the branch boundaries are algebraic and describes the partition at a fixed centre, §A.2 the comparison of algebraic numbers it rests on, and §A.3 the certificate and the vocabulary that goes with it. The certificate is about the partition and nothing else: it says nothing about cylinders or about complete periodicity. The small finite checks of proved statements that are not branch counts are collected in §A.4.

A.1. Branch boundaries, and the partition at a fixed centre.

Two facts about the development underlie the computation: branches are intervals, and their endpoints are algebraic.

Lemma A.1 (interval / nesting)**.** *For any finite reflection word $W = (e_0, \dots, e_{m-1})$, the set*

$$S_W = \{s \in (0,1) : \text{the ray exits } T_i \text{ through the copy of } e_i \text{ for every } i < m\}$$

is an open subinterval of $(0,1)$. Hence the branch partition is a partition into intervals, and the tree of words is a nested-interval tree, each split occurring at a single point.

Proof. Induction on m. For $m = 0$, $S_\emptyset = (0,1)$. For the step, let $I = S_{W'}$ be the interval for the prefix $W' = (e_0, \dots, e_{m-2})$. By Lemma 3.1(3)

the isometry g_{m-1} placing T_{m-1} depends only on W', so T_{m-1} is *one fixed triangle* for all $s \in I$. For a fixed triangle the horizontal line $\{y = s\}$ meets it precisely when s lies between its extreme vertex heights, and the pair (entry edge, exit edge) is constant on each of the two height ranges separated by the middle vertex height. The entry edge is forced by $s \in I$, so it determines the exit edge on each range, and $\{s \in I : \text{ exit } = e_{m-1}\}$ is the intersection of I with one vertex-height range — an interval. □

Remark A.2 (the slope law). Within a fixed word the development is a fixed isometry, so the position at which the ray $\{y = s\}$ crosses the copy e met at step j, measured along e, is affine in s: $C_j \cdot s + D_j$, with rate $C_j = 1/|\cos\varphi_j|$, φ_j the angle of e from the vertical, and D_j fixed by the word (Lemma 3.1(3)). So staying off both endpoints of e is a band of launch heights of width $|e| \cdot |\cos\varphi_j|$, and a branch is the intersection of the bands of its orbit.

Theorem A.3 (three-vertex projection)**.** *Every branch boundary — a launch height whose orbit exactly grazes a vertex of T — is the height of a developed image of one of the three vertices of T, and lies in the ring $\mathbb{Z}[\cos 2\alpha]$: it is an integer polynomial in $\cos 2\alpha$.*

Proof. By Lemma A.1 a branch boundary is a height at which the developed ray $\{y = s\}$ passes through a developed vertex $g_W(v)$, $v \in \{O, R, A\}$, and by Lemma 3.1(3) that vertex does not depend on s. So the grazing condition is simply $s = y(g_W(v))$ — the projection the name refers to.

It remains to place the height of a developed vertex in $\mathbb{Z}[\cos 2\alpha]$. The rotations of $\mathcal{G}$ are products of two side reflections, so they turn by multiples of 2α and of π, and the second row of every linear part $\mathrm{Lin}(g)$ is $\pm(\sin 2m\alpha, \pm\cos 2m\alpha)$ for an integer m. The vertices of T have abscissae 0 and $\cot\alpha$ and heights 0 and 1. The reflections in L_2 and H fix O and have no translation part, while the reflection in L_1 has translation part $(2\cot\alpha, 0)$. By $t(g \circ h) = \mathrm{Lin}(g) \cdot t(h) + t(g)$ the height of $g_W(v)$ is therefore an integer combination of the numbers $\cos 2m\alpha$ and $\sin 2m\alpha \cdot \cot\alpha$. Both are integer polynomials in $\cos 2\alpha$: $\cos 2m\alpha = T_m(\cos 2\alpha)$, and $\sin 2m\alpha \cdot \cot\alpha = U_{m-1}(\cos 2\alpha) \cdot \sin 2\alpha \cdot \cot\alpha = U_{m-1}(\cos 2\alpha)(1 + \cos 2\alpha)$, with T_m, U_m the Chebyshev polynomials (signs absorb negative m). □

Every branch boundary is therefore an exact algebraic number; at a fixed centre it is an element of $\mathbb{Z}[\zeta_{4Q}]$ up to a fixed scale, and the computation works in that ring.

Unfold the beam as in Lemma 3.1, so that it becomes the horizontal line $\Im z = y$ moving in the $-x$ direction, and scale T by $2\sin\alpha$: with $\zeta = e^{i\pi/2Q}$ its vertices are

$$O = 0, \qquad R = \zeta^P + \zeta^{-P}, \qquad A = 2\zeta^P,$$

and the launch height is $y = 2s\sin\alpha \in (0, \Im A)$.

Every developed copy of T is the image of this one under an isometry $z \mapsto t + \zeta^r z$ or $z \mapsto t + \zeta^r \bar{z}$, and the reflection in a side of direction ζ^k is $z \mapsto X + \zeta^{2k}(\bar{z} - \bar{X})$. So every developed vertex is an **integer** combination of powers of ζ, and one reflection acts on its coefficient vector by an index shift, an index reversal and two additions: no division, no rounding, nothing approximated.

The computation carries whole intervals of launch heights at once. An interval enters a copy of T through a side $[U, V]$ with $\Im U < y < \Im V$; let W be the third vertex. The beam leaves through $[U, W]$ if $y < \Im W$ and through $[V, W]$ if $y > \Im W$, and $y = \Im W$ is a vertex graze — a branch boundary. So the interval is carried across unchanged when $\Im W$ lies outside it and split at $\Im W$ when it lies strictly inside. The prefix of a word ends at the first head-on hit, when the exit side is vertical in the developed frame; by Lemma 2.1 the word is that prefix followed, unless the hit is on L_1, by its own reversal. **One step of the computation is therefore one comparison of $\Im W$ with the ends of the interval — the sign of the imaginary part of an element of $\mathbb{Z}[\zeta_{4Q}]$ — and every branch boundary is the imaginary part of a developed vertex.**

The sign is decided exactly. For $v \in \mathbb{Z}[\zeta_{4Q}]$, the element $v - \bar{v}$ represents $2i \cdot \Im v$, and reduced modulo the cyclotomic polynomial Φ_{4Q} (§A.2) it is zero exactly when $\Im v = 0$. Otherwise the sign of $\Im v$ is read off a floating-point evaluation when that value exceeds a bound on its own rounding error by a factor of 1024, and off an integer fixed-point evaluation with an explicit error bound, at doubling precision, when it does not. No decision rests on an unchecked float, and no width floor is applied: a branch of any width is found, because nothing is sampled. Coefficients are held as machine integers under a guard at 2^{40} that raises rather than overflows; by Remark A.4 they grow only linearly with the length of a passage, and the guard has never fired.

The check of Theorem 5.3 in §A.4 is the same computation run for the beam perpendicular to each side of T, with the frame relabelled; nothing else changes.

Remark A.4 (coefficient growth). In the $\cos(2k\alpha)$ basis a boundary polynomial has an integer coefficient vector, and its coefficients grow at most **linearly** with the number of reflections m of the adjacent orbit, so their bit-length grows logarithmically — which is why the exact integer arithmetic above remains feasible at the depths the counts require.

Integrality by itself would not give this, and it is worth saying why: a product of m integer matrices can have entries growing exponentially in m. What rules that out here is that the development accumulates by *addition*. In the notation of Lemma 3.1 the isometry placing the m-th copy is $g_m = R_{e_0} \circ \cdots \circ R_{e_{m-1}}$, and translation parts compose by $t(g \circ h) = \mathrm{Lin}(g) \cdot t(h) + t(g)$,

so

$$t(g_m) = \sum_{i<m} \mathrm{Lin}(g_i) \cdot t(R_{e_i}),$$

a sum of m vectors drawn from the *finite* set $\{\mathrm{Lin}(g) \cdot t(R_e) : g \in \mathcal{G},\ e \text{ a side of } T\}$ — finite because $\mathrm{Lin}(\mathcal{G})$ is the finite group of Lemma 3.2 and T has three sides. Each member of that set has a bounded coefficient vector, so the coefficients of $t(g_m)$ are bounded by a constant multiple of m; and $\mathrm{Lin}(g_m)$ contributes no growth at all, being one of finitely many matrices. The boundary equation of Theorem A.3 is assembled from these, so its coefficients inherit the bound. The argument does not depend on the basis, and it applies equally to the coefficient vectors in $\mathbb{Z}[\zeta_{4Q}]$ carried above. The constant is small in practice: measured in those vectors — integer vectors of length $4Q$ over the powers of ζ — the longest passage at $2\alpha/\pi = 11/41$, of $397,153$ reflections, ends with largest coefficient $53,492$, a ratio to m of 0.135; over the end copies of every returning branch at that centre the ratio is at most 0.204.

A.2. **Comparison by cyclotomic value, never by coefficient vector.** The numbers being compared are cyclotomic, and they must be compared *by value*. The powers ζ^k are $\mathbb{Q}$-linearly dependent — their minimal relation is Φ_{4Q} — so two equal numbers can carry different coefficient vectors, and comparing raw coefficients reports distinct values that are in fact equal; this is not a rare degeneracy, since the $4Q$ powers satisfy Φ_{4Q} at every Q.

Each element is therefore reduced to its canonical image in $\mathbb{Z}[x]/(\Phi_{4Q}(x))$, an integer vector of dimension $\varphi(4Q)$, with no floating point anywhere, for which value-equality holds if and only if the reduced vectors agree. Standard properties of that ring, in particular the integral basis and the degree $\varphi(4Q)$, are as in Washington [Was97]. A boundary's launch height is $s = y/(2\sin\alpha)$ with y the height of a developed vertex, and Theorem A.3 places it in $\mathbb{Z}[\cos 2\alpha]$; the comparisons are made on y, and a fixed positive scale changes no sign and no equality.

A.3. **The completeness certificate.** The computation walks until every interval has closed, and a result is reported only when none is left open. **A partition with no open interval is the complete partition:** its leaves tile the launch leg by construction, each with its exact endpoints, and the count is the number of leaves. Nothing is filtered and nothing is deduplicated after the fact, because two leaves are separated only by a vertex graze the computation found exactly.

The certificate is essential rather than cosmetic, and the reason defeats the obvious alternative. Odd-P counts *plateau* across long stretches of the computation before climbing: $3/29$, for instance, sits at $n = 4$ for a long time before rising through $5, 8, 10, 12$ to 19. Any "stable over two depths" test would have accepted 4. Only completeness, open $= 0$, is reliable.

What the certificate does **not** supply is a bound on the work. Termination is guaranteed — Proposition 3.4 makes the partition finite, and every interval

that has not split ends at a head-on hit after the finite prefix length of its branch — but the number of steps is not bounded in advance, and a step cap is applied as a guard. At a centre where the cap is reached the computation reports the centre as open and records no count; it is resumed from its saved state at a higher cap, never restarted. Every script that reads the record also recomputes a slice of it from scratch and compares.

The vocabulary, fixed here once. A centre is **certified** when its partition closed with no open interval; *computed completely* is a synonym and we prefer *certified*. The partition record deposited with this paper is drawn from the ranges the paper's statements use — every coprime (P, Q) with $P \le 9$ and $Q \le 60$, the rows $P = 1, 2, Q-2, Q-1$ to $Q = 51$, and $P = 1, 2$ to $Q = 201$, 595 centres — together with three centres outside them, 11/41 (the passage of §A.1), 17/22 and 19/24: 598 centres in all. Of these 572 are certified, the three extra ones among them; the other 26 are still open, all in the ranges, with $P \in \{3, 4, 6, 8\}$ and $33 \le Q \le 59$; nothing in this paper quotes a count at any of them.

How a leaf is recognised as an orphan. Orphan-hood is read off the branch word, not off the geometry: by Lemma 2.1 a branch is the orphan exactly when its word is a palindrome about a single head-on letter, equivalently when $J = \mathrm{id}$ on it. This is a test on the exact word the computation emits, so the check of Theorem 4.7 in §A.4 means that no second palindromic branch appears in any certified partition. That is a check with no exceptions; the proof of uniqueness is Theorem 4.7. The same test confirms existence and uniqueness across the record: of the 572 certified centres, the 394 with P odd each have exactly one palindromic branch, 245 of them at a Q that is not a prime power, and the 178 with P even have none.

A.4. **Consistency checks.** Each item below checks a statement the body proves; none is its basis, so none needs the certificate of §A.3. The ranges differ because each check was run as far as its own cost allowed; no range marks where a statement stops holding.

- Proposition 4.5's count, by exact computation at every coprime (P, Q) with $5 \le Q < 60$: no mismatch.
- Theorem 5.3's count, by exact enumeration of §5.2's congruence $(*)$ at every coprime (P, Q) with $Q \le 60$, and the identification of the developed model with the billiard by tracing the three perpendicular beams at the 116 coprime centres with $4 \le Q \le 19$: no exception.
- Theorem 4.7, at each centre computed: no second palindromic branch appears in any certified partition.
- The lift of Proposition 7.1's base against Theorem 6.1's cone data, at every coprime (P, Q) with $Q \le 40$, in both parities: no mismatch.
- Proposition 7.2's count of $2g+2$ fixed points, at every coprime (P, Q) with $Q \le 200$, building the surface from the gluing of §6 and computing the fixed points of τ on it rather than comparing the two formulas.

- The correspondence with McMullen's Weierstrass criterion [McM25] on the slice $\min(P, Q-P) = 1$ (§7.5), over $Q = 4, \dots, 15$.

Code availability. The implementation — the exact partition computation, the cyclotomic arithmetic, the construction of the unfolded surface, one script per statement, the partition record, and the script generating the figures, which re-derives every quantity a caption states — is deposited at `https://github.com/josh-boorstein/billiards`, directories `lib/` and `orphan/`. The version used here is release v1.0, archived at Zenodo as doi:10.5281/zenodo.24037744.

Appendix B. The trajectories of Lemmas 4.22 and 4.23

These are the two prongs followed by hand at $Q = 4p$ (Theorem 4.24). §4.5 states the lemmas and the idea of each.

Proof of Lemma 4.22. The cell $Q-2P$ lies in $(0, Q/3)$. Its prong leaves R in the direction $\pi - (Q-2P)\pi/Q = 4\alpha$. Put $\eta := (Q/2-P)/Q \in (0, 1/6)$. Put R at the origin, so that $O = (-L, 0)$ and $A = (0, 1)$, and write $s_1 := \sin\alpha$, $c_1 := \cos\alpha$. Billiards in T is billiards in the rhombus $\mathcal{R} = OAO'A'$, $O' = -O$, $A' = -A$, folded by the reflections in the lines of L_1 and L_2. A crossing of the diagonal AA' (resp. OO') is a collision with L_1 (resp. L_2), head-on exactly when it is perpendicular. The sides of $\mathcal{R}$ are copies of H.

The sides OA and $O'A'$ are parallel, each at distance c_1 from R. Measure ν along $O \to A$ from the foot of the perpendicular from R. Then OA spans $\nu \in [-c_1^2/s_1, s_1]$ and $O'A'$ spans $[-s_1, c_1^2/s_1]$. Unfolding $\mathcal{R}$ alternately across these two sides gives a strip with walls ℓ_k at normal distance $(2k-1)c_1$, $k \geq 1$. Odd walls are copies of OA, even walls copies of $O'A'$, each with the span of its original, since a reflection in a wall preserves ν. The outward normal of OA has direction $\pi/2 + \alpha$, and $4\alpha = \pi/2 + \alpha + \delta$ with

$$\delta := 3\alpha - \pi/2 \in (0, \pi/4),$$

turned toward O. So the unfolded prong is $\nu = -v\tan\delta$, v the normal coordinate, and it meets wall k at $\nu_k = -(2k-1)c_1\tan\delta$.

Which wall it misses. Wall k is crossed in its interior iff $|\nu_k| < c_1^2/s_1$ (odd k) or $|\nu_k| < s_1$ (even k). Let $m := \min\{m \geq 1 : (4m-1)\tan\delta > \tan\alpha\}$. Wall 1 is crossed, since $\tan\delta < \cot\alpha$. Now suppose an even wall $2m'$ is crossed. Then $3\tan\delta < \tan\alpha$. The function $3\tan\delta - \tan\alpha = -3\cot 3\alpha - \tan\alpha$ is increasing on $(\pi/6, \pi/4)$, its derivative being $9\csc^2 3\alpha - \sec^2\alpha > 0$, and it vanishes where $\tan^4\alpha + 6\tan^2\alpha - 3 = 0$. So $\tan^2\alpha < 2\sqrt{3} - 3 < 3/5$. It follows that $(4m'+1)\tan\delta < \tan\alpha + 2\tan\delta < (5/3)\tan\alpha < \cot\alpha$, and wall $2m'+1$ is crossed too. Hence walls $1, \dots, 2m-1$ are crossed in their interiors, and the prong passes wall $2m$ beyond its end $\nu = -s_1$, the copy of A'.

The exit. In the last copy of $\mathcal{R}$ the prong enters through the interior of the copy of OA, so it cannot pass through O. It leaves through the copy of $A'O$, missing A' by the strictness below. Folded back into $\mathcal{R}$, after $2m-1$ reflections in sides parallel to OA, its direction is $\psi_1 = 2\alpha - 4\alpha = -2\alpha$. The

side $A'O$ has direction $\pi - \alpha$, so the reflected direction is $2(\pi - \alpha) + 2\alpha \equiv 0$: **exactly horizontal**, to the right. The collision with $A'O$ is not head-on, since -2α is not normal to $A'O$. From a point of $A'O$ other than its ends, a horizontal ray to the right crosses the triangle ROA' and meets AA' perpendicularly at a height in $(-1, 0)$. That is a head-on collision with L_1 at an interior point.

Nothing earlier. Before the exit, every segment of the folded orbit has direction 4α or -2α, up to the reflections $\theta \mapsto -\theta$, $\pi - \theta$ in the diagonals. That is $\pi(1 - 2\eta)$ or $-\pi/2 + \pi\eta$, and neither lies in $(\pi/2)\mathbb{Z}$. The walls are crossed at angle $\delta \neq 0$ to their normal. So there is no earlier head-on collision. The vertices of each copy of $\mathcal{R}$ are the ends of its walls, which are avoided strictly, and its centre, a copy of R at $\nu = 0$, while the prong has $\nu \neq 0$ there.

Strictness: $(4i - 1)\tan\delta \neq \tan\alpha$ *for every* $i \geq 1$. An equality at $i < m$ would be a vertex hit at wall $2i$. Now $e^{2i\alpha} = u$ and $e^{2i\delta} = -u^3 = w^{3P+Q}$. When $p \neq 3$ both are primitive $8p$-th roots of unity, since $3P + Q$ is odd and prime to p. As $8p$ is not a prime power, $1 \pm \zeta$ is a unit for every primitive $8p$-th root ζ. So $\tan\alpha / \tan\delta$ is a unit of $\mathbb{Z}[w]$, and no odd integer $4i - 1 \geq 3$ is a unit. At $p = 3$ the only case is $P = 5$, where $m = 1$ and there is nothing to check. □

Proof of Lemma 4.23. Now $\pi/8 < \alpha < \pi/6$. The prongs leave R in the directions $\pi - 2\alpha$ (cell P) and $3P\pi/Q = 6\alpha$ (cell $Q - 3P$). For this proof only, scale T so that $|OA| = 1$, so that $|OR| = \cos\alpha$ and $|RA| = \sin\alpha$. Write $s_k \coloneqq \sin k\alpha$ and $c_k \coloneqq \cos k\alpha$. Rotate the plane by $-(\pi/2 + 2\alpha)$ about R, and put R at the origin. Then L_1 leaves R in the direction -2α and L_2 in the direction $\pi/2 - 2\alpha$. The two prongs leave R in the directions $-\theta$ and $+\theta$, where

$$\theta \coloneqq 4\alpha - \pi/2 \in (0, \pi/6), \quad 0 < \theta < 2\alpha, \quad \theta < \pi/2 - 2\alpha.$$

Put $S \coloneqq s_1 + s_3 = 2s_2c_1$, $a \coloneqq s_1s_2$ and $c \coloneqq c_1c_2$. Then $c - a = c_3 > 0$ and $c + a = c_1$.

The development. $T_0 \coloneqq T$ has vertices $R_0 = (0, 0)$, $O_t \coloneqq (S/2, c) = c_1(s_2, c_2)$ and $A_0 \coloneqq ((s_3 - s_1)/2, -a) = s_1(c_2, -s_2)$. Reflect successively in H, L_2 and H about O_t. Since $|O_tR| = c_1$ and $|O_tA| = 1$, this gives a fan of four copies whose far vertices lie at directions $-\pi/2 - 2\alpha, \ldots, -\pi/2 + 2\alpha$ from O_t:

$$R_0, \quad A_0, \quad R_1 = (S/2, -a), \quad A_2 = (S/2 + s_1, -a), \quad R_3 = (S, 0).$$

Reflect in the L_1-side R_3A_2. This gives a copy with apex $O_b \coloneqq (3S/2, -c)$. Reflect successively in L_2, H, L_2, H, L_2 about O_b. This gives a fan of six copies whose far vertices lie at directions $\pi/2 + 3\alpha, \ldots, \pi/2 - 3\alpha$ from O_b:

$$A_2, \quad R_3, \quad A_5 = (3S/2 - s_1, a), \quad R_6 = (3S/2, a),$$
$$A_7 = (3S/2 + s_1, a), \quad R_8 = (2S, 0), \quad A_9 = (3S/2 + s_3, -a).$$

The reflection of the last copy in its L_1-side R_8A_9 is $T_0 + \mathbf{d}$ with $\mathbf{d} \coloneqq (2S, 0)$, so the chain of copies continues periodically. Every vertex of the chain has height 0, $\pm a$ or $\pm c$. None lies in the open strip $\mathcal{S} \coloneqq \{-a < y < 0\}$.

Through the strip. Every side of the chain that meets $\mathcal{S}$ therefore spans it. These spanning sides are:

- the interior rays of the fans, namely the rays from O_t to A_0, R_1, A_2, and from O_b to R_3, A_5, R_6, A_7, R_8, with their translates;
- the L_1-sides between fans: R_0A_0 slanting down to the right, then R_3A_2 slanting down to the left, and their translates.

They are pairwise disjoint in $\mathcal{S}$, so they are ordered left to right, in the same order at the top $y = 0$ and at the bottom $y = -a$. On $y = -a$ the L_1-sides cut out alternately two kinds of interval. A **wall** $[A_0, A_2] + n\mathbf{d}$ is made of two L_1-sides meeting at R_1. A **gate** $G_n \coloneqq (A_2, A_9) + n\mathbf{d}$ is a chord of the fan about $O_b + n\mathbf{d}$. The gate has length $2s_3$, and its centre is $(2n + 3/2)S$.

Let Γ be the segment from R_0 in the direction $-\theta$. It starts in T_0, because $-2\alpha < -\theta$. It is itself spanning, and meets $y = -a$ at $X = (x^*, -a)$, where $x^* \coloneqq a \cot\theta$. So it crosses exactly the spanning sides with top > 0 and bottom $< x^*$, once each, in order. If $X \in G_n$, Γ meets no vertex up to X. It then enters the triangle $\Delta_n \coloneqq O_bA_lA_r + n\mathbf{d}$, where $A_l \coloneqq A_2$ and $A_r \coloneqq A_9$. That triangle is convex, contains no vertex in its interior, and is cut by the fan rays from O_b into pieces of consecutive copies.

The exit. The fan about O_b has its rays at steps of α. So the copy T_f beyond the side O_bA_r has its R, call it R_f, on the ray from O_b in the direction $\pi/2 - 4\alpha = -\theta$. Its $L_2 = R_fO_b$ is parallel to Γ, and its $L_1 = R_fA_r$, of length s_1, is perpendicular to Γ. Measure the offset t of a line of direction $-\theta$ from the line O_bR_f, positively toward A_r. For $0 < t < s_1$, the line crosses the open side O_bA_r and then meets R_fA_r perpendicularly at an interior point. For X on $y = -a$,

$$t(X) = s_1 - (x_{A_r} - x_X)\sin\theta, \quad \sin\theta = -c_4 > 0,$$

so $t = s_1$ at A_r, and $t = s_1 + 2s_3c_4 = s_7$ at A_l; also $t(O_b) = 0$. On the open gate t takes values strictly between s_7 and s_1. On the side O_bA_l it takes values between 0 and s_7. So if $t(X) > 0$, the line of Γ leaves Δ_n through the open side O_bA_r. Consequently, if $X \in G_n$ and $t(X) > 0$, the prong of q_P meets the L_1 of T_f head-on at an interior point, and meets no vertex before that.

It remains to see that no side crossed earlier is a boundary side of D. A side crossed transversally is a boundary side only if it is perpendicular to Γ, that is, of direction $\equiv -4\alpha \pmod{\pi}$. The crossed sides have directions $\pi/2 + k\alpha \pmod{\pi}$ with $-3 \le k \le 2$ (fan rays, and the side O_bA_r), or $\pm 2\alpha$ (L_1-sides). The first kind is perpendicular only if $(k + 4)\alpha \equiv \pi/2 \pmod{\pi}$ with $1 \le k + 4 \le 6$, and the second only if 6α or 2α lies in $\pi\mathbb{Z}$. None of these happens for $\pi/8 < \alpha < \pi/6$. The final L_1 is perpendicular, hence vertical, hence $\bar{\Lambda}$.

The other prong. The glide $g(x,y) \coloneqq (x+S,-y)$ carries T_0 onto the second copy of the O_b-fan, the triangle $R_3O_bA_5$, keeping the labels of the vertices. It carries the prong from R_0 in the direction $+\theta$ onto the segment from R_3 in the direction $-\theta$. That segment starts in the copy $R_3O_bA_5$, because $\theta < \pi/2 - 2\alpha$. The unfolding of an orbit is determined by its first copy and its direction, so the previous two paragraphs apply to it, with x^* replaced by $S + x^*$. Perpendicularity is preserved by g. So the prong of q_{Q-3P} lands on $\bar{\Lambda}$ if $S + x^* \in G_n$ for some $n \geq 0$ and $t > 0$ there.

The cover. Put $\xi \coloneqq x^*/2S$ and $\varpi \coloneqq s_3/S$. From $s_1 + s_3 = 4s_1c_1^2$ and $s_3 = s_1(4c_1^2 - 1)$ we get $\varpi = 1 - 1/(4c_1^2) > 1/2$. The gate conditions read:

- q_P: $\xi \in (n + 3/4 - \varpi/2, n + 3/4 + \varpi/2)$;
- q_{Q-3P}: $\xi \in (n + 1/4 - \varpi/2, n + 1/4 + \varpi/2)$, $n \geq 0$.

From $\cot\theta = -s_4/c_4$,

$$\xi - 1/4 = -(s_1s_4 + c_1c_4)/(4c_1c_4) = -c_3/(4c_1c_4) > 0.$$

If $\alpha \leq \pi/7$, then $s_7 \geq 0$. Since t is affine along a gate, with value s_7 at A_l and s_1 at A_r, we get $t > 0$ on every open gate. The windows have centres $1/4, 3/4, 5/4, \ldots$ and length $\varpi > 1/2$, so their union is $(1/4 - \varpi/2, \infty)$, which contains ξ.

If $\pi/7 < \alpha < \pi/6$, take q_{Q-3P} with $n = 0$. Then $\xi > 1/4$. Next, $\xi < 1/4 + \varpi/2 \iff 2s_1c_1c_3 < -c_4s_3 \iff s_2c_3 + s_3c_4 < 0 \iff s_6c_1 < s_1$, using $4c_1^2 - 1 = s_3/s_1$ and the product formulas. The last inequality holds, because $6\alpha \in (6\pi/7, \pi)$ gives $\sin 6\alpha < \sin(\pi/7) < \tan(\pi/7) < \tan\alpha$. Finally, X lies to the right of the gate's centre, where $t = (s_1 + s_7)/2 = s_4c_3 > 0$, and t increases toward A_r. So $t(X) > 0$. □

Use of AI

This paper — its text, its proofs and the code behind Appendix A — was drafted by an AI system (Claude, Anthropic) working under the author's direction, and checked by independent AI review passes and by the exact computations described in Appendix A. The author has read parts of it in detail and takes responsibility for its content.

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