hexagon:2611.00190v2

POSITIVITY OF CANONICAL BASIS MULTIPLICATION IN MODIFIED QUANTUM GROUPS

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Abstract. We prove the multiplication part of Lusztig's positivity conjecture for symmetric Cartan data.

1. Introduction

Lusztig constructed the canonical basis of a half of a quantum group in finite type A, D, E by comparing the bases of Poincaré–Birkhoff–Witt type attached to reduced expressions of the longest Weyl group element [14]. His geometric construction used intersection cohomology on varieties of quiver representations. In [15] he used induction and restriction of equivariant sheaves to construct the basis for arbitrary symmetric Cartan data. In that construction, the coefficients of multiplication and twisted comultiplication record the multiplicities of shifted simple perverse sheaves. Consequently they are Laurent polynomials with nonnegative integer coefficients [15, 11.5]. Kashiwara constructed global crystal bases using the crystal at parameter zero and its lift to the quantum group [11]. Grojnowski and Lusztig compared the global and geometric bases in symmetric type [9].

The two halves act together on integrable weight modules. In finite type A, Beilinson, Lusztig and MacPherson obtained an idempotented algebra from the stabilization of convolution algebras of pairs of partial flags [1]. Lusztig constructed the canonical basis of the modified quantum group in general by studying tensor products of a simple integrable lowest-weight module and a simple integrable highest-weight module [16]. The quasi-R-matrix defines a bar on such a tensor product, and the canonical vectors are its fixed lifts of ordinary tensors of canonical vectors. When both dominant weight parameters increase with their difference fixed, the surviving canonical vectors give compatible labels for a basis of the corresponding right weight column of the modified algebra.

In [16, §11] Lusztig asked whether the multiplication and comultiplication coefficients for this basis have the same positivity as those of the half algebra. The question appears as Conjecture 25.4.2 in [17]. Write v for the indeterminate, $\dot{\mathbf{U}}$ for the modified quantum group over $\mathbb{Q}(v)$, and $\dot{\mathbf{B}}$ for

Date: November 7, 2026.

2020 *Mathematics Subject Classification.* Primary 17B37. Secondary 20G42.

Key words and phrases. Canonical basis, modified quantum group, quiver Hecke algebra, positivity.

its canonical basis. The modification uses mutually orthogonal idempotents 1_ν indexed by weights ν. The column $\dot{\mathbf{U}}1_\nu$ consists of elements with right weight ν. The conjecture asserts that multiplication in $\dot{\mathbf{B}}$ and every weight component of the modified comultiplication have coefficients in $\mathbb{N}[v, v^{-1}]$, the Laurent polynomials with nonnegative integer coefficients.

The flag realization supplies geometric models for multiplication in type A. In affine type A, Schiffmann and Vasserot established compatibility of the canonical basis with the geometric bases in affine Schur algebras [20]. Fu and Shoji used the affine Schur algebras and the positive half of a larger affine quantum group to prove multiplication positivity in the original modified algebra [8]. Fan and Li proved positivity for its modified comultiplication [7].

A categorical approach begins with the quiver Hecke algebras of Khovanov and Lauda and Rouquier [12, 19]. For geometric parameters in characteristic zero, Varagnolo and Vasserot identified the classes of normalized indecomposable projectives with the half-algebra canonical basis [21]. Induction makes the product coefficients graded direct-summand multiplicities. For the modified algebra, Li constructed a geometric realization using double-framed quivers and formulated the identification of its geometric labels with the canonical basis as a separate conjecture [13, Conjecture 6.11]. The diagrammatic category of the modified algebra also has an indecomposable basis. Webster identified it with Lusztig's basis for finite type A, D, E and obtained multiplication positivity in those types [22, Corollary B and Corollary 8.10]. For arbitrary symmetric Cartan matrices his tensor-product identification applies to tensor products of highest-weight modules [22, Theorem 8.7]. The finite-type identification of the modified basis uses the longest Weyl group element to pass from lowest–highest tensor products to that setting.

Fang and Lan used framed diagrams to recover tensor-product canonical bases of highest-weight modules from the canonical basis for a larger quantum group [6]. Fang and He adjoin one framing vertex at each original vertex and realize the canonical bases of lowest–highest Demazure tensor subspaces using the canonical basis of the negative half of the enlarged quantum group [5, Theorems 3.5 and 3.7]. They use this realization to prove the comultiplication part of the conjecture and the related positivity of the bilinear form for every symmetric Cartan datum. Their multiplication theorem applies when one factor belongs to a parabolic subalgebra with a finite-type positive part, or to its image under exchange of the two halves [5, Theorem 6.6].

Theorem 1.1. *Let $\dot{\mathbf{U}}$ be associated with a symmetric Cartan datum and any root datum of that type. For $b, b' \in \dot{\mathbf{B}}$, the coefficients of the finite product expansion satisfy*

$$bb' = \sum_{c \in \dot{\mathbf{B}}} m^c_{b,b'}(v)c, \qquad m^c_{b,b'}(v) \in \mathbb{N}[v, v^{-1}].$$

Corollary 1.2. *Lusztig's Conjecture 25.4.2 holds for every symmetric Cartan datum and every root datum of that type.*

Proof. Theorem 1.1 gives multiplication positivity. Fang and He's comultiplication theorem gives the other assertion for the same Cartan and root data [5, Theorem 6.3 and §6.6]. □

For a given simple root, the raising functor acts at the Demazure stages where that root is a left descent. These stages are cofinal in Bruhat order. We construct the functor on a quotient by an ideal of morphisms at each such stage and prove compatibility with transition functors. This gives the action of the full diagrammatic quantum group on the colimit. Specializing the symmetric dot polynomials on the framing strands identifies the Hom spaces in its positive-word corner with cyclotomic Hom spaces. This identification gives full evaluation, so an indecomposable in the diagrammatic category evaluates to a single shifted canonical label or to zero. The fixed-weight-difference comparison with Lusztig's tensor bases then identifies the diagrammatic basis in every right column.

2. Weights and quiver Hecke categories

Let I be a finite set and $A = (a_{ij})_{i,j\in I}$ a symmetric generalized Cartan matrix, so $a_{ii} = 2$ and a_{ij} is a nonpositive integer for $i \neq j$. A root datum has a weight lattice X, its dual lattice $Y = \mathrm{Hom}(X, \mathbb{Z})$, and simple roots $\alpha_i \in X$ and coroots $h_i \in Y$ with $\langle h_i, \alpha_j\rangle = a_{ij}$. Here $\langle\ ,\ \rangle$ is the evaluation pairing. A weight is dominant if all its simple-coroot pairings are nonnegative. We use the Bruhat order on the Weyl group W with simple reflections $s_i(\lambda) = \lambda - \langle h_i, \lambda\rangle\alpha_i$. In this order $w \leq v_0$ means that a reduced expression for v_0 contains a reduced expression for w as a subword.

Set $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. For an integer h put

$$[h] = \frac{v^h - v^{-h}}{v - v^{-1}}, \qquad [a]! = \prod_{r=1}^{a} [r] \quad (a \geq 0).$$

The empty product is 1. Let $\mathbf{f}$ be the negative-half algebra over $\mathbb{Q}(v)$, with generators θ_i and the quantum Serre relations

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \theta_i^{(r)} \theta_j \theta_i^{(1-a_{ij}-r)} = 0 \quad (i \neq j), \qquad \theta_i^{(r)} = \theta_i^r/[r]!.$$

Its integral form ${}_{\mathcal{A}}\mathbf{f}$ is generated by the divided powers. Write $\mathbf{B}$ for Lusztig's canonical basis of this form. The positive and negative halves $\mathbf{U}^+$ and $\mathbf{U}^-$ of the quantum group are copies of $\mathbf{f}$, with generators E_i and F_i, respectively. For $y \in \mathbf{f}$, y^- denotes its image in $\mathbf{U}^-$. The idempotent 1_ν on a weight module projects onto weight ν. The modified algebra is the direct sum of the spaces $1_\mu \dot{\mathbf{U}} 1_\nu$ with orthogonal weight idempotents 1_μ. Its integral form ${}_{\mathcal{A}}\dot{\mathbf{U}}$ is generated by the weight idempotents and divided powers $E_i^{(a)} = E_i^a/[a]!$

and $F_i^{(a)} = F_i^a/[a]!$ for $a \geq 0$. The generators E_i, F_i change weight by $\alpha_i, -\alpha_i$, respectively. We use the convention

$$(E_iF_j - F_jE_i)1_\nu = \delta_{ij}[\langle h_i, \nu\rangle]1_\nu.$$

We use the free abelian group on the root symbols α_i to record formal degrees, even if their images in X are dependent. The degrees of E_i, F_i in this group are $\alpha_i, -\alpha_i$. For a homogeneous element $y \in \mathbf{f}$, its content $|y|$ is the negative of the formal degree of y^-, so $|\theta_i| = \alpha_i$ and content is additive under products. We use the same convention for the enlarged set of root symbols below.

2.1. **Framing and a fixed right weight.** Adjoin $I' = \{i' : i \in I\}$. Each i' is joined by a single edge to i, and there are no edges among the primed vertices. For $n = |I|$ the enlarged matrix and the chosen weight lattice are

$$\widetilde{A} = \begin{pmatrix} A & -I_n \\ -I_n & 2I_n \end{pmatrix}, \qquad P = \mathbb{Z}^{I \sqcup I'}.$$

The simple coroots on P are the coordinate functions and the simple roots are the columns of $\widetilde{A}$. Using the lower-right block $2I_n$, we obtain $\det \widetilde{A} = 2^n \det(A - \frac{1}{2}I_n) = \det(2A - I_n) \equiv 1 \pmod 2$. Thus the simple roots are independent, and we identify the formal root group with their integral span in P. The symmetric rational form on P with matrix $\widetilde{A}^{-1}$ has $(\alpha_i, \nu) = \langle h_i, \nu\rangle$ for all vertices. We first prove the theorem for the datum with weight lattice P and simple roots and coroots indexed by I.

Write $\widetilde{\mathbf{f}}$ and $\widetilde{\mathbf{B}}$ for the enlarged half and its canonical basis. Fix a right weight $\zeta \in P$. For a vector $m = (m_i)_{i\in I}$ of nonnegative integers define

$$\theta_m = \prod_{i\in I} \theta_{i'}^{(m_i)}, \qquad \gamma_m = \sum_{i\in I} m_i\alpha_{i'}, \qquad \eta_m = (0, m) \in P.$$

The factors in θ_m commute. Choose $m_i \geq \max(0, \langle h_i, \zeta\rangle)$ for each $i \in I$. The weights $\lambda_2 = -\gamma_m$ and $\lambda_1 = -\zeta - \gamma_m$ are dominant for I, with simple-coroot pairings m_i and $m_i - \langle h_i, \zeta\rangle$, respectively, and satisfy $\lambda_2 - \lambda_1 = \zeta$. For $w \in W$ put $\mu_w = -w\lambda_1$. The subspace used for the thickening is

$$F_m = \mathbf{f}\theta_m\mathbf{f} \ \subseteq \ \widetilde{\mathbf{f}},$$

where the right side denotes the linear span of the indicated products.

2.2. **Geometric projectives.** All categories are over $k = \mathbb{C}$. Choose a total order on $I \sqcup I'$ and orient all edges from smaller to larger vertices, with $-a_{ij}$ edges between i, j for distinct vertices. Use the entries of $\widetilde{A}$ when a vertex is primed. Write $R(\beta)$ for the quiver Hecke algebra of nonnegative formal content β for these vertices. A word of content β is a sequence of colors whose root symbols sum to β. For such a word $\mathbf{i} = (i_1, \ldots, i_N)$, write $e(\mathbf{i})$ for the word idempotent, x_r for the dot at position $1 \leq r \leq N$, and τ_r for the crossing at $1 \leq r < N$ which exchanges positions $r, r + 1$. Their

respective degrees on that word are $0, 2, -(\alpha_{i_r}, \alpha_{i_{r+1}})$. We use the geometric KLR relations in [21, Remark 3.3(c)], with

$$\tau_r^2 e(\mathbf{i}) = Q_{i_r i_{r+1}}(x_r, x_{r+1}) e(\mathbf{i}),$$
$$Q_{ij}(u_1, u_2) = (-1)^{\#\{\text{arrows } i \to j\}} (u_1 - u_2)^{-a_{ij}} \quad (i \neq j), \qquad Q_{ii} = 0.$$

Thus $Q_{ij}(u_1, u_2) = Q_{ji}(u_2, u_1)$. We use the dot slides for distinct colors, the nilHecke relations for equal colors, and the braid relation whose correction term is zero when the first and last colors of the three strands differ.

For contents β_1, β_2, let $e(\beta_1, \beta_2)$ be the sum of word idempotents whose first block has content β_1 and second block has content β_2. Concatenation embeds $R(\beta_1) \otimes_k R(\beta_2)$ into this corner of $R(\beta_1 + \beta_2)$. For projectives P_1, P_2 at the two contents their induction $P_1 P_2$ is

$$R(\beta_1 + \beta_2) e(\beta_1, \beta_2) \otimes_{R(\beta_1) \otimes_k R(\beta_2)} (P_1 \otimes_k P_2).$$

The restriction $\mathrm{Res}_{\beta_1, \beta_2}(M)$ is $e(\beta_1, \beta_2) M$, viewed as a module over the two block algebras. Both functors preserve finitely generated projectives. For restriction this follows from the KLR PBW basis, grouping permutations into a permutation within each block and a shuffle of the blocks. The category $\mathscr{P}_{\widetilde{I}}$ is the direct sum over β of the categories of finitely generated graded projectives. The subcategory $\mathscr{P}_I$ uses only unprimed content. Write $|P_1|$ for the content of a homogeneous object P_1, F_{P_1} for induction by P_1 on the left, and F_i for $F_{R(\alpha_i)}$. We use F_{P_1} also for the functor defined by the corresponding negative diagrams in an idempotent-complete 2-representation, by splitting the idempotent for each projective summand. A morphism $f : P_1 \to P_2$ induces the natural transformation $F_f : F_{P_1} \to F_{P_2}$.

For $d, j \in \mathbb{Z}$, the shift satisfies $(M\langle d \rangle)_j = M_{j-d}$. Ordinary morphisms have degree zero, and $\mathrm{HOM}^d(M, N) = \mathrm{Hom}(M\langle d \rangle, N)$ consists of maps raising degree by d. Write HOM for the direct sum of these spaces over d. We use the split Grothendieck group K_0, generated by isomorphism classes with the direct-sum relations. Multiplication by v means shifting by $\langle 1 \rangle$.

The geometric categorification gives normalized projectives P_b, indexed by $b \in \widetilde{\mathbf{B}}$, and an algebra isomorphism

$$K_0(\mathscr{P}_{\widetilde{I}}) \longrightarrow {}_{\mathcal{A}}\widetilde{\mathbf{f}}, \qquad [P_b] \longmapsto b.$$

For each β, [21, Theorem 3.6] identifies the algebra with the Ext algebra of the sum S_β of flag pushforwards, in the derived category equivariant for changes of basis at the vertices. The flag complexes are shifted by the dimensions of the varieties of pairs consisting of a representation and a compatible flag. The shift in that paper is $M[r]_j = M_{j+r}$. Its action of v on the group is therefore our shift by $\langle -1 \rangle$. Compose their map from the projective Grothendieck group to the half algebra with the algebra bar, which is semilinear for $v \mapsto v^{-1}$ and fixes θ_i. This gives an $\mathcal{A}$-linear map for our shift and retains every label b, since each b is bar invariant. The normalization means that all indecomposable projectives are the $P_b\langle d \rangle$, with no duplicate labels or shifts.

The projective P_b corresponds to $\mathrm{Ext}^\bullet(S_\beta, \mathrm{IC}_b)$ for a simple equivariant perverse sheaf IC_b [21, §§4.6–4.7]. Each such perverse sheaf is a summand of a shift of S_β. Graded Yoneda on those shifts and their summands identifies

$$\mathrm{HOM}^d(P_b, P_c) \cong \mathrm{Ext}^d(\mathrm{IC}_b, \mathrm{IC}_c).$$

The right side is zero for $d < 0$. For $d = 0$ it is k if $b = c$ and zero otherwise. The geometric calculation includes parallel edges. In the Euler-class calculation in [21, Lemma 3.8(b)], a torus weight is counted once per occurrence among the arrow-space summands. The inversion-set identity used there matches multiplicities for each arrow separately, hence matches the full products of weights. For $\epsilon = 1, -1$, let r_ϵ send θ_i to $\theta_i \otimes 1 + 1 \otimes \theta_i$, with tensor multiplication twisted by $v^{\epsilon(|x_2|,|y_1|)}$ on $(x_1 \otimes x_2)(y_1 \otimes y_2)$. With the present convention, restriction induces r_{-1}. In the Mackey filtration for the restriction of a product the crossing of the middle blocks of contents $|x_2|$ and $|y_1|$ has degree $-(|x_2|, |y_1|)$. This gives the indicated twist, and restriction of one strand gives the formula on θ_i. For $a \in \widetilde{\mathbf{f}}$ and homogeneous a_1, a_2, the two coproducts are related by

$$r_{-1}(a) = \tau(r_1(a)), \qquad \tau(a_1 \otimes a_2) = v^{-(|a_1|,|a_2|)} a_2 \otimes a_1.$$

For homogeneous a, b, c, d, the exponent of v obtained by applying τ to the r_1-twisted product of $a \otimes b$ and $c \otimes d$ is

$$(|b|, |c|) - (|a| + |c|, |b| + |d|) = -(|a|, |b|) - (|c|, |d|) - (|a|, |d|).$$

This is the exponent of the r_{-1}-twisted product of $\tau(a \otimes b)$ and $\tau(c \otimes d)$. Since τ fixes the primitive coproduct of each θ_i, the two maps agree as asserted.

Lemma 2.1. *Suppose a graded k-linear additive category has objects which are finite sums of shifts of objects P_b with $\mathrm{HOM}^d(P_b, P_c) = 0$ for $d < 0$ and $\mathrm{HOM}^0(P_b, P_c) = \delta_{bc} k$. For a homogeneous, shift-stable ideal of morphisms, the surviving $P_b\langle d \rangle$ in the quotient are pairwise nonisomorphic indecomposables. The idempotent completion adds no other indecomposable labels.*

Proof. An object whose identity belongs to the ideal is zero in the quotient. Delete such summands from a given finite sum and order the remaining summands by shift. A degree-zero map from $P_b\langle d \rangle$ to $P_c\langle e \rangle$ has degree $d - e$ between the unshifted objects. Hence the endomorphism algebra of the sum in degree zero is block triangular, with a matrix block for each repeated label at a common shift. An element of the ideal with a nonzero diagonal block entry would put the identity of that summand in the ideal, by composing with its inclusion and projection and scaling. Thus the quotient endomorphism algebra modulo its strictly triangular ideal is the product of those matrix blocks. The strictly triangular ideal in the quotient is nilpotent, since each factor in a nonzero composite in it strictly changes the shift.

For an idempotent e in the quotient, conjugate its diagonal matrices to matrices projecting onto coordinate subspaces by a block-diagonal automorphism. The corresponding coordinate-summand projection p has $e - p$ in the

nilpotent strictly triangular ideal. Then $u = ep + (1-e)(1-p)$ is congruent to 1 modulo that ideal, so it is invertible, and $eu = up$. This identifies the image of e with the coordinate summand. For individual surviving shifts, the degree-zero endomorphism algebra is k. Between distinct shifts at least one Hom direction vanishes, and between different labels at a common shift both directions vanish. These facts give the claimed indecomposability and distinguish the labels. □

2.3. **Cyclotomic projectives.** For a dominant weight λ write Λ_λ for the simple integrable highest-weight module and ${}^\omega\Lambda_\lambda$ for its twist interchanging E_i, F_i and negating weights. The cyclotomic algebra $R^\lambda(\beta)$ is the quotient of $R(\beta)$ by the relations $x_N^{\langle h_i,\lambda\rangle} e(\mathbf{i}) = 0$ on words $\mathbf{i}$ of length $N > 0$ with last color i. Set $R^\lambda(0) = R(0) = k$. The last strand is adjacent to the cyclic object in the induction order. The cyclotomic theorem identifies the split Grothendieck group of its projectives with the integral form of Λ_λ, taking the content-zero projective to the highest vector [10, Theorem 6.2]. The lowest-weight action is obtained by composing the two diagram reflections in [18, Propositions 2.6.4–2.6.5]. The two reversals of composition cancel, the weights change sign, and the lowest-weight induction operators are the E_i. Its split Grothendieck group is the integral form of ${}^\omega\Lambda_\lambda$ with the class at content zero the lowest vector [4, Remark 4.22].

The normalizations follow from the half-algebra projectives. Base change along $R(\beta) \to R^\lambda(\beta)$ is surjective on graded morphisms between projectives, by the lifting property of a projective source. Every projective over the quotient is a summand of a sum of shifts of the image of $R(\beta)$. Thus its projective category is the idempotent completion of this Hom quotient, and Lemma 2.1 gives its indecomposables. On classes the quotient map is $b \mapsto b^- \eta_\lambda$, where η_λ is the highest vector. Indeed the quotient is generated by induction on the content-zero projective, with each word acting as the corresponding product of divided powers and generators under the categorification isomorphism. Lusztig's based projection theorem [17, Theorem 14.4.11] identifies the nonzero images with the distinct normalized basis labels of Λ_λ. Reading the induction words at the lowest vector and applying the twist gives the same conclusion there.

Write $\mathscr{L}(-\lambda_1)$ for the original lowest-weight projective category. Let L_w be its normalized projective of class ξ_{μ_w}, the extremal canonical vector of weight μ_w. Write $L^+_{\eta_m}$ for the highest-weight projective of the enlarged module $\widetilde{\Lambda}_{\eta_m}$. If e is the identity in W, then L_e is the projective over $R(0) = k$.

Along a length-increasing step $u \mapsto s_i u$ in a reduced expression, the extremal vector at μ_u is at the lower end of an i-string with pairing $-r$, where $r \geq 0$. The normalized nilHecke summands $E_i^{(r)}$ and $F_i^{(r)}$ carry L_u to $L_{s_i u}$ and back. The rank-one divided-power formula gives the equality of classes, and freeness of the split group on indecomposable shifts gives the

isomorphisms. At this weight the shift $r(r+\langle h_i,\mu_u\rangle)$ of the left adjoint $F_i^{(r)}$ of $E_i^{(r)}$ is zero [18, Lemma 2.6.14]. Therefore

$$\mathrm{HOM}(L_{s_iu},L_{s_iu})\cong\mathrm{HOM}(F_i^{(r)}L_{s_iu},L_u)\cong\mathrm{HOM}(L_u,L_u).$$

Induction gives $\mathrm{HOM}(L_w,L_w)=k$ in degree zero. If $s_iw<w$, the extremal vector at μ_w is at the upper end of its i-string. Thus $\langle h_i,\mu_w\rangle\geq 0$, and $E_iL_w=0$ since its class is zero.

2.4. **The thickening maps.** For $w\in W$ let ${}^{\omega}V_w(\lambda_1)=\mathbf{U}^-\xi_{\mu_w}$ be the opposite Demazure submodule of ${}^{\omega}\Lambda_{\lambda_1}$. Here and below E_i is used on this submodule only when $s_iw<w$, in which case the submodule is stable under E_i. Denote by η_{λ_2} the highest vector of Λ_{λ_2}. The tensor coproduct used here has

$$\Delta(F_i)=F_i\otimes K_{-h_i}+1\otimes F_i,\qquad \Delta(E_i)=E_i\otimes 1+K_{h_i}\otimes E_i,$$

where K_h acts on weight ν by $v^{\langle h,\nu\rangle}$. The tensor bar applies the two module bars and then Lusztig's quasi-R-matrix [17, §24.1]. The latter is the intertwiner from the factorwise bar of the coproduct action to the action with the algebra bar, normalized to have constant term $1\otimes 1$. The tensor canonical vector of an ordinary tensor label is fixed by the tensor bar and is congruent to that label modulo v^{-1} times the $\mathbb{Z}[v^{-1}]$-span of the ordinary tensor basis.

To apply Fang–He's thickening results with the lattice P, send a weight to its tuple of original simple-coroot pairings. This gives the weights of their simply connected original datum. The highest- and lowest-weight module constructions depend on these pairings: the first cyclic vector is assigned the weight, subsequent weights are obtained by root shifts, and the E_i,F_i actions use only the resulting simple-coroot pairings. The module bars, tensor bar and canonical vectors are transported by these identifications. In the enlarged Verma realization the original E_i,F_i actions likewise depend only on the original coroot pairings of the Verma highest weight. Thus we may retain the full weights in P throughout the construction.

The subspace F_m is spanned by $\widetilde{\mathbf{B}}\cap F_m$ by [6, Proposition 4.3]. By [5, Proposition 1.5 and Theorems 3.5, 3.7], the maps

$$a_w:\mathbf{f}\longrightarrow{}^{\omega}V_w(\lambda_1),\quad y\longmapsto y^-\xi_{\mu_w},\qquad \Phi_w:F_m\longrightarrow{}^{\omega}V_w(\lambda_1)\otimes\Lambda_{\lambda_2}$$

send canonical basis elements to canonical basis elements or zero. In both cases the maps are surjective and the killed labels span the kernel. Thus each target label has a unique preimage label: two distinct surviving labels with the same image would have a difference in the kernel. Let M_{μ_w} be the Verma module with highest vector u of weight μ_w, identified as a vector space with $\mathbf{f}$ by $y\mapsto y^-u$. Before the Demazure projection, the thickening map is an isomorphism $F_m\to M_{\mu_w}\otimes\Lambda_{\lambda_2}$ [5, Lemma 2.2]. Its inverse ψ satisfies the following formulas, where $\mathrm{wt}(x)$ is the weight of a homogeneous

vector $x \in \Lambda_{\lambda_2}$:

$$\psi(y^- u \otimes \eta_{\lambda_2}) = \theta_m y, \tag{1}$$

$$\psi(y^- u \otimes F_i x) = F_i \psi(y^- u \otimes x) - v^{-\langle h_i, \mathrm{wt}(x)\rangle} \psi(F_i y^- u \otimes x). \tag{2}$$

On the right F_i acts by left multiplication by θ_i. The second identity follows by applying the inverse map to the coproduct action of F_i on $y^- u \otimes x$, then moving the $F_i y^- u \otimes K_{-h_i} x$ term to the other side. The map Φ_w sends θ_m to $\xi_{\mu_w} \otimes \eta_{\lambda_2}$. Its value on $x\theta_m y$, for $x, y \in \mathbf{f}$, is obtained by applying x^- to $y^- \xi_{\mu_w} \otimes \eta_{\lambda_2}$.

Write u_{η_m} for the enlarged highest vector. The vector $\theta_m^- u_{\eta_m}$ is nonzero: the primed vertices are pairwise orthogonal, and the primed highest-weight module generated by u_{η_m} is the product of the simple rank-one strings of lengths m_i. Each factor in θ_m^- takes its string to the lower endpoint. The original E_i commute with every primed F_j, so $\theta_m^- u_{\eta_m}$ is highest for I. Its weight $\eta_m - \gamma_m$ has original pairings m_i. It generates the simple integrable highest-weight module for those pairings. Sending η_{λ_2} to $\theta_m^- u_{\eta_m}$ therefore defines an injection $\iota : \Lambda_{\lambda_2} \to \widetilde{\Lambda}_{\eta_m}$ intertwining the original E_i, F_i, and shifting weights by η_m.

Lemma 2.2. *The labels in $\widetilde{\mathbf{B}} \cap F_m$ surviving evaluation on u_{η_m} are exactly $b\theta_m$ with $b \in \mathbf{B}$ and $b^- \eta_{\lambda_2} \neq 0$.*

Proof. For $b \in \mathbf{B}$, multiply on the right successively by $\theta_{i'}^{(m_i)}$. At each step the input has zero content at i'. The right-string formula [17, §14.4] gives a canonical label of right string length m_i plus terms of greater right string length. The latter vanish because the whole product has i'-content m_i. Here right string length at i' is the largest r for which the label lies in the subspace $\widetilde{\mathbf{f}}\theta_{i'}^{(r)}$. Thus the final product is a canonical label.

Every original right factor of positive content annihilates u_{η_m}. Thus the image of F_m is exactly $\mathbf{U}^- \theta_m^- u_{\eta_m} = \iota(\Lambda_{\lambda_2})$, where the action uses original colors. The images of the specified $b\theta_m$ form a basis of that image by the based projection for Λ_{λ_2}. The enlarged highest-weight projection sends distinct surviving canonical labels to distinct canonical basis vectors. Hence the listed labels exhaust the survivors. □

In the framing content of F_m, a label cannot lie in $\widetilde{\mathbf{f}}\theta_{i'}^{(m_i+1)}$ since its i'-content is m_i. Thus the highest-weight projection kills exactly the labels lying in some $\widetilde{\mathbf{f}}\theta_i$ for $i \in I$, and Lemma 2.2 gives the comparison used in [5, Equation (3.4)].

For an original content α and enlarged content γ, write $(r_1(z))_{\gamma,\alpha}$ for the projection onto the tensor factors of these respective contents. Define the map on that component by

$$a \otimes b \longmapsto b^- \xi_{\mu_w} \otimes a^- u_{\eta_m}.$$

Summing its values on $(r_1(z))_{\gamma,\alpha}$ gives $(1 \otimes \iota)\Phi_w(z)$ by [5, Lemma 2.1]. The powers in this formula follow from the map $b \otimes a \mapsto b^- \otimes K_{-|b|} a^-$, where

$K_{-\alpha} = K_{-\sum_i n_i h_i}$ for $\alpha = \sum_i n_i \alpha_i$. It sends the primitive coproduct of each generator to $\Delta(F_i)$. For homogeneous a, b, c, d, the identity

$$(K_{-|b|}a^-)(K_{-|c|}d^-) = v^{-(|a|,|c|)}K_{-(|b|+|c|)}(ad)^-$$

shows that it intertwines the r_{-1}-twisted product with the ordinary tensor product. The flip τ from r_1 to r_{-1} supplies $v^{-(\alpha,\gamma)}$. This cancels $v^{(\alpha,\gamma)}$ from $K_{-\alpha}$ on $a^- u_{\eta_m}$, because the original coroots pair to zero with η_m. On $x\theta_m y$ the surviving terms put y in the first evaluated factor and θ_m in the second, followed by the tensor action of x^-.

3. The Demazure quotients

Define $\mathrm{ev}_w(P) = F_P L_w$ for $P \in \mathscr{P}_I$, with the action $F_f L_w$ on a map f. Its graded kernel I_w is stable under shifts. Let $\widetilde{I}_w$ be the shift-stable homogeneous ideal of $\mathscr{P}_{\widetilde{I}}$ generated by $\mathrm{id}_Z f$ for $f \in I_w$ and Z a homogeneous object of $\mathscr{P}_{\widetilde{I}}$, with juxtaposition denoting induction also on maps. Set

$$\mathcal{A}_w = \mathrm{Kar}(\mathscr{P}_{\widetilde{I}}/\widetilde{I}_w).$$

Here Kar denotes idempotent completion. Give an object of content β weight $\mu_w - \beta$. Let $\mathcal{C}_w$ be the full additive subcategory closed under shifts and summands generated by the images of P_b for $b \in \widetilde{\mathbf{B}} \cap F_m$.

Proposition 3.1. *The normalized indecomposable objects of $\mathcal{C}_w$ are labeled by the canonical basis of ${}^{\omega}V_w(\lambda_1) \otimes \Lambda_{\lambda_2}$ via Φ_w.*

Proof. The kernel $\ker a_w$ is a left ideal of $\mathbf{f}$ spanned by its canonical labels. Under $M_{\mu_w} \cong \mathbf{f}$, the kernel of the Demazure tensor projection is $(\ker a_w) \otimes \Lambda_{\lambda_2}$. For $y \in \ker a_w$, the base case (1) gives $\theta_m y \in \widetilde{\mathbf{f}} \ker a_w$. In (2), the first term stays in this left ideal and the second does as well because $\theta_i y \in \ker a_w$. Induction on the length of negative words applied to η_{λ_2} gives $\ker \Phi_w \subseteq \widetilde{\mathbf{f}} \ker a_w$. A canonical label b in $\ker \Phi_w$ must then occur in some product zc with $z \in \widetilde{\mathbf{B}}$ and $c \in \mathbf{B} \cap \ker a_w$. Thus P_b up to shift is a summand of $P_z P_c$. Since $F_{P_c} L_w = 0$, the ideal kills $P_z P_c$ and hence P_b.

For the converse, for each pair of output contents (α, γ) with α unprimed, consider the additive functor

$$D_{w;\alpha,\gamma} = (\mathrm{ev}_w \boxtimes \mathrm{ev}_{\eta_m}) \mathrm{Res}_{\alpha,\gamma},$$

where $\mathrm{ev}_{\eta_m}(Q) = F_Q L^+_{\eta_m}$. The domain of $D_{w;\alpha,\gamma}$ consists of projectives of content $\alpha + \gamma$. Its target is the idempotent completion of sums of external products in the two cyclotomic projective categories, at the two weights $\mu_w - \alpha$ and $\eta_m - \gamma$. The functor $\boxtimes$ applies the two evaluations to the factors and takes their tensor product on graded Hom spaces. It extends to summands by splitting idempotents. The external products of the normalized cyclotomic projectives have vanishing negative Hom and scalar degree-zero endomorphisms, because the Hom spaces in the two factors have those properties. Distinct products are orthogonal in degree zero. Lemma 2.1, with

zero ideal, shows that these labels and their shifts account for all summands. The split group therefore has the basis of ordinary tensor labels. Write D_w for the direct sum over output contents, finite at each input content.

Take $f : P_1 \to P_2$ in I_w of unprimed content ρ. The Mackey filtration of $\mathrm{Res}_{\alpha,\gamma}(ZP_j)$ has steps indexed by minimal double-coset representatives. Each associated graded piece is a natural direct sum over compatible matrices of contents with row sums $|Z|, \rho$ and column sums α, γ [18, Proposition 2.4.4]. If any of ρ enters the second output, that output evaluates to zero because its rightmost unprimed factor annihilates $L^+_{\eta_m}$. Thus the only possible surviving matrix is

$$\begin{pmatrix} \alpha - \rho & \gamma \\ \rho & 0 \end{pmatrix}.$$

This matrix occurs only if $\alpha - \rho$ is nonnegative. Its block heights determine its double-coset representative. Its summand has P_j in the first factor and the content-zero unit in the second. The induced map on the first factor contains $F_f L_w = 0$. The subquotients are projective by PBW restriction and projective induction. Taking a linear extension of Bruhat order gives short exact sequences that split in graded modules. Evaluation preserves these sequences. At the unique possibly nonzero step, the preceding object evaluates to zero, and every following inclusion evaluates to an isomorphism. Naturality identifies the evaluated map on the whole filtration with the map on the surviving summand. It is zero, so D_w annihilates $\widetilde{I}_w$.

The class of $D_{w;\alpha,\gamma}(P_b)$ is obtained from $(r_{-1}(b))_{\alpha,\gamma}$ by applying $a \otimes b' \mapsto a^- \xi_{\mu_w} \otimes (b')^- u_{\eta_m}$. The coproduct identity $r_{-1} = \tau r_1$ gives, with the weight components relative to these two extremal vectors,

$$[D_{w;\alpha,\gamma}(P_b)] = v^{-(\alpha,\gamma)} \big((1 \otimes \iota) \Phi_w(b) \big)_{\alpha,\gamma}.$$

The two factor weights distinguish the formal contents because the enlarged simple roots are independent in P. Thus $\Phi_w(b) \neq 0$ implies $D_w(P_b) \neq 0$, so the identity of P_b survives. Lemma 2.1 now identifies the indecomposable objects. □

3.1. **Transition functors.** The restriction of $\widetilde{I}_w$ to $\mathscr{P}_I$ is exactly I_w. Indeed content is nonnegative, so an ideal generator with primed content cannot contribute to the unprimed subcategory. The kernel I_w is already stable under unprimed left induction. Denote the pure unprimed subcategory of $\mathcal{A}_w$ by $\mathcal{A}^I_w$. Its evaluation into $\mathscr{L}(-\lambda_1)$ is faithful.

For $w \leq v_0$ in W, the inclusion of opposite Demazure modules sends ξ_{μ_w} to a vector in ${}^\omega V_{v_0}(\lambda_1)$. By the based projection a_{v_0}, there is a unique $d_{v_0 w} \in \mathbf{B}$ whose action sends $\xi_{\mu_{v_0}}$ to ξ_{μ_w}. Its root content is $\mu_{v_0} - \mu_w$. Equality of classes in the cyclotomic projective category gives a degree-zero isomorphism

$$\tau_{v_0 w} : F_{P_{d_{v_0 w}}} L_{v_0} \xrightarrow{\sim} L_w.$$

Choose one such isomorphism, taking the identity for $v_0 = w$. If $f \in I_w$, then $f \,\mathrm{id}_{P_{d_{v_0 w}}} \in I_{v_0}$ by naturality of F_f with respect to $\tau_{v_0 w}$. Thus a generator

$\mathrm{id}_Z f$ of $\widetilde{I}_w$ is sent to $\mathrm{id}_Z(f\,\mathrm{id}_{P_{d_{v_0w}}}) \in \widetilde{I}_{v_0}$. Right induction defines $T_{v_0w} : \mathcal{A}_w \to \mathcal{A}_{v_0}$. It commutes with left induction and preserves weights because $|P_{d_{v_0w}}| = \mu_{v_0} - \mu_w$. On F_m one has

$$\Phi_{v_0}(zd_{v_0w}) = \mathrm{inc}(\Phi_w(z)),$$

where inc is the tensor of Demazure inclusion with the identity. It suffices to check this on $z = x\theta_m y$, where it follows from the action of d_{v_0w} on the extremal vector. The products zd_{v_0w} remain in F_m, so their projectives decompose into shifts of labels in F_m. Proposition 3.1 identifies the class of such a projective after taking the quotient with the displayed tensor image. The Demazure inclusions are based [5, Proposition 1.6]. For a surviving label this image is a single normalized label, so equality in the split group gives the corresponding degree-zero isomorphism in $\mathcal{C}_{v_0}$.

For $w \le v_0 \le u$, consider the two pure objects $P_{d_{v_0w}}P_{d_{uv_0}}$ and $P_{d_{uw}}$ in $\mathcal{A}_u^I$. The image of each at L_u is isomorphic to L_w by the chosen τ's. Pure evaluation at L_u sends each surviving normalized pure projective to a distinct normalized Demazure basis label by the based map a_u. It follows that the two pure objects are isomorphic in $\mathcal{A}_u^I$, since in each case the decomposition into surviving shifts has one term, of shift zero and with the same label. There is a unique isomorphism

$$c_{u,v_0,w} : P_{d_{v_0w}}P_{d_{uv_0}} \longrightarrow P_{d_{uw}}$$

such that

$$\tau_{uw} \circ \mathrm{ev}_u(c_{u,v_0,w}) = \tau_{v_0w} \circ F_{P_{d_{v_0w}}}(\tau_{uv_0}).$$

Indeed the degree-zero isomorphisms of the evaluated objects form a $k^\times$-torsor because the degree-zero endomorphism algebra of L_w is k. Rescale any chosen isomorphism to get this equality. Faithfulness of pure evaluation gives uniqueness. Left induction by an object of $\mathcal{A}_w$ applied to $c_{u,v_0,w}$ gives a natural comparison between $T_{uv_0}T_{v_0w}$ and T_{uw}, with units fixed by our choice of τ_{ww}.

For $w \le v_0 \le u \le t$, the two comparison maps from $P_{d_{v_0w}}P_{d_{uv_0}}P_{d_{tu}}$ to $P_{d_{tw}}$ are

$$c_{t,u,w} \circ (c_{u,v_0,w}\,\mathrm{id}_{P_{d_{tu}}}),$$
$$c_{t,v_0,w} \circ (\mathrm{id}_{P_{d_{v_0w}}}\, c_{t,u,v_0}).$$

After evaluating at L_t and composing with τ_{tw}, both maps are

$$\tau_{v_0w} \circ F_{P_{d_{v_0w}}}(\tau_{uv_0}) \circ F_{P_{d_{v_0w}}}F_{P_{d_{uv_0}}}(\tau_{tu}).$$

For the first comparison, use naturality of the map on the first two factors with respect to τ_{tu}, then use the defining equality for $c_{u,v_0,w}$. For the second, use the defining equality for c_{t,u,v_0} inside $F_{P_{d_{v_0w}}}$. The two comparisons agree by faithfulness. Inducing them on the left gives the associativity equation for the transition functors.

4. Single-root actions on the quotients

Fix a raising color $i \in I$. At an object X of weight λ put $h = \langle h_i, \lambda \rangle$. The degree-zero right adjoint of F_i on weight λ is $E_i\langle 1+h\rangle$. With the shifts suppressed, its unit and counit are

$$\eta_X : X \longrightarrow E_iF_iX, \qquad \varepsilon_X : F_iE_iX \longrightarrow X$$

of degrees $1-h$ and $1+h$. In particular the shift of the right adjoint on F_iX is $\langle h-1\rangle$. Throughout this section diagrams are composed from right to left.

Write $F = F_i$, $E = E_i$ and $T = F_j$, with j any lowering color, including a primed color in the enlarged algebra. Let $\tau : FT \to TF$ be the negative crossing. Its mixed mate is

$$\sigma_X = (ET\varepsilon_X) \circ (E\tau_{EX}) \circ \eta_{TEX} : TEX \longrightarrow ETX. \tag{3}$$

At TEX the coroot pairing is $h + 2 - a_{ij}$. The three degrees in (3) are $-1-h+a_{ij}, -a_{ij}, 1+h$, so σ_X has degree zero. For $i \neq j$ it is required to be invertible. For $n \geq 0$ write

$$[n]X = \bigoplus_{r=0}^{n-1} X\langle n-1-2r\rangle, \qquad [0]X = 0.$$

For $i = j$ the required isomorphisms are

$$\begin{cases} M_X : FEX \oplus [h]X \longrightarrow EFX, & h \geq 0, \\ N_X : FEX \longrightarrow EFX \oplus [-h]X, & h < 0. \end{cases} \tag{4}$$

Their $FEX \to EFX$ component is σ_X. Let $x : F \to F$ denote the dot of degree two. The unit column of M_X for $0 \leq r < h$ is $E(x_X^{h-1-r}) \circ \eta_X$, of degree $h-1-2r$ before shifting its source. The counit row of N_X for $0 \leq r < -h$ is $\varepsilon_X \circ x_{EX}^r$, of degree $1+h+2r$ before shifting its target. These are the maps of the dual presentation in [18, Theorem 2.6.7(2)].

Lemma 4.1. *Let two graded additive categories have negative quiver Hecke actions with common weights and parameters, and the shifted adjunction and isomorphisms (3)–(4) for a fixed color i. Denote the target functors by F_j', E', and its unit and counit by η', ε'. Let H be a graded k-linear additive functor preserving weights and shifts. Suppose it has degree-zero natural isomorphisms $a_j : HF_j \to F_j'H$ compatible with shifts and intertwining negative dots and crossings. Suppose the source is generated by an object V under negative words, finite sums, shifts and summands, and $EV = E'HV = 0$. Then the right mate $HE \to E'H$ of a_i^{-1} is invertible and makes H preserve the adjunction and the specified isomorphisms.*

Proof. Write $a = a_i$ and $F' = F_i'$. The mate at X is

$$\rho_X = E'H(\varepsilon_X) \circ E'(a_{EX}^{-1}) \circ \eta'_{HEX}.$$

Its two adjunction maps have degrees $1+h$ and $-1-h$, since EX has weight $\lambda+\alpha_i$. By the triangle identities and naturality,

$$\begin{aligned} \varepsilon'_{HX}\circ F'(\rho_X)\circ a_{EX} &= H(\varepsilon_X),\\ E'(a_X)\circ\rho_{FX}\circ H(\eta_X) &= \eta'_{HX}. \end{aligned} \tag{5}$$

The first identity is the defining transpose of ρ_X. For the second, substitute the formula for ρ_{FX}. Naturality of η' moves $H(\eta_X)$ past that unit, and naturality of a gives

$$H(\varepsilon_{FX})\circ a^{-1}_{EFX}\circ F'H(\eta_X) = H(\varepsilon_{FX}\circ F(\eta_X))\circ a_X^{-1} = a_X^{-1}.$$

Applying E' and then composing with $E'(a_X)$ gives the second identity.

Put $T' = F'_j$ and $b = a_j$. Write σ' for the target mixed mate. Then

$$E'(b_X)\circ\rho_{TX}\circ H(\sigma_X) = \sigma'_{HX}\circ T'(\rho_X)\circ b_{EX}. \tag{6}$$

To see this, transpose the left-hand side by $F' \dashv E'$ at the indicated weights. Naturality of the counit, the first equation of (5), and naturality of a give the transpose

$$b_X\circ H(\varepsilon_{TX}\circ F(\sigma_X))\circ a^{-1}_{TEX}.$$

Since $\varepsilon_{TX}\circ F(\sigma_X) = T(\varepsilon_X)\circ\tau_{EX}$, naturality of b rewrites this as

$$T'H(\varepsilon_X)\circ b_{FEX}\circ H(\tau_{EX})\circ a^{-1}_{TEX}.$$

Negative crossing compatibility says

$$T'(a_{EX})\circ b_{FEX}\circ H(\tau_{EX}) = \tau'_{HEX}\circ F'(b_{EX})\circ a_{TEX}.$$

Substituting it and then (5) gives

$$T'(\varepsilon'_{HX})\circ T'F'(\rho_X)\circ\tau'_{HEX}\circ F'(b_{EX}) = T'(\varepsilon'_{HX})\circ\tau'_{E'HX}\circ F'T'(\rho_X)\circ F'(b_{EX}).$$

The last equality is naturality of τ'. By the definition of the target mate, the last expression is the transpose of the right-hand side of (6).

For $j\neq i$, invertibility of both mixed maps in (6) gives invertibility of ρ_{F_jX} whenever ρ_X is invertible. For $j=i$, put

$$\chi_X = E'(a_X)\circ\rho_{FX},\qquad \psi_X = F'(\rho_X)\circ a_{EX}.$$

Equation (6) compares the $FEX\to EFX$ components. For a unit column, naturality of ρ at x^r_X and $a_X\circ H(x_X) = x'_{HX}\circ a_X$ give

$$\begin{aligned} \chi_X\circ H(E(x^r_X)\circ\eta_X) &= E'((x'_{HX})^r)\circ E'(a_X)\circ\rho_{FX}\circ H(\eta_X)\\ &= E'((x'_{HX})^r)\circ\eta'_{HX}. \end{aligned}$$

For a counit row, the same dot relation at EX, naturality of x' at ρ_X, and the first identity of (5) give

$$\varepsilon'_{HX}\circ(x'_{E'HX})^r\circ\psi_X = \varepsilon'_{HX}\circ\psi_X\circ H(x^r_{EX}) = H(\varepsilon_X\circ x^r_{EX}).$$

Consequently

$$\begin{aligned} \chi_X\circ H(M_X) &= M'_{HX}\circ(\psi_X\oplus 1) && (h\geq 0),\\ (\chi_X\oplus 1)\circ H(N_X) &= N'_{HX}\circ\psi_X && (h<0). \end{aligned}$$

These identities imply invertibility of χ_X from invertibility of ρ_X. In the second case the first diagonal component of the inverse of $\chi_X \oplus 1$ is an inverse of χ_X. Thus ρ_{FX} is invertible as well.

At V, ρ_V is an isomorphism between zero objects. Induction gives invertibility on negative words, and naturality gives it on shifts, sums and summands. The comparison preserves the unit, counit and required isomorphisms by the identities above.

If H, K have negative comparisons a^H, a^K and a degree-zero natural transformation $\xi : H \to K$ intertwines them, then $(E'\xi_X) \circ \rho^H_X = \rho^K_X \circ \xi_{EX}$. The first transpose is $\xi_X \circ H(\varepsilon_X) \circ (a^H_{EX})^{-1}$. Naturality of ξ at ε_X and $a^K_{EX} \circ \xi_{FEX} = F'(\xi_{EX}) \circ a^H_{EX}$ rewrite it as $K(\varepsilon_X) \circ (a^K_{EX})^{-1} \circ F'(\xi_{EX})$, the other transpose. For composable functors, the consecutive mate comparisons preserve the counit of the composite by applying (5) twice. They therefore agree with the mate of the composite negative comparison. □

Fix $w \in W$ and $i \in I$ with $s_i w < w$. Put $n = \langle h_i, \mu_w \rangle$. Then $n \geq 0$, $E_i L_w = 0$, and $\mathrm{HOM}(L_w, L_w)$ is k in degree zero. By (4), $E_i F_i L_w \simeq [n] L_w$. The shifted adjunction identifies

$$\mathrm{HOM}(F_i L_w, F_i L_w) \cong \mathrm{HOM}(L_w, [n] L_w \langle n-1 \rangle).$$

For $n = 0$ the right-hand side vanishes, so $F_i L_w = 0$. Otherwise its degrees are $2n - 2 - 2r$ for $0 \leq r < n$, because $\mathrm{HOM}(L_w, L_w)$ is concentrated in degree zero. The dot x_i on $F_i L_w$ has degree two, so its n-th power is zero.

For either the pure or the thickened quiver Hecke algebra, write A_β for the quotient imposing this relation at the rightmost strand:

$$A_\beta = R(\beta) \big/ \big(R(\beta) x_N^n e(\beta - \alpha_i, i) R(\beta) \big) \qquad (N = \mathrm{ht}\, \beta \geq 1). \tag{7}$$

Here $\mathrm{ht}\, \beta$ is the number of simple-root summands with multiplicity, x_N is the dot in the last position, and $e(\beta - \alpha_i, i)$ sums the idempotents for words of content β ending in i. Set that idempotent to zero if the set is empty, and set $A_0 = k$. On projectives, F_j is induction along the left-end inclusion selecting words beginning in j. On a projective X of content β and weight $\mu_w - \beta$,

$$E_i X = e(i, \beta - \alpha_i) X \langle -1 - \langle h_i, \mu_w - \beta \rangle \rangle. \tag{8}$$

The idempotent selects words beginning in i, with the other strands carrying the $A_{\beta - \alpha_i}$-action. The functor is zero if there is no such strand.

For this choice of parameters, [18, Definition 2.8.2 and Theorems 2.8.16–2.8.18] give the indicated functors on finitely generated graded projectives, the negative diagrams by right multiplication, the induction–restriction adjunction with the shift in (8), and the invertibility in (3)–(4). The cyclotomic exponent is prescribed only at i. Thus all the mixed maps with $j \neq i$, including primed j in the thickened algebra, are covered. The projectivity of the restriction bimodule follows from the intertwiner sequence and its monic central annihilator, in the left-end induction convention of [18, Proposition 2.8.14 and Theorems 2.8.15–2.8.16]. The displayed maps of

Theorem 2.8.18 are the counit $FE \to 1$ and the unit $1 \to EF$ for the shifted adjunction.

Lemma 4.2. *The adjunction and (3)–(4) descend to $\mathcal{A}_w$, and E_i preserves $\mathcal{C}_w$. Evaluation on the pure corner and transitions between stages satisfying $s_i w < w$ preserve the descended data.*

Proof. On pure words, evaluation at L_w annihilates the two-sided ideal in (7). The generating relation is the n-th dot power on $F_i L_w$, with any preceding negative word applied on the left. Thus evaluation factors on word projectives through the quotient by the relation. By taking finite sums, shifts and splitting idempotents in the target, it extends to all A_β-projectives. The negative comparisons persist on these summands. The source vacuum is the projective k of content zero and is killed by E_i. Its image is L_w, also killed by E_i. Lemma 4.1 gives the E_i-comparison.

For a pure map $f \in I_w$, apply E_i to its image in the pure A_β-projective category. The resulting objects lift to pure negative projectives. Indeed, extension of scalars from $R(\beta)$ to A_β is surjective on Hom between original projectives: a map to $A_\beta \otimes_{R(\beta)} Q$ lifts through the surjection from Q by projectivity of its source, also in each degree. Its essential image contains the word projectives and all their shifts and finite sums. The idempotent completion of the image is all projectives over the quotient, and Lemma 2.1 gives a lift for each of its objects. Consequently $E_i f$ lifts to a pure map between these lifts. That map evaluates to zero by the E_i-comparison, so it lies in I_w.

In the thickened A_β-projective categories, let J be the image ideal of $\widetilde{I}_w$. The relation defining A_β is already in $\widetilde{I}_w$, because it comes from x_i^n in the pure corner by left induction and composition. The preceding lift argument shows that E_i sends the pure generators of J into J. The maps $f \in J$ with $E_i f \in J$ form a graded ideal. To see that it is closed under F_j, for $j \neq i$ conjugate $E_i F_j f$ to $F_j E_i f$ by the mixed isomorphism. For $j = i$, if the current weight has coroot pairing $h \geq 0$, conjugate $E_i F_i f$ by M to $F_i E_i f \oplus [h] f$. If $h < 0$, N presents $E_i F_i f$ as a component of $F_i E_i f$ after inclusion and projection. All these maps belong to J, since it is closed under left induction and is an ideal. Every left induction by a projective is a summand of a finite sum of shifted word inductions. Closure under F_j, shifts and compositions therefore gives the same closure under projectives. The graded ideal contains every generator of J, so equals J. Thus E_i and its specified natural transformations descend, and extend to the idempotent completion.

Write $[k]$ for the class of the content-zero projective at weight μ_w. For a word $j_1, \ldots, j_N$ on thickened colors, (3) gives commutation on classes with the letters $j_r \neq i$. The class identities from (4) give $E_i F_i - F_i E_i = [\langle h_i, \lambda \rangle]$

at weight λ. Commuting E_i through the word until it kills $[k]$ yields

$$E_i(F_{j_1}\cdots F_{j_N}[k]) = \sum_{\substack{1\le r\le N\\ j_r=i}} [\langle h_i, \mu_w - \textstyle\sum_{s>r}\alpha_{j_s}\rangle]\, F_{j_1}\cdots \widehat{F_{j_r}}\cdots F_{j_N}[k].$$

A hat omits that factor. This is also the deletion formula on the negative half of the enlarged Verma module at μ_w. On $x\theta_m y$ each omitted letter lies in x or y, so this formula preserves F_m. On a homogeneous $y \in \mathbf{f}$, the formula at $\theta_m y$ only deletes letters in y, with the same coefficients as on $y^-\xi_{\mu_w}$. The map Φ_w therefore intertwines E_i there, since $\Phi_w(\theta_m y) = y^-\xi_{\mu_w}\otimes\eta_{\lambda_2}$ and $E_i\eta_{\lambda_2} = 0$. The source and target weights there are both $\mu_w - \gamma_m - |y|$. The equality propagates through left multiplication by original generators using the commutator and the fact that Φ_w intertwines F_j. This gives the E_i-action on the full Demazure tensor subspace. The surviving normalized projectives form a basis of the split Grothendieck group. Because E_i preserves the span of the labels from F_m, its value on such a projective has no summand labeled outside F_m. This proves stability of $\mathcal{C}_w$. For F_j with $j \in I$, stability follows from left multiplication on F_m.

Suppose $w \le v_0$, $s_i w < w$ and $s_i v_0 < v_0$. The transition $T_{v_0 w}$ sends the vacuum to the pure object represented by $P_{d_{v_0 w}}$, with evaluation isomorphic to L_w. The pure evaluation at v_0 is faithful and preserves E_i by the first part of this proof. Since $E_i L_w = 0$, the identity of $E_i P_{d_{v_0 w}}$ evaluates to zero, so $E_i P_{d_{v_0 w}} = 0$ in the quotient. The hypothesis on the image vacuum in Lemma 4.1 now holds, and the transition preserves E_i and the specified maps. The coherence proved in that lemma applies to the natural comparisons for composable transitions and to the comparisons with pure evaluation. All comparisons are degree zero, since every transition preserves the full weight $\mu_w - \beta$ and hence the adjunction shifts determined by that weight. □

5. The action on the Bruhat colimit

The Bruhat order on W is directed. The Demazure product $w * v_0$, obtained by multiplying w on the right by successive simple reflections in a reduced word for v_0 whenever the length increases, dominates w and v_0 by the subword description of Bruhat order. For each $i \in I$, the subset $W_i^- = \{w \in W : s_i w < w\}$ is cofinal and directed. An upper bound u can be kept if $s_i u < u$ and replaced by $s_i u$ if $s_i u > u$.

Form $\mathcal{A}_\infty$ with objects (w, P), $P \in \mathcal{A}_w$, and Hom spaces

$$\mathrm{HOM}_{\mathcal{A}_\infty}\big((w,P),(v_0,Q)\big) = \varinjlim_{u\ge w, v_0} \mathrm{HOM}_{\mathcal{A}_u}(T_{uw}P, T_{uv_0}Q).$$

A map $f : T_{uw}P \to T_{uv_0}Q$ goes at $t \ge u$ to

$$(\mathrm{id}_Q\, c_{t,u,v_0}) \circ T_{tu}(f) \circ (\mathrm{id}_P\, c_{t,u,w})^{-1}.$$

The associativity of the comparisons c makes this a directed system. Composition, sums and shifts are computed at a common stage. For an idempotent,

move a representative to a stage where its idempotent equation holds and split it there. The pure corner $\mathcal{A}^I_\infty$ and the full subcategory $\mathcal{C}_\infty$ on the $\mathcal{C}_w$ have the same construction. The weight of (w, P) is that of P in $\mathcal{A}_w$, preserved by the transitions.

The negative diagrams act on the colimit because the transitions intertwine all of them. To define E_i on an object, choose its representative at a stage in W_i^- and apply E_i there. For a morphism, move the chosen source and target representatives and the morphism to a common stage in W_i^-, then apply E_i to that morphism. The comparison maps in Lemma 4.2 identify the resulting objects and transport the morphism back to the chosen representatives. Their composition compatibility makes the result independent of the stage. It also ensures preservation of identities and composition by using a single common stage for any finite diagram. Units, counits and mixed and same-color inverses pass to the colimit by the same argument. In particular the mixed maps with the primed lowering colors remain invertible.

For the original colors use the dual presentation of the categorified quantum group $\mathcal{U}$: the negative quiver Hecke diagrams, the shifted adjunctions $F_i \dashv E_i$, and the inverses in (3)–(4). This is the full-color case of [18, Theorem 2.6.7(2)], compared with the Rouquier–Brundan and cyclic presentations in [3, 2]. We use the native adjunctions of that presentation throughout. In particular the upward dot and crossing operators mean the right mates of the downward dot and crossing, with the adjunction shifts removed. Each inverse requirement involves one raising color i and holds at a stage in W_i^-. The negative quiver Hecke relations hold at every stage. Thus the data define the full original-color action. The pure evaluation comparisons preserve the defining generators by Lemma 4.2. Their coherence gives a functor $\mathcal{A}^I_\infty \to \mathscr{L}(-\lambda_1)$ intertwining the action. It is faithful: a pure representative with zero evaluation belongs to I_w at the stage where it is represented.

The subcategory $\mathcal{C}_\infty$ is stable under the original action. Its normalized labels are the union of the canonical bases of ${}^\omega V_w(\lambda_1) \otimes \Lambda_{\lambda_2}$. The union is the canonical basis of ${}^\omega \Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2}$. Indeed the Demazure modules exhaust the lowest-weight module and their tensor inclusions are based, so every tensor vector is a linear combination of labels at one stage. At a common stage two normalized labels have no negative-degree morphisms. In degree zero the Hom space is k times the identity for identical labels and zero for distinct labels. The transition maps preserve each surviving identity, so taking the directed limit preserves these assertions. Every object is a finite sum of shifts of these labels, which therefore form a basis of the split Grothendieck group over $\mathcal{A}$. The calculation in Lemma 4.2 identifies its original E_i, F_i-action with the tensor action.

6. Specialization of the framing center

Fix the framing multiplicities m_i. Let $\Theta = P_{\theta_m}$ be the product of the normalized primed divided-power projectives, and use the same symbol for left induction by this projective. For a pure projective P, the class $[\Theta P] = \theta_m[P]$ lies in F_m. This subspace is spanned by a subset of canonical labels, so each summand of ΘP has a label in F_m. Thus ΘP lies in $\mathcal{C}_w$ after passing to the quotient. Put

$$S_m = \bigotimes_{i\in I} k[z_{i,1},\dots,z_{i,m_i}]^{\mathfrak{S}_{m_i}}, \qquad S_{m,+} = \bigoplus_{d>0}(S_m)_d.$$

Here $z_{i,a}$ denotes the dot on the a-th strand of color i', of degree 2, and $\mathfrak{S}_{m_i}$ permutes these variables. The factor with $m_i = 0$ is k. The ideal $S_{m,+}$ is the kernel of evaluation at zero. A polynomial in S_m acts on each word by evaluating it on the indicated dots. The dot slides commute with this action across distinct colors, and the nilHecke divided difference of a symmetric polynomial is zero across equal colors. Thus S_m acts centrally.

The transition functors append only original strands, so they preserve this central action. The induction bimodule for F_i and the restriction bimodule for E_i carry identical left and right actions of the primed symmetric dot polynomials. For E_i this follows from the idempotent description (8), since the removed strand has original color i. These identities descend from the single-root stages to the colimit. Thus F_i, E_i preserve the ideal of maps $S_{m,+}\,\mathrm{HOM}(X,Y)$, and the original 2-action descends to

$$\mathcal{C}^0 = \mathcal{C}_\infty/(S_{m,+}).$$

Write $\mathcal{C}^0_w$ for the same specialization at stage w. If X is a normalized indecomposable, the degree-zero part of $S_{m,+}\,\mathrm{HOM}(X,X)$ vanishes: a homogeneous polynomial of positive degree would have to multiply a negative-degree endomorphism of X. Lemma 2.1 implies that these quotients are idempotent complete and retain the same normalized labels, with the same degree-zero orthogonality and negative-degree vanishing.

Lemma 6.1. *For $P, Q \in \mathscr{P}_I$, left induction induces an isomorphism*

$$\mathrm{HOM}_{\mathscr{P}_I}(P,Q)/I_w(P,Q) \xrightarrow{\sim} \mathrm{HOM}_{\mathcal{C}^0_w}(\Theta P, \Theta Q), \qquad f \longmapsto \mathrm{id}_\Theta\, f.$$

These isomorphisms commute with the transition functors and give a fully faithful functor $\Theta : \mathcal{A}^I_\infty \to \mathcal{C}^0$.

Proof. Both Hom spaces vanish if P, Q are homogeneous of distinct formal contents, and the general case is obtained by finite sums. Consider the same-content case, with common original content β. Before taking any quotient, the map given by juxtaposition of diagrams identifies

$$\mathrm{HOM}_{\mathscr{P}_{\tilde{I}}}(\Theta P, \Theta Q) \cong S_m \otimes_k \mathrm{HOM}_{\mathscr{P}_I}(P,Q).$$

In fact, for word projectives with the primed block on the left, every permutation matching the boundary colors preserves the two blocks. Choose

reduced expressions for these permutations inside the blocks. The KLR PBW basis then consists of products of the two block bases. Cutting by the primed divided-power idempotents replaces the primed nilHecke algebra by its symmetric polynomial corner S_m. Taking summands and shifts of the original word projectives gives the displayed identification.

Apply the Mackey filtration of D_w to ΘP. All primed strands must enter the second factor, and any original strand sent to that factor would occur to the right of the primed block and annihilate $L^+_{\eta_m}$. The only surviving piece has all of P in the first factor and all of Θ in the second factor. The shuffle crosses contents γ_m and β, so that piece is

$$\big(F_P L_w \boxtimes F_\Theta L^+_{\eta_m}\big)\langle -(\gamma_m, \beta)\rangle.$$

The analogous piece for Q has the same shift. The primed colors are pairwise disconnected, and their cyclotomic projective category is the external tensor product of the rank-one projective categories at highest weights m_i. The divided power of length m_i takes the highest projective to the opposite extremal projective, with graded endomorphisms k in degree zero. This follows from the rank-one extremal equivalence used in Section 2. Hence $F_\Theta L^+_{\eta_m}$ has graded endomorphisms k, and S_m acts on it by the augmentation. On the surviving Mackey piece a juxtaposed map $s \otimes f$ acts by the augmentation of s on the primed factor and by $\mathrm{ev}_w(f)$ on the original factor. This follows from the naturality of the Mackey piece for block maps. Evaluation kills the other pieces, so the surviving piece computes the entire map.

By construction, $S_m \otimes I_w(P, Q)$ lies in $\widetilde{I}_w(\Theta P, \Theta Q)$. Since D_w kills $\widetilde{I}_w$, the corner of $\widetilde{I}_w$ lies in the kernel of augmentation tensored with ev_w, which is

$$S_{m,+} \otimes_k \mathrm{HOM}_{\mathscr{P}_I}(P, Q) + S_m \otimes_k I_w(P, Q).$$

Specialization adds the first summand to the ideal. Thus the ideal in the specialized corner is exactly this sum, which proves the formula. Left induction commutes with right induction by each transition tail, and the isomorphism is induced by left induction. Passing to the colimit proves full faithfulness. □

Pure evaluation gives an equivalence $\mathcal{A}^I_\infty \to \mathscr{L}(-\lambda_1)$. It is faithful by the definition of I_w. Positive words on L_e, their shifts, sums and summands generate the lowest-weight cyclotomic projective category. The Hom spaces between words are corners of cyclotomic quiver Hecke algebras, spanned by upward diagrams [4, Remark 4.22]. The presentation comparison rescales the dot and crossing generators by units, so the mates of Section 5 also generate these spaces. The action comparison lifts them to the pure colimit. The induced isomorphism on endomorphisms of word sums lifts idempotents, which split in the colimit. Composing lifts on word sums with the lifted idempotents gives fullness between the summands. We also write L_e for the empty word at stage e of the pure colimit.

7. Fullness of the evaluation

We compare the upward diagrams on L_e and $X_0 = \Theta L_e$ before specializing the center, in the colimit $\mathcal{A}_\infty$ with all framing degrees. The weight of X_0 is $\mu_e - \gamma_m = \zeta$. Let $P = F_{i_1} \cdots F_{i_\ell}$ be a negative word on the original colors with root content $\beta = \sum_{a=1}^{\ell} \alpha_{i_a}$, and let T be a negative word on the primed colors with root content γ. Composition applies the rightmost factor first. Crossing the two blocks gives

$$b_{P,T} : PT \longrightarrow TP, \qquad \deg b_{P,T} = -(\beta, \gamma).$$

For an original negative KLR diagram $f : P \to Q$ and a primed one $g : T \to T'$, the dot-slide and braid relations give

$$b_{Q,T}(fT) = (Tf)b_{P,T}, \qquad (gP)b_{P,T} = b_{P,T'}(Pg). \tag{9}$$

For a generating crossing, the correction in the braid relation vanishes because its first and last colors lie in disjoint color sets. This applies also when the two strands of the generating crossing have the same color.

Let R_P be the shifted right adjoint of P, with degree-zero unit η_P and counit ε_P. Juxtaposition with functors denotes whiskering, and compositions are read from right to left. The right mate of the block crossing is

$$\check{b}_{P,T} = (R_P T \varepsilon_P)(R_P b_{P,T} R_P)(\eta_P T R_P) : TR_P \longrightarrow R_P T.$$

Equivalently its transpose under $P \dashv R_P$ is characterized by

$$(\varepsilon_P T)(P\check{b}_{P,T}) = (T\varepsilon_P)(b_{P,T}R_P). \tag{10}$$

For $f : P \to Q$, its right mate is

$$f^r = (R_P \varepsilon_Q)(R_P f R_Q)(\eta_P R_Q) : R_Q \longrightarrow R_P.$$

Its transpose satisfies $\varepsilon_P(Pf^r) = \varepsilon_Q(fR_Q)$. We claim that

$$\check{b}_{P,T}(Tf^r) = (f^r T)\check{b}_{Q,T}, \qquad (R_P g)\check{b}_{P,T} = \check{b}_{P,T'}(gR_P). \tag{11}$$

Under $P \dashv R_P$, the first left-hand side transposes to

$$\begin{aligned}(T\varepsilon_P)(b_{P,T}R_P)(PTf^r) &= (T\varepsilon_Q)(TfR_Q)(b_{P,T}R_Q)\\ &= (T\varepsilon_Q)(b_{Q,T}R_Q)(fTR_Q).\end{aligned}$$

The first equality uses naturality of the crossing in its argument and the transpose identity for f^r. The second is (9). The first right-hand side transposes to

$$(\varepsilon_Q T)(fR_Q T)(P\check{b}_{Q,T}) = (\varepsilon_Q T)(Q\check{b}_{Q,T})(fTR_Q),$$

which is the same by (10). The two sides for g transpose to $g(T\varepsilon_P)(b_{P,T}R_P)$ and $(T'\varepsilon_P)(b_{P,T'}R_P)(PgR_P)$. These agree by the second identity of (9) and naturality of g.

For an original negative word Q and a primed negative word U, the chosen block crossings satisfy

$$b_{PQ,T} = (b_{P,T}Q)(Pb_{Q,T}), \qquad b_{P,TU} = (Tb_{P,U})(b_{P,T}U).$$

The composite right adjoint is $R_{PQ} = R_Q R_P$, with counit $\varepsilon_P(P\varepsilon_Q R_P)$. Then

$$\check{b}_{PQ,T} = (R_Q \check{b}_{P,T})(\check{b}_{Q,T} R_P), \qquad \check{b}_{P,TU} = (\check{b}_{P,T} U)(T\check{b}_{P,U}). \tag{12}$$

For the first identity, transposing its right-hand side under $PQ \dashv R_Q R_P$ gives

$$\begin{aligned}
&(\varepsilon_P T)(P\varepsilon_Q R_P T)(PQR_Q\check{b}_{P,T})(PQ\check{b}_{Q,T}R_P)\\
&= (\varepsilon_P T)(P\check{b}_{P,T})(P\varepsilon_Q T R_P)(PQ\check{b}_{Q,T}R_P)\\
&= (T\varepsilon_P)(b_{P,T}R_P)(PT\varepsilon_Q R_P)(Pb_{Q,T}R_Q R_P)\\
&= (T\varepsilon_{PQ})(b_{PQ,T}R_Q R_P).
\end{aligned}$$

The second equality uses (10) for P, T and Q, T. The other equalities use naturality and the factorization of the block crossing. For the second identity, the transpose under $P \dashv R_P$ is

$$\begin{aligned}
(\varepsilon_P TU)(P\check{b}_{P,T}U)(PT\check{b}_{P,U}) &= (T\varepsilon_P U)(b_{P,T}R_P U)(PT\check{b}_{P,U})\\
&= (T\varepsilon_P U)(TP\check{b}_{P,U})(b_{P,T}UR_P)\\
&= (TU\varepsilon_P)(b_{P,TU}R_P).
\end{aligned}$$

This proves both identities.

Put $P^\vee = E_{i_\ell}\cdots E_{i_1}$. At input weight λ the degree-zero right adjoint of F_i is $E_i\langle 1 + \langle h_i, \lambda\rangle\rangle$. Hence

$$R_P 1_\lambda = P^\vee 1_\lambda \langle c_\beta(\lambda)\rangle,$$

$$c_\beta(\lambda) = \sum_{a=1}^{\ell}\Big(1 + \big(\alpha_{i_a}, \lambda + \sum_{b<a}\alpha_{i_b}\big)\Big) = \frac{(\beta,\beta)}{2} + (\beta,\lambda).$$

The unit and counit used in the mate have degree zero, so $\check{b}_{P,T}$ has degree $-(\beta,\gamma)$. The target shift minus the source shift is $c_\beta(\lambda-\gamma) - c_\beta(\lambda) = -(\beta,\gamma)$. Removing the shifts gives a degree-zero map

$$s_{P,T} : TP^\vee \longrightarrow P^\vee T.$$

For original i and primed r, let $\tau_{ir} : F_i F_r \to F_r F_i$ be the downward crossing and let η_i, ε_i be the unshifted unit and counit for i from Section 4. Then

$$s_{F_i,F_r} = (E_i F_r \varepsilon_i)(E_i \tau_{ir} E_i)(\eta_i F_r E_i).$$

This is the mixed isomorphism (3). The two factorizations in (12) also hold for s. For original contents β, δ, the shifts compose according to $c_{\beta+\delta}(\lambda) = c_\beta(\lambda) + c_\delta(\lambda+\beta)$. Thus each $s_{P,T}$ is a composite of mixed isomorphisms. Empty words give identities.

For a negative diagram $f : P \to Q$ the words have the same content. Removing the common shifts from f^r gives $f^\vee : Q^\vee \to P^\vee$, and the first identity in (11) becomes

$$s_{P,T}(Tf^\vee) = (f^\vee T)s_{Q,T}.$$

We use the upward dots and crossings in the native mate normalization of Section 5. They are the $f^\vee$ for negative generating dots and crossings. Applying the second identity in (11) to the primed divided-power idempotents, and applying their normalization shifts, gives

$$\kappa_{E_\alpha} : \Theta E_\alpha \longrightarrow E_\alpha \Theta$$

for every positive word E_α, where α denotes its word of original colors. The preceding identity intertwines the upward diagrams with respect to these isomorphisms. All identities were formed from the negative diagrams and the adjunctions preserved by the right-tail comparisons. They hold in the colimit, and on pure input they lie in the fixed framing and survive its center specialization.

Lemma 7.1. *For positive words E_α, E_β on original colors, evaluation of upward KLR diagrams at X_0 surjects onto* $\mathrm{HOM}_{\mathcal{C}^0}(E_\alpha X_0, E_\beta X_0)$.

Proof. Lemma 6.1 and the pure evaluation identify this Hom space with

$$\mathrm{HOM}_{\mathscr{L}(-\lambda_1)}(E_\alpha L_e, E_\beta L_e).$$

The map from the latter space sends f to $\kappa_{E_\beta}\Theta(f)\kappa_{E_\alpha}^{-1}$. For an upward diagram the mate identities identify this image with the action of the same diagram at X_0. The cyclotomic KLR corners of the pure category are spanned by upward diagrams, giving the surjection. □

Proposition 7.2. *Evaluation at X_0 is a full functor*

$$\mathrm{ev}_{\zeta,m} : \dot{\mathcal{U}} 1_\zeta \longrightarrow \mathcal{C}^0.$$

Here $\dot{\mathcal{U}} 1_\zeta$ is the category of 1-morphisms with source ζ in the idempotent completion of the original 2-category.

Proof. Consider two words $F_\alpha E_\gamma$ and $F_\beta E_\delta$, with each subscript denoting a word of original colors. Write R_{F_α} for the actual shifted right adjoint of F_α. Adjunction identifies the Hom space of their evaluations with

$$\mathrm{HOM}_{\mathcal{C}^0}(E_\gamma X_0, R_{F_\alpha} F_\beta E_\delta X_0).$$

In the source, $R_{F_\alpha} F_\beta E_\delta$ is a summand of a finite sum of shifted words of the form $F_u E_v$. Order words by their number of pairs consisting of a positive letter and a negative letter to its right, followed by their length. At an adjacent such pair, exchange the letters using the following relations. Distinct colors exchange by the mixed isomorphism. For equal colors and current weight pairing $h \geq 0$, $E_i F_i$ is $F_i E_i$ plus the prescribed shifted identity summands. For $h < 0$, $E_i F_i$ is a specified summand of $F_i E_i$. The exchanged term has one fewer such pair, and deleting the pair to form an identity term also decreases this order. Induction gives the stated presentation with inclusions and projections in the source.

For a term $F_u E_v \langle d \rangle$ of the sum, write L_{F_u} for the degree-zero left adjoint of F_u. For one letter F_i at target weight λ it is $E_i \langle -1 - \langle h_i, \lambda \rangle \rangle$ [18, Lemma

2.6.14]. For a word, compose the left adjoints in reverse order, taking each shift at the weight at which its functor is applied. Adjunction gives

$$\mathrm{HOM}_{\mathcal{C}^0}(E_\gamma X_0, F_u E_v X_0\langle d\rangle) \cong \mathrm{HOM}_{\mathcal{C}^0}(L_{F_u} E_\gamma X_0, E_v X_0\langle d\rangle).$$

Both words on the right are positive up to shift. The shifts are represented in the source, so Lemma 7.1 lifts every homogeneous map in these spaces. Composing with the source adjunctions and the source projections and inclusions gives a lift between the original words. Arbitrary words are summands of finite sums of words in the same order $F_\alpha E_\gamma$ by the same reordering. Fullness extends to sums, shifts and idempotent summands by composing lifted maps with the source idempotents. □

8. Identification of the canonical basis

The diagrammatic Grothendieck group surjection is

$$\gamma : {}_{\mathcal{A}}\dot{\mathbf{U}} \longrightarrow K_0(\dot{\mathcal{U}}).$$

It sends divided powers and weight idempotents to their diagrammatic classes and uses the shift $\langle 1\rangle$ for multiplication by v [4, §3.5, (3.16)]. The split Grothendieck group is the direct sum over pairs of boundary weights, with product induced by composition and zero for incompatible weights.

Proposition 8.1. *The map γ is an isomorphism. Each original canonical basis element corresponds to the class of an indecomposable 1-morphism, uniquely up to isomorphism with the shift fixed by that class.*

Proof. The normalized labels of $\mathcal{C}^0$ identify its split Grothendieck group with the integral lattice spanned by the canonical basis of ${}^{\omega}\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2}$. Denote the composite of γ and evaluation on the column ${}_{\mathcal{A}}\dot{\mathbf{U}}1_\zeta$ by $\pi_{\zeta,m}$. It has value

$$\pi_{\zeta,m}(x) = x(\xi_{-\lambda_1} \otimes \eta_{\lambda_2}).$$

The action of E_i, F_i on classes was computed at the descent stages, and the class of X_0 corresponds to the displayed tensor vector. Specialization preserves the normalized classes and intertwines the action. Thus the formula holds on words in the generators. After scalar extension to $\mathbb{Q}(v)$ it also holds on divided powers by division by $[r]!$. The target split group is free over $\mathcal{A}$, so the integral formula follows.

For a nonzero $a \in {}_{\mathcal{A}}\dot{\mathbf{U}}1_\zeta$, choose all m_i sufficiently large that the finitely many canonical basis elements in its support have distinct nonzero tensor canonical images. Lusztig's separation theorem applies since $\lambda_2 - \lambda_1 = \zeta$ and all original simple-coroot pairings of both weights grow with m [17, Theorem 25.2.1]. These images are linearly independent, so $\pi_{\zeta,m}(a) \neq 0$ and $\gamma(a) \neq 0$. This proves injectivity on each column and hence on their direct sum.

The diagrammatic spanning set consists of finitely many reduced boundary diagrams decorated with dots and positive-degree bubble monomials [12, Proposition 3.11]. Dots and bubble variables have positive degrees,

with finitely many bubble variables in each degree. This gives a lower degree bound and finite dimension in each degree for fixed boundaries. The same bounds hold for summands. The degree-zero endomorphism algebra of each object is finite-dimensional, so a decomposition of its identity into primitive orthogonal idempotents gives a finite decomposition into objects with local endomorphism rings. The source is therefore Krull–Schmidt. For an indecomposable object, fullness makes the degree-zero endomorphism algebra of its image a quotient of its local endomorphism algebra. If the image is nonzero, the quotient is local and the image is indecomposable.

Now let P be indecomposable in the source column and expand

$$\gamma^{-1}[P] = \sum_{b \in T} a_b(v) b$$

with T the finite canonical support and $a_b(v) \in \mathcal{A}$ nonzero for $b \in T$. Choose m so that the elements of T have distinct nonzero images. The class of the evaluation of P is nonzero, and Proposition 7.2 implies that its evaluation is indecomposable. Thus it is a shift of one normalized tensor canonical label c, giving

$$\sum_{b \in T} a_b(v) \pi_{\zeta,m}(b) = v^s c$$

for some $s \in \mathbb{Z}$. Comparing coefficients gives $T = \{b\}$ and $a_b(v) = v^s$. Shift orbits of indecomposables freely generate the source Grothendieck group over $\mathcal{A}$. A nontrivial periodic shift would give endomorphisms in arbitrarily negative degrees and is excluded by the degree bound. If two distinct shift orbits corresponded to the same canonical element, suitable shifts of their classes would be equal, contradicting freeness. If a canonical element were omitted, it could not be in the span of the images of the shift orbits, contradicting surjectivity. Thus the correspondence is bijective. Shifting the representative by $-s$ gives the asserted normalization. □

For composable canonical basis elements b, b', let $P_b, P_{b'}$ be the normalized indecomposables from Proposition 8.1. Their composite has a finite decomposition

$$P_b P_{b'} \cong \bigoplus_{c,s} P_c \langle s \rangle^{\oplus n_{c,s}}, \qquad n_{c,s} \in \mathbb{N},$$

where c ranges over canonical basis elements with the resulting boundaries and s over integers. Taking classes gives $\sum_s n_{c,s} v^s$ as the coefficient of c in bb'. If the boundaries are incompatible, $bb' = 0$. This proves multiplication positivity for the regular realization.

9. Change of root datum

Let X now be the weight lattice of an arbitrary root datum with Cartan matrix A. For a formal root degree $\delta = \sum_{i \in I} n_i \alpha_i$, write $\rho_X(\delta)$ and $\rho_P(\delta)$ for the respective sums of the corresponding simple roots in X and P. Fix

$\nu \in X$ and choose $\widehat{\nu} \in P$ with $\langle h_i, \widehat{\nu}\rangle = \langle h_i, \nu\rangle$ for every $i \in I$. Such a choice exists because the original coroots on P are coordinate functions.

The simply connected weight lattice for A is $\mathbb{Z}^I$, with simple coroots the coordinate functions and simple roots the columns of A. Both ν and $\widehat{\nu}$ have the same image in this lattice, given by their coroot pairings. The change-of-datum isomorphism on a fixed right column [17, 23.2.5 and 25.2.2–25.2.4] sends each element $u^+u^-1_\nu$ to the same expression with the image of ν as right weight, and sends the canonical label indexed by a pair of half-algebra canonical vectors to the label indexed by the same pair [5, §6.6]. Here u^+ and u^- belong to the positive and negative halves, respectively, which depend only on A. Composing the comparison from X with the inverse comparison from P gives a basis-preserving linear isomorphism

$$T_{\nu,\widehat{\nu}} : \dot{\mathbf{U}}1_\nu \longrightarrow \dot{\mathbf{U}}_P 1_{\widehat{\nu}}.$$

The subscript P indicates the original-color datum on P. In particular $T_{\nu,\widehat{\nu}}(u^+u^-1_\nu) = u^+u^-1_{\widehat{\nu}}$.

This isomorphism intertwines left multiplication by each E_i, F_i. To check it, put a word in the two kinds of generators in positive-then-negative order. Each exchange of F_i, E_j either commutes them or contributes a quantum integer determined by the weight at that position in the word. If the suffix at that position has formal degree δ, the two weights there are $\nu+\rho_X(\delta)$ and $\widehat{\nu} + \rho_P(\delta)$. Their simple coroot pairings agree because the two simple-root maps give the same Cartan integers. The exchanges therefore give identical ordered expressions in the two columns. Since these expressions span, the claim follows for any left multiplication by a word in the E_i, F_i.

Proof of Theorem 1.1. Section 8 proves the theorem for P. If the right weight of b differs from the left weight of b', their product is zero. Otherwise let b' have right weight ν, left weight λ, and formal degree δ'. Canonical basis elements are homogeneous for the formal root grading [17, 25.2.1(e) and 25.2.4], so $\lambda = \nu + \rho_X(\delta')$. Take $\widehat{\lambda} = \widehat{\nu} + \rho_P(\delta')$. Write $b = u1_\lambda$ with u a linear combination of words in the E_i, F_i. Then $T_{\lambda,\widehat{\lambda}}(b) = u1_{\widehat{\lambda}}$, and $T_{\nu,\widehat{\nu}}(b')$ has left weight $\widehat{\lambda}$. The intertwining just proved yields

$$T_{\nu,\widehat{\nu}}(bb') = T_{\lambda,\widehat{\lambda}}(b)\, T_{\nu,\widehat{\nu}}(b').$$

The column isomorphism sends distinct canonical labels to distinct canonical labels. The product coefficients therefore agree with those for P.

For a symmetric Cartan matrix presented with a positive integral symmetrizing integer d on a connected component, the quantum integers in that component are those for the parameter v^d. The presentations and canonical-basis characterizations on the component are obtained from the normalized ones by the scalar substitution $v \mapsto v^d$. In the simply connected datum, right columns factor over the connected components. Their canonical bases factor as well, by the tensor-product characterization of the column bases and the factorization of the quasi-R-matrix over the components. Each product

coefficient is thus a product of nonnegative Laurent polynomials after the substitutions for its components. The column comparison transfers those coefficients to X. □

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