THE TANGENT LINE AT A MARKED POINT ON $M_{0,n}/\mathfrak{S}_{n-1}$, AFTER A TABLE OF GROTHENDIECK

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Abstract. Let L_n be the line bundle on $M_{0,n}$ whose fibre is the tangent line to $\mathbb{P}^1$ at the last marked point, and L the tensor product of the tangent lines at all of them. We show that, in characteristic zero, the Picard group of the quotient stack $[M_{0,n}/\mathfrak{S}_{n-1}]$ is cyclic of order $(n-1)(n-2)$ for every $n \geq 3$, generated by L_n, and that L has order $n-1$ in $\mathrm{Pic}[M_{0,n}/\mathfrak{S}_n]$. For even $n \geq 6$ the group was known, through the integral Chow ring computed by Edidin and Hu; the odd case does not seem to be in print. The question comes from a table in Grothendieck's working notes, which gives these orders for four points and only guesses for five and six. The proof takes a few lines once the stack is written as monic squarefree polynomials modulo the affine group. An elementary version, with explicit trivialisations and fixed configurations, is formalised in Lean.

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1. Introduction

Let k be a field of characteristic zero and $n \geq 3$. Let $M_{0,n}$ be the moduli space of n distinct ordered points $x_1, \dots, x_n$ on $\mathbb{P}^1$, up to PGL_2. For each i the tangent lines $T_{x_i}\mathbb{P}^1$ form a line bundle L_i on $M_{0,n}$, and we put $L = L_1 \otimes \cdots \otimes L_n$. The symmetric group $\mathfrak{S}_n$ acts on $M_{0,n}$ by renumbering the points and acts compatibly on L; the subgroup $\mathfrak{S}_{n-1}$ fixing the last point acts on L_n. This defines classes

$$[L] \in \mathrm{Pic}[M_{0,n}/\mathfrak{S}_n], \qquad [L_n] \in \mathrm{Pic}[M_{0,n}/\mathfrak{S}_{n-1}],$$

and the question is their order: the least $e \geq 1$ for which $L^{\otimes e}$, or $L_n^{\otimes e}$, has a nowhere-vanishing section compatible with the action.

Theorem 1. *For every $n \geq 3$:*

(1) $\mathrm{Pic}[M_{0,n}/\mathfrak{S}_{n-1}]$ *is cyclic of order $(n-1)(n-2)$, generated by $[L_n]$;*
(2) $[L]$ *has order $n-1$ in* $\mathrm{Pic}[M_{0,n}/\mathfrak{S}_n]$.

For $n = 4, 5, 6$ the orders are $3, 4, 5$ for L and $6, 12, 20$ for L_n.

The source. The University of Montpellier holds the mathematical papers that Alexandre Grothendieck left to Jean Malgoire, and publishes their facsimiles. Page 70 of its folder 146, *Groupoïdes de Teichmüller*, has a table which gives, for $n = 4, 5, 6$, the best exponent that makes L and the L_i trivial compatibly with the symmetric group. Its first row gives $(n-1)(n-2) = 6, 12, 20$. In the column $n = 4$ it gives 3 for L and 6 for L_i, and both are right. The columns $n = 5$ and $n = 6$ hold only queried values: *12 ?* for both bundles when $n = 5$, and *20 ?* for L and *60 ?* for L_i when $n = 6$. Below the table, a computation for $n = 5$ ends with $(L^4)^3 \simeq \mathcal{O}$, that is, $L^{\otimes 12}$ is trivial. Page 72 builds sections indexed by the three-element subsets of the points. For $n = 4$ it writes $L^{\otimes 6} \simeq 1$, which is true and weaker than the order 3. For $n = 5$ it writes $L^{\otimes 6} \simeq 1$ again (our transcription marks the words before the formula as an uncertain reading); this is false, since by Theorem 1 L then has order 4. The theorem completes the table. For $n = 5$ the queried 12 is right for L_5, but it is not minimal for L, whose order is

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4. For $n = 6$ the orders are 5 and 20, not 20 and 60. In every column the order of L_n is the number in the first row.

TABLE 1. The table of folder 146, page 70, and the orders given by Theorem 1.

	$n=4$		$n=5$		$n=6$	
	L	L_n	L	L_n	L	L_n
Page 70	3	6	12 ?	12 ?	20 ?	60 ?
Theorem 1	3	6	4	12	5	20

What was known. The stack $[M_{0,n}/\mathfrak{S}_n]$ is the stack of smooth binary forms of degree n up to scalars, modulo PGL_2. Arsie and Vistoli computed the Picard groups of stacks of smooth cyclic covers of projective spaces from such presentations by forms [1, Th. 5.1]. For $n = 2g + 2$ with $g \geq 2$, Gorchinskiy and Viviani identify $[M_{0,n}/\mathfrak{S}_n]$ with the stack $\mathcal{D}_{2g+2}$ of $\mathbb{P}^1$-bundles with an étale divisor of degree $2g+2$, and find $\operatorname{Pic}\mathcal{D}_{2g+2} \cong \mathbb{Z}/2(n-1)$ [5, Prop. 3.4, Th. 3.6]. The order of $[L]$ follows from these methods in a few lines (Remark 3), but we have not found it stated. For $n = 2g + 2$ and $g \geq 2$, Edidin and Hu compute the integral Chow ring of $[M_{0,2g+2}/\mathfrak{S}_{2g+1}]$, which they present as binary forms with one marked root, as $\mathbb{Z}[l]/(2g(2g+1)\,l)$ [3, Th. 1.3], building on Pernice's description of the stack of 1-pointed hyperelliptic curves [8]. So $\operatorname{Pic}[M_{0,n}/\mathfrak{S}_{n-1}]$ is cyclic of order $(n-1)(n-2)$ for even $n \geq 6$. Their generator is written in terms of characters of a torus [3, Th. 5.8], and its identification with L_n is not stated. For $n = 4$, sending an elliptic curve E to $E/\{\pm 1\} \cong \mathbb{P}^1$, with the origin as last point, makes $\mathcal{M}_{1,1}$ a μ_2-gerbe over $[M_{0,4}/\mathfrak{S}_3]$. Pulling back along it is injective on Picard groups, and its image consists of the classes on which μ_2 acts trivially. Since $\operatorname{Pic}\mathcal{M}_{1,1} = \mathbb{Z}/12$ is generated by the Hodge bundle λ, on which -1 acts by -1 [7, 4], the image is $2\mathbb{Z}/12 \cong \mathbb{Z}/6$, and L_4 pulls back to $T_0E^{\otimes 2} \cong \lambda^{-2}$, of order 6. For odd n we found no printed statement. D. Edidin confirms that their paper treats only the even case and that he knows no reference for the odd one [2]. The author asked the question on MathOverflow (question 516243) and posted the argument of Section 2 as an answer.

2. The stack as polynomials modulo the affine group

Put $m = n - 1 \geq 2$. Since PGL_2 acts transitively on $\mathbb{P}^1$, we may send the last point to ∞; its stabiliser is the affine group $\mathrm{Aff} = \mathbb{G}_a \rtimes \mathbb{G}_m$, acting on $\mathbb{A}^1$ by $x \mapsto ax + b$, and the other m points, unordered, are the roots of a monic squarefree polynomial of degree m. Taking the slice $x_n = \infty$ gives an isomorphism of stacks

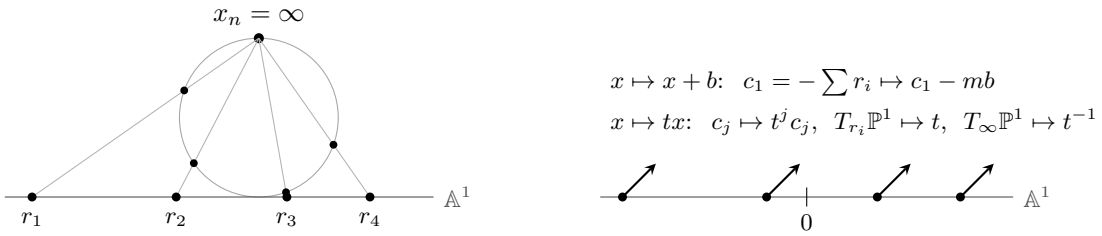
$$[M_{0,n}/\mathfrak{S}_{n-1}] \;\cong\; [U_m/\mathrm{Aff}],$$

where $U_m \subset \mathbb{A}^m$ is the open set of monic squarefree polynomials $x^m + c_1x^{m-1} + \cdots + c_m$, on which Aff acts by substitution (Figure 1).

The translations $\mathbb{G}_a$ act freely, and every orbit meets the closed set $c_1 = 0$ exactly once, since m is invertible in k: replacing $f(x)$ by $f(x - c_1/m)$ centres the roots. So

$$[U_m/\mathrm{Aff}] \;\cong\; [V_m/\mathbb{G}_m], \qquad V_m = \{x^m + c_2x^{m-2} + \cdots + c_m \text{ squarefree}\} \subset \mathbb{A}^{m-1},$$

where $t \in \mathbb{G}_m$ acts by $x \mapsto tx$, that is, on the coefficient c_j with weight j. The complement of V_m in $\mathbb{A}^{m-1}$ is the discriminant hypersurface $\mathrm{disc} = 0$. As a function of the roots, $\mathrm{disc} = \prod_{i<j}(r_i - r_j)^2$ is homogeneous of degree $m(m-1)$, so it is a semi-invariant of weight $m(m-1)$. It is irreducible in $k[c_2, \ldots, c_m]$. Indeed, the discriminant of the generic monic polynomial is irreducible in $k[c_1, \ldots, c_m]$ and invariant under translation of the roots. Since m is invertible, $k[c_1, \ldots, c_m] = k[c_1, c'_2, \ldots, c'_m]$, where the c'_j are the coefficients of the centred polynomial $f(x - c_1/m)$, and the discriminant is the same polynomial in $c'_2, \ldots, c'_m$, so it stays irreducible there.



(a) $\mathbb{P}^1$, projected from x_n (b) centred roots, $\sum r_i = 0$

FIGURE 1. The slice of Section 2 for $n = 5$. (a) With the last point at ∞, the other points become four roots on the affine line. (b) Translations move c_1 freely, so one may centre the roots; the scalings that remain act on the coefficient c_j with weight j, on each tangent line at a root (arrows) with weight 1, and on the tangent line at ∞ with weight -1.

Lemma 2. $\mathrm{Pic}[V_m/\mathbb{G}_m] \cong \mathbb{Z}/m(m-1)$, *generated by the line bundle attached to the character* $t \mapsto t$.

Proof. Since V_m is open in an affine space, $\mathrm{Pic}\, V_m = 0$, so a $\mathbb{G}_m$-linearised line bundle on V_m is the trivial bundle with a linearisation. A linearisation is a unit $\phi(t, x)$ of $\mathcal{O}(\mathbb{G}_m \times V_m) = \mathcal{O}(V_m)[t^{\pm 1}]$ satisfying the cocycle rule. As disc is irreducible, the units of $\mathcal{O}(V_m)$ are the $c\,\mathrm{disc}^j$ with $c \in k^\times$, so $\phi = c\,\mathrm{disc}(x)^j\, t^a$, and $\phi(1, x) = 1$ forces $c = 1$ and $j = 0$. So every linearisation is a character $t \mapsto t^a$. Two characters t^a and t^b give isomorphic bundles exactly when there is a unit u on V_m with $u(t \cdot x) = t^{b-a}u(x)$, that is, a semi-invariant unit of weight $b - a$. The units $c\,\mathrm{disc}^j$ have weights $j\,m(m-1)$. Hence $\mathrm{Pic}[V_m/\mathbb{G}_m] = \mathrm{Pic}^{\mathbb{G}_m}(V_m) \cong \mathbb{Z}/m(m-1)$, generated by the character of weight 1. □

Proof of Theorem 1. In the model $[U_m/\mathrm{Aff}]$ the last point is ∞. In the coordinate $u = 1/x$ at ∞, the map $x \mapsto ax + b$ is $u \mapsto u/(a + bu)$, whose derivative at $u = 0$ is a^{-1}; so $\mathbb{G}_m$ acts on $T_\infty\mathbb{P}^1$ through the character of weight -1, and $[L_n]$ corresponds to $-1 \in \mathbb{Z}/m(m-1)$. It generates the group, which proves (1).

At a finite root r, the derivative of $x \mapsto ax + b$ is a, so each $T_{r_i}\mathbb{P}^1$ has weight $+1$, and the pull-back of L to $[M_{0,n}/\mathfrak{S}_{n-1}]$ has weight $m - 1 = n - 2$. Its order in $\mathbb{Z}/(n-1)(n-2)$ is $n - 1$, so the order of $[L]$ is a multiple of $n - 1$. Conversely, write the points as vectors $v_i \in k^2 \smallsetminus \{0\}$ and $[v_i v_j] = \det(v_i, v_j)$. Since $T_{\langle v\rangle}\mathbb{P}^1 \cong \mathrm{Hom}(\langle v\rangle, k^2/\langle v\rangle) \cong \langle v\rangle^{\otimes -2} \otimes \det k^2$, and since the centre of GL_2 acts trivially on $\langle v\rangle^{\otimes -2} \otimes \det k^2$, descent along the $\mathbb{G}_m^n$-torsor of vectors over configurations shows that trivialising $L^{\otimes(n-1)}$ compatibly with the action amounts to giving a nowhere-vanishing function of the v_i, homogeneous of degree $2(n-1)$ in each v_i, transforming under $g \in \mathrm{GL}_2$ by $\det(g)^{n(n-1)}$ and invariant under renumbering. The function

$$\prod_{i \neq j} [v_i v_j] \;=\; (-1)^{n(n-1)/2} \prod_{i<j} [v_i v_j]^2$$

is such a function: each v_i occurs in $2(n-1)$ brackets, it vanishes nowhere on $M_{0,n}$, and the right-hand side is invariant under $\mathfrak{S}_n$. This proves (2). □

Remark 3. For $n = 3$ the stack $[M_{0,3}/\mathfrak{S}_2]$ is $B(\mathbb{Z}/2)$, whose Picard group is $\mathbb{Z}/2$, as the theorem says. The same method computes $\mathrm{Pic}[M_{0,n}/\mathfrak{S}_n]$. Let B_n be the space of binary forms of degree n, and let $H = (\mathrm{GL}_2 \times \mathbb{G}_m)/\{(t, t^n)\}$ act on it by $(g, s) \cdot f = s\,(f \circ g^{-1})$; then $[M_{0,n}/\mathfrak{S}_n] \cong [B_n^{\mathrm{sm}}/H]$. The characters of H are the $\det^a s^b$ with $2a + nb = 0$, a group isomorphic to $\mathbb{Z}$. The discriminant, irreducible of degree $2(n-1)$ [1, §5], has character $\det^{-n(n-1)} s^{2(n-1)}$, which is $2(n-1)$ times a generator when n is even and $n - 1$ times a generator when n is odd. The argument of Lemma 2 then shows that $\mathrm{Pic}[M_{0,n}/\mathfrak{S}_n]$ is cyclic of order $2(n-1)$ for even n, as in [5, Th. 3.6], and of order $n - 1$ for odd n. By Theorem 1(2), $[L]$ generates this group when n is odd, and generates its subgroup of index 2 when n is even.

3. An elementary version

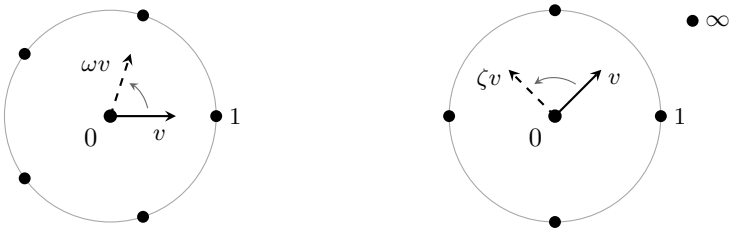
The argument can be phrased without stacks, in the vector model used above. In the same way, a trivialisation of $L_n^{\otimes e}$ compatible with $\mathfrak{S}_{n-1}$ is a nowhere-vanishing function $F(v_1, \dots, v_n)$ which is homogeneous of degree $2e$ in v_n and of degree 0 in the other v_i, transforms by $\det(g)^e$ under GL_2, and is symmetric in $v_1, \dots, v_{n-1}$.

Upper bound. The function

$$F = \Big(\prod_{i<n} [v_i v_n]\Big)^{2(n-2)} \Big/ \prod_{\substack{i\neq j \\ i,j<n}} [v_i v_j]$$

has these properties for $e = (n-1)(n-2)$.

Lower bound. A trivialisation over k gives one over an algebraic closure, so we may assume that k contains the roots of unity used below. Let ω be a primitive $(n-1)$-th root of unity, and consider the configuration with $x_n = 0$ and $x_1, \dots, x_{n-1}$ the $(n-1)$-th roots of unity. Write $0 = \langle (0,1) \rangle$ and the other points as $\langle (z,1) \rangle$. The matrix $g = \mathrm{diag}(\omega, 1)$ fixes $(0,1)$ and permutes the vectors $(z,1)$, so $F(gv) = F(v)$ by symmetry, while $F(gv) = \det(g)^e F(v) = \omega^e F(v)$. Hence $\omega^e = 1$. With ζ a primitive $(n-2)$-th root of unity and the configuration formed by $x_n = 0$, ∞ and the $(n-2)$-th roots of unity, the same argument with $\mathrm{diag}(\zeta, 1)$ gives $\zeta^e = 1$. This matrix also multiplies the vector $(1,0)$ of ∞ by ζ, which F does not see, since it has degree 0 in that variable. As $n-1$ and $n-2$ are coprime, $(n-1)(n-2)$ divides e (Figure 2).



$x_6 = 0$, $x_1, \dots, x_5 \in \mu_5$ rotated by $z \mapsto \omega z$

$x_6 = 0$, $x_1 = \infty$, $x_2, \dots, x_5 \in \mu_4$ rotated by $z \mapsto \zeta z$

Figure 2. The two configurations of the lower bound for $n = 6$. The last point is 0. A rotation permutes the other points and turns the tangent vector v at 0 into ωv or ζv, with $\omega = e^{2\pi i/5}$ and $\zeta = i$. A trivialisation compatible with the renumbering must take the same value at both, so $5 \mid e$ and $4 \mid e$, hence $20 \mid e$ for L_6.

Both directions, for every n, and the corresponding statements for L, are formalised in Lean 4 with Mathlib, over any field containing the relevant roots of unity, with F allowed to be any function on the points over that field (theorems `trivL_iff` and `trivLi_iff` of `Grothendieck.Folder146`, in the repositories Commutator-IO/grothendieck-project and Commutator-IO/grothendieck-project-palomar; they belong to the Lean appendix of [6]). The dictionary between equivariant line bundles on $M_{0,n}$ and such functions is not formalised, that is, the identification of the equivariant Picard group with the first cohomology of $\mathfrak{S}$ with values in the units; Mathlib does not have the spaces $M_{0,n}$.

Statement on the use of artificial intelligence

The transcription of folder 146, the search of the literature, the general formula, its proofs, the Lean formalisation and a first draft of this text were produced with language models (Claude Opus 5 and Opus 5.5, Anthropic), under the author's direction. The author checked the proofs and the references, and takes full responsibility for the content. The formal proofs are checked mechanically by Lean.

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