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AN EXPLICIT COUNTEREXAMPLE OF DEGREE 29 440 TO SMALE'S MEAN VALUE PROBLEM WITH CONSTANT ONE

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Abstract. For a complex polynomial p with $p(0) = 0$ and $p'(0) = 1$, let $S(p,0) = \min_{p'(c)=0} |p(c)/c|$. Adamczewski and Colbrook recently gave an explicit polynomial of degree 29 984 with $S(p,0) > 1$, certified by interval arithmetic, built on the counterexample found by GPT-6 Astra. Within their family of polynomials, with re-chosen coefficients, we give an explicit $p \in \mathbb{Q}(i)[z]$ of degree 29 440 with

$$1.00000093069254310 < S(p,0) < 1.00000093069254311.$$

Jatar and Ng have shown that Smale's constant $\sup_p S(p,0)$ exceeds 1.1018, through polynomials of unspecified degree; the point here is an explicit polynomial with a certified bound. The bound is certified twice: by a ball-arithmetic Krawczyk certificate covering all 29 439 critical points, and, independently, by the Adamczewski–Colbrook verifier, which confirms 29 418 of them and is inconclusive (insufficient working precision) on the other 21. The exact coefficients and both verification programs are provided as ancillary files.

1. Introduction

Let p be a complex polynomial of degree $n \geq 2$ and $a \in \mathbb{C}$ with $p'(a) \neq 0$. Smale [5] proved that

$$S(p,a) = \min_{p'(c)=0} \left| \frac{p(c) - p(a)}{(c-a)\,p'(a)} \right| \leq 4$$

and asked whether the constant 4 can be replaced by 1, or even by $(n-1)/n$. The constant-one version was disproved by GPT-6 Astra, with a formal Lean 4 proof [1]. Adamczewski and Colbrook [2] explained the underlying mechanism and gave an explicit polynomial $p \in \mathbb{Q}(i)[z]$ of degree 29 984 with $S(p,0) > 1.00000049595801$, certified by interval arithmetic [3]. Independently, Jatar and Ng [4] reduced the determination of Smale's mean value constant $\mathcal{M} = \sup S(p,0)$, the supremum over normalized polynomials, to an extremal problem for admissible pairs of a univalent function and a finite Blaschke product, and showed that $\mathcal{M} > 1.1018$; their polynomials come from an approximation argument, and no explicit degree is given. Thus the value $1.00000093\ldots$ below is not a bound on $\mathcal{M}$; what is new here is only the degree of an explicit, certified example.

This note records a smaller explicit counterexample. It uses *the same construction* as [2, Appendix A]: the same lemniscate coordinate, model anchor, Blaschke parameter, correction centres and correction structure. Only the truncation level N, the size of the high-degree tail and the coefficient tables are changed. The reduction in degree is modest (about 1.8%); no new mechanism is claimed.

2. The polynomial

We keep the notation of [2, Appendix A]: $m = 16$, $A = 1061603/250000$, $S = A + 1/A$, $q(z) = (z^{16} + iA)/S$, the model anchor a, the Blaschke parameter r and the truncated model h_L of [2,

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(A.5)]. We take

$$N = 1840, \qquad L = N - 2 = 1838, \qquad \text{tail blocks } t = 48,$$

and set

$$P(z) = (z-a)\frac{h_L(z)}{h_L(a)} + \Delta_0(z) + X(z) + H(z),$$

$$\Delta_0 = (z-a)^2\Big(KR_0 + \sum_{j=0}^{3} \phi_j (z-\mu)^j\Big), \qquad K = (z-\xi_1)^2(z-\xi_2)^2, \quad \mu = \tfrac{\xi_1+\xi_2}{2},$$

$$X = (z-a)^2 \sum_{\ell=0}^{127}\sum_{k=0}^{15} x_{\ell k}\Big(\frac{5z}{6}\Big)^k q(z)^\ell,$$

$$H = \omega\Big(\gamma\, q(z)^N + q(z)^{N-t}\sum_{\ell=0}^{t-1}\sum_{k=0}^{15} e_{\ell k}\Big(\frac{5z}{6}\Big)^k q(z)^\ell\Big),$$

with R_0 the correction polynomial of [2] (its 1024 coefficients re-chosen), rational ω with $|\omega| = 1$ exactly, rational $\gamma > 0$, and a rational base point $\hat{a}$. The normalized polynomial is

$$p(x) = \frac{P(\hat{a}+x) - P(\hat{a})}{P'(\hat{a})}.$$

All data are exact elements of $\mathbb{Q}$ or $\mathbb{Q}(i)$ and are listed in the ancillary file `polynomial_N1840.json`, which follows the format of [3] (field `defining_formula`). In particular $p \in \mathbb{Q}(i)[z]$.

Only γq^N reaches degree $16N = 29\,440$; all other terms have degree at most $29\,439$, and the leading coefficient $\omega\gamma/S^N$ is nonzero. Hence $\deg p = 29\,440$, and $p(0) = 0$, $p'(0) = 1$ by construction.

Theorem 1. *The polynomial $p \in \mathbb{Q}(i)[z]$ above has degree $29\,440$, its $29\,439$ critical points are simple, and*

$$1.00000093069254310 < S(p,0) < 1.00000093069254311.$$

In particular p is a counterexample to Smale's mean value conjecture with constant one.

Remark 2. Adamczewski and Colbrook note [2, §1.1] that perturbations εz^n of an explicit counterexample, together with Rouché's theorem, give counterexamples in every larger degree. Applied to p, the same argument would give counterexamples in every degree $n \geq 29\,440$; we do not repeat it here, and Theorem 1 only concerns degree $29\,440$.

3. Verification

Since $S(p,0)$ is unchanged by the affine normalization, it suffices to work with P and $\hat{a}$: $S(p,0) = \min_{P'(c)=0} |Q(c)|$ with $Q(c) = \dfrac{P(c) - P(\hat{a})}{(c-\hat{a})P'(\hat{a})}$.

Certificate 1 (ball arithmetic). Arithmetic is done with complex balls (Arb, via `python-flint`) at 1280 bits, starting from the exact rationals of the specification. For each of $29\,439$ approximate critical points:

(1) three Newton steps for P' give a centre z;
(2) on the square box D of half-side ρ about z, an enclosure of $P''(D)$ is obtained from $P''(z)$ and a Cauchy bound $\sup|P'''| \leq 6B/(R-\rho\sqrt{2})^3$, where B is a rigorous majorant of $|P|$ on the disc of radius R about z (term-by-term triangle inequality);
(3) the Krawczyk operator $K(D) = z - P'(z)/P''(z) + \big(1 - P''(D)/P''(z)\big)(D - z)$ is shown to lie in the interior of D, which proves that D contains exactly one zero of P', and that it is simple;

(4) $|Q|$ is bounded below on all of D by a second-order Taylor enclosure of P with the same remainder bound, and a point counts as certified only if this lower bound exceeds 1.

All 29 439 points pass. The boxes have half-side at most $7.4 \cdot 10^{-40}$, the refined centres differ from the supplied ones by at most $7.7 \cdot 10^{-12}$, and any two supplied centres are at distance at least $6.05 \cdot 10^{-5}$; hence the boxes are pairwise disjoint. They thus contain $29\,439 = \deg P'$ distinct zeros of P', i.e. all critical points, each simple. The smallest certified value is

$$\min_{P'(c)=0} |Q(c)| \;\geq\; 1.0000009306925431017438\ldots,$$

attained in a single box, which also gives the upper bound of Theorem 1.

Certificate 2 (Adamczewski–Colbrook verifier). The program `verify.cpp` of [3] (outward-rounded interval bounds with MPFR-directed endpoints, Rouché test on critical disks) was recompiled with `N_PARAM=1840`, `TAIL_BLOCKS_PARAM=48`, the expected parameters in `make_parameters.py` being read from the specification; the specification passes all its exact-rational consistency checks. Run disk by disk on the 29 439 critical disks, it certifies $|Q| > 1$ on 29 418 of them, with smallest lower bound $1.000000930692128\ldots$, on the same critical point as Certificate 1. On the remaining 21 disks (indices, in the order of `critical_disks_N1840.csv`, listed in the ancillary summary) its fixed working precision does not suffice to conclude; these are covered by Certificate 1 only.

	[2]	this note
N	1874	1840
degree	29 984	29 440
tail blocks	8	48
certified lower bound for $S(p, 0)$	1.00000049595801	1.00000093069254310

4. Ancillary files

- `polynomial_N1840.json`: exact specification ($\mathbb{Q}$ and $\mathbb{Q}(i)$ data).
- `centres_N1840.csv`: the 29 439 supplied critical-point centres, sorted by increasing $|Q|$.
- `verify_ball_N1840.py`: Certificate 1 for any range of critical points (Python with `python-flint` and `numpy`).
- `certificate_N1840.log`: summary of both certificates.
- `critical_disks_N1840.csv`, `make_parameters_N1840.py`: input disks and the modified parameter script used to run the verifier of [3] (Apache License 2.0).

Status and acknowledgements

Theorem 1 is a computer-assisted result: its proof is the pair of certificates above, which anyone can rerun from the ancillary files. The construction is entirely that of Adamczewski and Colbrook [2], building on the counterexample found by GPT-6 Astra [1]; only parameters were changed. The computations were carried out with the assistance of Claude (Anthropic).

References

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