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COUNTEREXAMPLES TO SOURCE–SINK RECURSIONS FOR DENOMINATOR VECTORS

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Abstract. Reading and Stella's source–sink recursions for denominator vectors, Conjectures 2.7 and 2.8, fail for an infinite family of four-vertex exchange matrices. The member realized by an unpunctured annulus with one marked point on one boundary and three on the other also refutes their Theorem 2.11.

1. Introduction

Fomin and Zelevinsky introduced cluster algebras in their study of total positivity and canonical bases [3]. The Laurent phenomenon [3, Theorem 3.1] expresses each cluster variable as a Laurent polynomial in any cluster. The denominator vector records the negatives of the least exponents in that expansion. In finite type, Fomin and Zelevinsky identified the denominator vectors for a distinguished initial cluster with the simple-root coordinates of the positive roots and of the negatives of the simple roots [4, Theorem 1.9]. Ceballos and Pilaud described the vectors with respect to arbitrary initial clusters in finite type by compatibility degrees, integers attached to pairs of almost positive roots [1, Corollary 3.2]. For cluster algebras from marked surfaces, Fomin, Shapiro, and Thurston expressed the coordinates as intersection numbers of tagged arcs [2, Theorem 8.6].

With the initial cluster fixed, mutation of the target seed replaces one variable whose denominator vector is to be computed. Fomin and Zelevinsky gave a recursion for this operation [5, Section 7, Eq. (7.7)]. Mutating the initial seed with the target variable fixed changes the variables in which its Laurent expansion is written. Fomin and Zelevinsky discussed the difficulty of finding a simple recursion for this change after Remark 7.7 of [5].

Reading and Stella [7], whose first preprint appeared in 2014, observed that the denominator coordinate for the replacement variable equals the highest exponent of the old variable in the original Laurent expansion [7, Proposition 2.5]. They showed that an initial-seed recursion, a formula for highest exponents in terms of denominator vectors, and a denominator-matrix duality are equivalent as identities for all seeds [7, Theorem 2.1]. These identities hold in rank two, in finite type, and for the surfaces classified in [7, Theorems 2.2–2.4], including the unpunctured annuli with at most two marked points on each boundary. In the presence of at least three marks on one boundary, the correction for a loop, an arc whose ends coincide, can make the intersection number asymmetric [7, Fig. 5, right].

Reading and Stella's Conjectures 2.7 and 2.8 concern initial mutation at a source or sink, where the arrows at the vertex all point out or all point in. Rupel and Stella established the recursions when the initial exchange matrix is acyclic, using reflection functors [8, Corollary 10 and the discussion following it]. Reading and Stella asserted both recursions for all marked surfaces in [7, Theorem 2.11].

Fu and Keller's three-vertex example [6, Example 6.7] refutes the proposed identification of denominator vectors with degrees of the associated F-polynomials in the individual coefficient variables [5, Conjecture 7.17]. These polynomials are obtained from cluster variables with principal coefficients by setting the initial cluster variables to one. Yurikusa

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realized the example on an unpunctured annulus with three marked points and computed the correction to the crossing count for a loop [9, Examples 2.14, 3.4, and 4.4].

We attach a fourth vertex to the third by m parallel arrows, where m is a positive integer. An initial mutation at the fourth vertex gives denominator coordinate $2m$ for the target variable constructed below, whereas both conjectures give m. For $m = 1$, the added vertex is represented by the arc along which one triangle is attached to Yurikusa's annulus.

2. Initial-seed recursions

Fix a positive integer n. A coefficient-free seed $(\mathbf{x}, B)$ consists of a cluster $\mathbf{x} = (x_1, \dots, x_n)$, an ordered transcendence basis of the ambient rational function field over $\mathbb{Q}$, and an integral $n \times n$ exchange matrix $B = (b_{ij})$. The matrix is skew-symmetrizable, so there are positive integers $s_1, \dots, s_n$ with $s_i b_{ij} = -s_j b_{ji}$ for all $1 \le i, j \le n$. In particular, $b_{ii} = 0$. Write $[a]_+ = \max(a, 0)$ for any integer a.

For $1 \le k \le n$, mutation μ_k replaces B by the matrix with entries

$$(\mu_k B)_{ij} = \begin{cases} -b_{ij}, & i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [-b_{ik}]_+[-b_{kj}]_+, & i, j \ne k, \end{cases} \tag{1}$$

and replaces x_k by x'_k, where

$$x_k x'_k = \prod_{i=1}^{n} x_i^{[b_{ik}]_+} + \prod_{i=1}^{n} x_i^{[-b_{ik}]_+}. \tag{2}$$

The other variables retain their positions. A product with all exponents zero is 1.

The matrix mutation preserves the weights s_i. For $i, j \ne k$, we have

$$s_i[b_{ik}]_+[b_{kj}]_+ = s_j[-b_{jk}]_+[-b_{ki}]_+,$$

using $s_i[b_{ik}]_+ = s_k[-b_{ki}]_+$ and $s_k[b_{kj}]_+ = s_j[-b_{jk}]_+$. Reversing all signs yields the equality for the other correction product. The two corrections at transposed positions therefore have opposite weighted sums. For $i = k$ or $j = k$, negating both transposed entries preserves the weighted skew relation.

The right-hand side of (2) is nonzero and belongs to the field generated by the unchanged variables. Dividing it by x'_k recovers x_k, so x'_k is transcendental over that field and the new tuple generates the same ambient field. A second mutation at k restores the incident entries and interchanges the two correction products at every other position. It also interchanges the monomials on the right-hand side of (2), so the second mutation restores x_k. Hence μ_k is involutive. The entries of the clusters reached by finite mutation sequences are the cluster variables. They generate the coefficient-free cluster algebra over $\mathbb{Z}$.

By the Laurent phenomenon [3, Theorem 3.1], each cluster variable X is a Laurent polynomial in $\mathbf{x}$. Its denominator vector $\mathbf{d} = (d_1, \dots, d_n)^{\mathsf{T}}$ is defined by taking d_i to be the negative of the least exponent of x_i among the monomials with nonzero coefficient. If a polynomial numerator contains a term omitting each variable in turn, its least exponent in each variable is zero. Dividing such a numerator by a Laurent monomial gives a denominator vector equal to the exponent vector of that monomial. For example, the denominator vector of x_i has entry -1 at i and zero elsewhere.

Fix an ordered target cluster containing X. Its denominator matrix has the denominator vectors relative to the initial cluster as columns, in the order of the target variables.

Suppose the entries in row k of B are all nonnegative or all nonpositive. This is the source–sink condition, with nonnegative row entries corresponding to a source in the convention that $b_{ij} > 0$ gives an arrow from i to j. Denote by $\mathbf{d}' = (d'_1, \dots, d'_n)^{\mathsf{T}}$ the denominator vector of the same X relative to the cluster of $\mu_k(\mathbf{x}, B)$. Keep the target cluster and its order fixed. Both conjectures in [7, p. 186] predict $d'_i = d_i$ for $i \ne k$. Their

predictions for the column of X at coordinate k are

$$d'_k = -d_k + \left[\sum_{j=1}^{n} |b_{kj}| d_j\right]_+ \qquad \text{(Conjecture 2.7)}, \tag{3}$$

$$d'_k = -d_k + \sum_{j=1}^{n} |b_{kj}| [d_j]_+ \qquad \text{(Conjecture 2.8)}. \tag{4}$$

3. A family of counterexamples

Take $n = 4$. For each integer $m \geq 1$, let the initial seed have cluster (x_1, x_2, x_3, x_4) and exchange matrix

$$B_m = \begin{pmatrix} 0 & 2 & -1 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & -1 & 0 & -m \\ 0 & 0 & m & 0 \end{pmatrix}. \tag{5}$$

This matrix is skew-symmetric. Its nonzero off-diagonal entries connect vertices $1, 2, 3$ in a triangle and join 4 to 3. Expansion along row 4 and then the last column of its minor gives $\det B_m = (-m)(-m)\big(0 \cdot 0 - 2(-2)\big) = 4m^2$. Let X_m be the variable created by the third mutation in the sequence $3, 2, 1$, taken in chronological order.

Theorem 1. *For every integer $m \geq 1$, the denominator vector of X_m is $(1, 1, 1, 0)^{\mathsf{T}}$ in the original initial cluster and $(1, 1, 1, 2m)^{\mathsf{T}}$ in the initial cluster obtained by mutation at 4. Both (3) and (4) fail for this mutation.*

Proof. Column 3 of B_m is $(-1, 1, 0, m)^{\mathsf{T}}$. After mutation at 3, the incident entries are negated. For the three upper-triangular positions off row and column 3, the two correction products in (1) give

$$(1,2)\colon 2 + 0 - 1 = 1, \qquad (1,4)\colon 0 + 0 - m = -m, \qquad (2,4)\colon 0 + 0 - 0 = 0.$$

Thus the matrix after the first step is

$$\begin{pmatrix} 0 & 1 & 1 & -m \\ -1 & 0 & -1 & 0 \\ -1 & 1 & 0 & m \\ m & 0 & -m & 0 \end{pmatrix}.$$

The second mutation uses column 2, which is $(1, 0, 1, 0)^{\mathsf{T}}$. It negates the entry -1 at $(2, 1)$. The two correction products at $(3, 1)$ vanish because the entries at $(3, 2), (2, 1)$ have opposite signs, and those at $(4, 1)$ vanish because the entry at $(4, 2)$ is zero. Thus column 1 after the second mutation is $(0, 1, -1, m)^{\mathsf{T}}$.

Let u and v be the variables created at the first and second mutations. The three columns just computed give the exchange equations

$$x_3 u = x_1 + x_2 x_4^m, \qquad x_2 v = 1 + x_1 u, \qquad x_1 X_m = u + x_4^m v.$$

Substituting the first two equations into the third and collecting terms gives

$$\begin{aligned} X_m &= \frac{x_2(x_1 + x_2 x_4^m) + x_4^m\big(x_3 + x_1(x_1 + x_2 x_4^m)\big)}{x_1 x_2 x_3} \\ &= \frac{x_1 x_2 x_4^{2m} + (x_1^2 + x_2^2 + x_3) x_4^m + x_1 x_2}{x_1 x_2 x_3}. \end{aligned} \tag{6}$$

All coefficients in the numerator are positive. The terms $x_2^2 x_4^m$, $x_1^2 x_4^m$, and $x_1 x_2$ omit x_1, x_2, and both x_3 and x_4, respectively. Thus the denominator vector is $(1, 1, 1, 0)^{\mathsf{T}}$.

Let z be the variable that replaces x_4 in the initial mutation at 4. Since column 4 has sole nonzero entry $-m$ in row 3, (2) gives $x_4 z = 1 + x_3^m$. The target cluster (X_m, v, u, x_4)

is reached from the new initial seed by $4, 3, 2, 1$, since the first step reverses the initial mutation. Substituting $x_4 = (1 + x_3^m)/z$ into (6) gives

$$X_m = \frac{x_1x_2(1 + x_3^m)^{2m} + (x_1^2 + x_2^2 + x_3)(1 + x_3^m)^m z^m + x_1x_2z^{2m}}{x_1x_2x_3z^{2m}}. \tag{7}$$

The constant term in z of the numerator is $x_1x_2(1 + x_3^m)^{2m} \neq 0$. The terms $x_2^2z^m$, $x_1^2z^m$, and x_1x_2 occur with positive coefficients and omit x_1, x_2, and x_3, respectively. Hence the new denominator vector is $(1, 1, 1, 2m)^\mathsf{T}$.

Row 4 of B_m is $(0, 0, m, 0)$ and is nonnegative. With $d_4 = 0$ and $d_3 = 1$, (3) predicts $[m \cdot 1]_+ = m$, and (4) predicts $m[1]_+ = m$. The new coordinate is $2m \neq m$. □

4. The marked annulus

In an annulus with boundary marked points, an arc is a curve with endpoints at marks and interior off the boundary. It is embedded except that its endpoints may coincide. Curves are taken up to isotopy fixing the marks. Arcs isotopic to a boundary segment and curves cutting out an unpunctured monogon or digon are excluded [2, Definition 2.2]. An ideal triangulation is a maximal collection of distinct arcs admitting representatives with pairwise disjoint interiors [2, Definition 2.6]. When all triangles have distinct sides, the signed adjacency matrix is indexed by the interior arcs. Each triangle adds 1 at (i, j) and -1 at (j, i) if interior side j immediately follows interior side i clockwise in its list of three sides. Pairs involving boundary segments are ignored [2, Definition 4.1].

Proposition 2. *At $m = 1$, the matrix (5) is the signed adjacency matrix of a triangulation of an unpunctured annulus with one marked point on one boundary and three on the other. The two recursions fail for this triangulation, contradicting* [7, Theorem 2.11].

Proof. Take four oriented triangles with clockwise side lists

$$(1, 2, a), \qquad (1, 2, 3), \qquad (4, 3, b), \qquad (4, c, d).$$

Glue the two copies of each numbered side by reversing their boundary directions and leave the lettered sides unglued. Label the corners of the four triangles A_i, C_i, D_i, E_i, respectively, with $i = 0, 1, 2$ in clockwise order. The first side in each list runs from corner 0 to corner 1. Each corner contributes an interval to the link of its image vertex. In the following chains, the integer above each $\sim$ indicates which side gluing joins the two corner intervals. The left column names the quotient vertices.

$$\begin{array}{c|l} P & A_0 \overset{1}{\sim} C_1 \overset{2}{\sim} A_2 \\ Q & D_2 \overset{3}{\sim} C_2 \overset{2}{\sim} A_1 \overset{1}{\sim} C_0 \overset{3}{\sim} D_1 \overset{4}{\sim} E_0 \\ R & D_0 \overset{4}{\sim} E_1 \\ S & E_2 \end{array}$$

All four links are intervals, giving half-disk neighborhoods at the vertices. Away from the vertices, each glued edge joins two half-disks and each unglued edge has one half-disk. The quotient is thus a surface. It is connected by the numbered side gluings and oriented because they reverse boundary direction. Its boundary is the circle a at P and the cycle b, c, d through Q, R, S in that order. Its Euler characteristic is $4 - 8 + 4 = 0$, from four vertices, eight edges, and four faces. A connected compact orientable surface with two boundary components and Euler characteristic zero is an annulus. Its four marked points lie on the boundary.

The numbered edges have pairwise disjoint interiors in the surface. Edges $1, 2$ join the two boundary components, from P to Q. By the first triangle, traversing one and returning along the other gives a loop homotopic to a up to orientation. This loop is essential, so $1, 2$ are distinct up to isotopy relative to the marks.

Edge 3 is a loop at Q. Cutting along it separates the first two triangles from the last two. The first pair has two vertices, four edges, and two faces, and its boundary circles are $a, 3$.

The links of its two vertices are intervals, given by the chains A_0, C_1, A_2 and C_2, A_1, C_0. It is an annulus. The last pair is glued along 4 and forms a disk with sides $3, b, c, d$. Thus 3 is essential in the full annulus, since it is homotopic to a through the first pair. Its disk side has four sides, so it cuts out neither a monogon nor a digon. No single boundary segment at Q is a loop.

Edge 4 joins R to Q. It cuts off the triangle with sides $4, c, d$, with S between c and d. The rest of the surface contains the annular part, so neither side of edge 4 is a monogon or a digon. The only boundary segment with the same endpoints as 4 is b. Traversing 4 and returning along b gives a loop homotopic up to orientation to 3 by the triangle $(4, 3, b)$. Thus 4 is an arc distinct from b, and its endpoints distinguish it from $1, 2, 3$.

Cutting along the four arcs gives the four original triangles, each with three distinct sides. Any additional arc with interior disjoint from the four arcs would lie in one cut triangle. In that disk it would contract if its endpoints lift to the same corner and be isotopic to a side if they lift to distinct corners. This proves maximality.

The first triangle adds 1 at $(1, 2)$. The second adds 1 at each of $(1, 2), (2, 3), (3, 1)$. The third adds 1 at $(4, 3)$, and the fourth has only one numbered side and contributes zero. Each addition is accompanied by a subtraction at the transposed position. The signed adjacency matrix is therefore B_1. By the surface cluster algebra construction [2, Section 5], this matrix is the initial exchange matrix for the coefficient-free choice. Theorem 1 gives the two failures for this seed. □

Deleting the triangle $(4, c, d)$ and treating 4 as a boundary segment recovers Yurikusa's triangulation with three interior arcs, up to relabeling [9, Examples 2.14 and 4.4]. On p. 201 of [7], the proof of Theorem 2.11 discards the loop correction on the premise that the arcs sharing a triangle with the source have distinct endpoints. In this triangulation, 4 is a source since $b_{43} = 1$ is the only nonzero entry in row 4. The triangle $(4, 3, b)$ contains both 4 and the loop 3, whose two endpoints are identified at Q.

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